Fourier Transform

Fourier series is a powerful tool in treating various problems involving periodic function. In many practical. problems the function is non-periodic. For this ~~in~~ a suitable form of Fourier series where the fundamental period is infinite. Then the Fourier series became Fourier integral. is known as Fourier transform which transform the given non-periodic function $f(t)$ in time domain into a function $F(\lambda)$ in frequency domain.

Fourier transform is useful in solving boundary value problems such as Heat equation, wave eqn, theory of communication etc.

Complex exponential form of Fourier Series

If $f(x)$ is a periodic function of period $2L$ defined in $-L \le x \le L$ and satisfies Dirichlet's conditions {$f(x)$ must be integrable on $[-L, L]$, $f(x)$ must have finite numbers of maxima and minima in the interval, $f(x)$ must have finite no. of discontinuities in the interval and $f(x)$ must be bounded}

then $f(x)$ ~~can be~~ is represented as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{--- } \mathrm{I}$$

where $a_0 = \frac{1}{L}\int_{-L}^{L} f(u)\,du$, $a_n = \frac{1}{L}\int_{-L}^{L} f(u)\cos\left(\frac{n\pi u}{L}\right) du$

and $b_n = \frac{1}{L}\int_{-L}^{L} f(u)\sin\left(\frac{n\pi u}{L}\right) du$

Using $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$, $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = -\frac{i}{2}\left(e^{i\theta} - e^{-i\theta}\right)$

Substituting in I we get,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n}{2}\left[e^{in\pi x/L} + e^{-in\pi x/L}\right] + b_n\left[\frac{-i}{2}\right]\left[e^{in\pi x/L} - e^{-in\pi x/L}\right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\frac{a_n - ib_n}{2}\right] e^{in\pi x/L} + \left[\frac{a_n + ib_n}{2}\right] e^{-in\pi x/L}$$

If we define $c_0 = \frac{a_0}{2}$, $c_n = \frac{a_n - ib_n}{2}$, $c_{-n} = \frac{a_n + ib_n}{2}$

The above series can be written as

$$f(x) = c_0 + \sum_{n=1}^{\infty} c_n e^{in\pi x/L} + c_{-n} e^{-in\pi x/L}$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L} \quad \text{where } c_0 = \frac{a_0}{2}$$

$$C_n = \frac{1}{2}\left(a_n - i b_n\right)$$

$$= \frac{1}{2}\left[\frac{1}{L}\int_{-L}^{L} f(u)\cos\left(\frac{n\pi u}{L}\right) du - i\,\frac{1}{L}\int_{-L}^{L} f(u)\sin\left(\frac{n\pi u}{L}\right) du\right]$$

$$= \frac{1}{2L}\int_{-L}^{L} f(u)\left(\cos\left(\frac{n\pi u}{L}\right) - i\sin\left(\frac{n\pi u}{L}\right)\right) du$$

$$\therefore\ C_n = \frac{1}{2L}\int_{-L}^{L} e^{-in\pi u/L} f(u)\, du.$$

Simillarly, $C_{-n} = \frac{1}{2L}\int_{-L}^{L} f(u)\, e^{in\pi u/L}\, du.$

The index n is +ve, −ve or zero, C_n is correctly given by the single formula.

$$C_n = \frac{1}{2L}\int_{-L}^{L} f(u)\, e^{-in\pi u/L}\, du.$$

$\therefore$ The complex exponential form of Fourier series is given by,

$$f(x) = \sum_{n=-\infty}^{\infty} C_n\, e^{in\pi x/L} \quad \text{———} \circledast$$

where $C_n = \frac{1}{2L}\int_{-L}^{L} f(u)\, e^{-in\pi u/L}\, du.$

Fourier Integral Theorem

Consider Complex exponential form of a Fourier series $\circledast$

Substitute value of C_n in $f(x)$

$$\therefore\ f(x) = \sum_{n=-\infty}^{\infty}\left[\frac{1}{2L}\int_{-L}^{L} f(u)\, e^{-in\pi u/L}\, du\right] e^{in\pi x/L}$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{2L} \int_{-L}^{L} f(u)\, e^{-in\pi u/L + in\pi x/L}\, du$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{2L} \int_{-L}^{L} f(u)\, e^{-in\pi(u-x)/L}\, du$$

multiply the neumerator & Denominator by π

$$= \sum_{n=-\infty}^{\infty} \frac{1}{2\pi}\left(\frac{\pi}{L}\right) \int_{-L}^{L} f(u)\, e^{-in\pi(u-x)/L}\, du$$

Let us denote $\lambda = \frac{n\pi}{L}$

$\therefore$ difference in the frequency distribution i.e,

$$\Delta\lambda = \frac{(n+1)\pi}{L} - \frac{n\pi}{L}$$

$$= \frac{\pi}{L}$$

$$\therefore f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_{-L}^{L} f(u)\, e^{-in\pi(u-x)/L}\, du\, \Delta\lambda$$

Now if $L \to \infty$, $\Delta L = \frac{\pi}{L} \to 0$.

$$\therefore f(x) = \lim_{L\to\infty} \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \left\{ \int_{-L}^{L} f(u)\, e^{-in\pi(u-x)/L}\, du \right\} \Delta\lambda$$

$$= \lim_{\Delta\lambda\to 0} \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \left\{ \int_{-\infty}^{\infty} f(u)\, e^{-in\pi(u-x)/L}\, du \right\} \Delta\lambda$$

[by using defination of definite integral as limit of sum

$$\int_a^b y\, dx = \lim_{\delta x\to 0} \sum_{x=a}^{b} y\, \delta x \quad]$$

$$\boxed{f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(u)\, e^{-i\lambda(u-x)}\, du \right] d\lambda}$$

This is known as Fourier Integral Theorem

Where $-\infty < x < \infty$.

Equivalent form of Fourier Integral theorem

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\, e^{-i\lambda(u-x)}\, du\, d\lambda$$

Can be written as.

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\left[\cos\lambda(u-x) - i\sin\lambda(u-x)\right] du\, d\lambda$$

$$= \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\cos\lambda(u-x)\, du\, d\lambda - i\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\sin\lambda(u-x)\, du\, d\lambda\right\}$$

Since $\sin\lambda(u-x)$ is an odd function of λ

$$\Rightarrow \int_{-\infty}^{\infty} \sin\lambda(u-x)\, d\lambda = 0.$$

$$\therefore f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\cos\lambda(u-x)\, du\, d\lambda$$

$\cos\lambda(u-x)$ is even function of λ in $-\infty < \lambda < \infty$

$$\therefore \int_{-\infty}^{\infty} \cos\lambda(u-x)\, d\lambda = 2\int_{0}^{\infty} \cos\lambda(u-x)\, d\lambda.$$

$$\therefore f(x) = \frac{1}{\pi}\int_{\lambda=0}^{\infty}\int_{u=-\infty}^{\infty} f(u)\cos\lambda(u-x)\, du\, d\lambda.$$

expanding $\cos\lambda(u-x)$ we get,

$$= \frac{1}{\pi}\int_{+0}^{\infty}\left[\int_{-\infty}^{\infty} f(u)\left(\cos\lambda u\cos\lambda x + \sin\lambda u\sin\lambda x\right)\right] du\, d\lambda.$$

If a function defined in the interval $-\infty < x < \infty$ is either an even function or an odd function then

Case 1] $f(x)$ is an even function then.

$f(u)\sin\lambda u$ is odd function $\Rightarrow \int_{-\infty}^{\infty} f(u)\sin\lambda u \, du = 0$.

$$\therefore \boxed{f(x) = \frac{2}{\pi}\int_0^{\infty}\int_0^{\infty} f(u)\cos\lambda u \cos\lambda x \, du \, d\lambda} \quad \text{— (II)}$$

~~as f(u) sin λu~~ as $\int_{-\infty}^{\infty} f(u)\cos\lambda u \, du = 2\int_0^{\infty} f(u)\cos\lambda u \, du$.

This eq^n (II) is known as Fourier cosine integral of $f(x)$

Case 2] If $f(x)$ is an odd function then.

$f(u)\cos\lambda u$ is odd function $\Rightarrow \int_{-\infty}^{\infty} f(u)\cos\lambda u \, du = 0$.

& $f(u)\sin\lambda u$ is an even function $\Rightarrow \int_{-\infty}^{\infty} f(u)\sin\lambda u \, du = 2\int_0^{\infty} f(u)\sin\lambda u \, du$

$$\therefore \boxed{f(x) = \frac{2}{\pi}\int_0^{\infty}\int_0^{\infty} f(u)\sin\lambda u \sin\lambda x \, du \, d\lambda} \quad \text{— (III)}$$

This is known as Fourier Sine integral of $f(x)$.

Now we define Fourier Transform

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\, e^{-i\lambda(u-x)} \, du \, d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u} \, du\right\} e^{i\lambda x} \, d\lambda .$$

we write $F(\lambda) = \int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u} \, du$. is known as Fourier transform.

and Inverse Fourier transform.

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)\, e^{i\lambda x} \, d\lambda$$

Fourier Sine Transform (If the function is odd in $-\infty < x < \infty$)

The equation (III) can be rewritten as.

$$f(x) = \frac{2}{\pi} \int_0^\infty \left\{ \int_0^\infty f(u) \sin \lambda u \, du \right\} \sin \lambda x \, d\lambda$$

we write Fourier sine transform $F_s(\lambda)$ as

$$F_s(\lambda) = \int_0^\infty f(u) \sin \lambda u \, du.$$

and Inverse Fourier sine transform as,

$$f(x) = \frac{2}{\pi} \int_0^\infty F_s(\lambda) \sin \lambda x \, d\lambda.$$

Fourier Cosine Transform (If the function is even defined in $-\infty < x < \infty$)

Use equation (II)

we write Fourier cosine transform $F_c(\lambda)$ as.

$$F_c(\lambda) = \int_0^\infty f(u) \cos \lambda u \, du$$

and Inverse Fourier cosine transform as,

$$f(x) = \frac{2}{\pi} \int_0^\infty F_c(\lambda) \cos \lambda x \, d\lambda.$$

Important formulae

1] If the given function $f(x)$, $-\infty \leq x \leq \infty$ is neither even nor odd, we use general Fourier transform

$$F(\lambda) = \int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u}\, du$$

Inverse Fourier transform,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)\, e^{i\lambda x}\, d\lambda$$

2] If $f(x)$ is an even function defined in $0 \leq x \leq \infty$ then we use Fourier cosine integral.

Fourier cosine transform, $F_C(\lambda) = \int_0^{\infty} f(u) \cos\lambda u\, du$

Inverse Fourier cosine transform is,

$$f(x) = \frac{2}{\pi}\int_0^{\infty} F_C(\lambda) \cos\lambda u\, d\lambda .$$

3] If $f(x)$ is an ~~even~~ odd function $0 \leq x \leq \infty$ then use Fourier sine integral.

Fourier sine transform is $F_S(\lambda) = \int_0^{\infty} f(u) \sin\lambda u\, du$

Invere fourier sine transform $\Rightarrow$ $f(x) = \frac{2}{\pi}\int_0^{\infty} F_S(\lambda) \sin\lambda x\, d\lambda$

Important Results (formulae)

1] $\int e^{ax}\sin bx\,dx = \dfrac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$

2] $\int e^{ax}\cos bx\,dx = \dfrac{e^{ax}}{a^2+b^2}\left(a\cos bx + b\sin bx\right)$

3] $\int uv = uv_1 - u'v_2 + u''v_3 - u'''v_4 + u^{iv}v_5 - - -$

Here dash $\to$ derivative

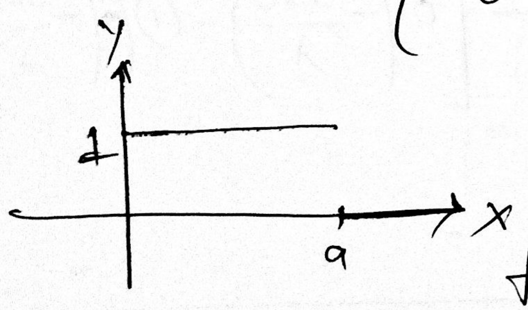
suffix $\to$ Integration

This rule is known as integration by parts generalised rule.

[Type – I]

Q-1] Find the sine transform of

$$f(x) = \begin{cases} 1, & 0 < x \le a \\ 0, & x \ge a \end{cases}$$



This is your pulse,

We use Fourier sine transform formula,

$$F_s(\lambda) = \int_0^{\infty} f(u)\sin\lambda u\,du$$

$$= \int_0^{a} 1 \sin\lambda u\,du = \int_0^{a}\sin\lambda u\,du$$

$$= \left[-\cos\lambda u\right]_0^a$$

$$= -(\cos\lambda a - \cos 0)$$

$$\boxed{F_s(\lambda) = 1 - \cos\lambda a}$$

Q-1)
(ii) If $f(x) = \begin{cases} x & , 0 \le x \le a \\ 0 & , x > a \end{cases}$

$\rightarrow$ Fourier sine transform,

$$F_s(\lambda) = \int_0^\infty f(u) \sin\lambda u \, du$$

$$= \int_0^a \underline{u} \, \underline{\underline{\sin\lambda u}} \, du + \int_a^\infty 0 \, du$$

$$= \left[u\left(-\frac{\cos\lambda u}{\lambda}\right) - (1)\left(-\frac{\sin\lambda u}{\lambda^2}\right)\right]_0^a$$

↳ Here use integration by parts generalised rule

$$= \left\{\left[a\left(-\frac{\cos\lambda a}{\lambda}\right) - (1)\left(-\frac{\sin\lambda a}{\lambda^2}\right)\right] - \left[0\left(-\frac{\cos 0}{\lambda}\right) - (1)\left(-\frac{\sin 0}{\lambda^2}\right)\right]\right.$$

$$F_s(\lambda) = -\frac{a\cos\lambda a}{\lambda} + \frac{\sin\lambda a}{\lambda^2}$$

HW:- For earlier both problems find Fourier cosine transform.

Q-2] Using Fourier Integral representation, show that

$$\frac{2}{\pi}\int_0^\infty \frac{(\lambda^2+2)\cos\lambda x}{(\lambda^4+4)}\,d\lambda = e^{-x}\cos x, \quad x>0$$

$\longrightarrow$ Here we have to show the Fourier cosine transform of $f(x) = e^{-x}\cos x$ is $\frac{\lambda^2+2}{\lambda^4+4}$

Fourier Cosine transform $F_c(\lambda) = \int_0^\infty f(u)\cos\lambda u\,du$

$$= \int_0^\infty e^{-u}\cos u\cos\lambda u\,du$$

$$= \int_0^\infty e^{-u}\left(\frac{1}{2}\left[\cos(1+\lambda)u + \cos(1-\lambda)u\right]\right)du$$

Use Result:- $\cos\alpha\cos\beta = \frac{1}{2}\left(\cos(\alpha+\beta)+\cos(\alpha-\beta)\right)$

$$= \frac{1}{2}\int_0^\infty e^{-u}\cos(1+\lambda)u\,du + \frac{1}{2}\int_0^\infty e^{-u}\cos(1-\lambda)u\,du.$$

$$= \frac{1}{2}\left[\frac{e^{-u}}{1+(1+\lambda)^2}\left(-\cos(1+\lambda)u+(1+\lambda)\sin\lambda u\right)\right]_0^\infty +$$

$$\frac{1}{2}\left[\frac{e^{-u}}{1+(1-\lambda)^2}\left(-\cos(1-\lambda)u+(1-\lambda)\sin\lambda u\right)\right]_0^\infty$$

$$= \frac{1}{2}\left[\frac{1}{1+(1+\lambda)^2}\right] + \frac{1}{2}\left[\frac{1}{1+(1-\lambda)^2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{1+1+\lambda^2+2\lambda} + \frac{1}{1+1+\lambda^2-2\lambda}\right] = \frac{1}{2}\left[\frac{1}{(\lambda^2+2)+2\lambda} + \frac{1}{(\lambda^2+2)-2\lambda}\right]$$

$$= \frac{1}{2}\left[\frac{\lambda^2+2-\cancel{2\lambda}+\lambda^2+2+\cancel{2\lambda}}{(\lambda^2+2)^2-(2\lambda)^2}\right] = \frac{1}{2}\left[\frac{2\lambda^2+4}{\lambda^4+4+\cancel{4\lambda^2}-\cancel{4\lambda^2}}\right]$$

$$F_c(\lambda) = \frac{\lambda^2+2}{\lambda^4+4}$$ Now we use Inverse Fourier cosine transform,

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(\lambda)\cos\lambda x\,d\lambda$$

$$\boxed{e^{-x}\cos x = \frac{2}{\pi}\int_0^\infty \frac{\lambda^2+2}{\lambda^4+4}\cos\lambda x\,d\lambda} \longrightarrow \text{Ans}$$

Find Fourier Sine transform of

$$f(x) = \begin{cases} 1 & , 0 \le x \le 1 \\ 0 & , x > 0 \end{cases}$$

Hence evaluate $\displaystyle\int_0^\infty \frac{\sin^3 x}{x}\, dx$.

$\rightarrow$ $$F_s(\lambda) = \int_0^\infty f(u) \sin\lambda u\, du = \int_0^1 \sin\lambda u\, du$$

$$= \left[\frac{-\cos\lambda u}{\lambda}\right]_0^1 = \frac{1-\cos\lambda}{\lambda}$$

$$= \frac{2\sin^2 \lambda/2}{\lambda}.$$

Using Inverse Fourier Sine transform,

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(\lambda) \sin\lambda x\, d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \frac{2\sin^2 \lambda/2}{\lambda} \sin\lambda x\, d\lambda$$

Put $\lambda/2 = t \Rightarrow d\lambda = 2\,dt$ $\begin{array}{c|c|c} \lambda & 0 & \infty \\ \hline t & 0 & \infty \end{array}$.

$$= \frac{4}{\pi}\int_0^\infty \frac{\sin^2 t}{2t} \sin(2tx)\,(2\,dt)$$

$$f(x) = \frac{4}{\pi}\int_0^\infty \frac{\sin^2 t \sin 2tx}{t}\, dt$$

Put $x = 1/2$

$$f(\tfrac{1}{2}) = \frac{4}{\pi}\int_0^\infty \frac{\sin^2 t \sin t}{t}\, dt$$

$$\Rightarrow \int_0^\infty \frac{\sin^3 t}{t}\, dt = \frac{\pi}{4} f(\tfrac{1}{2})$$

$$= \frac{\pi}{4} \quad \text{since } f(\tfrac{1}{2}) = 1$$

Q-3] Find the Fourier cosine integral representation for $f(x) = e^{-x} + e^{-2x}, \; x \geq 0$

$\rightarrow$ Apply Fourier cosine transform,

$$F_c(\lambda) = \int_0^\infty f(u) \cos \lambda u \, du.$$

$$= \int_0^\infty \left(e^{-u} + e^{-2u}\right) \cos \lambda u \, du$$

$$= \left[\frac{e^{-u}}{1+\lambda^2}\left(-\cos\lambda u + \lambda \sin\lambda u\right)\right]_0^\infty + \left[\frac{e^{-2u}}{4+\lambda^2}\left(-2\cos\lambda u + \lambda \sin\lambda u\right)\right]_0^\infty$$

$$= \left[\frac{1}{1+\lambda^2}\right] + \left[\frac{1}{4+\lambda^2}\right]$$

$$= \frac{(4+\lambda^2) + (1+\lambda^2)}{(1+\lambda^2)(4+\lambda^2)}$$

$$F_c(\lambda) = \frac{2\lambda^2+5}{(4+4\lambda^2+\lambda^2+\lambda^4)} = \frac{2\lambda^2+5}{\lambda^4+5\lambda^2+4}$$

Apply Inverse Fourier cosine transform,

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(\lambda) \cos\lambda x \, d\lambda$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \left(\frac{2\lambda^2+5}{\lambda^4+5\lambda^2+4}\right) \cos\lambda x \, d\lambda.$$

$\hookrightarrow$ This is known as Fourier cosine Integral representation.

Q-4] Find the Fourier sine transform of
$f(x) = \dfrac{e^{-ax} - e^{-bx}}{x}$, $x > 0$.

$\longrightarrow$ Fourier Sine transform

$$F_s(\lambda) = \int_0^\infty f(u) \sin \lambda u \, du .$$

$$= \int_0^\infty \left(\frac{e^{-au} - e^{-bu}}{u} \right) \sin \lambda u \, du .$$

[Note:- To eliminate the denominator function 'u', we use DUIS Rule, when we differentiate $\sin \lambda u$ with respect to 'λ' then we get $u \cos \lambda u$, so that 'u' gets cancelled out]

$$I(\lambda) = \int_0^\infty \left(\frac{e^{-au} - e^{-bu}}{u} \right) \sin \lambda u \, du \quad \text{———} \quad (I)$$

Use DUIS Rule,

$$\frac{dI}{d\lambda} = \int_0^\infty \frac{\partial}{\partial \lambda} \left[\left(\frac{e^{-au} - e^{-bu}}{u} \right) \sin \lambda u \right] du .$$

$$\frac{dI}{d\lambda} = \int_0^\infty \left(\frac{e^{-au} - e^{-bu}}{\cancel{u}} \right) (\cancel{u} \cos \lambda u) \, du$$

$$\frac{dI}{d\lambda} = \int_0^\infty e^{-au} \cos \lambda u \, du - \int_0^\infty e^{-bu} \cos \lambda u \, du .$$

$$\frac{dI}{d\lambda} = \left[\frac{e^{-au}}{a^2+\lambda^2} (-a \cos \lambda u + \lambda \sin \lambda u) \right]_0^\infty - \left[\frac{e^{-bu}}{b^2+\lambda^2} (-b \cos \lambda u + \lambda \sin \lambda u) \right]_0^\infty$$

$\dfrac{dI}{d\lambda} = \dfrac{a}{a^2+\lambda^2} - \dfrac{b}{b^2+\lambda^2}$ Now Integrating,

$dI = \displaystyle\int \frac{a}{a^2+\lambda^2} d\lambda - \int \frac{b}{b^2+\lambda^2} d\lambda \Rightarrow I(\lambda) = a \frac{1}{a} \tan^{-1}\left(\frac{\lambda}{a}\right) - \tan^{-1}\left(\frac{\lambda}{b}\right) + c$

$I(\lambda) = \tan^{-1}\left(\frac{\lambda}{a}\right) - \tan^{-1}\left(\frac{\lambda}{b}\right) + c$ Now to eliminate 'c', put $\lambda = 0$.

$\therefore c = I(0)$ Now put $\lambda = 0$ in (I) we get, $I(0) = 0$. therefore

$\boxed{I(\lambda) = \tan^{-1}(\lambda/a) - \tan^{-1}(\lambda/b)} \longrightarrow$ Ans

Q-5] By considering Fourier Sine integrals of e^{-mx} $(m>0)$, Prove that

$$\int_0^\infty \frac{\lambda \sin \lambda x}{\lambda^2+m^2}\, d\lambda = \frac{\pi}{2} e^{-mx}, \quad m>0,\ x>0.$$

$\longrightarrow$ Here $f(x) = e^{-mx}$, $m>0$.

Apply Fourier Sine transform,

$$F_s(\lambda) = \int_0^\infty f(u) \sin \lambda u \, du.$$

$$= \int_0^\infty e^{-mu} \sin \lambda u \, du$$

$$= \left[\frac{e^{-mu}}{m^2+\lambda^2} (-m \sin \lambda u - \lambda \cos \lambda u) \right]_0^\infty$$

$$= \left[(0) - \left(\frac{1}{m^2+\lambda^2} (-\lambda) \right) \right] = \frac{\lambda}{m^2+\lambda^2}.$$

Apply Inverse Fourier Sine transform,

$$f(x) = \frac{2}{\pi} \int_0^\infty F_s(\lambda) \sin \lambda x \, d\lambda$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{\lambda}{m^2+\lambda^2} \sin \lambda x \, d\lambda.$$

$$\therefore \int_0^\infty \frac{\lambda}{\lambda^2+m^2} \sin \lambda x \, d\lambda = \frac{\pi}{2} f(x)$$

$$\Rightarrow \boxed{\int_0^\infty \frac{\lambda}{\lambda^2+m^2} \sin \lambda x \, d\lambda = \frac{\pi}{2} e^{-mx}} \longrightarrow \text{proved}.$$

Show that the Fourier transform of $e^{-|x|}$ is $\frac{2}{1+\lambda^2}$

$\rightarrow$ Here $f(x) = e^{-|x|}$

$$F(\lambda) = \int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u}\, du = \int_{-\infty}^{\infty} e^{-|u|}\, e^{-i\lambda u}\, du.$$

$$= \int_{-\infty}^{\infty} e^{-|u|} (\cos \lambda u - i \sin \lambda u)\, du.$$

$$= \int_{-\infty}^{\infty} e^{-|u|} \cos \lambda u\, du - i \int_{-\infty}^{\infty} e^{-|u|} \sin \lambda u\, du$$

Since 2nd integrand is odd function $\Rightarrow \int_{-\infty}^{\infty} e^{-|u|} \sin \lambda u\, du = 0.$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-|u|} \cos \lambda u\, du$$

$$= 2 \int_{0}^{\infty} e^{-u} \cos \lambda u\, du$$

$$= 2 \left[\frac{e^{-u}}{1+\lambda^2} (-\cos \lambda u + \lambda \sin \lambda u) \right]_0^{\infty} = \frac{2}{1+\lambda^2}$$

Q-6] Show that the Fourier transform of $f(x) = e^{-x^2/2}$ is $e^{-\lambda^2/2}$

[Note:- The Fourier transform of $e^{-x^2/2}$ is itself]

$\longrightarrow$ $f(x) = e^{-x^2/2}$ is an even function.

$\therefore$ Apply Fourier cosine transform.

$$F_c(\lambda) = \int_0^\infty f(u)\cos\lambda u \, du = \int_0^\infty e^{-u^2/2}\cos\lambda u \, du.$$

(To add function we use DUIS Rule)

$$I(\lambda) = \int_0^\infty e^{-u^2/2}\cos\lambda u \, du.$$

$$\frac{dI}{d\lambda} = \int_0^\infty \frac{\partial}{\partial\lambda}\left(e^{-u^2/2}\cos\lambda u\right) du = +\int_0^\infty \underbrace{\left(-u\, e^{-u^2/2}\right)}_{v}\underbrace{\sin\lambda u}_{u} \, du.$$

Use Integration by parts,

$$\left[\int\left(-u\, e^{-u^2/2}\right) du \quad \text{Put } -u^2/2 = t \Rightarrow u^2 = -2t \Rightarrow 2u\,du = -2\,dt \Rightarrow u\,du = -dt. \quad \int\left(-u\, e^{-u^2/2}\right)dt = \int e^t\,dt = e^t = e^{-u^2/2}\right] +$$

$$\frac{dI}{d\lambda} = \left\{\left[(\sin\lambda u)\int\left(-u\, e^{-u^2/2}\right)du\right]_0^\infty - \int_0^\infty \frac{d}{du}(\sin\lambda u)\left[\int\left(-u\, e^{-u^2/2}\right)du\right]du\right\}$$

$$\frac{dI}{d\lambda} = \left\{\left[e^{-u^2/2}\sin\lambda u\right]_0^\infty - \int_0^\infty(\lambda\cos\lambda u)\, e^{-u^2/2}\,du\right\}$$

$$\frac{dI}{d\lambda} = \left\{[0-0] - \lambda\int_0^\infty e^{-u^2/2}\cos\lambda u \, du\right\}$$

$$\frac{dI}{d\lambda} = -\lambda I \Rightarrow \frac{dI}{I} = -\lambda\,d\lambda \Rightarrow \log I = -\lambda^2/2 + c.$$

$$\Rightarrow I = e^{-\lambda^2/2}e^c \Rightarrow I(\lambda) = A\,e^{-\lambda^2/2}$$

To eliminate A, put $\lambda = 0 \Rightarrow A = I(0)$

$I(0) = \int_0^\infty e^{-u^2/2}du$ Now Put $u^2/2 = t \Rightarrow u\,du = dt$ & $du = \frac{dt\, t^{-1/2}}{\sqrt{2}}$

$$\therefore I(0) = \int_0^\infty e^{-t}\frac{1}{\sqrt{2}}t^{-1/2}dt = \frac{1}{\sqrt{2}}\int_0^\infty e^{-t}t^{1/2-1}dt = \frac{1}{\sqrt{2}}\Gamma\tfrac{1}{2} = \frac{\sqrt{\pi}}{\sqrt{2}}$$

$\therefore A = \sqrt{\pi/2} \Rightarrow \boxed{I(\lambda) = \sqrt{\frac{\pi}{2}}\,e^{-\lambda^2/2}}$ Apply ~~Inverse~~ fourier cosine transf.

$\cancel{f_c(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c(\lambda)\cos\lambda x\,d\lambda = \sqrt{\frac{2}{\pi}}\sqrt{\frac{\pi}{2}}\int_0^\infty}$

$$F_c(\lambda) = \sqrt{\frac{2}{\pi}}\sqrt{\frac{\pi}{2}}\,e^{-\lambda^2/2}$$
$$= e^{-\lambda^2/2} \rightarrow \text{Required.}$$

Using inverse sine transform, find $f(x)$ if

$$F_s(\lambda) = \frac{1}{\lambda} e^{-a\lambda}$$

$\rightarrow$

$$f(x) = \frac{2}{\pi} \int_0^\infty F_s(\lambda) \sin \lambda x \, d\lambda$$

$$= \frac{2}{\pi} \int_0^\infty \frac{1}{\lambda} e^{-a\lambda} \sin \lambda x \, d\lambda$$

Let $I(x) = \int_0^\infty \frac{1}{\lambda} e^{-a\lambda} \sin \lambda x \, d\lambda$

$$\frac{dI}{dx} = \int_0^\infty \frac{\partial}{\partial x} \frac{e^{-a\lambda}}{\lambda} \sin \lambda x \, d\lambda$$

$$= \int_0^\infty \frac{e^{-a\lambda}}{\lambda} \lambda \cos \lambda x \, d\lambda$$

$$= \int_0^\infty e^{-a\lambda} \cos \lambda x \, d\lambda$$

$$= \left[\frac{e^{-a\lambda}}{a^2+x^2} (-a \cos \lambda x + x \sin \lambda x) \right]_0^\infty$$

$$= \frac{1}{a^2+x^2} (+a) = \frac{a}{a^2+x^2}$$

$$\frac{dI}{dx} = \frac{a}{a^2+x^2}$$

$$\int dI = a \int \frac{1}{a^2+x^2} dx + c$$

$$I(x) = a \cdot \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c.$$

$$I(x) = \tan^{-1}\left(\frac{x}{a}\right) + c.$$

Put $x = 0 \Rightarrow I(0) = c$

$I(0) = \int_0^\infty \frac{1}{\lambda} e^{-a\lambda} \sin 0 \, d\lambda = 0. \quad \therefore \boxed{c = 0}$

$\therefore f(x) = \frac{2}{\pi} I(x) = \frac{2}{\pi} \tan^{-1} \frac{x}{a}$

Solve the integral equation

$$\int_0^\infty f(x) \sin\lambda x \, dx = \begin{cases} 1-\lambda & , 0 \le \lambda \le 1 \\ 0 & , \lambda > 1 \end{cases}$$

$\longrightarrow$ since $\sin\lambda x$ term is present, the LHS is $F_s(\lambda)$ given

$$\therefore \quad F_s(\lambda) = \begin{cases} 1-\lambda & , 0 \le \lambda \le 1 \\ 0 & , \lambda > 1 \end{cases}$$

Using Inverse Fourier sine transform,

$$f(x) = \frac{2}{\pi} \int_0^\infty F_s(\lambda) \sin\lambda x \, d\lambda$$

$$= \frac{2}{\pi} \int_0^1 (1-\lambda) \sin\lambda x \, d\lambda$$

$$= \frac{2}{\pi} \left[(1-\lambda)\left(-\frac{\cos\lambda x}{x}\right) - (-1)\left(-\frac{\sin\lambda x}{x^2}\right) \right]_0^1$$

$$= \frac{2}{\pi} \left[\left(0 - \frac{\sin x}{x^2}\right) - \left(-\frac{1}{x}\right) \right]$$

$$= \frac{2}{\pi} \left(\frac{1}{x} - \frac{\sin x}{x^2} \right) = \frac{2}{\pi x^2} (x - \sin x)$$

Type I : Problems on Fourier integral representation :

Ex. 1 : Find the Fourier integral representation of the function

$$f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

and hence (a) evaluate $\int_0^{\infty} \frac{\sin\lambda \cos\lambda x}{\lambda} d\lambda$ (b) deduce the value of $\int_0^{\infty} \frac{\sin\lambda}{\lambda} d\lambda$.

(c) Find the value of above integrals at $|x| = 1$, which are points of discontinuity of f(x).

(Dec. 91, May 93)

Sol. : Here the given function f(x) is

$$f(x) = \begin{cases} 1, & -1 < x < 1 \\ 0, & |x| > 1 \end{cases} \qquad \text{... (i)}$$

This shows that f(–x) = f(x) i.e. f(x) is an even function in the interval $-\infty < x < \infty$ [See Fig. 5.2].

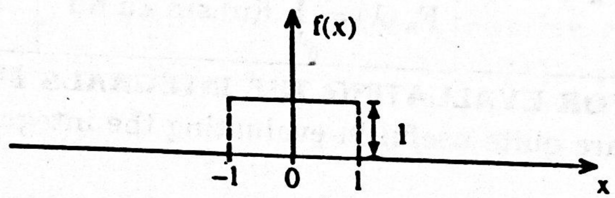


Fig. 5.2

Hence by result (10), the Fourier cosine transform for even function f(x) in the interval $-\infty < x < \infty$ is given by

$$F_c(\lambda) = \int_0^{\infty} f(u) \cos\lambda u \, du = \int_0^{1} \cos\lambda u \, du \qquad \text{[from (i)]}$$

$$= \left[\frac{\sin\lambda u}{\lambda}\right]_0^1 = \frac{\sin\lambda}{\lambda} \qquad \text{... (ii)}$$

By using inverse transform [result (11)], the Fourier integral representation is given by

$$f(x) = \frac{2}{\pi}\int_0^{\infty} F_c(\lambda)\cos\lambda x \, d\lambda = \frac{2}{\pi}\int_0^{\infty} \frac{\sin\lambda}{\lambda}\cos\lambda x \, d\lambda \qquad \text{[substituting } F_c(\lambda) \text{ from (ii)]}$$

$$= \frac{2}{\pi}\int_0^{\infty} \frac{\sin\lambda \cos\lambda x}{\lambda} dx \qquad \text{... (iii)}$$

which is the required Fourier integral representation.

The result (iii) can be expressed as

$$\int_0^\infty \frac{\sin \lambda \cos \lambda x}{\lambda}\, d\lambda = \frac{\pi}{2}\, f(x), \text{ where } f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

$$\therefore \quad \int_0^\infty \frac{\sin \lambda \cos \lambda x}{\lambda}\, d\lambda = \begin{cases} \frac{\pi}{2}, & |x| < 1 \\ 0, & |x| > 1 \end{cases} \qquad \text{... (iv)}$$

Now if we put $x = 0$ (which lies in $-1 < x < 1$) in (iv), we have

$$\int_0^\infty \frac{\sin \lambda}{\lambda}\, d\lambda = \frac{\pi}{2} \qquad \text{... (v)}$$

At $|x| = 1$, which are points of discontinuity, the value of the Fourierier integral equals to average of left-hand and right-hand limit of $f(x)$ at $|x| = 1$.

Thus,

$$\left[\int_0^\infty \frac{\sin \lambda \cos \lambda x}{\lambda}\, dx\right]_{|x|=1} = \frac{\frac{\pi}{2} + 0}{2} = \frac{\pi}{4}.$$

Ex. 9 : Find the Fourier cosine transform of the function

$$f(x) = \begin{cases} \cos x, & 0 < x < a \\ 0, & x > a \end{cases}$$

Sol. : Using result (10), cosine transform of f(x) is

$$F_c(\lambda) = \int_0^\infty f(u) \cos \lambda u \, du = \int_0^a \cos u \cos \lambda u \, du + \int_a^\infty (0) \cos \lambda u \, du$$

$$= \frac{1}{2} \int_0^a [\cos(\lambda + 1) u + \cos(\lambda - 1) u] \, du$$

$$= \frac{1}{2} \left[\frac{\sin(\lambda + 1) u}{\lambda + 1} + \frac{\sin(\lambda - 1) u}{\lambda - 1} \right]_0^a$$

$$= \frac{1}{2} \left[\frac{\sin(\lambda + 1) a}{\lambda + 1} + \frac{\sin(\lambda - 1) a}{\lambda - 1} \right]$$

Q-7] Find $f(x)$ if $F_s(\lambda) = e^{-\pi\lambda}$

$\longrightarrow$ $f(x) = \frac{2}{\pi} \int_0^\infty F_s(\lambda) \sin\lambda x \, d\lambda$

$$f(x) = \frac{2}{\pi} \int_0^\infty e^{-\pi\lambda} \sin\lambda x \, d\lambda$$

$$f(x) = \frac{2}{\pi} \left[\frac{e^{-\pi\lambda}}{x^2+\pi^2} \left(-\pi \sin\lambda x - x \cos\lambda x\right) \right]_0^\infty$$

$$f(x) = \frac{2}{\pi} \left[(0) - \left(\frac{1}{x^2+\pi^2} (-x) \right) \right]$$

$$f(x) = \frac{2x}{\pi(x^2+\pi^2)}$$

Question Bank Fourier Transform

Q-1] Find Fourier transform on

(a) $f(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$ (b) $f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & x < 0 \end{cases}$

(c) $f(x) = \begin{cases} 2+x, & x > 0 \\ 0, & x < 0 \end{cases}$ (d) $f(x) = \begin{cases} 1-x^2, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$

(e) $f(x) = \begin{cases} x^2, & x > 0 \\ 0, & x < 0 \end{cases}$ (f) $f(x) = \begin{cases} x - x^2, & x > 0 \\ 0, & x < 0 \end{cases}$

(g) $f(x) = \begin{cases} \sin x, & x > 0 \\ 0, & x < 0 \end{cases}$ (h) $f(x) = \begin{cases} 2, & |x| \le 1 \\ 0, & |x| \geq 1 \end{cases}$

Q-2] Find Fourier sine transform of following functions. Also represent it in Fourier integral representation

(a) $f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases}$ (b) $f(x) = \begin{cases} x, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$

(c) $f(x) = \begin{cases} x^2, & 0 \le 1 \le 1 \\ 0, & x > 1 \end{cases}$ (d) $f(x) = \begin{cases} 5, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$

(e) e^{-mx}, $m > 0$ (f) $f(x) = 2e^{-5x} + 5e^{-2x}$, $x > 0$

(g) $f(x) = \begin{cases} \pi/2, & 0 < x < \pi \\ 0, & x > 0 \end{cases}$ (h) $f(x) = \begin{cases} 1-x^2, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$

(i) $f(x) = e^{-|x|}$, $-\infty < x < \infty$ (j) $f(x) = \begin{cases} \sin x, & 0 < x < \pi \\ 0, & x > \pi \end{cases}$

Q-3] Find Fourier cosine transform of the following functions. Also represent it in Fourier integral representation

(a) $f(x) = \begin{cases} \cos x, & 0 < x < \pi \\ 0, & x > \pi \end{cases}$ (b) $f(x) = e^{-mx}$, $m > 0$, $x > 0$

(c) $f(x) = e^{-5x} + e^{-2x}$, $x > 0$ (d) $f(x) = \begin{cases} x^2, & 0 < x < a \\ 0, & x > a \end{cases}$

(e) $f(x) = \begin{cases} 2+x, & 0<x<2 \\ 0, & x>2 \end{cases}$

(f) $f(x) = \begin{cases} x, & 0<x<a \\ 0, & x>a \end{cases}$

(g) $f(x) = \begin{cases} 1, & 0<x\le 1 \\ 2, & 1<x\le 2 \\ 0, & x>0 \end{cases}$

(h) $f(x) = e^{-x}\cos x, \quad x>0$

(i) $f(x) = \begin{cases} x, & 0<x<\frac{1}{2} \\ 1-x, & \frac{1}{2}<x<1 \\ 0, & x>1 \end{cases}$

(k) $f(x) = \begin{cases} 0, & 0\le x<a \\ x, & a\le x\le b \\ 0, & x>b \end{cases}$

Q-4] Using inverse Fourier cosine transform, find $f(x)$ if (1) $F_s(\lambda) = \lambda/1+\lambda^2$

Q-5] find $f(x)$ if $F_c(\lambda) = \begin{cases} \sqrt{2/\pi}\left(a-\lambda/2\right), & \lambda \not> 2a \\ 0, & \lambda>2a \end{cases}$

Q-6] Find Fourier sine transform of $e^{-|x|}$, Hence evaluate

$$\int_0^\infty \frac{x \sin mx}{1+x^2}\,dx.$$

Q-7] Solve the integral equation

(1) $\int_0^\infty f(x)\cos\lambda x\,dx = e^{-\lambda}, \quad \lambda>0$

(2) $\int_0^\infty f(x)\sin\lambda x\,dx = \begin{cases} 1, & 0\le\lambda\le 1 \\ 0, & \lambda>1 \end{cases}$

(3) $\int_0^\infty f(x)\sin\lambda x = \begin{cases} 1-\lambda, & 0\le\lambda\le 1 \\ 0, & \lambda>1 \end{cases}$

(4) $\int_0^\infty f(x)\sin\lambda x\,dx = \begin{cases} 1, & 0\le\lambda\le 1 \\ 2, & 1\le\lambda<2 \\ 0, & \lambda\ge 2 \end{cases}$

Q-8] Find inverse Fourier cosine transform of $F_c(\lambda) = \dfrac{\sin a\lambda}{\lambda}$

Q-9] Find inverse sine transform of $F_s(\lambda) = \dfrac{e^{-a\lambda}}{\lambda}$

Q-10] Find Fourier sine transform of $\dfrac{e^{-ax}}{x}$ & hence evaluate $\int_0^\infty \tan^{-1}\left(\frac{x}{a}\right)\sin x\,dx$

Q-2] Solve the integral equation

1) $\int_0^\infty f(x)\cos\lambda x\,dx = e^{-\lambda}, \quad \lambda > 0$

2] $\int_0^\infty f(x)\sin\lambda x\,dx = \begin{cases} 1-\lambda, & 0 \le \lambda \le 1 \\ 0, & \lambda > 1 \end{cases}$

3] $\int_0^\infty f(x)\cos\lambda x\,dx = \begin{cases} \sqrt{\frac{2}{\pi}}\left(a - \frac{\lambda}{2}\right), & \lambda \le 2a \\ 0, & \lambda > 2a. \end{cases}$

4] $\int_0^\infty f(x)\cos\lambda x\,dx = \begin{cases} 1, & 0 < \lambda < 1 \\ 2, & 1 < \lambda < 2 \\ 0, & \lambda > 2 \end{cases}$

5] $\int_0^\infty f(x)\sin\lambda x\,dx = \frac{\lambda}{\lambda^2 + k^2}, \quad \lambda > 0$

Q-8] Find $f(x)$ if $F_c(\lambda) = \begin{cases} \frac{1}{2\pi}\left(a - \lambda/2\right) & , \lambda \le 2a \\ 0 & , \lambda > 2a. \end{cases}$

Ans → $\dfrac{2 \sin^2 a x}{\pi^2 x^2}$

Q-9] Find Fourier transform of

$$f(x) = \begin{cases} \sqrt{\dfrac{2\pi}{2a}} & , |x| \le a \\ 0 & , |x| > a. \end{cases}$$

Ans → $\dfrac{\sin \lambda a}{\lambda a}$

Q-3] Using Fourier integral representation, show that

1] $\frac{2}{\pi}\int_0^\infty \frac{k\cos\lambda x}{\lambda^2+k^2}\,d\lambda = e^{-kx}, \quad x>0$

2] $\frac{2}{\pi}\int_0^\infty \frac{\lambda}{\lambda^2+k^2}\sin\lambda x\,d\lambda = e^{-kx}, \quad x>0$

3] $\frac{2}{\pi}\int_0^\infty \left(\frac{\lambda^2+2}{\lambda^4+4}\right)\cos\lambda x\,d\lambda = e^{-x}\cos x, \quad x>0$

4] $\frac{2}{\pi}\int_0^\infty \left(\frac{1-\cos k\lambda}{\lambda}\right)\sin\lambda x\,d\lambda = \begin{cases} 1, & 0<x<k \\ \frac{1}{2}, & x=k \\ 0, & x>k \end{cases}$

5] $\frac{2}{\pi}\int_0^\infty \frac{\sin\lambda\pi\,\sin\lambda x}{1-\lambda^2}\,d\lambda = \begin{cases} \sin x, & |x|\le\pi \\ 0, & |x|>\pi \end{cases}$

6] $\frac{12}{\pi}\int_0^\infty \frac{\lambda\sin\lambda x}{(\lambda^2+4)(\lambda^2+16)}\,d\lambda = e^{-3x}\sinh x.$

$\left(\text{Here Use } \sinh x = \frac{e^x - e^{-x}}{2}\right)$

7] $\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda = \begin{cases} \pi/2, & 0\le x\le 1 \\ \pi/4, & x=1 \\ 0, & x>1 \end{cases}$