State Space Analysis

- 2 approaches to analyse and design in control system

a) T.F approach

b) state variable approach.

Comparison betn T.F approach and state variable approach

Transfer function approach	state variable approach.
1) Also called classical approach	1) modern approach.
2) applicable only to LTI, SISO system. For MIMO calculation are complex	2) applicable to linear as well as non linear time variant, time invariant, SISO, MIMO.
3) based on i/p o/p relationship.	3) based on system equations. eqn can be 1st order differential eqns.
4) Initial conditions are neglected	4) Initial conditions are considered.

T.F	state variable.
5) Frequency domain approach	5) Time domain approach.
6) only i/p, o/p, error signals are important	6) state variables are considered. may not represent physical variables.
7) i/p, o/p variables must be measurable	7) neednot be measurable or observable.
8) requires Laplace transform for continuous data control and Z transform for descrete data control systems.	8) formulates both continuous data control and descrete. data control in same way.
9) Internal variables ~~are~~ cannot be fed back	9) state variables can be fed back.
10) T.F is unique for a give system	11) state model is not unique.

Important def^n^s

State: State of a dynamic system is the smallest set of variables such that knowledge of these variables at $t = t_0$, along with knowledge of i/p for $t \geq t_0$ completely determine the behaviour of the system.

State variables: are smallest set of variables . .

. — . – – –

- State variables need not be physically measurable or observable quantities.
- Variables which are non measurable and observable can also be chosen as state variables.

State vector: Vector whose components are state variables are called state vector.

- denoted by $x(t)$.
- If n state variables. $x_1, \ldots x_n$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

State space : n ~~dimens~~ dimesional space whose coordinate axes consist of x_1 axis, x_2 axis . . x_n axis where $x_1, x_2 \cdot\cdot x_n$ are state variables is called state space.

State space Equations

- In state space analysis, three types of variables are involved.

a) Input variables

b) output "

c) State variables.

i) those variables that memorize the values of i/p. for $t \geq t_0$.

o/p of integrators serve as state variables.

Physical Variable form Electrical circuit

- current in the inductor at time

$t=0\ (t_0)$ $i_L(t_0)$.

- voltage across capacitor at $t=0\ (t_0)$ $V_c(t_0)$.

- state of RLC circuit at time $t=0\ (t_0)$.

is defined by $i_L(t_0)$ & $V_c(t_0)$

- physical quantities $i_L(t_0)$ & $V_c(t_0)$ is called

initial state of circuit

- At time t

$i_L(t)$ & $V_c(t)$ is called state of the n/w.

i_L, V_c — state variable of circuit.

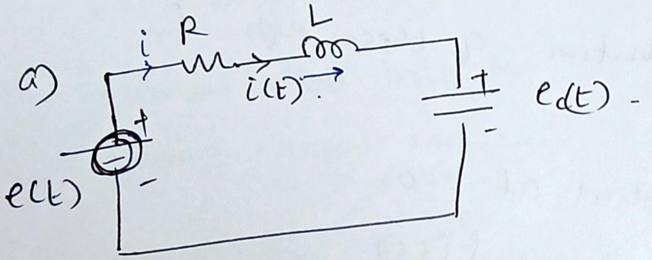
1) Using physical variables Electrical n/w (RLC).

1) Assign all inductor currents and capacitor voltages as state variables.

2). select a set of loop currents and write the relationship betⁿ state variable and their derivative in terms of these loop currents

state variable - i_L & v_c.

a)



$$i_L = i_c = i = c\frac{dv_c}{dt}.$$

$$L\frac{di_L}{dt} + Ri_L + v_c = v_i.$$

$$\frac{dv_c}{dt} = \frac{i}{c}.$$

$$\frac{di_L}{dt} = -\frac{Ri}{L} - \frac{v_c}{L} + \frac{v_i}{L}.$$

$$R i_L(t) + L\frac{di_L(t)}{dt} + e_c(t) = e(t). \quad -(1)$$

$$c\frac{de_c(t)}{dt} = i_L(t) \quad -(2)$$

$\left[e_c(t) = \frac{1}{c}\int i(t)\,dt. \right.$ differentiating both sides $\left.\right]$

from (1) & (2)

$$\frac{di(t)}{dt} = \frac{1}{L}\left[e(t) - e_c(t) - R\,i(t)\right]$$

$$= \frac{1}{L}e(t) - \frac{1}{L}e_c(t) - \frac{R}{L}i(t). \quad -(3)$$

$$\frac{de_c(t)}{dt} = \frac{1}{C} i(t) \quad -(4).$$

From (3) & (4). state vector matrix,

$$\begin{bmatrix} \frac{de_c(t)}{dt} \\ \frac{di(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} e_c(t) \\ i(t) \end{bmatrix} +$$

$$\begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} e(t)$$

The solution of these eqn for

~~y(t)~~ =

a) i/p $e(t)$ applied at $t=0$.

b) initial state $e_c(0)$ & $i(0)$.

gives state variables $[e_c(t), i(t)]$ for $t>0$.

<u>o/p of system</u>

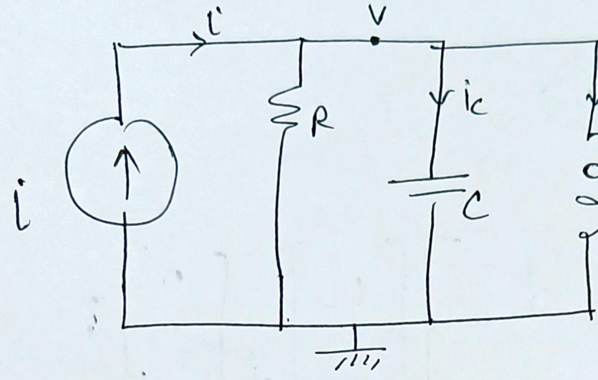
$y(t) = e_c(t)$ → <u>o/p equation</u>.

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} e_c(t) \\ i(t) \end{bmatrix}.$$

state eqn & o/p equations together are called dynamic equations of system.

* Also called state model of the system.

(2) Write the state variable formulation of parallel R-L-C circuit. The current thro' inductor and the voltage across the capacitor are the o/p variables.



Let state variables be i

current thro' inductor i_L

voltage across capacitor v:

o/p variables be

$i_0 = i_L$

$v_0 = v$

writing KCL

$$i = i_R + i_C + i_L$$

$$\cancel{\frac{v}{R} + \frac{1}{L}\int v\,dt}$$

$$= \frac{v}{R} + C\frac{dv}{dt} + i_L$$

$$\frac{dv}{dt} = \frac{-1}{C} i_L - \frac{v}{RC} + \frac{1}{C} i \quad — \quad (1)$$

voltage across inductor

$$v = L\frac{di}{dt}$$

$$\frac{di_L}{dt} = \frac{v}{L} \quad — \quad (2)$$

$$\underbrace{\begin{bmatrix} \frac{di_L}{dt} \\ \frac{dv}{dt} \end{bmatrix}}_{} = \underbrace{\begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{RC} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} i_L \\ v \end{bmatrix}}_{X(t)} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{C} \end{bmatrix}}_{B} \underbrace{i}_{u(t)}$$

$$\cancel{\begin{bmatrix} i_o \\ v_o \end{bmatrix}} = \cancel{\begin{bmatrix} 0 & \frac{1}{L} \end{bmatrix}}$$

$$\begin{bmatrix} i_o \\ v_o \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} i_L \\ v \end{bmatrix}}_{X(t)} \qquad \begin{bmatrix} i_o = i_L \\ v_o = v \end{bmatrix}.$$

28/10 - 3, 9, 11, 12, 13, 14, 1~~6~~, 17, 18, ~~19~~. 21, 24, 25, 31, 32, 34, 37, 40, 44, 47, 47, 49, 51, 52, 53, 54, 58, 62, 66.

11

2/11 2nd 12/10 3rd remaining

State space using phase variables.

(T.F given in differential form)

Phase variables : are defined as those particular state variables, which are obtained from one of the system variables and its derivatives.

- In this system variable is system o/p. and remaining variables are derivatives of o/p.

state variables = o/p variables and derivative of o/p variables.

- disadvantage: phase variables are not physical variables of the system ∴ not available for measurement and control purpose.

- For linear time invariant system, change of state doesn't depend on initial time, but depends only on the length of time for which control force is applied.

Let there are n state variables.

$x_1, x_2, \ldots \, x_n$.

<u>Steps</u>:

1) Let

$$x_1 = y.$$
$$x_2 = \dot{y}$$
$$x_3 = \ddot{y}$$
$$x_n = y^{(n-1)}.$$

These eqns canbe reduced to 1st order differential eqns as.

$$x_1 = y$$

2) ⇒ <u>write</u>

$$y = x_1$$
$$\dot{y} = \boxed{\dot{x}_1 = x_2}$$
$$\ddot{y} = \ddot{x}_1 = \boxed{\dot{x}_2 = x_3}$$
$$\dddot{y} = \dddot{x}_1 = \ddot{x}_2 = \boxed{\dot{x}_3 = x_4}$$

$$\boxed{y = x_1}$$
$$x_1 = y$$
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = x_4$$
$$\vdots$$

3) last eqn is obtained by equating the highest order derivative term of the o/p in differential eqn to all other terms.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & \cdots & & & -a_1 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ b \end{bmatrix}}_{B} u$$

$$\dot{x} = Ax + Bu.$$

A – has all 1s in upper of diagonal.

- Last row of -ve coeff. of diff. eqn. and all other elements zero.

- bush form or companion form.

B – all zero except last row.

o/p $y = x_1$.

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

$$y(t) = Cx(t).$$

1) obtain state model for the system described as

$$T(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^3+6s^2+10s+5}.$$

Step 1 :

$$U(s) = (s^3+6s^2+10s+5)\,Y(s).$$

Taking laplace inverse, $\left[L\, 5\frac{d^2y}{dx^2} = 5s^2\right].$

$$u = \dddot{y} + 6\ddot{y} + 10\dot{y} + 5y. \quad — (1)$$

Step 1 : let $y = x_1$.

$\dot{y} = \boxed{\dot{x}_1 = x_2}$

$\ddot{y} = \ddot{x}_1 = \boxed{\dot{x}_2 = x_3.}$

~~$\ddddot{y} = \dddot{x}_1 = \ddot{x}_2 = \boxed{\dot{x}_3 = x_4}$~~

write upto (n-1) derivative.
→ prob. has 3rd der. write upto 2nd derivative.

$\dddot{y} = \dddot{x}_1 = \ddot{x}_2 = \dot{x}_3$

$\boxed{\dddot{y} = \dot{x}_3}$ — (2)

(2) ~~$\dddot{y}$/~~ from eq^n (1).

$$\dddot{y} = u - 6\ddot{y} - 10\dot{y} - 5y.$$

$$= -6\ddot{y} - 10\dot{y} - 5y + u.$$

(3) i.e. from eq (2) substitute $\dddot{y} = \dot{x}_3$.

$$\dot{x}_3 = -6.$$

Step:
(2) let

$$y = x_1.$$
$$\dot{y} = \boxed{\dot{x}_1 = x_2}$$
$$\ddot{y} = \ddot{x}_1 = \boxed{\dot{x}_2 = x_3}$$
$$\dddot{y} = \dddot{x}_1 = \ddot{x}_2 = \dot{x}_3$$
$$\boxed{\dddot{y} = \dot{x}_3}$$

eq^n (2).

Step 3: From eq^n (1) & (2).

$$\dddot{y} = -6\ddot{y} - 10\dot{y} - 5y + u.$$

put $\dddot{y} = \dot{x}_3$ and correspondiy all derivatives in terms of state variables.

$$\dot{x}_3 = -6x_3 - 10x_2 - 5x_1 + u.$$

from (2)

$$\left[\begin{array}{l} \ddot{y} = x_3 \\ \dot{y} = x_2. \\ y = x_1 \end{array}\right]$$

State space model is.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -10 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u.$$

$\dot{x} = Ax + Bu.$, $y = x_1.$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(2) $\frac{y(s)}{v(s)} =$

(2) $\dfrac{Y(s)}{U(s)} = \dfrac{1}{s^3+6s^2+11s+6}$.

done

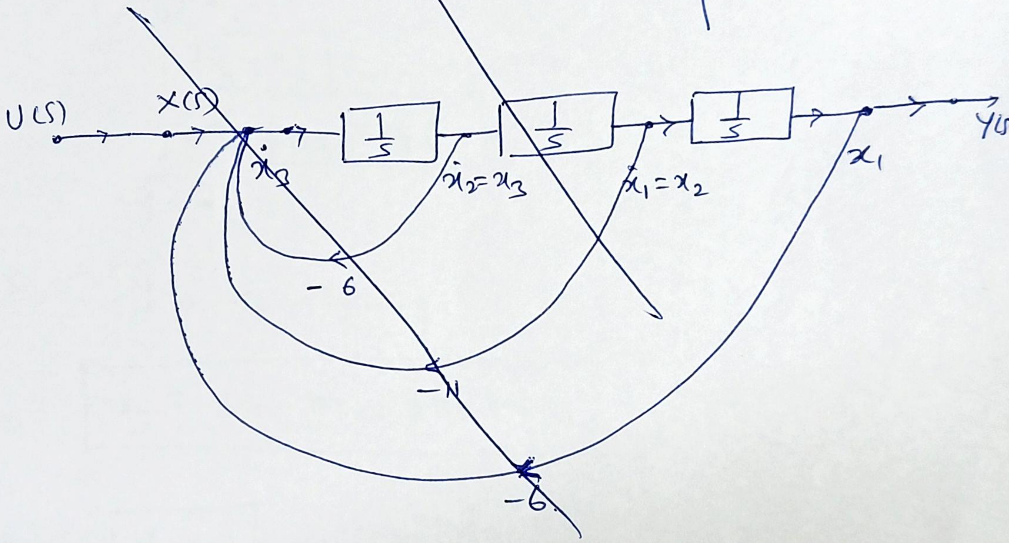
solve by differential eqn

$\dddot{y}+6\ddot{y}+11\dot{y}+6y=u$.

$$\frac{Y(s)}{U(s)} = \frac{1}{1+\frac{6}{s}+\frac{11}{s^2}+\frac{6}{s^3}} \cdot \frac{X(s)}{X(s)}$$

$y = X(s) = u(s)$.

$X(s) = U(s) - \frac{6}{s} - \frac{11}{s^2} - \frac{6}{s^3}$.



$\dot{x}_1 = x_2$

$\dot{x}_2 = x_3$.

$\dot{x}_3 = -6x_1 - 11x_2 - 6x_3 + u$.

$[y] = [x_1$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u.$$

Phase variable

3) $\frac{Y(s)}{U(s)} = \frac{5}{s^3+6s+7}$

$Y(s)(s^3+6s+7) = 5U(s).$

Step 1 Taking laplace inverse.

$\dddot{y} + 6\dot{y} + 7y = 5u.$

Step 2 :

$y = x_1$

$\dot{y} = \boxed{\dot{x}_1 = x_2}$

$\ddot{y} = \ddot{x}_1 = \boxed{\dot{x}_2 = x_3}$

$\dddot{y} = \dot{x}_3$

Step 3 : $\dddot{y} = 5u - 6\dot{y} - 7y.$

$\boxed{\dot{x}_3 = 5u - 7x_1 - 6x_2 + 0x_3.}$

$\boxed{y = x_1}$

Step 4 :

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7 & -6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]u.$$

(2) $\ddddot{y} + 5\dddot{y} + 7\dot{y} + 9y = 8u.$

Step 1 :

Step : 2 :
$$y = x_1$$
$$\dot{y} = \boxed{\dot{x}_1 = x_2}$$
$$\ddot{y} = \ddot{x}_1 = \boxed{\dot{x}_2 = x_3}$$
$$\dddot{y} = \dddot{x}_1 = \ddot{x}_2 = \boxed{\dot{x}_3 = x_4}.$$
$$\ddddot{y} = \ddddot{x}_1 = \dddot{x}_2 = \ddot{x}_3 = \dot{x}_4.$$

Step 3 : $\ddddot{y} = 8u - 5\dddot{y} - 7\dot{y} - 9y$

put all derivatives in terms of state Variables.

$$\boxed{\dot{x}_4 = 8u - 5x_4 + 0x_3 - 7x_2 - 9x_1} \qquad \boxed{y = x_1}$$

Step 4 : state space model.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -9 & -7 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 8 \end{bmatrix} u(t).$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \cancel{d} \begin{bmatrix} 0 \end{bmatrix} u.$$

State space representation for given transfer function

1) Direct decomposition
2) Cascade "
3) Canonical or diagonal form. (Parallel decomposition)
4) Jordan's cannonical form
5). By signal flow graph.

1) <u>Direct decomposition</u> | signal flow graph.

- <u>T.F is not in factored form</u> → form is K/as ~~observable~~ controllable cannonical form (CCF)

1) $\frac{Y(s)}{U(s)} = \frac{2s^2+3s+1}{s^3+5s^2+6s+7}$

cont

<u>Step 1</u>: ~~multiply~~ Divide & N & D by highest power i.e s^3.

$T.F = \frac{M_K \Delta_K}{1-(\ \ )}$.

$$\frac{Y(s)}{U(s)} = \frac{\frac{2}{s} + \frac{3}{s^2} + \frac{1}{s^3}}{1 + \frac{5}{s} + \frac{6}{s^2} + \frac{7}{s^3}}.$$

<u>Step 2</u>: Multiply & divide by dummy variable $X(s)$.

$$\frac{Y(s)}{U(s)} = \frac{\left(\frac{2}{s} + \frac{3}{s^2} + \frac{1}{s^3}\right) X(s)}{\left(1 + \frac{5}{s} + \frac{6}{s^2} + \frac{7}{s^3}\right) X(s)}.$$

<u>Step 3</u>: equating N & D on either side.

$$Y(s) = \left(\frac{2}{s} + \frac{3}{s^2} + \frac{1}{s^3}\right) X(s) \quad \text{---} (1)$$

$$U(s) = \left(1 + \frac{5}{s} + \frac{6}{s^2} + \frac{7}{s^3}\right) X(s) \quad \text{---} (2)$$

step 4: from eqn (1) + (2)

$$\boxed{Y(s) = \frac{2}{s} X(s) + \frac{3}{s^2} X(s) + \frac{1}{s^3} X(s)}$$

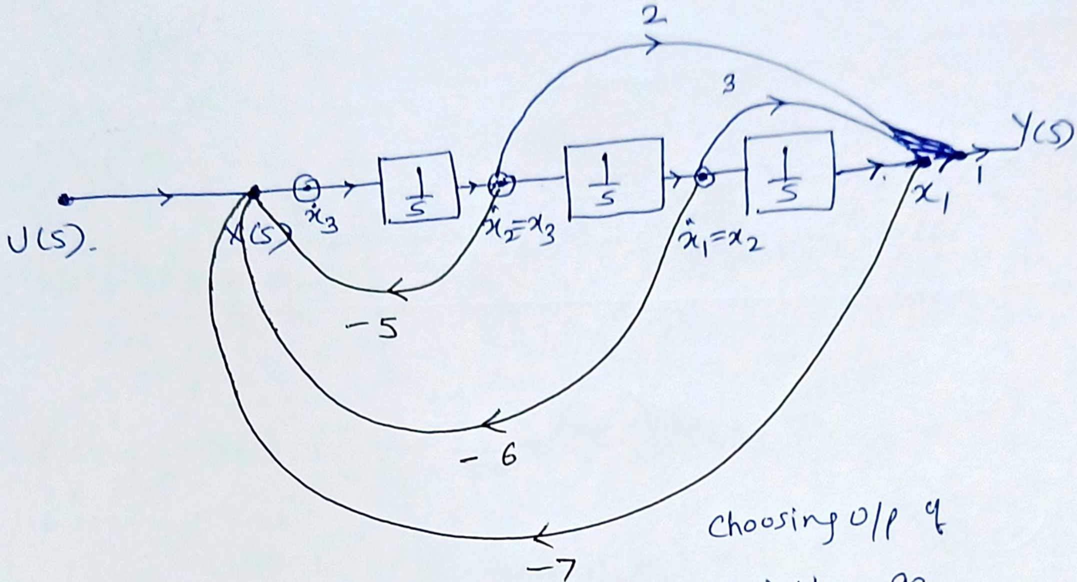
$$U(s) = X(s) + \frac{5}{s} X(s) + \frac{6}{s^2} X(s) + \frac{7}{s^3} X(s)$$

$$\Rightarrow \boxed{X(s) = U(s) - \frac{5}{s} X(s) - \frac{6}{s^2} X(s) - \frac{7}{s^3} X(s)}$$

step 5: since power of T.F is 3. there are 3. integrators

The diff eqn is of 3rd order so, there are 3 summing point and 3 integrators.

after signal flow graph.



Choosing o/p & Variables as state variables.

$\dot{x}_1 = x_2$

$\dot{x}_2 = x_3$

$\dot{x}_3 = -7x_1 - 6x_2 - 5x_3 + u.$

$y = x_1 + 3x_2 + 2x_3$

state space form.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7 & -6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u.$$

$$y = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u.$$

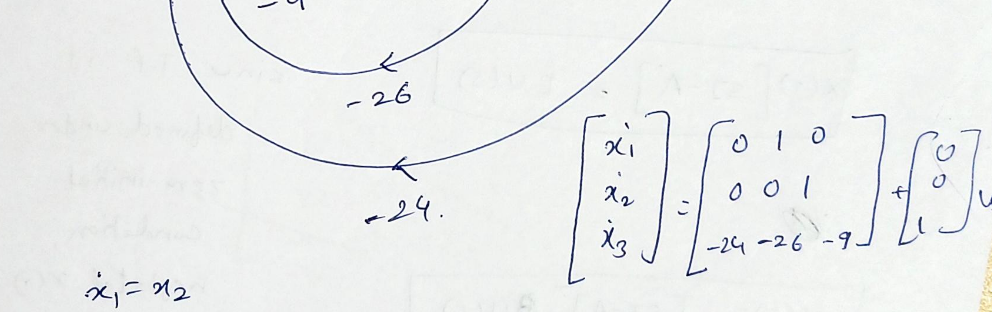
Direct decomposition

2) $\frac{C(s)}{R(s)} = \frac{24}{s^3+9s^2+26s+24}.$

1) $\frac{\frac{24}{s^3}}{1+\frac{9}{s}+\frac{26}{s^2}+\frac{24}{s^3}}$ $\quad \frac{X(s)}{X(s)} = \frac{Y(s)}{U(s)}.$

$U(s) = 1+\frac{9}{s}+\frac{26}{s^2}+\frac{24}{s^3} \; X(s) \Rightarrow X(s) = U(s) - \frac{9}{s}X(s) - \frac{24}{s^3}X(s)$

$Y(s) = \frac{24}{s^3} \cdot Y(s).$



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$\dot{x}_1 = x_2$

$\dot{x}_2 = x_3$

$\dot{x}_3 = -24x_1 - 26x_2 - 9x_3 + u.$ $\quad y = \begin{bmatrix} 24 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$

$y = 24x_1$

~~Controlling Problem~~

Transfer function from state model.

We have seen conversion from T-F to state space

- now from state space to T.F

Given :

$\dot{x}(t) = Ax(t) + Bu(t)$.

$y(t) = Cx(t) + Du(t)$.

Taking laplace transform,

$sX(s) - x(0) = AX(s) + BU(s)$.

$sX(s) - AX(s) = BU(s) + x(0)$.

$\boxed{X(s)[sI - A] = BU(s)}$

since T-F is defined under zero initial condition, neglected $x(0)$.

$\boxed{X(s) = [sI - A]^{-1} \cdot BU(s).}$

similarly.

$Y(s) = CX(s) + DU(s)$

$\boxed{Y(s) = C[sI - A]^{-1} \cdot B + D.}$

$$\therefore \frac{Y(s)}{U(s)} = \frac{C[sI-A]^{-1}B\,U(s) + D\,U(s)}{U(s)}$$

$$= C[sI-A]^{-1}B + D.$$

<u>example</u>:

Obtain T.F of a system.

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\Rightarrow sI - A = s \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}$$

$$sI - A = s\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}.$$

$$= \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{bmatrix}$$

$$[sI - A]^{-1} =$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

$$A^{-1} = \frac{1}{|A|} \left\| \begin{matrix} \cancel{a_{22}} & \cancel{a_{23}} \\ \cancel{a_{32}} & \cancel{a_{33}} \end{matrix} \right\| \quad \cancel{\text{adj } A}.$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$\text{adj } A = \begin{bmatrix} \text{cofactors} \end{bmatrix}$$

$$= \begin{bmatrix} \text{cofactor} \end{bmatrix}^T.$$

$$[SI + A]^{-1} = \frac{1}{|SI - A|} \begin{bmatrix} & C & \end{bmatrix}$$

$$C = \frac{\begin{bmatrix} s(s+3)+2 & (s+3)(0)+1 & 0-s \\ -1(s+3)+0 & s(s+3)+0 & 2s+2. \\ 1 & -s & s^2 \end{bmatrix}^T}{s(s^2+3s+2)+1(+1)+0.}$$

$$= \frac{\begin{bmatrix} s^2+3s+2 & s+3 & -s \\ -s-3. & s^2+3s & 2s+2 \\ 1 & -s & s^2 \end{bmatrix}^T}{s^3+3s^2+2s+1.}$$

$$= \frac{\begin{bmatrix} s^2+3s+2 & -s-3 & 1 \\ s+3 & s^2+3s & -s. \\ -s & 2s+2 & s^2 \end{bmatrix}}{s^3+3s^2+2s+1.}$$

2) $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$T(s) = C\left[sI - A\right]^{-1} B + D.$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \left\{ \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \right\}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Big/ |sI - A|$$

$$= \frac{\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+3 & -2 \\ 1 & s \end{bmatrix}^{T} \begin{bmatrix} 0 \\ 1 \end{bmatrix}}{s(s+3) + 1(2)}$$

$$= \frac{\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}}{s^2 + 3s + 2}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ s \end{bmatrix} \Big/ (s+1)(s+2) = \boxed{\frac{1}{s^2 + 3s + 2}}$$

(3) $$\dot{x} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$T(s) = \frac{s+3}{(s+1)(s+2)(s+3)}.$$

State transition matrix

- It is defined as transition of states from initial time $t=0$, to any time t or final time t_f when inputs are zero.

- sol^n of linear homogeneous eq^n
  - response due to initial vector $X(0)$
  - dependent on initial state and not i/p vector
  - K/as zero i/p response since i/p is zero.
  -

Solution of state equations (homogeneous eqn, i/p = 0)

$$\dot{x} = Ax(t)$$

$$\dot{x}(t) = Ax(t)$$

$$\frac{dx}{dt} = Ax$$

$$\frac{dx}{x} = A\,dt.$$

$$\int_{x(0)}^{x(t)} \frac{dx}{x} = \int_{t=0}^{t} A\,dt$$

$$(\log x)_{x(0)}^{x(t)} = At.$$

$$\log_e x(t) - \log_e x(0) = At$$

$$\log_e \left(\frac{x(t)}{x(0)}\right) = At$$

taking anti log.

$$\frac{x(t)}{x(0)} = e^{At}$$

$$\boxed{x(t) = e^{At}.\, x(0)}$$

$\phi(t) = e^{At}$ - state transition matrix

Properties of state transition matrix

$\phi(t) = e^{At}$.

1) $\phi(0) = e^{0} = I$ - identity matrix.

2) $e^{At} = (e^{-At})^{-1} = [\phi(-t)]^{-1}$.

3) $e^{A(t_1+t_2)} = e^{At_1} \cdot e^{At_2} = \phi(t_1)\phi(t_2)$.

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Laplace method to find state transition matrix

$$\phi(s) = [sI - A]^{-1} = \frac{\text{Adj}[sI - A]}{|sI - A|}$$

$\dot{x} = Ax$.

$sX(s) - X(0) = AX(s)$. $\Rightarrow sX(s) - AX(s) = X(0)$

$[sI - A]X(s) = X(0)$.

$$X(s) = [sI - A]^{-1} X(0)$$

Taking laplace inverse.

$$X(t) = \mathcal{L}^{-1}[sI - A]^{-1} X(0).$$

$$=$$

Comparing

$$X(t) = e^{At} X(0).$$

$$e^{At} = \mathcal{L}^{-1}[sI - A]^{-1}.$$

$$= \mathcal{L}^{-1} \phi(s).$$

$$e^{At} = \mathcal{L}^{-1} \phi(s).$$

$$= \mathcal{L}^{-1} \underbrace{(sI - A)^{-1}}_{\text{resolvant matrix.}}$$

$$\phi(s) = [sI - A]^{-1} = \frac{\text{Adj}\,[sI - A]}{|sI - A|}.$$

$$e^{At} = \mathcal{L}^{-1}[sI - A]^{-1}$$

$$= \mathcal{L}^{-1} \frac{\text{Adj}\,[sI - A]}{|sI - A|}.$$

obtain state transition matrix

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A = \begin{bmatrix} -3 & 1 \\ 0 & -1 \end{bmatrix}.$$

STM

$$e^{At} = \phi(t) = \mathcal{L}^{-1}[sI-A]^{-1}. \quad \text{— } (1)$$

$$sI-A = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -3 & 1 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} s-3 & -1 \\ 0 & s+1 \end{bmatrix}.$$

$$[sI-A]^{-1} = \frac{Adj[sI-A]}{|sI-A|}.$$

$$\frac{Adj[sI-A]}{|sI-A|} = \frac{\begin{bmatrix} s+1 & 0 \\ -1 & s+3 \end{bmatrix}^T}{(s-3)(s+1)}$$ → divide by each term.

$$= \begin{bmatrix} \frac{\cancel{s+1}}{\cancel{(s+1)}(s+3)} & \frac{1}{(s+1)(s+3)} \\ 0 & \frac{\cancel{s+3}}{(s+1)\cancel{(s+3)}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s+3} & \frac{1}{(s+1)(s+3)} \\ 0 & \frac{1}{s+1} \end{bmatrix}$$

$$[sI-A]^{-1} = \begin{bmatrix} \frac{1}{s+3} & \frac{0.5}{s+1} - \frac{0.5}{s+3} \\ 0 & \frac{1}{s+1} \end{bmatrix}$$

$$\mathcal{L}^{-1} = \begin{bmatrix} e^{-3t} & 0.5e^{-t} - 0.5e^{-3t} \\ 0 & e^{-t} \end{bmatrix}$$

Controllability and observability

Kalman's test.

$\dot{x} = A\, x(t) + Bu(t).$

$A = n \times n$ matrix

$u(t) = m \times 1.$

$x(t) = n \times 1.$

rank of Q_c is n. Q_c should be nonsingular.

$|Q_c| \neq 0.$

$Q_c = [B : AB : A^2B \ldots A^{n-1}B]$

↳ completely controllable.

Observable

$Q_o = [C^T : A^T C^T \ldots (A^T)^{n-1} C^T].$

Rank $Q_o = n.$

$$\dot{x} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t).$$

1

$$\downarrow$$

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$$Q_c = \cancel{\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \end{bmatrix}}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}.$$

$|Q| \neq 0.$

$\therefore$ rank $= 2$

Controllable.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

2 $y = [1 \quad 0]\, x.$

$$A = \begin{bmatrix} 3 & 0 \\ 2 & 4 \end{bmatrix}_{2\times 2} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = [1 \quad 0] \qquad n = 2$$

$$Q_c = \begin{bmatrix} 0 & 0 \\ 1 & 4 \end{bmatrix}$$

$|Q| = 0.$ rank $= 1.$ any cofactor non zero.

$\neq n.$

not completely controllable.

$$Q_0 = \begin{bmatrix} C^T & A^T C^T \end{bmatrix}$$

$$C^T \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad A^T = \begin{bmatrix} 3 & 2 \\ 0 & 4 \end{bmatrix}$$

$$Q_0 = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$$

$|Q_0| = 0$ rank $= 1$

$\neq n$

Unobservable