Some of the important partial differential equations involving two independent variables and one dependent variable which occur in the study of Engineering and Physical problems are :

I. The Wave Equation :

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

which occurs in the problems involving *vibrations of a stretched string*. It is also called as *one-dimensional wave equation.*

II. Diffusion Equation in One Dimension (One-dimensional heat flow equation) :

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

which occurs in the conduction of heat flow along a bar.

III. Laplace's Equation in Two dimensions (Two-dimensional heat flow equation) :

(a) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{(Cartesian form)}$$

which occurs in the conduction of heat in a plate in steady state. The equation is also satisfied by electrostatic potential (ϕ).

(b) $$r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0 \quad \text{(Polar form)}$$

There are various methods of solving PDEs. However, in what follows, we shall consider solution of linear PDE by the method of separation of variables. This method in general is used to reduce the PDE to the solutions of a set of ordinary differential equations each of which involves only one of the variables.

Note : We know that if an ordinary differential equation is linear and homogeneous then from known solutions, we can obtain further solutions by superposition. For a homogeneous linear partial differential equation, the situation is quite similar.

If y_1 and y_2 are any solutions of a linear homogeneous partial differential equation in some region R, then $\boxed{y = c_1 y_1 + c_2 y_2}$ where c_1 and c_2 are any constants, is also a solution of that equation in R.

11.3 MODELING OF VIBRATIONS OF A STRETCHED STRING (ONE-DIMENSIONAL WAVE EQUATION)

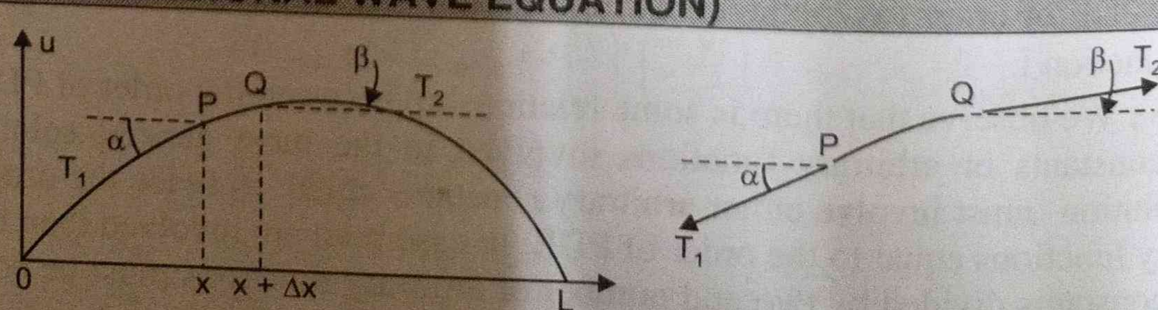

Fig. 11.1 : Deflected String at Fixed Time t

Let us derive the equation governing small transverse vibrations of an elastic string (such as a violin string). We stretch the string to length l and fix it at the ends. We then distort it and at some instant, say, $t = 0$, we release it and allow it to vibrate. The problem is to determine the vibrations of the string, that is, to find its deflection $u(x, t)$ at any point x and at any time $t > 0$.

While deriving a differential equation corresponding to a given physical problem, we usually have to make simplifying assumptions to ensure that the resulting equation does not become too complicated.

We assume the following *physical assumptions* :

(a) The string is perfectly elastic and does not offer any resistance to bending.

(b) The mass of the string per unit length is constant.

(c) The tension caused by stretching the string before fixing it at the ends is so large that the action of the gravitational force on the string can be neglected.

(d) The string performs small transverse motions in a vertical plane; that is, every particle of the string moves strictly vertically and so that the deflection and the slope at every point of the string always remain small in absolute value.

Under these assumptions, we may expect that the solution $u(x, t)$ of the differential equation to be obtained will reasonably well describe small vibrations of the physical non-idealized string of small homogeneous mass under large tension.

Consider the forces acting on a small portion of the string. Since the string does not offer resistance to bending, the tension is tangential to the curve of the string at each point. Let T_1 and T_2 be the tensions at the end points P and Q of that portion. Since there is no motion in horizontal direction, the horizontal components of the tension must be constant.

$$\therefore \quad T_1 \cos\alpha = T_2 \cos\beta = T = \text{constant} \qquad \ldots (1)$$

In vertical direction, we have two forces, the vertical components $-T_1 \sin\alpha$ and $T_2 \sin\beta$ of T_1 and T_2 (minus sign appears because that component at P is directed downward.).

By Newton's second law, the resultant of these two forces is equal to the mass "m δx" of the portion times the acceleration $\frac{\partial^2 u}{\partial t^2}$, evaluated at some point between x and $x + \delta x$, where, m = mass of the undeflected string per unit length, δx = the length of the portion of the undeflected string.

$$\therefore \quad T_2 \sin\beta - T_1 \sin\alpha = m\,\delta x \cdot \frac{\partial^2 u}{\partial t^2}$$

By using equation (1), we can divide this by $T_2 \cos\beta = T_1 \cos\alpha = T$,

$$\frac{T_2 \sin\beta}{T_2 \cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \left(\frac{m\,\delta x}{T}\right)\frac{\partial^2 u}{\partial t^2}$$

$$\frac{T}{m}\,\frac{(\tan\beta - \tan\alpha)}{\delta x} = \frac{\partial^2 u}{\partial t^2}$$

Now tan α and tan β are the slopes of the string at x and x + δx.

$$\therefore \quad \tan\alpha = \left(\frac{\partial u}{\partial x}\right)_x \quad \text{and} \quad \tan\beta = \left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$$

Note : Here we write partial derivatives because u also depends on t.

$$\therefore \quad \frac{T}{m}\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x}\right] = \frac{\partial^2 u}{\partial t^2}$$

As $\delta x \to 0$,

$$\frac{T}{m} \cdot \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

Let $c^2 = \frac{T}{m}$

$$\therefore \quad \boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}$$

This is called one-dimensional wave equation. One-dimensional indicates that the equation involves only one space variable, x.

The notation c^2 (instead of c) for the physical constant $\frac{T}{m}$ has been chosen to indicate that this constant is positive.

Note : Vibration in membrane or drumhead, oscillations induced in a guitar or violin string is governed by wave equation.

11.4 SOLUTION OF WAVE EQUATION BY METHOD OF SEPARATION OF VARIABLES

The vibrations of an elastic string are governed by the one-dimensional wave equation :

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}} \qquad \dots (1)$$

where, u (x, t) is the deflection of the string.

To find out how the string moves, we determine a solution u (x, t) of (1) that also satisfies the conditions imposed by the physical system.

Since the string is fixed at the ends x = 0 and x = l, we have two **boundary conditions.**

$$\boxed{\textbf{B.C.} : u(0, t) = 0, \; u(l, t) = 0 \;\; \text{for all } t} \qquad \dots (2)$$

The form of the motion of the string will depend on the initial deflection, say f(x) (deflection at $t = 0$) and on the initial velocity, say g(x) (velocity at $t = 0$).

$\therefore$ We obtain the two **initial conditions.**

$$\boxed{\textbf{I.C. : } u(x, 0) = f(x), \quad \left(\frac{\partial u}{\partial t}\right)_{t=0} = g(x)} \quad \dots (3)$$

Our problem is now to find a solution of (1) satisfying the conditions (2) and (3).

We take the following important steps :

Step 1 : By applying the so-called method of separation of variables (or product method), we shall obtain two ordinary differential equations.

Step 2 : We shall determine solutions of those two equations that satisfy the boundary conditions.

Step 3 : Using Fourier series, we shall compose those solutions, in order to get a solution of the wave equation (1) that also satisfies the given initial conditions.

Let us consider these steps one by one.

Step 1 : Two ordinary differential equations : In the method of separating variables (or product method), we determine solution of the wave equation (1) of the form :

$$\boxed{u(x, t) = F(x) \cdot G(t)} \quad \dots (4)$$

where, F (x) is a function of x alone and G (t) is a function of t alone.

$$\frac{\partial u}{\partial t} = F(x)\frac{dG}{dt} \quad \therefore \quad \frac{\partial^2 u}{\partial t^2} = F(x) \cdot \frac{d^2G}{dt^2}$$

$$\frac{\partial u}{\partial x} = \frac{dF}{dx} \cdot G(t) \quad \therefore \quad \frac{\partial^2 u}{\partial x^2} = \frac{d^2F}{dx^2} \cdot G(t)$$

But
$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$\therefore$
$$F(x) \cdot \frac{d^2G}{dt^2} = c^2 \frac{d^2F}{dx^2} \cdot G(t)$$

Or
$$F G'' = c^2 F'' G$$

$\therefore$
$$\frac{G''}{c^2 G} = \frac{F''}{F}$$

i.e.
$$\frac{F''}{F} = \frac{G''}{c^2 G}$$

Here L.H.S. is a function of x alone and R.H.S. is a function of t alone, hence above two expressions are independent of each other. Therefore we can equate to any constant, say k.

$\therefore$
$$\frac{F''}{F} = \frac{G''}{c^2 G} = k$$

This yields immediately two ordinary linear differential equations :

$$\frac{F''}{F} = k \Rightarrow F'' - kF = 0 \Rightarrow \frac{d^2F}{dx^2} - kF(x) = 0$$

Also,
$$\frac{G''}{c^2 G} = k \Rightarrow G'' - c^2 kG = 0 \Rightarrow \frac{d^2G}{dt^2} - c^2 kG(t) = 0$$

Step 2 : Satisfying the boundary conditions : The boundary conditions are :

$$u(0, t) = 0, \; u(l, t) = 0 \quad \text{for all } t.$$

But $\quad u(x, t) = F(x) \cdot G(t)$

$\therefore \quad u(0, t) = 0 \Rightarrow F(0) \cdot G(t) = 0$

$\quad u(l, t) = 0 \Rightarrow F(l) \cdot G(t) = 0$

If $G(t) = 0$ then $u = 0$ is a trivial solution. Thus $G(t) \neq 0$, and hence $F(0) = F(l) = 0$

Case (i) : Let $\quad k = 0$

$$\therefore \quad F'' = 0 \quad \Rightarrow \quad \frac{d^2F}{dx^2} = 0$$

whose solution is $F(x) = c_1 x + c_2$.

$$F(0) = 0 \Rightarrow c_2 = 0 \;;\; F(l) = 0 \Rightarrow c_1 = 0$$

$\therefore \quad F(x) = 0$ which is of no interest because then $u = 0$.

Hence we reject the case $k = 0$.

Case (ii) : Let $k > 0$ i.e. $k = m^2$ (say)

$$\frac{F''}{F} = \frac{G''}{c^2 G} = m^2$$

$$\frac{F''}{F} = m^2 \Rightarrow F'' - m^2 F = 0$$

$$\frac{d^2F}{dx^2} - m^2 F = 0 \quad \left(\text{let } D \equiv \frac{d}{dx}\right)$$

$$D^2 F - m^2 F = 0, \quad (D^2 - m^2) F = 0$$

$$\text{A.E. : } D^2 - m^2 = 0 \Rightarrow D = \pm m.$$

$$\therefore \quad F(x) = c_1 e^{mx} + c_2 e^{-mx}$$

Now
$$F(0) = 0 \quad \Rightarrow \quad c_1 + c_2 = 0$$
$$F(l) = 0 \quad \Rightarrow \quad c_1 e^{ml} + c_2 e^{-ml} = 0$$

Solving we get $c_1 = 0, \; c_2 = 0 \Rightarrow f(x) = 0 \; \therefore \; u = 0$

Hence we reject the case $k > 0$ too.

Case (iii) (Important) : Let $k < 0$ i.e. $k = -m^2$ (say).

$$\frac{F''}{F} = \frac{G''}{c^2 G} = -m^2$$

$$\frac{F''}{F} = -m^2 \Rightarrow F'' + m^2 F = 0$$

$$\frac{d^2F}{dx^2} + m^2 F = 0 \qquad \left(\text{let } D \equiv \frac{d}{dx}\right)$$

$$D^2 F + m^2 F = 0, \ (D^2 + m^2) F = 0$$

A.E. : $D^2 + m^2 = 0 \Rightarrow D = \pm\, im$

$\therefore$ $F(x) = c_1 \cos mx + c_2 \sin mx.$

Now $F(0) = 0 \Rightarrow c_1 = 0$

$\therefore$ $F(x) = c_2 \sin mx,$

and $F(l) = 0 \Rightarrow 0 = c_2 \sin ml$.

Now, $c_2 \neq 0$ since otherwise $F = 0$, and hence $u = 0$.

$\therefore$ $\sin ml = 0 \Rightarrow ml = n\pi, \ n = 1, 2, 3, 4 \ldots$ [Since $\sin n\pi = 0$ for all n]

$\therefore$ $m = \frac{n\pi}{l}$

We thus obtain infinitely many solutions $F(x) = F_n(x)$, where

$$F_n(x) = c_2 \sin \frac{n\pi x}{l}, \ n = 1, 2, 3, \ldots\ldots$$

Also, $\frac{G''}{c^2 G} = -m^2 \qquad \left(\text{where } m = \frac{n\pi}{l}\right)$

$$G'' + c^2 m^2 G = 0 \qquad \Rightarrow \frac{d^2G}{dt^2} + c^2 m^2 G = 0$$

Let $D \equiv \frac{d}{dt}$ $\quad\therefore D^2 G + c^2 m^2 G = 0 \Rightarrow (D^2 + c^2 m^2) G = 0$

A.E. : $D^2 + c^2 m^2 = 0 \Rightarrow D = \pm\, i\,(cm)$

$\therefore$ $G(t) = c_3 \cos cmt + c_4 \sin cmt$

Or $G(t) = G_n(t) = c_3 \cos \frac{n\pi ct}{l} + c_4 \sin \frac{n\pi ct}{l} \qquad \left(\because m = \frac{n\pi}{l}\right)$

Now, $u(x, t) = F(x) \cdot G(t) = F_n(x) \cdot G_n(t) = u_n(x, t) \ \ldots$ (say)

$\therefore$ $u_n(x, t) = \left(c_2 \sin \frac{n\pi x}{l}\right)\left[c_3 \cos \frac{n\pi ct}{l} + c_4 \sin \frac{n\pi ct}{l}\right]$

$$u_n(x, t) = \left[A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l}\right] \sin \frac{n\pi x}{l}$$

where, $A_n = c_2 c_3, \ B_n = c_2 c_4, \ n = 1, 2, 3, \ldots\ldots\ldots\ldots$

Step 3 : Solution of the entire problem : It is evident that no single solution of $u_n(x, t)$ can satisfy the initial conditions. However, the given P.D.E. is linear, principle of superimposition is valid meaning thereby that if, *we have several solutions then their sum is also a solution.*

Hence we take, $u(x, t) = \sum_{n=1}^{\infty} u_n(x, t)$

$$\boxed{u(x, t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l}\right) \sin \frac{n\pi x}{l}} \quad \dots (5)$$

as a *most general solution* which may yield a solution satisfying the initial conditions.

From (3) : $u(x, 0) = f(x)$, (5) becomes $u(x, 0) = f(x) = \sum_{n=1}^{\infty} (A_n) \cdot \sin \frac{n\pi x}{l}, \ 0 < x < l.$

We must choose A_n so that $u(x, 0)$ becomes the half range Fourier sine series of $f(x)$.

$$\therefore \quad \boxed{A_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx}$$

To determine B_n, we have

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \left(-\frac{n\pi c}{l} A_n \sin \frac{n\pi ct}{l} + \frac{n\pi c}{l} B_n \cos \frac{n\pi ct}{l}\right) \sin \frac{n\pi x}{l}$$

From (3), $\left(\frac{\partial u}{\partial t}\right)_{t=0} = g(x).$

$$g(x) = \sum_{n=1}^{\infty} \left(\frac{n\pi c}{l} B_n\right) \sin \frac{n\pi x}{l}$$

We must choose B_n so that $\left(\frac{\partial u}{\partial t}\right)_{t=0}$ becomes the half range Fourier sine series of $g(x)$.

$$\therefore \quad \frac{n\pi c}{l} B_n = \frac{2}{l} \int_0^l g(x) \cdot \sin \frac{n\pi x}{l} \, dx$$

$$\therefore \quad \boxed{B_n = \frac{2}{n\pi c} \int_0^l g(x) \cdot \sin \frac{n\pi x}{l} \, dx}$$

Substituting values of A_n and B_n in (5), we get the required solution.

Result : Our discussion shows that u (x, t) given by (5) with coefficients A_n and B_n is a solution of (1) that satisfies all the conditions (2) and (3) of our problem, provided the series (5) converges.

Summary : To solve the one-dimensional wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{subject to the conditions}$$

$$\left.\begin{aligned} &1. \quad u(0, t) = 0 \\ &2. \quad u(l, t) = 0 \end{aligned}\right\} \text{Boundary conditions}$$

$$\left.\begin{aligned} &3. \quad u(x, 0) = f(x) \\ &4. \quad \left(\frac{\partial u}{\partial t}\right)_{t=0} = g(x) \end{aligned}\right\} \text{Initial conditions}$$

The most general solution is given by

$$\boxed{u(x, t) = (c_1 \cos mx + c_2 \sin mx).(c_3 \cos cmt + c_4 \sin cmt)} \quad \ldots(6)$$

In obtaining solutions of the problems on vibration of tightly stretched string, we should directly assume the solution given in (6).

ILLUSTRATIONS

Ex. 1 : *If* $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ *represents the vibrations of a string of length l fixed at both ends, find the solution with boundary conditions,*

(i) $y(0, t) = 0,$

(ii) $y(l, t) = 0$

and initial conditions,

(iii) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$

(iv) $y(x, 0) = k(lx - x^2),\ 0 \le x \le l.$ **(Dec. 2008)**

Sol. : Given $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$. The most general solution is given by

$$y(x, t) = (c_1 \cos mx + c_2 \sin mx)\ (c_3 \cos cmt + c_4 \sin cmt)$$

Applying condition (i), $y(0, t) = 0,$

$$0 = [c_1(1) + c_2(0)]\ [c_3 \cos cmt + c_4 \sin cmt] \qquad \therefore \boxed{c_1 = 0}$$

$$\therefore \quad y(x, t) = (c_2 \sin mx)\ [c_3 \cos cmt + c_4 \sin cmt]$$

To apply condition (iii), $\left(\dfrac{\partial y}{\partial t}\right)_{t=0} = 0$, we first obtain $\dfrac{\partial y}{\partial t}$.

$$\therefore \quad \frac{\partial y}{\partial t} = (c_2 \sin mx)\ [-cm\, c_3 \sin cmt + cm\, c_4 \cos cmt]$$

$$0 = (c_2 \sin mx)\ [0 + cm\, c_4]$$

Here, $\quad c_2 \neq 0, \ \sin mx \neq 0 \quad \therefore \ \boxed{c_4 = 0}$

The most general solution will be

$$y(x, t) = (c_2 \sin mx)\ (c_3 \cos cmt)$$

$$y(x, t) = c_5 \sin mx \ \cos cmt \quad \dots (1)$$

Applying condition (ii), $y(l, t) = 0$

$$0 = c_5 \ \sin ml \,.\, \cos cmt$$

Now, $\quad c_5 \neq 0 \quad$ (otherwise $y(x, t) = 0$ will become trivial solution.)

$$\cos cmt \neq 0$$

$$\therefore \quad \sin ml = 0 \Rightarrow ml = n\pi, \ \therefore \ m = \frac{n\pi}{l}, \ n = 1, 2, 3 \ \dots\dots\dots$$

$\therefore$ Solution (1) becomes

$$y(x, t) = c_5 \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}, \ n = 1, 2, \dots\dots$$

Combining all these solutions, we get

$$\boxed{\mathbf{y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}}} \quad \dots (2)$$

Applying condition (iv),

$$y(x, 0) = k(lx - x^2), \ 0 \leq x \leq l$$

$$k(lx - x^2) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}, \ 0 \leq x \leq l$$

This is Fourier's half range sine series for $f(x) = k(lx - x^2)$ in $0 \leq x \leq l$.

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l}\, dx = \frac{2}{l} \int_0^l k(lx - x^2) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2k}{l} \left\{ (lx - x^2) \left(-\frac{l}{n\pi} \cos \frac{n\pi x}{l} \right) - (l - 2x) \left(-\frac{l^2}{n^2\pi^2} \sin \frac{n\pi x}{l} \right) + (-2) \left(\frac{l^3}{n^3\pi^3} \cos \frac{n\pi x}{l} \right) \right\}_0^l$$

$$= \frac{2k}{l} \left(-\frac{2l^3}{n^3\pi^3} \right) \left[\cos \frac{n\pi x}{l} \right]_0^l$$

$$b_n = \frac{4kl^2}{\pi^3}\left(\frac{1-(-1)^n}{n^3}\right)$$

Substituting in (2), we get the required most general solution.

$$y(x,t) = \frac{4kl^2}{\pi^3}\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n^3}\right)\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \quad \text{... Ans.}$$

Note : $1-(-1)^n = 2$; if n is odd

$= 0$; if n is even

$\therefore$ The above solution can be written as

$$y(x,t) = \frac{4kl^2}{\pi^3}\left\{\frac{2}{1^3}\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l} + 0 + \frac{2}{3^3}\sin\frac{3\pi x}{l}\cos\frac{3\pi ct}{l} + 0 + \frac{2}{5^3}\sin\frac{5\pi x}{l}\cos\frac{5\pi ct}{l} + \dots\dots\right\}$$

$$y(x,t) = \frac{4kl^2}{\pi^3}\sum_{n=1}^{\infty}\frac{2}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{l}\cos\frac{(2n-1)\pi ct}{l}$$

$$\boxed{y(x,t) = \frac{8kl^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{l}\cos\frac{(2n-1)\pi ct}{l}}$$

Ex. 2 : *A string is stretched and fastened to two points l apart. Motion is started by displacing the string in the form $u = a\sin\frac{\pi x}{l}$ from which it is released at time $t = 0$. Find the displacement u (x, t) from one end. $\left(\text{Use wave equation } \frac{\partial^2 u}{\partial t^2} = c^2 \cdot \frac{\partial^2 u}{\partial x^2}\right)$.*

(Dec. 2004)

Sol. : Given $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

Subject to the conditions

(i) $u(0,t) = 0, \ \forall t$

(ii) $u(l,t) = 0, \ \forall t$

(iii) $\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0$?

(iv) $u(x,0) = a\sin\frac{\pi x}{l}$

The general solution is

$$u(x,t) = (c_1\cos mx + c_2\sin mx)(c_3\cos cmt + c_4\sin cmt)$$

Condition (i) $\Rightarrow c_1 = 0$

Condition (iii) $\Rightarrow c_4 = 0$.

$\therefore$ Solution becomes

$$u(x, t) = (c_2 \sin mx)\ (c_3 \cos cmt)$$

$$u(x, t) = c_5 \sin mx \cos cmt \qquad \ldots (1)$$

Applying condition (ii), we get,

$$0 = c_5 \sin ml \cos cmt$$

$$c_5 \neq 0, \quad \cos cmt \neq 0 \qquad \therefore \sin ml = 0$$

$$ml = n\pi \qquad \therefore m = \frac{n\pi}{l},\ n = 1, 2, 3, \ldots\ldots$$

$\therefore$ Substituting in (1), we get

$$u(x, t) = c_5 \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l},\ n = 1, 2, 3\ \ldots\ldots\ldots$$

Combining these solutions, we get,

$$\boxed{u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}} \qquad \ldots (2)$$

Applying condition (iv), we get

$$u(x, 0) = a \sin \frac{\pi x}{l}$$

$$a \sin \frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$a \sin \frac{\pi x}{l} = b_1 \sin \frac{\pi x}{l} + b_2 \sin \frac{2\pi x}{l} + \ldots\ldots$$

$\therefore$ $b_1 = a$, $b_2 = 0 = b_3 = \ldots\ldots = b_n = \ldots\ldots\ldots\ldots$

$\therefore$ (2) will become

$$u(x, t) = b_1 \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l} + b_2 \sin \frac{2\pi x}{l} \cos \frac{2\pi ct}{l} + \ldots\ldots$$

$$\boxed{u(x, t) = a \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l}}$$

which is the required general solution.

Ex. 3 : *A string is stretched tightly between $x = 0$, $x = l$ and both ends are given displacement $y = a \sin pt$ perpendicular to the string. If the string satisfies the differential equation $\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2}$, prove that the oscillations of the string are given by*

$$y = a \sec \frac{pl}{2c} \cos\left(\frac{px}{c} - \frac{pl}{2c}\right) \sin pt.$$

(May 2005)

Sol. : We have the G.S.,

$$y = (c_1 \cos mx + c_2 \sin mx)\ (c_3 \cos cmt + c_4 \sin cmt) \qquad \ldots \text{(i)}$$

The condition $y = a \sin pt$ for $x = 0$ gives

$$a \sin pt = c_1 (c_3 \cos cmt + c_4 \sin cmt)$$

This implies that $c_1 c_3 = 0$ ... (ii)

$$c_1 c_4 = a \qquad \ldots \text{(iii)}$$

and $cm = p$ ... (iv)

From (ii), (iii), $c_3 = 0$ and from (iv), $m = \dfrac{p}{c}$

Substituting in (i), we get

$$y = \left(c_1 \cos \frac{px}{c} + c_2 \sin \frac{px}{c}\right) \cdot c_4 \sin pt$$

$$= \left(c_1 c_4 \cos \frac{px}{c} + c_2 c_4 \sin \frac{px}{c}\right) \sin pt$$

$$\Rightarrow \quad y = \left(a \cos \frac{px}{c} + c_2 c_4 \sin \frac{px}{c}\right) \sin pt \qquad \ldots \text{(v)}$$

The condition $y = a \sin pt$ for $x = l$ gives

$$\boxed{a \sin pt = \left(a \cos \frac{pl}{c} + c_2 c_4 \sin \frac{pl}{c}\right) \sin pt}$$

$$\Rightarrow a\left(1 - \cos \frac{pl}{c}\right) = c_2 c_4 \sin \frac{pl}{c}$$

$$\Rightarrow \quad c_2 c_4 = a \cdot \frac{2 \sin^2 \frac{pl}{2c}}{2 \sin \frac{pl}{2c} \cos \frac{pl}{2c}} = a \cdot \frac{\sin \frac{pl}{2c}}{\cos \frac{pl}{2c}}$$

Substituting in (v), we get

$$y = \left(a \cos \frac{px}{c} + a \frac{\sin \frac{pl}{2c}}{\cos \frac{pl}{2c}} \sin \frac{px}{c}\right) \sin pt$$

$$= a \cdot \frac{\cos \frac{px}{c} \cos \frac{pl}{2c} + \sin \frac{px}{c} \sin \frac{pl}{2c}}{\cos \frac{pl}{2c}} \sin pt$$

$$\Rightarrow \quad \boxed{y = a \sec \frac{pl}{2c} \cos\left(\frac{px}{c} - \frac{pl}{2c}\right) \sin pt.}$$

Ex. 4 : *A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by $y(x, 0) = y_o \sin^3\left(\frac{\pi x}{l}\right)$. If it is released from rest from this position, find the displacement y at any distance x from one end and at any time t.*

(Dec. 06, May 07, May 2011)

Sol. : The differential equation satisfied by y is $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$. The initial and boundary conditions are given by :

(i) $y(0, t) = 0$, (ii) $y(l, t) = 0$, (iii) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$,

(iv) $y(x, 0) = y_o \sin^3\left(\frac{\pi x}{l}\right)$

The most general solution is given by :

$$y(x, t) = (c_1 \cos mx + c_2 \sin mx)(c_3 \cos cmt + c_4 \sin cmt)$$

Condition (i) $\Rightarrow c_1 = 0$

Condition (iii) $\Rightarrow c_4 = 0$

$\therefore$ The most general solution will become

$$\boxed{y(x, t) = c_5 \sin mx \cdot \cos cmt} \quad \text{... (I)}$$

Condition (ii) $\Rightarrow 0 = c_5 \sin ml \cdot \cos cmt$

$\therefore \quad \sin ml = 0, \quad ml = n\pi$

$\therefore \quad m = \frac{n\pi}{l}, \; n = 1, 2, \ldots\ldots\ldots$

$\therefore$ Solution (I) becomes :

$$y(x, t) = c_5 \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}, \; n = 1, 2, \ldots\ldots\ldots$$

Combining all these solutions, we get

$$\boxed{y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}} \quad \text{... (II)}$$

Applying condition (iv),

$$y(x, 0) = y_o \sin^3 \frac{\pi x}{l} = \frac{3y_o}{4} \sin \frac{\pi x}{l} - \frac{y_o}{4} \sin \frac{3\pi x}{l}$$

$$\left(\text{by using } \sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta\right)$$

$$\therefore \quad y_o \sin^3 \frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$\frac{3y_0}{4}\sin\frac{\pi x}{l} - \frac{y_0}{4}\sin\frac{3\pi x}{l} = b_1 \sin\frac{\pi x}{l} + b_2 \sin\frac{2\pi x}{l} + b_3 \sin\frac{3\pi x}{l} + \ldots$$

Comparing we get, $b_1 = \frac{3y_0}{4}$; $b_2 = 0$; $b_3 = -\frac{y_0}{4}$; $b_4 = 0 = b_5 = b_6 = \ldots = b_n = \ldots$

Substituting in (II), we get,

$$y(x, t) = b_1 \sin\frac{\pi x}{l}\cos\frac{\pi ct}{l} + b_2 \sin\frac{2\pi x}{l}\cos\frac{2\pi ct}{l} + b_3 \sin\frac{3\pi x}{l}\cos\frac{3\pi ct}{l} + \ldots\ldots$$

$$\boxed{y(x, t) = \frac{3y_0}{4}\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l} - \frac{y_0}{4}\sin\frac{3\pi x}{l}\cos\frac{3\pi ct}{l}}$$

Ex. 5 : *An elastic string is stretched between two fixed points at a distance l apart, one end is taken at the origin and at a distance $\frac{2l}{3}$ from this end the string is displaced a distance "a" transversely and is released from rest when in this position. Find y (x, t), if satisfies the equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$.* **(Dec. 2006)**

Sol. :

$$\text{Slope of OB} = \frac{a}{2l/3} = \frac{3a}{2l}$$

$$\text{Equation of OB is } y = \frac{3a}{2l}\, x$$

$$\text{Slope of BA} = \frac{a-0}{\frac{2l}{3} - l} = -\frac{3a}{l}$$

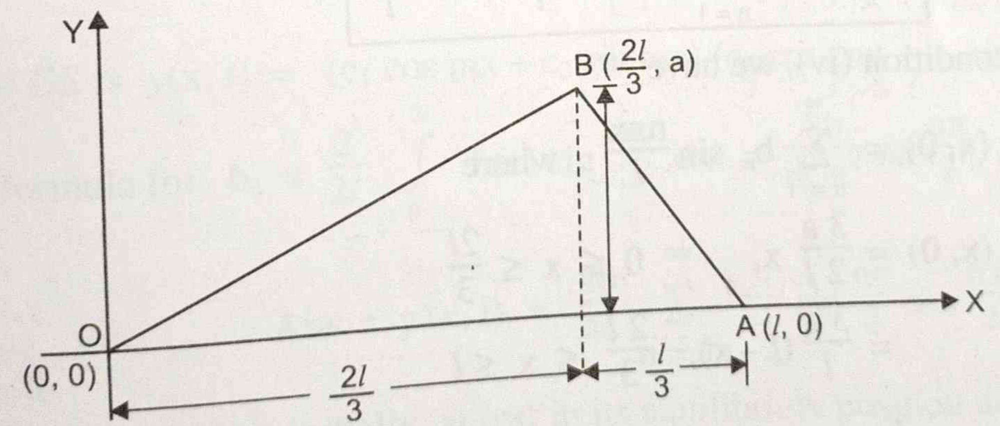

Fig. 11.2

$$\text{Equation of BA is } y - 0 = -\frac{3a}{l}(x - l)$$

$$\therefore \quad y = \frac{3a}{l}(l - x)$$

We have to solve $\quad \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$

The boundary conditions are

(i) $y(0, t) = 0$

(ii) $y(l, t) = 0$

Initial conditions are

(iii) $\left(\dfrac{\partial y}{\partial t}\right)_{t=0} = 0$

(iv) $y(x, 0) = \dfrac{3a}{2l} x, \quad 0 \le x \le \dfrac{2l}{3}$

$= \dfrac{3a}{l}(l - x), \quad \dfrac{2l}{3} \le x \le l$

The most general solution is

$$y(x, t) = (c_1 \cos mx + c_2 \sin mx)\,(c_3 \cos cmt + c_4 \sin cmt)$$

Condition (i) $\Rightarrow$ $c_1 = 0$

Condition (iii) $\Rightarrow$ $c_4 = 0$

$$\therefore \quad \boxed{y(x, t) = c_5 \sin mx \cos cmt} \qquad \ldots \text{(I)}$$

Condition (ii) $\Rightarrow$ $0 = c_5 \sin ml \cos cmt$

$$\sin ml = 0, \; ml = n\pi \quad \therefore \quad m = \frac{n\pi}{l}, \; n = 1, 2, \ldots\ldots\ldots$$

$$y(x, t) = c_5 \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}, \; n = 1, 2, \ldots\ldots$$

Combining all these solutions, we get

$$\boxed{y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}} \qquad \ldots \text{(II)}$$

Applying condition (iv), we have

$$y(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}, \text{ where}$$

$$f(x) = y(x, 0) = \frac{3a}{2l} x, \quad 0 \le x \le \frac{2l}{3}$$

$$= \frac{3a}{l}(l - x), \quad \frac{2l}{3} \le x \le l$$

$$\therefore \quad b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l} \left\{ \int_0^{\frac{2l}{3}} \frac{3ax}{2l} \sin \frac{n\pi x}{l}\, dx + \int_{\frac{2l}{3}}^{l} \frac{3a}{l}(l - x) \sin \frac{n\pi x}{l}\, dx \right\}$$

$$= \frac{6a}{l^2}\left\{\left[\left(\frac{x}{2}\right)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right)-\left(\frac{1}{2}\right)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_0^{\frac{2l}{3}}\right.$$

$$\left.+\left[(l-x)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right)-(-1)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_{\frac{2l}{3}}^{l}\right\}$$

$$= \frac{6a}{l^2}\left\{-\frac{l^2}{3n\pi}\cos\frac{2n\pi}{3}+\frac{l^2}{2n^2\pi^2}\sin\frac{2n\pi}{3}+\frac{l^2}{3n\pi}\cos\frac{2n\pi}{3}+\frac{l^2}{n^2\pi^2}\sin\frac{2n\pi}{3}\right\}$$

$$= \frac{6a}{l^2}\left(\frac{3}{2}\frac{l^2}{n^2\pi^2}\sin\frac{2n\pi}{3}\right) = \frac{9a}{\pi^2}\frac{1}{n^2}\sin\frac{2n\pi}{3}$$

∴ Substituting in (II), we get,

$$\boxed{y(x,t) = \frac{9a}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{2n\pi}{3}\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}}$$

EXERCISE 11.1

1. A taut string of a length $2l$ is fastened at both ends. The mid point of the string is taken to a height b and then released from rest in that position. Obtain the displacement.

Hint : B.C. : (i) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$, (ii) $y(2l, t) = 0$, Use $\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$

I.C. : (iii) $y(0, t) = 0$; (iv) $y(x, 0) = \begin{cases}\frac{bx}{l}, & 0 \le x \le l\\ \frac{b}{l}(2x - x), & l \le x \le 2l\end{cases}$

The most GS is $y(x, t) = (c_1 \cos mx + c_2 \sin mx)(c_3 \cos cmt + c_4 \sin cmt)$

and use formula for $b_n = \frac{2}{2l}\int_0^{2l} f(x)\sin\frac{n\pi x}{2l}dx = \frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2}$.

Ans. : $y(x,t) = \frac{8b}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{2l}\cos\left(\frac{n\pi ct}{2l}\right)$

2. If a string of length l is initially at rest in its equilibrium position and each of its point is given a velocity v (x) such that

$$v(x) = \begin{cases} cx, & 0 < x \le \frac{l}{2}\\ c(l-x), & \frac{l}{2} \le x \le l\end{cases}$$

Obtain the displacement y (x, t) at any time t.

Hint : B.C. : (i) y (0, t) = 0, (ii) y (*l*, t) = 0, Use $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

(iii) I.C. : $\left(\frac{\partial y}{\partial t}\right)_{t=0} = \begin{cases} cx, & 0 \le x \le \frac{l}{2} \\ c(l-x), & \frac{l}{2} \le x \le l \end{cases}$, (iv) y (x, 0) = 0

and use $b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\, dx = \frac{4cl}{n^2\pi^2}\sin\frac{n\pi}{2}$.

Ans. : $y(x, t) = \frac{4cl^2}{a\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$

3. A tightly stretched string with fixed ends x = 0 and x = *l* is initially at rest in its equilibrium position. If it is set vibrating giving each point a velocity 3x (*l* – x) for 0 < x < *l*, find the displacement.

Hint : BC. : (i) y (0, t) = 0, (ii) y (*l*, t) = 0;

I.C. : (iii) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 3x(l-x)$, (iv) y (x, 0) = 0.

and use $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx = \frac{12l^2}{n^3\pi^3}(1-\cos n\pi)$

Ans. : $y(x, t) = \frac{24\, l^3}{\pi^4 c}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\sin\frac{(2n-1)\pi x}{l}\sin\frac{(2n-1)\pi ct}{l}$

4. Work exercise 3, given that velocity is $v_o \sin^3\frac{\pi x}{l}$, $0 \le x \le l$

Ans. : $y(x, t) = \frac{3v_o\, l}{4\pi c}\left[\sin\frac{\pi x}{l}\sin\frac{\pi ct}{l} - \frac{1}{9}\sin\frac{3\pi x}{l}\sin\frac{3\pi ct}{l}\right]$

5. A string is stretched and fastened to two points distance *l* apart is displaced into the form y (x, 0) = 3 (*l*x – x²) from which it is released at t = 0. Find the displacement of the string at a distance x from one end.

Hint : B.C. : (i) y (0, t) = 0, (ii) y (*l*, t) = 0,

I.C. : (iii) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$, (iv) $y(x, 0) = 3(lx - x^2)$.

and $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx = \frac{18l^2}{n^3\pi^3}(1-\cos n\pi)$

Ans. : $\frac{24l^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{l}\cos\frac{(2n-1)\pi ct}{l}$

$n = 1$

11. A uniform string stretched between the points $x = 0$ and $x = l$ is given the initial displacement $y(x, 0) = \sin \frac{\pi x}{l}$, $0 < x < l$ and initial velocity,

$$v(x) = \begin{cases} 0, & 0 < x < \frac{l}{4} \\ a, & \frac{l}{4} < x < \frac{3l}{4} \\ 0, & \frac{3l}{4} < x < l \end{cases}$$

Find subsequent displacement.

$$\textbf{Ans.:}\ y(x, t) = \sin \frac{\pi x}{l} \cos \frac{\pi a t}{l} + \frac{4l}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{2} \sin \frac{n\pi}{4} \sin \frac{n\pi x}{l} \sin \frac{n\pi a t}{l}$$

12. A string of length l fixed at its ends satisfies the wave equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$.

Find the solution if the string has initial triangular deflection given by :

$$y(x, 0) = \begin{cases} \frac{2k}{l} x, & 0 \le x \le \frac{l}{2} \\ \frac{2k}{l} (l - x), & \frac{l}{2} \le x \le l \end{cases}$$

and initial velocity zero.

$$\textbf{Ans.:}\ y(x, t) = \frac{8k}{\pi^2} \sum_{1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin \frac{(2n-1)\pi x}{l} \cos \frac{(2n-1)\pi c t}{l}$$

13. Find the deflection $u(x, t)$ of a vibrating string $\left(\text{length } l = \pi, \text{ ends fixed and } c^2 = \frac{T}{\rho} = 1\right)$ corresponding to zero velocity and initial deflection $0.01(\pi - x)$. **(May 2008)**

$$\textbf{Ans.:}\ u(x, t) = 0.02 \sum_{1}^{\infty} \frac{1}{n} \sin nx \cos nt.$$

14. A flexible string of length π is tightly stretched between $x = 0$, $x = \pi$, on x-axis, its ends being fixed at these points. When set into small transverse vibration, the displacement $y(x, t)$ from x–axis of any point x at time t is given by

$$\frac{\partial^2 y}{\partial t^2} = 4 \frac{\partial^2 y}{\partial x^2}.$$

Find the solution of the equation which satisfies (i) $y(0, t) = 0$, (ii) $y(\pi, t) = 0$, (iii) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$ and (iv) $y(x, 0) = 0.1 \sin x + 0.01 \sin 4x$ for $0 \le x \le \pi$.

Hint : $c^2 = 4$ $\therefore$ $c = 2$, $b_1 = 0.1$, $b_2 = 0$, $b_3 = 0$, $b_4 = 0.01$, $b_5 = b_6 = \ldots = 0$.

Ans. : $y(x, t) = 0.1 \sin x \cos 2t + 0.01 \sin 4x \cos 8t$

11.5 MODELING OF ONE-DIMENSIONAL HEAT FLOW

Derivation of Equation :

We make use of following experimental facts or empirical laws :

(i) Heat flows from higher temperature to lower temperature.

(ii) The rate of flow of heat through an area is proportional to the area and to the temperature gradient in degrees per unit distance $\left(\frac{\partial u}{\partial t}\right)$, where $u(x, t)$ is temperature distribution normal to the area. Constant of proportionality is called the *thermal conductivity* of the material and denoted generally by k.

(iii) The amount of heat required to change the temperature through a given range is proportional to the mass of the body and the change of temperature. The constant of proportionality is termed as specific heat and generally denoted by S.

Consider a homogeneous bar of uniform cross-section, sides coated with insulating material. It is assumed that the loss of heat from the sides by conduction or radiation is negligible. One end of the bar is treated as the origin and the direction of heat flow as positive X-axis. Let ρ be the density (gm/cm³), 'S' the specific heat (cal/gm deg) and k the thermal conductivity (cal/cm-deg.sec). The temperature at any point of the bar depends on the distance x of the point from one end and time 't' and is denoted by u(x, t) or u. Also the temperature distribution through a cross-section is same.

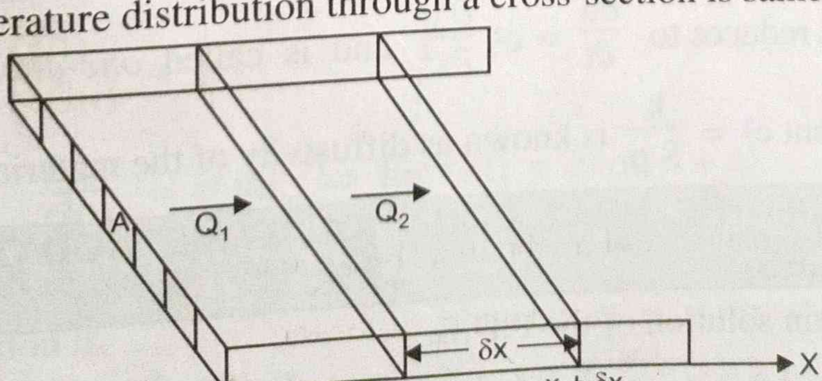

Fig. 11.3

Now, as the quantity of heat crossing any section of the bar is proportional to the area and the temperature gradient normal to the area, the quantity 'Q_1' flowing into the section at a distance x is,

$$Q_1 = -kA\left(\frac{\partial u}{\partial x}\right)_x$$

The quantity 'Q_2' flowing out of the section at a distance $x + \delta x$ is,

$$Q_2 = -kA\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$$

∴ Quantity of heat retained by the slab with thickness δx is,

$$Q_1 - Q_2 = kA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right] \quad \text{... (1)}$$

But the rate of increase of heat in the slab

$$= S\rho A\,\delta x\,\frac{\partial u}{\partial t} \quad \text{... (2)}$$

∴ From equations (1) and (2),

$$S\rho A\,\delta x\,\frac{\partial u}{\partial t} = kA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right]$$

$$\therefore \quad S\rho\,\frac{\partial u}{\partial t} = k\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x}\right]$$

Taking limit as $\delta x \to 0$

$$S\rho\,\frac{\partial u}{\partial t} = k\lim_{\delta x \to 0}\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x}\right]$$

Or $$\frac{\partial u}{\partial t} = \frac{k}{S\rho}\,\frac{\partial^2 u}{\partial x^2}$$

For $\frac{k}{S\rho} = c^2$, it reduces to $\frac{\partial u}{\partial t} = c^2\,\frac{\partial^2 u}{\partial x^2}$ and is called *one-dimensional heat flow equation*. The constant $c^2 = \frac{k}{S\rho}$ is known as diffusivity of the material of the bar.

11.6 SOLUTION OF THE HEAT EQUATION BY METHOD OF SEPARATION OF VARIABLES

We have to obtain solution of the P.D.E.

$$\boxed{\frac{\partial u}{\partial t} = c^2\,\frac{\partial^2 u}{\partial x^2}} \quad \text{... (1)}$$

Let $u(x, t) = F(x) \cdot G(t)$ be the solution. ... (2)

$$\therefore \quad \frac{\partial u}{\partial t} = F(x) \cdot G'(t) \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = F''(x) \cdot G(t)$$

Substituting in (1), we get

$$F(x) \cdot G'(t) = c^2 F''(x) \cdot G(t)$$

$$\frac{G'(t)}{c^2 \cdot G(t)} = \frac{F''(x)}{F(x)}$$

Since L.H.S. is a function of 't' alone and R.H.S. is a function of x alone, therefore both sides are independent of each other, hence can be equated to any arbitrary constant, say k.

$$\therefore \quad \frac{G'(t)}{c^2 \cdot G(t)} = \frac{F''(x)}{F(x)} = k$$

Case (i) : Let $k = 0 \Rightarrow G'(t) = 0 \Rightarrow G(t) = c_1$

and $F''(x) = 0 \Rightarrow F(x) = c_2 x + c_3$

$\therefore$ Complete solution is

$$u(x, t) = (c_2 x + c_3) \cdot c_1$$

Or $\boxed{u(x, t) = c_4 x + c_5}$ ... (3)

Case (ii) : Let $k > 0$ (say $k = m^2$)

$$\frac{F''(x)}{F(x)} = m^2 \Rightarrow F''(x) - m^2 F(x) = 0$$

$$\frac{d^2F}{dx^2} - m^2 F = 0 \quad \left(\text{Let } D \equiv \frac{d}{dx}\right)$$

$$(D^2 - m^2) F = 0 \Rightarrow F(x) = c_1 e^{mx} + c_2 e^{-mx}$$

Also, $$\frac{G'(t)}{c^2 \cdot G(t)} = m^2$$

$$\frac{G'(t)}{G(t)} = c^2 m^2 \Rightarrow \log G(t) = c^2 m^2 t + A$$

$$\therefore \quad G(t) = c_3 e^{c^2 m^2 t}$$

Complete solution is

$$u(x, t) = (c_1 e^{mx} + c_2 e^{-mx}) \; c_3 \cdot e^{c^2 m^2 t}$$

Or $\boxed{u(x, t) = (c_4 e^{mx} + c_5 e^{-mx}) \, e^{c^2 m^2 t}}$ ... (4)

Case (iii) : Let $k < 0$ (say $(k = -m^2)$

$$\frac{F''(x)}{F(x)} = -m^2 \Rightarrow F''(x) + m^2 F(x) = 0$$

$$\frac{d^2F}{dx^2} + m^2 F = 0 \qquad \left(\text{Let } D \equiv \frac{d}{dx}\right)$$

$$(D^2 + m^2) F = 0 \Rightarrow F(x) = c_1 \cos mx + c_2 \sin mx$$

Also, $$\frac{G'(t)}{c^2 \cdot G(t)} = -m^2 \Rightarrow \frac{G'(t)}{G(t)} = -c^2 m^2$$

$$\log G(t) = -c^2 m^2 t + A$$

$$G(t) = c_3 \, e^{-c^2 m^2 t}$$

$\therefore$ Complete solution is

$$u(x, t) = (c_1 \cos mx + c_2 \sin mx) \; c_3 \, e^{-c^2 m^2 t}$$

Or $$\boxed{u(x, t) = (c_4 \cos mx + c_5 \sin mx) \, e^{-c^2 m^2 t}} \qquad \dots (5)$$

Again the question arises which solution we must adopt. In present case, we are concerned with conduction of heat from a source which has finite temperature. Naturally, the temperature u (x, t) cannot become unbounded as t increases. Naturally the solution given by (4) is therefore to be rejected and solution given by (5) is suitable in present case. Solution corresponding to $k = 0$ does not involve any t, hence it can be considered as steady-state solution i.e. solution when temperature no longer varies with time t.

Hence in obtaining solution of the problems of one-dimensional heat flow, we will always begin with solution given in (5) and consider solution (3) under steady-state conditions.

Note : Insulated boundary or end means no heat is flowing from it, meaning thereby $\frac{\partial u}{\partial x} = 0$ at that boundary or end.

ILLUSTRATIONS

Ex. 1 : *Solve* $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ *if (i) u is finite* $\forall t$, *(ii)* $u = 0$ *when* $x = 0, \pi \, \forall t$, *(iii)* $u = \pi x - x^2$ *when* $t = 0$ *and* $0 \le x \le \pi$.

Sol. : Given equation is $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$. The boundary conditions are given as :

(i) $u(x, t)$ is bounded $\forall \, t$

(ii) $u(0, t) = 0, \; \forall \, t$

(iii) $u(\pi, t) = 0, \; \forall \, t$

(iv) $u(x, 0) = \pi x - x^2, \; 0 \le x \le \pi$ **(Dec. 2006)**

The most general solution is

$$\therefore \quad u(x, t) = (c_4 \cos mx + c_5 \sin mx) \, e^{-m^2 t} \qquad \dots (1)$$

Applying the second condition :

$$u(0, t) = 0 \Rightarrow c_4 = 0$$

$$\therefore \quad u(x, t) = c_5 \sin mx \, e^{-m^2 t}$$

Applying the third condition $u(\pi, t) = 0$

$$\Rightarrow \quad 0 = c_5 \sin m\pi \, e^{-m^2 t}$$

Now since $c_5 \neq 0$

and $e^{-m^2 t} \neq 0$ $\quad \therefore \sin m\pi = 0 \Rightarrow m\pi = n\pi$

$\therefore \quad m = n$ for $n = 1, 2, 3, 4 \ldots\ldots$

$\therefore$ Solution becomes $u(x, t) = c_5 \sin nx \, e^{-n^2 t}$ for $n = 1, 2, 3 \ldots\ldots\ldots$

Taking $n = 1, 2, 3 \ldots\ldots\ldots$ and varying the constant c_5 for each n, we see that the general solution is

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin nx \, e^{-n^2 t} \quad \ldots (2)$$

Using aforesaid condition (iv) in (2), we get

$$\pi x - x^2 = u(x, 0) = \sum_{n=1}^{\infty} b_n \sin(nx); \quad 0 \leq x \leq \pi$$

In order to treat this as half range Fourier sine series of $(\pi x - x^2)$ in $0 \leq x \leq \pi$, b_n should be chosen as

$$b_n = \frac{2}{\pi} \int_0^{\pi} (\pi x - x^2) \sin nx \, dx$$

$$= \frac{2}{\pi} \left\{ (\pi x - x^2) \left(-\frac{\cos nx}{n} \right) - (\pi - 2x) \left(-\frac{\sin nx}{n^2} \right) + (-2) \left(\frac{\cos nx}{n^3} \right) \right\}_0^{\pi}$$

$$= \frac{4}{\pi n^3} (1 - \cos n\pi) = \begin{cases} 0, & \text{for n even} \\ 8/\pi n^3, & \text{for n odd} \end{cases}$$

$\therefore$ Solution (2) becomes

$$u(x, t) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^3} \sin(nx) \cdot e^{-n^2 t}$$

Or $u(x, t) = \dfrac{8}{\pi} \displaystyle\sum_{r=0}^{\infty} \frac{1}{(2r+1)^3} \sin[(2r+1)x] \cdot e^{-(2r+1)^2 t}$ which is the required solution.

Ex. 2 : *Solve* $\dfrac{\partial V}{\partial t} = k \dfrac{\partial^2 V}{\partial x^2}$ *if* **(May 2011)**

(i) $V \neq \infty$ as $t \to \infty$ (ii) $\left(\dfrac{\partial V}{\partial x}\right)_{x=0} = 0, \ \forall t$

(iii) $V(l, t) = 0, \ \forall t$ (iv) $V(x, 0) = v_o$, for $0 < x < l$. **(Dec. 2004)**

Sol. : The most general solution is

$$V(x, t) = (c_4 \cos mx + c_5 \sin mx)\ e^{-m^2 kt}$$

$$\frac{\partial V}{\partial x} = (-m c_4 \sin mx + m c_5 \cos mx)\ e^{-m^2 kt}$$

Condition (ii) $\Rightarrow c_5 = 0$

$$\therefore \quad V(x, t) = c_4 . \cos mx . e^{-m^2 kt}$$

Condition (iii) $\Rightarrow 0 = c_4 \cos ml \ e^{-m^2 kt}$

$$\therefore \quad \cos ml = 0 \Rightarrow ml = \frac{n\pi}{2}, \text{ (n is odd)}$$

or $m = \dfrac{n\pi/2}{l}$, (n is odd) Or $m = \dfrac{(2n+1)\pi/2}{l}$, $n = 0, 1, 2, \ldots\ldots$

$$V(x, t) = c_4 \cos \frac{[(2n+1)\pi/2]\, x}{l}\ e^{\frac{-[(2n+1)^2 \pi^2/4]\, kt}{l^2}}, \quad n = 0, 1, 2, \ldots$$

Taking n = 0, 1, 2, … and combining all these solutions, we have the general solution

$$\text{as } V(x, t) = \sum_{n=0}^{\infty} a_{2n+1} \cos \frac{[(2n+1)\pi/2]\, x}{l}\ e^{-\frac{[(2n+1)^2 \pi^2/4]\, kt}{l^2}}$$

Note : Notation a_{2n+1}, n = 0, 1, 2, … is used instead a_n because n is odd.

Applying condition (iv), we have

$$v_o = \sum_{n=0}^{\infty} a_{2n+1} \cos \frac{[(2n+1)\pi/2]\, x}{l}$$

which is nothing but half range Fourier cosine series for f (x) = v_o in (0, l) with $a_0 = 0$.

$$\therefore \quad a_{2n+1} = \frac{2}{l} \int_0^l v_o \cos \frac{[(2n+1)\pi/2]\, x}{l}\, dx = \frac{2 v_o}{l} \left[\frac{2l}{(2n+1)\pi} \sin \frac{(2n+1)\pi x}{2l}\right]_0^l$$

$$a_{2n+1} = \frac{4 v_o}{\pi} \frac{1}{(2n+1)} \sin (2n+1)\frac{\pi}{2} = \frac{4 v_o}{\pi} \frac{(-1)^n}{2n+1}$$

$$\therefore \quad V(x, t) = \frac{4 v_o}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cos \frac{(2n+1)\pi x}{2l}\ e^{\frac{-(2n+1)^2 \pi^2 kt}{4l^2}}$$

Ex. 3 : *The equation for the conduction of heat along a bar of length l is* $\dfrac{\partial \theta}{\partial t} = k \dfrac{\partial^2 \theta}{\partial x^2}$, *neglecting radiation. Find an expression for θ if the ends of the bar are maintained at* zero temperature and if initially the temperature is T at the centre of the bar and falls uniformly to zero at its ends.

Sol. :

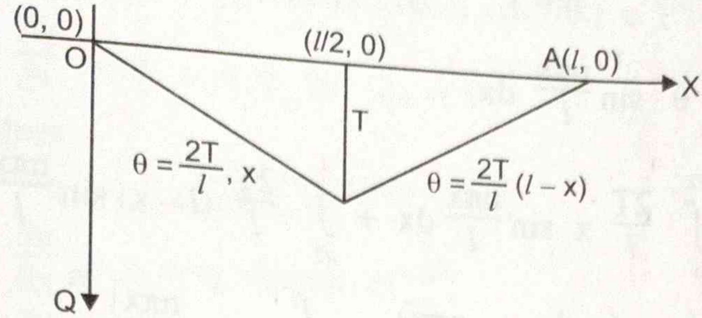

Fig. 11.4

Given equation is $\frac{\partial \theta}{\partial t} = k \frac{\partial^2 \theta}{\partial x^2}$ and the boundary condition is given as :

(i) $\theta(0, t) = 0$

(ii) $\theta(l, t) = 0$

(iii) $\theta(x, 0) = \frac{2T}{l} x$ for $0 \le x \le \frac{l}{2}$

$= \frac{2T}{l}(l - x)$ for $\frac{l}{2} \le x \le l$

The most general solution is

$$\theta(x, t) = (c_4 \cos mx + c_5 \sin mx)\, e^{-k m^2 t} \qquad \dots (1)$$

Applying condition (i), $\theta(0, t) = 0 \Rightarrow c_4 = 0$

$$\therefore \quad \theta(x, t) = c_5 \sin mx \, e^{-k m^2 t}$$

Applying second condition, $\theta(l, t) = 0$

$$\Rightarrow \quad \sin ml = 0 \quad \Rightarrow \quad ml = n\pi$$

i.e. $m = \frac{n\pi}{l}$ for $n = 1, 2, 3, \dots\dots$

$\therefore$ Solution becomes

$$\theta(x, t) = c_5 \sin \frac{n\pi x}{l}\, e^{\frac{-k n^2 \pi^2 t}{l^2}}$$

Putting n = 1, 2, 3 .... and varying constant c_5 for each n, we have the general solution as

$$\theta(x, t) = \sum_{n=1}^{\infty} b_n \frac{n\pi x}{l}\, e^{\frac{-k n^2 \pi^2 t}{l^2}} \qquad \dots (2)$$

Applying the last condition, at t = 0

$$\theta = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$\text{where } b_n = \frac{2}{l} \int_0^l \theta \cdot \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l} \left[\int_0^{l/2} \frac{2T}{l} x \sin \frac{n\pi x}{l}\, dx + \int_{l/2}^{l} \frac{2T}{l} (l - x) \sin \frac{n\pi x}{l}\, dx \right]$$

$$= \frac{4T}{l^2} \left[\left\{ (x) \left(-\frac{l}{n\pi} \cos \frac{n\pi x}{l} \right) + \frac{l^2}{n^2\pi^2} \sin \frac{n\pi x}{l} \right\}_0^{l/2} \right.$$

$$\left. + \left\{ (l - x) \left(-\frac{l}{n\pi} \cos \frac{n\pi x}{l} \right) - \frac{l^2}{n^2\pi^2} \sin \frac{n\pi x}{l} \right\}_{l/2}^{l} \right]$$

$$\therefore \quad b_n = \frac{4T}{l^2} \left[-\frac{l^2}{2n\pi} \cos \frac{n\pi}{2} + \frac{l^2}{n^2\pi^2} \sin \frac{n\pi}{2} + \frac{l^2}{2n\pi} \cos \frac{n\pi}{2} + \frac{l^2}{n^2\pi^2} \sin \frac{n\pi}{2} \right]$$

$$= \frac{8T}{n^2\pi^2} \sin \frac{n\pi}{2}$$

Substituting in (2), we get

$$\theta(x, t) = \frac{8T}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{2} \cdot \sin \frac{n\pi x}{l}\, e^{\frac{-k n^2 \pi^2 t}{l^2}}$$

$$\theta(x, t) = \frac{8T}{\pi^2} \left[\sin \frac{\pi x}{l}\, e^{\frac{-k\pi^2 t}{l^2}} - \frac{1}{3^2} \sin \frac{3\pi x}{l}\, e^{\frac{-9k\pi^2 t}{l^2}} + \ldots\ldots \right]$$

Ex. 4 : *Solve* $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$ *for the conduction of heat along a rod without radiation, subject to the following conditions :*

(i) *u is not infinite as* $t \to \infty$

(ii) $\frac{\partial u}{\partial x} = 0$ *for* $x = 0,\ x = l$ *(i.e. ends are insulated i.e. no heat flows through the ends) and (iii)* $u = lx - x^2$ *for* $t = 0$ *between* $x = 0,\ x = l$.

Sol. : We have

$$u(x, t) = (c_4 \cos mx + c_5 \sin mx)\, e^{-k m^2 t} \qquad \ldots (1)$$

Now applying the second condition,

$$\frac{\partial u}{\partial x} = 0 \quad \text{for } x = 0$$

$$\frac{\partial u}{\partial x} = (-m\, c_4 \sin mx + m\, c_5 \cos mx)\, e^{-k m^2 t}$$

$$\therefore \quad \frac{\partial u}{\partial x} = 0,\ x = 0 \Rightarrow c_5 = 0$$

$\therefore$ Solution becomes

$$u = c_4 \cos mx\, e^{-k m^2 t}$$

Also,

$$\frac{\partial u}{\partial x} = -m\, c_4 \sin mx\, e^{-k m^2 t}$$

$$\frac{\partial u}{\partial x} = 0,\ x = l \Rightarrow 0 = -m\, c_4 \sin ml\, e^{-k m^2 t}$$

$$c_4 \neq 0 \qquad \therefore \sin ml = 0 \Rightarrow ml = n\pi$$

$$\therefore \quad m = \frac{n\pi}{l}$$

$\therefore$ Solution is $u(x, t) = c_4 \cos \frac{n\pi x}{l}\, e^{\frac{-k n^2 \pi^2 t}{l^2}}$ for $n = 1, 2, 3, \ldots\ldots\ldots$

i.e.

$$u(x, t) = \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}\, e^{\frac{-k n^2 \pi^2 t}{l^2}} \qquad \ldots (2)$$

Applying the third condition $u = lx - x^2$ for $t = 0$ between $x = 0$, $x = l$.

By putting $t = 0$,

$$\therefore \quad u = \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

which is represented by Fourier half range cosine series for $lx - x^2$ in $(0, l)$ where

$$a_0 = \frac{1}{l}\int_0^l (lx - x^2)\, dx = \frac{l^2}{6}$$

$$a_n = \frac{2}{l}\int_0^l (lx - x^2) \cos \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\left[(lx - x^2)\left(\frac{l}{n\pi} \sin \frac{n\pi x}{l}\right) - (l - 2x)\left(-\frac{l^2}{n^2\pi^2} \cos \frac{n\pi x}{l}\right) + (-2)\left(-\frac{l^3}{n^3\pi^3} \sin \frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{2}{l}\left[-\frac{l^3}{n^2\pi^2} \cos n\pi - \frac{l^3}{n^2\pi^2}\right]$$

$$= -\frac{2l^2}{n^2\pi^2}(1 + \cos n\pi) = \begin{cases} 0 & \text{for n odd} \\ -\dfrac{4l^2}{n^2\pi^2} & \text{for n even} \end{cases}$$

Let $\quad n = 2p$

$$a_{2p} = -\frac{4\,l^2}{4\,p^2\,\pi^2} = -\frac{l^2}{p^2\,\pi^2} \qquad \text{for } p = 1, 2, 3, \ldots\ldots$$

Solution (2) becomes

$$u = a_0 + \sum_{p=1}^{\infty} a_{2p} \cos\frac{2p\pi x}{l}\, e^{\frac{-k\,4\,p^2\,\pi^2\,t}{l^2}}$$

i.e.
$$u = \frac{l^2}{6} - \frac{l^2}{\pi^2}\sum_{p=1}^{\infty}\frac{1}{p^2}\cos\frac{2\,p\,\pi\,x}{l}\, e^{\frac{-4\,k\,p^2\,\pi^2\,t}{l^2}}$$

which is the required solution.

Ex. 5 : *Solve* $\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}$ *if*

(i) $u(0, t) = 0$

(ii) $u_x(l, t) = 0$

(iii) $u(x, t)$ *is bounded and*

(iv) $u(x, 0) = \frac{u_o\,x}{l}$ *for* $0 \le x \le l$.

Sol. : The most general solution is

$$u(x, t) = (c_4 \cos mx + c_5 \sin mx)\; e^{-k\,m^2\,t}$$

Applying condition (i) $\Rightarrow$ $c_4 = 0$

$$u(x, t) = c_5 \sin mx\; e^{-k\,m^2\,t}$$

$$u_x(l, t) = 0 \;\Rightarrow\; \left(\frac{\partial u}{\partial x}\right)_{x=l} = 0$$

$$\frac{\partial u}{\partial x} = m\,.\,c_5 \cos mx\; e^{-k\,m^2\,t}$$

$$0 = m\,.\,c_5\,.\cos ml\; e^{-k\,m^2\,t}$$

$$\cos ml = 0, \qquad ml = \frac{n\pi}{2}\ (n = \text{odd})$$

$$m = \frac{n\pi}{2l} \quad \text{or} \quad m = \frac{(2n+1)\,\pi}{2l}, \qquad n = 0, 1, 2, \ldots\ldots$$

$$u(x, t) = c_5 \sin\frac{(2n+1)\,\pi\,x}{2l}\; e^{\frac{-k\,(2n+1)^2\,\pi^2\,t}{4\,l^2}}$$

Or
$$u(x, t) = \sum_{n=0}^{\infty} b_n \sin\frac{(2n+1)\,\pi\,x}{2\,l}\,.\,e^{\frac{-k\,(2n+1)^2\,\pi^2\,t}{4\,l^2}}$$

Applying condition (iv), we get

$$\frac{u_o \cdot x}{l} = \sum_{n=0}^{\infty} b_n \sin \frac{(2n+1)\pi x}{2l}$$

where
$$b_n = \frac{2}{l}\int_0^l \frac{u_o \cdot x}{l} \sin \frac{(2n+1)\pi x}{2l}\, dx$$

$$b_n = \frac{2u_o}{l^2}\left\{(x)\left(-\frac{2l}{(2n+1)\pi}\cos\frac{(2n+1)\pi x}{2l}\right) - (1)\left(-\frac{4l^2}{(2n+1)^2\pi^2}\sin\frac{(2n+1)\pi x}{2l}\right)\right\}_0^l$$

$$= \frac{8u_o}{\pi^2}\frac{1}{(2n+1)^2}\sin(2n+1)\frac{\pi}{2}$$

$$= \frac{8u_o}{\pi^2}\frac{1}{(2n+1)^2}\sin\left(n\pi + \frac{\pi}{2}\right)$$

$$b_n = \frac{8u_o}{\pi^2}\frac{1}{(2n+1)^2}\cdot(-1)^n$$

$\therefore$ The complete solution is

$$u(x, t) = \frac{8u_o}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\sin\frac{(2n+1)\pi x}{2l}\cdot e^{\frac{-(2n+1)^2\pi^2 kt}{4l^2}}$$

Ex. 6 : *A homogeneous rod of conducting material of length 100 cm has its ends kept at zero temperature and the temperature initially is*

$$u(x, 0) = x \quad , \quad 0 \le x \le 50$$
$$= 100 - x \, , \quad 50 \le x \le 100$$

Find the temperature $u(x, t)$ at any time.

Sol. : We have to solve $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$, subject to conditions

(i) $u(0, t) = 0$

(ii) $u(100, t) = 0$

(iii) $u(x, 0) = x \quad , \ 0 \le x \le 50$
$\qquad = 100 - x, \ 50 \le x \le 100$

(iv) $u(x, t)$ is finite $\forall\, t$

The most general solution is

$$u(x, t) = (c_4 \cos mx + c_5 \sin mx)\, e^{-c^2m^2t}$$

Condition (i) $\Rightarrow$ $c_4 = 0$

$$u(x, t) = c_5 \sin mx \; e^{-c^2m^2t} \quad \ldots (1)$$

Condition (ii) $\Rightarrow$ $0 = c_5 \sin 100\, m \; e^{-c^2m^2t}$

$$\sin(100\,m) = 0 \Rightarrow \quad 100\,m = n\pi$$

$$m = \frac{n\pi}{100}, \quad n = 1, 2, 3, \ldots\ldots\ldots\ldots$$

Solution (1) becomes

$$u(x, t) = c_5 \sin \frac{n\pi x}{100} \; e^{-\frac{n^2 \pi^2 c^2 t}{1000,00}}, \; n = 1, 2, \ldots\ldots\ldots\ldots\ldots$$

Combining all these solutions

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{100} \; e^{-\frac{n^2 \pi^2 c^2 t}{100,00}} \qquad \ldots (2)$$

Applying condition (iii), we have

$$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{100}$$

and is half range sine series for u (x, 0).

$$\therefore \quad b_n = \frac{2}{100} \int_0^{100} u(x, 0) \sin \frac{n\pi x}{100}\, dx$$

$$= \frac{1}{50} \left[\int_0^{50} u(x, 0) \sin \frac{n\pi x}{100}\, dx + \int_{50}^{100} u(x, 0) \sin \frac{n\pi x}{100}\, dx \right]$$

$$= \frac{1}{50} \left[\int_0^{50} x \sin \frac{n\pi x}{100}\, dx + \int_{50}^{100} (100 - x) \sin \frac{n\pi x}{100}\, dx \right]$$

$$= \frac{1}{50} \left[\left\{ (x) \times \left(\frac{-100}{n\pi} \cos \frac{n\pi x}{100} \right) - (1) \left(-\frac{100^2}{n^2 \pi^2} \sin \frac{n\pi x}{100} \right) \right\}_0^{50} \right.$$

$$\left. + \left\{ (100 - x) \left(-\frac{100}{n\pi} \cos \frac{n\pi x}{100} \right) - (-1) \left(\frac{-100^2}{n^2 \pi^2} \sin \frac{n\pi x}{100} \right) \right\}_{50}^{100} \right]$$

$$= \frac{1}{50} \left[-\frac{100}{n\pi} \left(50 \cos \frac{n\pi}{2} - 0 \right) + \frac{100^2}{n^2\pi^2} \left(\sin \frac{n\pi}{2} - \sin 0 \right) - \frac{100}{n\pi} \left(0 - 50 \cos \frac{n\pi}{2} \right) \right.$$

$$\left. - \frac{100^2}{n^2\pi^2} \left(\sin n\pi - \sin \frac{n\pi}{2} \right) \right]$$

$$\therefore \; b_n = \frac{1}{50} \left(\frac{100^2}{n^2\pi^2} \right) 2 \sin \frac{n\pi}{2} = \frac{400}{n^2\pi^2} \sin \frac{n\pi}{2} = \begin{cases} 0, & \text{if n is even} \\ (-1)^n \dfrac{400}{n^2\pi^2}, & \text{if n is is odd} \end{cases}$$

Replace $n \rightarrow 2n + 1$

$$\therefore \quad b_n = (-1)^n \cdot \frac{400}{(2n+1)^2 \pi^2}$$

$\therefore$ Required solution is

$$u(x, t) = \sum_{n=0}^{\infty} \frac{(-1)^n \times 400}{(2n+1)^2 \pi^2} \sin \frac{(2n+1)\pi x}{100} \cdot e^{\frac{-(2n+1)^2 \pi^2 c^2 t}{100^2}}$$

Ex. 7 : *A bar with insulated sides is initially at temperature 0°C throughout. The end x = 0 is kept at 0°C for all time and the heat is suddenly applied so that* $\frac{\partial u}{\partial x} = 10$ *at* $x = l$ *for all time. Find the temperature function u (x, t).*

Sol. : We have to solve the P.D.E.

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

Let
$$u(x, t) = F(x) \cdot G(t)$$

$$FG' = a^2 F'' G$$

$$\frac{F''}{F} = \frac{G'}{a^2 G} = -m^2 \text{ (say)}$$

then the solution is

$$u(x, t) = (c_1 \cos mx + c_2 \sin mx)\, e^{-a^2 m^2 t} \quad \dots \text{(I)}$$

Also,
$$\frac{F''}{F} = \frac{G'}{a^2 G} = 0$$

then the solution is

$$u(x, t) = c_6 + c_7 \cdot x \quad \dots \text{(II)}$$

Conditions are

(i) $u(x, 0) = 0$

(ii) $u(0, t) = 0$

(iii) $\left(\frac{\partial u}{\partial x}\right)_{x=l} = 10$, for all t.

Since the above conditions of the problem are such that any one of the above solutions (i.e. (I) and (II)) does not satisfy them. We use the combinations of the solutions to satisfy the given conditions i.e. sum of (I) and (II).

$$\therefore \quad u(x, t) = c_6 + c_7 x + (c_1 \cos mx + c_2 \sin mx)\, e^{-m^2 a^2 t}$$

Now (ii) $\Rightarrow c_6 = 0, \quad c_1 = 0$

$$u(x, t) = c_7 x + c_2 \sin mx \cdot e^{-m^2 a^2 t}$$

Now (iii) $\Rightarrow \frac{\partial u}{\partial x} = c_7 + m c_2 \cos mx \ e^{-m^2 a^2 t}$

$$10 = c_7 + m c_2 \cos ml \ e^{-m^2 a^2 t}$$

$$\Rightarrow c_7 = 10$$

$$\cos ml = 0 \qquad ml = \frac{(2n+1)\pi}{2}$$

$$u(x, t) = 10x + \sum_{n=0}^{\infty} c_{2n+1} \cdot \sin \frac{(2n+1)\pi x}{2l} \cdot e^{\frac{-a^2(2n+1)^2\pi^2 t}{4l^2}}$$

Now (i) $\Rightarrow$ $t = 0, \ u = 0$

$$-10x = \sum_{n=0}^{\infty} c_{2n+1} \sin \frac{(2n+1)\pi x}{2l}$$

$$c_{2n+1} = \frac{2}{l}\int_0^l (-10x) \sin \frac{(2n+1)\pi x}{2l} \cdot dx$$

$$= -\frac{20}{l}\left[(x)\left(-\frac{2l}{(2n+1)\pi}\cos\frac{(2n+1)\pi x}{2l}\right) + \frac{4l^2}{(2n+1)^2\pi^2}\sin\frac{(2n+1)\pi x}{2l}\right]_0^l$$

$$c_{2n+1} = -\frac{80l}{(2n+1)^2\pi^2}\sin\frac{(2n+1)\pi}{2}$$

$\therefore$ The complete solution is

$$u(x, t) = 10x - \frac{80l}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\sin\frac{(2n+1)\pi}{2} \cdot \sin\frac{(2n+1)\pi x}{2l} \ e^{\frac{-a^2(2n+1)^2\pi^2 t}{4l^2}}$$

Ex. 8 : *A rod of length l has its ends A and B maintained at 20°C and 40°C respectively until steady-state conditions prevail. The temperature at A is suddenly raised to 50°C while that at B is lowered to 10°C and maintained thereafter. Find the subsequent temperature distribution of the rod.*

Sol. : Here initial conditions of temperature distribution are not explicitly given. We will first obtain the same. Initially steady-state conditions prevail, the temperature depends on x only, let it be $u_{s_1}(x) = Ax + B$.

$x = 0, \ u_{s_1}(0) = 20°\,C, \ x = l, \ u_{s_1}(l) = 40°C, \ 20 = B, \ \ 40 = Al + 20, \ A = \frac{20}{l}$

$$u_{s_1}(x) = \frac{20x}{l} + 20$$

Hence, we have to solve the P.D.E.

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2},$$ subject to the boundary conditions

(i) $u(0, t) = 50°C$

(ii) $u(l, t) = 10°C$ and initial condition

$$u(x, 0) = u_{s_1}(x) = \frac{20x}{l} + 20$$

Since the boundary conditions are not zero, we cannot proceed directly.

Note that after certain time, the temperature distribution of the rod has to reach steady state which implies that the solution has two parts :

(i) Steady state part and

(ii) Transient part which ultimately becomes zero.

We denote these as $u_s(x)$ and $u_t(x, t)$.

$$\therefore \quad u(x, t) = u_s(x) + u_t(x, t)$$

where, $u_s(x)$ satisfies the P.D.E. under steady-state conditions i.e.

$$u_s(x) = cx + d$$

where, $u_s(0) = 50°C$ and $u_s(l) = 10°C$

$$50 = 0 + d \quad \therefore d = 50$$

$$10 = lc + 50 \quad \therefore \quad c = -\frac{40}{l}$$

$$\therefore \quad u_s(x) = -\frac{40x}{l} + 50$$

$$\therefore \quad u(x, t) = -\frac{40x}{l} + 50 + u_t(x, t).$$

Our problem reduces to obtain $u_t(x, t)$

where $\quad u_t(x, t) = u(x, t) + \dfrac{40x}{l} - 50$

Substituting $u_t(x, t)$ in the P.D.E.

$$\frac{\partial u_t}{\partial t} = c^2 \frac{\partial^2 u_t}{\partial x^2}$$

For obtaining boundary and initial conditions

$$u_t(0, t) = u(0, t) + 0 - 50$$

$$= 50 - 50 \qquad (\because u(0, t) = 50)$$

$$\boxed{u_t(0, t) = 0}$$

$$u_t(l, t) = u(l, t) + 40 - 50$$

$$= 10 + 40 - 50 \qquad (\because u(l, t) = 10)$$

$$\boxed{u_t(l, t) = 0}$$

$$u_t(x, 0) = u(x, 0) + \frac{40x}{l} - 50$$

$$= \frac{20x}{l} + 20 + \frac{40x}{l} - 50$$

$$\boxed{u_t(x, 0) = \frac{60x}{l} - 30}$$

Solution is $u_t(x, t) = (c_1 \cos mx + c_2 \sin mx)\ e^{-c^2 m^2 t}$

$$u_t(0, t) = 0 \Rightarrow c_1 = 0$$

$$u_t(x, t) = c_2 \sin mx\ e^{-c^2 m^2 t}$$

$$u_t(l, t) = 0 \Rightarrow 0 = c_2 \sin ml\ e^{-c^2 m^2 t}$$

$$\sin ml = 0 \Rightarrow m = \frac{n\pi}{l},\ n = 1, 2, \ldots\ldots\ldots$$

$$u_t(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}\ e^{-\frac{c^2 n^2 \pi^2 t}{l^2}}$$

$$u_t(x, 0) = \sum_{1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$b_n = \frac{2}{l} \int_0^l \left(\frac{60\,x}{l} - 30\right) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l} \left\{\left(\frac{60\,x}{l} - 30\right)\left(-\frac{l}{n\pi} \cos \frac{n\pi x}{l}\right) - \left(\frac{60}{l}\right)\left(-\frac{l^2}{n^2\pi^2} \sin \frac{n\pi x}{l}\right)\right\}_0^l$$

$$= \frac{2}{l}\left\{-\frac{30\,l}{n\pi}(-1)^n - \frac{30\,l}{n\pi}\right\}$$

$$b_n = -\frac{60}{n\pi}\ [(-1)^n + 1]$$

$$u_t(x, t) = -\frac{60}{\pi} \sum_{1}^{\infty} \frac{(-1)^n + 1}{n} \sin \frac{n\pi x}{l}\ e^{-\frac{c^2 n^2 \pi^2 t}{l^2}}$$

$$u_t(x, t) = -\frac{60}{\pi} \times 2 \cdot \sum_{1}^{\infty} \frac{1}{2n} \sin \frac{2n\pi x}{l}\ e^{-\frac{4 c^2 n^2 \pi^2 t}{l^2}}$$

$$\therefore \quad \boxed{u(x, t) = 50 - \frac{40\,x}{l} - \frac{60}{\pi} \sum_{1}^{\infty} \frac{1}{n} \sin \frac{2n\pi x}{l}\ e^{-\frac{4 c^2 n^2 \pi^2 t}{l^2}}}$$

Ex. 9 : *Solve* $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ *if*

(i) *u is finite for all t*

(ii) $u(0, t) = 0, \ \forall t$

(iii) $u(l, t) = 0, \ \forall t$

(iv) $u(x, 0) = u_0$ *for* $0 \le x \le l$, *where l being the length of the bar.* **(May 2008)**

Sol. : (i) The most general solution is

$$u(x, t) = (c_4 \cos mx + c_5 \sin mx)\, e^{-m^2 c^2 t}$$

(ii) $\Rightarrow$

$$c_4 = 0$$

$$u(x, t) = c_5 \sin mx\, e^{-m^2 c^2 t}$$

(iii) $\Rightarrow$

$$0 = c_5 \sin ml\, e^{-m^2 c^2 t}$$

$$\sin(ml) = 0, \ ml = n\pi$$

$$m = \frac{n\pi}{l}, \ n = 1, 2, \ldots\ldots\ldots\ldots\ldots$$

$$u(x, t) = c_5 \sin \frac{n\pi x}{l}\, e^{-\frac{n^2 \pi^2 c^2 t}{l^2}}, \ n = 1, 2, \ldots\ldots$$

Or

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}\, e^{-\frac{n^2 \pi^2 c^2 t}{l^2}} \quad \ldots \text{(I)}$$

Applying condition (iv),

$$u_o = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$b_n = \frac{2}{l} \int_0^l u_o \sin \frac{n\pi x}{l}\, dx = \frac{2u_o}{l} \left(-\frac{l}{n\pi} \cos \frac{n\pi x}{l}\right)_0^l$$

$$b_n = \frac{2\, u_o}{\pi} \left(\frac{1 - (-1)^n}{n}\right)$$

$$u(x, t) = \frac{2\, u_o}{\pi} \sum_{n=1}^{\infty} \left(\frac{1 - (-1)^n}{n}\right) \sin \frac{n\pi x}{l}\, e^{-\frac{n^2 \pi^2 c^2 t}{l^2}}$$

Ex. 10 : *Solve the equation* $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$ *where u (x, t) satisfies the following conditions :*

(i) $u(0, t) = 0$

(ii) $u(l, t) = 0$ *for all t*

(iii) $u(x, 0) = x$ *in* $0 < x < l$

(iv) $u(x, \infty)$ *is finite.* **(Dec. 2004, May 2009)**

Sol. : The most general solution is

$$u(x, t) = (c_4 \cos mx + c_5 \sin mx)\, e^{-a^2 m^2 t}$$

(i) $\Rightarrow$ $c_4 = 0$

$$u(x, t) = c_5 \sin mx\, e^{-a^2 m^2 t}$$

(ii) $\Rightarrow$ $0 = c_5 \sin ml\, e^{-a^2 m^2 t}$

$$\sin ml = 0, \qquad ml = n\pi$$

$$m = \frac{n\pi}{l},\ n = 1, 2, 3, \ldots\ldots\ldots$$

$$u(x, t) = c_5 \sin \frac{n\pi x}{l}\, e^{-\frac{a^2 n^2 \pi^2 t}{l^2}},\ n = 1, 2, \ldots\ldots\ldots$$

Combining all these solutions, we get

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}\, e^{-\frac{a^2 n^2 \pi^2 t}{l^2}}$$

(iii) $\Rightarrow$

$$x = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l},\ 0 < x < l$$

which is nothing but half range sine series for $f(x) = x$ in $(0, l)$.

$$\therefore \quad b_n = \frac{2}{l} \int_0^l x \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\left\{(x)\left(-\frac{l}{n\pi} \cos \frac{n\pi x}{l}\right) - (1)\left(-\frac{l^2}{n^2\pi^2} \sin \frac{n\pi x}{l}\right)\right\}_0^l$$

$$= \frac{2}{l}\left\{-\frac{l^2}{n\pi}(-1)^n\right\} = -\frac{2l}{\pi}\left(\frac{(-1)^n}{n}\right)$$

$$\therefore \quad u(x, t) = \frac{2l}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l}\, e^{-\frac{a^2 n^2 \pi^2 t}{l^2}}$$

Ex. 11 : *The temperature at any point of the insulated metal rod of one metre length is governed by the differential equation* $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$. *Find u(x, t), subject to the following conditions :*

(i) $u(0, t) = 0°C$

(ii) $u(l, t) = 0°C$

(iii) $u(x, 0) = 50°C$ *and hence find the temperature in the middle of the rod at any subsequent time.*

Sol. : The most general solution is

$$u(x, t) = (c_4 \cos mx + c_5 \sin mx)\, e^{-c^2 m^2 t}$$

(i) $\Rightarrow$ $c_4 = 0$

$$u(x, t) = c_5 \sin mx\, e^{-c^2 m^2 t}$$

(ii) $\Rightarrow$ $0 = c_5 \sin m\, e^{-c^2 m^2 t}$

$$\sin m = 0 \qquad \therefore\ m = n\pi, \quad n = 1, 2, \ldots\ldots\ldots\ldots$$

$$\therefore \quad u(x, t) = c_5 \sin n\pi x\ e^{-c^2 n^2 \pi^2 t}, \quad n = 1, 2, \ldots\ldots\ldots\ldots$$

Combining all these solutions, we get

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin n\pi x\ e^{-c^2 n^2 \pi^2 t}$$

Applying condition (iii), we get

$$50 = \sum_{n=1}^{\infty} b_n \sin n\pi x, \; 0 < x < 1$$

which is represented by half range Fourier sine series for $f(x) = 50$ in $(0, 1)$

$$\therefore \quad b_n = 2 \int_0^1 50 \sin n\pi x\ dx$$

$$= 100 \left[-\frac{\cos n\pi x}{n\pi} \right]_0^1 = \frac{100}{\pi} \left(\frac{1-(-1)^n}{n} \right)$$

$$\therefore \quad u(x, t) = \frac{100}{\pi} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \sin n\pi x\, e^{-c^2 n^2 \pi^2 t}$$

Now the temperature in the middle of the rod at any subsequent time is

$$u\left(\frac{1}{2}, 0\right) = \frac{100}{\pi} \sum_{n=1}^{\infty} \frac{1-(-n)^n}{n} \sin \frac{n\pi}{2}$$

$$= 0, \quad \text{if n is even}$$

$$= \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n}, \quad \text{if n is odd.}$$

Solve the one-dimensional heat flow equation $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$ for function u(x, t), subject to following conditions :

1. (i) $u(0, t) = 0,$ (ii) $u(l, t) = 0$, for all t

(iii) $u(x, 0) = x, \quad 0 < x < l$ (iv) $u(x, \infty)$ is finite.

Hint : $b_n = \frac{2l}{n\pi}(-1)^{n-1}$

$$\textbf{Ans. :} \quad u(x, t) = \frac{2l}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} e^{-\frac{n^2 a^2 \pi^2 t}{l^2}} \sin \frac{n\pi x}{l}$$

2. (i) $u(0, t) = 0,$ (ii) $\frac{\partial}{\partial x} u(l, t) = 0$, for all t

(iii) $u(x, 0) = x$ (iv) $u(x, \infty)$ is finite

$$\textbf{Ans. :} \quad u(x, t) = \frac{8l}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)^2} \sin \frac{(2n+1)\pi x}{2l} e^{-\frac{a^2 (2n+1)^2 \pi^2 t}{4l^2}}$$

3. (i) $u(0, t) = 0,$ (ii) $u(\pi, t) = 0$, for all t

(iii) $u(x, 0) = \pi x - x^2, \ 0 < x < \pi$ (iv) $u(x, \infty)$ is finite.

$$\textbf{Ans. :} \quad u(x, t) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)^3} e^{-a^2(2n-1)^2 t}$$

4. (i) $\frac{\partial}{\partial x} u(0, t) = 0$, and (ii) $\frac{\partial}{\partial x}(l, t) = 0$, for all t

(iii) $u(x, 0) = x^2, \ 0 < x < l$ (iv) $u(x, \infty)$ is finite.

$$\textbf{Ans. :} \quad u(x, t) = \frac{l^2}{3} + \frac{4l^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \frac{n\pi x}{l} e^{-\frac{n^2 a^2 \pi^2 t}{l^2}}$$

5. Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ if

(i) u is finite, $\forall$ t (ii) $u(0, t) = 0$

(iii) $u(\pi, t) = 0$ (iv) $u(x, 0) = \pi x - x^2, \ 0 \le x \le \pi$

$$\textbf{Ans. :} \quad u(x, t) = \frac{8}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \sin(2n+1)x \ e^{-(2n+1)^2 t}$$

6. Solve $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$ if

(i) u (x, t) is bounded (ii) $u(0, t) = 0$

(iii) $u(l, t) = 0$ (iv) $u(x, 0) = \frac{u_o x}{l}, \quad 0 \le x \le l$

Ans. : $u(x, t) = \frac{2 u_o}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l} e^{-\frac{k n^2 \pi^2 t}{l^2}}$

7. The equation for the conduction of heat along a bar of length l is $\frac{\partial \theta}{\partial t} = k \frac{\partial^2 \theta}{\partial t^2}$, neglecting radiation. Find an expression for θ if the ends of the bar are maintained at zero temperature and if initially the temperature is T at the centre of the bar and falls uniformly to zero at its ends.

Hint : (i) $\theta(0, t) = 0$, (ii) $\theta = (l, 0) = 0$, (iii) $\theta(x, 0) = \begin{cases} \frac{2T}{l} x, & 0 \le x \le l/2 \\ \frac{2T}{l}(-x), & l/2 \le x \le l \end{cases}$

Ans. : $\theta(x, t) = \frac{8T}{\pi^2} \sum_{1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{l} e^{-\frac{k n^2 \pi^2 t}{l^2}}$

8. Solve $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$, subject to the following boundary conditions :

(i) $u(0, t) = 0$, (ii) $u(l, t) = 0$, (iii) $u(x, 0) = \begin{cases} x, & 0 < x \le i/2 \\ l - x, & l/2 \le x < l \end{cases}$

Ans. : $u(x, t) = \frac{4l}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^2} \sin \frac{(2n-1)\pi x}{l} e^{-\frac{(2n-1)^2 \pi^2 c^2 t}{l^2}}$

9. The temperatures at the ends x = 0 and x = 50 cm in length of a rod are held at 0°C and 50°C respectively until steady-state conditions prevail. The two ends of the rod are suddenly insulated. Find the temperature distribution of the rod assuming that the surface of the rod is impervious to heat.

Ans. : $u(x, t) = 25 - \frac{200}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x}{50} e^{\frac{-(2n-1)^2 a^2 \pi^2 t}{2500}}$

10. A rod of length l is insulated along its length so that no heat is transformed from its sides, the uniform temperature of the rod is 50° C. Suddenly the end $x = 0$ is cooled to 0° C and the end $x = l$ heated to 100° C and these are maintained afterwards. Find the subsequent temperature distribution of the rod.

$$\textbf{Ans. :}\ u(x, t) = \frac{100\,x}{l} + \frac{100}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{2n\pi x}{l}\, e^{-\frac{4n^2 a^2 \pi^2 t}{l^2}}$$

11. A rod of length l has its ends A and B kept at 0° C and 75° C, until steady-state conditions prevail. If the temperature of A is suddenly raised to 75° C and that of B to 175°C and maintained thereafter, find the subsequent temperature distribution of the rod.

$$\textbf{Ans. :}\ u(x, t) = 75 + \frac{100\,x}{l} - \frac{300}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} \sin \frac{(2n-1)\,\pi x}{l}\, e^{\frac{-(2n-1)^2 a^2 \pi^2 t}{l^2}}$$

12. A rod of length l has one end kept at 0° C and other end B at 100° C until steady-state conditions prevail. The temperature of A is suddenly raised to 50° C while the end B is insulated. These conditions are maintained thereafter, find the subsequent temperature distribution of the rod.

$$\textbf{Ans. :}\ u(x, t) = 50 + \sum_{n=1}^{\infty} \left[\frac{(-1)^{n-1}\, 800}{(2n-1)^2 \pi^2} - \frac{200}{(2n-1)\,\pi}\right] \sin \frac{(2n-1)\,\pi x}{2l}\, e^{\frac{-(2n-1)^2 a^2 \pi^2 t}{4l^2}}$$

13. A rod of length l has its ends A and B maintained at 20°C and 40°C respectively until steady-state conditions prevail. The temperature at A is suddenly raised to 50°C while that at B is lowered to 10°C and maintained thereafter. Find the subsequent temperature distribution of the rod.

$$\textbf{Ans. :}\ u(x, y) = 50 - \frac{40}{l}\,x - \frac{60}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{2n\pi x}{l}\, e^{-\frac{4n^2 a^2 \pi^2 t}{l^2}}$$

14. A uniform rod of length l whose surface is thermally insulated, is initially at temperature θ_o. At time $t = 0$, one end is suddenly cooled to temperature 0°C and subsequently maintained at this temperature and at the same time, the other end is thermally insulated. Find the temperature at end $x = l$ at any time t.

15. The ends A and B of a insulated rod of length l, have their temperatures at 20°C and 80°C respectively until steady-state conditions prevail. The temperatures at these ends are changed suddenly to 40°C and 60°C respectively. Find the temperature distribution of the rod at time t.

$$\textbf{Ans. :}\ u(x, t) = \frac{20\,x}{l} + 40 - \frac{40}{\pi} \sum_{1}^{\infty} \frac{1}{n} \sin \frac{2n\pi x}{l}\, e^{-\frac{4c^2 n^2 \pi^2 t}{l^2}}$$

11.7 MODELING OF TWO-DIMENSIONAL HEAT FLOW

Consider the flow of heat in a metal plate in XOY plane. If the temperature at a point does not depend upon z-coordinate and it depends only on x, y, and t, then the flow is called two-dimensional and the heat flow lies in XOY plane only and is zero along the normal to XOY plane.

Consider a rectangular element of the plate with sides δx and δy and thickness 'h'. As discussed in one-dimensional heat flow along a bar, the quantity of heat that enters the plate per second from the sides AB and AD is given by $-kh\,\delta x\left(\frac{\partial u}{\partial y}\right)_y$ and $-kh\,\delta y\left(\frac{\partial u}{\partial x}\right)_x$ respectively.

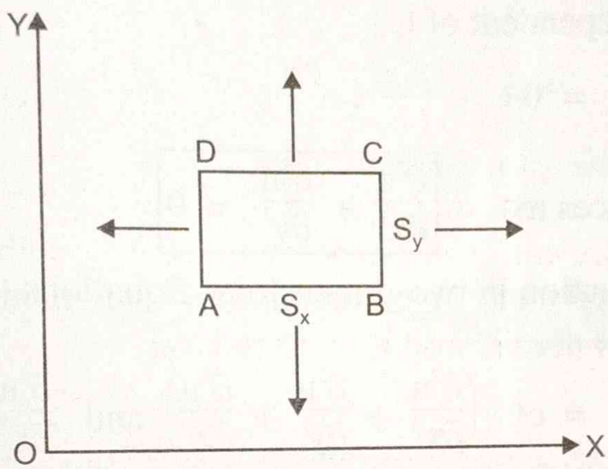

Fig. 11.5

Heat flowing out through sides CD and BC per second is $-kh\,\delta x\left(\frac{\partial u}{\partial y}\right)_{y+\delta y}$ and $-kh\,\delta y\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$ respectively. Therefore, the total gain of heat by rectangular plate ABCD per second

$$= -kh\,\delta x\left(\frac{\partial u}{\partial y}\right)_y - kh\,\delta y\left(\frac{\partial u}{\partial x}\right)_x + kh\,\delta x\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} + kh\,\delta y\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$$

$$= kh\,\delta x\,\delta y\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}-\left(\frac{\partial u}{\partial x}\right)_x}{\delta x}+\frac{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y}-\left(\frac{\partial u}{\partial y}\right)_y}{\delta y}\right] \quad \ldots (1)$$

The rate of gain of heat by the plate is also given by,

$$S\rho h\,\delta x\,\delta y\,\frac{\partial u}{\partial t} \quad \ldots (2)$$

where, S = specific heat and ρ = density of the metal plate

$\therefore$ Equating equations (1) and (2), we get

$$kh\,\delta x\,\delta y\,.\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}-\left(\frac{\partial u}{\partial x}\right)_x}{\delta x}+\frac{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y}-\left(\frac{\partial u}{\partial y}\right)_y}{\delta y}\right] = S\rho h\,\delta x\,\delta y\,\frac{\partial u}{\partial t}$$

Dividing by h $\delta x\ \delta y$ and taking limit as $\delta x \to 0$, $\delta y \to 0$, we get

$$k\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) = S\rho\,\frac{\partial u}{\partial t}$$

$$\therefore \quad \frac{\partial u}{\partial t} = \frac{k}{S\rho}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

Or $$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \quad \ldots (3)$$

where $\dfrac{k}{S\rho} = c^2$ is the diffusivity

Equation (3) represents temperature distribution of the plate in the transient state. For steady state when u is independent of t,

$$\frac{\partial u}{\partial t} = 0$$

$\therefore$ Equation (3) reduces to $$\boxed{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0} \quad \ldots (4)$$

and is called Laplace's equation in two-dimensions. Equations (3) and (4) can be extended to three-dimensional solids as,

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) \text{ and } \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial u^2}{\partial z^2} = 0$$

in a similar way and is called Laplace's equation in three-dimensions.

11.8 SOLUTION OF LAPLACE'S EQUATION IN TWO-DIMENSIONS BY THE METHOD OF SEPARATION OF VARIABLES

Laplace's equation in two-dimensions is $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0 \quad \ldots (1)$$

Let $u = XY$ ... (2)

where, X is a function of x alone and Y is a function of y alone, be a solution of equation (1).

Then, $$\frac{\partial^2 u}{\partial x^2} = X''Y \text{ and } \frac{\partial^2 u}{\partial y^2} = XY''$$

Substituting these values in equation (1), we get

$$X''Y + XY'' = 0$$

$$\therefore \quad \frac{X''}{X} = \frac{-Y''}{Y} \quad \ldots (3)$$

As L.H.S. is being a function of 'x' alone and R.H.S. being a function of 'y' alone and x and y being independent variables, equation (3) will hold good only if both sides reduce to a constant, say 'k'.

$\therefore$ Equation (3) leads to,

$$\frac{X''}{X} = k \text{ and } \frac{-Y''}{Y} = k$$

Or $\quad X'' - kX = 0$ and $Y'' + kY = 0$

$\therefore \quad (D^2 - k)\,X = 0$ and $(D^2 + k)\,Y = 0 \quad \ldots (4)$

Case (i) : Let $\quad k = 0$

$$D^2 X = 0 \;\Rightarrow\; X = c_1 x + c_2$$

$$D^2 Y = 0 \;\Rightarrow\; Y = c_3 y + c_4$$

$\therefore$ Complete solution is

$$\boxed{u(x, y) = (c_1 x + c_2)\,(c_3 y + c_4)} \quad \ldots (5)$$

Case (ii) : Let $k > 0$ i.e. $k = m^2$.

From equation (4),

$$(D^2 - m^2)\,X = 0 \;\Rightarrow\; X = c_1 e^{mx} + c_2 e^{-mx}$$

$$(D^2 + m^2)\,Y = 0 \;\Rightarrow\; Y = c_3 \cos my + c_4 \sin my$$

$\therefore$ Complete solution is

$$\boxed{u(x, y) = (c_1 e^{mx} + c_2 e^{-mx})\,(c_3 \cos my + c_4 \sin my)} \quad \ldots (6)$$

Case (iii) : Let $k < 0$, i.e. $k = -m^2$.

From equation (4),

$$(D^2 + m^2)\,X = 0 \;\Rightarrow\; X = c_1 \cos mx + c_2 \sin mx$$

$$(D^2 - m^2)\,Y = 0 \;\Rightarrow\; Y = c_3 e^{my} + c_4 e^{-my}$$

$\therefore$ Complete solution is

$$\boxed{u(x, y) = (c_1 \cos mx + c_2 \sin mx)\,(c_3 e^{my} + c_4 e^{-my})} \quad \ldots (7)$$

The most benefitting solution will be the one, consistent with the physical nature and the boundary conditions of the problem.

Note : To select the appropriate "most general solution", we will adopt the following procedure :

1. If in a physical problem, the plate subjected to steady temperature extends to infinity in the positive y-direction, we should take the constant $k = -m^2$ to represent each side of equation (3) i.e. if the condition given in a physical problem is $u = 0$ for $y = \infty$ for $\forall\, x$ between $(0, l)$ say or $u(x, \infty) = 0$, $\forall\, x$ in $(0, l)$, we always select the most suitable general solution of this nature as

$$\boxed{u(x, y) = (c_1 \cos mx + c_2 \sin mx)\,(c_3 e^{my} + c_4 e^{-my})}$$

2. If in a physical problem, the plate subjected to steady temperature extends to infinity in the positive x-direction, we should take the constant $k = m^2$ to represent each

side of equation (3) i.e. if the condition given in a physical problem is $u = 0$ for $x = \infty$ for $\forall\, y$ between $(0, l)$ (say) or $u(\infty, y) = 0, \forall\, y$ in $(0, l)$, we always select the most suitable general solution of this nature as

$$\boxed{\mathbf{u(x, y) = (c_1 e^{mx} + c_2 e^{-mx})\ (c_3 \cos my + c_4 \sin my)}}$$

3. The constant $k = 0$ is ruled out in physical applications.
4. When the constant $k = m^2$, the solution (6) can be written as

$$\boxed{\mathbf{u(x, y) = (c_1 \cosh mx + c_2 \sinh mx)\ (c_3 \cos my + c_4 \sin my)}}$$

5. When the constant $k = -m^2$, the solution (7) can be written as

$$\boxed{\mathbf{u(x, y) = (c_1 \cos mx + c_2 \sin mx)\ (c_3 \cosh my + c_4 \sinh my)}}$$