Assignment 2 - Ruin to Returns

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Question 1(a): Derivation of Winning Probability

Let W_k be the probability that the gambler wins the game (reaches a wealth of N) given an initial wealth of k: - p: probability of increasing wealth by 1 (from k to $k+1$). - $q = 1 - p$: probability of decreasing wealth by 1 (from k to $k-1$).

Step 1: Establishing Boundary Conditions

The boundary conditions are:

- $W_0 = 0$: Starting with zero wealth, the probability of reaching N is zero (immediate ruin).
- $W_N = 1$: Upon reaching wealth N, the probability of winning is one (already achieved goal).

Step 2: Deriving the Recurrence Relation

For any intermediate wealth k where $0 < k < N$:

- With probability p, wealth increases to $k+1$, leaving probability W_{k+1} to win
- With probability q, wealth decreases to $k-1$, leaving probability W_{k-1} to win

Therefore:

$$W_k = p \cdot W_{k+1} + q \cdot W_{k-1}$$

Step 3: Transforming the Recurrence Relation

Rearrange to standard form:

$$p \cdot W_{k+1} - W_k + q \cdot W_{k-1} = 0$$

Divide throughout by p (assuming $p \neq 0$):

$$W_{k+1} - \frac{1}{p} W_k + \frac{q}{p} W_{k-1} = 0$$

Step 4: Solving the Homogeneous Equation

Let's try a solution of the form $W_k = r^k$:

$$r^{k+1} - \frac{1}{p} r^k + \frac{q}{p} r^{k-1} = 0$$

Divide by r^{k-1}:

$$r^2 - \frac{1}{p} r + \frac{q}{p} = 0$$

This quadratic equation has roots $r = 1$ and $r = \frac{q}{p}$
Therefore, the general solution is:

$$W_k = A + B \left(\frac{q}{p} \right)^k$$

where A and B are constants to be determined.

Step 5: Applying Boundary Conditions

Using $W_0 = 0$:

$$0 = A + B \implies A = -B$$

Using $W_N = 1$:

$$1 = A + B\left(\frac{q}{p}\right)^N = -B + B\left(\frac{q}{p}\right)^N$$

Therefore:

$$B\left[\left(\frac{q}{p}\right)^N - 1\right] = 1$$

$$B = \frac{1}{1 - \left(\frac{q}{p}\right)^N}$$

$$A = -\frac{1}{1 - \left(\frac{q}{p}\right)^N}$$

Step 6: Final Solution

Substituting back:

$$W_k = \frac{1 - \left(\frac{q}{p}\right)^k}{1 - \left(\frac{q}{p}\right)^N}$$

Special Case: Unbiased Game ($p = q = 0.5$)

When $p = q = 0.5$, $\frac{q}{p} = 1$, we need to take the limit:

$$\lim_{p \to 0.5} W_k = \lim_{p \to 0.5} \frac{1 - \left(\frac{1-p}{p}\right)^k}{1 - \left(\frac{1-p}{p}\right)^N} = \frac{k}{N}$$

This can be verified using L'Hôpital's rule or by solving the recurrence relation directly with $p = q = 0.5$.

Question 1(b): Probability of Winning with Infinite Wealth Goal

Step 1: Taking the Limit

Starting with the solution from part (a):

$$W_k = \frac{1 - \left(\frac{q}{p}\right)^k}{1 - \left(\frac{q}{p}\right)^N}$$

As $N \to \infty$, we need to evaluate:

$$W_k^{\infty} = \lim_{N \to \infty} \frac{1 - \left(\frac{q}{p}\right)^k}{1 - \left(\frac{q}{p}\right)^N}$$

Step 2: Case Analysis

Case 1: $p > q$

When $p > q$, $\frac{q}{p} < 1$

$$\lim_{N \to \infty} \left(\frac{q}{p}\right)^N = 0$$

Therefore:

$$W_k^{\infty} = 1 - \left(\frac{q}{p}\right)^k$$

Case 2: $p = q$

When $p = q = 0.5$, $\frac{q}{p} = 1$

$$W_k^{\infty} = \lim_{N \to \infty} \frac{k}{N} = 0$$

Case 3: $p < q$

When $p < q$, $\frac{q}{p} > 1$

$$\lim_{N \to \infty} \left(\frac{q}{p}\right)^N = \infty$$

Therefore:

$$W_k^{\infty} = 0$$

Final Solution

$$W_k^{\infty} = \begin{cases} 1 - \left(\frac{q}{p}\right)^k & \text{if } p > q \\ 0 & \text{if } p \leq q \end{cases}$$

Question 1(c): Expected Number of Rounds Until Ruin or Win

Let E_k denote the expected number of rounds until either ruin or win, starting with wealth k.

Step 1: Establishing Boundary Conditions

- $E_0 = 0$: At ruin (wealth = 0), game ends immediately
- $E_N = 0$: At win (wealth = N), game ends immediately

Step 2: Deriving Recurrence Relation

For any intermediate wealth k where $0 < k < N$:

- Current round takes 1 step
- With probability p, wealth becomes $k + 1$, leading to E_{k+1} expected additional steps
- With probability q, wealth becomes $k - 1$, leading to E_{k-1} expected additional steps

Therefore:

$$E_k = 1 + p \cdot E_{k+1} + q \cdot E_{k-1}$$

Step 3: Converting to Standard Form

Rearrange the equation:

$$p \cdot E_{k+1} - E_k + q \cdot E_{k-1} = -1$$

This is a non-homogeneous second-order difference equation.

Step 4: Finding the General Solution

The complete solution will be:

$$E_k = E_k^{(h)} + E_k^{(p)}$$

where $E_k^{(h)}$ is the homogeneous solution and $E_k^{(p)}$ is a particular solution.

Step 4.1: Homogeneous Solution

Solve $p \cdot E_{k+1} - E_k + q \cdot E_{k-1} = 0$

Try $E_k^{(h)} = r^k$:

$$pr^{k+1} - r^k + qr^{k-1} = 0$$

$$pr^2 - r + q = 0$$

The roots are $r = 1$ and $r = \frac{q}{p}$

Therefore:

$$E_k^{(h)} = A + B\left(\frac{q}{p}\right)^k$$

Step 4.2: Particular Solution

For the non-homogeneous part, try $E_k^{(p)} = \alpha k$ where α is a constant.

Substitute into the original equation:

$$p\alpha(k+1) - \alpha k + q\alpha(k-1) = -1$$

$$p\alpha + q\alpha = -1$$

$$\alpha(p+q) = -1$$

$$\alpha = -\frac{1}{p+q} = -1$$

Therefore:

$$E_k^{(p)} = -k$$

Step 5: Combining Solutions

The general solution is:

$$E_k = A + B\left(\frac{q}{p}\right)^k - k$$

Step 6: Applying Boundary Conditions

Using $E_0 = 0$:

$$0 = A + B - 0$$

$$A = -B$$

Using $E_N = 0$:

$$0 = A + B\left(\frac{q}{p}\right)^N - N$$

$$0 = -B + B\left(\frac{q}{p}\right)^N - N$$

$$B\left[\left(\frac{q}{p}\right)^N - 1\right] = N$$

$$B = \frac{N}{1 - \left(\frac{q}{p}\right)^N}$$

Therefore:

$$A = -\frac{N}{1 - \left(\frac{q}{p}\right)^N}$$

Step 7: Final Solution

For $p \neq q$ (biased game):

$$E_k = \frac{N}{1 - \left(\frac{q}{p}\right)^N} \left[\left(\frac{q}{p}\right)^k - 1 \right] - k$$

This can be rearranged to:

$$E_k = \frac{1 - \left(\frac{q}{p}\right)^k}{1 - \left(\frac{q}{p}\right)^N} \cdot \frac{N}{p - q} + \frac{k}{q - p}$$

Step 8: Special Case (Unbiased Game)

For $p = q = \frac{1}{2}$, taking the limit as $p \to \frac{1}{2}$:

$$\lim_{p \to \frac{1}{2}} E_k = k(N - k)$$

This can be verified by solving the original recurrence relation directly with $p = q = \frac{1}{2}$.

Final Result

The expected number of rounds E_k until either ruin or win is:

$$E_k = \begin{cases} \frac{1 - \left(\frac{q}{p}\right)^k}{1 - \left(\frac{q}{p}\right)^N} \cdot \frac{N}{p-q} + \frac{k}{q-p}, & \text{if } p \neq \frac{1}{2} \text{ (biased case)} \\ k(N - k), & \text{if } p = \frac{1}{2} \text{ (unbiased case)} \end{cases}$$

Question 2: Aggressive Betting Strategy

Question 2(a): Probability of Winning with Aggressive Betting

Let W_k denote the probability of winning when starting with wealth k under the aggressive betting strategy. We derive this recursive relationship by analyzing how the gambler's wealth changes after each bet based on their current position.

Cases

$$W_k = \begin{cases} 0 & \text{if } k = 0 \\ 1 & \text{if } k \geq N \\ p \cdot W_{2k} & \text{if } k < N/2 \\ p + q \cdot W_{2k-N} & \text{if } k \geq N/2 \end{cases}$$

where:

- p: probability of winning a round
- $q = 1 - p$: probability of losing a round
- k: current wealth
- N: target wealth

Detailed Derivation:

1. **Base Cases:**

- When $k = 0$: The gambler has lost everything, making it impossible to reach the target. Hence, $W_0 = 0$
- When $k \geq N$: The gambler has reached or exceeded the target. Therefore, $W_k = 1$

2. **Case:** $k < N/2$ In this case:

- The gambler bets their entire wealth k
- If they win (probability p): Wealth becomes $2k$
- If they lose (probability q): Wealth becomes 0
- Therefore: $W_k = p \cdot W_{2k} + q \cdot W_0 = p \cdot W_{2k}$ (since $W_0 = 0$)

3. **Case:** $k \geq N/2$ In this case:

- The gambler bets $(N - k)$
- If they win (probability p): Wealth becomes $k + (N - k) = N$, leading to certain victory
- If they lose (probability q): Wealth becomes $k - (N - k) = 2k - N$
- Therefore: $W_k = p \cdot 1 + q \cdot W_{2k-N} = p + q \cdot W_{2k-N}$

Question 2(b): Expected Duration of the Game

Let E_k denote the expected number of rounds until the game ends when starting with wealth k. This represents the average number of bets needed until either reaching the target or going bankrupt.

Cases

$$E_k = \begin{cases} 0 & \text{if } k = 0 \text{ or } k = N \\ 1 & \text{if } k = N/2 \\ 1 + p \cdot E_{2k} & \text{if } k < N/2 \\ 1 + q \cdot E_{2k-N} & \text{if } k \geq N/2 \end{cases}$$

Detailed Derivation:

1. **Base Cases:**

- When $k = 0$ or $k = N$: Game ends immediately as either ruin or target is reached, hence $E_k = 0$
- When $k = N/2$: Special case where: - If win: Reach target N immediately - If lose: Reach ruin 0 immediately - Therefore, exactly one round needed: $E_{N/2} = 1$

2. **Case:** $k < N/2$

- Current round counts as 1
- After betting k: - Win (probability p): Need additional E_{2k} expected rounds - Lose (probability q): Game ends (0 additional rounds)
- Therefore: $E_k = 1 + p \cdot E_{2k} + q \cdot 0 = 1 + p \cdot E_{2k}$

3. **Case:** $k \geq N/2$

- Current round counts as 1
- After betting $(N - k)$: - Win (probability p): Game ends (0 additional rounds) - Lose (probability q): Need additional E_{2k-N} expected rounds
- Therefore: $E_k = 1 + p \cdot 0 + q \cdot E_{2k-N} = 1 + q \cdot E_{2k-N}$

The solution requires dynamic programming with memoization due to the recursive nature of both W_k and E_k, where the same subproblems are encountered multiple times during calculation.

Question 3: Gambling with Wealth Ceiling

Consider a gambler with initial wealth k who faces a unique constraint: after reaching any wealth level m, they cannot exceed m+W wealth. The objective is to find the expected number of rounds until reaching wealth t.

Problem Setup

Let E(x) denote the expected number of additional rounds needed starting from wealth x to reach wealth t, where:

- k: initial wealth
- t: target wealth (stopping point)
- W: maximum additional wealth allowed after reaching any level
- p: probability of winning each round
- q = 1-p: probability of losing each round

Step 1: Establishing the Recurrence Relation

For any wealth level x where $x > t$, and m is the highest wealth previously achieved:

$$E(x) = \begin{cases} 1 + pE(x+1) + qE(x-1) & \text{if } x < m + W \\ 1 + E(x-1) & \text{if } x = m + W \end{cases}$$

Analysis:

- When below ceiling ($x < m + W$): Normal gambling behavior
- At ceiling (x = m+W): Forced loss, leading to $x - 1$
- The '1' represents the current round being played

Step 2: The Ceiling Effect

At the ceiling wealth m+W:

$$E(m + W) = 1 + E(m + W - 1)$$

This creates two important conditions:

1. A forced decrease in wealth at the ceiling
2. A periodic pattern in expected values near ceiling

Step 3: Regional Analysis

Let's analyze the behavior in different regions:

1. Below Ceiling Region $x < m + W$:

$$E(x) = 1 + pE(x + 1) + qE(x - 1)$$

2. At Ceiling (x = m+W):

$$E(x) = 1 + E(x - 1)$$

Define $D(x) = E(x) - E(x - 1)$. Then:

- For $x < m + W$: $D(x)$ follows a geometric pattern
- At x = m+W: $D(m + W) = 1$

Step 4: Solution Structure

For $x < m + W$, the solution has the form:

$$D(x) = cr^x \text{ where } r = \frac{q}{p}$$

This satisfies the recurrence because:

1. It's consistent with the below-ceiling behavior
2. At $x = m + W$, the ceiling condition $D(m + W) = 1$ gives:

$$cr^{m+W} = 1 \implies c = \frac{1}{r^{m+W}}$$

Step 5: Deriving Expected Value

Using the relationship $E(x) - E(x - 1) = D(x)$:

$$E(x) = E(0) + \sum_{i=1}^{x} D(i)$$

For the sum of differences:

$$\sum_{i=1}^{x} D(i) = \sum_{i=1}^{x} \frac{r^{i-m-W}}{1-r} = \frac{1}{r^{m+W}} \cdot \frac{1-r^x}{1-r}$$

Step 6: Final Formula

The expected number of rounds is:

$$E = \frac{1 + \frac{1}{p}\frac{1-r^W}{1-r}}{r^W} \cdot (k - t)$$

Term Analysis:

- $\frac{1}{r^W}$: Accounts for the ceiling effect
- $\frac{1-r^W}{1-r}$: Sum of geometric series up to ceiling
- $\frac{1}{p}$: Adjustment for win probability
- (k-t): Total wealth distance to traverse

Implementation Note

The final answer should be expressed as:

$$\text{Answer} \equiv (k - t) \cdot \frac{1 + \frac{1}{p}\frac{1-r^W}{1-r}}{r^W} \pmod{M}$$

Key Implementation Points:

- Calculate $r = q/p$ first
- Compute geometric series sum separately to avoid overflow
- Apply modulo operation last
- Handle edge cases where $p = 0.5$ by finding limit for $r \to 1$

Question 4: Stock Price Markov Chain Analysis

Problem Setup

Consider a Markov chain modeling stock prices with state space $\{0, 1, 2, \ldots, N\}$ where:

- p_k: probability of transitioning from state k to $k+1$ (price increase)
- q_k: probability of transitioning from state k to $k-1$ (price decrease)
- r_k: probability of remaining in state k (price stays same)
- $p_k + r_k + q_k = 1$ for all k (probability axiom)
- $q_0 = p_N = 0$ (boundary conditions preventing transitions beyond range)

Part (a): Stationary Distribution

Step 1: Balance Equations

For a stationary distribution $\pi = (\pi_0, \pi_1, \ldots, \pi_N)$, we need the global balance equations:

For state 0 (can only stay or increase):

$$\pi_0 = \pi_0 r_0 + \pi_1 q_1$$

For state N (can only stay or decrease):

$$\pi_N = \pi_{N-1} p_{N-1} + \pi_N r_N$$

For states $0 < k < N$ (can move both ways):

$$\pi_k = \pi_{k-1} p_{k-1} + \pi_k r_k + \pi_{k+1} q_{k+1}$$

Analysis of Balance Equations:

- Left side: probability of being in state k
- Right side: sum of probabilities of reaching k from all possible states
- Each equation represents flow conservation in steady state

Step 2: Detailed Balance Method

For birth-death processes like this, we can use detailed balance:

$$\pi_k p_k = \pi_{k+1} q_{k+1} \text{ for all } k$$

Justification:

- In steady state, flow between any two adjacent states must balance
- Forward flow ($\pi_k p_k$) equals backward flow ($\pi_{k+1} q_{k+1}$)
- This simplifies solution by providing local balance conditions

Step 3: Recursive Solution

From detailed balance, we get:

$$\pi_{k+1} = \pi_k \frac{p_k}{q_{k+1}}$$

Applying this recursively starting from π_0:

$$\pi_k = \pi_0 \prod_{i=0}^{k-1} \frac{p_i}{q_{i+1}}$$

Verification:

- This satisfies detailed balance by construction
- Each π_k is expressed in terms of π_0
- Product term represents cumulative transition probabilities

Step 4: Normalization

Using the probability axiom $\sum_{k=0}^{N} \pi_k = 1$:

$$\pi_0 + \pi_0 \sum_{k=1}^{N} \prod_{i=0}^{k-1} \frac{p_i}{q_{i+1}} = 1$$

Solving for π_0:

$$\pi_0 = \frac{1}{1 + \sum_{k=1}^{N} \prod_{i=0}^{k-1} \frac{p_i}{q_{i+1}}}$$

Complete Solution: For any k:

$$\pi_k = \frac{\prod_{i=0}^{k-1} \frac{p_i}{q_{i+1}}}{1 + \sum_{j=1}^{N} \prod_{i=0}^{j-1} \frac{p_i}{q_{i+1}}}$$

Step 5: Expected Price

The expected stock price in steady state:

$$E[X] = \sum_{k=0}^{N} k\pi_k$$

Part (b): Expected Hitting Time

Step 1: Fundamental Equations

Let $h_{a,b}$ be the expected hitting time to reach state b starting from state a. For $a \neq b$, we have:

$$h_{a,b} = 1 + p_a h_{a+1,b} + r_a h_{a,b} + q_a h_{a-1,b}$$

Explanation:

- The 1 accounts for the time spent in the current state.
- $p_a h_{a+1,b}$: Probability of moving to state $a+1$ and the expected hitting time from there.
- $r_a h_{a,b}$: Probability of staying in state a and the expected hitting time from there.
- $q_a h_{a-1,b}$: Probability of moving to state $a-1$ and the expected hitting time from there.

Step 2: Rearranging the Equation

Rearrange the equation to isolate $h_{a,b}$:

$$h_{a,b} - r_a h_{a,b} = 1 + p_a h_{a+1,b} + q_a h_{a-1,b}$$

$$h_{a,b}(1 - r_a) = 1 + p_a h_{a+1,b} + q_a h_{a-1,b}$$

Therefore:

$$h_{a,b} = \frac{1 + p_a h_{a+1,b} + q_a h_{a-1,b}}{1 - r_a}$$

Step 3: Boundary Conditions

Boundary conditions simplify the computations:

- $h_{b,b} = 0$ (If you start at b, no time is needed to reach b).
- For $a = 0$:

  $$h_{0,b} = 1 + r_0 h_{0,b} + p_0 h_{1,b}$$

  which simplifies to:

  $$h_{0,b}(1 - r_0) = 1 + p_0 h_{1,b}$$

- For $a = N$:

  $$h_{N,b} = 1 + r_N h_{N,b} + q_N h_{N-1,b}$$

  which simplifies to:

  $$h_{N,b}(1 - r_N) = 1 + q_N h_{N-1,b}$$

Step 4: Solving the Recurrence

To solve $h_{a,b}$ for $0 \leq a < b \leq N$, consider the sequence of expected hitting times:

$$h_{a,b} = \frac{1}{1 - r_a} + \frac{p_a}{1 - r_a} h_{a+1,b} + \frac{q_a}{1 - r_a} h_{a-1,b}$$

Using induction or iterative methods, we compute these values from boundary conditions towards the desired state.

Step 5: Expressing in Terms of Stationary Distribution

In the case of large N or a general a, we use the stationary distribution to simplify:

$$h_{a,b} = \sum_{k=a}^{b-1} \frac{1}{p_k \pi_k} \sum_{j=0}^{k} \pi_j$$

Derivation:

- This formula accounts for cumulative expected times by summing the probabilities and transitions from each state up to $b - 1$.
- The outer sum $\sum_{k=a}^{b-1}$ accumulates the expected hitting time starting from each state k to b.
- The inner sum $\sum_{j=0}^{k} \pi_j$ integrates the stationary probabilities from state 0 to k.
- The term $\frac{1}{p_k \pi_k}$ normalizes by the forward transition probability, ensuring that the expected time accounts for the likelihood of moving forward.

Step 6: Practical Computation

- First, compute the stationary distribution π_k for all states.
- Use dynamic programming to compute $h_{a,b}$ efficiently for different values of a and b.
- Ensure numerical stability, especially for large N, by maintaining precision in recursive and summation steps.

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