THE Z-TRANSFORM

4.1 INTRODUCTION

Use of Z-transform is very prominent in the analysis of linear time-invariant systems. Linear time invariant (LTI) systems are characterised either by their Z-transform or by the Fourier transform and their characteristics are related to the location of the poles or zeroes of their system functions. We have already discussed Fourier Transforms in details in Chapter 5.

Number of systems of practical importance such as economical systems, population systems and many other systems occurring in statistical studies are discrete in nature. While the modern technological development has made it possible to consider many systems occurring in Engineering fields as discrete. Number of important types of digital systems including resonators, notch filters, comb filters, all-pass filters and oscillators use Z-transforms for their analysis.

Discrete systems give rise to difference equations, and their solutions as well as their analysis are carried out by using transform techniques. Z-transform plays important role in these aspects. Its role in analysis of discrete systems is same as that of Laplace transform and Fourier transform in continuous systems.

4.2 BASIC PRELIMINARY

I. SEQUENCE

An ordered set of real or complex numbers is called a sequence. It is denoted by $\{f(k)\}$ or $\{f_k\}$. The sequence $\{f(k)\}$ is represented in two ways.

1. The most elementary way is to list all the numbers of the sequence; such as :

(i) $$\{f(k)\} = \{15, 13, 10, \underset{\uparrow}{8}, 5, 2, 0, 3\} \qquad \dots (1)$$

In this representation, a vertical arrow indicates the position corresponding to $k = 0$.

$\therefore$ $f(0) = 8$, $f(1) = 5$, $f(2) = 2$, $f(3) = 0$, $f(4) = 3$.

$$f(-1) = 10$$
$$f(-2) = 13$$
$$f(-3) = 15$$

(ii) For the sequence

$$\{f(k)\} = \{15, 13, \underset{\uparrow}{10}, 8, 5, 2, 0, 3\} \qquad \dots (2)$$

$f(-2) = 15$, $f(-1) = 13$, $f(0) = 10$, $f(1) = 8$, $f(2) = 5$, $f(3) = 2$, $f(4) = 0$, $f(5) = 3$.

Note : The sequences given in (1) and (2) are having the same listing but they are not treated as identical, since k = a corresponds to different terms in these sequences.

The method of representation, as discussed above, is appropriate only for a sequence with finite number of terms.

When vertical arrow $\uparrow$ is not given, then the starting or left hand end term of the sequence denotes the position corresponding to k = 0.

In the sequence :

$$\{f(k)\} = \{9, 7, 5, 3, 1, -2, 0, 2, 4\} \qquad \dots (3)$$

the zeroeth term is 9, the left hand term.

$$\therefore \qquad f(0) = 9, \quad f(1) = 7, \quad f(2) = 5 \quad \dots \text{etc.}$$

2. The second way of specifying the sequence is to define the general term of the sequence (if possible) as a function of position i.e. k.

e.g. The sequence $\{f(k)\}$ where $\{f(k)\} = \frac{1}{4^k}$ (k is any integer) represents the sequence

$$\left\{\frac{1}{4^{-8}}, \frac{1}{4^{-7}}, \dots\dots, \frac{1}{4^{-1}}, \underset{\uparrow}{1}, \frac{1}{4}, \frac{1}{4^2} \dots\dots\right\} \qquad \dots (4)$$

Here $f(0) = 1$, $f(1) = \frac{1}{4}$, $f(2) = \frac{1}{4^2}$, etc.

If $\quad f(k) = \frac{1}{4^k}; \qquad -3 \le k \le 5 \quad$ then it represents the sequence

$$\left\{4^3, 4^2, 4, \underset{k=0}{\overset{1}{\uparrow}}, \frac{1}{4}, \frac{1}{4^2}, \frac{1}{4^3}, \frac{1}{4^4}, \frac{1}{4^5}\right\}$$

Hence a sequence $\{f(k)\}$ can be written as :

$\{f(k)\} = \{\dots\dots f(-3), f(-2), f(-1), f(0), f(1), f(2), f(3), f(4), \dots\}$

having f (0) as the zeroeth term.

OR $\qquad \{f(k)\} = \{f(0), f(1), f(2), \dots\dots\}$

OR $\qquad \{f(k)\} = \{f(-2), f(-1), f(0), f(1), f(2), f(3)\}$

II. CAUSAL SEQUENCE

ILLUSTRATION

Any sequence whose terms corresponding to k < 0 are all zero is called causal sequence.

Ex. 1 : *What sequence is generated when*

$$f(k) = \begin{cases} 0, & k < 0 \\ \cos\frac{k\pi}{3}, & k \ge 0 \end{cases}$$

Sol. : We have $\{f(k)\} = \left\{\ldots 0,\ 0,\ \underset{k=0}{\overset{1}{\uparrow}},\ \cos\frac{\pi}{3},\ \cos\frac{2\pi}{3},\ \cos\pi,\ \ldots\ldots\right\}$

From application point of view, it is sometimes convenient to consider finite sequences to be of infinite length by appending additional zeroes to each.

e.g. $\{f(k)\} = \{8, 6, 4, 2, 0, 2, 4, 6, 8, 10\}$

$\therefore$ $\{f(k)\} = \{\ldots 0, 0, \ldots, 0, 0, 8, 6, 4, 2, 0, 2, 4, 6, 8, 10, 0, 0, \ldots, 0, 0, \ldots\ldots\}$

III. BASIC OPERATIONS ON SEQUENCES

1. **Addition :** If $\{f(k)\}$ and $\{g(k)\}$ are the two sequences with same number of terms, then the addition of these sequences is a sequence given by $\{f(k) + g(k)\}$ i.e.

$$\{f(k)\} + \{g(k)\} = \{f(k) + g(k)\}$$

2. **Scaling :** If a is a scalar, then

$$a\{f(k)\} = \{a f(k)\}$$

3. **Linearity :** If a and b are scalars, then

$$\{a f(k) + b g(k)\} = \{a f(k)\} + \{b g(k)\} = a\{f(k)\} + b\{g(k)\}$$

ILLUSTRATION

Ex. 1 : *Write the sequence* $\frac{1}{2}\{f(k)\}$, *where* $\{f(k)\}$ *is given by* $f(k) = \frac{1}{2^k}$.

Sol. :
$$\frac{1}{2}\{f(k)\} = \left\{\frac{1}{2} f(k)\right\} = \left\{\frac{1}{2}\cdot\frac{1}{2^k}\right\} = \left\{\frac{1}{2^{k+1}}\right\}$$

Ex. 2 : *Write the sequence* $\{f(k) + g(k)\}$, *where* $\{f(k)\}$ *is given by* $f(k) = \frac{1}{2^k}$ *and* $\{g(k)\}$ *is given by*

$$g(k) = \begin{cases} 0, & k < 0 \\ 3, & k \geq 0 \end{cases}$$

Sol. : $\{f(k) + g(k)\} = \{f(k)\} + \{g(k)\} = \{h(k)\}$

where
$$h(k) = \begin{cases} \frac{1}{2^k}, & k < 0 \\ \frac{1}{2^k} + 3, & k \geq 0. \end{cases}$$

Ex. 3 : If $\{f(k)\}$ is given by $f(k) = \begin{matrix} 0, & k<0 \\ 2, & k\geq 0 \end{matrix}$ find $\frac{1}{3}\{f(k)\}$.

Sol. :
$$\frac{1}{3}\{f(k)\} = \left\{\frac{1}{3} f(k)\right\} = \left\{\ldots 0,\ 0,\ \ldots\ldots 0,\ \frac{2}{3},\ \frac{2}{3},\ \ldots\ldots\right\}$$

IV. ADDITIONAL RESULTS

1. $$\frac{1}{1+y} = 1 - y + y^2 - y^3 + y^4 - \ldots\ldots, |y| < 1.$$

2. $$\frac{1}{1-y} = 1 + y + y^2 + y^3 + \ldots\ldots, |y| < 1.$$

3. $$(1+y)^n = 1 + ny + \frac{n(n-1)}{2!} y^2 + \frac{n(n-1)(n-2)}{3!} y^3 + \ldots, |y| < 1.$$
$$= \sum_{r=0}^{n} {}^nC_r y^r$$

4. $$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots\ldots,$$

5. Since S_∞ of the Geometric Progression G.P.

$$a + ar + ar^2 + ar^3 + \ldots\ldots = \frac{a}{1-r}$$

and it is convergent if $|r| < 1$

where a = First term

r = Common ratio.

6. If $z = x + iy$, then $|z| = \sqrt{x^2 + y^2}$

Also $|z| = 1$ represents $\sqrt{x^2+y^2} = 1$

i.e. $x^2 + y^2 = 1$ a circle. (see Fig. 4.1)

$|z| > 4 \Rightarrow \sqrt{x^2+y^2} > 4$

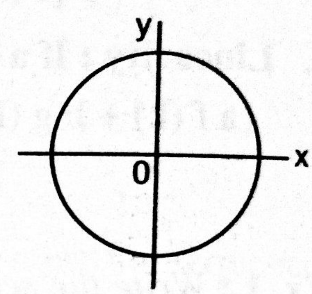

Fig. 4.1

$x^2 + y^2 > 16$ i.e. collection of points which lie outside the circle $x^2 + y^2 = 16$ (see Fig. 4.2)

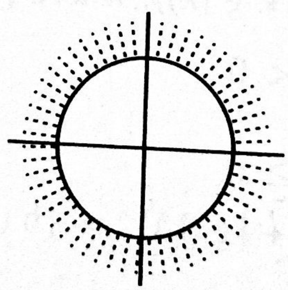

Fig. 4.2

Similarly $|z| < 1$ represents the collection of points which lie inside the unit circle $|z| = 1$ i.e. $x^2 + y^2 = 1$ (see Fig. 4.3).

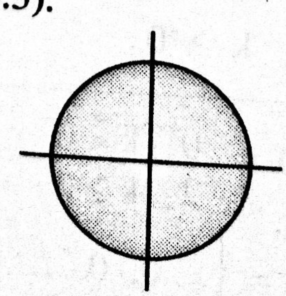

Fig. 4.3

4.3 Z-TRANSFORMS

Definition :

1. The Z-transform of a sequence $\{f(k)\}$, symbolically denoted by $Z\{f(k)\}$ is defined as :

$$\boxed{Z\{f(k)\} = F(z) = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k} = \sum_{k=-\infty}^{\infty} \frac{f(k)}{z^k}}$$

where, $z = x + i\,y$ is a complex number. Z is a Z-transform operator and F (z), the Z-transform of $\{f(k)\}$.

2. For a finite sequence $\{f(k)\}$, $m \le k \le n$, its Z-transform is,

$$Z\{f(k)\} = F(z) = \sum_{k=m}^{n} f(k)\, z^{-k}$$

$$Z\{f(k)\} = f(m)\, z^{-m} + f(m+1)\, z^{-(m+1)} + \ldots + f(n)\, z^{-n}$$

The Z-transform of $\{f(k)\}$ exists if the sum of the series on R.H.S. exists i.e. the series on R.H.S. converges absolutely.

3. Z-transform of a causal sequence :

$$\{f(k)\} = \{0, 0, \ldots 0, 0, f(0), f(1), \ldots\ldots\}$$

which is defined for positive integers k, is defined as

$$\boxed{Z\{f(k)\} = F(z) = \sum_{k=0}^{\infty} f(k)\, z^{-k}}$$

Note :

1. To obtain Z-transform of a sequence we multiply each term by *negative power of z of the order of that term* and take the sum.
2. $Z\{f(k)\}$ is a function of a complex variable z and is defined only if the sum is finite i.e. *if the infinite series* $\sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$ *is absolutely convergent.*

ILLUSTRATIONS

Ex. 1 : For $\{f(k)\}$ if

$$f(k) = \{8,\ 6,\ 4,\ 2,\ -1,\ \underset{\uparrow}{0},\ 1,\ 2,\ 3\}$$

$$\{f(k)\} = \{f(-5), f(-4), f(-3), f(-2), f(-1), f(0), f(1), f(2), f(3)\}$$

we have $\quad F(z) = Z\{f(k)\} = \sum_{k=-5}^{3} f(k)\, z^{-k}$

$$= f(-5)\, z^5 + f(-4)\, z^4 + f(-3)\, z^3 + f(-2)\, z^2 + f(-1)\, z + f(0)\, z^0 + f(1)\, z^{-1} + f(2)\, z^{-2} + f(3)\, z^{-3}$$

$$F(z) = Z\{f(k)\} = 8z^5 + 6z^4 + 4z^3 + 2z^2 - 1(z) + 0 + 1(z^{-1}) + 2z^{-2} + 3z$$

$$F(z) = 8z^5 + 6z^4 + 4z^3 + 2z^2 - z + 0 + \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3}$$

Ex. 2 : For $\{f(k)\}$ if $f(k) = \{4, 2, 0, \underset{\uparrow}{-2}, -4, -6\}$

$$F(z) = Z\{f(k)\} = \sum_{k=-3}^{2} f(k)\, z^{-k}$$

$$= 4z^3 + 2z^2 + 0z^1 - 2z^0 - 4z^{-1} - 6z^{-2}$$

$$= 4z^3 + 2z^2 + 0 - 2 - \frac{4}{z} - \frac{6}{z^2}$$

Ex. 3 : For $\{f(k)\}$, if $f(k) = \left\{ \ldots\ldots \frac{1}{2^{-2}}, \frac{1}{2^{-1}}, \underset{\uparrow}{1}, \frac{1}{2}, \frac{1}{2^2}, \ldots \right\}$

$$F(z) = Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\,.\,z^{-k}$$

$$= \ldots\ldots + 2^2 \cdot z^2 + 2 \cdot z + 1(z^0) + \frac{1}{2} z^{-1} + \frac{1}{2^2} z^{-2} + \ldots\ldots$$

$$F(z) = \ldots\ldots + 2^2 z^2 + 2 \cdot z + 1 + \frac{1}{2z} + \frac{1}{2^2 z^2} + \ldots\ldots$$

4.4 INVERSE Z-TRANSFORM

The operation of obtaining the sequence $\{f(k)\}$ from $F(z)$ is defined as inverse Z-transform and is denoted as :

$$\boxed{Z^{-1}[F(z)] = \{f(k)\}}$$

where, Z^{-1} is inverse Z-transform operator.

4.5 Z-TRANSFORM PAIR

Sequence $\{f(k)\}$ and its Z-transform $F(z)$ are together termed as Z-transform pair and denoted as $\{f(k)\} \longleftrightarrow F(z)$.

i.e. $\quad Z\{f(k)\} = F(z)$

and $\quad Z^{-1}[F(z)] = \{f(k)\}$

4.6 UNIQUENESS OF INVERSE Z-TRANSFORM : REGION OF ABSOLUTE CONVERGENCE (ROC)

Consider the two sequences $\{f(k)\}$ and $\{g(k)\}$

where $\quad f(k) = \begin{cases} 0, & k < 0 \\ a^k, & k \geq 0 \end{cases}; \quad g(k) = \begin{cases} -b^k, & k < 0 \\ 0, & k \geq 0 \end{cases}$

$\therefore$ Z-transform of the sequence $\{f(k)\}$ is

$$Z\{f(k)\} = F(z) = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k} = \sum_{k=-\infty}^{-1} f(k)\, z^{-k} + \sum_{k=0}^{\infty} f(k)\, z^{-k}$$

$$= 0 + \sum_{k=0}^{\infty} a^k z^{-k}$$

$$= \sum_{k=0}^{\infty} (a z^{-1})^k = 1 + (a z^{-1}) + (a z^{-1})^2 + (a z^{-1})^3 + \ldots$$

which is an infinite G.P.

$$\therefore \qquad S_\infty = \frac{a}{1-r}, \quad |r| < 1$$

Here, $\quad a$ = first term = 1

r = common ratio = $a z^{-1}$.

$$Z\{f(k)\} = \frac{1}{1 - a z^{-1}} \text{ provided } |a z^{-1}| < 1$$

$$= \frac{1}{1 - \frac{a}{z}}, \quad |a| < |z|$$

$$F(z) = \frac{z}{z-a}, \quad |z| > |a|$$

but $\qquad z = x + i y \qquad \therefore \quad |z| = \sqrt{x^2 + y^2}$

$$|z| > |a| \Rightarrow \sqrt{x^2 + y^2} > a$$

i.e. $x^2 + y^2 > a^2$, which represents exterior of circle $x^2 + y^2 = a^2$ [refer Fig. 4.4 (a)].

Now consider the Z-transform of the sequence $\{g(k)\}$,

$$Z\{g(k)\} = G(z) = \sum_{k=-\infty}^{\infty} g(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{\infty} g(k)\, z^{-k} + \sum_{k=0}^{\infty} g(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{\infty} - b^k z^{-k} + 0$$

Let $\qquad k = -r$ when

$k = -1$	$r = 1$
$k = -\infty$	$r = \infty$

and $\sum\limits_{k=-\infty}^{-1} = \sum\limits_{r=\infty}^{1} = \sum\limits_{r=1}^{\infty}$

$$G(z) = -\sum_{r=1}^{\infty} b^{-r} z^{r} = -\sum_{r=1}^{\infty} (b^{-1}z)^{r}$$

$$= -b^{-1} z \, (b^{-1}z)^{2} - (b^{-1}z)^{3} \; \dots\dots\dots$$

$$= -\frac{b^{-1} z}{1-(b^{-1}z)}, \quad |b^{-1} z| < 1$$

$$= -\frac{\frac{z}{b}}{1-\frac{z}{b}}, |z| < |b| = \frac{z}{z-b}, \; |z| < |b|$$

but $\quad z = x + iy \quad |z| = \sqrt{x^2+y^2}$

$$|z| < |b| \Rightarrow \sqrt{x^2+y^2} < b$$

i.e. $x^2 + y^2 < b^2$ which represents the interior of circle $x^2 + y^2 = b^2$ [Refer Fig. 4.4 (b)]

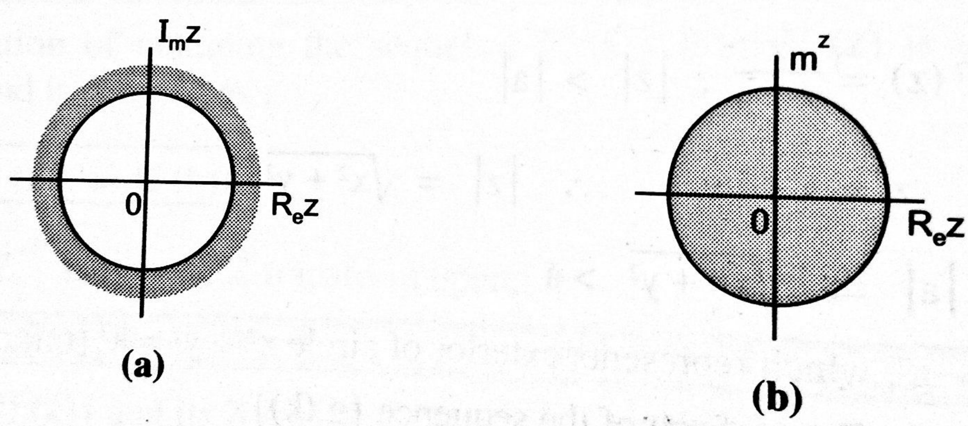

Fig. 4.4

Now if a = b then

$$F(z) = G(z) = \frac{z}{z-a}$$

i.e. $\quad$ for $a = b$, $\quad Z\{f(k)\} = Z\{g(k)\} = \frac{z}{z-a}$

If a = b, then two sequences $\{f(k)\}$ and $\{g(k)\}$ have the same Z-transform, therefore inverse Z-transform of $\frac{z}{z-a}$ will be two different sequences $\{f(k)\}$ and $\{g(k)\}$, indicating that *inverse Z-transform is not unique.*

However, if we specify the region, interior or exterior of circle $x^2 + y^2 = a^2$ known as *region of convergence*, then we get exactly

$$Z^{-1}\{F(z)\} = \{f(k)\} \quad \text{and} \quad Z^{-1}\{G(z)\} = \{g(k)\}.$$

This implies that *Z – transform and its inverse are uniquely related in the specified region of convergence.*

Note :

1. In case of one-sided sequences (i.e. causal sequences or sequences for which $f(k) = 0$ for $k < 0$), then there is no necessity of specifying the ROC.
2. By the term "region of convergence", we will mean the "region of absolute convergence". This term will be abbreviated to ROC.

Now, consider the sequence $\{f(k)\}$, where

$$f(k) = \begin{cases} -b^k, & k < 0 \\ a^k, & k \geq 0 \end{cases}$$

$$\therefore \quad Z\{f(k)\} = \sum_{k=-\infty}^{-1} f(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} f(k)\, z^{-k} + \sum_{k=0}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} -b^k\, z^{-k} + \sum_{k=0}^{\infty} a^k\, z^{-k}$$

Put $k = -m \quad \therefore k = -\infty \Rightarrow m = \infty, \; k = -1 \Rightarrow m = 1$

$$Z\{f(k)\} = -\sum_{m=1}^{\infty} b^{-m}\, z^m + \sum_{k=0}^{\infty} (a z^{-1})^k$$

$$= -\sum_{m=1}^{\infty} (b^{-1} z)^m + \sum_{k=0}^{\infty} (a z^{-1})^k$$

$$= -\left(\frac{b^{-1} z}{1 - b^{-1} z}\right) + \frac{1}{1 - az^{-1}}$$

provided $\left|b^{-1} z\right| < 1$ and $\left|az^{-1}\right| < 1$

i.e. $|z| < |b|$ and $|a| < |z|$

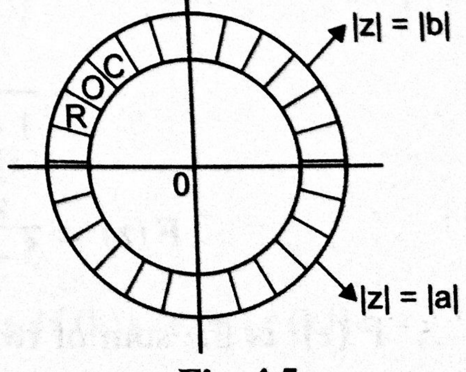

Fig. 4.5

$$Z\{f(k)\} = \frac{z}{z-b} + \frac{z}{z-a}$$

provided $|a| < |z| < |b|$

Note that the Z - transform exists only if |b| > |a| and does not exist for |b| = |a| or |b| < |a|.

Note : In general, Z-transform of sum of sequences is the sum of corresponding transforms with region of absolute convergence consisting of those values of z for which all of the individual transforms converge absolutely i.e. the region of absolute convergence of sum of transforms is the intersection of the individual regions of absolute convergence.

Note : For finite sequence, Z-transform exists for all values of z except for $z = 0$ and $z = \infty$.

Note : The region lying between two concentric circles is called an annulus.

e.g. Consider $\{f(k)\}$

where
$$f(k) = 5^k, \quad \text{for } k < 0$$
$$= 3^k, \quad \text{for } k \geq 0$$

$$\therefore \quad Z\{f(k)\} = \sum_{k=-\infty}^{-1} f(k)\, z^{-k} + \sum_{k=0}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} 5^k\, z^{-k} + \sum_{k=0}^{\infty} 3^k\, z^{-k}$$

$k = -r$ when $k = -\infty \quad r = \infty$

$k = -1 \quad r = 1$

$$Z\{f(k)\} = \sum_{r=1}^{\infty} 5^{-r}\, z^{r} + \sum_{k=0}^{\infty} (3z^{-1})^k$$

$$= \sum_{r=1}^{\infty} (5^{-1}z)^r + \sum_{k=0}^{\infty} (3z^{-1})^k$$

$$= \frac{5^{-1}z}{1 - 5^{-1}z} + \frac{1}{1 - 3z^{-1}}, \quad |5^{-1}z| < \text{and } |3z^{-1}| < 1$$

$$F(z) = \frac{z}{5 - z} + \frac{z}{z - 3}, \quad |z| < 5 \text{ and } 3 < |z|$$

$\therefore$ **F (z) is the sum of two infinite series both of which are G.P. The first series is absolutely (and therefore uniformly) convergent if $|5^{-1}z| < 1$ and the second one if $|3z^{-1}| < 1$.**

Thus F (z) is defined iff $|z| > 3$ and $|z| < 5$ i.e. Z lies in the annulus $3 < |z| < 5$.

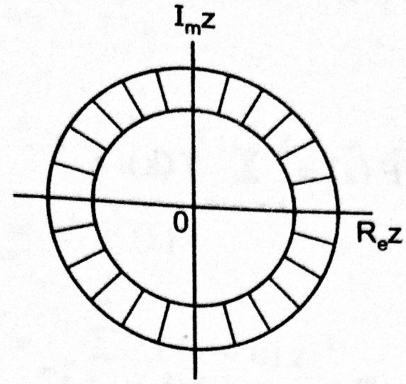

Fig. 4.6

4.7 PROPERTIES OF Z-TRANSFORMS

1. Linearity : If {f(k)} and {g(k)} are such that they can be added and 'a' and 'b' are constants, then

$$Z\{a\,f(k) + b\,g(k)\} = a\,Z\{f(k)\} + b\,Z\{g(k)\}$$

Proof : We have

$$Z\{a\,f(k) + b\,g(k)\} = \sum_{k=-\infty}^{\infty} Z\{a\,f(k) + b\,g(k)\}\, z^{-k}$$

$$= \sum_{k=-\infty}^{\infty} [a\,f(k)\, z^{-k} + b\,g(k)\, z^{-k}]$$

$$= a \sum_{k=-\infty}^{\infty} f(k)\, z^{-k} + b \sum_{k=-\infty}^{\infty} g(k)\, z^{-k}$$

$$= a\,F(z) + b\,G(z) = a \cdot Z\{f(k)\} + b \cdot Z\{g(k)\}$$

2. If $Z\{f(k)\} = F(z)$ and $Z\{g(k)\} = G(z)$ and 'a' and 'b' are constants, then $Z^{-1}[a\,F(z) + b\,G(z)] = a\,Z^{-1}[F(z)] + b\,Z^{-1}[G(z)]$

Proof : We have,

$$Z\{a\,f(k) + b\,g(k)\} = a\,Z\{f(k)\} + b\,Z\{g(k)\}$$

$$= a\,F(z) + b\,G(z)$$

$$\therefore \quad Z^{1}\{a\,F(z) + b\,G(z)] = \{a\,f(k) + b\,g(k)\} = a\,\{f(k)\} + b\,\{g(k)\}$$

$$= a\,Z^{1}[F(z)] + b\,Z^{1}[G(z)]$$

i.e. operator Z^{1} is a linear operator.

3. Change of scale : If $Z\{f(k)\} = F(z)$ then $Z\{a^k f(k)\} = F\left(\frac{Z}{a}\right)$

Proof : We have,

$$Z\{f(k)\} = F(z) = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

Replacing z by $\frac{z}{a}$, we get

$$F\left(\frac{z}{a}\right) = \sum_{k=-\infty}^{\infty} f(k)\left(\frac{z}{a}\right)^{-k} = \sum_{k=-\infty}^{\infty} a^k f(k)\, z^{-k}$$

$$F\left(\frac{z}{a}\right) = Z\{a^k f(k)\}$$

$$\therefore \quad Z\{a^k f(k)\} = F\left(\frac{z}{a}\right)$$

4. If $Z\{f(k)\} = F(z)$ then $Z\{e^{-ak} f(k)\} = F(e^a z)$

Proof : We have,

$$Z\{f(k)\} = F(z) = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

$$\therefore \quad Z\{e^{-ak} f(k)\} = \sum_{-\infty}^{\infty} e^{-ak} \cdot f(k)\, z^{-k} = \sum_{-\infty}^{\infty} f(k)\,(e^a z)^{-k}$$

$$= F(e^a z)$$

5. Shifting Property :

(a) If $Z\{f(k)\} = F(z)$ then $Z\{f(k+n)\} = z^n F(z)$ and $Z\{f(k-n)\} = z^{-n} F(z)$

Proof : We have, $Z\{f(k)\} = F(z) = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$

$$\therefore \quad Z\{f(k+n)\} = \sum_{k=-\infty}^{\infty} f(k+n)\, z^{-k} = \sum_{k=-\infty}^{\infty} f(k+n)\, z^{-(k+n)} \cdot z^n$$

$$= z^n \sum_{k=-\infty}^{\infty} f(k+n)\, z^{-(k+n)}$$

For $k + n = r$ if $k = -\infty$, $r = -\infty$

$k = \infty$, $r = \infty$

$$\therefore \quad Z\{f(k+n)\} = z^n \sum_{r=-\infty}^{\infty} f(r)\, z^{-r} = z^n F(z)$$

$$\therefore \quad Z\{f(k+n)\} = z^n F(z)$$

Similarly, $Z\{f(k-n)\} = \sum_{k=-\infty}^{\infty} f(k-n)\, z^{-k}$

$$= \sum_{k=-\infty}^{\infty} f(k-n)\, z^{-(k-n)} \cdot z^{-n}$$

$$= z^{-n} \sum_{k=-\infty}^{\infty} f(k-n)\, z^{-(k-n)}$$

For $k - n = r$; if $k = -\infty$, $r = -\infty$ and $k = \infty$, $r = \infty$

$$= z^{-n} \sum_{k=-\infty}^{\infty} f(r)\, z^{-r} = z^{-n} F(z)$$

$$\therefore \quad Z\{f(k-n)\} = z^{-n} F(z)$$

(b) For one sided Z-transform defined as $Z\{f(k)\} = \sum_{k=0}^{\infty} f(k)\, z^{-k}$

(i.e Z - transform for $k \geq 0$), we have

$$Z\{f(k+n)\} = z^n F(z) - \sum_{r=0}^{n-1} f(r)\, z^{n-r}$$

and $$Z\{f(k-n)\} = z^{-n} F(z) + \sum_{r=-n}^{-1} f(r)\, z^{-(n+r)}$$

Proof : We have for $k \geq 0$,

$$Z\{f(k)\} = \sum_{k=0}^{\infty} f(k)\, z^{-k}$$

$$\therefore \quad Z\{f(k+n)\} = \sum_{k=0}^{\infty} f(k+n)\, z^{-k} = \sum_{k=0}^{\infty} f(k+n)\, z^{-(k+n)} \cdot z^n$$

For $k + n = r$, when $k = 0$,

$k = \infty$, $\quad r = \infty$

$$\therefore \qquad \text{R.H.S.} = z^n \sum_{r=n}^{\infty} f(r)\, z^{-r}$$

Now $r = n$ to ∞ means ($r = 0$ to ∞) $-$ ($r = 0$ to $n-1$)

$$\therefore \qquad Z\{f(k+n)\} = z^n \sum_{r=0}^{\infty} f(r)\, z^{-r} - z^n \sum_{r=0}^{n-1} f(r)\, z^{-r}$$

$$Z\{f(k+n)\} = z^n F(z) - z^n \sum_{r=0}^{n-1} f(r)\, z^{-r}$$

$$= z^n F(z) - \sum_{r=0}^{n-1} f(r)\, z^{n-r}$$

Now, $\quad Z\{f(k-n)\} = \sum_{k=0}^{\infty} f(k-n)\, z^{-k}$

$$= \sum_{k=0}^{\infty} f(k-n)\, z^{-(k-n)} \cdot z^{-n}$$

$$= z^{-n} \sum_{k=0}^{\infty} f(k-n)\, z^{-(k-n)}$$

For $k - n = r$, when $k = 0$, $\quad r = -n$

$k = \infty$, $\quad r = \infty$

$$\therefore \qquad Z\{f(k+n)\} = z^{-n} \sum_{r=-n}^{\infty} f(r)\, z^{-r}$$

Now $r = -n$ to ∞ is ($r = -n$ to -1) + ($r = 0$ to ∞)

$$\therefore \qquad Z\{f(k-n)\} = z^{-n} \sum_{r=-n}^{-1} f(r)\, z^{-r} + z^{-n} \sum_{r=0}^{\infty} f(r)\, z^{-r}$$

$$\therefore \qquad Z\{f(k-n)\} = z^{-n} F(z) + \sum_{r=-n}^{-1} f(r)\, z^{-(n+r)}$$

Additional ...

1. If $\{f(k)\}$ is causal sequence then

$$Z\{f(k-n)\} = z^{-n} F(z)$$

because $f(-1)\ f(-2)\ f(-3)\ \ldots\ldots\ f(-n)$ are all zero.

2.
$$Z\{f(k-n)\} = z^{-n} F(z)$$

For $n = 1$, $Z\{f(k-1)\} = z^{-1} F(z), f(-1) = 0$

$$Z\{f(k-2)\} = z^{-2} F(z),\ f(-1) = 0,\ f(-2) = 0$$

$$Z\{f(k+1)\} = z F(z) - z f(0)$$

$$Z\{f(k+2)\} = z^2 F(z) - z^2 f(0) - z f(1)$$

Shifting properties are very useful in Z-transforming linear difference equations, from which the solution is obtained by inverse transforming.

6. Multiplication by k :

If $Z\{f(k)\} = F(z)$ then $Z\{k f(k)\} = -z \dfrac{d}{dz} F(z)$

$\therefore$ **In general $Z\{k^n f(k)\} = \left(-z \dfrac{d}{dz}\right)^n F(z)$**

Proof : We have,
$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k} = F(z)$$

$$\therefore \quad Z\{k f(k)\} = \sum_{k=-\infty}^{\infty} k f(k) z^{-k}$$

Multiply and divide by $(-z)$ on R.H.S.

$$= \sum_{k=-\infty}^{\infty} -k f(k) z^{-k-1} (-z)$$

$$= -z \sum_{k=-\infty}^{\infty} f(k) \{-k z^{-k-1}\}$$

$$= -z \sum_{k=-\infty}^{\infty} f(k) \left(\frac{d}{dz} z^{-k}\right) = -z \frac{d}{dz} \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$

$$\therefore \quad Z\{k f(k)\} = -z \frac{d}{dz} F(z)$$

$\therefore \quad Z\{k^2 f(k)\} = Z\{k \,.\, k f(k)\} = \left(-z\dfrac{d}{dz}\right)\left(-z\dfrac{d}{dz}F(z)\right)$

$$= \left(-z\frac{d}{dz}\right)^2 F(z)$$

$\therefore$ **On generalizing, we get**

$$Z\{k^n f(k)\} = \left(-z\frac{d}{dz}\right)^n F(z)$$

Note :

$\left(-z\dfrac{d}{dz}\right)^2 \neq z^2\dfrac{d^2}{dz^2}$ but it is a repeated operator $\left(-z\dfrac{d}{dz}\right)^2 = \left(-z\dfrac{d}{dz}\right)\left(-z\dfrac{d}{dz}\right)$

Note : Let $k \geq 0$ and let $f(k) = 1$.

$$Z\{f(k)\} = F(z) = \sum_{k=-\infty}^{\infty} 1 \,.\, z^{-k}$$

$$Z\{1\} = 1 + z^{-1} + z^{-2} + \ldots\ldots$$

$$= \frac{1}{1-z^{-1}} \qquad |z^{-1}| < 1$$

$$Z\{1\} = (1-z^{-1})^{-1} \qquad |z| > 1.$$

$\therefore$

$$Z\{k\} = Z\{k \,.\, 1\} = \left(-z\frac{d}{dz}\right) F(z)$$

$$= \left(-z\frac{d}{dz}\right)\left[(1-z^{-1})^{-1}\right] = -z\left\{-1 \,.\, (1-z^{-1})^{-2}\right\} \times z^{-2}$$

$$Z\{k\} = z^{-1}(1-z^{-1})^{-2}, \quad |z| > 1$$

Similarly,

$$Z\{k^n\}\ (k \geq 0) = \left(-z\frac{d}{dz}\right)^n (1-z^{-1})^{-1}, \quad |z| > 1$$

7. Division by k :

If $Z\{f(k)\} = F(z)$ then $Z\left\{\dfrac{f(k)}{k}\right\} = -\displaystyle\int^{z} z^{-1} F(z)\, dz.$

Proof : We have,

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

$$\therefore \quad Z\left\{\frac{f(k)}{k}\right\} = \sum_{k=-\infty}^{\infty} \frac{f(k)}{k}\, z^{-k}$$

As $\int z^{-k-1}\, dz = \frac{z^{-k}}{-k}$

we have, $Z\left\{\frac{f(k)}{k}\right\} = -\sum_{k=-\infty}^{\infty} f(k)\, \frac{z^{-k}}{-k}$

$$= -\sum_{k=-\infty}^{\infty} f(k) \int^{z} z^{-k-1}\, dz$$

$$= -\sum_{k=-\infty}^{\infty} \int^{z} f(k)\, z^{-k}\, z^{-1}\, dz$$

$$= -\int^{z} z^{-1}\left(\sum_{k=-\infty}^{\infty} f(k)\, z^{-k}\right) dz$$

$$Z\left[\left\{\frac{f(k)}{k}\right\}\right] = -\int^{z} z^{-1}\, F(z)\, dz.$$

8. Initial Value Theorem (One sided sequence) :

If $Z\{f(k)\} = F(z)$ then $f(0) = \lim_{z\to\infty} F(z)$.

Proof : We have,

$$F(z) = Z\{f(k)\} = \sum_{k=0}^{\infty} f(k)\, z^{-k}$$

$\therefore$ $F(z) = f(0) + f(1)\, z^{-1} + f(2)\, z^{-2} + f(3)\, z^{-3} + \ldots\ldots$

$\therefore$ $\lim_{z\to\infty} F(z) = \lim_{z\to\infty} [f(0) + f(1)\, z^{-1} + f(2)\, z^{-2} + f(3)\, z^{-3} + \ldots\ldots]$

As $\lim_{z\to\infty} z^{-n} = 0$

$\therefore$ R.H.S. = f (0) + all vanishing terms

$\therefore$ $f(0) = \lim_{z\to\infty} F(z).$

9. Final Value Theorem (One sided sequence) :

$$\lim_{k \to \infty} \{f(k)\} = \lim_{z \to 1} (z-1)\ F(z), \text{ if limit exists.}$$

Proof : We have,

$$Z\{[f(k+1) - f(k)]\} = \sum_{k=0}^{\infty} [f(k+1) - f(k)]\ z^{-k}$$

$$\therefore Z\{f(k+1)\} - Z\{f(k)\} = \sum_{k=0}^{\infty} [f(k+1) - f(k)]\ z^{-k} \quad \ldots (A)$$

For causal sequence, we have

$$Z\{f(k+n)\} = z^n\ F(z) - \sum_{r=0}^{n-1} f(r)\ z^{n-r}$$

$\therefore$ For $n = 1$, $Z\{f(k+1)\} = z\,F(z) - f(0)$.

$\therefore$ From equation (A),

$$z\,F(z) - f(0) - F(z) = \lim_{n \to \infty} \sum_{k=0}^{n} [f(k+1) - f(k)]\ z^{-k}$$

$$\therefore \quad \lim_{z \to 1} (z-1)\ F(z) = f(0) + \lim_{z \to 1} \lim_{n \to \infty} \sum_{k=0}^{n} [f(k+1) - f(k)]\ z^{-k}$$

$$= f(0) + \lim_{n \to \infty} \sum_{k=0}^{n} [f(k+1) - f(k)] \lim_{z \to 1} z^{-k}$$

$$= f(0) + \lim_{n \to \infty} [f(1) - f(0) + f(2) - f(1) + f(3) - f(2) + \ldots + f(n+1) - f(n)]$$

$$= \lim_{n \to \infty} [f(0) + f(1) - f(0) + f(2) - f(1) + f(3) - f(2) + \ldots + f(n+1) - f(n)]$$

$$= \lim_{n \to \infty} f(n+1) = \lim_{k \to \infty} f(k)$$

For $k = n + 1$, when $n \to \infty$, $k \to \infty$

$$\therefore \quad \lim_{z \to 1} (z-1)\ F(z) = \lim_{k \to \infty} f(k).$$

10. Partial Sum :

If $Z\{f(k)\} = F(z)$ then $Z\left[\left\{\sum_{m=-\infty}^{k} f(m)\right\}\right] = \dfrac{F(z)}{1-z^{-1}}$

Proof : Form $\{g(k)\}$ such that $g(k) = \sum_{m=-\infty}^{k} f(m)$.

Hence we have to obtain $Z[\{g(k)\}]$.

We have, $$g(k) - g(k-1) = \sum_{m=-\infty}^{k} f(m) - \sum_{m=-\infty}^{k-1} f(m) = f(k)$$

$$\therefore \quad Z[\{g(k) - g(k-1)\}] = Z[\{f(k)\}] = F(z)$$

$$\therefore \; Z[\{g(k)\}] - Z[\{g(k-1)\}] = F(z)$$

$$\therefore \quad G(z) - z^{-1} G(z) = F(z) \Rightarrow (1 - z^{-1})\; G(z) = F(z)$$

$$\therefore \quad \sum_{m=-\infty}^{k} f(m) = G(z) = \frac{F(z)}{1-z^{-1}}$$

Alternative :

$$Z\left[\left\{\sum_{m=-\infty}^{k} f(m)\right\}\right] = \sum_{k=-\infty}^{\infty}\left[\sum_{m=-\infty}^{k} f(m)\right] z^{-k}$$

$$= \sum_{k=-\infty}^{\infty} [\ldots + f(k-3)\, z^{-k} + f(k-2)\, z^{-k} + f(k-1)\, z^{-k} + f(k)\, z^{-k}]$$

$$= \sum_{k=-\infty}^{\infty} [\ldots + f(k-3)\, z^{-(k-3)} z^{-3} + f(k-2)\, z^{-(k-2)} . z^{-2}$$
$$+ f(k-1)\, z^{-(k-1)} z^{-1} + f(k)\, z^{-k}]$$

$$= \sum_{k=-\infty}^{\infty} \sum_{r=0}^{\infty} f(k-r)\, z^{-(k-r)} z^{-r}$$

$$= \sum_{r=0}^{\infty} z^{-r} \sum_{k=-\infty}^{\infty} f(k-r)\, z^{-(k-r)}, \quad \text{(let } k - r = p\text{)}$$

$$= \sum_{r=0}^{\infty} z^{-r} \sum_{p=-\infty}^{\infty} f(p)\, z^{-p} = \sum_{r=0}^{\infty} F(z)\, z^{-r}$$

$$= F(z) \sum_{r=0}^{\infty} z^{-r}$$

$$= F(z)\,(1 + z^{-1} + z^{-2} + \ldots)$$

$$= F(z) \frac{1}{1 - z^{-1}}, \quad |z^{-1}| < 1$$

$$Z\left[\left\{\sum_{m=-\infty}^{k} f(m)\right\}\right] = \frac{F(z)}{1 - z^{-1}}, \quad |z| > 1.$$

Remark : $\lim_{k \to \infty} g(k) = \lim_{k \to \infty} \sum_{m=-\infty}^{k} f(m) = \sum_{m=-\infty}^{k} f(m).$

By final value theorem,

$$\lim_{k \to \infty} g(k) = \lim_{z \to 1} (z - 1) \left(\frac{F(z)}{1 - z^{-1}}\right) \text{ (by using property 10) .}$$

$$= \lim_{z \to 1} (z - 1) \frac{F(z)}{z - 1} \cdot z = F(1).$$

$$\therefore \quad \boxed{\sum_{m=-\infty}^{\infty} f(m) = F(1)}$$

11. Convolution :

I. General Case

Convolution of two sequences $\{f(k)\}$ and $\{g(k)\}$ denoted as $\{f(k)\} * \{g(k)\}$, is defined as :

$$\{h(k)\} = \{f(k)\} * \{g(k)\}$$

where $h(k) = \sum_{m=-\infty}^{\infty} f(m)\, g(k - m)$ (Replacing dummy index m by k – m)

$$= \sum_{m=-\infty}^{\infty} g(m)\, f(k - m)$$

$$= \{g(k)\} * \{f(k)\}$$

Taking Z-transform of both sides, we get

$$Z[\{h(k)\}] = \sum_{k=-\infty}^{\infty} \left[\sum_{m=-\infty}^{\infty} f(m)\ g(k-m) \right] z^{-k}$$

Since the power series converges absolutely, it converges uniformly also within the ROC, this allows us to interchange the order of summation, we get

$$Z[\{h(k)\}] = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{k} f(m)\ g(k-m)\ z^{-k}$$

$$= \sum_{m=-\infty}^{\infty} f(m)\ z^{-m} \sum_{k=-\infty}^{\infty} g(k-m)\ z^{-(k-m)}$$

$$= \left[\sum_{m=-\infty}^{\infty} f(m)\ z^{-m} \right] G(z)$$

$$H(z) = F(z)\ G(z)$$

ROC of H (z) is common region of convergence of F (z) and G (z).

We have $\{f(k)\} * \{g(k)\} \leftrightarrow F(z)\ G(z)$.

II. Convolution of Causal Sequences

In this case, f (k) and g (k) are zero for negative values of k, due to this

$$h(k) = \sum_{m=-\infty}^{\infty} f(m)\ g(k-m) \text{ becomes}$$

$$= \sum_{m=0}^{k} f(m)\ g(k-m)$$

Because for negative values of m, f (m) is zero and for values of m > k, g (k – m) becomes zero.

The Z-transform of $\{h(k)\} = Z[\{f(k)\} * \{g(k)\}]$

$$= F(z) \cdot G(z)$$

remains unchanged.

4.8 Z-TRANSFORM OF SOME STANDARD SEQUENCES

1. **Unit Impulse :**

$$\delta(k) = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

$$\therefore\ Z\{\delta(k)\} = \sum_{k=-\infty}^{\infty} \delta(k)\, z^{-k} = \sum_{k=-\infty}^{\infty} (0 + 0 + 0 \ldots + 1 + 0 + 0 \ \ldots)\, z^{-k}$$

$$\therefore\ Z\{\delta(k)\} = 1 \text{ as } z^{-k} = z^0 = 1 \quad \text{for } k = 0.$$

2. **Discrete Unit Step :**

$$U(k) = \begin{cases} 0, & k < 0 \\ 1, & k \geq 0 \end{cases}$$

$$\therefore \quad Z\{U(k)\} = \sum_{k=-\infty}^{\infty} U(k) \cdot z^{-k} = \sum_{k=-\infty}^{-1} 0 \,.\, z^{-k} + \sum_{k=0}^{\infty} 1\,(z^{-k})$$

$$= \left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} \ \ldots\ldots\right) + \text{an infinite G.P.}$$

$$= \frac{1}{1 - \frac{1}{z}} \qquad \text{for } \left|\frac{1}{z}\right| < 1$$

$$= \frac{z}{z-1} \qquad \text{for } 1 < |z|$$

$$\therefore \quad Z\{U(k)\} = \frac{z}{z-1} \qquad \text{for } |z| > 1$$

$$\therefore \quad Z^{-1}\left\{\frac{z}{z-1}\right\} = \{U(k)\} \qquad \text{for } |z| > 1.$$

3. **$f(k) = a^k$, $k \geq 0$**

$$Z\{f(k)\} = \sum_{k=0}^{\infty} f(k)\, z^{-k} = \sum_{k=0}^{\infty} a^k\, z^{-k} = \sum_{k=0}^{\infty} (a\, z^{-1})^k$$

$$= 1 + a\,z^{-1} + (a\,z^{-1})^2 + \ldots \text{ an infinite G.P.}$$

$$= \frac{1}{1 - az^{-1}} \qquad \text{provided } |a\,z^{-1}| < 1$$

$$= \frac{z}{z-a} \qquad \text{if } |a| < |Z|$$

$$\therefore \quad Z\{a^k\} = \frac{z}{z-a} \qquad \text{for } |z| > |a|$$

$$\therefore \quad Z^{-1}\left[\frac{z}{z-a}\right] = a^k, \qquad k \geq 0 \text{ provided } |z| > |a|$$

4. $f(k) = a^k,\ k < 0$

$$Z\{f(k)\} = \sum_{-\infty}^{-1} f(k)\ z^{-k} = \sum_{-\infty}^{-1} a^k\ z^{-k}$$

Replacing $k \to -k, -\infty \le k \le -1 \Rightarrow \infty \ge -k \ge 1$

$$\therefore \quad Z\{f(k)\} = \sum_{1}^{\infty} a^{-k}\ z^k = \sum_{1}^{\infty} (a^{-1}\ z)^k$$

$$= a^{-1} z + (a^{-1} z)^2 + (a^{-1} z)^3 + \ldots\ldots \text{ an infinite G.P.}$$

$$= \frac{a^{-1} z}{1 - a^{-1} z} \qquad \text{provided } |a^{-1}\ z| < 1$$

$$\underset{k<0}{Z\{a^k\}} = \frac{z}{a - z} \qquad \text{for } |z| < |a|$$

$$\therefore \quad Z^{-1}\left\{\frac{z}{a - z}\right\} = a^k \qquad \text{for } k < 0 \text{ if } |z| < |a|$$

5. $f(k) = \left\{a^{|k|}\right\}$ for all k

$$\therefore \quad Z\{f(k)\} = \sum_{-\infty}^{\infty} f(k)\ z^{-k} = \sum_{-\infty}^{-1} f(k)\ z^{-k} + \sum_{0}^{\infty} f(k)\ z^{-k}$$

$$= \sum_{-\infty}^{-1} a^{|k|}\ z^{-k} + \sum_{0}^{\infty} a^{|k|}\ z^{-k}$$

$$= \sum_{1}^{\infty} a^{|-k|}\ z^{+k} + \sum_{0}^{\infty} a^k\ z^{-k} = \sum_{1}^{\infty} (a\,z)^k + \sum_{0}^{\infty} (a\,z^{-1})^k$$

$$= \underset{\text{infinite G.P.}}{[az + (az)^2 + (az)^3 + \ldots]} + \underset{\text{infinite G.P.}}{[1 + (az^{-1}) + (az^{-1})^2 + \ldots]}$$

$$= \frac{az}{1 - az} + \frac{1}{1 - az^{-1}}, \quad |az| < 1 \text{ and } |a\,z^{-1}| < 1$$

$$\therefore \quad |z| < \frac{1}{|a|} \text{ and } |a| < |z|$$

$$\therefore Z\left\{a^{|k|}\right\} = F(z) = \left(\frac{az}{1 - az} + \frac{z}{z - a}\right) \text{ for } |a| < |z| < \frac{1}{|a|}$$

6. $f(k) = \cos \alpha k, \ (k \geq 0)$

We have, $\cos \alpha k = \dfrac{e^{i\alpha k} + e^{-i\alpha k}}{2}$ (by Euler's formula)

$$\therefore Z\{\cos \alpha k\} = \sum_{k=0}^{\infty} \frac{(e^{i\alpha k} + e^{-i\alpha k})}{2} z^{-k}$$

$$= \frac{1}{2}\left[\sum_{k=0}^{\infty} e^{i\alpha k} z^{-k} + \sum_{k=0}^{\infty} e^{-i\alpha k} z^{-k}\right]$$

$$= \frac{1}{2}\left[\sum_{k=0}^{\infty} (e^{i\alpha} z^{-1})^k + \sum_{k=0}^{\infty} (e^{-i\alpha} z^{-1})^k\right]$$

Both are infinite G.P. $\quad \therefore \quad S_\infty = \dfrac{a}{1-r}$

$$= \frac{1}{2}\left[\frac{1}{1-e^{i\alpha} z^{-1}} + \frac{1}{1-e^{-i\alpha} z^{-1}}\right], \ |e^{i\alpha} z^{-1}| < 1 \text{ and } |e^{-i\alpha} z^{-1}| < 1$$

$$= \frac{1}{2}\left[\frac{1-e^{-i\alpha} z^{-1} + 1 - e^{i\alpha} z^{-1}}{1-e^{i\alpha} z^{-1} - e^{-i\alpha} z^{-1} + z^{-2}}\right] = \frac{1}{2}\left[\frac{2-(e^{i\alpha}+e^{-i\alpha}) z^{-1}}{1-(e^{i\alpha}+e^{-i\alpha}) z^{-1} + z^{-2}}\right]$$

$$= \frac{1}{2}\left[\frac{2-(2\cos\alpha) z^{-1}}{1-(2\cos\alpha) z^{-1} + z^{-2}}\right] = \frac{(z-\cos\alpha)/z}{(z^2 - 2z\cos\alpha + 1)/z^2}$$

$$\therefore F(z) = Z\{\cos \alpha k\} = \frac{z(z-\cos\alpha)}{z^2 - 2z\cos\alpha + 1}, \ |z| > 1.$$

Note :

$$e^{i\alpha} = \cos\alpha + i\sin\alpha, \quad e^{-i\alpha} = \cos\alpha - i\sin\alpha$$

$$|e^{i\alpha}| = |\cos\alpha + i\sin\alpha|$$

$$= \sqrt{\cos^2\alpha + \sin^2\alpha} = 1.$$

$$|e^{-i\alpha}| = 1$$

$$|e^{i\alpha} z^{-1}| < 1$$

$$|z^{-1}| < 1 \quad \text{i.e.} \quad |z| > 1$$

$$|e^{-i\alpha} z^{-1}| < 1 \Rightarrow |z| > 1.$$

$e^{i\alpha} + e^{-i\alpha} = 2\cos\alpha$
$e^{i\alpha} - e^{-i\alpha} = 2i\sin\alpha$

7. $f(k) = \{\sin \alpha k\},\ k \geq 0$

We have $\sin \alpha k = \dfrac{e^{i\alpha k} - e^{-i\alpha k}}{2i}$

$$\therefore \quad Z\{\sin \alpha k\} = \sum_{k=0}^{\infty} \frac{(e^{i\alpha k} - e^{-i\alpha k})}{2i} \cdot z^{-k}$$

$$= \frac{1}{2i}\left[\sum_{k=0}^{\infty} e^{i\alpha k} z^{-k} - \sum_{k=0}^{\infty} e^{-i\alpha k} z^{-k}\right]$$

$$= \frac{1}{2i}\left[\sum_{k=0}^{\infty} (e^{i\alpha} z^{-1})^{k} - \sum_{k=0}^{\infty} (e^{-i\alpha} z^{-1})^{k}\right]$$

Both are infinite G.P., $S_\infty = \dfrac{a}{1-r}$

$$= \frac{1}{2i}\left[\frac{1}{1 - e^{i\alpha} z^{-1}} - \frac{1}{1 - e^{-i\alpha} z^{-1}}\right],\ |e^{i\alpha} z^{-1}| < 1 \text{ and } |e^{-i\alpha} z^{-1}| < 1$$

$$= \frac{1}{2i}\left[\frac{1 - e^{-i\alpha} z^{-1} - 1 + e^{i\alpha} z^{-1}}{1 - (e^{i\alpha} + e^{-i\alpha}) z^{-1} + z^{-2}}\right],\ |z| > 1$$

$$= \frac{1}{2i}\left[\frac{\dfrac{(e^{i\alpha} - e^{-i\alpha})}{z}}{\dfrac{z^2 - (e^{i\alpha} + e^{-i\alpha}) z + 1}{z^2}}\right] = \frac{1}{2i}\ \frac{z\,(2i \sin \alpha)}{z^2 - 2z \cos \alpha + 1}$$

$$\therefore \quad F(z) = z\ \{\sin \alpha k\} = \frac{z \sin \alpha}{z^2 - 2z \cos \alpha + 1},\ |z| > 1.$$

8. $\{f(k)\} = \{\cosh \alpha k\},\ k \geq 0$

We have, $\cosh \alpha k = \dfrac{e^{\alpha k} + e^{-\alpha k}}{2}$

$$\therefore \quad Z\{\cosh k\} = \sum_{k=0}^{\infty} \frac{(e^{\alpha k} + e^{-\alpha k})}{2}\ z^{-k}$$

$$= \frac{1}{2}\left[\sum_{k=0}^{\infty} e^{\alpha k} z^{-k} + \sum_{k=0}^{\infty} e^{-\alpha k} z^{-k}\right]$$

$$= \frac{1}{2}\left[\sum_{k=0}^{\infty} (e^{\alpha} z^{-1})^{k} + \sum_{k=0}^{\infty} (e^{-\alpha} z^{-1})^{k}\right]$$

$$= \frac{1}{2}\left[\frac{1}{1-e^{\alpha} z^{-1}} + \frac{1}{1-e^{-\alpha} z^{-1}}\right], \; |e^{\alpha} z^{-1}| < 1 \text{ and } |e^{-\alpha} z^{-1}| < 1$$

$$= \frac{1}{2}\left[\frac{1-e^{-\alpha} z^{-1} + 1 - e^{\alpha} z^{-1}}{1-(e^{\alpha} + e^{-\alpha}) z^{-1} + z^{-2}}\right], |z| > \max. (|e^{\alpha}| \text{ or } |e^{-\alpha}|)$$

$$= \frac{1}{2}\left[\frac{2-(e^{\alpha}+e^{-\alpha}) z^{-1}}{1-(e^{\alpha}+e^{-\alpha}) z^{-1} + z^{-2}}\right]$$

$$= \frac{1}{2}\left[\frac{\frac{2z-2\cosh\alpha}{z}}{\frac{z^2-2z\cosh\alpha+1}{z^2}}\right] \qquad (\because e^{\alpha}+e^{-\alpha} = 2\cosh\alpha)$$

$$Z\{\cosh\alpha k\} = \frac{z(z-\cosh\alpha)}{z^2-2z\cosh\alpha+1}, \quad |z| > \max. (|e^{\alpha}| \text{ or } |e^{-\alpha}|)$$

9. $\{f(k)\} = \{\sinh\alpha k\}$, $k \geq 0$

We have, $\sinh\alpha k = \dfrac{e^{\alpha k} - e^{-\alpha k}}{2}$

$$\therefore \quad Z\{\sinh\alpha k\} = \sum_{0}^{\infty} \frac{(e^{\alpha k} - e^{-\alpha k})}{2} z^{-k}$$

$$= \frac{1}{2}\left[\sum_{0}^{\infty} (e^{\alpha} z^{-1})^{k} - \sum_{0}^{\infty} (e^{-\alpha} z^{-1})^{k}\right]$$

$$= \frac{1}{2}\left[\frac{1}{1-e^{\alpha} z^{-1}} + \frac{1}{1-e^{-\alpha} z^{-1}}\right]$$

$$= \frac{1}{2}\left[\frac{1-e^{-\alpha} z^{-1} - 1 + e^{\alpha} z^{-1}}{1-(e^{\alpha}+e^{-\alpha}) z^{-1} + z^{-2}}\right]$$

$$= \frac{1}{2}\left[\frac{(e^{\alpha}-e^{-\alpha}) z^{-1}}{1-(e^{\alpha}+e^{-\alpha}) z^{-1} + z^{-2}}\right]$$

$$= \frac{1}{2}\left[\frac{\frac{2\sinh\alpha}{z}}{\frac{z^2-2z\cosh\alpha+1}{z^2}}\right] \qquad (\because e^{\alpha}-e^{-\alpha} = 2\sinh\alpha)$$

$$\therefore F(z) = Z\{\sinh\alpha k\} = \frac{z\sinh\alpha}{z^2-2z\cosh\alpha+1}, \quad |z| > \max. (|e^{\alpha}| \text{ or } |e^{-\alpha}|)$$

10. $f(k) = \{{}^nC_k\}$, $(0 \le k \le n)$

Since ${}^nC_k = 0$ if $k > n$ i.e. $0 \le k \le n$

$$\therefore \quad Z\{{}^nC_k\} = \sum_{k=0}^{\infty} {}^nC_k\, z^{-k} = {}^nC_0 + {}^nC_1\, z^{-1} + {}^nC_2\, z^{-2} + \ldots$$

$$= (1 + z^{-1})^n$$

$$\therefore \quad Z\{{}^nC_k\} = (1 + z^{-1})^n, \; |z| > 0$$

11. $f(k) = {}^kC_n$ $(k \ge n)$

Since ${}^kC_n = 0$ if $k < n$

$$\therefore \quad Z\{{}^kC_n\} = \sum_{k=n}^{\infty} {}^kC_n\, z^{-k}$$

Put $k = n + r$

$\therefore$ When $k = n$, $r = \infty$

$k = \infty$, $r = \infty$

$$\therefore \quad Z\{{}^kC_n\} = \sum_{r=0}^{\infty} {}^{n+r}C_n\, z^{-(n+r)}$$

As ${}^nC_r = {}^nC_{n-r}$, it follows that ${}^{n+r}C_n = {}^{n+r}C_r$

$$\therefore \quad Z\{{}^kC_n\} = \sum_{r=0}^{\infty} {}^{n+r}C_r\, z^{-(n+r)}$$

$$= \sum_{r=0}^{\infty} {}^{n+r}C_r \cdot z^{-r} \cdot z^{-n}$$

$$= z^{-n}\left[{}^nC_0 + {}^{n+1}C_1\, z^{-1} + {}^{n+1}C_2\, z^{-2} \ldots\ldots\right]$$

$$Z\{{}^kC_n\} = z^{-n}(1 - z^{-1})^{-(n+1)}, \; |z| > 1.$$

12. $\{f(k)\} = \{{}^{(k+n)}C_n\}$

$$Z\{{}^{(k+n)}C_n\} = \sum_{k=-\infty}^{\infty} {}^{(k+n)}C_n\, z^{-k}$$

${}^{(k+n)}C_n = 0$ if $k + n < n$ i.e. if $k < 0$

$$\therefore \quad Z\left[\left\{ {}^{(k+n)}C_n \right\}\right] = \sum_{k=0}^{\infty} {}^{(k+n)}C_n \, z^{-k}$$

$$= \sum_{k=0}^{\infty} {}^{(k+n)}C_k \, z^{-k}$$

$$= \left[1 + \frac{n+1}{1} z^{-1} + \frac{(n+2)(n+1)}{1.2} z^{-2} + \ldots\right]$$

$$= (1 - z^{-1})^{-(n+1)}, \qquad |z| > 1$$

$$\therefore \quad Z\left[\left\{ {}^{(k+n)}C_n \right\}\right] (k \geq 0) \leftrightarrow (1 - z^{-1})^{-(n+1)}, \qquad |z| > 1.$$

13. $\{f(k)\} = \left\{ {}^{k+n}C_n \, a^k \right\}$

$$Z\left[\left\{ {}^{k+n}C_n \, a^k \right\}\right] = \sum_{k=0}^{\infty} {}^{k+n}C_n \, (az^{-1})^k$$

$$= (1 - az^{-1})^{-(n+1)}, \qquad |z| > |a|$$

$\therefore$ By putting $n = 1$, we have

$$Z\left[\{(k+1)\, a^k\}\right] = (1 - az^{-1})^{-2} = \frac{z^2}{(z-a)^2}, \qquad |z| > |a|$$

By putting $n = 2$, we have

$$Z\left[\left\{\frac{(k+1)\,(k+2)}{2!} a^k\right\}\right] = Z\left[\left\{ {}^{k+2}C_2 \, a^k \right\}\right] = (1 - az^{-1})^{-3}$$

$$= \frac{z^3}{(z-a)^3} \qquad |z| > |a|$$

By putting $n - 1$ in place of n, we have

$$Z\left[\left\{\frac{(k+1)\ldots(k+n-1)}{(n-1)\,!}\right\}\right] = (1 - az^{-1})^{-n}, \qquad |z| > |a|$$

$$= \frac{z^n}{(z-a)^n}, \qquad |z| > |a|$$

$$\therefore \qquad [(k+1)\, a^k] \leftrightarrow \frac{z^2}{(z-a)^2}, \qquad |z| > |a|$$

$$\left\{\frac{(k+1)\,(k+2)}{2!} a^k\right\} \leftrightarrow \frac{z^3}{(z-a)^3}, \qquad |z| > |a|$$

$$\left\{\frac{(k+1)\,(k+2)\ldots\left(k+\overline{n-1}\right)}{(n-1)\,!}\right\} \leftrightarrow \frac{z^n}{(z-a)^n}, \qquad |z| > |a|$$

These results are very useful in obtaining inverse Z-transform.

14. $\{f(k)\} = \left\{\dfrac{a^k}{k!}\right\}, k \geq 0$

$$Z\{f(k)\} = Z\left\{\frac{a^k}{k!}\right\} = \sum_{k=0}^{\infty} \frac{a^k}{k!} z^{-k}$$

$$= \sum_{k=0}^{\infty} \frac{(a z^{-1})^k}{k!}$$

$$= 1 + \frac{(az^{-1})}{1!} + \frac{(az^{-1})^2}{2!} + \frac{(az^{-1})^3}{3!} + \ldots\ldots\ldots\ldots$$

$$= e^{(az^{-1})} = e^{a/z}$$

$$\therefore \quad Z\left\{\frac{a^k}{k!}\right\} = e^{a/z} \qquad \left[\text{Since, } e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots\right.$$

$$\left. e^{az^{-1}} = 1 + az^{-1} + \frac{(az^{-1})^2}{2!} + \frac{(az^{-1})^3}{3!} + \ldots\ldots\ldots\ldots \right]$$

15. $\{f(k)\} = \{c^k \cos \alpha k\}, k \geq 0$

We have

$$Z\{\cos \alpha k\} = \frac{z(z - \cos \alpha)}{z^2 - 2z\cos\alpha + 1}, \quad |z| > 1$$

$$= F(z)$$

By using change of scale property,

$$Z\{f(k)\} = F(z) \text{ then } Z\{c^k f(k)\} = F\left(\frac{z}{c}\right)$$

$$\therefore \quad Z\{c^k \cos \alpha k\} = \frac{\frac{z}{c}\left(\frac{z}{c} - \cos\alpha\right)}{\left(\frac{z}{c}\right)^2 - 2\left(\frac{z}{c}\right)\cos\alpha + 1}, \quad \text{provided } \left|\frac{z}{c}\right| > 1$$

$$= \frac{z(z - c\cos\alpha)}{z^2 - 2cz\cos\alpha + c^2}, \quad \text{provided } |z| > |c|$$

16. $\{f(k)\} = \{c^k \sin \alpha k\},\ k \geq 0$

$$Z\{\sin \alpha k\} = \frac{z \sin \alpha}{z^2 - 2z\cos\alpha + 1}, \qquad |z| > 1$$

$$\therefore \quad Z\{c^k \sin \alpha k\} = \frac{\left(\frac{z}{c}\right)\sin\alpha}{\left(\frac{z}{c}\right)^2 - 2\left(\frac{z}{c}\right)\cos\alpha + 1}, \qquad \left|\frac{z}{c}\right| > 1$$

$$= \frac{c\,z \sin\alpha}{z^2 - 2\,c\,z\cos\alpha + c^2}, \qquad |z| > |c|$$

17. $\{f(k)\} = \{c^k \cosh \alpha k\},\ k \geq 0$

$$\because \quad Z\{\cosh \alpha k\} = \frac{z(z - \cosh\alpha)}{z^2 - 2z\cosh\alpha + 1}$$

provided $|z| > \max.\left(|e^{\alpha}| \text{ or } |e^{-\alpha}|\right)$

$$Z\{c^k \cosh \alpha k\} = \frac{\frac{z}{c}\left(\frac{z}{c} - \cosh\alpha\right)}{\left(\frac{z}{c}\right)^2 - 2\left(\frac{z}{c}\right)\cosh\alpha + 1}$$

provided $|z| > \max.\left(|c\,e^{\alpha}| \text{ or } |c\,e^{-\alpha}|\right)$

$$= \frac{z(z - c\cosh\alpha)}{z^2 - 2\,c\,z\cosh\alpha + c^2}$$

18. $\{f(k)\} = \{c^k \sinh \alpha k\},\ k \geq 0$

Proceeding in the same manner as $c^k \cosh \alpha k$

$$Z\{c^k \sinh \alpha k\} = \frac{cz\sinh\alpha}{z^2 - 2\,cz\cosh\alpha + c^2}, \quad |z| > \max.\left(|c\,e^{\alpha}| \text{ or } |c\,e^{-\alpha}|\right)$$

Ex. 1 : *Find the Z – transform and its ROC of*

(i) $2^k, \quad k \geq 0,$ (ii) $3^k, \quad k < 0$

(iii) $\left(\frac{1}{3}\right)^k, \quad k \geq 0$ (iv) $\left(\frac{1}{5}\right)^k, \quad k < 0$

Sol. : (i) $f(k) = 2^k, \; k \geq 0$

$$Z\{2^k\} = \sum_{k=0}^{\infty} 2^k z^{-k} = \sum_{k=0}^{\infty} (2z^{-1})^k$$

$$= 1 + (2z^{-1}) + (2z^{-1})^2 + \ldots\ldots\ldots$$

$$= \frac{1}{1-2z^{-1}}, \text{ if } |2z^{-1}| < 1$$

$$Z\{2^k\} = \frac{z}{z-2}, \qquad |z| > 2$$

$$\{2^k\} \leftrightarrow \frac{z}{z-2}, \qquad k \geq 0$$

(ii) $f(k) = 3^k, \; k < 0$

$$Z\{3^k\} = \sum_{k=-\infty}^{-1} 3^k z^{-k}$$

Put $k = -r$

$k = -\infty \qquad r = \infty$

$k = -1 \qquad r = 1$

$$= \sum_{r=1}^{\infty} 3^{-r} z^r = \sum_{r=1}^{\infty} (3^{-1} z)^r = (3^{-1} z) + (3^{-1} z)^2 + \ldots\ldots$$

$$= \frac{3^{-1} z}{1-3^{-1} z}, \quad \text{if } |3^{-1} z| < 1$$

$$= \frac{z}{3-z}, \qquad \text{if } |z| < 3$$

$$\underset{(k<0)}{\{3^k\}} \leftrightarrow \frac{z}{3-z}$$

(iii) $f(k) = \left(\frac{1}{3}\right)^k, \; k \geq 0$

$$Z\{(1/3)^k\} = \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k z^{-k} = \sum_{k=0}^{\infty} \left(\frac{1}{3} z^{-1}\right)^k$$

$$= 1 + \left(\frac{1}{3} z^{-1}\right) + \left(\frac{1}{3} z^{-1}\right)^2 + \ldots\ldots$$

$$= \frac{1}{1-\frac{1}{3}z^{-1}}, \quad \text{if} \left|\frac{1}{3}z^{-1}\right| < 1$$

$$= \frac{z}{z-\frac{1}{3}}, \quad \text{if} |z| > \frac{1}{3}$$

$$\left(\frac{1}{3}\right)^k \leftrightarrow \frac{z}{z-\frac{1}{3}}, \quad k \geq 0$$

(iv) $f(k) = \left(\frac{1}{5}\right)^k, \; k < 0$

$$Z\left\{\left(\frac{1}{5}\right)^k\right\} = \sum_{k=-\infty}^{-1} \left(\frac{1}{5}\right)^k z^{-k}$$

Put $k = -r$

$$= \sum_{r=1}^{\infty} \left(\frac{1}{5}\right)^{-r} z^r = \sum_{r=1}^{\infty} \left[\left(\frac{1}{5}\right)^{-1} z\right]^r$$

$$= \sum_{r=1}^{\infty} (5z)^r = 5z + (5z)^2 + \ldots\ldots$$

$$= \frac{5z}{1-5z}, \quad |5z| < 1$$

$$\left(\frac{1}{5}\right)^k \leftrightarrow \frac{5z}{1-5z}, \quad |z| < \frac{1}{5}$$

$(k < 0)$

Ex. 2 : *Find $Z\{f(k)\}$*

where $\quad f(k) = 3^k, \quad k < 0$

$\qquad\qquad = 2^k, \quad k \geq 0$ **(Dec. 2010)**

Sol. :

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} 3^k z^{-k} + \sum_{k=0}^{\infty} 2^k z^{-k}$$

$$= \sum_{r=1}^{\infty} 3^{-r} z^r + \sum_{k=0}^{\infty} 2^k z^{-k}$$

$$= \sum_{r=1}^{\infty} (3^{-1} z)^r + \sum_{k=0}^{\infty} (2 z^{-1})$$

$$= \frac{3^{-1} z}{1 - 3^{-1} z} + \frac{1}{1 - 2 z^{-1}}$$

provided $|3^{-1} z| < 1$ and $|2 z^{-1}| < 1$

$$F(z) = \frac{z}{3 - z} + \frac{z}{z - 2}, \quad |z| < 3 \text{ and } 2 < |z|$$

$$= \frac{z}{(3 - z)(z - 2)} \quad \text{if } 2 < |z| < 3$$

Ex. 3 : *Find Z {f (k)} if* $f(x) = \left(\frac{1}{4}\right)^{|k|}$ *for all k.* **(May 2006, 2010)**

Sol. :

$$Z\left\{\left(\frac{1}{4}\right)^{|k|}\right\} = \sum_{k=-\infty}^{\infty} \left(\frac{1}{4}\right)^{|k|} z^{-k}$$

$$= \sum_{k=-\infty}^{-1} \left(\frac{1}{4}\right)^{-k} z^{-k} + \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^{k} z^{-k}$$

$$= \sum_{r=1}^{\infty} \left(\frac{1}{4}\right)^{r} z^{r} + \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^{k} z^{-k} = \frac{\frac{1}{4} z}{1 - \frac{1}{4} z} + \frac{1}{1 - \frac{1}{4 z}}$$

provided $\left|\frac{1}{4} z\right| < 1$ and $\left|\frac{1}{4 z}\right| < 1$

$$F(z) = \frac{1}{4} \frac{z}{1 - \frac{z}{4}} + \frac{z}{z - \frac{1}{4}}, \quad \frac{1}{4} < |z| < 4$$

Ex. 4 : *Find Z {f (k)} if (i)* $f(k) = \frac{1}{k}$, $k \geq 1$, *(ii)* $f(k) = \frac{a^k}{k}$, $k \geq 1$.

Sol. : (i) $\qquad f(k) = \frac{1}{k}, \quad k \geq 1$

Assuming $f(k) = 0$ for $k \leq 0$

$$Z\{f(k)\} = Z\left\{\frac{1}{k}\right\} = \sum_{k=1}^{\infty} \frac{1}{k} z^{-k}$$

$$= z^{-1} + \frac{(z^{-1})^2}{2} + \frac{(z^{-1})^3}{3} + \ldots\ldots\ldots = -\log(1-z^{-1})$$

Applying D'Alembert's Ratio test, we find that the series is convergent if $|z| > 1$.

Note :

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots$$

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\ldots\ldots$$

(ii) $$f(k) = \frac{a^k}{k}, \quad k \geq 1$$

Assuming $f(k) = 0, \ k \leq 0$

$$Z\left\{\frac{a^k}{k}\right\} = \sum_{k=1}^{\infty} \frac{a^k}{k} z^{-k}$$

$$= a z^{-1} + \frac{(a z^{-1})^2}{2} + \frac{(a z^{-1})^3}{3} + \ldots\ldots\ldots$$

$$= -\log(1 - a z^{-1})$$

Applying D'Alembert's Ratio test, we find that the series is convergent if $|a z^{-1}| < 1$

i.e. $|a| < |z|$ or $|z| > |a|$.

Ex. 5 : *Find $Z\{f(k)\}$ where (i) $f(k) = \frac{2^k}{k!}$ $k \geq 0$* **(May 2011, 2012; Dec. 2012)**

(ii) $f(k) = e^{-ak}$, $k \geq 0$. **(May 2009, Dec. 2010)**

Sol. : (i) $$Z\left\{\frac{2^k}{k!}\right\} = \sum_{k=0}^{\infty} \frac{2^k}{k!} z^{-k} = \sum_{k=0}^{\infty} \frac{(2z^{-1})^k}{k!}$$

$$= \frac{1}{0!} + \frac{(2z^{-1})}{1!} + \frac{(2z^{-1})^2}{2!} + \ldots\ldots = e^{2z^{-1}} = e^{2/z}$$

where, ROC is all of Z-plane.

(ii)

$$Z\{e^{-ak}\} = \sum_{k=0}^{\infty} e^{-ak} z^{-k} = \sum_{k=0}^{\infty} (e^{-a} z^{-1})^k$$

$$= 1 + (e^{-a} z^{-1}) + (e^{-a} z^{-1})^2 + \ldots\ldots$$

$$= \frac{1}{1 - e^{-a} z^{-1}}, \qquad |e^{-a} z^{-1}| < 1$$

$$= \frac{z}{z - e^{-a}}, \qquad |z| > |e^{-a}|$$

Ex. 6 : ***Find Z {f (k)}, where***

$$f(k) = \begin{cases} 2^k, & k < 0 \\ \left(\frac{1}{2}\right)^k, & k = 0, 2, 4, 6, \ldots\ldots \\ \left(\frac{1}{3}\right)^k, & k = 1, 3, 5, 7, \ldots\ldots \end{cases}$$

(Dec. 2005)

Sol. : $Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k} = \sum_{k=-\infty}^{\infty} f(k) z^{-k} \sum_{k=0}^{\infty} f(k) z^{-k}$

$$= \sum_{k=-\infty}^{-1} f(k) z^{-k} + \sum_{k=0}^{2n} f(k) z^{-k} + \sum_{k=1}^{2n-1} f(k) z^{-k}$$

for n = 1, 2, 3, ...

$$= \sum_{k=-\infty}^{-1} 2^k z^{-k} + \sum_{k=0}^{2n} \left(\frac{1}{2}\right)^k z^{-k} + \sum_{k=1}^{2n-1} \left(\frac{1}{3}\right)^k z^{-k}$$

$$= \sum_{r=1}^{\infty} 2^{-r} z^r + \sum_{k=0}^{2n} \left(\frac{1}{2} z^{-1}\right)^k + \sum_{k=1}^{2n-1} \left(\frac{1}{3} z^{-1}\right)^k$$

$$= \frac{2^{-1} z}{1 - 2^{-1} z} + \left[1 + \left(\frac{1}{2} z^{-1}\right)^2 + \left(\frac{1}{2} z^{-1}\right)^4 + \ldots\ldots\ldots\right]$$

$$+ \left[\frac{1}{3} z^{-1} + \left(\frac{1}{3} z^{-1}\right)^3 + \left(\frac{1}{3} z^{-1}\right)^5 + \ldots\ldots\right]$$

$$= \frac{z}{2 - z} + \frac{1}{1 - \left(\frac{1}{2} z^{-1}\right)^2} + \frac{\frac{1}{3} z^{-1}}{1 - \left(\frac{1}{3} z^{-1}\right)^2} = \frac{z}{2 - z} + \frac{1}{1 - \frac{1}{4 z^2}} + \frac{\frac{1}{3 z}}{1 - \frac{1}{9 z^2}}$$

$$F(z) = \frac{z}{2 - z} + \frac{4z^2}{4z^2 - 1} + \frac{3z}{9z^2 - 1}$$

provided $|2^{-1} z| < 1$

$$|2^{-1}z| < 1; \quad \left|\left(\frac{1}{2}z^{-1}\right)^2\right| < 1; \left|\left(\frac{1}{3}z^{-1}\right)^2\right| < 1$$

$$|z| < 2; \frac{1}{2} < |z|; \frac{1}{3} < |z|$$

$\therefore$ ROC is $\frac{1}{2} < |z| < 2$

Ex. 7 : *Find Z {f (k)} where*

(i)
$$f(k) = \begin{cases} -\left(-\frac{1}{3}\right)^k, & k < 0 \\ \left(-\frac{1}{4}\right)^k, & k \geq 0 \end{cases}$$

(ii)
$$f(k) = 4^k + 5^k, \; k \geq 0.$$

Sol. : (i)
$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{\infty} -\left(-\frac{1}{3}\right)^k z^{-k} + \sum_{k=0}^{\infty} \left(-\frac{1}{4}\right)^k z^{-k}$$

$$= -\sum_{r=1}^{\infty} \left(-\frac{1}{3}\right)^{-r} z^r + \sum_{k=0}^{\infty} \left(-\frac{1}{4}z^{-1}\right)^k$$

$$= -\frac{\left(-\frac{1}{3}\right)^{-1} z}{1-\left(-\frac{1}{3}\right)^{-1} z} + \frac{1}{1-\left(-\frac{1}{4}z^{-1}\right)}$$

$$= \frac{3z}{1+3z} + \frac{4z}{4z+1}$$

provided $\left|\left(-\frac{1}{3}\right)^{-1} z\right| < 1$ and $\left|-\frac{1}{4}z^{-1}\right| < 1$

$$|z| < \frac{1}{3} \quad \text{and} \quad \frac{1}{4} < |z|$$

$\therefore$ ROC is $\frac{1}{4} < |z| < \frac{1}{3}$

(ii) $Z\{4^k + 5^k\} = Z\{4^k\} + Z\{5^k\}$

$$= \sum_{k=0}^{\infty} 4^k z^{-k} + \sum_{k=0}^{\infty} 5^k z^{-k}$$

$$= \frac{1}{1-4z^{-1}} + \frac{1}{1-5z^{-1}}, \quad |4z^{-1}| < 1 \text{ and } |5z^{-1}| < 1$$

$$= \frac{z}{z-4} + \frac{z}{z-5}, \quad 4 < |z| \text{ and } 5 < |z|$$

ROC is $|z| > 5$.

Ex. 8 : *Find Z {f (k)} if*

(i) $$f_k = \left(-\frac{1}{2}\right)^{k+1} + 3\left(\frac{1}{2}\right)^{k+1}, \; k \geq 0$$

(ii) $$f_k = \begin{cases} 2^k, & k \geq 0 \\ \left(\frac{1}{3}\right)^k, & k < 0 \end{cases}$$ **(Dec. 2007)**

Sol. : (i) $$f(k) = \left(-\frac{1}{2}\right)^{k+1} + 3\left(\frac{1}{2}\right)^{k+1}, \; k \geq 0$$

$$= \left(-\frac{1}{2}\right)^k \left(-\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right) = -\frac{1}{2}\left(-\frac{1}{2}\right)^k + \frac{3}{2}\left(\frac{1}{2}\right)^k.$$

$$Z\{f(k)\} = Z\left\{-\frac{1}{2}\left(-\frac{1}{2}\right)^k + \frac{3}{2}\left(\frac{1}{2}\right)^k\right\}$$

$$= -\frac{1}{2} Z\left\{\left(-\frac{1}{2}\right)^k\right\} + \frac{3}{2} \cdot Z\left\{\left(\frac{1}{2}\right)^k\right\}$$

$$= -\frac{1}{2} \sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^k z^{-k} + \frac{3}{2} \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k z^{-k}$$

$$= -\frac{1}{2} \sum_{k=0}^{\infty} \left(-\frac{1}{2} z^{-1}\right)^k + \frac{3}{2} \sum_{k=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^k$$

$$= -\frac{1}{2} \frac{1}{1-\left(-\frac{1}{2}z^{-1}\right)} + \frac{3}{2} \frac{1}{1-\frac{1}{2}z^{-1}}$$

$$= -\frac{1}{2} \cdot \left(\frac{z}{z+\frac{1}{2}}\right) + \frac{3}{2}\left(\frac{z}{z-\frac{1}{2}}\right)$$

provided $\left|-\frac{1}{2}z^{-1}\right| < 1$ and $\left|\frac{1}{2}z^{-1}\right| < 1$

$$\frac{1}{2} < |z| \text{ or } |z| > \frac{1}{2}.$$

$\therefore$ ROC is $|z| > \frac{1}{2}$.

(ii) $f_k = \begin{cases} 2^k, & k \geq 0 \\ \left(\frac{1}{3}\right)^k, & k < 0 \end{cases}$

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\,z^{-k} = \sum_{k=-\infty}^{-1}\left(\frac{1}{3}\right)^k z^{-k} + \sum_{k=0}^{\infty} 2^k\,z^{-k}$$

$$= \sum_{r=1}^{\infty}\left(\frac{1}{3}\right)^{-r} z^r + \sum_{k=0}^{\infty}(2z^{-1})^k$$

$$= \frac{\left(\frac{1}{3}\right)^{-1} z}{1-\left(\frac{1}{3}\right)^{-1} z} + \frac{1}{1-2z^{-1}} = \frac{3z}{1-3z} + \frac{z}{z-2}$$

provided $\left|\left(\frac{1}{3}\right)^{-1} z\right| < 1$ and $|2z^{-1}| < 1$

$$|z| < \frac{1}{3} \text{ and } 2 < |z|$$

$\therefore$ ROC is $2 < |z| < \frac{1}{3}$.

Ex. 9 : *Find $Z\{f(k)\}$ if $f(k) = \left(\frac{1}{2}\right)^{|k|}$ for all k.* **(May 2005)**

Sol. :

$$Z\{f(k)\} = \sum_{k=-\infty}^{-1}\left(\frac{1}{2}\right)^{|k|} z^{-k} + \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^{|k|} z^{-k}$$

$$= \sum_{k=-\infty}^{-1}\left(\frac{1}{2}\right)^{-k} z^{-k} + \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^{k} z^{-k}$$

$$= \sum_{r=1}^{\infty} \left(\frac{1}{2}\right)^r z^r + \sum_{k=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^k$$

$$= \frac{\frac{1}{2} z}{1 - \frac{1}{2} z} + \frac{1}{1 - \frac{1}{2} z^{-1}} = \frac{z}{2 - z} + \frac{z}{z - \frac{1}{2}}$$

provided $\left|\frac{1}{2} z\right| < 1$ and $\left|\frac{1}{2} z^{-1}\right| < 1$

i.e. $|z| < 2$ and $\frac{1}{2} < |z|$

$\therefore$ ROC is $\frac{1}{2} < |z| < 2$.

Ex. 10 : *Find Z {f (k)} if f (k) = a cos kα + b sin kα, k ≥ 0.*

Sol. : $Z\{a \cos \alpha k + b \sin \alpha k\} = a Z\{\cos \alpha k\} + bZ\{\sin \alpha k\}$

(by using linearity property)

$$= a \cdot \frac{z(z - \cos \alpha)}{z^2 - 2z \cos \alpha + 1} + b \cdot \frac{z \sin \alpha}{z^2 - 2z \cos \alpha + 1}, \ |z| > 1$$

$$= \frac{a z^2 + z(b \sin \alpha - a \cos \alpha)}{z^2 - 2z \cos \alpha + 1}, \ |z| > 1$$

Ex. 11 : *Find Z {f (k)} if*

(i) $f(k) = \frac{\sin ak}{k}$, $k > 0$ **(May 2005, 2012; Dec. 2010, 2012)**

(ii) $f(k) = \frac{2^k}{k}$, $k \geq 1$ **(Dec. 2006)**

Sol. : (i) $f(k) = \frac{\sin ak}{k}$, $k > 0$

$$Z\{\sin ak\} = \frac{z \sin a}{z^2 - 2z \cos a + 1}$$

$$Z\left\{\frac{\sin ak}{k}\right\} = \int_z^{\infty} \frac{1}{z} \frac{z \sin a}{z^2 - 2z \cos a + 1} dz = \int_z^{\infty} \frac{\sin a}{z^2 - 2z \cos a + 1} dz$$

$$= \sin a \int_z^{\infty} \frac{dz}{(z - \cos a)^2 + \sin^2 a}$$

$$= \sin a \left[\frac{1}{\sin a} \tan^{-1}\left(\frac{z - \cos a}{\sin a}\right)\right]_z^{\infty}$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{z - \cos a}{\sin a}\right) = \cot^{-1}\left(\frac{z - \cos a}{\sin a}\right).$$

(ii) $$f(k) = \frac{2^k}{k}, \quad k \geq 1$$

$$Z\{2^k\} = \sum_{k=1}^{\infty} 2^k z^{-k} = \frac{2z^{-1}}{1 - 2z^{-1}}, \quad |z| > 2 = \frac{2}{z-2}$$

$$Z\left\{\frac{2^k}{k}\right\} = \int_z^{\infty} z^{-1} \frac{2}{z-2} dz = 2\int_z^{\infty} \frac{1}{z(z-2)} dz$$

$$= 2\int_z^{\infty} \left(-\frac{1/2}{z} + \frac{1/2}{z-2}\right) dz = \int_z^{\infty} \left(-\frac{1}{z} + \frac{1}{z-2}\right) dz$$

$$= [-\log z + \log(z-2)]_z^{\infty}$$

$$= -\log\frac{z-2}{z} = -\log(1 - 2z^{-1}), \quad |z| > 2$$

Ex. 12 : *Find $Z\{f(k)\}$ where*

(i) $f(k) = \sin\left(\frac{k\pi}{4} + \alpha\right), \quad k \geq 0$

(ii) $f(k) = \cos\left(\frac{k\pi}{4} + \alpha\right), \quad k \geq 0$ **(Dec. 2005)**

Sol. : (i) $$\sin\left(\frac{k\pi}{4} + \alpha\right) = \sin\frac{k\pi}{4} \cdot \cos\alpha + \cos\frac{k\pi}{4} \cdot \sin\alpha$$

$$Z\left\{\sin\left(\frac{k\pi}{4} + \alpha\right)\right\} = \cos\alpha \cdot Z\left\{\sin\left(\frac{k\pi}{4}\right)\right\} + \sin\alpha \cdot Z\left\{\cos\left(\frac{k\pi}{4}\right)\right\}$$

$$= \cos\alpha \cdot \frac{z\sin\frac{\pi}{4}}{z^2 - 2z\cos\frac{\pi}{4} + 1} + \sin\alpha \cdot \frac{z\left(z - \cos\frac{\pi}{4}\right)}{z^2 - 2z\cos\frac{\pi}{4} + 1}$$

$$= \frac{\cos\alpha \dfrac{z}{\sqrt{2}}}{z^2 - \dfrac{2z}{\sqrt{2}} + 1} + \sin\alpha \frac{z\left(z - \dfrac{1}{\sqrt{2}}\right)}{z^2 - \dfrac{2z}{\sqrt{2}} + 1}$$

$$= \frac{z}{\sqrt{2}}\left[\frac{\cos\alpha + \sin\alpha\,(\sqrt{2}\,z - 1)}{z^2 - \sqrt{2}\,z + 1}\right],\ |z| > 1$$

(ii) $\cos\left(\frac{k\pi}{4} + \alpha\right) = \cos\frac{k\pi}{4}\cos\alpha - \sin\frac{k\pi}{4}\sin\alpha.$

$$Z\left\{\cos\left(\frac{k\pi}{4} + \alpha\right)\right\} = \cos\alpha \cdot Z\left\{\cos\frac{k\pi}{4}\right\} + \sin\alpha \cdot Z\left\{\sin\frac{k\pi}{4}\right\}$$

$$= \cos\alpha \cdot \frac{z\left(z - \cos\dfrac{\pi}{4}\right)}{z^2 - 2z\cos\dfrac{\pi}{4} + 1} - \sin\alpha \cdot \frac{z\sin\dfrac{\pi}{4}}{z^2 - 2z\cos\dfrac{\pi}{4} + 1}$$

$$= \frac{\cos\alpha z\left(z - \dfrac{1}{\sqrt{2}}\right)}{z^2 - \dfrac{2z}{\sqrt{2}} + 1} - \frac{\sin\alpha\,.\,(z/\sqrt{2})}{z^2 - \dfrac{2z}{\sqrt{2}} + 1}$$

$$= \frac{z}{\sqrt{2}}\left[\frac{\cos\alpha\,(\sqrt{2}\,z - 1) - \sin\alpha}{z^2 - \sqrt{2}\,z + 1}\right]$$

Ex. 13 : *Find $Z\{f(k)\}$ if*

(i) $f(k) = e^{-ak}\cos bk,\ k \geq 0$

(ii) $f(k) = e^{-ak}\sin bk,\ k \geq 0$

(iii) $f(k) = e^{-3k}\cos 4k,\ k \geq 0$ **(May 2011)**

Sol. : Here we will make use of property No. 4 i.e.

if $Z\{f(k)\} = F(z)$ then $Z\{e^{-ak} f(k)\} = F(e^a z)$

i.e. replace z by e^a z.

(i) $$Z\{\cos bk\} = \frac{z(z - \cos b)}{z^2 - 2z\cos b + 1}$$

$$Z\{e^{-ak}\cos bk\} = \frac{(e^a z)(e^a z - \cos b)}{(e^a z)^2 - 2e^a z\cos b + 1} = \frac{z(z - e^{-a}\cos b)}{z^2 - (2e^{-a}\cos b)z + e^{-2a}}$$

(ii) $$Z\{\sin bk\} = \frac{z \sin b}{z^2 - 2z\cos b + 1}$$

$$Z\{e^{-ak}\sin bk\} = \frac{(e^a z)\sin b}{(e^a z)^2 - 2(e^a z)\cos b + 1} = \frac{z e^{-a}\sin b}{z^2 - 2e^{-a}\cos bz + e^{-2a}}$$

(iii) Left as an exercise [refer part (i)].

Ex. 14 : *Find $Z\{f(k)\}$ if*

(i) $f(k) = 2^k \cos(3k+2), \quad k \geq 0$

(ii) $f(k) = 4^k \sin(2k+3), \quad k \geq 0$ **(May 2006)**

(iii) $f(k) = 3^k \sinh \alpha k, \quad k \geq 0$ **(May 2010)**

(iv) $f(k) = 2^k \cosh \alpha k, \quad k \geq 0$ **(Dec. 2009, May 2009)**

Sol. : Here we will use property No. 3 (change of scale) i.e.

if $Z\{f(k)\} = F(z)$ then $Z\{a^k f(k)\} = F\left(\frac{z}{a}\right)$

(i) $$\cos(3k+2) = \cos 3k \cos 2 - \sin 3k \sin 2$$

$$Z\{\cos(3k+2)\} = \cos 2 Z\{\cos 3k\} - \sin 2 Z\{\sin 3k\}$$

$$= \cos 2 \frac{z(z-\cos 3)}{z^2 - 2z\cos 3 + 1} - \frac{\sin 2(z\sin 3)}{z^2 - 2z\cos 3 + 1}$$

$$= \frac{z[z.\cos 2 - (\cos 3 . \cos 2 + \sin 3 . \sin 2)]}{z^2 - 2z\cos 3 + 1}$$

$$= \frac{z[z\cos 2 - \cos(3-2)]}{z^2 - 2z\cos 3 + 1} = \frac{z(z\cos 2 - \cos 1)}{z^2 - 2z\cos 3 + 1}$$

$$Z\{2^k\cos(3k+2)\} = \frac{\frac{z}{2}\left(\frac{z}{2}\cos 2 - \cos 1\right)}{\left(\frac{z}{2}\right)^2 - 2.\frac{z}{2}.\cos 3 + 1} = \frac{z(z\cos 2 - 2\cos 1)}{z^2 - 4z\cos 3 + 4}$$

(ii) $$Z\{\sin(2k+3)\} = \cos 3 Z\{\sin 2k\} + \sin 3 Z\{\cos 2k\}$$

$$= \cos 3 \frac{z \sin 2}{z^2 - 2z\cos 2 + 1} + \sin 3 . \frac{z(z-\cos 2)}{z^2 - 2z\cos 2 + 1}$$

$$= \frac{z[\cos 3 \sin 2 - \sin 3\cos 2 + z\sin 3]}{z^2 - 2z\cos 2 + 1}$$

$$= \frac{z[z\sin 3 - \sin 1]}{z^2 - 2z\cos 2 + 1}$$

$$Z\{4^k \sin(2k+3)\} = \frac{\frac{z}{4}\left(\frac{z}{4}\sin 3 - \sin 1\right)}{\left(\frac{z}{4}\right)^2 - 2\frac{z}{4}\cos 2 + 1} = \frac{z(z\sin 3 - 4\sin 1)}{z^2 - 8z\cos 2 + 16}$$

(iii) $$Z\{\sinh \alpha k\} = \frac{z \sinh\alpha}{z^2 - 2z\cosh\alpha + 1}$$

$$Z\{3^k \sinh \alpha k\} = \frac{\frac{z}{3}\sinh\alpha}{\left(\frac{z}{3}\right)^2 - 2\frac{z}{3}\cosh\alpha + 1} = \frac{z\sinh\alpha}{z^2 - 6z\cosh\alpha + 9}$$

(iv) $$Z\{\cosh \alpha k\} = \frac{z(z - \cosh\alpha)}{z^2 - 2z\cosh\alpha + 1}$$

$$Z\{2^k \cosh \alpha k\} = \frac{\frac{z}{2}\left(\frac{z}{2} - \cosh\alpha\right)}{\left(\frac{z}{2}\right)^2 - 2\frac{z}{2}\cosh\alpha + 1} = \frac{z(z - 2\cosh\alpha)}{z^2 - 4z\cosh\alpha + 4}$$

Ex. 15 : *Find $Z\{f(k)\}$ if*

(i) $f(k) = k, \quad k \geq 0$

(ii) $f(k) = k\,5^k, \quad k \geq 0$

(iii) $f(k) = (k+1)\,a^k, \quad k \geq 0$ **(Dec. 2007, May 2011, Dec. 2012)**

Sol. : Here we will use property No. 6 (multiplication by k) i.e. if $Z\{f(k)\} = F(z)$ then, $Z\{k\,f(k)\} = \left(-z\frac{d}{dz}\right)(f(z))$.

(i) Let $f(k) = 1$

$$Z\{f(k)\} = Z\{1\} = \frac{z}{z-1} = (1 - z^{-1})^{-1}$$

$$\therefore \quad Z\{k\} = Z\{k\cdot 1\} = -z\frac{d}{dz}\left[(1-z^{-1})^{-1}\right]$$

$$= -z\left\{-(1-z^{-1})^{-2}\cdot z^{-2}\right\}$$

$$= \frac{z^{-1}}{(1-z^{-1})^2} = \frac{z}{(z-1)^2}$$

(ii) $$Z\{5^k\} = \frac{z}{z-5} = (1 - 5z^{-1})^{-1}$$

$$Z\{k\,5^k\} = -z\frac{d}{dz}\left[(1-5z^{-1})^{-1}\right]$$

$$= -z\left\{-(1-5z^{-1})^{-2}\,5z^{-2}\right\} = \frac{5z^{-1}}{(1-5z^{-1})^2} = \frac{5z}{(z-5)^2}$$

(iii) $$Z\{(k+1)a^k\} = Z\{k\,a^k + a^k\} = Z\{k\,a^k\} + Z\{a^k\}$$

$$= -z\frac{d}{dz}(1-az^{-1})^{-1} + \frac{z}{z-a}$$

$$= -z\left[-(1-az^{-1})^{-2}\cdot az^{-2}\right] + \frac{z}{z-a}$$

$$= \frac{a\,.\,z^{-1}}{(1-az^{-1})^2} + \frac{z}{z-a}$$

$$= \frac{a\,z}{(z-a)^2} + \frac{z}{z-a} = \frac{az+z(z-a)}{(z-a)^2} = \frac{z^2}{(z-a)^2}$$

Ex. 16 : *Find $Z\{f(k)\}$ if*

(i) $f(k) = k^2\, e^{-ak}$, $k \geq 0$ **(Dec. 2010)**

(ii) $f(k) = k^2\, a^{k-1}$, $k \geq 0$

(iii) $f(k) = k^2\, a^{k-1}$, $U(k-1)$ **(May 2006)**

Sol. : (i) $$Z\{e^{-ak}\} = \frac{z}{z-e^{-a}}$$

$$Z\{k\,e^{-ak}\} = -z\frac{d}{dz}\left[(1-e^{-a}z^{-1})^{-1}\right]$$

$$= -z\left[-(1-e^{-a}z^{-1})^{-2}e^{-a}z^{-2}\right]$$

$$= \frac{e^{-a}z^{-1}}{(1-e^{-a}z^{-1})^2} = \frac{z\,e^{-a}}{(z-e^{-a})^2}$$

$$Z\{k^2 e^{-ak}\} = Z\{k\,.\,k\,e^{-ak}\}$$

$$= \left(-z\frac{d}{dz}\right)\cdot\left(\frac{z\,e^{-a}}{(z-e^{-a})^2}\right) = -z\,e^{-a}\left\{\frac{d}{dz}\left(\frac{z}{(z-e^{-a})^2}\right)\right\}$$

$$= (-z\,e^{-a})\left\{\frac{(z-e^{-a})^2 - z\,.\,2(z-e^{-a})}{(z-e^{-a})^4}\right\}$$

$$= (-z\,e^{-a})\frac{[z-e^{-a}-2z]}{(z-e^{-a})^3} = \frac{z\,e^{-a}(e^{-a}+z)}{(z-e^{-a})^3},\quad |z| > |e^{-a}|$$

(ii) We know that if $\{f(k)\}$ is causal sequence, then

$$Z\{f(k-n)\} = z^{-n}F(z)$$

$$Z\{f(k-1)\} = z^{-1}F(z)$$

$$\therefore \quad Z\{a^k\} = \frac{z}{z-a}$$

$$Z\{a^{k-1}\} = z^{-1}\left(\frac{z}{z-a}\right) = \frac{1}{z-a}$$

$$Z\{k^2 a^{k-1}\} = \left(-z\frac{d}{dz}\right)\left(-z\frac{d}{dz}\right)\left(\frac{1}{z-a}\right) = \left(-z\frac{d}{dz}\right)\cdot(-z)\left(\frac{-1}{(z-a)^2}\right)$$

$$= -z\frac{d}{dz}\left(\frac{z}{(z-a)^2}\right) = -z\cdot\left[\frac{(z-a)^2 - z.2(z-a)}{(z-a)^4}\right]$$

$$= \frac{-z[z-a-2z]}{(z-a)^3} = \frac{z(z+a)}{(z-a)^3}, \quad |z| > |a|$$

(iii)
$$Z\{U(k)\} = \frac{z}{z-1}$$

$$Z\{a^k U(k)\} = \frac{z/a}{z/a-1} = \frac{z}{z-a}$$

$$Z\{a^{k-1}U(k-1)\} = z^{-1}\left(\frac{z}{z-a}\right) = \frac{1}{z-a}$$

$$Z\{k^2 a^{k-1}U(k-1)\} = \left(-z\frac{d}{dz}\right)^2\left(\frac{1}{z-a}\right) = \frac{z(z+a)}{(z-a)^3}, \quad |z| > |a|$$

Ex. 17 : *Find Z {f (k)} if*

(i) $f(k) = (k+1)(k+2)\,2^k,\ k \geq 0$ **(Dec. 2012)**

(ii) $f(k) = \frac{1}{2!}(k+1)(k+2)\,a^k,\ k \geq 0$ **(Dec. 2006, May 2009))**

Sol. : (i)
$$Z\{2^k\} = \frac{z}{z-2} = (1-2z^{-1})^{-1}$$

$$Z\{k\,2^k\} = -z\frac{d}{dz}\left[(1-2z^{-1})^{-1}\right]$$

$$= -z\left[-(1-2z^{-1})^{-2}(2z^{-2})\right]$$

$$= \frac{2z^{-1}}{(1-2z^{-1})^2} = 2z^{-1}(1-2z^{-1})^{-2}$$

$$\therefore \quad Z\{(k+1)\,2^k\} = Z\{k\,2^k\} + Z\{2^k\}$$

$$= 2z^{-1}(1-2z^{-1})^{-2} + (1-2z^{-1})^{-1}$$

$$= (2z^{-1} + 1 - 2z^{-1})(1-2z^{-1})^{-2}$$

$$= (1-2z^{-1})^{-2}$$

$$= -z \frac{d}{dz}(1-2z^{-1})^{-2}$$

$$= -z\left[-2(1-2z^{-1})^{-3}(2z^{-2})\right]$$

$$= 4z^{-1}(1-2z^{-1})^{-3}$$

$$Z\{(k+1)(k+2)2^k\} = Z\{k(k+1)2^k\} + Z\{2(k+1)2^k\}$$

$$= 4z^{-1}(1-2z^{-1})^{-3} + 2(1-2z^{-1})^{-2}$$

$$= (1-2z^{-1})^{-3}\left[4z^{-1} + 2(1-2z^{-1})\right]$$

$$= 2(1-2z^{-1})^{-3}$$

(ii) From (i), $Z\{(k+1)(k+2)a^k\} = 2(1-az^{-1})^{-3}$

$$\therefore \quad Z\left\{\frac{1}{2!}(k+1)(k+2)a^k\right\} = (1-az^{-1})^{-3}$$

Ex. 18 : *Find* $Z\,|x_k|$ *if*

$$x_k = \frac{1}{1^k} * \frac{1}{2^k} * \frac{1}{3^k}, \quad k \geq 0$$

(Dec. 2008)

Sol. : Let

$$A(k) = \frac{1}{k}$$

$$Z\{A(k)\} = \frac{z}{z-1}, \quad |z| > 1$$

$$B(k) = \frac{1}{2^k}$$

$$Z\{B(k)\} = \sum_{k=0}^{\infty} \frac{1}{2^k} z^{-k} = \sum_{k=0}^{\infty} (2^{-1}z^{-1})^k$$

$$= 1 + (2^{-1}z^{-1}) + (2^{-1}z^{-1})^2 + \ldots\ldots\ldots$$

$$= \frac{1}{1-2^{-1}z^{-1}}, \quad |2^{-1}z^{-1}| < 1 = \frac{2z}{2z-1}, \quad |z| > \frac{1}{2}$$

$$C(k) = \frac{1}{3^k}$$

$$Z\{C(k)\} = Z\left\{\frac{1}{3^k}\right\} = \frac{3z}{3z-1}, \quad |z| > \frac{1}{3}$$

By using convolution property,

$$Z\{x_k\} = Z\{A(k) * B(k) * C(k)\}$$

$$= Z\{A(k)\} \cdot Z\{B(k)\} \cdot Z\{C(k)\}$$

$$= \left(\frac{z}{z-1}\right)\left(\frac{2z}{2z-1}\right)\left(\frac{3z}{3z-1}\right), \quad |z| > 1.$$

Ex. 19 : *Verify convolution theorem for $f_1(k) = k$ and $f_2(k) = k$.*

Sol. :

$$Z\{k\} = Z\{k \cdot 1\} = -z \frac{d}{dz}\left(\frac{z}{z-1}\right)$$

$$Z\{f_1(k)\} = F_1(z) = \frac{z}{(z-1)^2}$$

$$\therefore \quad Z\{f_2(k)\} = F_2(z) = \frac{z}{(z-1)^2}$$

$$\therefore \quad F_1(z)\, F_2(z) = \frac{z^2}{(z-1)^4} \qquad \ldots (I)$$

$$\{F_1(k) * F_2(k)\} = \sum_{m=0}^{\infty} f_1(m)\, f_2(k-m) = \sum_{m=0}^{\infty} m(k-m)$$

$$= k \sum_{m=0}^{\infty} m - \sum_{m=0}^{\infty} m^2$$

$$= k\, \frac{k(k+1)}{2} - \frac{k(k+1)(2k+1)}{6}$$

$$= \frac{k(k+1)}{6}(3k - 2k - 1) = \frac{k}{6}(k^2 - 1)$$

$$Z\{f_1(k) * f_2(k)\} = Z\left\{\frac{k(k^2-1)}{6}\right\}$$

$$= \frac{1}{6}[Z\{k^3\} - Z\{k\}]$$

$$= \frac{1}{6}\left(-z\frac{d}{dz}\right)^3 (1-z^{-1})^{-1} - \left(-z\frac{d}{dz}\right)(1-z^{-1})^{-1}$$

$$= \frac{1}{6}\, \frac{z(z^2+4z+1)}{(z-1)^4} - \frac{z}{(z-1)^2}$$

$$= z\left[\frac{z^2 + 4z + 1 - z^2 + 2z - 1}{6(z-1)^4}\right]$$

$$= \frac{z^2}{(z-1)^4} \qquad \ldots (II)$$

From (I) and (II), convolution theorem is verified.

EXERCISE 4.1

For each of the following sequences, evaluate corresponding Z-transforms specifying ROC of the transform.

1. $f(k) = 3^k, \quad k \geq 0$ **Ans.** $\frac{z}{z-3}, \ |z| > 3$

2. $f(k) = 2, \quad k \geq 0$ **Ans.** $\frac{2z}{z-1}, \ |z| > 1$

3. $f(k) = \left(\frac{1}{3}\right)^k, \quad k \geq 0$ **Ans.** $\frac{z}{z-\frac{1}{3}}, |z| > \frac{1}{3}$

4. $f(k) = \frac{1}{3^k}, \quad k \geq 0$ **Ans.** $\frac{3z}{3z-1}, |z| > \frac{1}{3}$

5. $f(k) = 4^k, \quad k < 0$ **Ans.** $\frac{z}{4-z}, |z| < 4$

6. $f(k) = \left(\frac{1}{3}\right)^k, \quad k < 0$ **Ans.** $\frac{3z}{1-3z}, |z| < \frac{1}{3}$

7. $f(k) = 3\left(\frac{1}{4}\right)^k + 4\left(\frac{1}{5}\right)^k, \quad k \geq 0$ **Ans.** $\frac{12z}{4z-1} + \frac{20z}{5z-1}, |z| > \frac{1}{4}$

8. $f(k) = 4^k + 5^k, \quad k \geq 0$ **Ans.** $\frac{z}{z-4} + \frac{z}{z-5}, |z| > 5$

9. $f(k) = 5^k, \quad k < 0$ **Ans.** $\frac{2z}{(5-z)(z-3)}, 3 < |z| < 5$

10. $f(k) = \frac{5^k}{k}, \quad k > 1$ **Ans.** $-\log(1-5z^{-1}), |z| > 5$

$= 3^k, \quad k \geq 0$ **Ans.** $-\log(1-5z^{-1}), |z| > 5$

11. $f(k) = \left(\frac{1}{2}\right)^{|k|}$ for all k **Ans.** $\frac{z}{2-z} + \frac{2z}{2z-1}, \frac{1}{2} < |z| < 2$

12. $f(k) = 2^k + \left(\frac{1}{2}\right)^k, \quad k \geq 0$ (Dec. 2012) **Ans.** $\frac{z}{z-2} + \frac{z}{z-\frac{1}{2}}, |z| > 2$

13. $f(k) = 3^k, \quad k < 0$

$= \left(\frac{1}{3}\right)^k, \quad k = 0, 2, 4, 6, \ldots$

$= \left(\frac{1}{2}\right)^k, \quad k = 1, 3, 5, 7, 9, \ldots$ **Ans.** $\frac{z}{3-z} + \frac{9z^2}{9z^2-1} + \frac{2z}{4z^2-1}, \frac{1}{2} < |z| < 3$

14. $f(k) = \frac{3^k}{k!}, \quad k \geq 0$ **Ans.** $e^{3/z}$, ROC – z plane

15. $f(k) = e^{k\alpha}, \quad k \geq 0$ **Ans.** $\frac{z}{z-e^{\alpha}}, |z| > |e^{\alpha}|$

16. $f(k) = \cos\left(\frac{k\pi}{8} + \alpha\right), \quad k \geq 0$ **Ans.** $\frac{z^2 \cos\alpha - z\cos\left(\frac{\pi}{8} - \alpha\right)}{z^2 - 2z\cos\frac{\pi}{8} + 1}, |z| > 1$

17. $f(k) = \sin 4k, \quad k \geq 0$ Ans. $\dfrac{z \sin 4}{z^2 - 2z\cos 4 + 1}, \ |z| > 1$

18. $f(k) = \sin(3k + 5), \quad k \geq 0$ (May 08, Dec. 12)

Ans. $\dfrac{z^2 \sin 5 - z \sin 2}{z^2 - 2z\cos 3 + 1}, \ |z| > 1$

19. $f(k) = \cos(7k + 2), \ k \geq 0$ Ans. $\dfrac{z^2 \cos 2 - z \cos 5}{z^2 - 2z\cos 7 + 1}, \ |z| > 1$

20. $f(k) = \cos\left(\dfrac{k\pi}{2} + \dfrac{\pi}{4}\right), \quad k \geq 0$ Ans. $\dfrac{z^2 - z}{\sqrt{2}\,(z^2 + 1)}, \ |z| > 1$

21. $f(k) = \sin\left(\dfrac{k\pi}{2} + \alpha\right), \ k \geq 0$ Ans. $\dfrac{z^2 \sin\alpha + z\cos\alpha}{z^2 + 1}, \ |z| > 1$

22. $f(k) = \cosh\left(\dfrac{k\pi}{2}\right), \quad k \geq 0$ Ans. $\dfrac{z\left(z - \cosh\dfrac{\pi}{2}\right)}{z^2 - 2z\cosh\dfrac{\pi}{2} + 1},$

$|z| > \max.\left(|e^{\pi/2}|, |e^{-\pi/2}|\right)$

23. $f(k) = \sinh\dfrac{k\pi}{2}, \quad k \geq 0$ Ans. $\dfrac{z\sinh\dfrac{\pi}{2}}{z^2 - 2z\cosh\dfrac{\pi}{2} + 1}, \ |z| > \max.\left(|e^{\pi/2}|, |e^{-\pi/2}|\right)$

24. $f(k) = \cosh\left(\dfrac{k\pi}{2} + \alpha\right), \ k \geq 0$ Ans. $\dfrac{z^2\cosh\alpha - z\cosh\left(\dfrac{\pi}{2} - \alpha\right)}{z^2 - 2z\cosh\dfrac{\pi}{2} + 1}$

25. $f(k) = 2^k \cos(3k + 2)$ Ans. $\dfrac{z^2 \cos 2 - 2z\cos 1}{z^2 - 4z\cos 3 + 4}, \ |z| > 2$

26. $f(k) = \begin{cases} -\left(-\dfrac{1}{4}\right)^k, & k < 0 \\ \left(-\dfrac{1}{5}\right)^k, & k \geq 0 \end{cases}$ Ans. $\dfrac{4z}{4z + 1} + \dfrac{5z}{5z + 1}, \ \dfrac{1}{5} < |z| < \dfrac{1}{4}$

27. $f(k) = e^{-3k} \sin 4k, \ k \geq 0$ Ans. $\dfrac{z e^{-3} \sin 4}{z^2 - 2e^{-3} z\cos 4 + e^{-6}}, \ |z| > |e^{-3}|$

28. $f(k) = k e^{-ak}, \quad k \geq 0$ (May 2012) Ans. $\dfrac{z e^{-a}}{(z - e^{-a})^2}, \ |z| > |e^{-a}|$

29. $f(k) = k^2, \ k \geq 0$ Ans. $\dfrac{z(z + 1)}{(z - 1)^3}, \ |z| > 1$