Subject - LADC
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Unit 1. Theory of Matrices

Q 1) Find the rank of the Matrices using Echelon form.

① $\begin{bmatrix} 4 & 2 & -1 & 2 \\ 1 & -1 & 2 & 1 \\ 2 & 2 & -2 & 0 \end{bmatrix}$

$\rightarrow R_2 - \frac{R_1}{4} \rightarrow R_2$ $\begin{bmatrix} 4 & 2 & -1 & 2 \\ 0 & -3/2 & 9/4 & 1/2 \\ 2 & 2 & -2 & 0 \end{bmatrix}$

Now,

$R_3 - \frac{R_1}{R_2} \rightarrow R_3$ $\begin{bmatrix} 4 & 2 & -1 & 2 \\ 0 & -3/2 & 9/4 & 1/2 \\ 0 & 1 & -3/2 & 1 \end{bmatrix}$

Now,

$R_3 - \left(\frac{-2}{3}\right) R_2 \rightarrow R_3$ $\begin{bmatrix} 4 & 2 & -1 & 2 \\ 0 & -3/2 & 9/4 & 1/2 \\ 0 & 0 & 0 & -2/3 \end{bmatrix}$

$\boxed{\text{Rank} = 3}$

② $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{bmatrix}$

$R_1 \leftrightarrow R_4$
$R_2 \leftrightarrow R_1$ $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & -2 & 0 \\ 3 & 1 & 0 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix}$

$\rightarrow$

$$\xrightarrow[\substack{R_3 - 3R_1 \\ R_4 - R_2 \\ R_3 - R_2}]{R_2 - R_1} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So, $\boxed{\text{Rank}(A) = 2}$

3) $\begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$

$$R_2 - 2R_1 \to R_2 \quad \begin{bmatrix} 2 & -2 & 0 & 6 \\ 0 & 6 & 0 & -10 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$$

Now,

$$\underline{R_3 - R_1/2 \to R_3}$$
$$R_4 - R_1/2 \to R_4 \quad \begin{bmatrix} 2 & -2 & 0 & 6 \\ 0 & 6 & 0 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \end{bmatrix}$$

$$\underline{R_4 - (-1/6) R_2 \to R_4}$$
$$R_4 \leftrightarrow R_3$$

$$\begin{bmatrix} 2 & -2 & 0 & 6 \\ 0 & 6 & 0 & -10 \\ 0 & 0 & 1 & -8/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ So, $\boxed{\text{Rank} = 3}$

Q 2) Find the rank of Matrices using normal form

① $\begin{bmatrix} 1 & -1 & 2 & 3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$

$\xrightarrow[R_3 - 3/6 R_2]{R_2 - 4R_1} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 5 & -8 & -10 \\ 0 & 0 & 29/5 & 10 \\ 0 & 1 & 0 & 2 \end{bmatrix}$

$\xrightarrow{R_4 - R_2/6} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 5 & -8 & -10 \\ 0 & 0 & 29/5 & 10 \\ 0 & 0 & 8/5 & 4 \end{bmatrix}$

$\xrightarrow{R_4 - 8/29 R_3} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 5 & -8 & -10 \\ 0 & 0 & 29/5 & 10 \\ 0 & 0 & 0 & 36/29 \end{bmatrix}$

$\therefore \boxed{\text{Rank} = 4}$

② $\begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$

$\xrightarrow[R_3 - R_1 \to R_3]{R_2 - (-2)R_1} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & -2 & 1 & 8 \end{bmatrix}$

$$R_3 - \left(\frac{-1}{4}\right) R_2 \rightarrow R_3 \longrightarrow \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & 0 & 9/4 & -8 \end{bmatrix}$$

$\boxed{Rank = 3}$

(3) $$\begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$$

$$\xrightarrow{R_2 - 2/3 R_1} \begin{bmatrix} 6 & 1 & 3 & 8 \\ 0 & 4/3 & 4 & -19/3 \\ 0 & 4/3 & 4 & -19/3 \\ 16 & 4 & 12 & 15 \end{bmatrix}$$

$$\xrightarrow{R_4 - 8/3 R_1} \begin{bmatrix} 6 & 1 & 3 & 8 \\ 0 & 4/3 & 4 & -19/3 \\ 0 & 4/3 & 4 & -19/3 \\ 0 & 4/3 & 4 & -19/3 \end{bmatrix}$$

$$\xrightarrow[R_4 - R_2]{R_3 - R_2} \begin{bmatrix} 6 & 1 & 3 & 8 \\ 0 & 4/3 & 4 & -19/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\boxed{Rank = 2}$

Q3) Examine for consistency and if consistent then solve it.

① Given - $4x - 2y + 6z = 8$
$x + y - 3z = -1$
$15x - 3y + 9z = 21$

The system can be written in the matrix form as.

$$\begin{bmatrix} 4 & -2 & 6 \\ 1 & 1 & -3 \\ 15 & -3 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ 21 \end{bmatrix} \quad \rho(A) = 2$$

$$\rho(A,B) = 1 \Rightarrow \left[\begin{array}{ccc|c} 4 & -2 & 6 & 8 \\ 1 & 7 & -3 & -1 \\ 15 & -3 & 9 & 21 \end{array}\right]$$

$$\therefore \left[\begin{array}{ccc|c} 4 & -2 & 6 & 2 \\ 0 & 3/2 & -9/2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

System is consistent
$\rho(A) = \rho(A,B) = 2 < 3$ posses infinite soln.

$4x - 2y + 6z = 8$ then $x = 1$
$\frac{3}{2}y - \frac{9}{2}z = -3$ $z = t$, $y = 3t$

Then, $x = 1$
$y = 3t - 2$
$z = t$

② Given - $2x + z = 4$
$x - 2y + 2z = 7$
$3x + 2y = 1$

The system can be written in matrix form as

$$\begin{bmatrix} 2 & 0 & 1 \\ 1 & -2 & 2 \\ 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 1 \end{bmatrix} \quad \rho(A) = 2$$

$$(A, B) = \left[\begin{array}{ccc|c} 2 & 0 & 1 & 4 \\ 1 & -2 & 2 & 7 \\ 3 & 2 & 0 & 1 \end{array}\right]$$

$\therefore$

$$\left[\begin{array}{ccc|c} 2 & 0 & 1 & 4 \\ 0 & -2 & 3/2 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\rho(A, B) = 2$ system is consistent
$\rho(A) = \rho(A, B) = 2 < 3$
so not,
By $R_1 \rightarrow 2x + z = 4$ — Let $z = t$ then
$R_2 \rightarrow 2y + 3/2 z = 5$ — $x = 4 - t/2$
So, solnare - — $y = \frac{-5}{2} + \frac{x}{4}$
$x = t$
$x = \frac{4-t}{2}, \quad y = \frac{-5}{2} + \frac{3t}{4}$

(3) Given:-
$$2x_1 + x_2 - 5x_3 + x_4 = 8$$
$$x_1 + 3x_2 - 6x_4 = 15$$
$$0x_1 + 2x_2 - x_3 + 2x_4 = -5$$
$$x_1 + 4x_2 - 7x_3 + 6x_4 = 0$$

The system can be written in matrix form is

$$\begin{bmatrix} 2 & 1 & -5 & 1 \\ 1 & 3 & 0 & -6 \\ 0 & 2 & -1 & 2 \\ 1 & 4 & -7 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ -15 \\ -5 \\ 0 \end{bmatrix}$$

$\rho(A) = 3 + 1 = 4$

$$(A, B) = \left[\begin{array}{cccc|c} 2 & 1 & -5 & 1 & 8 \\ 1 & 3 & 0 & -6 & -15 \\ 0 & 2 & -1 & 2 & -5 \\ 1 & 4 & -7 & 6 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cccc|c} 2 & 1 & -5 & 1 & 8 \\ 0 & 5/2 & 5/2 & -13/2 & -19 \\ 0 & 0 & -3 & 36/5 & 51/5 \\ 0 & 0 & 0 & -23/5 & -23/5 \end{array}\right]$$

By matrix:

$$\left[\begin{array}{cccc|c} 2 & 1 & -5 & 1 & 8 \\ 1 & 5/2 & 5/2 & -13/2 & -15 \\ 0 & 0 & -3 & 36/5 & 51/5 \\ 0 & 0 & 0 & -23/5 & -23/5 \end{array}\right]$$

$R_1 \rightarrow 2x_1 + x_2 - 5x_3 + x_4 = 8$

$$-3x_2 + \frac{36}{5} x_4 = \frac{51}{5}$$

$$-\frac{23}{5} x_4 = \frac{23}{6}$$

$x_4 = 1 \quad x_3 = -1 \quad x_2 = -4 \quad x_1 = 3$

(4) Given.
$$x_1 + x_2 + 2x_3 + x_4 = 5$$
$$2x_1 + 3x_2 - x_3 - 2x_4 = 2$$
$$4x_1 + 5x_2 + 3x_3 = 7$$

The system can be written in matrix form as

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 3 & -1 & -2 \\ 4 & 5 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 7 \end{bmatrix}$$

Hence the system is inconsistent as $\rho(A) \neq \rho(A,B)$

Q 4) Investigate for what values of a & b the system of simultaneous

$$2x - y + 3z = 2$$
$$x + y + 2z = 2$$
$$5x - y + az = b$$

→

(1) No solution
$a = 8,\ b = 6$
$P(A) = P(A,B)$

(2) Unique solution
$a = 8 \quad PA = P(A,B)$
$b \in R$

(3) Infinite solution
$a = 8,\ b = 6$

In metion form

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 2 \\ 5 & -1 & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ b \end{bmatrix}$$

by solving

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 0 & 3/2 & 1/2 & 1 \\ 0 & 0 & a-8 & b-6 \end{array}\right]$$

So $P(A) = P(A,B)$
$a = 8 \to 3$
$a = 8 \quad b = 6 \to 2$

Q5) Investigate for what values of k the quations Have infinite number of solutions? Hence, find solutions.

→ Given $x + y + z = 1$
$2x + y + 4z = k$
$4x + y + 10z = k^2$

for infinite soln. $\rho(A) = \rho(A,B) = r < 3$
i.e.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ 4 & 1 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ k \\ k^2 \end{bmatrix} \quad Ax = B$$

so, By solving weget.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 1 & 4 & k \\ 4 & 1 & 0 & k^2 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & k-2 \\ 0 & 0 & 0 & k^2-3k+2 \end{array}\right]$$

$\therefore k^2 - 3k + 2 \neq 0 \rightarrow$ No soln
& $k^2 - 3k + 2 = 0 \rightarrow \rho(2)$

so, $$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & k-2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore \boxed{k = 1, 2}$

Q6) Examine for Linear dependence or independence the following system of vectors, If dependent, find relation btw them

(1) Given $x_1 = (1,-1,1)$, $x_2 = (2,1,1)$ $x = (3,0,2)$

$$c_1x_1 + c_2x_2 + c_3x_3 = 0$$

$$c_1(1,-1,1) + c_2(2,1,1) + c_3(3,0,2) = 0$$

$$c_1 + 2c_2 + 3c_3 = 0$$

$$-c_1 + c_2 + 0 = 0$$

$$c_1 + c_2 + 2c_3 = 0$$

In matrix form $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$P(A) = P(A,2) = 2 = 2 < 3$ $\therefore \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

System posses a non-finite soln.

$$c_1 + 2c_2 + 3c_3 = 0$$

$$3c_2 + 3c_3 = 0$$

$$c_2 - c_1 = 0$$

$$c_2 = c_1 = t$$

$$c_3 = -t$$

c_1, c_2, c_3 are non-zero so mostly dependent.

$$x_1 + tx_2 + tx_3 = 0$$

$$\therefore \boxed{x_1 + x_2 = x_3}$$

(2) Given $x_1 = (1,1,1,3)$, $x_2 = (1,2,3,4)$, $x_3 = (2,3,4,7)$

$$c_1 x + c_2 x_2 + c_3 x_3 = 0$$

$$c_1(1,1,1,3) + c_2(1,2,3,4) + c_3(2,3,4,7)$$

$$c_1 + c_2 + 2c_3 = 0$$
$$c_1 + 2c_2 + 3c_3 = 0$$
$$c_1 + 3c_2 + 4c_3 = 0$$
$$3c_1 + 4c_2 + 7c_3 = 0$$

In matrix form

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \\ 3 & 4 & 7 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Here

$$P(A) = P(A,2) = 2 < 3$$

$\therefore$ system posses non-trival soln.

$$\text{so, } c_1 + c_2 + 2c_3 = 0$$
$$c_2 + c_3 = 0$$
$$c_1 - c_3 = 0$$
$$c_1 = c_3 = t,$$
$$t_1 x + t x_2 - t x_3 = 0$$

$$\therefore \boxed{x_1 + x_2 = x_3}$$

(3) $x_1 = (3, 1, -4)$, $x_2 = (2, 2, -3)$ $x_3 = (0, -4, 1)$
consider the matrix eqn

$$c_1 x_1 + c_2 x_2 + c_3 x_3 = 0$$

$$c_1(3, 1, -4) + c_2(2, 2, -3) + c_3(0, -4, 1)$$

$$3c_1 + 2c_2 + 0c_3 = 0$$

$$c_1 + 2c_2 - 4c_3 = 0$$

$$-4c_1 - 3c_2 + c_3 = 0$$

In matrix form

$$\begin{bmatrix} 3 & 2 & 0 \\ 1 & 2 & -4 \\ -4 & -3 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So, $\rho(A) = \rho(A, 2) = 2 < 3$ $\left[\begin{array}{ccc|c} 3 & 2 & 0 & 0 \\ 0 & 4/3 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

System possess a non trivial soln.

So, $3c_1 + 2c_2 + 0c_3 = 0$

$$4/3\, c_2 - 4c_3 = 0$$

So, $2t_1 - 3t_2 - t_3 = 0$

$$2x_1 = 3x_2 + x_3$$

$\therefore$ Ans is $\boxed{2x_1 = 3x_2 + x_3}$

(4) $x_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, x_2 = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}, x_3 = \begin{pmatrix} 1 \\ -6 \\ -6 \end{pmatrix}$

consider the matrix eqn $c_1x_1 + c_2x_2 + c_3x_3 = 0$

$$c_1\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + c_2\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + c_3\begin{pmatrix} 1 \\ -6 \\ 5 \end{pmatrix} = 0$$

$$c_1 + 3c_2 + c_3 = 0$$
$$2c_1 - 2c_2 - 6c_2 = 0$$
$$3c_1 + c_2 - 5c_3 = 0$$

In matrix form

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & -2 & -6 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So, $P(A) = P(A,2) = 2 < 3$

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & -8 & -8 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}$$

Here

system posses non-trival soln.

$$c_1 + 3c_2 + c_3 = 0$$
$$-8c_2 - 8c_3 = 0$$
$$c_2 - c_3 = 8t - t$$

So, $-2x_1 + x_2 - x_3 = 0$

$$2x_1 + x_3 = x_3$$

Q7) Find the coordinates (x_1, x_2, x_3) corresponding to $(2,3,0)$ in Y. for

$\rightarrow$ Transformation = $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & -3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$Y = Ax$ i.e. $Ax = y$

$$\begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & -3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$

This is non-homogenious system of eqn

$$|A| = \begin{vmatrix} 1 & -2 & 3 \\ 2 & 0 & -3 \\ 1 & 3 & 1 \end{vmatrix} \quad |A| = 19, \quad |A| = 19 \neq 0$$

System posses a unique soln $x = A^{-1}x$

$$A^{-1} = \frac{1}{|A|} \operatorname{Adj} A$$

$$\text{So, } A^{-1} = \begin{vmatrix} 3/19 & 5/19 & 6/19 \\ -5/19 & -2/19 & 9/19 \\ 2/15 & -3/19 & 4/19 \end{vmatrix}$$

$$x = \begin{vmatrix} 3/19 & 5/19 & 6/19 \\ -5/19 & -2/19 & 9/19 \\ 2/19 & -3/19 & 4/19 \end{vmatrix} \begin{vmatrix} 2 \\ 9 \\ 0 \end{vmatrix} = \begin{vmatrix} 21/19 \\ -16/19 \\ -9/19 \end{vmatrix}$$

So, $x_1 = \frac{21}{19}$, $x_2 = \frac{-16}{19}$, $x_3 = \frac{-5}{19}$

Q 8) Express each of the transformation $x_1 = 3y_1 + 5y_2$, $x_2 = -y_1 + 7y_2$ & $y_1 = z_1 + 3z_2$, $y_2 = 4z_1$ In the matrix form and find the composite transformation which expresses x_1, x_2 in terms of z_1, z_2.

→ The transformation $\left.\begin{matrix} x_1 = 5y_1 + 5y_2 \\ x_2 = -y_1 + 7y_2 \end{matrix}\right\}$ in the matrix form can be written as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$x = Ay$$

Also the transformation $= y_1 = z_1 + 3z_2$
$y_2 = 4z_1$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \quad y = Bz$$

∴ Required composite transformation.

$$x = Ay = \begin{bmatrix} 3 & 6 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

so

$$= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 23 & 9 \\ 27 & -3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$x_1 = 23z_1 + 9z_2$
$x_2 = 27z_1 - 3z_2$ is the required transformation.

Q 9) Verify whether the following matrices are orthogonal or not, if so write A^{-1}:

① $A = \begin{bmatrix} 1/\sqrt{3} & 0 & 2/\sqrt{3} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{3} \\ -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$

If A is orthogonal then $AA' = 2$

$$AA' = \begin{bmatrix} 1/\sqrt{3} & 0 & 2/\sqrt{3} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{3} \\ -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 2\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$ $\therefore$ A is Orthogonal

② $A = \frac{1}{3} \begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -1 & 2 \end{bmatrix}$ If A is orthogonal then $AA' = I$

$$= \frac{1}{3} \begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -1 & 2 \end{bmatrix} \times \frac{1}{3} \begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 2/5 \\ 0 & 1 & 1/5 \\ 2/5 & 1/5 & 2/3 \end{vmatrix}$$ So, $AA^{-1} = I$

Then A is not orthogonal &

$$A^{-1} = \frac{1}{7} \begin{bmatrix} 4 & -5 & 3 \\ 6 & 3 & -6 \\ 1 & 4 & 6 \end{bmatrix}$$

Q10) If $A = \begin{bmatrix} 1/3 & 2/3 & a \\ 2/3 & 1/3 & b \\ 2/3 & -2/3 & c \end{bmatrix}$ is orthogonal, Find a, b, c

→ Here A is orthogonal

$\therefore AA^{-1} = I$

$$AA^{-1} = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ a & b & c \end{bmatrix} \cdot \begin{bmatrix} 1/3 & 2/3 & a \\ 2/3 & 1/3 & b \\ 2/3 & -2/3 & c \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} 1 & 0 & a+2b+2c/2 \\ 0 & 1 & 2a+b-2c/3 \\ \frac{a+2b+2c}{3} & \frac{2a+b-2c}{3} & a^2+b^2+c^2 \end{bmatrix}$$

out $AA^{-1} = x$

$$\frac{a+2b+2c}{3} = 1 \Rightarrow \frac{a+2b}{3} = \frac{+2c}{3}$$

$$\frac{2a+b-2c}{3} = 1 \quad a+2b+2c = 3$$

$$\frac{4a^2+4b^2-8ab+c^2+4ac-4b}{3} = I$$

$$\therefore \boxed{a = \pm\frac{2}{3}, \ b = \pm\frac{2}{3}, \ c = \pm\frac{1}{3}}$$

Q 12) verify Cayley-Hamilton theorem for the following matrix and use it find inverse:-

① $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ $|A - \lambda I| = 0$ $\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-3 & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$

$$(2-\lambda)((2-\lambda)^2 - 1) + 1(-2+\lambda+1) + 1(1-2+3) = 0$$

$$-\lambda^3 + 6\lambda^2 - 9\lambda + 4 = 0$$

$$\text{so, } A^3 - 6A^2 + 9A - 4I = 0$$

$$A^2 = A \times A = \begin{vmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{vmatrix}$$

$$A^3 = \begin{pmatrix} 22 & -21 & 22 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{pmatrix} \quad A^3 - 6A^2 + 9A - 4I$$

$$\begin{bmatrix} 22 & -21 & 22 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6 \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9 \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} -$$

$$4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 1/6 & 5/6 & 1/2 \\ -1/6 & 1/6 & 1/2 \end{bmatrix}$$

② $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$ $|A - \lambda I| = 0$

$\begin{bmatrix} 1-\lambda & 0 & 2 \\ 2 & 2-\lambda & 4 \\ 0 & 0 & 2-\lambda \end{bmatrix} = 0$ $-\lambda^3 + 5\lambda^2 - 3\lambda + 4 = 0$

from this $A^3 - 5A^2 + 8A - 4 = 0$

$A^2 = A \cdot A = \begin{bmatrix} 1 & 0 & 6 \\ 6 & 4 & 20 \\ 0 & 0 & 4 \end{bmatrix}$

$A^3 = \begin{bmatrix} 1 & 0 & 14 \\ 14 & 8 & 68 \\ 0 & 0 & 8 \end{bmatrix}$ so, $A^3 - 5A^2 + 8A - 4 = 0$

$\begin{bmatrix} 1 & 0 & 14 \\ 14 & 8 & 68 \\ 0 & 0 & 8 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 6 \\ 6 & 4 & 20 \\ 0 & 0 & 4 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 & 2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

So, $A^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}$

→

Q3 13) Find A^4 with the help of Cayley Hamilton theorem.

→ The characteristics eqn of A is $|A - I\lambda| = 0$

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} \quad so \Rightarrow \begin{bmatrix} 1-\lambda & 0 & -1 \\ +1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix}$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

$$S_1 = 1 + 2 + 3 = 6$$

$$S_2 = \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix}$$

$$S_2 = 4 + 5 + 2 = 11$$

$$|A| = 6$$

Hence, the characteristics of eqn of $A = \lambda^3 - 6\lambda^2 + 11\lambda - 3 = 0$, By using Cayley Hamilton thm.

$$A^3 + 6A^2 + 11A + 6 = 0$$

So, $A^3 - 6A^2 + 11A - 6 = 0$

So, $A^2 = \begin{bmatrix} -1 & -2 & -4 \\ 5 & 6 & 4 \\ 10 & 10 & 9 \end{bmatrix}$ $A^3 = \begin{bmatrix} -11 & -12 & -13 \\ 19 & 20 & 13 \\ 28 & 38 & 27 \end{bmatrix}$

$$A^4 = 6A^3 - 11A^2 + 6A$$

$$A^4 = \begin{bmatrix} -49 & -50 & -40 \\ 65 & 66 & 40 \\ 130 & 130 & 81 \end{bmatrix}$$

Q(14) If $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$, then express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ in terms of A.

→ $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

then $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ in terms of A

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{bmatrix}$$

Then $A^5 - 4A^4 - 7A^3 + 11A^2 - A = 10I$

$$A - IA = \begin{bmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{bmatrix} \quad |A - \lambda I| = \lambda^2 - 4\lambda - 5 = 0$$

Now, we substitute the matrix (A) in the characteristics of eqn in accordance to cayley hamilton theorem. ∴

$$A^2 - 4A - 5I = 0$$

$$A^2 = A \times A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 3 & 17 \end{bmatrix}$$

$$4 \times A (4A) = 4 \times \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 16 \\ 8 & 18 \end{bmatrix}$$

$$5I = 5 \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

→

$$A^2-4A-5I=\begin{bmatrix}9&16\\8&17\end{bmatrix}-\begin{bmatrix}4&16\\8&17\end{bmatrix}-\begin{bmatrix}5&0\\5&0\end{bmatrix}$$

$$=\begin{bmatrix}5&0\\0&5\end{bmatrix}-\begin{bmatrix}5&0\\0&5\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$\therefore$ CayleyHamilton theorem has been verified for the given matrix 'A'

Now the given polynomial is

$$A^5-4A^4-7A^3+11A^2-A-10I$$

Let us divide this by the eqn obtained from cayley Hamilton theorem.

Reminder $=(A+5I)$

Quoitent $=A^2-2A+3I$

Thus, polynomial $=(A^2-2A+3I)\times(A^2-4A-5I)+(A+5I)$

$\therefore$ we know that $(A^2-4A-5I)=0$

So, Now we can express the poly as

$$(A^3-2A+3I)\times 0+(A+5I)=A+5I.$$

$\therefore$ The reauire solution is

$$\boxed{A+5I}$$