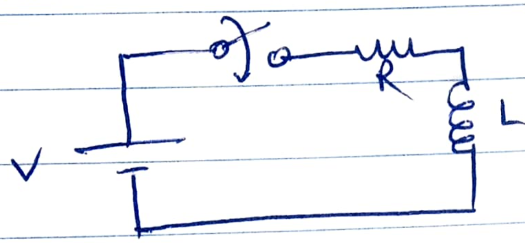
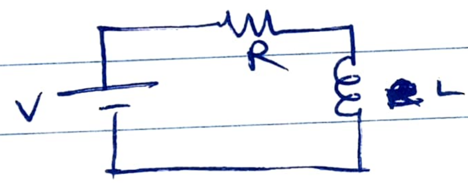
R-L circuit

Consider a series R-L circuit as shown,



The switch is closed at $t=0$.
The inductor in ckt is initially un-energized

Ckt at $t>0$ will be,



KVL eqn at $t>0$ will be,

$$V - Ri - L\frac{di}{dt} = 0$$

This is a linear differential eqn of first order.
It can be solved if variables can be separated.

$$(V-Ri)dt = L\,di$$

$$\frac{L\,di}{V-Ri} = dt$$

Integrating both sides,

$$-\frac{L}{R}\ln(V-Ri) = t+K \quad \text{--- (1)}$$

Where ~~ln~~ ln is log at base e & k is arbitrary constant. Value of k can be evaluated from the initial condition.

In the given ckt. the switch is closed at $t=0$ ie, before closing the switch, the current in the inductor is zero. Since the inductor doesnot allow ~~sudde~~ sudden change in current, at $t=0^+$ the current remains zero.

Setting $i=0$ at $t=0$,

$$-\frac{L}{R}\ln V = k \qquad \text{——— (2)}$$

Putting value of k from eq^n (2) in (1),

$$-\frac{L}{R}\ln(V-Ri) = t - \frac{L}{R}\ln V$$

$$-\frac{L}{R}\left[\ln(V-Ri) - \ln V\right] = t$$

$$\frac{V-Ri}{V} = e^{-R/L\,t}$$

$$V - Ri = V.e^{-R/L\,t}$$

$$Ri = V - V.e^{-R/L\,t}$$

$$i = \frac{V}{R} - \frac{V}{R}e^{-R/L\,t} \qquad \text{for } t>0$$

This response is composed of two parts, the steady state or response V/R and transient response $\frac{V}{R}e^{-R/L\,t}$.
or natural response

This transient response is a characteristic of the circuit.

Here the transient period is defined as the time ~~for~~ taken for the current to reach its final or steady state value from its initial value.

The term L/R is called time constant & is denoted by T:

$$T = L/R$$

- At one time constant, the current reaches 63.2% of its final value V/R.

- Voltage across the resistor is,

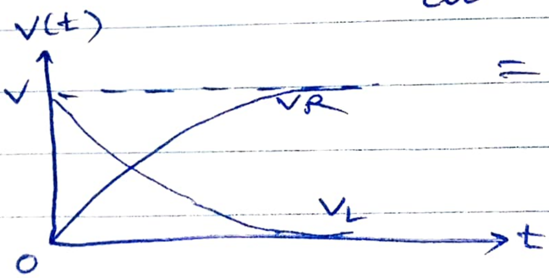
$$V_R = Ri = R \times \frac{V}{R}\left(1 - e^{-R/L\,t}\right)$$

$$= V\left(1 - e^{-R/L\,t}\right) \quad \text{for } t > 0$$

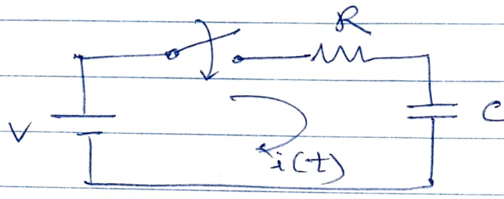
Similarly, voltage across inductor is,

$$V_L = L\frac{di}{dt} = L \cdot \frac{V}{R}\frac{d}{dt}\left(1 - e^{-R/L\,t}\right)$$

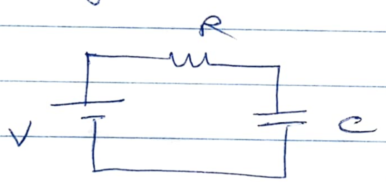
$$= V \cdot e^{-R/L\,t} \quad \text{for } t > 0$$



Resistor - Capacitor Circuit



Consider a series RC circuit.
The switch is closed at $t=0$.
The capacitor is initially uncharged.
Applying KVL to the circuit for $t>0$,



$$V - Ri - \frac{1}{C}\int_0^t i\,dt = 0$$

Differentiating above eq^n,

$$0 - R\frac{di}{dt} - \frac{i}{C} = 0$$

$$\therefore \frac{di}{dt} + \frac{1}{RC}i = 0$$

This is a linear differential eq^n of first order. The variables may be separated to solve the eq^n.

$$\frac{di}{i} = -\frac{dt}{RC}$$

Integrating both the sides,

$$\ln i = -\frac{1}{RC}t + K$$

The constant k can be evaluated from initial condition. In the ckt, the switch is closed at $t=0$. Since the capacitor never allows sudden change in voltage, it will act as s.c. at $t=0^+$. Hence, current in the circuit at $t=0^+$ is V/R.

Setting $i = V/R$ at $t=0$

$$\ln\left(\frac{V}{R}\right) = K$$

$$\ln i = -\frac{1}{RC}t + \ln\frac{V}{R}$$

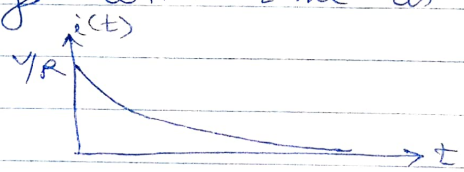
$$\ln i - \ln\frac{V}{R} = -\frac{1}{RC}t$$

$$\ln\left(\frac{i}{(V/R)}\right) = -\frac{1}{RC}t$$

$$\therefore \frac{i}{V/R} = e^{-\frac{1}{RC}t}$$

$$\therefore i = \frac{V}{R}\cdot e^{-\frac{1}{RC}t} \quad \text{for } t>0$$

When the switch is closed, the response decays with time as shown,



The term RC is called time constant & is denoted by T

$\therefore T = RC$

voltage across the resistor is,

$$V_R = Ri$$

$$= R \cdot \frac{V}{R} e^{-\frac{1}{RC}t}$$

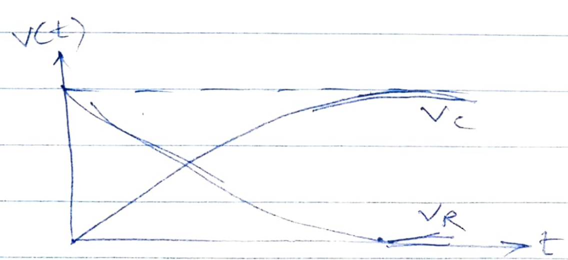
$$= V \cdot e^{-\frac{1}{RC}t} \quad \text{for } t > 0$$

Similarly voltage across capacitor is,

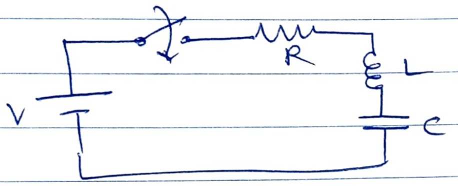
$$V_C = \frac{1}{C}\int_0^t i\,dt$$

$$= \frac{1}{C}\int_0^t \frac{V}{R} e^{-1/RC\,t}$$

$$= -V e^{-\frac{1}{RC}t} + k$$



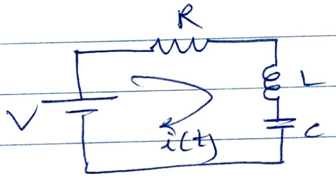
R-L-C circuit.



Consider a RLC series ckt. shown above.

- Switch is closed at $t=0$
- Capacitor & inductor are initially uncharged.

at $t>0$, the ckt. will be,



applying KVL at $t>0$,

$$V - Ri - L\frac{di}{dt} - \frac{1}{C}\int_0^t i\,dt = 0$$

Differentiating above eqn,

$$0 - R\frac{di}{dt} - L\frac{d^2i}{dt^2} - \frac{1}{C}i = 0$$

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = 0$$

This is a second order differential eqn. The auxiliary or characteristic eqn will be given by,

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

Let s_1 & s_2 be the roots of the eqn

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$= -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$= -\alpha + \beta$$

and

$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$= -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

$$= -\alpha - \beta$$

where $\alpha = \frac{R}{2L}$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$\beta = \sqrt{\alpha^2 - \omega_0^2}$$

The solution of above second order differential eqn will be given by

$$i(t) = k_1 \cdot e^{s_1 t} + k_2 e^{s_2 t}$$

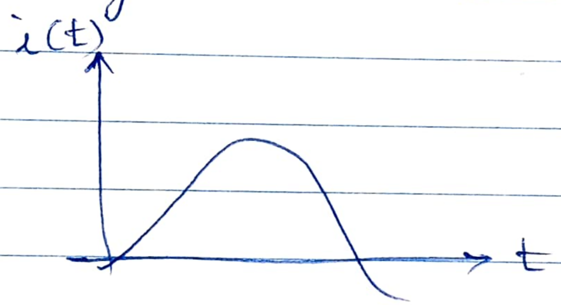
where k_1 and k_2 are constants to be determined and s_1 and s_2 are the roots of the eqn.

Now depending upon values of α & ω_0, there are 3 cases of response.

I] $\alpha > \omega_0$ ie,

$$\frac{R}{2L} > \frac{1}{\sqrt{LC}}$$

The roots are real & unequal and it gives an overdamped response.



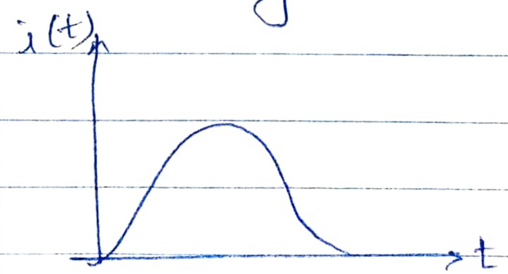
In this case sol^n is given by

$$i = e^{-\alpha t}(k_1 e^{\beta t} + k_2 e^{-\beta t})$$

or $i = k_1 e^{s_1 t} + k_2 e^{s_2 t}$ for $t > 0$.

II] $\alpha = \omega_0$ ie, $\frac{R}{2L} = \frac{1}{\sqrt{LC}}$

Roots are real & equal & it gives critically damped response.



In this case the solution is given by

$\cancel{i = e^{-\alpha t}(k_1 + k_2 t) \text{ for } t > 0}$

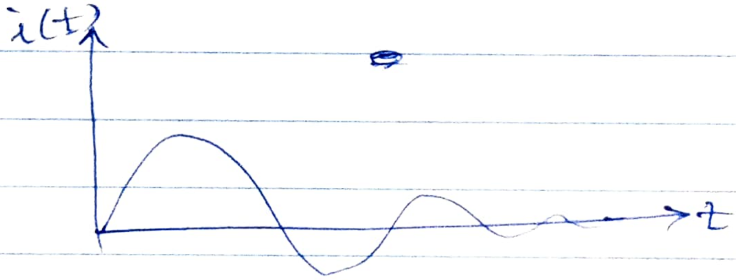
$$i = e^{-\alpha t}(k_1 + k_2 t) \text{ for } t >$$

III] $\alpha < \omega_0$ ie $\frac{R}{2L} < \frac{1}{\sqrt{LC}}$

The roots are complex conjugate and it gives an underdamped response.

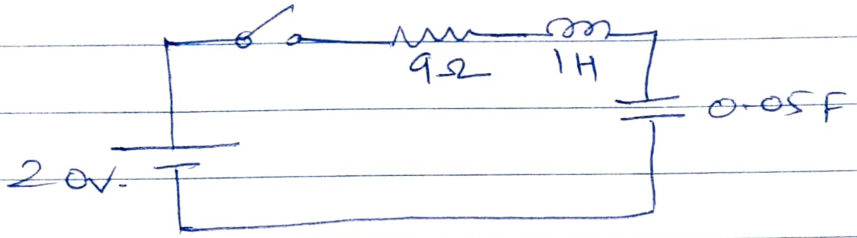
In this case the solution is given by

$$i(t) = K_1 e^{s_1 t} + k_2 e^{s_2 t}$$



Example:

1. In the network shown below, the switch is closed at $t=0$. Obtain expression for current $i(t)$ at $t>0$



Soln. At $t=0^-$ the switch is open

$\therefore \; i(0^-) = 0$

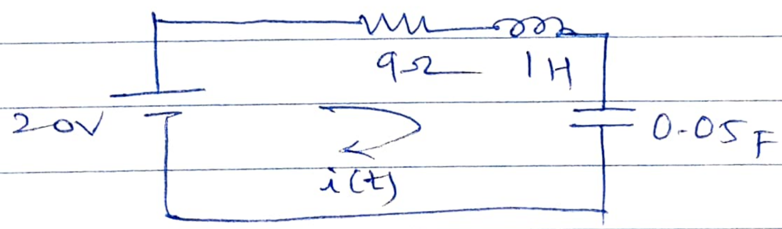
$V_c(0^-) = 0$

As the current through inductor & voltage across capacitor cannot change instantaneously,

$i(0^+) = 0$

$V_c(0^+) = 0$

At $t \geq 0$, the n/w is,



Applying KVL,

$$20 - 9i - 1\frac{di}{dt} - \frac{1}{0.05}\int_0^t i\,dt = 0 \quad \text{——} (1)$$

Differentiating above eqn,

$$0 - 9\frac{di}{dt} - \frac{d^2i}{dt^2} - 20\,i = 0$$

$$\therefore \frac{d^2 i}{dt^2} + 9\frac{di}{dt} + 20i = 0$$

$$D^2 + 9D + 20 = 0$$

$$\therefore D_1 = -4, \quad D_2 = -5$$

The solution for diff. eqn is given by

$$i(t) = k_1 \cdot e^{-4t} + k_2 e^{-5t} \quad \text{——— (2)}$$

Differentiating eqn (2),

$$\frac{di}{dt} = -4k_1 \cdot e^{-4t} - 5k_2 e^{-5t} \quad \text{——— (3)}$$

At $t = 0$, $i(0) = 0$.

$$\therefore 0 = k_1 + k_2 \quad \text{——— (4)}$$ From eqn (2)

$$\frac{di}{dt}(0) = -4k_1 - 5k_2 \quad \text{——— (5)}$$ From eqn (3)

Putting $t=0$ in eqn (1),

$$20 - 9i(0^+) - \frac{di}{dt}(0^+) - 0 = 0$$

$$\frac{di}{dt}(0^+) = 20 - 9i(0^+)$$

$$= 20 \text{ A/s.}$$

putting this value in eqn (5),

$$20 = -4k_1 - 5k_2 \quad \text{——— (6)}$$

Solving eqns (4) & (6)

$$k_1 = 20, \quad k_2 = -20$$

$$\therefore \boxed{i(t) = 20e^{-4t} - 20e^{-5t}} \text{ for } t > 0.$$

2. A RC circuit has $R = 20\Omega$ & $C = 400\mu F$. What is time constant?

$T = RC$

$= 20 \times 400 \times 10^{-6} = 8ms$

3. ~~A capacitor in RC circuit with $R = 25\Omega$ & $C = 50\mu F$ is being charged with initial zero voltage. What is time taken for the capacitor voltage to reach 40% of its steady state value?~~

$R = 25\ \Omega$

$C = 50\mu F$

$\therefore T = RC = 1.25 \times 10^{-3}\ s$

$\therefore \frac{1}{RC} = 800\ s^{-1}$

4. In a RL ckt with time const of 1.25 sec, inductor current increases from the initial value of zero to the final value of 1.2 A

a) Calculate inductor current at time 0.4 s, 0.8 s & 2 sec.

b) Find the time at which the inductor current reaches 0.3 A, 0.6 A & 0.9 A

$L/R = 1.25$

$\therefore R/L = 0.8$

$i_L(0) = 0$

$i_L(\infty) = 1.2\ A$

$$i_L = \frac{V}{R}\left(1 - e^{-R/L\,t}\right) \text{ A}$$

$$= 1.2\left(1 - e^{-0.8t}\right) \text{ A}$$

a]

at $t = 0.4$ s, $i_L = 1.2\left(1 - e^{-0.32}\right) = 0.3286$ A

$t = 0.8$ sec, $i_L = 0.5672$ A

$t = 2$ sec $i_L = 0.9577$ A

b]

$0.3 = 1.2\left(1 - e^{-0.8t_1}\right)$ $\therefore t_1 = 0.3596$ sec

$0.6 = 1.2\left(1 - e^{-0.8t_2}\right)$ $\therefore t_2 = 0.8664$ sec

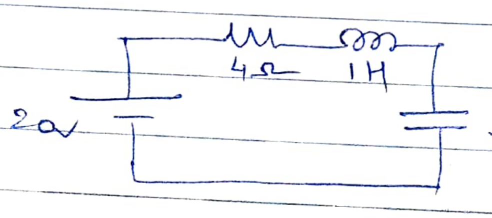
0.9 $t_3 = 1.7329$ sec.

4. Find expression for the current in a series RLC ckt. fed by dc voltage of 20V with $R = 4\Omega$, $L = 1H$ & $C = \frac{1}{4}$ F. Assume initial condn to be zero

Soln Assuming zero initial condn,

$i_L(0^-) = 0 = i_L(0^+)$ ——— (1)

$V_C(0^-) = 0 = V_C(0^+)$ ——— (2)



KVL eqn

$$20 - 4i - \frac{1}{1/4}\int_0^t i\,dt - 1\frac{di}{dt}$$

$$4i + 4\int_0^t i\,dt + \frac{di}{dt} = 20 \quad \text{——— (3)}$$

Differentiating,

$$4\frac{di}{dt} + \frac{d^2i}{dt^2} + 4i = \cancel{\ } 0$$

$$\therefore \frac{d^2i}{dt^2} + 4\frac{di}{dt} + 4i = \cancel{\ } 0 \quad \text{--- (4)}$$

$$s^2 + 4s + 4 = 0$$

Roots of the eqn are,

$$s_1 = s_2 = -2 \quad \text{--- (5)}$$

$\therefore$ Roots are real & equal

Hence soln is,

$$i(t) = \cancel{k} \; k_1 \cdot e^{-2t} + k_2 t e^{-2t} \quad \text{--- (6)}$$

at $t=0$, $i(t)=0$

Putting values,

$$0 = k_1 \cdot e^{-0} + k_2(0) \cdot e^{-0}$$

$$\therefore k_1 = 0 \quad \text{--- (7)}$$

$$\therefore i(t) = k_2 \cdot t \cdot e^{-2t} \quad \text{--- (8)}$$

Now putting $t=0$ in eqn (3),

$$4\, i(0) + \frac{di}{dt}(0) + 4\int(0) = 20$$

$$\therefore \frac{di}{dt} = 20 \quad \text{A/s.} \quad \text{--- (9)}$$

Differentiate eqn (8) w.r.t. t,

$$\frac{di}{dt} = k_2\left[t(-2)e^{-2t} + e^{-2t}(1)\right]$$

$$\therefore \frac{di}{dt} = k_2\left(-2te^{-2t}\right) + k_2\left(e^{-2t}\right) \quad \text{--- (10)}$$

At $t=0$, eqn (10) becomes,

$$\frac{di}{dt}(0) = 20 = k_2\left(-2te^{-2t}\right) + k_2\left(e^{-2t}\right)$$

$$= k_2\left(\underbrace{-2(0)e^{-0}}_{0}\right) + k_2 \cdot e^{-0}$$

$$\therefore K_2 = 20$$

$\therefore$ Expression for current,

$$i(t) = k_2 \cdot t \cdot e^{-2t}$$

$$\boxed{i(t) = 20te^{-2t}}$$