Transform Techniques

Analog System → Laplace
Analog System → Fourier

Digital system ⟶ Z-Transform.

Objective — To solve the differential eq^{ns} in the areas of Electric ckt., Beams & string problems. For difference eq^n in case of discrete f^n.

Laplace Transforms

If $f(t)$ is a f^n of t for all $t > 0$ Then

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

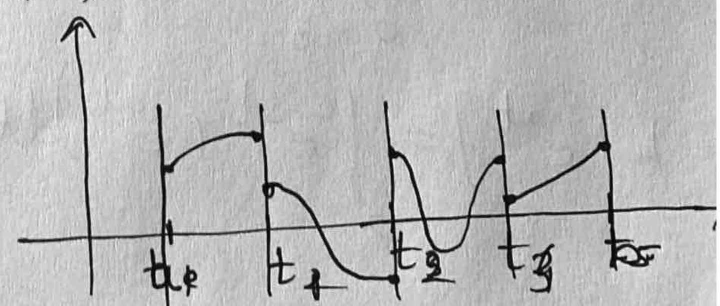
s is parameter (real or complex)

existence cond^n (sufficient):

If $f(t)$ is piecewise cont. in every finite interval in the range $t \geq 0$ & is of exponential order α then it's Laplace Transform exists for all $s > \alpha$

Piecewise cont. f^n — $f(t)$

$$\begin{aligned} f(t) &= f_1(t) & a \leq t \leq t_1 \\ &= f_2(t) & t_1 < t < t_2 \\ &= f_3(t) & t_2 < t < t_3 \\ & f_4(t) & t_3 < t < b \end{aligned}$$



Exponential order -

For constants α, M & N if

$$|f(t)| < Me^{\alpha t} \quad \text{for all } t > N$$

OR $\rightarrow$ if $e^{-\alpha t}|f(t)|$ remains bounded as $t \to \infty$

then $f(t)$ is of exp. order α.

(i.e. if a f^n is of exp-order α, its absolute value must not increase more rapidly than $Me^{\alpha t}$ as t increases.

Ex: (1) $f(t) = t^2$ is of exp-order 3, $\because |t^2| = t^2 < e^{3t} \; \forall t$

(2) sin at, cos at are of exp. order.

(3) $f(t) = e^{t^2}$ is not exp. order

$\because e^{t^2} > Me^{\alpha t}$ however large M

Note: The existence condn is only sufficient

$\Rightarrow$ even if the f^n $f(t)$ is not piecewise or exp. order its L.T. may exist.

Ex: $f(t) = \frac{1}{\sqrt{t}}$ (not cont. at $t=0$) but its Laplace Trans. Exists.

Laplace Trans. of standard f^{ns}:

① $L(1) = \int_0^\infty e^{-st}(1)\,dt = \left(\frac{e^{-st}}{-s}\right)_0^\infty = \frac{1}{s}$ if $s > 0$

② $L[e^{at}] = \int_0^\infty e^{-st}(e^{at})\,dt = \frac{1}{s-a}$ if $s > a$

③ $f(t) = \sin at \Rightarrow L[\sin at] = \frac{a}{s^2+a^2}$ if $s>0$

④ $L[\cos at] = \int_0^\infty e^{-st} \cos at \, dt = \frac{s}{s^2+a^2}$ if $s>0$

⑤ $L[\sinh at] = \int_0^\infty e^{-st} \sinh at \, dt = \int_0^\infty e^{-st} \left(\frac{e^{at}-e^{-at}}{2}\right) dt$

$= \frac{a}{s^2-a^2}$ if $s>|a|$

⑥ $L[\cosh at] = \frac{s}{s^2-a^2}$ if $s>|a|$

⑦ $L[t^n] = \int_0^\infty e^{-st} t^n \, dt$ put $st = y$
$\Rightarrow s\,dt = dy$

$= \frac{\Gamma(n+1)}{s^{n+1}}$ if $s>0,\ n>-1$

$= \frac{n!}{s^{n+1}}$ (if n is +ve integer)

Table of standard L.T.

$f(t)$	$F(s) = L[f(t)]$	
1	$\frac{1}{s}$	$s>0$
e^{at}	$\frac{1}{s-a}$	$s>a$
$\sin at$	$\frac{a}{s^2+a^2}$	$s>0$
$\cos at$	$\frac{s}{s^2+a^2}$	$s>0$
$\sinh at$	$\frac{a}{s^2-a^2}$	$s>\lvert a\rvert$
$\cosh at$	$\frac{s}{s^2-a^2}$	$s>\lvert a\rvert$
t^n	$\frac{\Gamma(n+1)}{s^{n+1}}$ or $\frac{n!}{s^{n+1}}$	$s>0$

Examples

① $L[5t - 7e^{-6t} + t^{5/2}]$

$= L[5t] - 7L[e^{-6t}] + L[t^{5/2}]$

$= 5\dfrac{(1!)}{s^2} - 7\dfrac{1}{s+6} + \dfrac{\Gamma(5/2+1)}{s^{5/2+1}}$

$= \dfrac{5}{s^2} - \dfrac{7}{s+6} + \dfrac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{s^{7/2}}$

$= \dfrac{5}{s^2} - \dfrac{7}{s+6} + \dfrac{15}{8}\sqrt{\dfrac{\pi}{s^7}} \qquad s > 0$

② $L[3e^{4t} + 6t^2 - 4\sin 3t + \cos 2t]$

$= 3L[e^{4t}] + 6L[t^2] - 4L[\sin 3t] + L[\cos 2t]$

$= \dfrac{3}{s-4} + 6\dfrac{2!}{s^3} - 4\dfrac{3}{s^2+9} + \dfrac{s}{s^2+4}$

③ if $f(t) = t \quad 0 < t < 4$

$\qquad = 5 \quad t > 4$

then $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = \int_0^4 e^{-st}\, t\,dt + \int_4^\infty e^{-st}\, 5\,dt$

$= \dfrac{1}{s^2} + e^{-4s}\left(\dfrac{1}{s} - \dfrac{1}{s^2}\right) \qquad s > 0$

④ $L[\sin^2 4t + 2^{3t}]$

$= L\left[\dfrac{1-\cos 8t}{2}\right] + L[e^{3t\log 2}]$

$= \dfrac{32}{s(s^2+64)} + \dfrac{1}{s - 3\log 2}$

Properties of L.T.

if $L[f(t)] = F(s)$ then

① $L[e^{-at} f(t)] = F(s+a)$ 1^{st} shifting

② $L\left[\begin{matrix} F(t) = f(t-a) & t > a \\ = 0 & t < a \end{matrix}\right] = e^{-as} F(s)$

2^{nd} shifting

③ $L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$ change of scale

④ $L[f'(t)] = sF(s) - f(0)$ derivative

$\Rightarrow L[f^{n}(t)] = s^{n}F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - - f^{n-1}(0)$

⑤ $L\left[\int_0^t f(u)\,du\right] = \frac{1}{s} F(s)$ Integral

⑥ $L[t f(t)] = (-1) \frac{d}{ds} F(s)$ Multiplication by t

$\Rightarrow L[t^{n} f(t)] = (-1)^{n} \frac{d^{n}}{ds^{n}} F(s)$

⑦ $L\left[\frac{f(t)}{t}\right] = \int_s^{\infty} F(s)\,ds$ division by t

⑧ $L\left[\int_0^t f(u)\, g(t-u)\,du\right] = F(s)\, G(s)$

Convolution.

x. 2 : Obtain the Laplace transform of each of the following functions :

(i) $at + bt^2 + ct^3$ (ii) $4t^3 + t^7 + t^{4/3}$ (iii) $(2t+3)^3$ (iv) $5t - 7e^{-6t} + t^{5/2}$

ol. : (i) $L[at + bt^2 + ct^3] = a\,L[t] + b\,L[t^2] + c\,L[t^3]$ [(Using the linearity property)]

$$= a\frac{1}{s^2} + b\frac{2!}{s^3} + c\frac{3!}{s^4}$$

$$= \frac{a}{s^2} + \frac{2b}{s^3} + \frac{6c}{s^4} \quad \text{where } s > 0$$

ii) $L[4t^3 + t^7 + t^{4/3}] = 4\,L[t^3] + L[t^7] + L[t^{4/3}]$

$$= 4\frac{3!}{s^4} + \frac{7!}{s^8} + \frac{\Gamma(4/3+1)}{s^{4/3+1}}$$

$$= \frac{24}{s^4} + \frac{5040}{s^8} + \frac{4/3 \cdot 1/3\,\Gamma(1/3)}{s^{7/3}}$$

ii) $L[(2t+3)^3] = L[(2t)^3 + 2(2t)^2(3) + 3(2t)(3)^2 + (3)^3]$

$$= 8\,L[t^3] + 36\,L[t^2] + 54\,L[t] + 27\,L[1]$$

$$= 8\frac{3!}{s^4} + 36\frac{2!}{s^3} + 54\frac{1!}{s^2} + 27\frac{1}{s}$$

$$= \frac{48}{s^4} + \frac{72}{s^3} + \frac{54}{s^2} + \frac{27}{s} \quad \text{where } s > 0$$

v) $L[5t - 7e^{-6t} + t^{5/2}] = 5\,L[t] - 7\,L[e^{-6t}] + L[t^{5/2}]$

$$= 5 \cdot \frac{1!}{s^2} - 7\frac{1}{s+6} + \frac{\Gamma(5/2+1)}{s^{5/2+1}}$$

$$= \frac{5}{s^2} - \frac{7}{s+6} + \frac{5/2 \cdot 3/2 \cdot 1/2\,\Gamma(1/2)}{s^{7/2}}$$

$$= \frac{5}{s^2} - \frac{7}{s+6} + \frac{15}{8}\sqrt{\frac{\pi}{s^7}} \quad \text{where } s > 0$$

x. 3 : Find the Laplace transform of each of the following functions :

) $2\sin 4t + 5\cos 2t$ (ii) $\sin 2t \cos 3t$ (iii) $\cos t \cos 2t$ (iv) $\cosh at - \cos bt$

ol. : (i) $L[2\sin 4t + 5\cos 2t] = 2\,L[\sin 4t] + 5\,L[\cos 2t]$

$$= 2\frac{4}{s^2 + 4^2} + 5\frac{s}{s^2 + 2^2}$$

$$= \frac{8}{s^2 + 16} + \frac{5s}{s^2 + 4} \quad \text{where } s > 0$$

(ii) $$L[\sin 2t \cos 3t] = L\left[\frac{1}{2}(2\cos 3t \sin 2t)\right] = L\left[\frac{1}{2}(\sin 5t - \sin t)\right]$$

$$= \frac{1}{2}\{L[\sin 5t] - L[\sin t]\} = \frac{1}{2}\left\{\frac{5}{s^2+5^2} - \frac{1}{s^2+1^2}\right\}$$

$$= \frac{2(s^2-5)}{(s^2+25)(s^2+1)} \quad \text{where } s > 0$$

(iii) $$L[\cos t \cos 2t] = L\left[\frac{1}{2}(2\cos 2t \cos t)\right] = L\left[\frac{1}{2}(\cos 3t + \cos t)\right]$$

$$= \frac{1}{2}\{L[\cos 3t] + L[\cos t]\} = \frac{1}{2}\left\{\frac{s}{s^2+3^2} + \frac{s}{s^2+1^2}\right\}$$

$$= \frac{s(s^2+5)}{(s^2+9)(s^2+1)} \quad \text{where } s > 0$$

(iv) $$L[\cosh at - \cos bt] = L[\cosh at] - L[\cos bt]$$

$$= \frac{s}{s^2-a^2} - \frac{s}{s^2+b^2} \quad \text{where } s > |a|$$

Ex. 4 : Find the Laplace transform of the following functions :

(i) $3\cos(4t+7)$ (ii) $5\sin(2t+3)$ (iii) $\sin^2 4t$ (iv) $\cos^3 2t$ (v) $\cosh^3 2t$

Sol. : (i) $$L[3\cos(4t+7)] = 3L[\cos(4t+7)]$$

Note : Here we first express $\cos(4t+7)$ as the difference of two terms.

$\therefore$ $$3L[\cos(4t+7)] = 3L[\cos 4t \cos 7 - \sin 4t \sin 7]$$

$$= 3\{\cos 7 L[\cos 4t] - \sin 7 L[\sin 4t]\}$$

$$= 3\left\{\cos 7 \frac{s}{s^2+4^2} - \sin 7 \frac{4}{s^2+4^2}\right\}$$

$$= \frac{3}{s^2+16}\{s\cos 7 - 4\sin 7\} \quad \text{where } s > 0$$

(ii) $$L[5\sin(2t+3)] = 5L[\sin(2t+3)]$$

$$= 5L[\sin 2t \cos 3 + \cos 2t \sin 3]$$

$$= 5\{\cos 3 L[\sin 2t] + \sin 3 L[\cos 2t]\}$$

$$= 5\left\{\cos 3 \frac{2}{s^2+2^2} + \sin 3 \frac{s}{s^2+2^2}\right\}$$

$$= \frac{5}{s^2+4}\{2\cos 3 + s\sin 3\} \quad \text{where } s > 0$$

(iii) $$L[\sin^2 4t] = L\left[\frac{1-\cos 8t}{2}\right] = \frac{1}{2}\{L[1] - L[\cos 8t]\}$$

$$= \frac{1}{2}\left\{\frac{1}{s} - \frac{s}{s^2+8^2}\right\} = \frac{32}{s(s^2+64)} \quad \text{where } s > 0$$

(iv)

$$L[\cos^3 2t] = L\left[\frac{\cos 6t + 3\cos 2t}{4}\right] \qquad \left\{\because \cos 3\theta = 4\cos^3\theta - 3\cos\theta\right\}$$

$$= \frac{1}{4}\{L[\cos 6t] + 3L[\cos 2t]\}$$

$$= \frac{1}{4}\left\{\frac{s}{s^2+6^2} + 3\frac{s}{s^2+2^2}\right\}$$

$$= \frac{s(s^2+28)}{(s^2+4)(s^2+36)} \quad \text{where } s > 0$$

(v)

$$L[\cosh^3 2t] = L\left[\frac{\cosh 6t + 3\cosh 2t}{4}\right] \quad \left\{\because \cosh 3\theta = 4\cosh^3\theta - 3\cosh\theta\right\}$$

$$= \frac{1}{4}\{L[\cosh 6t] + 3L[\cosh 2t]\}$$

$$= \frac{1}{4}\left\{\frac{s}{s^2-6^2} + 3\frac{s}{s^2-2^2}\right\}$$

$$= \frac{s(s^2-28)}{(s^2-4)(s^2-36)}$$

Ex. 5 : Obtain Laplace transforms of

(i) $3e^{4t} + 6t^2 - 4\sin 3t + \cos 2t$ (ii) $5e^{-t/2} + t^{-1/2} + 7\sin\frac{t}{2}$.

Sol. : (i) $L[3e^{4t} + 6t^2 - 4\sin 3t + \cos 2t]$

$$= 3L[e^{4t}] + 6L[t^2] - 4L[\sin 3t] + L[\cos 2t]$$

$$= 3\frac{1}{s-4} + 6\frac{2!}{s^3} - 4\frac{3}{s^2+3^2} + \frac{s}{s^2+2^2}$$

$$= \frac{3}{s-4} + \frac{12}{s^3} - \frac{12}{s^2+9} + \frac{s}{s^2+4} \quad \text{where } s > 4$$

(ii) $L\left[5e^{-t/2} + t^{-1/2} + 7\sin\frac{t}{2}\right] = 5L[e^{-(1/2)t}] + L[t^{-1/2}] + 7L\left[\sin\left(\frac{1}{2}\right)t\right]$

$$= 5\frac{1}{s+\frac{1}{2}} + \sqrt{\frac{\pi}{s}} + 7\frac{1/2}{s^2+\left(\frac{1}{2}\right)^2}$$

$$= \frac{5}{s+\frac{1}{2}} + \sqrt{\frac{\pi}{s}} + \frac{\frac{7}{2}}{s^2+\frac{1}{4}} \quad \text{where } s > 0$$

... functions

$$f(t) = \begin{cases} \sin 2t & 0 < t < \pi \\ 0 & t > \pi \end{cases}$$

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = \int_0^\pi e^{-st} \sin 2t\, dt + \int_\pi^\infty e^{-st} (0)\, dt$$

$$= \left[\frac{e^{st}}{s^2+4}(-s \sin 2t - 2\cos 2t)\right]_0^\pi \qquad \left\{ \begin{array}{l} \because \int e^{at} \sin bt\, dt \\ = \dfrac{e^{at}}{a^2+b^2}(a \sin bt - b \cos bt) \end{array} \right.$$

$$= \frac{1}{s^2+4} \left[e^{-s\pi}(-2) - (-2)\right]$$

$$= \frac{2}{s^2+4}(1 - e^{-s\pi}) \qquad \text{where } s > 0$$

EXERCISE 10.1

1. Find the Laplace transforms of each of the following functions :

(i) $2e^{3t} + 3e^{-2t}$ (ii) $(e^{-at} - e^{-bt})^2$ (iii) $(2e^{3t} + 5)^2$ (iv) $c^{at + b}$

Ans. (i) $\frac{2}{s-3} + \frac{3}{s+2}$ (ii) $\frac{1}{s+2a} + \frac{2}{s+(a+b)} + \frac{1}{s+2b}$

(iii) $\frac{4}{s-6} + \frac{20}{s-3} + \frac{25}{s}$ (iv) $\frac{c^b}{s - a \log c}$

2. Obtain the Laplace transforms of each of the following functions :

(i) $t^2 - 3t + 5$ (ii) $t^4 + 5t^3 + t^{1/2}$ (iii) $a + \frac{b}{\sqrt{t}}$ (iv) $(t+2)^3 + (e^{2t} + 3)^2$

Ans. (i) $\frac{5s^2 - 3s + 2}{s^3}$ (ii) $\frac{24}{s^5} + \frac{30}{s^4} + \frac{\sqrt{\pi}}{2s^{3/2}}$ (iii) $\frac{a}{s} + b\sqrt{\frac{\pi}{s}}$

(iv) $\frac{6}{s^4} + \frac{12}{s^3} + \frac{12}{s^2} + \frac{17}{s} + \frac{1}{s-4} + \frac{6}{s-2}$

3. Find the Laplace transforms of each of the following functions :

(i) $3\cos 2t - \sin 2t$ (ii) $\cosh 5t + \cos 5t$ (iii) $\cos 3t \cos 2t$ (iv) $\sin 2t \cos 5t$

Ans. (i) $\frac{3s}{s^2+4} - \frac{2}{s^2+4}$ (ii) $\frac{s}{s^2-25} + \frac{s}{s^2+25}$ (iii) $\frac{s(s^2+13)}{(s^2+1)(s^2+25)}$ (iv) $\frac{2(s^2-21)}{(s^2+9)(s^2+49)}$

4. Find the Laplace transforms of the following functions :

(i) $\cos(\omega t + \alpha)$ (ii) $\sin(\omega t + \alpha)$ (iii) $\cos^2 bt$ (iii) $\sin^3 2t$ (iv) $\sinh^3 2t$ (vi) $(\sin t - \cos t)^2$

Ans. (i) $\frac{s\cos\alpha - \omega\sin\alpha}{s^2+\omega^2}$ (ii) $\frac{\omega\cos\alpha + s\sin\alpha}{s^2+\omega^2}$ (iii) $\frac{s^2+2b^2}{s(s^2+4b^2)}$

(iv) $\frac{48}{(s^2+4)(s^2+36)}$ (v) $\frac{48}{(s^2-4)(s^2-36)}$ (vi) $\frac{s^2-2s+4}{s(s^2+4)}$

5. Obtain Laplace transforms of

(i) $e^{2t} + 4t^3 - 2\sin 3t + 3\cos 3t$ (ii) $4\cos 2t - 5t^2 + 2e^{3t}$

(iii) $3t^4 - 2t^3 + 4e^{-3t} - 2\sin 5t + 3\cos 2t$

Ans. (i) $\frac{1}{s-2} + \frac{24}{s^4} - \frac{6}{s^2+9} + \frac{3s}{s^2+9}$ (ii) $\frac{4s}{s^2+4} - \frac{10}{s^3} + \frac{2}{s-3}$

(iii) $\frac{72}{s^5} - \frac{12}{s^4} + \frac{4}{s+3} - \frac{10}{s^2+25} + \frac{3s}{s^2+4}$

6. Obtain Laplace transforms of each of the following functions :

(i) $f(t) = \begin{cases} a & 0 < t < b \\ 0 & t > b \end{cases}$ (ii) $f(t) = \begin{cases} \cos t & 0 < t < 2\pi \\ 0 & t > 2\pi \end{cases}$ (iii) $f(t) = \begin{cases} \cos t & 0 < t < \pi \\ \sin t & t > \pi \end{cases}$

Ex. 2 : Find $L[F(t)]$ if $F(t) = \begin{cases} (t-1)^2 & t > 1 \\ 0 & 0 < t < 1 \end{cases}$

Sol. : Here $\quad f(t-a) = (t-1)^2$ where $a = 1$

$\therefore \quad f(t) = t^2$ and $F(s) = \dfrac{2}{s^3}$

Hence by the second shifting theorem [result (10)], with a = 1, we have

$$L[F(t)] = e^{-as} F(s) = e^{-s}\left(\frac{2}{s^3}\right)$$

Ex. 3 : If $L[f(t)] = \dfrac{1}{s} e^{-1/s}$, find $L[e^{-t} f(3t)]$.

Sol. : We have given that

$$L[f(t)] = \frac{1}{s} e^{-1/s}$$

By change of scale theorem [(result 11)], with a = 3, we have

$$L[f(3t)] = \frac{1}{3}\left(\frac{1}{s/3} e^{-3/s}\right) = \frac{1}{s} e^{-3/s}$$

Hence by the first shifting theorem, we get

$$L[e^{-t} f(3t)] = \left\{\frac{1}{s} e^{-3/s}\right\}_{s \to s+1} = \frac{e^{-3/(s+1)}}{(s+1)}$$

$\sqrt{s}$

Ex. 5 : Given that $4 f''(t) + f(t) = 0$, $f(0) = 0$ and $f'(0) = 2$, show that $L[f(t)] = \frac{8}{4s^2+1}$.

Sol. : Taking Laplace transforms of both sides, we get

$$4 L[f''(t)] + L[f(t)] = L[0]$$

$$\therefore 4\{s^2 F(s) - s f(0) - f'(0)\} + F(s) = 0 \qquad \text{[By results (12) and (13)]}$$

$$\therefore 4\{s^2 F(s) - s(0) - (2)\} + F(s) = 0 \qquad [f(0) = 0 \text{ and } f'(0) = 2]$$

$$\therefore (4s^2 + 1) F(s) - 8 = 0$$

$$\therefore F(s) = L[f(t)] = \frac{8}{4s^2+1}$$

Ex. 6 : Use theorem on Derivative, to derive the following Laplace transforms :

(i) $L[e^{at}] = \frac{1}{s-a}$ (ii) $L[\sin at] = \frac{a}{(s^2+a^2)}$

Sol. : (i)Let $f(t) = e^{at}$. Then $f'(t) = a e^{at}$, $f(0) = 1$

Now $L[f'(t)] = s L[e^{at}] - f(0)$ [By result (12)]

$\therefore$ $L[ae^{at}] = s L[e^{at}] - 1$

or $a L[e^{at}] = s L[e^{at}] - 1$

or $L[e^{at}] = \frac{1}{s-a}$

(ii) Let $f(t) = \sin at$. Then $f'(t) = a \cos at$, $f''(t) = -a^2 \sin at$, $f(0) = 0$, $f'(0) = a$.

Now $L[f''(t)] = s^2 L[f(t)] - s f(0) - f'(0)$ [By result (13)]

$\therefore$ $L[-a^2 \sin at] = s^2 L[\sin at] - s(0) - a$

or $-a^2 L[\sin at] = s^2 L[\sin at] - a$

or $L[\sin at] = \frac{a}{s^2+a^2}$

Ex. 7 : Find $L\left[\int_0^t \sin 2u \, du\right]$.

Sol. : We have

$$L[\sin 2t] = \frac{2}{s^2+4}$$

$$\therefore \int_0^t \sin 2u \, du = \frac{1}{s}\left(\frac{2}{s^2+4}\right) \qquad \text{[By result (16)]}$$

Ex. 8 : Find the Laplace transform of the following functions :

(i) t cos at (iii) $\dfrac{t \sinh at}{2a}$ (iii) $t \sin^3 t$

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Sol. : (i) We have

$$L[\cos at] = \frac{s}{s^2 + a^2}$$

$$\therefore \quad L[t \cos at] = (-1)\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right) = -\left\{\frac{(s^2+a^2)(1) - s(2s)}{(s^2+a^2)^2}\right\} \qquad \text{[By result (18)]}$$

$$= \frac{s^2 - a^2}{(s^2+a^2)^2}$$

(ii) We have

$$L\left[\frac{\sinh at}{2a}\right] = \frac{1}{2a} L[\sinh at] = \frac{1}{2}\,\frac{1}{s^2 - a^2}$$

$$\therefore \quad L\left[\frac{t \sinh at}{2a}\right] = (-1)\frac{d}{ds}\left\{\frac{1}{2}\,\frac{1}{(s^2-a^2)}\right\} = (-1)\,\frac{1}{2}\,\frac{-2s}{(s^2-a^2)^2} \qquad \text{[By result (18)]}$$

$$= \frac{s^2 - a^2}{(s^2+a^2)^2}.$$

(iii) We have

$$L[\sin^3 t] = L\left[\frac{3}{4}\sin t - \frac{1}{4}\sin 3t\right] \qquad \{\because \sin 3t = 3\sin t - 4\sin^3 t\}$$

$$= \frac{3}{4}L[\sin t] - \frac{1}{4}L[\sin 3t] = \frac{3}{4}\left(\frac{1}{s^2+1} - \frac{1}{s^2+9}\right)$$

$$\therefore \quad L[t \sin^3 t] = (-1)\frac{d}{ds}\left[\frac{3}{4}\left(\frac{1}{s^2+1} - \frac{1}{s^2+9}\right)\right] = -\frac{3}{4}\left[\frac{(-2s)}{(s^2+1)^2} - \frac{(-2s)}{(s^2+9)^2}\right]$$

[By result (18)]

$$= \frac{3s}{2}\left[\frac{1}{(s^2+1)^2} - \frac{1}{(s^2+9)^2}\right]$$

Ex. 9 : Find the Lapl

(ii) $$L\left[\frac{1-\cos t}{t^2}\right] = L\left[\frac{1}{t}\left(\frac{1-\cos t}{t}\right)\right]$$

$$= \int_s^{\infty} \frac{1}{2} \log \frac{s^2+1}{s^2}\, ds = \frac{1}{2}\int_s^{\infty}\left(\log \frac{s^2+1}{s^2}\right)\cdot 1\, ds$$

[From result of (i) above]

Integrating by parts, we have

$$= \frac{1}{2}\left[\left(\log \frac{s^2+1}{s^2}\right)\cdot s - \int \left\{\frac{s^2}{s^2+1}\ \frac{s^2(2s)-(s^2+1)(2s)}{s^4}\right\}\cdot s\, ds\right]_0^{\infty}$$

$$= \frac{1}{2}\left[s \log \frac{s^2+1}{s^2} - \int \left\{\frac{s^2}{s^2+1}\left(\frac{-2}{s^3}\right)\right\} s\, ds\right]_s^{\infty}$$

$$= \frac{1}{2}\left[s \log \frac{s^2+1}{s^2} + 2\int \frac{1}{s^2+1}\, ds\right]_s^{\infty}$$

$$= \frac{1}{2}\left[s \log \frac{s^2+1}{s^2} + 2\tan^{-1} s\right]_s^{\infty}$$

$$= \frac{1}{2}\left[\left\{0 + 2\left(\frac{\pi}{2}\right)\right\} - \left\{s \log \frac{s^2+1}{s^2} + 2\tan^{-1} s\right\}\right]$$

$$\left\{\because \lim_{s\to\infty} s\log\left(1+\frac{1}{s^2}\right) = \lim_{s\to\infty} s\left(\frac{1}{s^2} - \frac{1}{2s^4} + \frac{1}{3s^6}\cdots\right) = 0\right\}$$

$$= \frac{1}{2}\left[\pi - s\log\frac{s^2+1}{s^2} - 2\tan^{-1} s\right]$$

$$= \frac{1}{2} s \log \frac{s^2}{s^2+1} + \cot^{-1} s \qquad \left(\because \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s\right)$$

EXERCISE 10.2

1. Find the Laplace transform of each of the following functions :

(i) $t^3 e^{-3t}$ (ii) $e^{at}(2\cos bt - 3\sin bt)$ (iii) $(1 + t e^{-t})^3$ (iv) $e^{-t}\{4t^3 + \cos(4t + 7)\}$

(v) $2e^t \sin 4t \cos 2t$ (vi) $\sinh \frac{t}{2} \sin \frac{\sqrt{3}}{2} t$ (vii) $\cos at \sinh at$ (viii) $e^{4t}\, t^{3/2}$

(ix) $\dfrac{\cosh at}{\sqrt{t}}$ (x) $e^{-t} \sin^3 t$

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Ans. (i) $\dfrac{6}{(s+3)^4}$ (ii) $\dfrac{2s - 2a - 2b}{(s-a)^2 + b^2}$ (iii) $\dfrac{1}{s} + \dfrac{3}{(s+1)^2} + \dfrac{6}{(s+2)^3} + \dfrac{6}{(s+3)^4}$

(iv) $\dfrac{24}{(s+1)^4} + \dfrac{(s+1)\cos 7 - 4\sin 7}{s^2 + 2s + 17}$ (vi) $\dfrac{6}{s^2 - 2s + 37} + \dfrac{2}{s^2 - 2s + 5}$

(vi) $\dfrac{\sqrt{3}}{2} \dfrac{s}{s^4 + s^2 + 1}$ (vii) $\dfrac{a(s^2 - 2a^2)}{s^4 + 4a^2}$ (viii) $\dfrac{3}{4} \dfrac{\sqrt{\pi}}{(s-4)^{5/2}}$

12. Obtain the Laplace transform of each of the following functions :

(i) $t e^{3t} \cos 2t$ (ii) $\int_0^t e^u u^3 \, du$ (iii) $\int_0^t e^x \cos x \, dx$ (iv) $\int_0^t \frac{e^t - \cos 2t}{t} \, dt$ (Ma

Ans. (i) $\frac{s^2 - 6s + 5}{(s^2 - 6s + 13)^2}$ (ii) $\frac{6}{s(s-1)^4}$ (iii) $\frac{s-1}{s(s^2 - 2s + 2)}$ (iv) $\frac{1}{s} \log \frac{\sqrt{s^2+4}}{(s-1)}$

13. Obtain Laplace transform of

(i) $t \int_0^t e^{-3t} \sin 2t \, dt$ (ii) $e^{-3t} \int_0^t t \sin 2t \, dt$ (iii) $\int_0^t t e^{-3t} \sin 2t \, dt$ (I

(iv) $\cosh t \int_0^t t \cosh t \, dt$

Ans. (i) $\frac{6s^2 + 24s + 26}{s^2 (s^2 + 6s + 13)^2}$ (ii) $\frac{4}{(s^2 + 6s + 13)^2}$ (iii) $\frac{1}{s} \frac{2(2s+6)}{(s^2 + 6s + 13)^2}$

(iv) $\frac{1}{2} \left[\frac{s^2 - 2s}{(s-1)(s^2 - 2s + 2)^2} + \frac{s^2 + 2s}{(s+1)(s^2 + 2s + 2)^2} \right]$.

14. Obtain Laplace transform of

(i) $\frac{e^{-3t} \sin 2t}{t}$ (ii) $\int_0^t \frac{e^{-3t} \sin 2t}{t} \, dt$ (iii) $e^{-3t} \int_0^t \frac{\sin 2t}{t} \, dt$

Ans. (i) $\cot^{-1} \frac{s+3}{2}$ (ii) $\frac{1}{s} \cot^{-1} \frac{s+3}{2}$ (iii) $\frac{1}{s+3} \cot^{-1} \frac{s+3}{2}$

15. Find the following convolutions :

(i) $1 * 1$ (ii) $1 * e^t$ (iii) $1 * \cos t$ (iv) $(e^{-t} - e^{2t}) * e^{-t}$

(v) $t * e^{at}$ (vi) $\cos t * \cos t$ (vii) $\sin t * \sin t$ (viii) $\sin t * \cos t$

Ans. (i) t (ii) e^t (iii) $\sin t$ (iv) $e^{-2t} + (t-1) e^{-t}$ (v) $(e^{at} - 1)/a^2 - t/a$

(vi) $\frac{1}{2} (t \cos t + \sin t)$ (vii) $\frac{1}{2} (\sin t - t \cos t)$ (viii) $\frac{1}{2} t \sin t$.

Hint : Find $f(t) * g(t) = \int_0^t f(u) g(t-u) \, du$ or $f(t) * g(t) \int_0^t f(t-u) g(u) \, du$.

16. Verify the convolution theorem for the pair of following functions :

(i) $f(t) = t^2$, $g(t) = e^{-at}$ (ii) $f(t) = t$, $g(t) = \cos t$.

Hint : Show that $L[f(t) * g(t)] = F(s) \, G(s)$.

$\sqrt{s^2 + 1}$

10.10 LAPLACE TRANSFORMS OF SPECIAL FUNCTIONS

In discussion of certain types of physical and engineering problems, number of situatio such as

(i) a force acting on the part of the system or voltage acting for finite interval of time.

(ii) a large force acting, for very short time (impulsive force) or over a very short area. F examples, heavy excitation is introduced by putting on and off switch in one actio concentrated load acting on a beam.

(iii) a periodic function or periodic voltage.

The analytical representation of such functions and the nature of their Laplace transfor are of great practical importance. Hence in the following articles, such functions and th Laplace transforms are discussed.

10.11 UNIT STEP FUNCTION

The unit step function also called *Heaviside's unit step function U(t),* is defined as

$$\boxed{U(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}}$$

This unit step function is a curve which has value zero at all points to the left of the o and is equal to 1 (unity) on the right of the origin (see Fig. 10.4).

The displaced unit step function $U(t-a)$ is defined as

$$\boxed{U(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}}$$

where $a \geq 0$.

The displaced unit step function $U(t - a)$ represents curve $U(t)$ which is dis (translated) a distance a units to the right along t-axis (see Fig. 10.5).

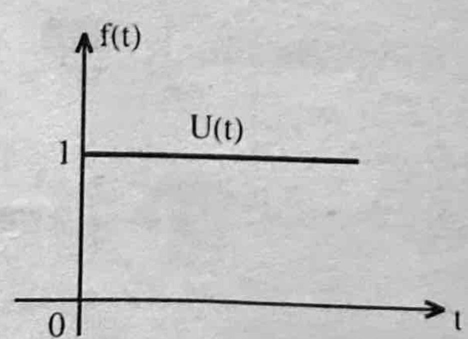

Fig. 10.4 : Unit step function U(t)

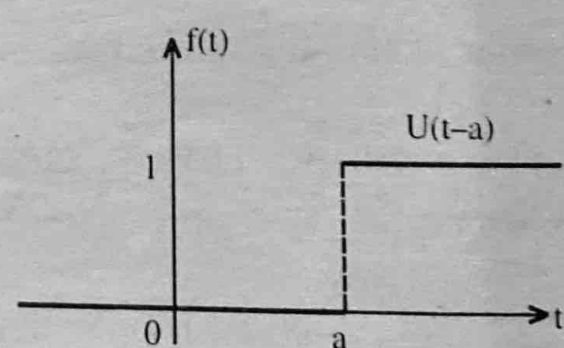

Fig. 10.5 : Displaced unit step function U(t –

Note 1 : Observe that at t = a, a step of unit height is formed, hence the name unit step function. Other notations to represent unit step functions are H(t), H(t – a), $u_0(t)$, $u_a(t)$.

Note 2 : Unit step functions U(t) and U(t – a) are extensively used to represent a portion of the curve of the function f(t) as explained in the following cases.

Case I : f(t) U(t) :

When the function f(t) is multiplied by unit step function U(t), the resultant function f(t) U(t) will represent the part of the function f(t) on the right of the origin, the part of f(t) on the left of the origin being cut off (i.e. vanishes for t < 0) (see Fig. 10.6) i.e.

$$f(t)\,U(t) = \begin{cases} 0 & t < 0 \\ f(t) & t \geq 0 \end{cases}$$

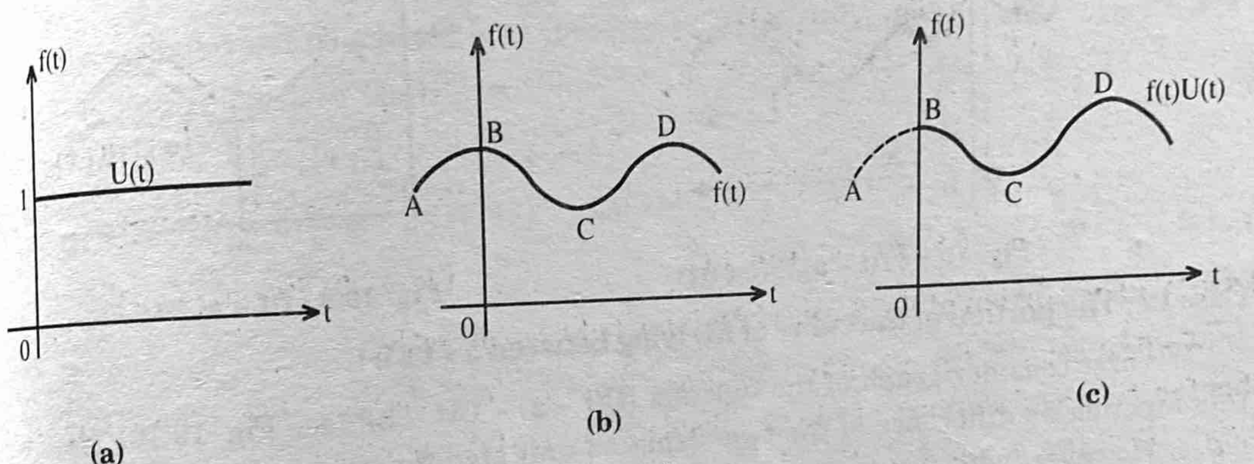

Fig. 10.6

In Fig. 10.6 (c), the part AB of the curve f(t) is cut off.

Case II : f(t) U(t – a) :

When the function f(t) is multiplied by displaced unit step function U(t – a), the resultant function f(t) U(t – a) will represent the part of the function f(t) on the right of t = a, the part of the function f(t) on the left of t = a is cut off (see Fig. 10.7) i.e.

$$f(t)\,U(t-a) = \begin{cases} 0 & t < a \\ f(t) & t \geq a \end{cases}$$

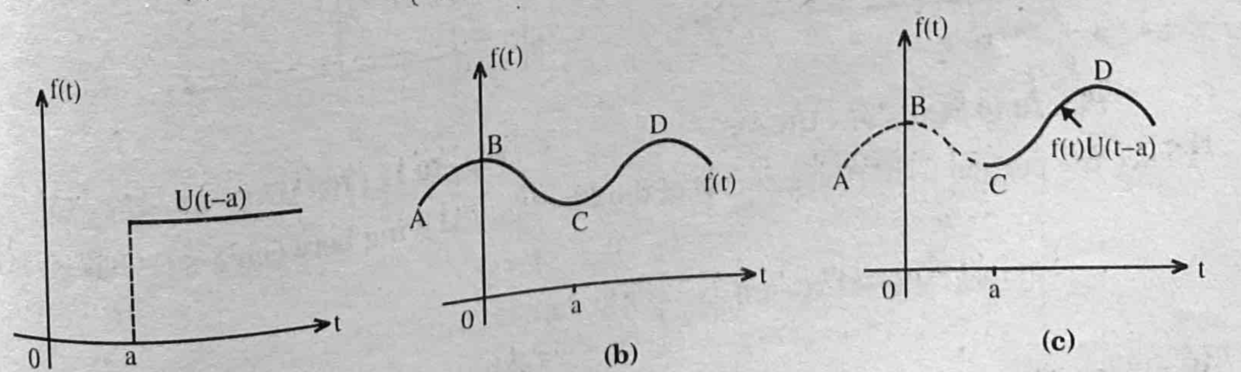

Fig. 10.7

(a)

In Fig. 10.7 (c), the part ABC of the function f(t) is cut off.

10.12 TRANSFORMS USING UNIT STEP FUNCTIONS

1. Laplace transform of Unit step function U(t) :

By definition of Laplace transform

$$L[U(t)] = \int_0^\infty e^{-st} U(t)\,dt = \int_0^\infty e^{-st}(1)\,dt \qquad \left[\because U(t) = \begin{cases} 0 & t<0 \\ 1 & t\ge 0 \end{cases}\right]$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s} \qquad \text{where } s>0$$

Hence $$\boxed{L[U(t)] = \frac{1}{s}} \qquad \ldots (30)$$

2. Laplace transform of Displaced unit step function U(t – a) :

By definition of Laplace transform

$$L[U(t-a)] = \int_0^\infty e^{-st} U(t-a)\,dt$$

$$= \int_0^a e^{-st}(0)\,dt + \int_a^\infty e^{-st}(1)\,dt \qquad \left[\because U(t-a) = \begin{cases} 0 & t<a \\ 1 & t\ge a \end{cases}\right]$$

$$= 0 + \left[\frac{e^{-st}}{-s}\right]_a^\infty = \frac{e^{-as}}{s} \qquad \text{where } s>0$$

Hence $$\boxed{L[U(t-a)] = \frac{e^{-as}}{s}} \qquad \ldots (31)$$

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3. Laplace transform of f(t – a) U(t – a) :

f(t – a) U (t – a) represents the part of f(t – a) to the right of t = a i.e.

$$f(t-a)\,U(t-a) = \begin{cases} 0 & t<a \\ f(t-a) & t\ge a \end{cases}$$

By definition of Laplace transform

$$L[f(t-a)\,U(t-a)] = \int_0^\infty e^{-st}[f(t-a)\,U(t-a)]\,dt$$

$$= \int_0^a e^{-st}[f(t-a)\,U(t-a)]\,dt + \int_a^\infty e^{-st}[f(t-a)\,U(t-a)]\,dt$$

$$= 0 + \int_a^\infty e^{-st} f(t-a)\,dt \qquad \left\{\begin{array}{l}\text{Since } f(t-a)\,U(t-a) \text{ vanishes} \\ \text{identically to the left of } t=a\end{array}\right.$$