Differential Calculus

The objective of these particular module is to prepare students for various partial differential equations in engineering fields which enable them to solve initial value problems and boundary value problems like one dimensional heat equation, one dimensional heat equation, two dimensional heat equation etc. as an outcome .

The partial derivative

Problem!

The slope of the $f(x,y)$ surface at (x,y) depends on which direction you move off in!

We have to think about slope in a particular direction.

The obvious directions are those along the $x-$ and $y-$axes.

Solution! To move off from (x,y) in the x direction, keep y fixed.

Define the partial derivative wrt x:

$$f_x = \left(\frac{\partial f}{\partial x}\right)_y = \lim_{\delta x \to 0} \left[\frac{f(x+\delta x, y) - f(x,y)}{\delta x}\right]$$

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More than two independent variables

If we are dealing with a function of more variables ...

... simply keep all but the one variable constant.

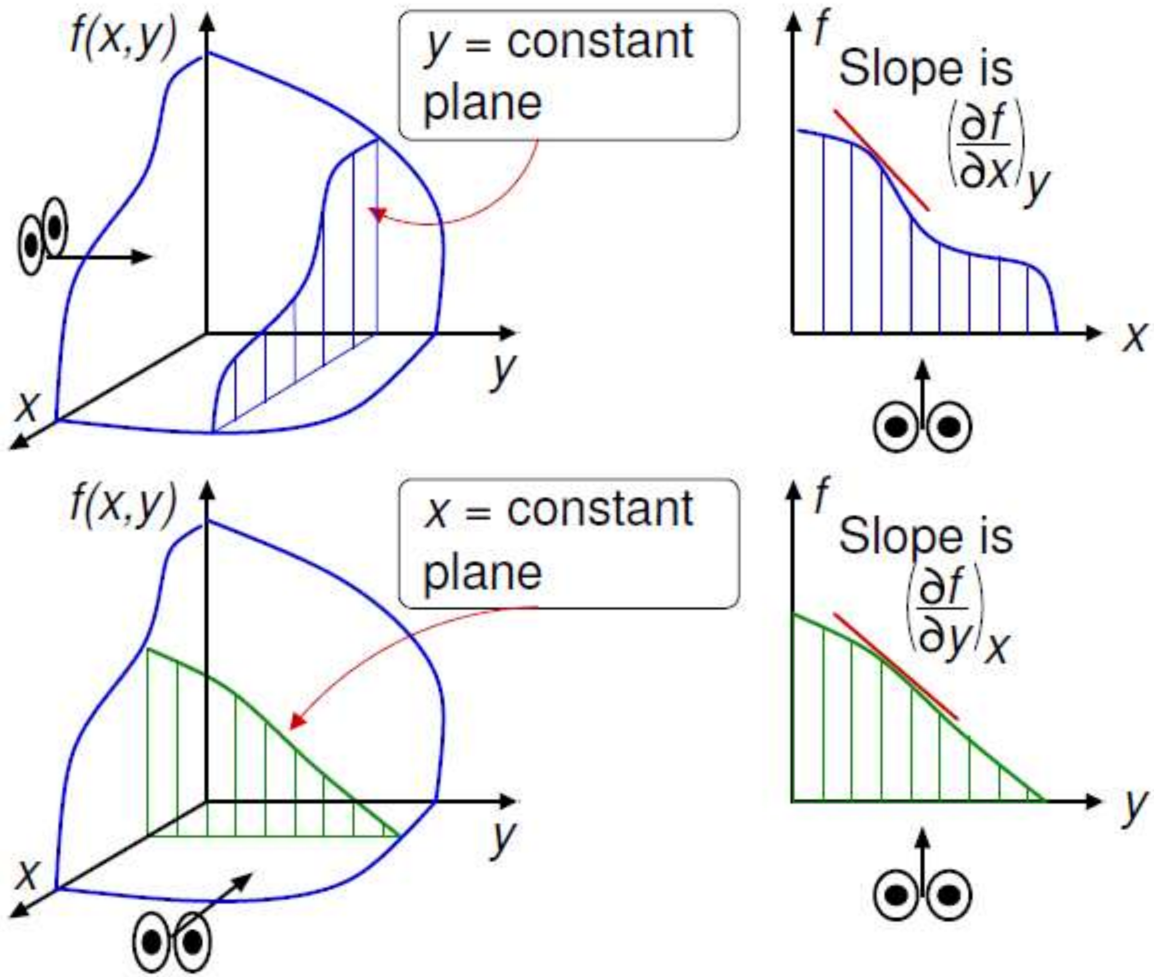
Eg for $f(x_1, x_2, x_3, ...)$ we have

The partial derivative again ...

$$\left(\frac{\partial f}{\partial x_3}\right)_{x_1 x_2 x_4 \ldots} = \lim_{\delta x_3 \to 0}\left[\frac{f(x_1, x_2, x_3 + \delta x_3, x_4, \ldots) - f(x_1, x_2, x_3, x_4, \ldots)}{\delta x_3}\right]$$

Given that you know the list of the variables, and know the one being varied, the "held constant" subscripts are superfluous and are often omitted.

Geometrical interpretation of the partial derivative



https://www.youtube.com/watch?v=GkB4vW16QHI

The mechanics of evaluating partial derivatives

Operationally, partial differentiation is exactly the same as normal differentiation with respect to one variable, **with all the others treated as constants**.

♣ **Example**

Suppose

$$f(x, y) = x^2y^3 - 2y^2$$

First assume y is a constant:

$$\frac{\partial f}{\partial x} = 2xy^3$$

Then x is a constant:

$$\frac{\partial f}{\partial y} = 3x^2y^2 - 4y$$

Higher partial derivatives

$\partial f/\partial x$ and $\partial f/\partial y$ are probably perfectly good functions of (x, y), so we can differentiate again.

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} f_x = f_{xx}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{xy}$$

$similarly\ we\ can\ define f_{yy}, f_{yx}, f_{xxx}, f_{xxy}$ and so on.

♣ **Example.**

$$\begin{aligned} f(x,y) &= x^2y^3 - 2y^2 \\ \partial f/\partial x &= 2xy^3 \\ \partial f/\partial y &= 3x^2y^2 - 4y \end{aligned}$$

Hence

$$\begin{aligned} \partial^2 f/\partial x^2 &= 2y^3 \\ \partial^2 f/\partial y^2 &= 6x^2y - 4 \ . \end{aligned}$$

But we should also consider

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = 6xy^2 \quad \text{and} \quad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = 6xy^2.$$

Ex. 1 : If $u = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$, Find u_{xy}.

Sol. : Step 1 : Let
$$u = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$$

We have
$$u_{xy} = \frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

Step 2 : First we will find $\frac{\partial u}{\partial y}$.

$\therefore$ Differentiating equation (1) partially with respect to y keeping x constant

$$\frac{\partial u}{\partial y} = x^2 \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) - 2y \tan^{-1}\frac{x}{y} - y^2 \frac{1}{1+\frac{x^2}{y^2}}\left(\frac{-x}{y^2}\right)$$

$$= \frac{x^3}{x^2+y^2} - 2y\tan^{-1}\frac{x}{y} + \frac{xy^2}{x^2+y^2} = \frac{x(x^2+y^2)}{x^2+y^2} - 2y\tan^{-1}\frac{x}{y}$$

$$\frac{\partial u}{\partial y} = x - 2y\tan^{-1}\frac{x}{y}$$

Step 3 :

$$\frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left[x - 2y\tan^{-1}\frac{x}{y}\right] = 1 - 2y\left(\frac{1}{1+\frac{x^2}{y^2}}\right)\left(\frac{1}{y}\right)$$

$$= 1 - \frac{2y^2}{x^2+y^2} = \frac{x^2+y^2-2y^2}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x\, \partial y} = \frac{x^2-y^2}{x^2+y^2}$$

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Ex. 1 : If $u = x^y$ then verify $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$.

Sol. : $u = x^y$, $\frac{\partial u}{\partial x} = y \cdot x^{y-1}$ ($\because$ y is constant)

$$\frac{\partial^2 u}{\partial x^2} = y\,(y-1)\,x^{y-2} \quad \text{(Again y is constant)}$$

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\,[y \cdot x^{y-1}]$$

$$\frac{\partial^2 u}{\partial y\,\partial x} = y \cdot x^{y-1} \log x + x^{y-1}\,(1)$$

(by using product rule keeping x constant)

Now,

$$u = x^y$$

$$\frac{\partial u}{\partial y} = x^y \log x \quad (\because \text{ x is constant}) \quad \left(\text{Recall that } \frac{d}{dy}(a^y) = a^y \log a\right)$$

$$\frac{\partial^2 u}{\partial y^2} = \log x \cdot [x^y \cdot \log x] \quad \text{(Again x is constant)}$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial y}\right] = \frac{\partial}{\partial x}\,[x^y \log x] = x^y\left(\frac{1}{x}\right) + \log x\,(y \cdot x^{y-1})$$

(by using product rule keeping y constant)

$$\frac{\partial^2 u}{\partial x\,\partial y} = x^{y-1} + y x^{y-1} \log x$$

We note that, $$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$$

Ex. 12 : If $v = (x^2 - y^2)\, f(xy)$ then show that $v_{xx} + v_{yy} = (x^4 - y^4)\, f''(xy)$.

Sol. :

$$v = (x^2 - y^2)\, f(xy)$$

Using product rule,

$$\frac{\partial v}{\partial x} = 2x\, f(xy) + (x^2 - y^2)\, f'(xy) \cdot y$$

$$\frac{\partial^2 v}{\partial x^2} = 2\, f(xy) + 2x\, f'(xy) \cdot y + y\, [2x\, f'(xy) + (x^2 - y^2)\, f''(xy)\, y]$$

$$= 2\, f(xy) + 2xy\, f'(xy) + 2xy\, f'(xy) + (x^2 - y^2)\, f''(xy)\, y^2 \qquad \dots (1)$$

Again,

$$\frac{\partial v}{\partial y} = -2y\, f(xy) + (x^2 - y^2)\, f'(xy) \cdot x$$

$$\frac{\partial^2 v}{\partial y^2} = -2\, f(xy) - 2y\, f'(xy)\, x + x\, [-2y\, f'(xy) + (x^2 - y^2)\, f''(xy) \cdot x]$$

$$= -2f\,(xy) - 2xy\, f'(xy) - 2xy\, f'(xy) + x^2\, (x^2 - y^2)\, f''(xy) \qquad \dots (2)$$

Adding equations (1) and (2), we get

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = (x^2 + y^2)\,(x^2 - y^2)\, f''(xy) = (x^4 - y^4)\, f''(xy)$$

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Ex. 15 : If $u = x^m y^n$, show that $\frac{\partial^3 u}{\partial x\, \partial y\, \partial x} = \frac{\partial^3 u}{\partial y\, \partial x^2}$. (Dec. 93)

Sol. :

$$u = x^m y^n$$

$$\frac{\partial u}{\partial x} = m\, x^{m-1} y^n, \qquad \frac{\partial^2 u}{\partial y\, \partial x} = m \cdot n \cdot x^{m-1} y^{n-1}$$

$$\frac{\partial^3 u}{\partial x\, \partial y\, \partial x} = m \cdot n\, (m-1)\, x^{m-2}\, y^{n-1} \quad \text{... (i)}$$

Also,

$$\frac{\partial^2 u}{\partial x^2} = m\, (m-1)\, x^{m-2} y^n$$

$$\frac{\partial^3 u}{\partial y\, \partial x^2} = m \cdot n\, (m-1)\, x^{m-2}\, y^{n-1} \quad \text{... (ii)}$$

From equations (i) and (ii),

$$\frac{\partial^3 u}{\partial x\, \partial y\, \partial x} = \frac{\partial^3 u}{\partial y\, \partial x^2}$$

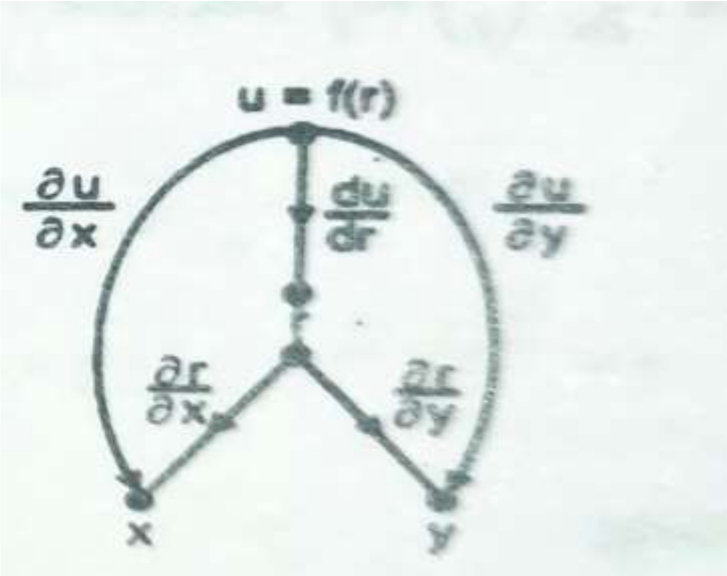
PPT-03

THE CHAIN RULE FOR COMPOSITE FUNCTIONS (CASE 1)

Let u = f(r) where r is again a function of two variables x and y.

i.e u → r → x , y then

$$\frac{\partial u}{\partial x} = \frac{du}{dr}\frac{\partial r}{\partial x} \quad , \quad \frac{\partial u}{\partial y} = \frac{du}{dr}\frac{\partial r}{\partial y}$$



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THE CHAIN RULE (CASE 2)

Let $z = f(x, y)$ where x and y are functions

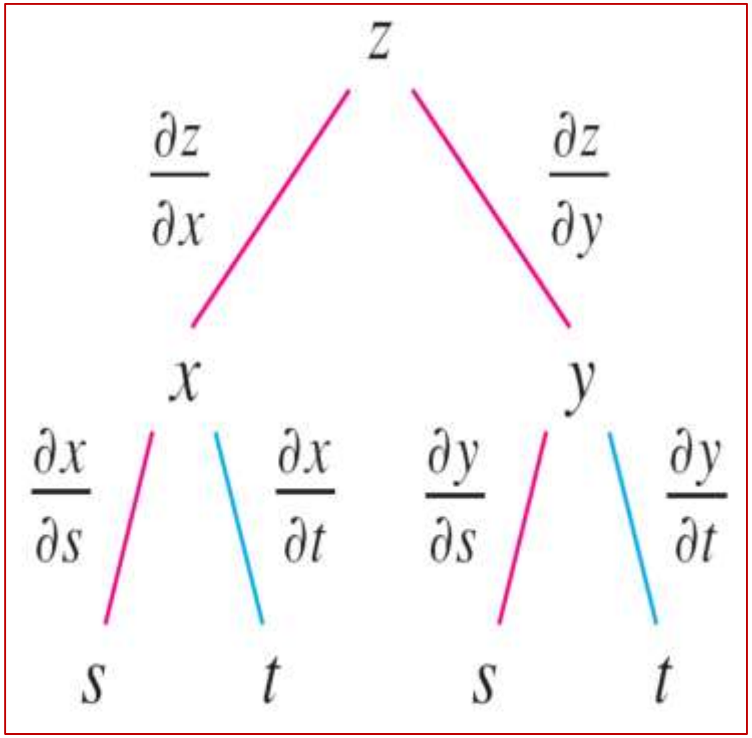
of s and t.

Then z is differntiable function

of s and t.

i.e $\quad z \rightarrow x, y \rightarrow s, t$

Therefore we have ,



$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

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THE CHAIN RULE (GEN. VERSION)

If u $\rightarrow x_1, x_2, \ldots x_n \rightarrow t_1, t_2, \ldots, t_n$

Then

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

for each $i = 1, 2, \ldots, m$.

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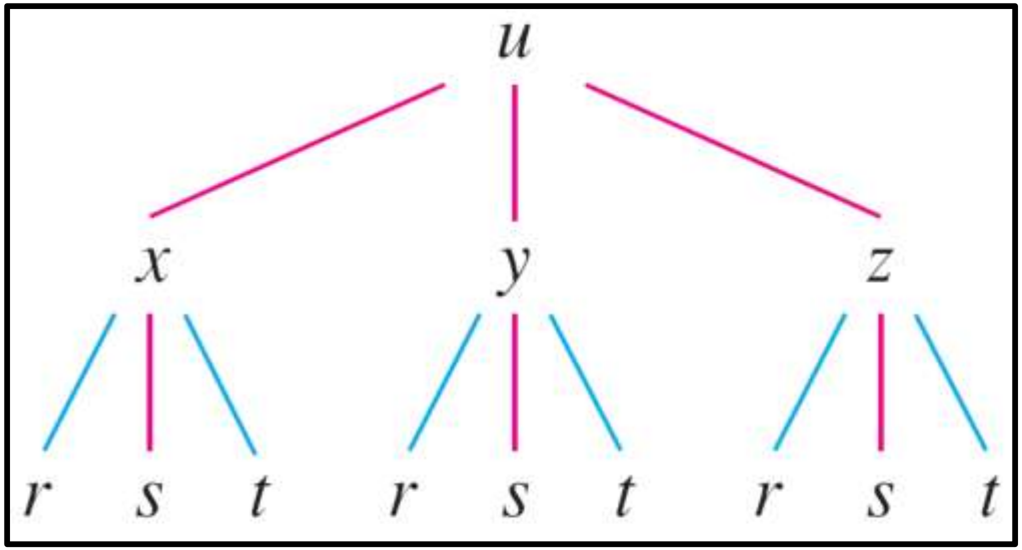

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$$

$$Simillarly\ we\ can\ find\ \frac{\partial u}{\partial r}, \frac{\partial u}{\partial t}$$

PPT-03

Ex. 1 : If $u = f(r)$ where $r = \sqrt{x^2 + y^2}$,

prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$ or $u_{xx} + u_{yy} = \frac{d^2 u}{dr^2} + \frac{1}{r}\frac{du}{dr}$. **(Dec. 2009)**

Sol. : Step 1 : Here $u \to r \to x, y$ i.e. u becomes composite function of x and y.

We have, $$\frac{\partial u}{\partial x} = \frac{du}{dr}\frac{\partial r}{\partial x}, \quad \frac{\partial u}{\partial y} = \frac{du}{dr}\frac{\partial r}{\partial y}$$

Step 2 : We have, $$r = \sqrt{x^2 + y^2}, \quad r^2 = x^2 + y^2$$

$$2r\frac{\partial r}{\partial x} = 2x \quad \therefore \quad \frac{\partial r}{\partial x} = \frac{x}{r}; \quad \frac{\partial r}{\partial y} = \frac{y}{r}$$

Make a note that for $u = f(r) \Rightarrow \frac{du}{dr}$ or $\frac{\partial f}{\partial r}$ or $f'(r)$ all mean the same thing.

Step 3 : $$\frac{\partial u}{\partial x} = \frac{du}{dr}\frac{x}{r} = f'(r)\frac{x}{r}$$

Carefully note the second order partial derivative.

Step 4 : $$\frac{\partial^2 u}{\partial x^2} = f'(r)\left\{\frac{r(1) - x \cdot \frac{\partial r}{\partial x}}{r^2}\right\} + \frac{x}{r} f''(r)\frac{\partial r}{\partial x}.$$

[**Note :** $\frac{\partial}{\partial x}\left(\frac{x}{r}\right) \neq \frac{1}{r}$ because r is a function of x and y. Also $\frac{\partial}{\partial x}[f'(r)] = \frac{d}{dr}[f'(r)]\frac{\partial r}{\partial x} = f''(r)\frac{\partial r}{\partial x}$].

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = f'(r)\left\{\frac{r - x \cdot \frac{x}{r}}{r^2}\right\} + \frac{x^2}{r^2} f''(r)$$

$$\frac{\partial^2 u}{\partial x^2} = f'(r)\left(\frac{r^2 - x^2}{r^3}\right) + \frac{x^2}{r^2} f''(r)$$

Similarly, $$\frac{\partial^2 u}{\partial y^2} = f'(r)\left(\frac{r^2 - y^2}{r^3}\right) + \frac{y^2}{r^2} f''(r)$$

Step 5 : $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{f'(r)}{r^3}[2r^2 - (x^2 + y^2)] + \frac{f''(r)}{r^2}(x^2 + y^2)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{f'(r)}{r^3}(2r^2 - r^2) + \frac{f''(r)}{r^2}(r^2) \quad (\because x^2 + y^2 = r^2)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r).$$

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Ex. 3 : If $u = f(r)$ where $r = \sqrt{x^2 + y^2 + z^2}$, prove that $u_{xx} + u_{yy} + u_{zz} = f''(r) + \frac{2}{r} f'(r)$.

Sol. : Here $u \to r \to x, y, z$ hence u becomes composite function of x, y, z.

$$\frac{\partial u}{\partial x} = \frac{du}{dr}\frac{\partial r}{\partial x} = f'(r) \cdot \frac{x}{r}$$

$$u_{xx} = f'(r)\left(\frac{r - x\frac{\partial r}{\partial x}}{r^2}\right) + \frac{x}{r} f''(r)\frac{\partial r}{\partial x}$$

$$u_{xx} = f'(r)\left(\frac{r^2 - x^2}{r^3}\right) + \frac{x^2}{r^2} f''(r)$$

Because of symmetry,

$$u_{yy} = f'(r)\left(\frac{r^2 - y^2}{r^3}\right) + \frac{y^2}{r^2} f''(r)$$

$$u_{zz} = f'(r)\left(\frac{r^2 - z^2}{r^3}\right) + \frac{z^2}{r^2} f''(r)$$

$$\therefore \quad u_{xx} + u_{yy} + u_{zz} = \frac{f'(r)}{r^2}[3r^2 - (x^2 + y^2 + z^2)] + \frac{f''(r)}{r^2}(x^2 + y^2 + z^2)$$

$$= \frac{f'(r)}{r^2}(2r^2) + \frac{f''(r)}{r^2}(r^2)$$

$$\therefore \quad u_{xx} + u_{yy} + u_{zz} = f''(r) + \frac{2}{r} f'(r)$$

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Ex. 4 : *If* $V = e^{\frac{r-x}{l}}$ *where* $r^2 = x^2 + y^2$; l *is a constant, show that* $V_{xx} + V_{yy} + \frac{2}{l} V_x = \frac{V}{lr}$.

Sol. : Given
$$V = e^{\frac{r-x}{l}}$$

$\therefore$
$$V_x = e^{\frac{r-x}{l}} \cdot \frac{1}{l}\left(\frac{\partial r}{\partial x} - 1\right)$$

$$V_x = \frac{V}{l}\left(\frac{x}{r} - 1\right)$$

$$V_{xx} = \frac{V_x}{l}\left(\frac{x}{r} - 1\right) + \frac{V}{l}\left(\frac{1}{r} - \frac{x}{r^2} \cdot \frac{\partial r}{\partial x}\right)$$

$$V_{xx} = \frac{V}{l^2}\left(\frac{x}{r} - 1\right)^2 + \frac{V}{lr} - \frac{V x^2}{lr^3}$$

$$V_y = e^{\frac{r-x}{l}} \cdot \frac{1}{l}\left(\frac{\partial r}{\partial y}\right) = \frac{V}{l}\left(\frac{y}{r}\right) = \frac{Vy}{lr}$$

$$V_{yy} = \frac{V}{lr} + \frac{yV_y}{lr} - \frac{Vy}{lr^2}\frac{\partial r}{\partial y} = \frac{V}{lr} + \frac{y^2 V}{l^2 r^2} - \frac{Vy^2}{lr^3}$$

$$V_{xx} + V_{yy} = \frac{V}{l^2}\left(\frac{x}{r} - 1\right)^2 + \frac{V}{lr} - \frac{Vx^2}{lr^3} + \frac{V}{lr} + \frac{y^2 V}{l^2 r^2} - \frac{V \cdot y^2}{lr^3}$$

$$= \frac{V}{l^2}\left(\frac{x}{r} - 1\right)^2 + \frac{2V}{lr} - \frac{V}{lr^3}(x^2 + y^2) + \frac{Vy^2}{l^2 r^2}$$

$$= \frac{V}{l^2}\left(\frac{x}{r} - 1\right)^2 + \frac{V}{lr} + \frac{Vy^2}{l^2 r^2}$$

$\therefore$
$$V_{xx} + V_{yy} + \frac{2}{l} V_x = \frac{V}{l^2}\left(\frac{x}{r} - 1\right)^2 + \frac{V}{lr} + \frac{Vy^2}{l^2 r^2} + \frac{2V}{l^2}\left(\frac{x}{r} - 1\right)$$

$$= \frac{V}{l^2}\left(\frac{x^2}{r^2} - \frac{2x}{r} + 1\right) + \frac{V}{lr} + \frac{Vy^2}{l^2 r^2} + \frac{2V \cdot x}{l^2 r} - \frac{2V}{l^2}$$

$$= \frac{V}{l^2 r^2}(x^2 + y^2) - \frac{V}{l^2} + \frac{V}{lr} = \frac{V}{lr}$$

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The total differential

A **differential** is a different from a **derivative**.

Suppose that we have a continuous function $f(x, y)$ in some region, and both $(\partial f/\partial x)$ and $(\partial f/\partial y)$ are continuous in that region.

The differential tells one by how much the value of the function changes as one moves infinitesimal amounts $\mathrm{d}x$ and $\mathrm{d}y$ in the $x-$ and $y-$directions.

The total or perfect differential of $f(x, y)$

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y$$

$\boldsymbol{Note}$:

$\mathbf{1}.\ \boldsymbol{If\ z \to x, y \to t}$ **then ,** $$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$\mathbf{2}.\ \boldsymbol{Also\ , if\ \ u \to x, y, z \to t\ \ then\ z\ is\ function\ of\ t.}$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

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Examples :

Ex. 6 : If $u = x^2 + y^2$, where $x = at^2$, $y = 2at$, find $\frac{du}{dt}$.

Sol. : Here u is composite function of t and $\frac{du}{dt}$, the total derivative is required. Hence

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

$$= (2x)(2at) + 2y(2a)$$

$$= 4at(x) + 4ay$$

$$= 4at(at^2) + 4a(2at)$$

$$= 4a^2 t^3 + 8a^2 t$$

Ex. 5 : If $u = \sin\frac{x}{y}$ and $x = e^t, y = t^2$, verify $\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$.

Sol. : By actual substitution, we have $u = \sin\frac{e^t}{t^2}$.

$$\therefore \quad \frac{du}{dt} = \left(\cos\frac{e^t}{t^2}\right)\cdot\frac{t^2 e^t - 2\,te^t}{t^4} = \left(\cos\frac{e^t}{t^2}\right)\cdot\left(\frac{1}{t^2}-\frac{2}{t^3}\right)e^t$$

Here $u \to xy \to t \Rightarrow u$ is composite function of t i.e.

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$$= \left(\cos\frac{x}{y}\right)\cdot\frac{1}{y}\cdot e^t + \left(\cos\frac{x}{y}\right)\cdot\frac{-x}{y^2}\cdot 2t$$

$$= \left(\cos\frac{x}{y}\right)\left[\frac{1}{y}e^t - \frac{2x}{y^2}t\right]$$

$$= \left(\cos\frac{e^t}{t^2}\right)\left[\frac{1}{t^2}e^t - \frac{2e^t}{t^4}\cdot t\right]$$

$$= \left(\cos\frac{e^t}{t^2}\right)\cdot\left(\frac{1}{t^2}-\frac{2}{t^3}\right)\cdot e^t$$

By (i) and (ii), the formula stands verified.

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Variables to be treated as constants :

Let $x = rcos\,\theta$, $y = rsin\,\theta$. Then $r^2 = x^2 + y^2$

Notation $\left(\frac{\partial r}{\partial x}\right)_\theta$ means the partial derivative of r with respect to x , treating θ constant in a relation expressing r as a function of x and θ only.

Thus $\left(\frac{\partial r}{\partial x}\right)_\theta = \sec\theta$ and $2r.\left(\frac{\partial r}{\partial x}\right)_y = 2x$. Hence $\left(\frac{\partial r}{\partial x}\right)_y = \frac{x}{r} = \cos\theta$

When no indication is given regarding the variable to be kept constant ,then by convention $\left(\frac{\partial}{\partial x}\right)$ always means $\left(\frac{\partial}{\partial x}\right)_y$ and $\left(\frac{\partial}{\partial y}\right)$ means $\left(\frac{\partial}{\partial y}\right)_x$.

Similarly $\left(\frac{\partial}{\partial r}\right)$ means $\left(\frac{\partial}{\partial r}\right)_\theta$ and $\left(\frac{\partial}{\partial \theta}\right)$ means $\left(\frac{\partial}{\partial \theta}\right)_r$.

Examples :

Ex. 2 : If $u = ax + by$, $v = bx - ay$, find the value of $\left(\frac{\partial u}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial u}\right)_v \cdot \left(\frac{\partial y}{\partial v}\right)_x \cdot \left(\frac{\partial v}{\partial y}\right)_u$.

Sol. : Given that $u = ax + by$, hence $\left(\frac{\partial u}{\partial x}\right)_y = a.$ ... (1)

and $v = bx - ay \quad \therefore \quad y = \frac{b}{a}x - \frac{v}{a} \quad \therefore \left(\frac{\partial y}{\partial v}\right)_x = -\frac{1}{a}$ ... (2)

To get $\left(\frac{\partial x}{\partial u}\right)_v$, express x in terms of u and v, so we eliminate y between the relation given.

$$\Rightarrow \qquad y = \frac{u - ax}{b} = \frac{bx - v}{a}$$

$$\Rightarrow \qquad au - a^2x = b^2x - bv \quad \Rightarrow x = \frac{au + bv}{a^2 + b^2}$$

Hence $$\left(\frac{\partial x}{\partial u}\right)_v = \frac{a}{a^2 + b^2} \qquad \text{... (3)}$$

Similarly to get $\left(\frac{\partial v}{\partial y}\right)_u$ eliminate x between the relation and get,

$$\frac{u - by}{a} = \frac{v + ay}{b} \quad \Rightarrow \quad bu - b^2y = av + a^2y$$

$$\Rightarrow \qquad v = \frac{bu - (a^2 + b^2)\,y}{a}, \text{ hence } \left(\frac{\partial v}{\partial y}\right)_u = -\left(\frac{a^2 + b^2}{a}\right) \qquad \text{... (4)}$$

From equations (1), (2), (3) and (4) on multiplication, we get,

$$\left(\frac{\partial u}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial u}\right)_v \cdot \left(\frac{\partial y}{\partial v}\right)_x \cdot \left(\frac{\partial v}{\partial y}\right)_u = a\left(\frac{a}{a^2 + b^2}\right) \cdot \left(-\frac{1}{a}\right)\left(-\frac{a^2 + b^2}{a}\right) = 1$$

Ex. 6 : If $u \cdot x + v \cdot y = 0$, $\frac{u}{x} + \frac{v}{y} = 1$, prove that

$$\frac{u}{x}\left(\frac{\partial x}{\partial u}\right)_v + \frac{v}{y}\left(\frac{\partial y}{\partial v}\right)_u = 0$$

Sol. : We have,

$$y = -\frac{u \cdot x}{v}$$

$$\therefore \quad \frac{u}{x} + v \cdot \left[-\frac{v}{u \cdot x}\right] = 1$$

$$\frac{u^2 - v^2}{u \cdot x} = 1 \qquad \therefore x = \frac{u^2 - v^2}{u}$$

$$\therefore \quad \left(\frac{\partial x}{\partial u}\right)_v = 1 + \frac{v^2}{u^2}$$

PPT-03

We have

$$x = -\frac{v \cdot y}{u}$$

$$\therefore \quad u\left(-\frac{u}{v \cdot y}\right) + \frac{v}{y} = 1$$

$$\therefore \quad y = \frac{v^2 - u^2}{v}$$

$$\left(\frac{\partial y}{\partial v}\right)_u = 1 + \frac{u^2}{v^2}$$

$$\frac{u}{x} \cdot \left(\frac{\partial x}{\partial u}\right)_v + \frac{v}{y}\left(\frac{\partial y}{\partial v}\right)_u = \frac{u}{x} + \frac{v^2}{xu} + \frac{v}{y} + \frac{u^2}{yv}$$

$$= \frac{u^2 + v^2}{xu} + \frac{v^2 + u^2}{yv}$$

$$= (u^2 + v^2)\left(\frac{yv + xu}{xu \cdot yv}\right)$$

$$= 0 \qquad \because \quad xu + yv = 0$$

PPT-03

Differntiation of Implicit functions:

Implicit Function: Let $f(x,y) = 0$ represents an implicit relation which can not be necessarily be solved for one of the variable say x in terms of other variable say y.

$$\text{e.g.}\ x^3 + y^3 - 3axy = 0$$

Write $z = f(x,y) = 0$ the $\frac{dz}{dx} = \frac{\partial z}{\partial x}\frac{dx}{dx} + \frac{\partial z}{\partial y}\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = -\frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

We now introduce notations

$$p = \frac{\partial f}{\partial x}; \quad q = \frac{\partial f}{\partial y}; \quad r = \frac{\partial^2 f}{\partial x^2}; \quad s = \frac{\partial^2 f}{\partial x \partial y}; \quad t = \frac{\partial^2 f}{\partial y^2}$$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{p}{q}$$

Differentiating above equation w. r. t. x again

$$\frac{d^2y}{dx^2} = -\left(\frac{q\frac{dp}{dx} - p\frac{dq}{dx}}{q^2}\right)$$

Which can be represented as $$q^3\frac{d^2y}{dx^2} = \det\begin{bmatrix} r & s & p \\ s & t & q \\ p & q & 0 \end{bmatrix}$$

Example . If $f(x, y) = 0$ and $\varphi(y, z) = 0$, show that $\frac{\partial f}{\partial y}\frac{\partial \varphi}{\partial z}\frac{dz}{dx} = \frac{\partial f}{\partial x}\frac{\partial \varphi}{\partial y}$

Sol. :

$$f(x, y) = 0$$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \quad \cdots\ (1)$$

and

$$\phi(y, z) = 0$$

$$\frac{dz}{dy} = -\frac{\frac{\partial \phi}{\partial y}}{\frac{\partial \phi}{\partial z}} \quad \cdots\ (2)$$

Multiplying the respective sides of (1) and (2), we have :

$$\Rightarrow \quad \frac{dy}{dx}\cdot\frac{dz}{dy} = \frac{\frac{\partial \phi}{\partial y}\cdot\frac{\partial f}{\partial x}}{\frac{\partial \phi}{\partial z}\cdot\frac{\partial f}{\partial y}}$$

$$\frac{dz}{dx}\cdot\frac{\partial \phi}{\partial z}\cdot\frac{\partial f}{\partial y} = \frac{\partial \phi}{\partial y}\cdot\frac{\partial f}{\partial x}$$

$$\Rightarrow \quad \frac{\partial f}{\partial y}\cdot\frac{\partial \phi}{\partial z}\cdot\frac{dz}{dx} = \frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial y} \quad \ldots \text{Proved.}$$

Ex. 5 : If $(\cos x)^y = (\sin y)^x$ then find $\frac{dy}{dx}$.

Sol. : Taking logarithm, we have

$$y \log \cos x = x \log \sin y$$

Let $$f(x, y) = y \log \cos x - x \log \sin y = 0$$

$$\therefore \quad \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

Now, $$\frac{\partial f}{\partial x} = y \cdot \frac{1}{\cos x}(-\sin x) - \log \sin y$$

$$= -y \tan x - \log \sin y$$

and $$\frac{\partial f}{\partial y} = \log \cos x - x \frac{1}{\sin y} \cos y$$

$$= \log \cos x - x \cot y$$

$\therefore$ From equation (1),

$$\frac{dy}{dx} = \frac{y \tan x + \log \sin y}{\log \cos x - x \cot y}$$

HOMOGENEOUS FUNCTION:

Definition: A function $f(x_1, x_2, \ldots x_n)$ is said to be homogeneous of degree k if $f(tx_1, tx_2, \ldots, tx_n) = t^k f(x_1, x_2, \ldots x_n)$

Or

f(x, y) is homogeneous function of degree n if it can be expressed as

$$x^n \phi\left(\frac{y}{x}\right) \quad \text{OR} \quad y^n \phi\left(\frac{x}{y}\right)$$

PPT-03

Examples :

1) $F(x,y) = x^2 \sin\left(\frac{x}{y}\right)$ is homogenous with degree 2 .

2) $F(x,y) = x^3 + xy^2 + y^3$ is homogenous with degree 3 .

3) $F(x,y) = \frac{x^3+y^3}{x^2+y^2}$ is homogenous with degree 1 .

4) $F(x,y) = \sin(\frac{x^3+y^3}{x^2+y^2})$ is not homogenous .

Euler's Theorem: If z is a homogeneous function of x and y of degree n then

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$$

Proof: As z is homogeneous function of degree n,

We have, $z = x^n f\left(\frac{y}{x}\right)$

$$\frac{\partial z}{\partial x} = x^n f'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) + nx^{n-1} f\left(\frac{y}{x}\right)$$

$$or\ \frac{\partial z}{\partial x} = -yx^{n-2}f'\left(\frac{y}{x}\right) + nx^{n-1}f\left(\frac{y}{x}\right)$$

$$\frac{\partial z}{\partial y} = x^n f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) = x^{n-1}f'\left(\frac{y}{x}\right)$$

$$x\frac{\partial z}{\partial x} + \mathrm{y}\frac{\partial z}{\partial y} = -yx^{n-1}f'\left(\frac{y}{x}\right) + nx^n f\left(\frac{y}{x}\right) + yx^{n-1}f'\left(\frac{y}{x}\right)$$

$$\therefore x\frac{\partial z}{\partial x} + \mathrm{y}\frac{\partial z}{\partial y} = nxf\left(\frac{y}{x}\right) = nz$$

Hence proof.

Euler's theorem for three variables: If f is a homogeneous function of three independent variables x, y, z of order n, then

$$\boxed{xf_x + yf_y + zf_z = nf}$$

Note:

- In Euler′s theorem n could be + ve, −ve real number or 0

- The operator $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ is called as differential operator which operates on the function f.

COROLLARY I:

If z = $f(x, y)$ is a homo. function of x and y of degree n,

then $$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = n(n-1)z$$

COROLLARY II:

If z is a Homogeneous function of degree n in the variables x and y and z = f(u) then

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n \frac{f(u)}{f'(u)}$$

COROLLARY 3:

If z is a homogeneous function of degree n in the variables x and y and z=f(u)

$$\boxed{x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial^2 u}{\partial x\partial y}=g(u)[g'(u)-1],}$$

where

$$\boxed{g(u)=n\frac{f(u)}{f'(u)}}$$

Ex. 1 : If $u = \sin^{-1}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$, show that $\frac{\partial u}{\partial x} = -\frac{y}{x}\cdot\frac{\partial u}{\partial y}$.

Sol. : $u = x^0 \cdot \sin^{-1}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right) = x^0 \sin^{-1}\left[\frac{1-\sqrt{\frac{y}{x}}}{1+\sqrt{\frac{y}{x}}}\right]$

$$= x^0 f\left(\frac{y}{x}\right)$$

Hence u is a homogeneous function of degree 0, hence by Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0\,(u) = 0$$

$$\Rightarrow \quad x\frac{\partial u}{\partial x} = -y\frac{\partial u}{\partial y}$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = -\frac{y}{x}\frac{\partial u}{\partial y}$$ Proved.

PPT-03

Ex. 9 : Verify Euler's theorem on homogeneous function for $u = \dfrac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}$.

Sol. : We have $$\frac{\partial u}{\partial x} = \frac{\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right) - (x+y+z)\cdot\dfrac{1}{2\sqrt{x}}}{\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)^2}$$

and similar expression for $\dfrac{\partial u}{\partial y}, \dfrac{\partial u}{\partial z}$.

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{(\sqrt{x}+\sqrt{y}+\sqrt{z})(x+y+z) - \dfrac{1}{2}(x+y+z)(\sqrt{x}+\sqrt{y}+\sqrt{z})}{(\sqrt{x}+\sqrt{y}+\sqrt{z})^2}$$

$$= \frac{\dfrac{1}{2}(x+y+z)(\sqrt{x}+\sqrt{y}+\sqrt{z})}{(\sqrt{x}+\sqrt{y}+\sqrt{z})^2}$$

$$= \frac{1}{2}\,\frac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}} = \frac{1}{2}u \qquad \ldots (1)$$

Also $$u = \frac{x\left(1+\dfrac{y}{x}+\dfrac{z}{x}\right)}{\sqrt{x}\left(1+\sqrt{\dfrac{y}{x}}+\sqrt{\dfrac{z}{x}}\right)} = x^{1/2} f\left(\frac{y}{x}, \frac{z}{x}\right)$$

which is a homogeneous function of x, y, z of degree $n = \dfrac{1}{2}$.

$\therefore$ By Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu = \frac{1}{2}u \qquad \ldots (2)$$

From equations (1) and (2), we see that Euler's theorem is verified for the given function.

PPT-03

Ex. 5 : If $u = \operatorname{cosec}^{-1}\sqrt{\dfrac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}$, show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{12}\left(\frac{13}{12} + \frac{\tan^2 u}{12}\right).$$

Sol. : Here

$$\operatorname{cosec} u = \sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}} = \sqrt{\frac{x^{1/2}\,[1+(y/x)^{(1/2)}]}{x^{1/3}\,[1+(y/x)^{(1/3)}]}}$$

$$= x^{1/12}\left[\frac{1+(y/x)^{1/2}}{1+(y/x)^{1/3}}\right]^{1/2} = x^{1/12}\,\phi\left(\frac{y}{x}\right)$$

$$= \text{a homogeneous function of } x, y \text{ of order } \frac{1}{12}$$

Also $f(u) = \operatorname{cosec} u$

$\therefore$ By formula,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = \frac{1}{12}\,\frac{\operatorname{cosec} u}{-\operatorname{cosec} u\cdot\cot u} = -\frac{1}{12}\tan u$$

Differentiating partially with respect to x and y respectively, we get

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x\,\partial y} = -\frac{1}{12}\sec^2 u\cdot\frac{\partial u}{\partial x}$$

and

$$x\frac{\partial^2 u}{\partial x\,\partial y} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = -\frac{1}{12}\sec^2 u\cdot\frac{\partial u}{\partial y}$$

Multiplying equation (2) by x and equation (3) by y and adding,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = -\frac{\sec^2 u}{12}\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$\Rightarrow \quad x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) \times \left(-\frac{\sec^2 u}{12} - 1\right)$$

$$= -\frac{1}{12}\tan u\left[-\left(\frac{1+\tan^2 u}{12}\right) - 1\right]$$

$$= \frac{\tan u}{12}\left(\frac{\tan^2 u}{12} + \frac{13}{12}\right)$$

PPT-03