4. Analysis of Transient Response in Circuit using Laplace Transform.

Equations describing behaviour of n/w consisting of element like resistances, Inductances & Capacitances are always of integro-differential type. In earlier chapter we have seen classical methods for solving such eqⁿs. It has been realized that the method of obtaining solⁿ of D.E. is little bit complicated. But using transform methods it becomes simpler. Here we are going to discuss Laplace Transform Method which transforms the time domain differential eqⁿs to freq. domain.

The Laplace Transform method provides very simple way of solving complicated integro-differential time domain eqⁿs, by converting them into simple algebraic eqⁿs in the freq. domain.

integro D-E → [Laplace Transform] → [Algebraic eqⁿ] → Algebraic operations → [Inverse Laplace Transform] → [Solⁿ]

Laplace Transform Method.

Laplace Transform.

If $f(t)$ is the time domain function, then its L.T. is denoted by $F(s)$ and is defined by the eqn,

$$L[f(t)] = F(s) = \int_{0^-}^{\infty} f(t)\, e^{-st}\, dt.$$

where s is the complex freq. variable and is given by

$$s = \sigma + j\omega$$

The real part σ is called attenuation constant or damping factor and ω which is imaginary part of s is the angular freq.

To take into account the possibility that $f(t)$ may be an impulse or one of its higher derivatives, the lower limit of integration is taken as 0^-. This includes time just before the instant $t = 0$.

~~For~~ In the case when no impulses or its higher order derivatives are involved, ie for continuous functions, $f(0^-) = f(0^+)$, then the integration can be effectively taken from 0 to ∞

$$L[f(t)] = F(s) = \int_{0}^{\infty} f(t)\, e^{-st}\, dt. \quad \text{... when } f(t) \text{ is continuous.}$$

The function $f(t)$ possesses a L.T. only if it satisfies the condⁿ given by,

$$\int_{0^-}^{\infty} |f(t)| e^{-\sigma t} dt < \infty \quad \text{for real, positive } \sigma$$

Similar to the L-T, the inverse L-T. giving time domain function from freq. domain is defined as,

$$f(t) = L^{-1}[F(s)]$$

$$= \frac{1}{2\pi j} \int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} F(s) e^{st} ds \quad \text{for real, positive } \sigma_1$$

Properties of L-T.

1. **Linearity**

$$L\{f_1(t) + f_2(t) + \dots f_n(t)\} = F_1(s) + F_2(s) + \dots F_n(s).$$

it can be further extended as,

$$L\{a_1f_1(t) + a_2f_2(t) + \dots a_nf_n(t)\} = a_1F_1(s) + a_2F_2(s) + \dots + a_nF_n(s)$$

where $a_1, a_2 \dots$ are constant.

2. **Scaling**

$$L\{K\,F(t)\} = K\,F(s)$$

3. **Real differentiation**

$$L\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0^-)$$

where $f(0^-)$ indicates value of $f(t)$ at $t = 0^-$

The theorem can be extended for n^{th} order derivative as,

$$L\left\{\frac{d^nf(t)}{dt^n}\right\} = s^nF(s) - s^{n-1}f(0^-) - s^{n-2}f'(0^-) - \dots - f^{(n-1)}(0^-)$$

for $n = 2$

$$L\left\{\frac{d^2f(t)}{dt^2}\right\} = s^2F(s) - s.f(0^-) - f'(0^-)$$

for $n = 3$

$$L\left\{\frac{d^3f(t)}{dt^3}\right\} = s^3F(s) - s^2f(0^-) - s\,f'(0^-) - f''(0^-)$$

& so on.

4. Real integration

$$L\left\{\int_0^t f(t)\,dt\right\} = \frac{F(s)}{s}$$

For multiple integrals, it can be extended as,

$$L\left\{\int_0^{t_1}\int_0^{t_2}\cdots\int_0^{t_n} f(t)\,dt_1\,dt_2\cdots dt_n\right\} = \frac{F(s)}{s^n}$$

5. Differentiation by s

If F(S) is L-T- of f(t) then the differentiation by s in the complex freq. domain corresponds to the multiplication by t in the time domain

$$L\left\{t\,f(t)\right\} = \frac{-F(s)}{ds}$$

6. Complex translation

$$L\left\{e^{at}f(t)\right\} = F(s-a)$$

$$L\left\{e^{-at}f(t)\right\} = F(s+a)$$

where a is a complex no.

7. Real translation (shifting theorem)

it is useful to obain L-T- of the shifted or delayed function of time.

If F(S) is the L-T- of f(t) then the L-T- of the function delayed by time T is,

$$L\{f(t-T)\} = e^{-Ts} F(s)$$

8. Initial value theorem

it is useful to find the initial value of the time function $f(t)$. Thus if $F(s)$ is the L.T. of $f(t)$ then,

$$f(0^+) = \lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s)$$

Only restriction that $f(t)$ must be continuous or at the most, a step discontinuity at $t=0$.

9. Final value theorem.

it is useful to find the final value of the time function $f(t)$

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

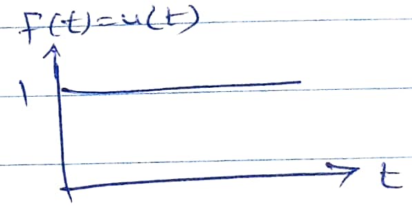
Table of L.T.

$f(t)$	$F(s)$
1	$1/s$
Constant K	K/s
t	$1/s^2$
t^n	$\frac{n!}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$
e^{at}	$\frac{1}{s-a}$
$e^{-at}t^n$	$\frac{n!}{(s+a)^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
$\sinh \omega t$	$\frac{\omega}{s^2-\omega^2}$
$\cosh \omega t$	$\frac{s}{s^2-\omega^2}$
$t \cdot e^{-at}$	$\frac{1}{(s+a)^2}$
$1-e^{-at}$	$\frac{a}{s(s+a)}$

L.T. of standard functions.

1. **Step function.**

Unit step fun is,



$$u(t) = 1 \quad \text{for } t \geq 0$$
$$= 0 \quad \text{for } t < 0$$

$$L\{u(t)\} = \frac{1}{s}$$

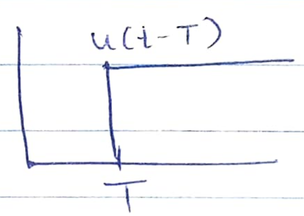
- If the step is of amplitude A,

$$L\{A\,u(t)\} = \frac{A}{s}$$

- If the unit step is delayed by T instants

$$f(t) = u(t-T)$$

Using shifting theorem,



$$L\{f(t-T)\} = e^{-Ts}F(s)$$

$$\therefore L\{u(t-T)\} = e^{-Ts}\cdot\frac{1}{s}$$

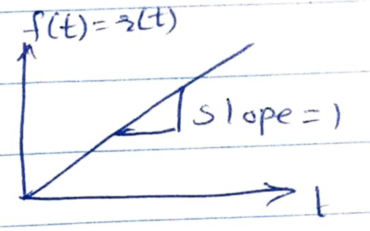
$$= \frac{e^{-Ts}}{s}$$

- The step fun is analogous to the switch which closes at $t=0$

2. <u>Ramp function</u>

The ramp function is defined as,

$$r(t) = t \quad \text{for } t \geq 0$$
$$= 0 \quad \text{for } t < 0$$



Unit rampfun.

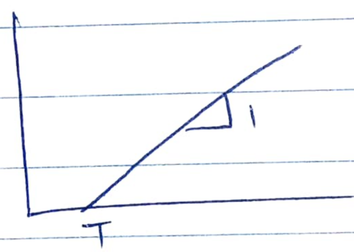
$$L\{t\,u(t)\} = \frac{1}{s^2}$$

- For ramp having slope, A,

$$L\{A\,t\,u(t)\} = A/s^2$$

- If the ramp is shifted by T,

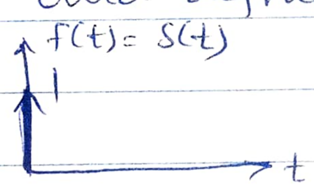
$$L\{r(t-T)\} = \frac{e^{-Ts}}{s^2}$$



3. Impulse function.

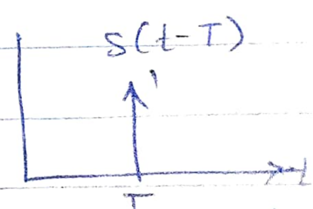
Unit impulse function is $\delta(t)$ and defined as,

$$\delta(t) = 1 \quad \text{for } t = 0$$
$$= 0 \quad \text{for } t \neq 0$$



$$L\{\delta(t)\} = 1$$

If there is delayed impulse, then,
function is $\delta(t-T)$



$$\therefore L\{\delta(t-T)\} = e^{-Ts}$$

Convolution theorem

$$L\{f_1(t) * f_2(t)\} = F_1(s) \cdot F_2(s)$$

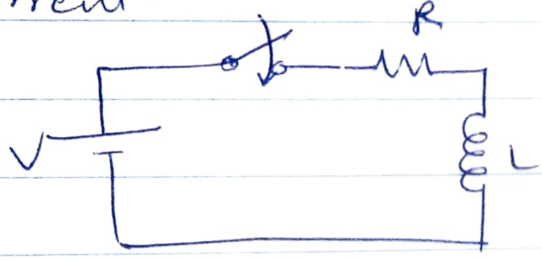
$f_1(t) * f_2(t)$ indicates convolution of $f_1(t)$ & $f_2(t)$.

Inverse laplace transform.

by using partial fraction method.

Ex:

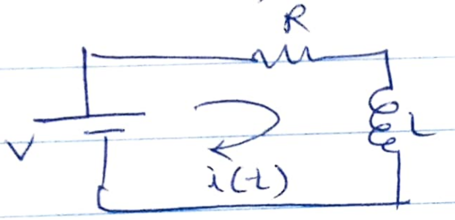
1. In the ckt shown, the switch is closed at $t=0$. Find expression for the resulting current.



Before closing switch, there is no current.

$$\therefore \; i(0^-) = 0 = i_L(0^-)$$

at $t>0$, the n/w is,



Applying KVL,

$$-i(t)R - L\frac{di(t)}{dt} + V = 0$$

$$\therefore \; V = i(t)R + L\frac{di(t)}{dt} \qquad \ldots V \text{ is const.}$$

Taking L.T. of above eqn,

$$I(s).R + L\left[s.I_s - i(0^-)\right] = \frac{V}{s}$$

$$I(s)\left[R + sL\right] = \frac{V}{s} \qquad \ldots \text{as } i(0^-) = 0$$

$$\therefore \; I(s) = \frac{V}{s(sL+R)} = \frac{V/L}{s(s+R/L)}$$

using partial fraction,

$$I(s) = \frac{A}{s} + \frac{B}{s+R/L}$$

where, after solving

$$\cancel{I(s) = A(s+R/L) + Bs}$$

at $s = 0$,

$$\frac{V/L}{R/L} = A = V/R$$

at $s = -R/L$

$$B = I(s)\left(s+\frac{R}{L}\right) \quad \text{at } t = -R/L.$$

$$= \frac{V/L}{-R/L} = -V/R$$

$$\therefore I(s) = \frac{V/R}{s} - \frac{V/R}{(s+R/L)}$$

Taking inverse L.T.,

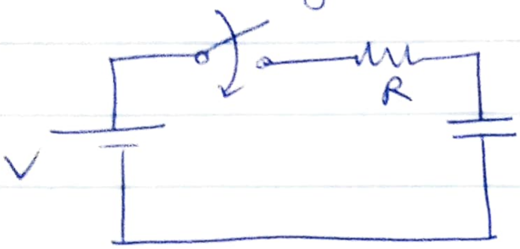
$$i(t) = L^{-1}[I(s)]$$

$$= L^{-1}\left[\frac{V/R}{s} - \frac{V/R}{s+R/L}\right]$$

$$= \frac{V}{R} - \frac{V}{R} \cdot e^{-\frac{R}{L}t}$$

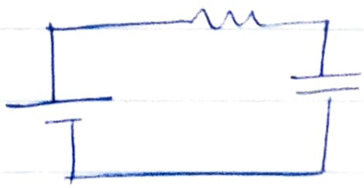
$$= \frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right).$$

2. In the R-C ckt. shown below, the switch is closed at $t=0$. Obtain expression for current using L.T.



$V_c(0^-) = 0V$

For $t>0$, KVL eqn is,



$$-i(t)R - \frac{1}{C}\int_{-\infty}^{t} i(t)\,dt + V = 0$$

$$\therefore \; i(t)R + \frac{1}{C}\int_{0}^{t} i(t)\,dt = V \quad \cdots \quad \text{as } \frac{1}{C}\int_{-\infty}^{0} i(t)dt = 0 = V_c(0^-) = 0$$

Taking L.T on both sides.

$$I(s)R + \frac{I(s)}{sC} + \underbrace{\frac{V_c(0^-)}{s}}_{= 0} = \frac{V}{s}$$

$$\therefore \; I(s)\left[R + \frac{1}{sC}\right] = \frac{V}{s}$$

$$\therefore \; I(s) = \frac{V \cdot sC}{s[1+sRC]} = \frac{VC}{1+sRC} = \frac{VC}{RC\left[s+\frac{1}{RC}\right]}$$

$$\therefore \; I(s) = \frac{V/R}{s+\frac{1}{RC}}$$

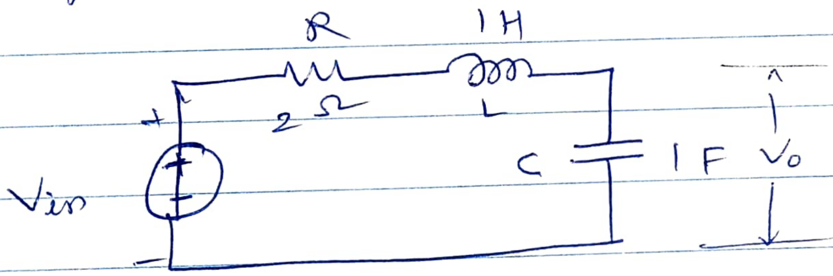
Taking inverse L.T.,

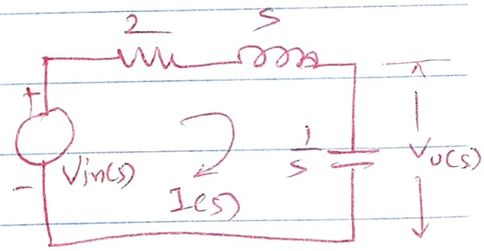
$$i(t) = L^{-1}[I(s)]$$

$$= L^{-1}\left[\frac{V/R}{s+\frac{1}{RC}}\right]$$

$$\boxed{i(t) = \frac{V}{R} \cdot e^{-t/RC} \quad A}$$

3. In series RLC ckt. shown below determine the response in time domain for unit impulse input.





Soln. Assuming initial conditions zero,

KVL eqn can be written as,

$$-2I(s) - sI(s) - \frac{1}{s}I(s) + V_{in}(s) = 0$$

$$\therefore I_s\left(2+s+\frac{1}{s}\right) = V_{in}(s) \quad \text{——— (1)}$$

while ~~Voc~~ $V_o(s) = I(s) \cdot \frac{1}{s}$ → Vltg. across capacitor.

$$\therefore I(s) = s \cdot V_o(s)$$

Substituting this in eqn (1),

$$s \cdot V_o(s) \frac{(s^2+2s+1)}{s} = V_{in}(s)$$

$$\frac{V_o(s)}{V_{in}(s)} = \frac{1}{s^2+2s+1} = \frac{1}{(s+1)^2}$$

Input is impulse function.

$\therefore V_{in}(t) = \delta(t) =$ unit impulse.

$$V_{in}(s) = L\{\delta(t)\} = 1$$

$$\therefore \frac{V_o(s)}{1} = \frac{1}{(s+1)^2}$$

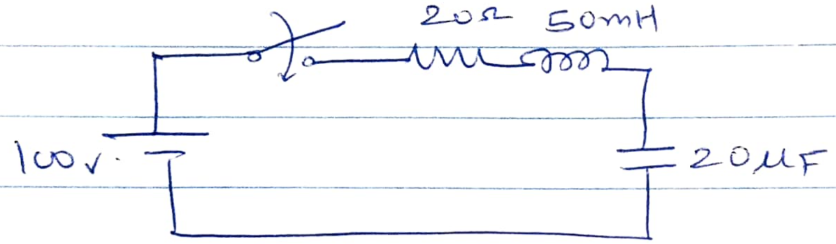
$$\therefore V_o(t) = L^{-1}[V_o(s)]$$

$$\therefore L^{-1}\left[\frac{1}{(s+1)^2}\right]$$

$$\therefore \boxed{V_o(t) = t\,e^{-t}} \quad \text{as} \quad L[t\,e^{-at}] = \frac{1}{(s+a)^2}$$

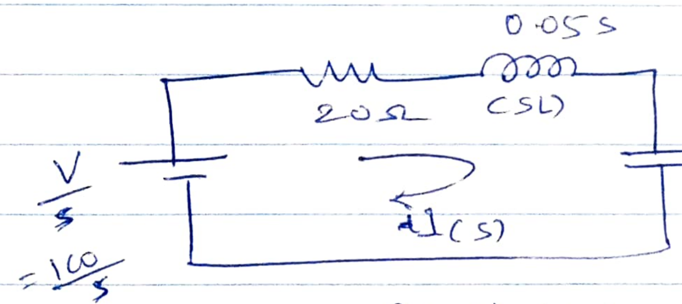
This is the reqd. impulse response of given RLC ckt.

3. The switch is closed at $t=0$. Assuming switch was open for long time before closing, determine current i for $t>0$. Also find voltage across inductor V_L for $t>0$.



Soln.

∴ Network in ~~time~~ freq. domain can be drawn as,



$1/sC$, $\frac{1}{s(20\times10^{-6})} = \frac{5\times10^{4}}{s}$

By ohm's law, $V = IR$

$$\frac{100}{s} = I(s)\left[20 + 0.05s + \frac{5\times10^{4}}{s}\right]$$

$$I(s) = \left[\frac{100/s}{20 + 0.05s + \frac{5\times10^{4}}{s}}\right]$$

$$= \frac{100}{0.05s^2 + 20s + (5\times10^{4})}$$

$$= \frac{2000}{s^2 + 400s + 10^6}$$

$$I(s) = \frac{2000}{(s+200)^2 + (979.79)^2}$$

$$\because \quad e^{-at} \sin \omega t = \frac{\omega}{(s+a)^2 + \omega^2}$$

$$= \frac{2000}{979.79} \cdot \frac{979.79}{(s+200)^2 + (979.79)^2}$$

Taking inverse laplace,

$$\boxed{i(t) = 2.04\, e^{-200t} \sin(979.79t)}$$

Voltage across inductor,

$$V_L(s) = I(s) \cdot Ls$$

$$= \frac{2000}{(s+200)^2 + (979.79)^2} \times 0.05\, s$$

$$= \frac{100s}{(s+200)^2 + (979.79)^2}$$

$$\because \quad e^{-at} \cos \omega t = \frac{s+a}{(s+a)^2 + \omega^2}$$

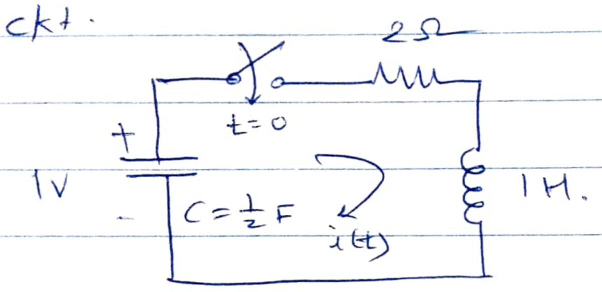
$$V_L(s) = \frac{100(s+200) - 20000}{(s+200)^2 + (979.79)^2}$$

$$= \frac{100(s+200)}{(s+200)^2 + (979.79)^2} - \frac{20000}{(s+200)^2 + (979.79)^2}$$

$$= \frac{100(s+200)}{(s+200)^2 + (979.79)^2} - \frac{20000}{979.79} \cdot \frac{979.79}{(s+200)^2 + (979.79)^2}$$

$$\boxed{V_L(t) = 100\, e^{-200t} \cos(979.79t) - 20.41\, e^{-200t} \sin(979.79t)}$$

4 In the given n/w capacitor is initially charged to 1V. Find current i(t) and draw the s-domain representation of the ckt.



Soln.

Writing KVL eqn.

$$-L\frac{di(t)}{dt} + V_c(0^+) - \frac{1}{C}\int i(t)\,dt - Ri(t) = 0$$

$$L\frac{di(t)}{dt} - V_c(0^+) + \frac{1}{C}\int i(t)\,dt + Ri(t) = 0$$

Taking L.T. of above eqn.

$$L\left[sI(s) - i(0^+)\right] - \frac{V_c(0^+)}{s} + \frac{1}{Cs}I(s) + RI(s) = 0$$

Put values, $L = 1H$
$R = 2\Omega$ $\quad V_c(0^+) = 1V.$
$C = 0.5F$

$i(0^+) = 0$ ... inductor behaves as O.C.

$$sI(s) - \frac{1}{s} + \frac{2}{s}I(s) + 2I(s) = 0$$

$$\therefore I(s)\left[s + \frac{2}{s} + 2\right] = \frac{1}{s}$$

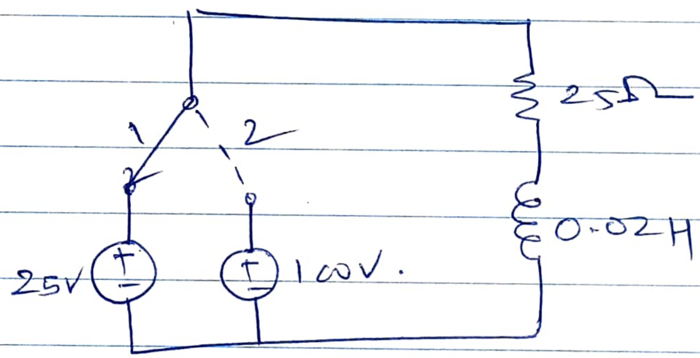
$$\therefore I(s) = \frac{1}{s^2+2s+2}$$

$$= \frac{1}{(s+1)^2+1}$$

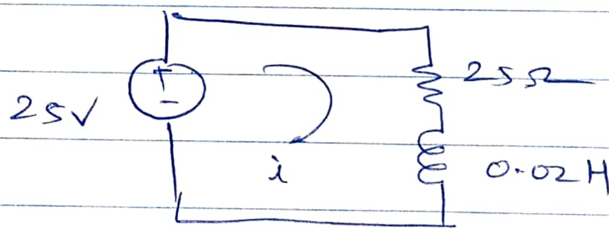
Taking inverse L.T., we get

$$\boxed{i(t) = e^{-t}\sin t}$$

5. In the R-L circuit shown below, the switch is in position 1 for long time enough to establish a steady state condn. At $t = 0$, the switch is thrown to position 2. Find the expression for the resulting current.



Soln. : at position 1 of switch, the ckt. will be



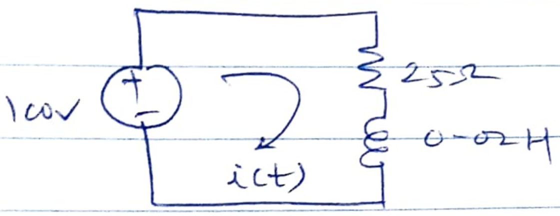
Under steady state condn, the inductor acts as s.c.

$$\therefore \ i = \frac{25}{25} = 1A.$$

This is initial current through ckt.

$\therefore\ i(0^-) = 1A$

When the switch is thrown to position 2, the ckt. will be



Applying KVL,

$$100 - 25\,i(t) - 0.02\frac{di(t)}{dt} = 0$$

$$\therefore\ 0.02\frac{di(t)}{dt} + 25 = 100$$

taking L.T.,

$$0.02\ s\ I(s) - 0.02\ i(0^-) + 25\ I(s) = \frac{100}{s}$$

$$\therefore\ 0.02\,s.I(s) - 0.02(1) + 25\ I(s) = \frac{100}{s}$$

$$\therefore\ I(s)\left[0.02s + 25\right] = \frac{100}{s} + 0.02$$

$$\therefore\ I(s) = \frac{100 + 0.02s}{s(25 + 0.02s)}$$

$$= \frac{0.02(s + 5000)}{0.02s(s+1250)}$$

$$= \frac{s + 5000}{s(s+1250)}$$

Let, $$\frac{s+5000}{s(s+1250)} = \frac{A}{s} + \frac{B}{s+1250}$$

at $s=0$, $A=4$

at $s=-1250$, $B=-3$

$$\therefore I(s) = \frac{4}{s} - \frac{3}{s+1250}$$

Taking inverse L-T,

$$i(t) = L^{-1}[I(s)]$$

$$\boxed{i(t) = 4 - 3e^{-1250t} \quad A}$$