Chapter 1

Fuzzy Sets♪

Sets♪

⌗ Elements of sets

- X : an universal set
- A : *a set A* in the universal set X ($A \subseteq X$)
- x : an element x is included in the set A ($x \in A$)
- For a set A, we define a membership function μ_A such as

$$\mu_A(x) = \begin{cases} 1 & \text{if and only if } x \in A \\ 0 & \text{if and only if } x \notin A \end{cases}$$

Sets♪

⌗ Relation between sets

- Family of sets $\{A_i \mid i \in I\}$
- $A \subseteq B$ iff (if and only if) $x \in A \Rightarrow x \in B$
- If $A \subseteq B$ and $B \subseteq A$ then $A = B$
- $A \subseteq B$ and $A \neq B$ then $A \subset B$ (A is called a **proper subset** of B)

Operation of Sets ♪

- Complement $B - A = \{x \mid x \in B, xA\}$
- Union $A \cup B = \{x \mid x \in A$ or $x \in B\}$
- Intersection $A \cap B = \{x \mid x \in A \text{ and } x \in B$
- *Partition* $\pi(A) = \{A_i \mid i \in I, A_i \subseteq A\}$

  (1) $A_i \neq \varnothing$

  (2) $A_i \cap A_j = \varnothing \quad i \neq j \quad i, j \in I$

  (3) $\bigcup_{i \in I} A_i = A$

Characteristics of Crisp Set ♪

(1) Involution♪	$(\bar{\bar{A}}) = A$
♪(2) Commutativity♪	$A \cup B = B \cup A$ $A \cap B = B \cap A$
♪(3) Associativity♪	$(A \cup B) \cup C = A \cup (B \cup C)$ $(A \cap B) \cap C = A \cap (B \cap C)$
♪(4) Distributivity♪	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
♪(5) Idempotency♪	$A \cup A = A, A \cap A = A$
♪(6) Law of contradiction♪	$A \cap \bar{A} = \varnothing$
♪(7) Law of excluded middle♪	$A \cup \bar{A} = X$
♪(8) De Morgan's law♪	$\overline{A \cap B} = \bar{A} \cup \bar{B}$ $\quad$ $\overline{A \cup B} = \bar{A} \cap \bar{B}$

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Characteristics of Crisp Set♪

- Convex set

The term convex is applicable to a set A in R^n (n-dimensional Euclidian ve ctor space) if the followings is satisfied;

For two arbitrary points s and r are defined in A, point t is involved in A where t is

$$t = (\lambda r_i + (1-\lambda)s_i \mid i \in N_n)$$

Characteristics of Crisp Set♪

- Convex set(example)

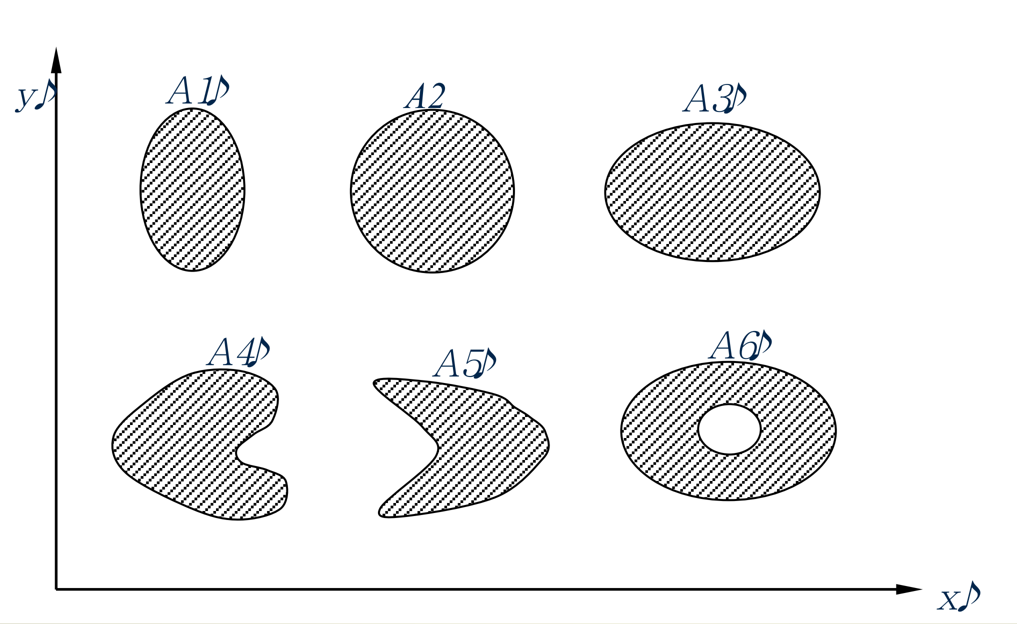


Fig 1.1 convex sets *A1, A2, A3* and non-convex sets *A4, A5, A6* in R^2

Definition of Fuzzy Set ♪

- **Definition (Membership function of fuzzy set)**

In fuzzy sets, each elements is mapped to [0,1]

by membership function.

$$\mu_A : X \rightarrow [0,\ 1]$$

Where [0,1] means real numbers between 0 and 1 (including 0 and 1).

Definition of fuzzy set

- Example 1.2

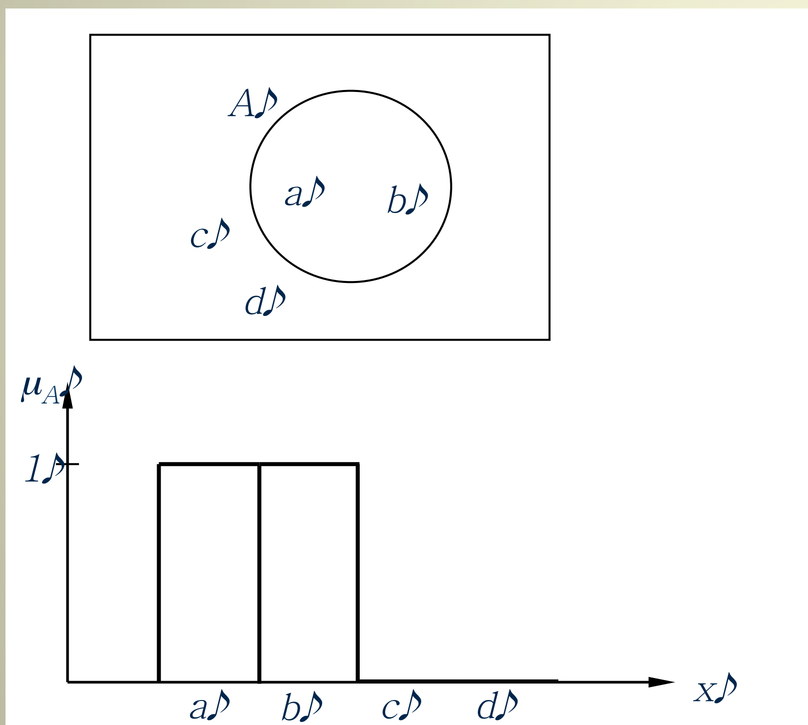


Fig 1.2 Graphical representation of crisp set

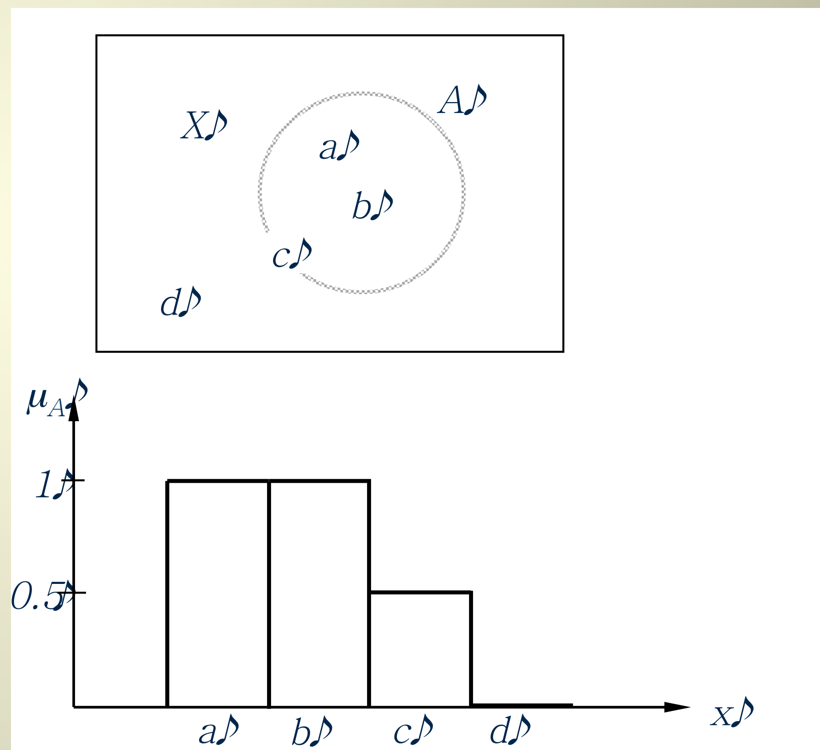


Fig 1.3 Graphical representation of fuzzy set

Examples of fuzzy set (1)♪

- A= "young" , B="very young"

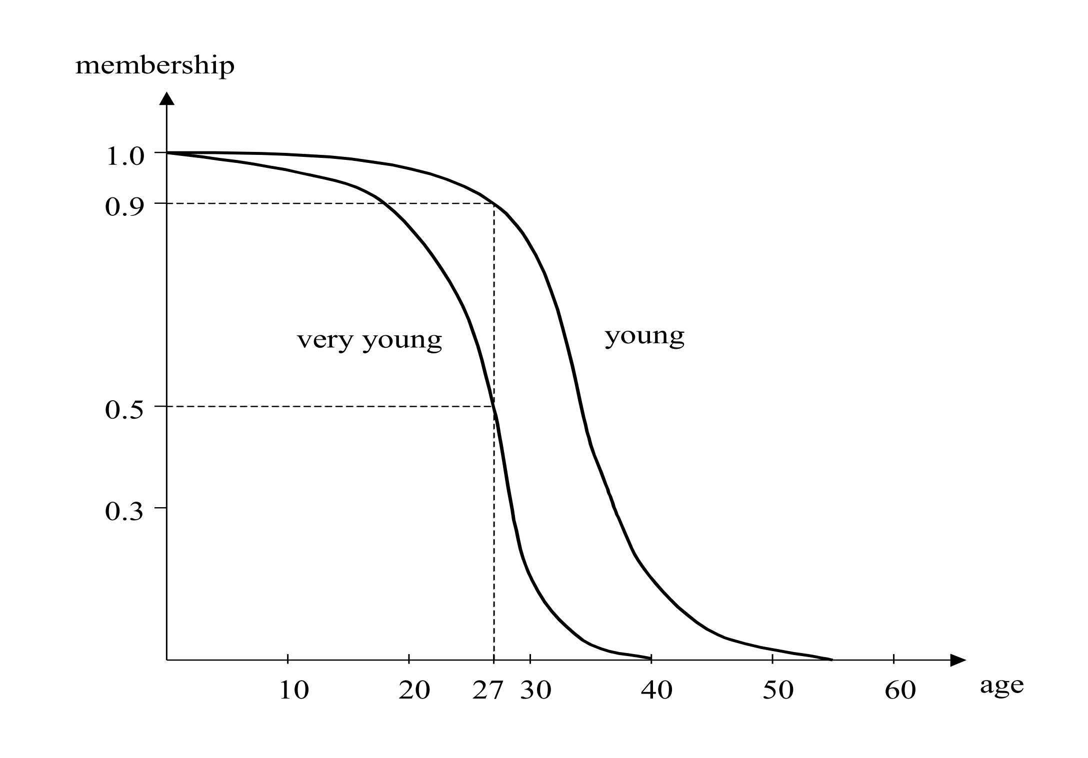


Fig 1.5 Fuzzy sets representing “young” and “very young”

Examples of fuzzy set ♪

- A ={real number near 0}

$$A = \int \mu_A(x)/x \quad \text{where } \mu_A(x) = \frac{1}{1+x^2}$$

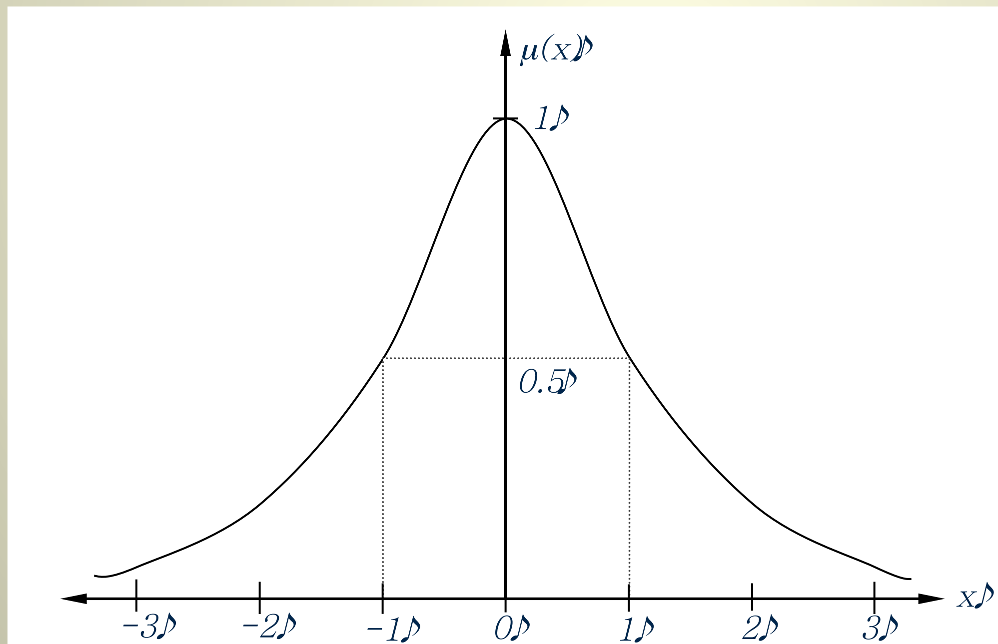


Fig 1.6 membership function of fuzzy set “real number near 0”

Examples of fuzzy set ♪

- B ={real number very near 0}, $\mu_B(x) = \left(\frac{1}{1+x^2}\right)^2$

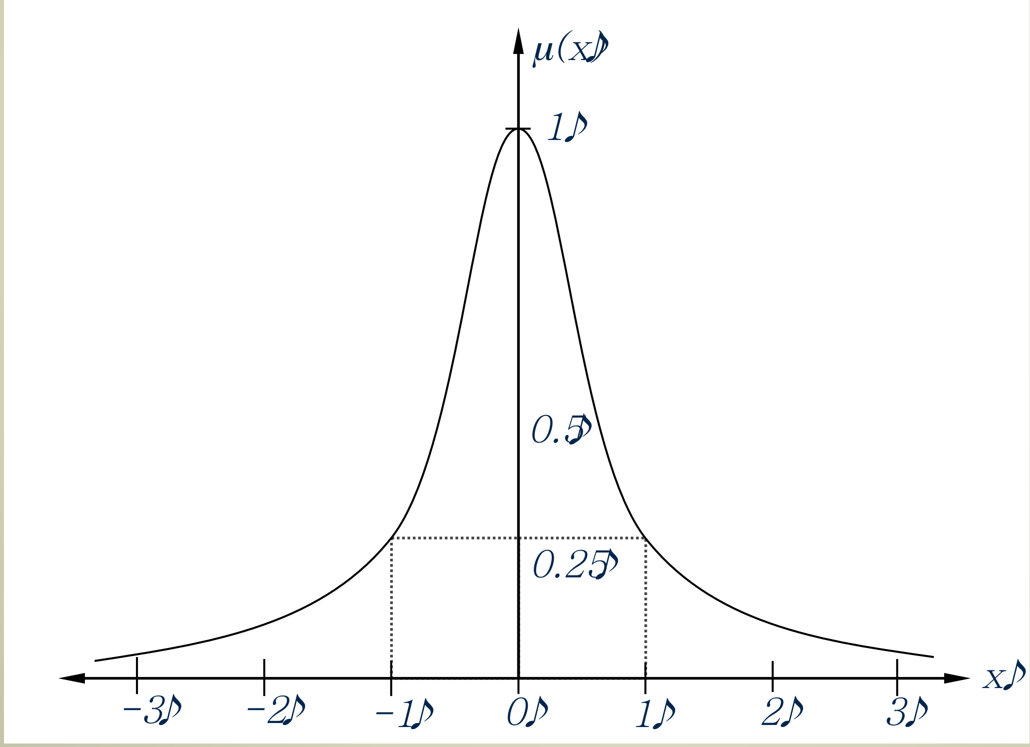


Fig 1.7 membership function for “real number very near to 0”

Examples of fuzzy set ♪

Example of Fuzzy Set

- $X = \{5, 15, 25, 35, 45, 55, 65, 75, 85\}$: age domain

age(element)♪	infant♪	young♪	adult♪	senior♪
5	0	0	0	0
15	0	0.2	0.1	0
25	0	1	0.9	0
35	0	0.8	1	0
45	0	0.4	1	0.1
55	0	0.1	1	0.2
65	0	0	1	0.6
75	0	0	1	1
85	0	0	1	1

Table 1.2 example of fuzzy set

Expanding Concepts of Fuzzy Set♪

- Support

  $Support(A) = \{x \in X \mid \mu_A(x) > 0\}$

  : a set that is made up of elements contained in *A*

  ex) $Support(youth) = \{15, 25, 35, 45, 55\}$

- Height : maximum value of the membership

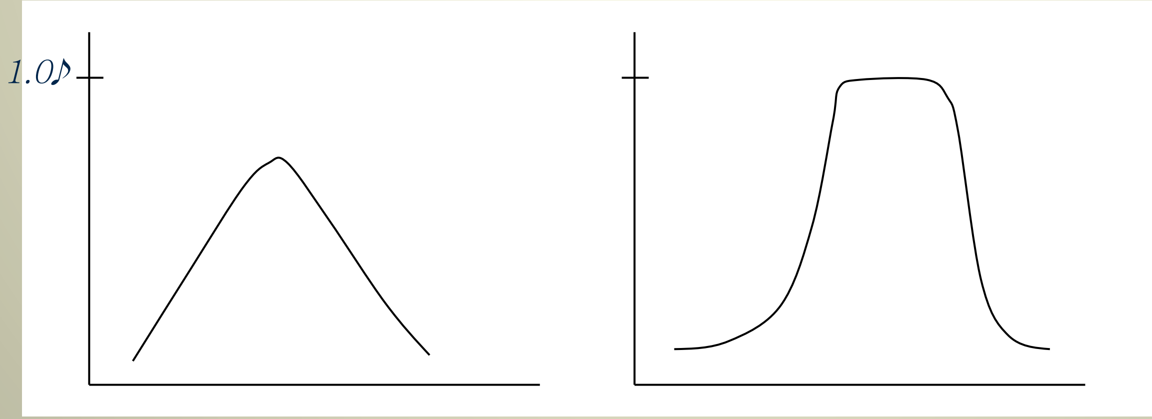


Fig 1.9 Non-Normalized Fuzzy Set and Normalized Fuzzy Set

α−Cut Set ♪

- α-cut set
  - $A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}$, α is an arbitrary real number in [0,1]
  - α-cut set is a crisp set
- Example 1.11
  - $Young_{0.2} = \{12, 25, 35, 45\}$
  - If α=0.4, $Young_{0.4} = \{25, 35, 45\}$
  - If α=0.8, $Young_{0.8} = \{25, 35\}$
  - $\Lambda_{\text{“young”}} = \{0, 0.1, 0.2, 0.4, 0.8, 1.0\}$

α−Cut Set♪

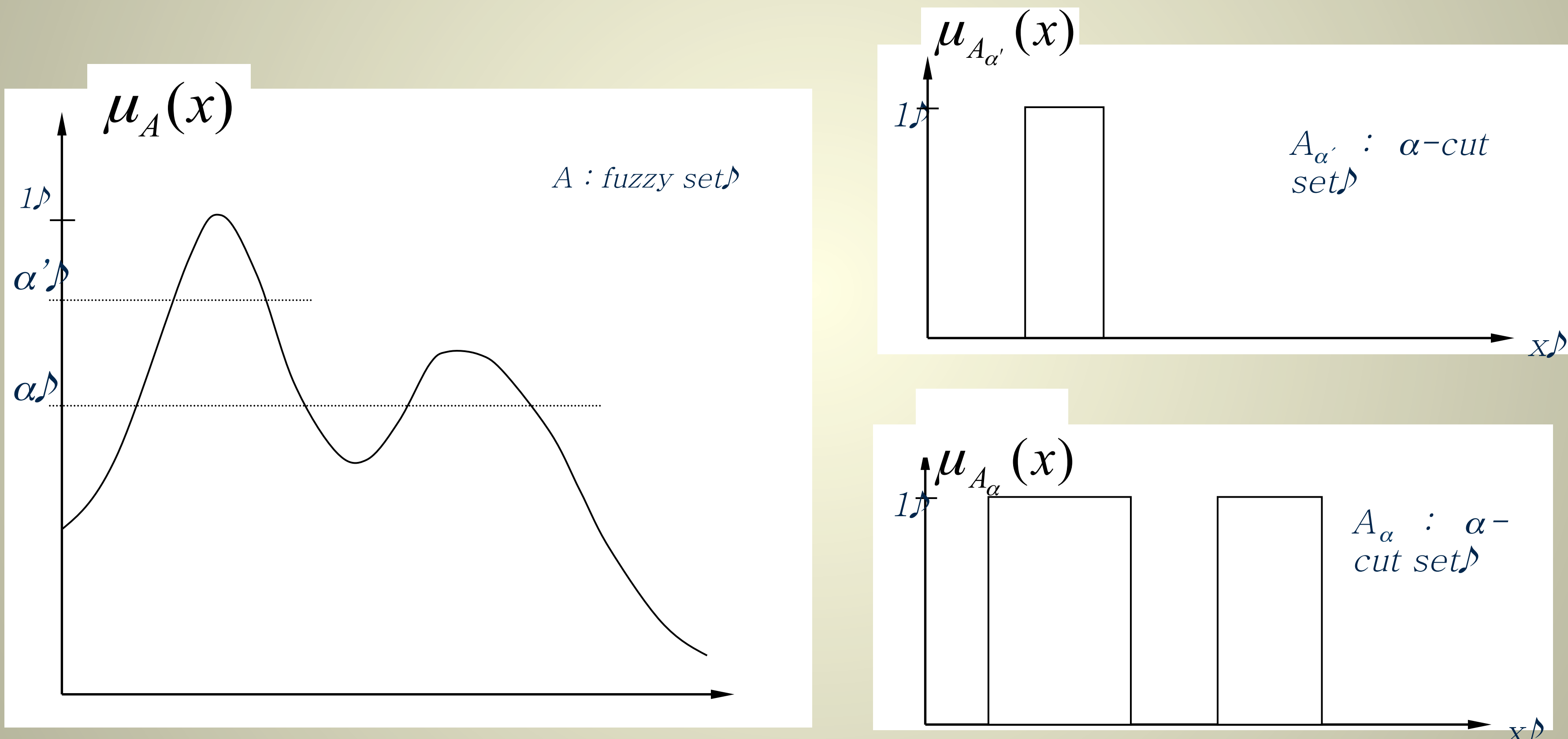


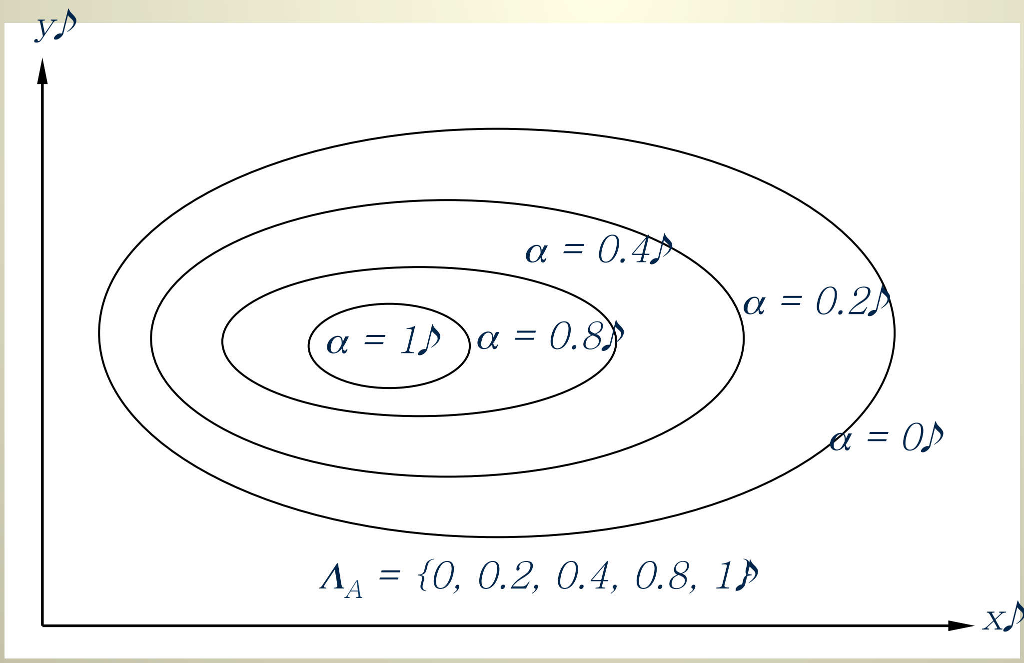
Fig 1.10 α-cut set $\alpha \leq \alpha'$, $A_\alpha \subseteq A_{\alpha'}$

Convex Fuzzy Set ♪

- Convex fuzzy set(1)

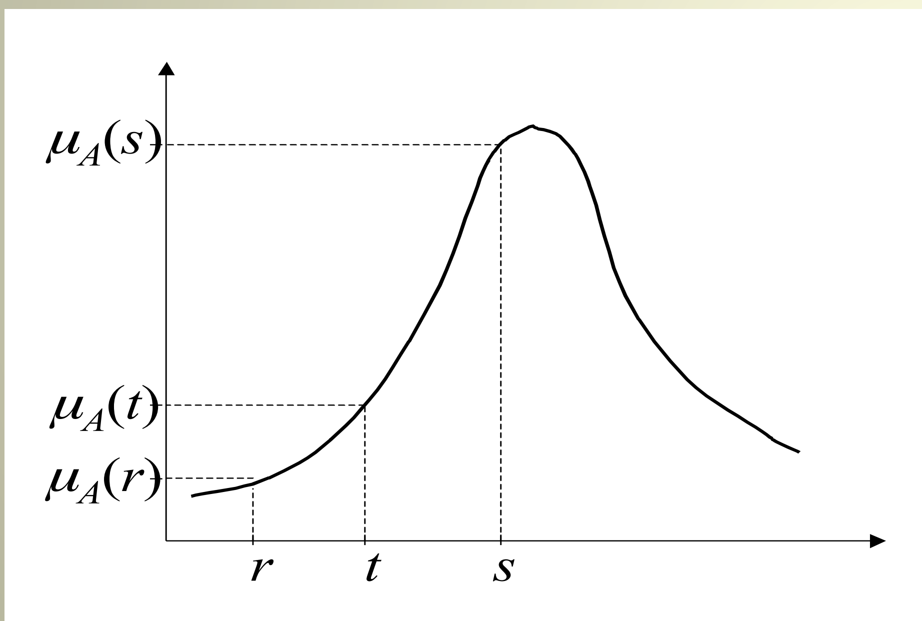
$$\mu_A(t) \geq \mathrm{Min}[\mu_A(r), \mu_A(s)]$$

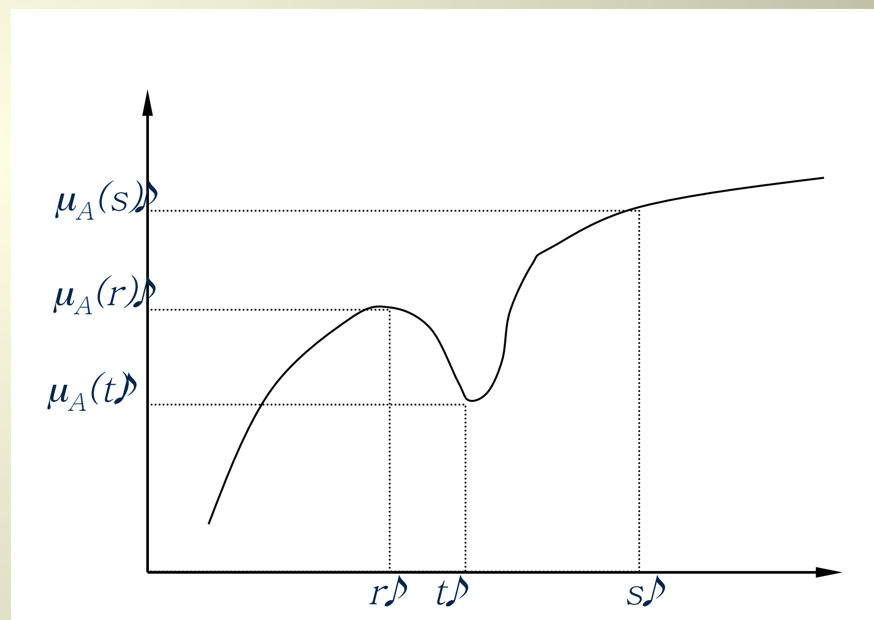
where $t = \lambda r + (1-\lambda)s$ $\quad r, s \in \Re^n, \lambda \in [0,1]$



Convex Fuzzy Set ♪

- Convex fuzzy set(2)





Convex Fuzzy Set $\mu_A(t) \geq \mu_A(r)$

Non-Convex Fuzzy Set $\mu_A(t) \leq \mu_A(r)$

The magnitude of fuzzy set ♪

- scalar cardinality $|A| = \sum_{x \in X} \mu_A(x)$
- relative cardinality $||A|| = \dfrac{|A|}{|X|}$

ex) $|senior| = 0.1 + 0.4 + 1 = 1.5$,

$|X| = 6,$

$||senior|| = 1.5/6 = 0.25$

The magnitude of Fuzzy set♪

- Fuzzy cardinality

$$\mu_{|A|}(|A_\alpha|) = \alpha, \quad \alpha \in \Lambda_A$$

where A_α is $\alpha-$cut set.

- ex) $\text{senior}_{0.1} = \{45, 55, 65, 75, 85\}$, $|\text{senior}_{0.1}| = 5$,

  $\text{senior}_{0.2} = \{55, 65, 75, 85\}$, $|\text{senior}_{0.2}| = 4$,

  $\text{senior}_{0.6} = \{65, 75, 85\}$, $|\text{senior}_{0.6}| = 3$,

  $\text{senior}_{1.0} = \{75, 85\}$, $|\text{senior}_{1.0}| = 2$.

  Fuzzy cardinality $|\text{senior}|_F = \{(5, 0.1), (4, 0.2), (3, 0.6), (2,1)\}$

Subset of fuzzy set ♪

- equivalence : $A = B$ iff $\mu_A(x) = \mu_B(x),\ \forall x \in X$
- subset : $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x),\ \forall x \in X$

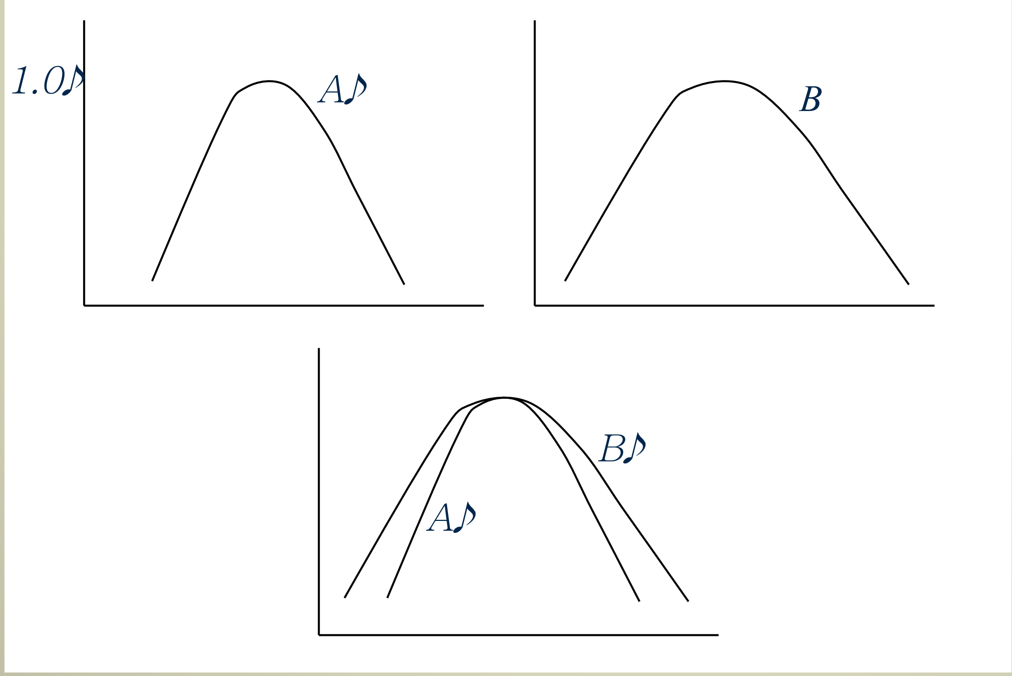


Fig 1.15 Subset $A \subset B$