Slides for Fuzzy Sets, Ch. 2 of Neuro-Fuzzy and Soft Computing

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Fuzzy Sets: Outline

Introduction

Basic definitions and terminology

Set-theoretic operations

MF formulation and parameterization

- MFs of one and two dimensions
- Derivatives of parameterized MFs

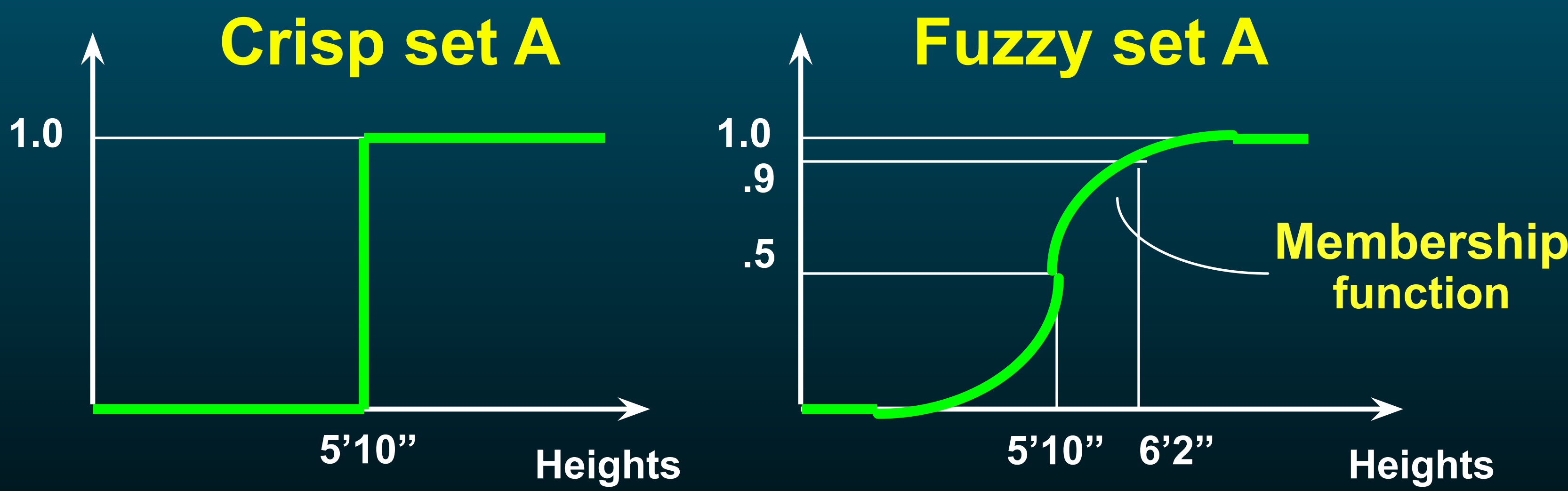
More on fuzzy union, intersection, and complement

- Fuzzy complement
- Fuzzy intersection and union
- Parameterized T-norm and T-conorm

Fuzzy Sets

Sets with fuzzy boundaries

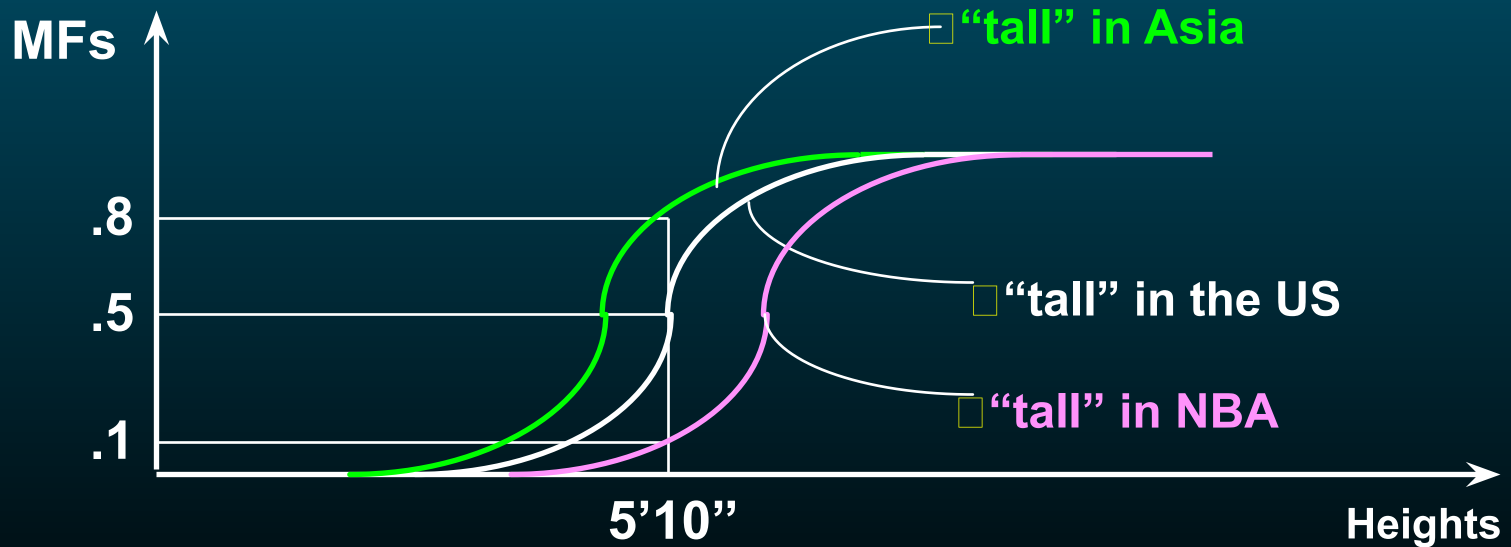
A = Set of tall people



Membership Functions (MFs)

Characteristics of MFs:

- Subjective measures
- Not probability functions



Fuzzy Sets

Formal definition:

A fuzzy set *A* in *X* is expressed as a set of ordered pairs:

$$A = \{(x, \underbrace{\mu_A(x)}_{\text{Membership function (MF)}}) \mid x \in X\}$$

Fuzzy set

Universe or universe of discourse

A fuzzy set is totally characterized by a membership function (MF).

Fuzzy Sets with Discrete Universes

Fuzzy set C = "desirable city to live in"

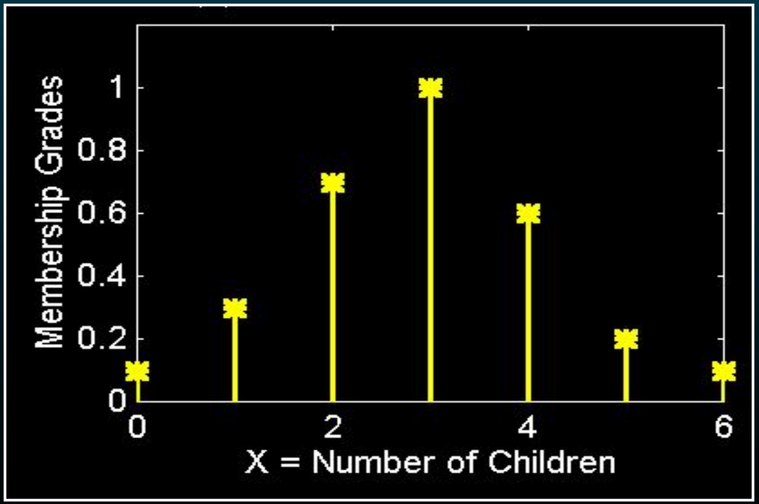
X = {SF, Boston, LA} (discrete and nonordered)

C = {(SF, 0.9), (Boston, 0.8), (LA, 0.6)}

Fuzzy set A = "sensible number of children"

X = {0, 1, 2, 3, 4, 5, 6} (discrete universe)

A = {(0, .1), (1, .3), (2, .7), (3, 1), (4, .6), (5, .2), (6, .1)}

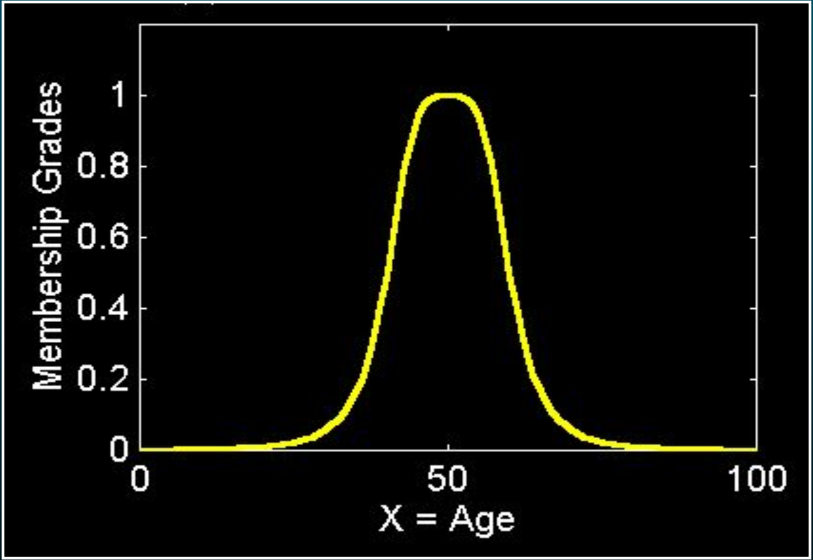


Fuzzy Sets with Cont. Universes

Fuzzy set B = "about 50 years old"

- X = Set of positive real numbers (continuous)
- B = {(x, μ_B(x)) | x in X}

$$\mu_B(x) = \frac{1}{1 + \left(\frac{x-50}{10}\right)^2}$$



Alternative Notation

A fuzzy set A can be alternatively denoted as follows:

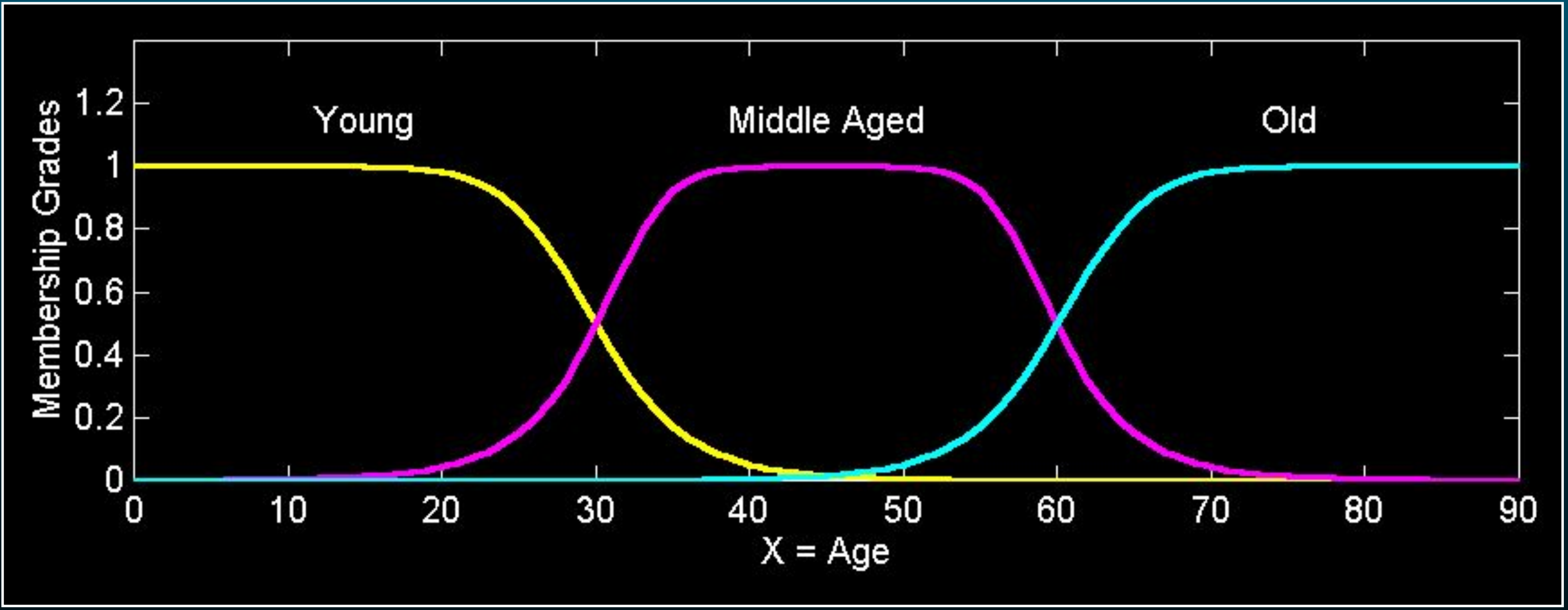
X is discrete $\rightarrow$ $$\boxed{A = \sum_{x_i \in X} \mu_A(x_i) / x_i}$$

X is continuous $\rightarrow$ $$\boxed{A = \int_X \mu_A(x) / x}$$

Note that Σ and integral signs stand for the union of membership grades; “/” stands for a marker and does not imply division.

Fuzzy Partition

Fuzzy partitions formed by the linguistic values "young", "middle aged", and "old":



lingmf.m

More Definitions

Support

Core

Normality

Crossover points

Fuzzy singleton

α-cut, strong α-cut

Convexity

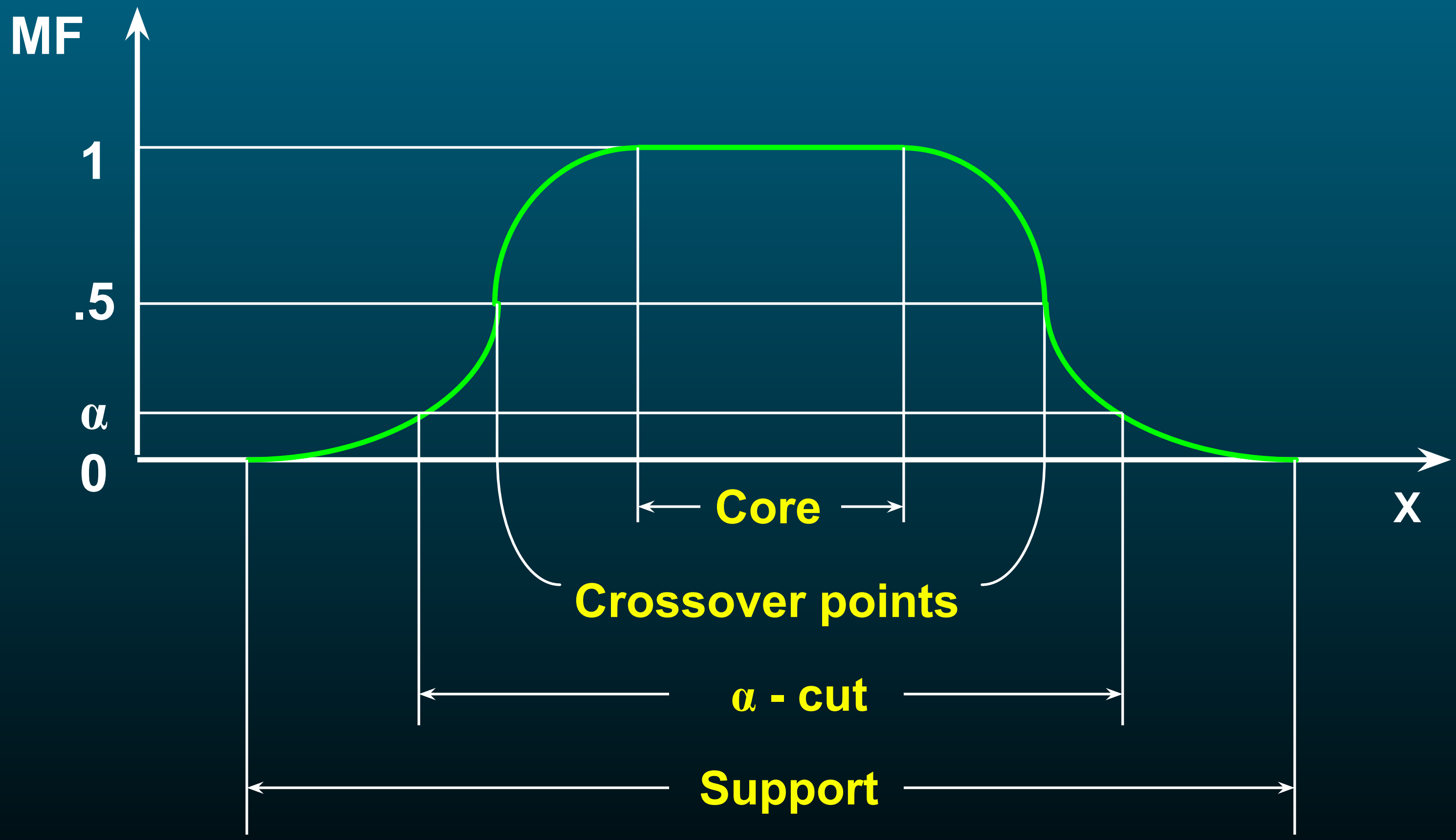
Fuzzy numbers

Bandwidth

Symmetricity

Open left or right, closed

MF Terminology

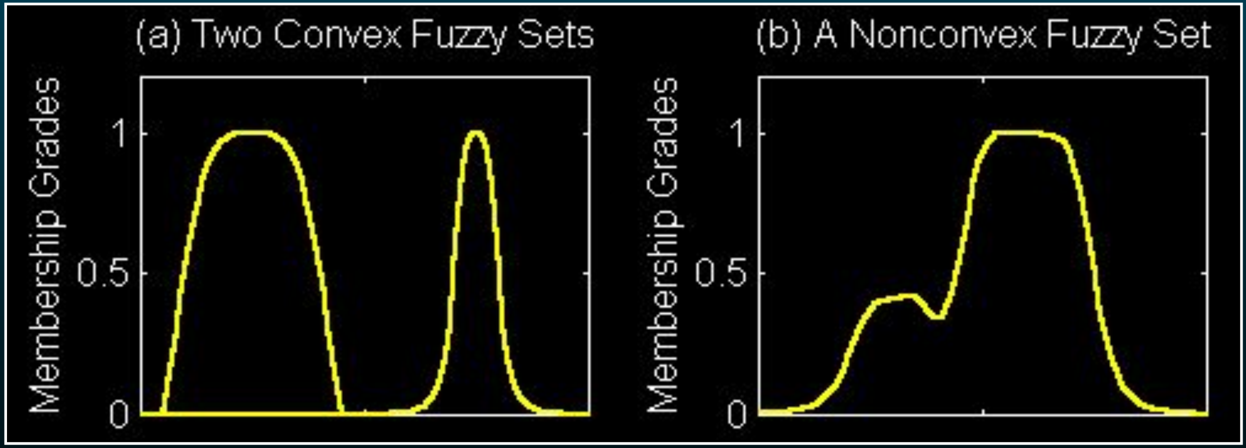


Convexity of Fuzzy Sets

A fuzzy set *A* is convex if for any λ in [0, 1],

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2))$$

Alternatively, *A* is convex is all its α-cuts are convex.



convexmf.m

Set-Theoretic Operations

Subset:

$$A \subseteq B \Leftrightarrow \mu_A \le \mu_B$$

Complement:

$$\overline{A} = X - A \Leftrightarrow \mu_{\overline{A}}(x) = 1 - \mu_A(x)$$

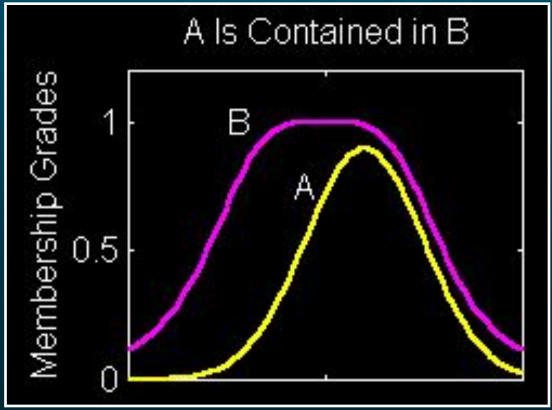
Union:

$$C = A \cup B \Leftrightarrow \mu_c(x) = \max(\mu_A(x), \mu_B(x)) = \mu_A(x) \vee \mu_B(x)$$

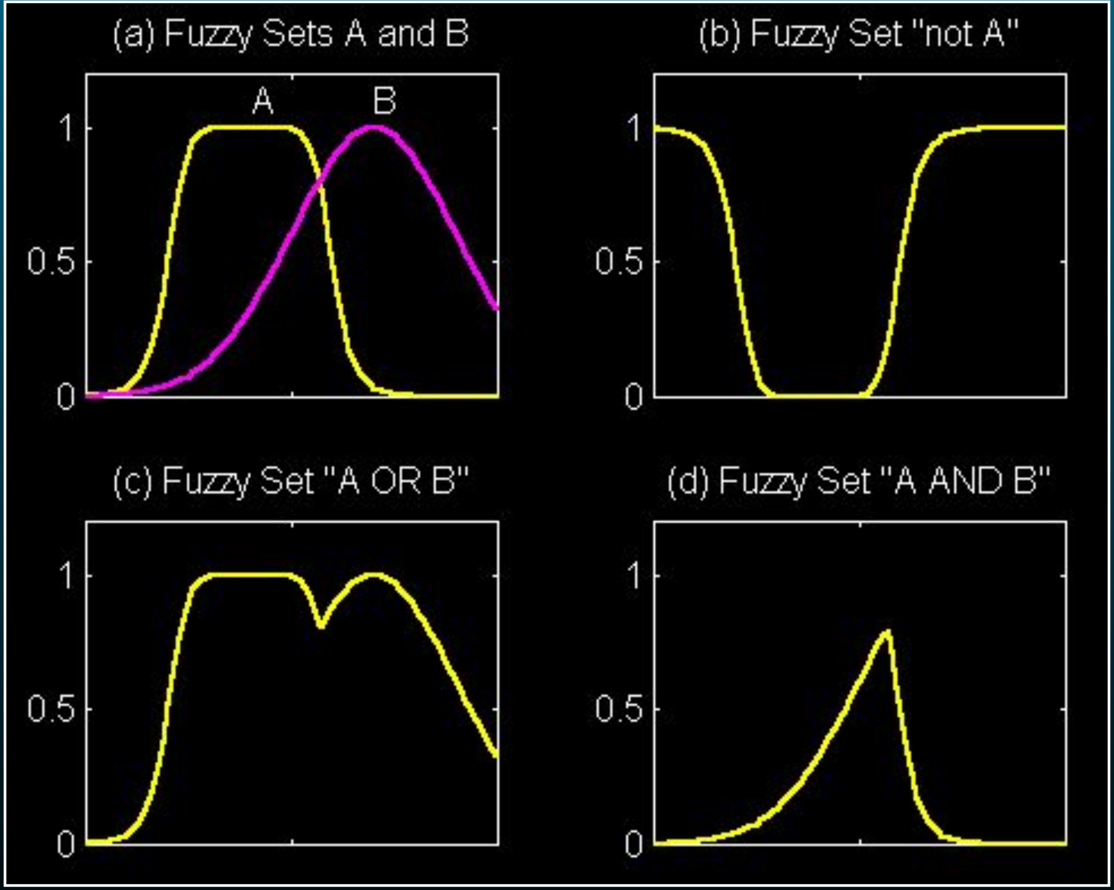
Intersection:

$$C = A \cap B \Leftrightarrow \mu_c(x) = \min(\mu_A(x), \mu_B(x)) = \mu_A(x) \wedge \mu_B(x)$$

Set-Theoretic Operations



subset.m



fuzsetop.m

MF Formulation

Triangular MF:

$$trimf(x;a,b,c) = \max\left(\min\left(\frac{x-a}{b-a}, \frac{c-x}{c-b}\right), 0\right)$$

Trapezoidal MF:

$$trapmf(x;a,b,c,d) = \max\left(\min\left(\frac{x-a}{b-a}, 1, \frac{d-x}{d-c}\right), 0\right)$$

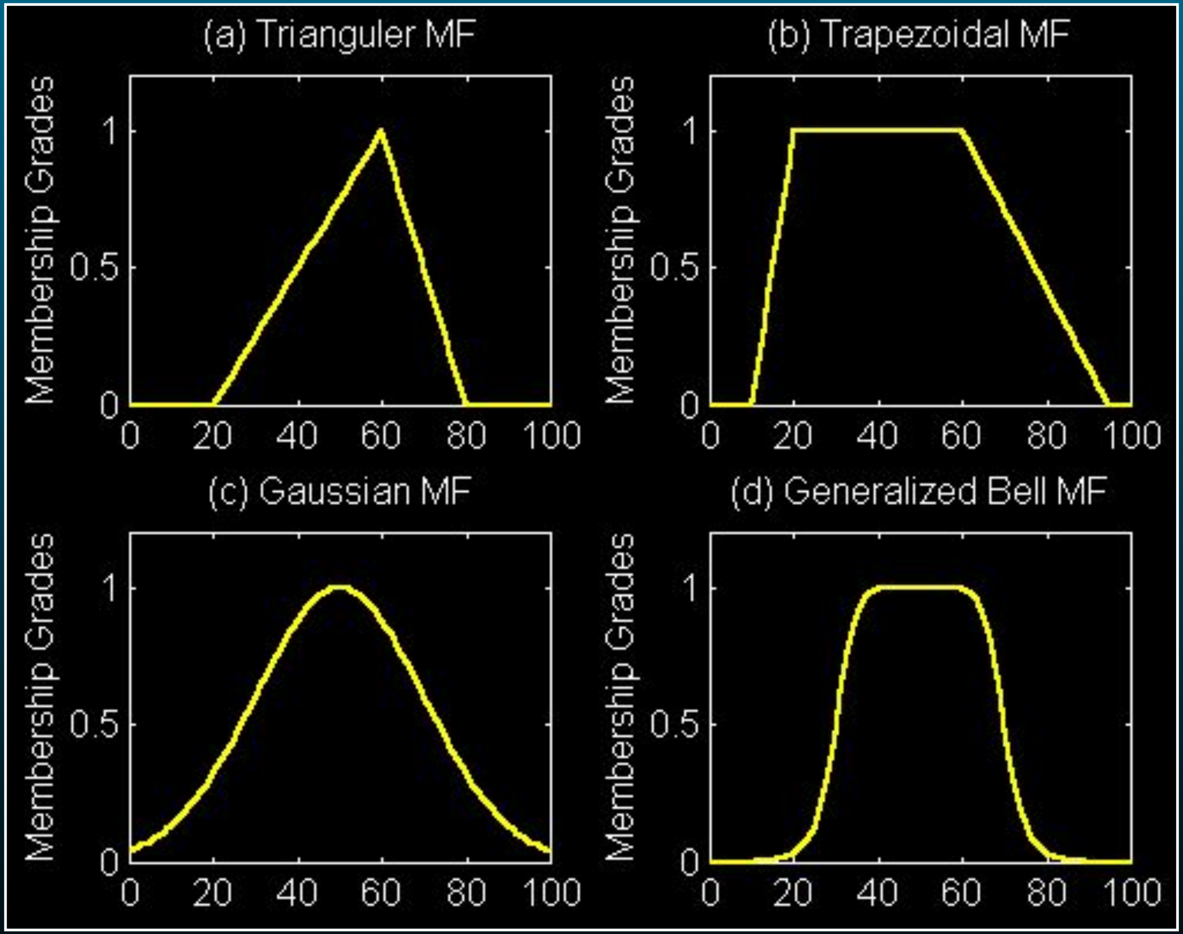
Gaussian MF:

$$gaussmf(x;a,b,c) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}$$

Generalized bell MF:

$$gbellmf(x;a,b,c) = \frac{1}{1+\left|\frac{x-c}{b}\right|^{2b}}$$

MF Formulation



disp_mf.m

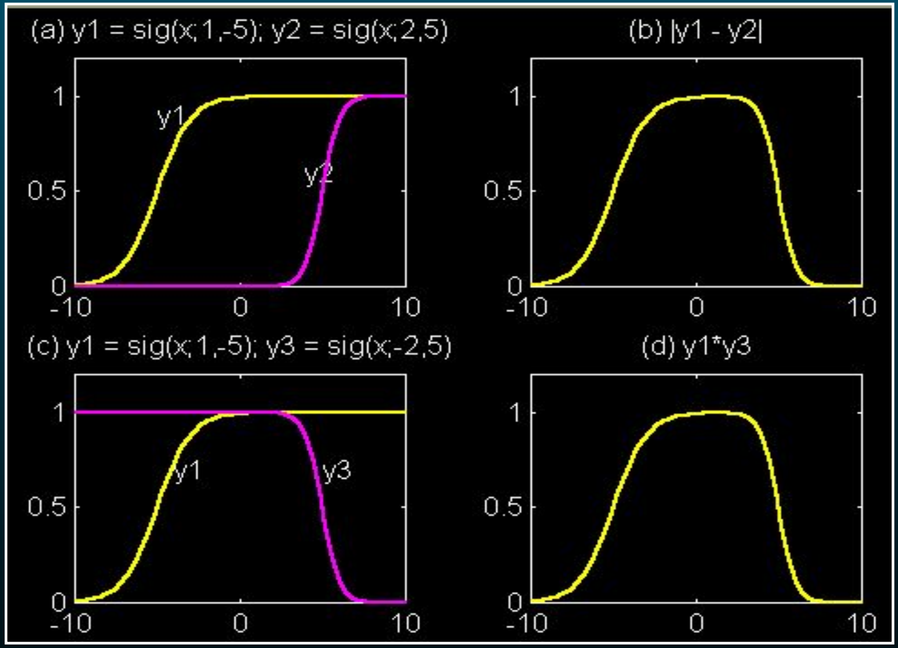
MF Formulation

Sigmoidal MF: $$sigmf(x;a,b,c)=\frac{1}{1+e^{-a(x-c)}}$$

Extensions:

Abs. difference of two sig. MF

Product of two sig. MF



disp_sig.m

MF Formulation

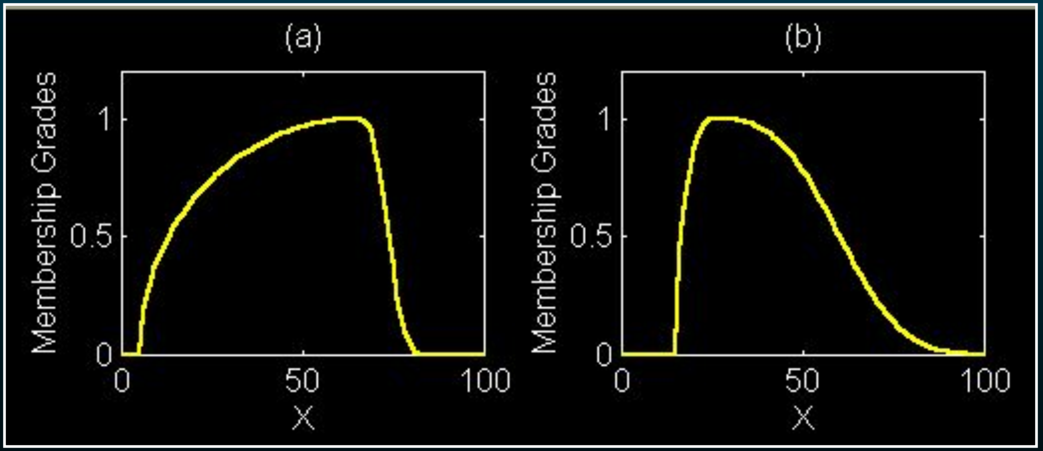
L-R MF:

$$LR(x;c,\alpha,\beta)=\begin{cases}F_L\left(\dfrac{c-x}{\alpha}\right), x<c\\ F_R\left(\dfrac{x-c}{\beta}\right), x\geq c\end{cases}$$

Example:

$$F_L(x)=\sqrt{\max(0,1-x^2)} \quad F_R(x)=\exp(-|x|^3)$$

c=65
a=60
b=10

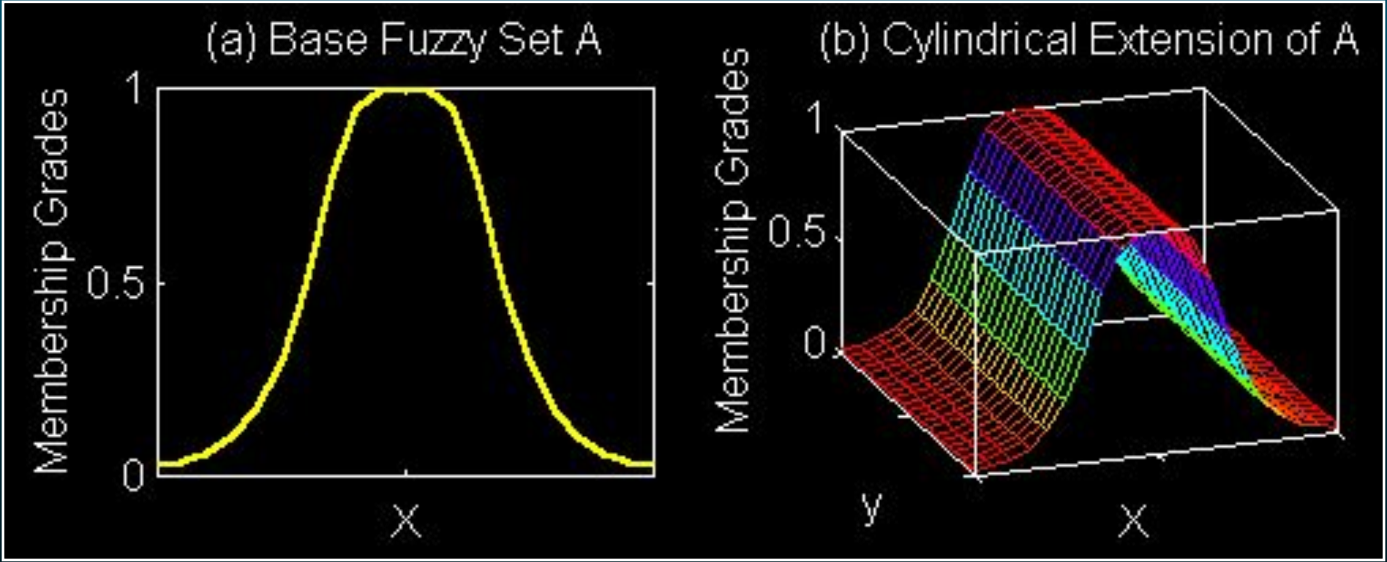


c=25
a=10
b=40

difflr.m

Cylindrical Extension

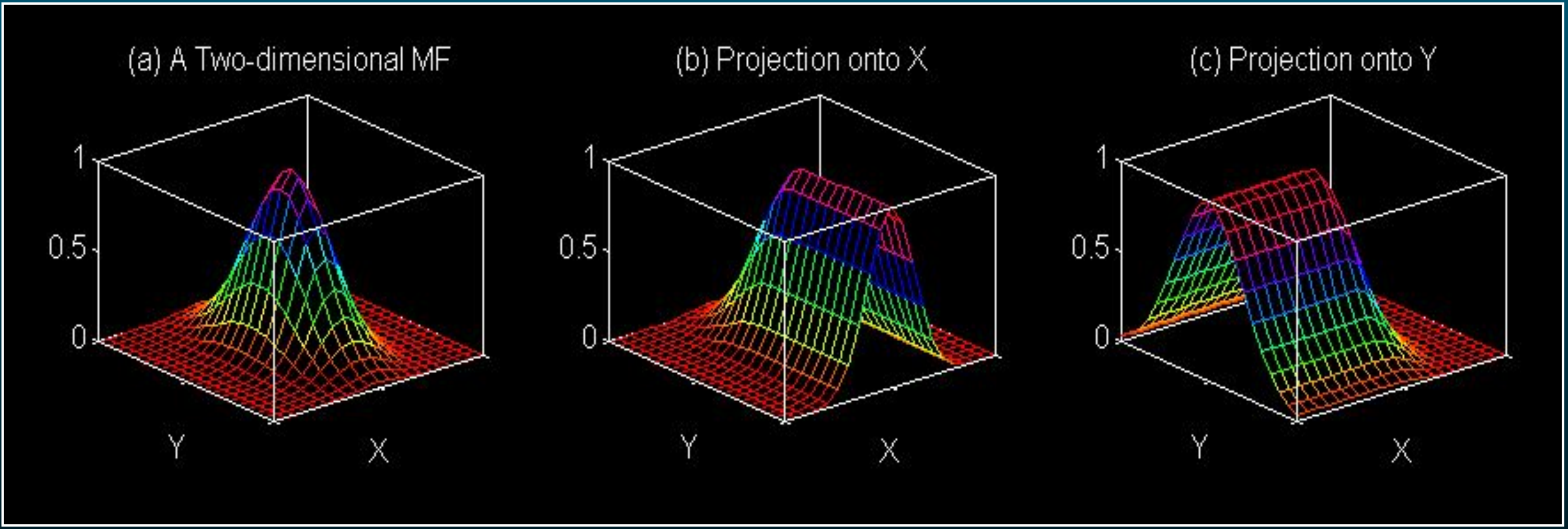
Base set A **Cylindrical Ext. of A**



cyl_ext.m

2D MF Projection

Two-dimensional MF | **Projection onto X** | **Projection onto Y**



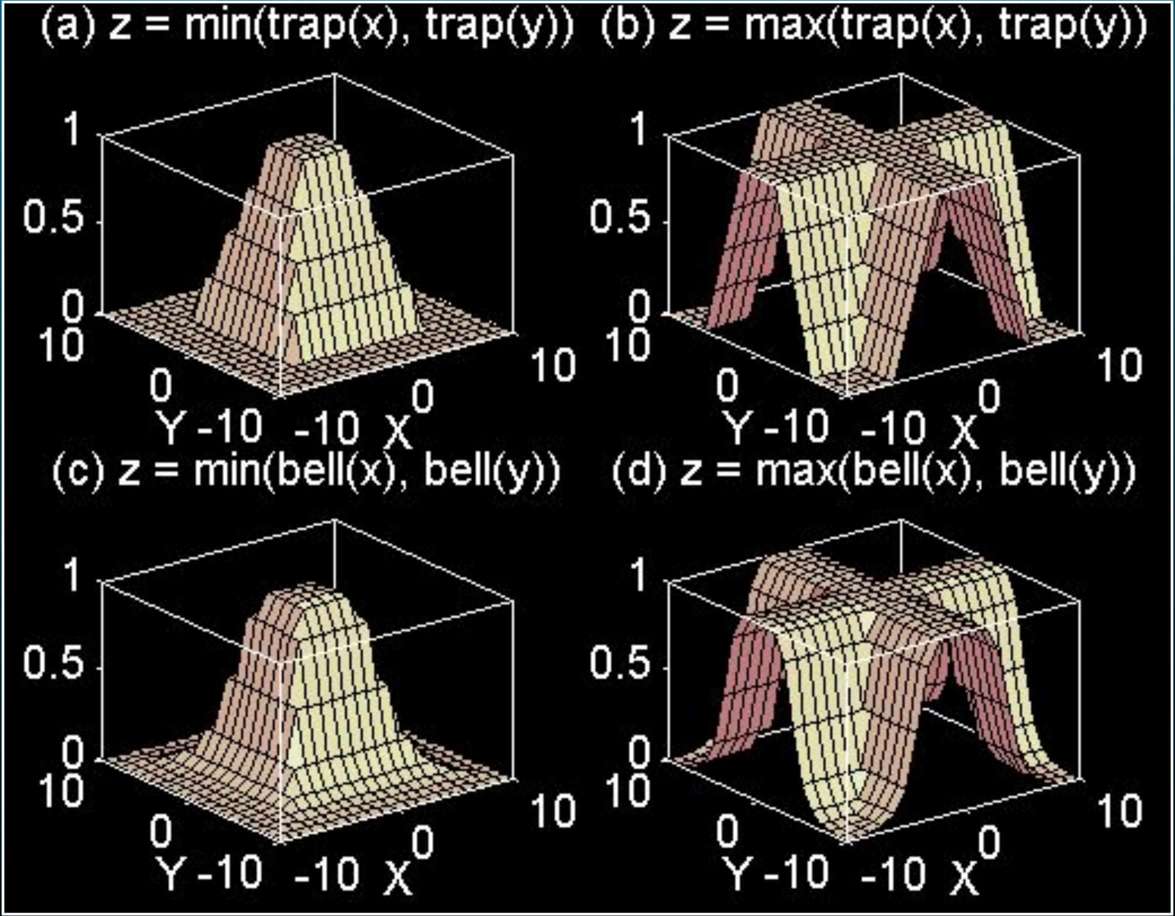
$$\mu_R(x, y)$$

$$\mu_A(x) = \max_y \mu_R(x, y)$$

$$\mu_B(y) = \max_x \mu_R(x, y)$$

project.m

2D MFs


(a) z = min(trap(x), trap(y))
(b) z = max(trap(x), trap(y))
(c) z = min(bell(x), bell(y))
(d) z = max(bell(x), bell(y))
1
0.5
0
10
0
Y -10
-10
X
0
10

2dmf.m

Fuzzy Complement

General requirements:

- Boundary: N(0)=1 and N(1) = 0
- Monotonicity: N(a) > N(b) if a < b
- Involution: N(N(a) = a

Two types of fuzzy complements:

- Sugeno's complement:

$$N_s(a) = \frac{1-a}{1+sa}$$

- Yager's complement:

$$N_w(a) = (1-a^w)^{1/w}$$

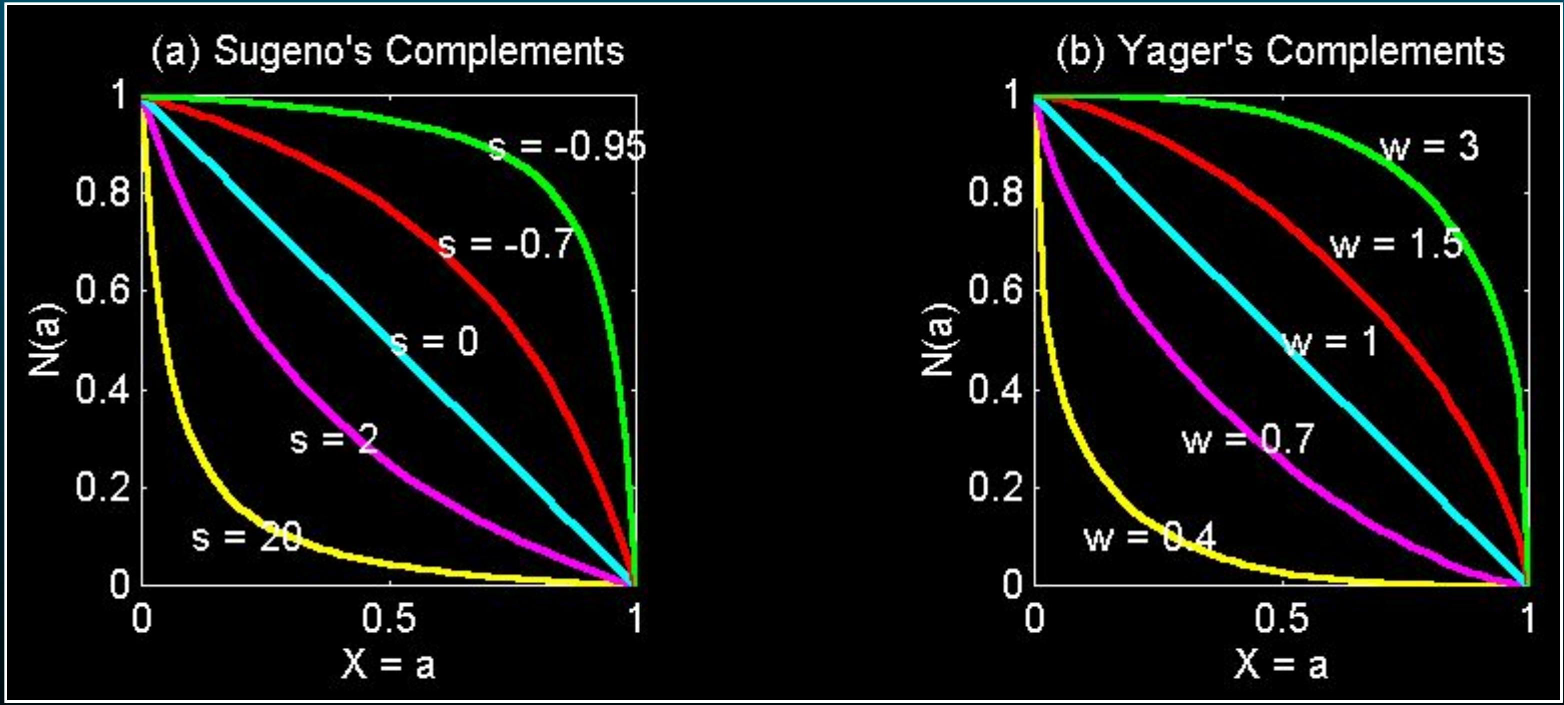
Fuzzy Complement

Sugeno's complement:

$$N_s(a) = \frac{1-a}{1+sa}$$

Yager's complement:

$$N_w(a) = (1-a^w)^{1/w}$$



negation.m

Fuzzy Intersection: T-norm

Basic requirements:

- Boundary: $T(0, 0) = 0$, $T(a, 1) = T(1, a) = a$
- Monotonicity: $T(a, b) < T(c, d)$ if $a < c$ and $b < d$
- Commutativity: $T(a, b) = T(b, a)$
- Associativity: $T(a, T(b, c)) = T(T(a, b), c)$

Four examples (page 37):

- Minimum: $T_m(a, b)$
- Algebraic product: $T_a(a, b)$
- Bounded product: $T_b(a, b)$
- Drastic product: $T_d(a, b)$

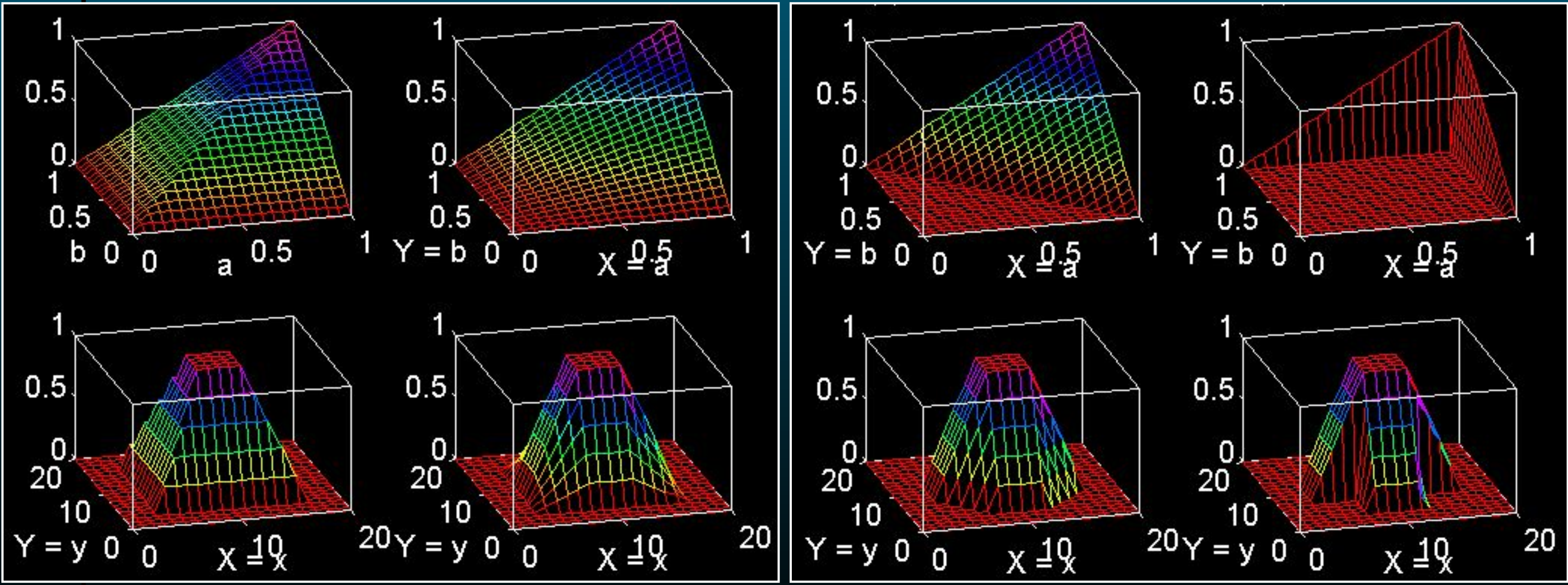
T-norm Operator

Minimum: $T_m(a, b)$

Algebraic product: $T_a(a, b)$

Bounded product: $T_b(a, b)$

Drastic product: $T_d(a, b)$



tnorm.m

Fuzzy Union: T-conorm or S-norm

Basic requirements:

- Boundary: $S(1, 1) = 1$, $S(a, 0) = S(0, a) = a$
- Monotonicity: $S(a, b) < S(c, d)$ if $a < c$ and $b < d$
- Commutativity: $S(a, b) = S(b, a)$
- Associativity: $S(a, S(b, c)) = S(S(a, b), c)$

Four examples (page 38):

- Maximum: $S_m(a, b)$
- Algebraic sum: $S_a(a, b)$
- Bounded sum: $S_b(a, b)$
- Drastic sum: $S_d(a, b)$

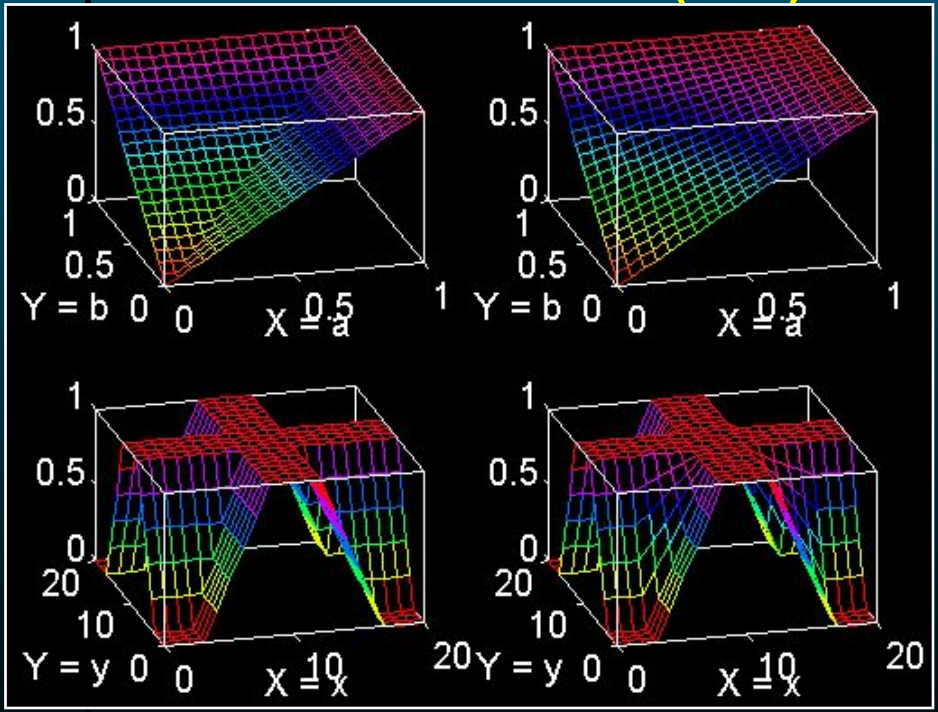
T-conorm or S-norm

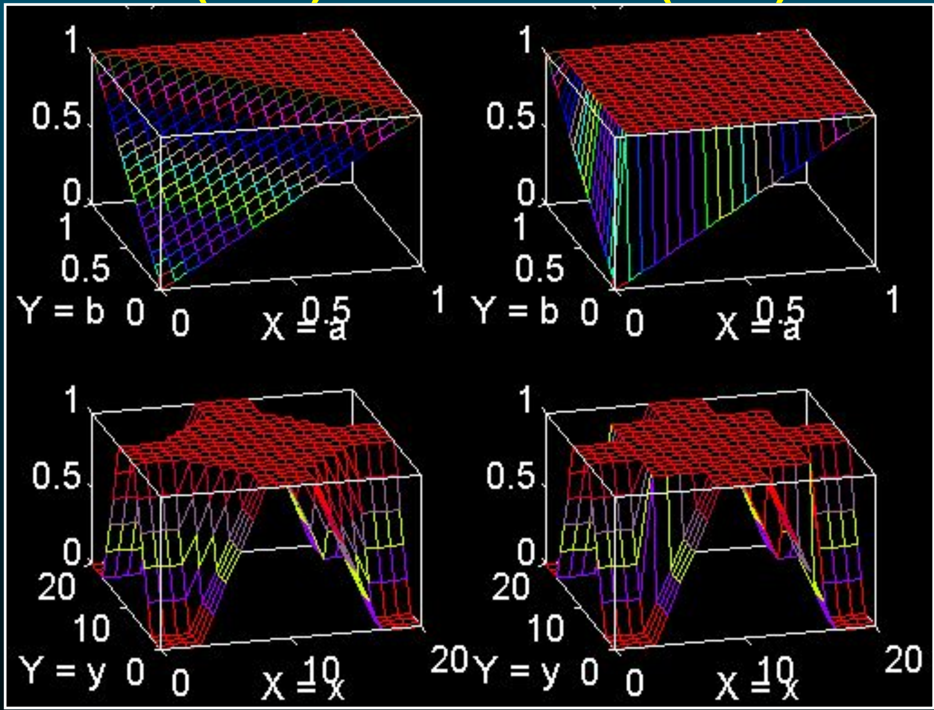
Maximum: $S_m(a, b)$

Algebraic sum: $S_a(a, b)$

Bounded sum: $S_b(a, b)$

Drastic sum: $S_d(a, b)$





tconorm.m

Generalized DeMorgan's Law

T-norms and T-conorms are duals which support the generalization of DeMorgan's law:

- $T(a, b) = N(S(N(a), N(b)))$
- $S(a, b) = N(T(N(a), N(b)))$

$T_m(a, b) \longleftrightarrow S_m(a, b)$

$T_a(a, b) \longleftrightarrow S_a(a, b)$

$T_b(a, b) \longleftrightarrow S_b(a, b)$

$T_d(a, b) \longleftrightarrow S_d(a, b)$

Parameterized T-norm and S-norm

Parameterized T-norms and dual T-conorms have been proposed by several researchers:

- Yager
- Schweizer and Sklar
- Dubois and Prade
- Hamacher
- Frank
- Sugeno
- Dombi