classmate
Date
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F I S - I D F I S
$u\int v - \int d(u)\int v\,dx$

Tutorial - 1

$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

Q.1 Find the roots of
1) $2D^2 - D - 10 = 0$
2) $D^3 - 6D^2 + 11D - 6 = 0$
3) $D^3 - 3D^2 + 3D - 1 = 0$
4) $D^4 + 1 = 0$

Solve
1) $\int e^x e^{e^x} dx$

2) $\int x^2 \sin 3x\,dx$

$$\begin{array}{r} D^2 - 5D + 6 \\ D-1 \,\big)\, D^3 - 6D^2 + 11D - 6 \\ \pm D^3 \mp D^2 \\ \hline -5D^2 + 11D - 6 \\ \mp 5D^2 \pm 5D \\ \hline 6D - 6 \end{array}$$

Solution:

Q.1 1. $D = \dfrac{-(-1) \pm \sqrt{1 - 4(2)(-10)}}{4}$

$= \dfrac{1 \pm \sqrt{1+80}}{4}$

$= \dfrac{1 \pm 9}{4}$

$D = \dfrac{10}{4}$ $\quad$ $D = -2$

$\boxed{D = 5/2}$ $\quad$ $\boxed{D = -2}$

General solⁿ: $y = C_1 e^{5/2 x} + C_2 e^{-2x}$

$D^2 - 5D + 6$
$D^2 - 6D + D + 6$
$D(D-6) + 1(D-6)$
$(D+1)(D-6)$
$D = 1, -1, 6.$ $D = 1$

$r - 3 + 3 - 1$

$$\begin{array}{r} D^2 - 2D + 1 \\ D-1 \,\big)\, D^3 - 3D^2 + 3D - 1 \\ \pm D^3 \mp D^2 \\ \hline -2D^2 + 3D - 1 \\ \mp 2D^2 \pm 2D \\ \hline D - 1 \end{array}$$

2. $D = 1, -1, 6$
Roots are real and distinct
General solⁿ: $y = C_1 e^{x} + C_2 e^{6x} + C_3 e^{-x}$

$D^2 - 2D + 1$
$D^2 - D - D + 1$
$D(D-1) - 1(D-1)$
D

3. $(D-1)(D-1)(D-1) = 0$
$D = 1$
General solⁿ: $y = e^{x}(C_1 + C_2 + C_3)$.

2) $x^2 \sin 3x \, dx$

$u = x^2 \quad v = \sin 3x$

$u\int v\,dx - \int u' \int v\,dx.$

$x^2 \int \sin 3x\,dx - \int 2x \int \sin 3x\,dx.$

$x^2\left(-\frac{\cos 3x}{3}\right) - \int(2x)\left(-\frac{\sin 3x}{9}\right) + 2\left(\frac{\cos 3x}{27}\right) + c.$

$= \cancel{x^2 - \frac{\cos 3x}{3} + \frac{1}{3}\int \frac{2x\cos 3x\,dx}{3}}$

$= x^2 \cancel{- \frac{\cos 3x}{3}}$ $\quad \int uv\,dx = uv_1 - u'v_2 + u''v_3 - \cdots$

3) $\int \frac{dx}{1+e^x}$

1) put $e^x = t$

$\therefore e^x dx = dt$

$\therefore \int e^t dt$

$= e^t + c$

$= e^{e^x} + c.$

1) General method for P.I.

Solve: $(D^2+D)y = \frac{1}{1+e^x}$

C.F. $\Rightarrow c_1 + c_2 e^{-x}$

$P.I = \frac{1}{D^2+D}\left(\frac{1}{1+e^x}\right)$

$= \frac{1}{D(D+1)} \frac{1}{1+e^x}$ —①

Consider $\frac{1}{D(D+1)} = \frac{A}{D} + \frac{B}{D+1}$

$N^r \Rightarrow 1 = A(D+1) + BD$

Put $D=0$ $\qquad D=-1$

$A=1$ $\qquad B=-1$

$\therefore \frac{1}{D(D+1)} = \frac{1}{D} - \frac{1}{D+1}$

$P.I. = \frac{1}{D}\left(\frac{1}{1+e^x}\right) - \frac{1}{D+1}\left(\frac{1}{1+e^x}\right)$

$P.I. = \int \frac{dx}{1+e^x} - e^{-x}\int e^x \frac{1}{1+e^x}\,dx$

$= \int \frac{e^{-x}dx}{e^{-x}+1} - e^{-x}\int \frac{e^x}{1+e^x}\,dx.$

Put $1+e^{-x} = t$ $\qquad 1+e^x = u$

$-e^{-x}dx = dt$ $\qquad e^x dx = du$

$= \int \frac{-dt}{t} - e^{-x}\int \frac{du}{u}$

$P.I. = -\log(1+e^{-x}) - e^{-x}(\log(1+e^x))$

2) Short-cut method.

Case I: $f(x) = e^{ax}$

$P.I. = \frac{1}{\phi(D)} e^{ax}$ put $D = a$.

$$= \frac{1}{\phi(a)} e^{ax} \quad \text{if } \phi(a) \neq 0.$$

if $\phi(a) = 0 \Rightarrow a$ is root of $\phi(D)$

$D-a$ is a factor of $\phi(D)$.

$$P.I. = \frac{1}{\phi'(D)} x e^{ax} \quad \text{then put } D = a.$$

$$= \frac{1}{\phi'(a)} x e^{ax} \quad \text{if } \phi'(a) \neq 0$$

if $\phi'(a) = 0$ then

$$P.I = \frac{1}{\phi''(a)} x^2 e^{ax} \ \ldots$$ continue till non-zero answer in D^r.

Note : i) $P.I. = \frac{1}{(D-a)} e^{ax} = \frac{x e^{ax}}{1!}$

$$P.I. = \frac{1}{(D-a)^2} e^{ax} = \frac{x^2 e^{ax}}{2!}$$

$$= \frac{1}{(D-a)^r} e^{ax} = \frac{x^r e^{ax}}{r!}$$

ii) for constant $k = e^{ax} \Rightarrow$ put $D=0$

iii) for $f(x) = a^x \Rightarrow e^{x \log a} \Rightarrow$ put $D = \log a$.

iv) for $f(x) = a^{-x} = e^{-x \log a} \Rightarrow$ put $D = -\log a$.

Ex: Solve $(D^2 + 3D + 2)y = e^{2x}$

C.F. $\Rightarrow C_1 e^{-x} + C_2 e^{-2x}$

$$P.I. = \frac{1}{(D+1)(D+2)} e^{2x} \quad \text{put } D = 2.$$

$$P.I. = \frac{1}{12} e^{2x}.$$

$(D^2 + 3D + 2)y = e^{-2x}$

C.F. $= C_1 e^{-x} + C_2 e^{-2x}$

$$P.I. = \frac{1}{D^2 + 3D + 2} e^{-2x} \quad \text{putting } D = -2$$

$D^r \Rightarrow 0$

$\therefore$ Diff. D^r & multiply by x.

$$= \frac{1}{2D+3} x e^{-2x}$$

$$P.I. = -xe^{-2x}$$

3) $(D^2+3D+2)y = e^{-2x} + 2^x + 3/2$

$$P.I. = \frac{1}{D^2+3D+2} e^{-2x} + \frac{1}{D^2+3D+2} 2^x + \frac{1}{D^2+3D+2}\left(\frac{3}{2}\right)$$

$$= -xe^{-2x} + \frac{1}{(\log 2)^2 + 3\log 2 + 2} 2^x + \frac{1}{2} \cdot \frac{3}{2}$$

H.W Solve

1) $(D^2-4D+4)y = e^{2x} + 3$

2) $(D+1)^2(D-2)y = e^{-x} + 3^{-x}$

Sol^n 1) $P.I. = \frac{1}{(D-2)^2} \cdot e^{2x} + \frac{1}{(D-2)^2} 3 \cdot e^{0x}$

Put $D=2$. put $D=0$.

$$= \frac{x^2}{2} e^{2x} + \frac{3}{4}$$

$\sin^2 x = \frac{1-\cos 2x}{2}$

$\cos^2 x = \frac{1+\cos 2x}{2}$

2) ~~$P.I. = \frac{1}{(D+1)}$~~

$\sin 2x = 2\sin x \cos x$

$\sin a \cos b = \frac{\sin(a+b) + \sin(a-b)}{2}$

Case II: If $f(x) = \sin(ax+b)$ or $\cos(ax+b)$ then replace D^2 of $\phi(D)$ as $(-a^2)$

$$P.I. = \frac{1}{\phi(D)} \sin(ax+b) \text{ or } \cos(ax+b)$$

$\cos a \cos b = \frac{\cos(a+b) + \cos(a-b)}{2}$

Put $D^2 = -a^2$

$$= \frac{1}{\phi(-a^2)} \sin(ax+b) \text{ or } \cos(ax+b). \text{ if } \phi(-a^2) \neq 0.$$

$\sin A \sin B = \frac{\cos(A-B) - \cos(A+B)}{2}$

if $\phi(-a^2) = 0 \Rightarrow$ diff. D^r & multiply by x.

Solve: $(D^2 + 2D + 1)y = \sin x$

$C.F. = (C_1 + C_2 x)e^{-x}$

$$P.I. = \frac{1}{D^2+2D+1}\sin x$$

$D^2 = -1$

$$= \frac{1}{-1+2D+1}\sin x$$

$$= \frac{1}{2D}\sin x$$

$$= \frac{1}{2}\int \sin x \, dx$$

$$P.I. = \frac{-\cos x}{2}$$

$G.S. = C.F. + P.I.$

$$= (C_1 + C_2 x)e^{-x} - \frac{\cos x}{2}$$

$D(D^2+4)$

$D^2 = 4$

$D = \pm 2$

$C_1 e^{-2x}$

$D^2 = -1$

$D =$

Note: If D^r contains $\frac{1}{mD+n}$ term in $\phi(D)$ then bring D^2 term by multiplying N^r & D^r by conjugate → $mD - n$.

Solve: $(D^3 + 4D)y = \cos 2x$

$C.F. = C_1 + (C_2 \cos 2x + C_3 \sin 2x)$

$$P.I. = \frac{1}{D^3+4D}\cos 2x$$

$D^2 = -4 \Rightarrow 0$

$$= \frac{1}{3D^2+4}x\cos 2x$$

Put $D^2 = -4$

$$P.I. = \frac{-1}{8}x\cos 2x.$$

$$G.S. = C_1 + (C_2\cos 2x + C_3 \sin 2x) - \frac{1}{8}x\cos 2x.$$

Solve: $(D^2+1)y = \sin 2x \cos x$

C.F. $= c_1 \cos x + c_2 \sin x$.

P.I. $= \frac{1}{D^2+1} \sin 2x \cos x$

$= \frac{1}{D^2+1}\left(\frac{\sin(3x)+\sin x}{2}\right)$

$= \frac{1}{D^2+1}\frac{\sin 3x}{2} + \frac{1}{D^2+1}\frac{\sin x}{2}$

$= -\frac{1}{16}\sin 3x + \frac{1}{2D}\frac{x\sin x}{2}$

$= -\frac{1}{16}\sin 3x + \frac{x}{4}(-\cos x)$

$= \frac{-1}{16}\sin 3x - \frac{x\cos x}{4}$

G.S. $= c_1\cos x + c_2 \sin x - \left(\frac{1}{16}\sin 3x + \frac{x\cos x}{4}\right)$

Case III: when $f(x) = \cosh(ax+b)$ or $\sinh(ax+b)$
Put $D^2 = a^2$ in $\phi(D)$.

Ex: $(D^2+1)y = \sinh(3x)$ $\quad$ or $\quad \sinh = \frac{e^x - e^{-x}}{2}$

$\Rightarrow$ P.I. $= \frac{1}{D^2+1}\sinh(3x)$ $\quad\quad \cosh = \frac{e^x + e^{-x}}{2}$.

$D^2 = 3^2 = 9$

$= \frac{1}{10}\sinh(3x)$.

H.W.

Solve 1) $(D^2+3D+2)y = \sin^2 x$

2) $(D^2-1)y = \cos x \cos 2x$

$D^2+2D+D+2$

$D(D+2)+1(D+2)$

$D=-1,-2$

Soln: C.F. $= c_1 e^{-x} + c_2 e^{-2x}$.

P.I. $= \frac{1}{(D+1)(D+2)}\sin^2 x$.

$= \frac{1}{D^2+3D+2}\sin^2 x$.

Case IV: If $f(x) = x^m$ in $\phi(D)y = f(x)$.

$$P.I. = \frac{1}{\phi(D)} x^m.$$

$$= [\phi(D)]^{-1} x^m.$$

$$= [\text{Expanding by binomial exp}^n]\, x^m.$$

adjust $\rightarrow (1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$

$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots$

Note: Depending upon $x^m \rightarrow m$ derivatives are possible. where x can be in form of $D = \frac{d}{dx}$

$\therefore$ Use the expansion upto the power m.

ex: $(D^2-1)y = x^3$

Sol^n: $P.I. = \frac{1}{D^2-1} x^3$

$$= \frac{1}{-1[1-D^2]} x^3$$

$$= -[1-D^2]^{-1} x^3$$

Using $(1-x)^{-1} = 1 + x + x^2 + \dots$

$$= -(1 + D^2 + (D^2)^2 + \dots)\, x^3$$

$$= -[1 + D^2 + D^4 + \dots]\, x^3$$

$\therefore D = \frac{d}{dx}$

neglecting the terms from D^4

$$= -(1+D^2)x^3$$

$$= -(x^3 + D^2(x^3))$$

$$= -(x^3 + 6x).$$

$D(x^3) = 3x^2$

$D^2(x^3) = 6x$

$D^3(x^3) = 6$

Solve: $(D^2 - D + 1)y = x^3 - 3x^2 + 1$

Sol^n: $P.I. = \frac{1}{D^2-D+1} x^3 - \frac{3}{D^2-D+1} x^2 + \frac{1}{D^2-D+1}$

$$= \frac{1}{D^2-D+1}(x^3 - 3x^2 + 1).$$

$$= (1 + (D^2 - D))^{-1} (x^3 - 3x^2 + 1)$$

$$= (1 - (D^2-D) + (D^2-D)^2 - (D^2-D)^3 + \dots)\, (x^3 - 3x^2 + 1)$$

$\because$ Maximum derivatives possible for x^3-3x^2+1 are 3.

$\therefore$ Discarding the terms from D^4 - - . .

$= [1-D^2+D+D^4-2D^3+2D^2+D^3-](x^3-3x^2+1)$

$= [1+D-2D^3](x^3-3x^2+1)$

$= (x^3-3x^2+1)+3x^2-6x-6$

Case V: $f(x)=e^{ax}.V$ (where v is any other f^n of x)

$$P.I. = \frac{1}{\phi(D)} e^{ax}.V$$

Replace $D = D+a$.

$$P.I. = e^{ax}.\frac{1}{\phi(D+a)}V.$$

Using any of the previous cases depending upon V, find P.I.

Q. Find P.I. for

$(D^2-4D+3)y = x^3e^{2x}$

$$P.I. = \frac{1}{D^2-4D+3}e^{2x}.x^3$$ here $V=x^3$, $a=2$

$D = D+2$

$$= e^{2x}\frac{1}{(D+2)^2-4(D+2)+3}x^3$$

$$= e^{2x}.\frac{1}{D^2+4D+4-4D-8+3}x^3$$

$$= e^{2x}\frac{1}{D^2-1}x^3$$ Using case IV for P.I.

$= -e^{2x}(1-D^2)^{-1}x^3$

$= -e^{2x}(1+D^2]x^3$

$= -e^{2x}[x^3+6x]$.

Q. Find P.I. for

$(D^2-4D+4)y = e^{x}\cos 2x$

$V = \cos 2x$

$a = 1$

$$P.I. = \frac{1}{D^2 - 4D + 4} \cdot e^x \cos 2x$$

$$D = D+1$$

$$= e^x \frac{1}{(D+1)^2 - 4(D+1) + 4} \cos 2x$$

$$= e^x \frac{1}{D^2 + 2D + 1 - 4D - 4 + 4} \cos 2x$$

$$= e^x \frac{1}{D^2 - 2D + 1} \cos 2x$$

$$D^2 = -4$$

$$= e^x \frac{1}{-4 - 2D + 1} \cos 2x$$

$$= e^x \frac{1}{-3 - 2D} \cos 2x$$

$$= e^x \frac{1 (2D-3) \cos 2x,}{-(2D+3) \times (2D-3)}$$

$$= -e^x \frac{(2D-3)}{4D^2 - 9} \cos 2x,$$

$$D^2 = -4$$

$$= -e^x \frac{2D - 3}{-25} \cos 2x$$

$$= \frac{e^x}{25} \left[(-4 \sin 2x) - 3 \cos 2x \right]$$

$$= \frac{e^x}{25} \left[-4 \sin 2x - 3 \cos 2x \right].$$

Case VI: $f(x) = x \cdot V$ $[V = \sin ax \text{ or } \cos ax]$

$$P.I. = \frac{1}{\phi(D)} x \cdot V.$$

$$= \left[x - \frac{\phi'(D)}{\phi(D)} \right] \frac{1}{\phi(D)} V$$

7/09/22

Tutorial - 2

Solve

① $(D^2 - 9D + 18)y = e^{e^{-3x}}$

② $(D^3 + D)y = \cos x$

③ $(D^3 - 5D^2 + 8D - 4)y = e^{2x} + 2^{-x} + 3$

④ $(D^3 + 8)y = x^4 + 2x + 1$

⑤ $(D^2 + 2D + 1)y = \dfrac{e^{-x}}{x+2}$

⑥ $(D^2 + 3D + 2)y = x\sin 2x$

Sol^n ①: $C.F = C_1 e^{6x} + C_2 e^{3x}$

$P.I. = \dfrac{1}{(D-6)(D-3)} e^{e^{-3x}}$

consider $\dfrac{1}{(D-6)(D-3)} = \dfrac{A}{D-6} + \dfrac{B}{D-3}$

$1 = A(D-3) + B(D-6)$

Put $D = 6$ Put $D = 3$

$A = \dfrac{1}{3}$ $B = \dfrac{-1}{3}$

$P.I. = \dfrac{1}{3(D-6)} e^{e^{-3x}} - \dfrac{1}{3(D-3)} e^{e^{-3x}}$

$P.I. = \dfrac{1}{3D-18} e^{e^{-3x}} - \dfrac{e^{e^{-3x}}}{3D-9}$

$= \dfrac{1}{3} e^{+6x} \int e^{-6x} e^{e^{-3x}} dx - \dfrac{1}{3} e^{+3x} \int e^{-3x} e^{e^{-3x}} dx$

$e^{-3x} = t$

$-3e^{-3x} dx = dt$

$$\frac{1}{3}e^{6x}\int \frac{(e^{t})^2 dt}{-3} - \frac{1}{3}e^{3x}\int \frac{e^t\,dt}{-3}$$

$$= \frac{-\cancel{\;}}{\cancel{\;}}\,\cancel{e^{6x}}\,e^{-3x}$$

$$P.I. = \frac{-1}{18}e^{6x}\,e^{2e^{-3x}} + \frac{1}{9}e^{3x}\,e^{e^{-3x}}$$

$$G.S. = C_1e^{6x} + C_2e^{3x} - \frac{1}{18}e^{6x}e^{2e^{-3x}} + \frac{1}{9}e^{3x}e^{e^{-3x}}$$

② $(D^3+D)y = \cos x$

$D^3 + D = 0$

$\cancel{D^2} = -D^2$

$D^2 = \cancel{\;}1.$

Sol^n: $C.F = C_1 + C_2\cos x + C_3\sin x,\ C_3$

$$P.I. = \frac{1}{D^3+D}\cos x$$

$$= \frac{1}{D(D^2+1)}\cos x$$

$D^2 = -a^2$

$D^2 = -1$

~~eeeeboe~~ $\therefore \phi(a) = 0$

$\therefore$ Differentiating

$$P.I. = \frac{1}{3D^2+1}x.\cos x$$

Putting $D^2 = -1$

$$\boxed{P.I. = \frac{1}{-2}x\cos x.}$$

$\therefore$ G.S. = C.F + P.I.

$$\boxed{G.S = C_1 + C_2\cos x + C_3\sin x - \frac{1}{2}x\cos x}$$

③ $(D^3 - 5D^2 + 8D - 4)y = e^{2x} + 2^{-x} + 3$

Sol^n: $D-1$ is a factor of $D^3 - 5D^2 + 8D - 4$

$$
\begin{array}{r}
D^2-4D+4 \\
D-1 \overline{) \cancel{D^3}-5D^2+8D-4} \\
\pm \cancel{D^3} \mp D^2 \\
\hline
-\cancel{4D^2}+8D-4 \\
\mp \cancel{4D^2} \pm 4D \\
\hline
4D-4 \\
\pm 4D \mp 4 \\
\hline
0 \\
\hline
\end{array}
$$

$\therefore (D^3-5D^2+8D-4) = (D-1)(D-2)(D-2)$

~~$P.I. = \frac{1}{(D-1)(D+2)(D+2)} e^{2x} + \frac{1}{(D-1)(D+2)(D-2)}$~~

$C.F = C_1 e^{x} + (C_2 + C_3)e^{2x}$

$P.I. = \frac{1}{(D-1)(D-2)(D-2)} e^{2x} + \frac{1}{(D-1)(D-2)(D-2)} 2^{-x} + \frac{3}{(D-1)(D-2)(D-2)}$

Putting $D = 2$

$\phi = 0$

$\therefore$ Differentiating

$= \frac{1}{3D^2-10D+8} x e^{2x} + \frac{1}{D^3-5D^2+8D-4} 2^{-x} + \frac{3}{D^3-5D^2+8D-4}$

Put $D=2$ Put $D = -\log 2$ Put $D = 0$

$=$ ~~e^{2x}~~

$\phi'(a) = 0$

$\therefore$ Differentiating again

$= \frac{1}{6D-10} x^2 e^{2x} + \frac{2^{-x}}{(\log 2)^3 - 5(\log)^2 + 8\log 2 - 4} - \frac{3}{4}$

Put $D=2$

$P.I. = \frac{1}{2} x^2 e^{2x} + \frac{2^{-x}}{(-\log 2)^3 - 5(\log 2)^2 - 8\log 2 - 4} - \frac{3}{4}$

$$\boxed{G.S = C_1 e^{x} + (C_2 + C_3)e^{2x} + \frac{1}{2} x^2 e^{2x} + \frac{2^{-x}}{(-\log 2)^3 - 5(\log 2)^2 - 8\log 2 - 4} - \frac{3}{4}}$$

④ $(D^3+8)y = x^4+2x+1$

Soln: C.F. $= C_1 e^{2x} + C_2 \cos x + C_3 \sin x$.

P.I. $= \frac{1}{D^3+8} x^4+2x+1$

$= \frac{1}{8\left(1+\frac{D^3}{8}\right)} (x^4+2x+1)$

$= \frac{1}{8}\left(1+\frac{D^3}{8}\right)^{-1} (x^4+2x+1)$

$= \frac{1}{8}\left[1-\frac{D^3}{8}+\left(\frac{D^3}{8}\right)^2 - \dots\right] (x^4+2x+1)$

$= \frac{1}{8}\left[1-\frac{D^3}{8}\right](x^4+2x+1)$ neglecting the terms from $\left(\frac{D^3}{8}\right)^2 \dots$

$= \left[\frac{1}{8}-\frac{D^3}{64}\right](x^4+2x+1)$

$= \frac{1}{8}(x^4+2x+1) - \frac{1}{64}(24x)$

P.I. $= \frac{1}{8}(x^4+2x+1) - \frac{3x}{8}$.

$$\boxed{G.S. = C_1 e^{2x} + C_2 \cos x + C_3 \sin x + \frac{1}{8}(x^4+2x+1) - \frac{3x}{8}}$$

$D^3 = -8$
$D = \sqrt[3]{-8}$
$4x^3+2 = D$
$12x^2 = D^2$
$24x = D^3$

⑤ $(D^2+2D+1)y = \frac{e^{-x}}{x+2}$

Soln: C.F. $= C_1 e^{-x} + C_2 e^{-x}$

C.F. $= (C_1+C_2)e^{-x}$

P.I. $= \frac{1}{D^2+2D+1} (x+2)^{-1} e^{-x}$

$V = (x+2)^{-1}$
$a = -1$

$D^2+D+D+1$
$D(D+1)+1$

Replace $D = D+a$

$= e^{-x} \frac{1}{(D-1)^2+2(D-1)+1} (x+2)^{-1}$

$= e^{-x} \frac{1}{D^2-2D+1+2D-2+1} \frac{1}{(x+2)}$

$$= e^{-x} \frac{1}{D^2 (x+2)}$$

$$= e^{-x} \frac{1}{D^2} \frac{1}{2(1+x/2)}$$

$$= \frac{e^{-x}}{2} \frac{1}{D^2} (1+x/2)^{-1}$$

⑤ $(D^2+2D+1)y = \frac{e^{-x}}{x+2}$

Soln: Roots of D^2+2D+1 are $\frac{-2 \pm \sqrt{4-1}}{2+1}$

$$C.F. = (C_1 + C_2 x) e^{-x}$$

$$P.I. = \frac{1}{D^2+2D+1} \left(\frac{e^{-x}}{x+2} \right)$$

Put $D = D-1$

$$\therefore P.I. = e^{-x} \frac{1}{(D-1)^2 + 2[2(D-1)+1]} \cdot \frac{1}{(x+2)}$$

$$= e^{-x} \frac{1}{D^2 - 2D + 1 + 2D - 2 + 1} \cdot \frac{1}{x+2}$$

$$= e^{-x} \frac{1}{D^2 + (x+2)}$$

$$= e^{-x} \frac{1}{D^2} \frac{1}{x+2}$$

$$= e^{-x} \frac{1}{D} \int \frac{1}{x+2}$$

$$= e^{-x} \frac{1}{D} \log(x+2) + C_3$$

$$= e^{-x} \int \log(x+2) + C_3$$

$$= e^{-x} [x \log(x+2) - 2x + \log(x+2)] + C_4$$

$$\therefore G.S. = (C_1 + C_2 x) e^{-x} + e^{-x} [x \log(x+2) - 2x + \log(x+2)] + C_4$$

⑥ $(D^2+3D+2)y = x\sin 2x$

Sol^n: $CF = C_1e^{-x} + C_2e^{-2x}$

$$P.I. = \frac{1}{D^2+3D+2}x\sin 2x$$

$$= \left[x - \frac{2D+3}{D^2+3D+2}\right]\frac{1}{D^2+3D+2}\sin 2x$$

$$= \left[x - \frac{2D+3}{D^2+3D+2}\right]\frac{1}{3D-2}\sin 2x$$

$$= \left[x - \frac{2D+3}{3D-2}\right]\frac{3D+2}{9D^2-4}\sin 2x$$

$$= \left[x - \frac{2D+3}{3D-2}\right]\frac{3D+2}{-40}\sin 2x$$

$$= \left[x - \frac{2D+3}{3D-2}\right]\times\left(\frac{-1}{40}\right)\times\left[3D\sin 2x + 2\sin 2x\right]$$

$$= \left[x - \frac{2D+3}{3D-2}\right]\times\left(\frac{-1}{40}\right)\times\left[6\cos 2x + 2\sin 2x\right]$$

$$= -\frac{6x\cos 2x}{40} - \frac{2x\sin 2x}{40} + \frac{6D^2+13D+6}{9D^2-4}\left[\frac{6\cos 2x + 2\sin 2x}{40}\right]$$

$$= -\frac{6x\cos 2x}{40} - \frac{2x\sin 2x}{40} + \frac{-24+13D+6}{-40}\left[\frac{6\cos 2x + 2\sin 2x}{40}\right]$$

$$= \frac{-6x\cos 2x}{40} - \frac{2x\sin 2x}{40} + \frac{13D-18}{40}\left[\frac{6\cos 2x + 2\sin 2x}{40}\right]$$

$$= \frac{1}{40}\left[-6x\cos 2x - 2x\sin 2x + 13D - 18(6\cos 2x + 2\sin 2x)\right]$$

$$= \frac{1}{40}\left[-6x\cos 2x - 2x\sin 2x + 78D\cos 2x + 26D\sin 2x - 108\cos 2x - 36\sin 2x\right]$$

$$= \frac{1}{40}\left[-6x\cos 2x - 2x\sin 2x - 156\sin 2x + 52\cos 2x - 108\cos 2x - 36\sin 2x\right]$$

$$= \frac{1}{40}\left[-6x\cos 2x - 2x\sin 2x - 192\sin 2x - 56\cos 2x\right]$$

$$P.I. = \frac{1}{40}\left[(-6x-56)\cos 2x + (-2x-192)\sin 2x\right]$$

$\therefore$ G.S. $= c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{40}\left[(-6x-56)\cos 2x + (-2x-192)\sin 2x\right]$

$\left(\frac{7-30x}{200}\right)\cos 2x + \left(\frac{12-5x}{100}\right)\sin 2x.$

(8) Method of variation of parameters for P.I.

Procedure:

If C.F. of L.D.E (2nd order)

$\Rightarrow Ay_1 + By_2$ where A, B arb. const. or will say parameters.

y_1 & y_1 are f^{ns} of x.

Varying the parameters A & B

Let P.I $= uy_1 + vy_2$.

where $u = \int \frac{-y_2 f(x)}{W}\,dx$ & $v = \int \frac{y_1 f(x)}{W}\,dx$.

where W = wronskian

$= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$

Ex: $(D^2+4)y = \sec 2x$

$D^2 = 4$

$D = \pm 2$

C.F. $=$ ~~$c_1e^{2x}+c_2$~~ $c_1\cos 2x + c_2\sin 2x.$

$= A\cos 2x + B\sin 2x.$

P.I. $= u\cos 2x + v\sin 2x.$ — (1)

$y_1 = \cos 2x \quad y_2 = \sin 2x$

$W = \cos 2x\cdot(2\cos 2x) - \sin 2x(-2\sin 2x)$

$= 2\cos^2 2x + 2\sin^2 2x$

$W = 2$

where $u = \int \frac{-y_2 f(x)}{W}\,dx$

$= \int -\frac{\sin 2x \cdot \sec 2x}{2}\,dx$

$u = -\frac{1}{2}\int \tan 2x\,dx$

$= \frac{1}{2}\frac{\log|\cos 2x|}{2}$

$u = \frac{1}{4}\log|\cos 2x|$

$$V = \int \frac{\cos 2x \cdot \sec 2x}{2}\,dx$$

$$V = \frac{x}{2}$$

$$P.I. = \frac{1}{4}\log|\cos 2x|\cos 2x + \sin 2x \cdot \frac{x}{2}$$

$$G.S. = C_1\cos 2x + C_2\sin 2x + \frac{1}{4}\log|\cos 2x|\cos 2x + \frac{x}{2}\sin 2x.$$

Solve: $(D^2 - 6D + 9)y = \frac{e^{3x}}{x^2}$

$D^2 - 6D + 9$

D^2

$\Rightarrow$ $C.F. = (C_1 + C_2x)e^{3x}$

$y_1 = e^{3x}$ $\quad y_2 = xe^{3x}$

$W = e^{3x}(xe^{3x}\cdot 3 + e^{3x}) - xe^{3x}\cdot e^{3x}\cdot 3$

$W = 3xe^{6x} + e^{6x} - 3xe^{6x}$

$W = e^{6x}.$

$P.I. = u\cdot e^{3x} + v\cdot xe^{3x}$

$$u = \int \frac{-xe^{3x}\cdot\frac{e^{3x}}{x^2}}{e^{6x}}\,dx$$

$$u = \int \frac{-1}{x}\,dx$$

$$u = -\log x$$

$$v = \int \frac{e^{3x}\cdot\frac{e^{3x}}{x^2}}{e^{6x}}\,dx$$

$$v = \int \frac{1}{x^2}\,dx.$$

$$v = \frac{-1}{x}$$

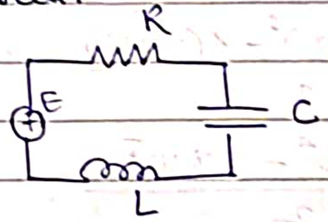
$$P.I. = -\log x\cdot e^{3x} - \frac{1}{x}xe^{3x}$$

$$= (-\log x)e^{3x} - e^{3x}$$

$$G.S. = (C_1 + C_2x)e^{3x} + (-\log x)e^{3x} - e^{3x}.$$

1) $(D^2+1)y = \operatorname{cosec} x$
2) $(D^2-2D+2)y = e^x \tan x.$

*) Applications to Electric circuit.
LRC Circuit.



To form LDE → Kirchoff's law.
Potential Diff. across R → Ri.
" " " L → $L\frac{di}{dt}$

C → $\frac{Q}{C}$.

By the Kirchoff's law
Algebraic sum of potential diff. across the components of the ckt = total emf in the ckt.

LDE ⇒ $L\frac{di}{dt} + Ri + \frac{Q}{C} = E(t).$ $\frac{dQ}{dt} = i.$

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E(t). \quad — ①.$$

① ⇒ 2nd order LDE in Q ⇒ soln ⇒ $Q(t)$ - f^n of t.

Q. A circuit with $L = 0.05\,H$, $R = 5\Omega$ & $C = 4\times10^{-4}F$. If $Q=0$, $i=0$ at $t=0$. Find $Q(t)$ & $i(t)$ when if there is constant emf of 110 V is applied.

Soln: Let LDE ⇒ $L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E$

$$0.05\frac{d^2Q}{dt^2} + 5\frac{dQ}{dt} + \frac{Q}{4\times10^{-4}} = 110.$$

÷ by 0.05

$$\frac{d^2Q}{dt^2} + 100\frac{dQ}{dt} + 5\times10^4 Q = 2200 \quad —①$$

To find C.P

Consider AE $\Rightarrow$ $D^2 + 100D + 50000 = 0$

$$D = \frac{-100 \pm \sqrt{(100)^2 - 4(50,000)}}{2}$$

$$= \frac{-100 \pm \sqrt{10^4 - 2\times10^5}}{2}$$

$$= -50 \pm 50\sqrt{19}\, i \qquad \alpha = -50 \quad \beta = 50\sqrt{19}$$

C.F. $= e^{-50t}\left[A\cos 50\sqrt{19}t + B\sin 50\sqrt{19}t\right]$

P.I. $= \frac{1}{D^2+100D+50,000}(2200)e^{0t}$

Put $D=0$

$$= \frac{1}{50,000}(2200)$$

$$= 0.044$$

$\therefore$ Sol^n $\Rightarrow$ $Q(t) =$ C.F. + P.I.

$$Q(t) = e^{-50t}\left[A\cos 50\sqrt{19}t + B\sin 50\sqrt{19}t\right] + 0.044 \quad —②$$

Initially when $t=0,\ i=0,\ Q=0$

$$0 = A + 0.044$$

$$A = -0.044$$

Diff. ② w.r.t t

$$\frac{dQ}{dt} = i(t) = \left[A\cos 50\sqrt{19}t + B\sin 50\sqrt{19}t\right]-50e^{-50t} + e^{-50t}\left[-A50\sqrt{19}\cos 50\sqrt{19} + B50\sqrt{19}\sin 50\sqrt{19}\right]$$

$$i(t) = A\left[-50e^{-50t}\cos 50\sqrt{19}t - e^{-50t}50\sqrt{19}\sin 50\sqrt{19}t\right] + B\left[-50e^{-50t}\sin\omega t + e^{-50t}50\sqrt{19}\cos 50\sqrt{19}t\right] \quad —③$$

At $t=0,\ i=0$

$$0 = -50A + B50\sqrt{19}$$

$$0 = 2.2 + B50\sqrt{19}$$

$$B = -0.01$$

Putting A & B in ② & ③

Q. Determine Q(t) & i(t) in LRC circuit with $L = 0.5H$ $R = 6\Omega$ $C = 0.02F$ & $E(t) = 24 \sin 10t$. when $t = 0$, $Q = 0$ & $I = 0$. Also find steady state condn.

Soln:
$$0.5 \frac{d^2Q}{dt^2} + 6 \frac{dQ}{dt} + \frac{Q}{0.02} = 24 \sin 10t.$$

$$\frac{d^2Q}{dt^2} + 12 \frac{dQ}{dt} + 100Q = 48 \sin 10t \quad — ①$$

To find C.F.

AE → $D^2 + 12D + 100 = 0.$

$$D = \frac{-12 \pm \sqrt{(12)^2 - 4(100)}}{2}$$

$$= \frac{-12 \pm \sqrt{144 - 400}}{2}$$

$$= \frac{-12 \pm 16i}{2}$$

$$D = -6 \pm 8i$$

$$C.F. = e^{-6t}\left[A \cos 8t + B \sin 8t\right]$$

$$P.I. = \frac{1}{D^2 + 12D + 100} 48 \sin 10t. \qquad 10000 - 1200 + 100$$

Put $D^2 = -100$

$$= \frac{1}{12D} 48 \sin 10t$$

$$= 4 \cdot \frac{1}{D} \sin 10t$$

$$= 4 \cdot \frac{-\cos 10t}{10}$$

$$= -\frac{2}{5} \cos 10t.$$

2×10^5

$Q(t) = C.F + P.I.$

$= e^{-6t}[A\cos 8t + B\sin 8t] - \frac{2}{5}\cos 10t.$ — ②

$t=0,\ i=0,\ Q=0$

$0 = A - \frac{2}{5}$

$\boxed{A = \frac{2}{5}}$

$i(t) = -6e^{-6t}[A\cos 8t + B\sin 8t] + \cancel{4\sin 10t}$
$e^{-6t}[-8A\sin 8t + 8B\cos 8t] + 4\sin 10t$ — ③

$t=0,\ i=0$

$0 = -6[A] + 8B$

$0 = -6 \times \frac{2}{5} + 8B$

$\boxed{B = 0.3}$

For steady state condition $t \to \infty$

$\Rightarrow e^{-\infty} = 0.$

$Q(t) =$ ∵ cosine func^n $\not\leftrightarrow$ $t \to \infty$ → cannot be determined

Q. Find Q & I in LRC circuit. $L=2$, $R=4$, $C=0.05$ & $E=100V$. Initially $t=0, Q=0, I=0$. Find steady state cond^n.

27/09/22

Partial Differentiation

1) One-dimensional Heat flow

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad -(1)$$

$\Rightarrow u(x,t) = ?$

Solution by method of separation of variables.

Let $u = F(x) . G(t)$. be sol^n of (1).

Diff. w.r.t t partially. & diff (1) partially w.r.t x. (2 times)

$$\frac{\partial u}{\partial t} = F(x)\, G'(t). \qquad \frac{\partial^2 u}{\partial x^2} = F''(x)\, G(t).$$

$\therefore$ From eq (1)

$$F(x)\,G'(t) = c^2 F''(x)\, G(t)$$

$$\frac{G'}{c^2 G} = \frac{F''}{F} = k$$

$$\frac{G'}{c^2 G} = k \quad \& \quad \frac{F''}{F} = k \quad -(2)$$

Depending upon k

Case I : $k=0$

$\Rightarrow G' = 0 \quad \& \quad F'' = 0$

$\rightarrow \boxed{G(t) = c_1} \Rightarrow \boxed{F(x) = c_2 x + c_3}$

$\therefore$ sol^n is $u(x,t) = F(x).G(t)$

$= c_1(c_2 x + c_3)$.

$u(x,t) = c_4 x + c_5$.

Case II : when $k>0$, let $k = m^2$

$\therefore (2) \rightarrow \frac{G'}{c^2 G} = m^2 \quad \& \quad \frac{F''}{F} = m^2$

Ordinary diff. eq^n. $\left[\begin{array}{l} G' - c^2 m^2 G = 0 \quad \& \quad F'' - m^2 F = 0 \\ \Rightarrow (D - c^2 m^2) G(t) = 0 \quad (D^2 - m^2) F(x) = 0 \end{array} \right.$

$G(t) = c_1 e^{c^2 m^2 t} \quad \& \quad F(x) = c_2 e^{mx} + c_3 e^{-mx}$

$\therefore$ From (1) sol^n $\Rightarrow u(x,t) = F(x).G(t)$

$u(x,t) = c_1 e^{c^2 m^2 t}\left[c_2 e^{mx} + c_3 e^{-mx}\right]$.

$u(x,t) = (c_4 e^{mx} + c_5 e^{-mx})\, e^{c^2 m^2 t} \quad -(\mathrm{II})$.

Case III: If $K<0$, say $K = -m^2$

$\therefore$ (2) becomes

$$\frac{G'}{c^2 G} = -m^2 \quad \& \quad \frac{F''}{F} = -m^2$$

$$G' + c^2 m^2 G = 0 \quad \& \quad F'' + m^2 F = 0$$

$$G(t) = C_1 e^{-c^2 m^2 t} \qquad F(x) = C_2 \cos mx + C_3 \sin mx.$$

$$u(x,t) = C_1 e^{-c^2 m^2 t} (C_2 \cos mx + C_3 \sin mx).$$

$$= (C_4 \cos mx + C_5 \sin mx) e^{-c^2 m^2 t}.$$

$\therefore$ As the problem is for heat conduction $\rightarrow$ as time $\uparrow$es temperature should decrease $\Rightarrow$ solution obtained for $K<0$ is an appropriate solⁿ for heat eqⁿ.

$$u(x,t) = (A \cos mx + B \sin mx) e^{-c^2 m^2 t}.$$

Q. Solve the heat eqⁿ for the boundary conditions

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

1) $u(0,t) = 0 \quad \forall t$

2) $u(1,t) = 0 \quad \forall t$

3) $u(x,0) = 50 \quad 0 \le x \le 1$

Solⁿ: Let the solⁿ be $u(x,t) = (A \cos mx + B \sin mx) e^{-c^2 m^2 t}$ —— (1)

From condⁿ (1) $u(0,t) = 0 \Rightarrow 0 = A e^{-c^2 m^2 t}$

$$\Rightarrow A = 0.$$

$\therefore$ (1) becomes $u(x,t) = B \sin mx \, e^{-c^2 m^2 t}$ —— (2).

Now $u(1,t) = 0 \Rightarrow 0 = B \sin m \, e^{-c^2 m^2 t}$

$$\Rightarrow \sin m = 0$$

$$m = n\pi$$

$$n = 1, 2, \ldots$$

$\therefore$ (2) becomes

$$u(x,t) = B \sin(n\pi) x \, e^{-c^2 n^2 \pi^2 t} \quad \text{—— (3)}$$

$\because$ condⁿ (3) $\Rightarrow u(x,0) = 50$

$\therefore$ put $t=0$ in (3)

$u(x,0) = 50 = B\sin(n\pi)x \quad n = \cancel{0}, 1, 2, \ldots$

or

$$\underset{f(x)}{50} = \sum_{n=1}^{\infty} b_n \sin(n\pi x) \quad -(4). \rightarrow \text{equivalent to half range Fourier sine series. in}$$

where $b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx. \quad 0 \le x \le 1$

where l = length of rod.

here $l = 1$.

$$b_n = 2\int_0^1 50\sin(n\pi x)\,dx.$$

$$b_n = 100\left[\frac{-\cos(n\pi x)}{n\pi}\right]_0^1$$

$$\therefore b_n = \frac{-100}{n\pi}\left[\cos n\pi - \cos 0\right]$$

$$b_n = \frac{-100}{n\pi}\left[(-1)^n - 1\right]$$

$$b_n = 100\left[\frac{1-(-1)^n}{n\pi}\right] \quad - (5)$$

From ~~(3) & (4)~~ (3) & (4) & (5)

$$u(x,t) = \sum_{n=1}^{\infty} 100\frac{[1-(-1)^n]}{n\pi}\sin(n\pi x)e^{-c^2n^2\pi^2 t}$$

$$u(x,t) = \frac{100}{\pi}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n}\sin(n\pi x)e^{-c^2n^2\pi^2 t}$$

If temperature required at half distance $\Rightarrow$ put $x = \frac{1}{2}$ m.

$$u\left(\frac{1}{2}, t\right) = \frac{100}{\pi}\sum\left[\frac{1-(-1)^n}{n}\right]\sin\left(\frac{n\pi}{2}\right)e^{-c^2n^2\pi^2 t}$$

$\hookrightarrow 0$ for n = even.

$\hookrightarrow (-1)^n$ for n = odd.

classmate

Date ____
Page ____

FIS-IDFIS

Q. Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ with

1) $u(0,t) = 0 \quad \forall t$
2) $u(\pi,t) = 0 \quad \forall t$
3) $u(x,0) = \pi x - x^2 \quad 0 \le x \le \pi$

Soln: let $u(x,t) = (A\cos mx + B\sin mx)\, e^{-m^2 t}$ — ①

from condn ① $u(0,t) = 0 \Rightarrow 0 = A e^{-m^2 t}$

$\Rightarrow A = 0.$

$\therefore$ ① becomes $u(x,t) = B\sin mx\, e^{-m^2 t}$ — ②

Now $u(\pi,t) = 0 \Rightarrow 0 = B\sin m\pi\, e^{-m^2 t}$

$\sin m\pi = 0$

$\sin m\pi = \sin n\pi$

$m = n.$

$\therefore$ ② becomes $u(x,t) = B\sin nx\, e^{-n^2 t}$ — ③

condn ③ $\Rightarrow u(x,0) = \pi x - x^2$

put $t=0$ in eqn ③

$\pi x - x^2 = B\sin nx$

$\pi x - x^2 = \sum_{n=1}^{\infty} b_n \sin nx$ — ④

$b_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$

$= \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2) \sin(nx)\, dx$

$= \frac{2}{\pi}\left[\pi x - x^2 \int \sin(nx) - \int (\pi - 2x) \int \sin nx\, dx\right]$

$= \frac{2}{\pi}\left[-\frac{\cos nx}{n}(\pi x - x^2) - \int (\pi - 2x)\left(\frac{-\cos nx}{n}\right) dx\right]$

$= \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{-\cos nx}{n}\right) + \int\left(\frac{\pi\cos nx}{n} - \frac{2x\cos nx}{n}\right) dx\right]$

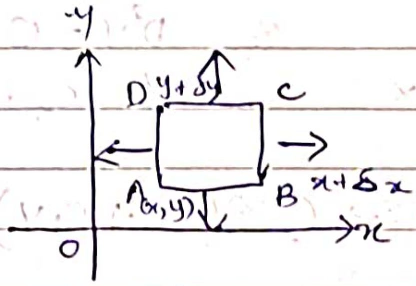
$\int uv\,dx = uv_1 - u'v_2 + u''v_3 \ldots$

$u', u'' \ldots$ derivative of 1st funcⁿ
$v_1, v_2 \ldots$ integrals. of 2nd

$$b_n = \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - (\pi - 2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$=$

*) 2-dimensional Heat flow
(Laplace Eqⁿ)



In steady state condⁿ (not depending upon time t)

Heat flows through metal plate ABCD entering through AB & AD & flowing out from CD & BC.

$\therefore$ As per law of heat conduction $\Rightarrow$ quantity of heat flowing is proportional to the area & rate of change of temp. w.r.t dist. normal to surface.

if h is thickness of plate $\rightarrow$ sides δx, δy.

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

$\Rightarrow$ is Laplace eqⁿ $\Rightarrow$ u is temp. depending upon x & y.

*) Solution of Laplace eqⁿ
[Method of separation of variables]

Let $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ —— ①

let $u = X \cdot Y$ be solⁿ of ① where X is funcⁿ of x alone.
Y is funcⁿ of Y alone.

$\therefore \frac{\partial^2 u}{\partial x^2} = X''Y$ & $\frac{\partial^2 u}{\partial y^2} = XY''$

$\therefore$ From ①

$$X''Y + Y''X = 0$$

$$\Rightarrow \frac{X''}{X} = \frac{-Y''}{Y} = K$$

Consider $\frac{X''}{X} = k$ — ② & $\frac{-Y''}{Y} = k$ — ③

$\therefore$ ② $\Rightarrow$ $X'' - kX = 0$ & ③ $\Rightarrow Y'' + kY = 0$

which are ordinary diff. eq^{ns}

$(D^2 - k)X = 0$ — ② $(D^2 + k)Y = 0$ — ③

Case I: when $k = 0$

$\Rightarrow D^2X = 0$ & $D^2Y = 0$

$X = c_1 + c_2 x$ $Y = c_3 + c_4 y$

$u = X \cdot Y$

$u(x,y) = (c_1 + c_2 x)(c_3 + c_4 y)$

Case II: when $k > 0$, say $k = m^2$

$\therefore$ ② $\Rightarrow X'' - m^2 X = 0$ ③ $\Rightarrow Y'' + m^2 Y = 0$

$(D^2 - m^2)X = 0$ $(D^2 + m^2)Y = 0$

$X = c_1 e^{mx} + c_2 e^{-mx}$ & $Y = c_3 \cos my + c_4 \sin my$

$\therefore u(x,y) = X \cdot Y$

$u(x,y) = (c_1 e^{mx} + c_2 e^{-mx})(c_3 \cos my + c_4 \sin my)$

Case III: when $k < 0$ say $k = -m^2$

② & ③

$(D^2 + m^2)X = 0$ $(D^2 - m^2)Y = 0$

$X = c_1 \cos mx + c_2 \sin mx$ $Y = c_3 e^{my} + c_4 e^{-my}$

$u(x,y) = X \cdot Y$

$u(x,y) = (c_1 \cos mx + c_2 \sin mx)(c_3 e^{my} + c_4 e^{-my})$

*) Note: As per the boundary $cond^{ns}$ appropriate sol^n is selected

1) if $u(x,\infty) = 0 \;\forall\; x \in (0, l) \Rightarrow$ select the sol^n for $k<0$

$u(x,y) = (c_1 \cos mx + c_2 \sin mx)(c_3 e^{my} + c_4 e^{-my})$

2) if $u(\infty, y) = 0 \;\forall\; y \in (0, l) \Rightarrow$ select the sol^n for $k>0$

$\therefore u(x,y) = (c_1 e^{mx} + c_2 e^{-mx})(c_3 \cos my + c_4 \sin my)$

Q. Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ Using boundary cond^{ns}.

1) $u(x,\infty) = 0$
2) $u(0,y) = 0$
3) $u(1,y) = 0$
4) $u(x,0) = x(1-x) \quad \forall \quad 0 < x < 1.$

Sol^n: Using sol^n $(K<0)$

$$u(x,y) = (C_1 \cos mx + C_2 \sin mx)(C_3 e^{my} + C_4 e^{-my}) \quad \text{—} (1)$$

$$(1) \Rightarrow \quad 0 = (C_1 \cos mx + C_2 \sin mx)\left(C_3 + \frac{C_4}{e^{my}}\right).$$

$C_3 = 0.$

$$u(x,y) = (C_1 \cos mx + C_2 \sin mx)\, C_4 e^{-my}$$
$$= (C_1 C_4 \cos mx + C_2 C_4 \sin mx) e^{-my}$$
$$u(x,y) = (C_5 \cos mx + C_6 \sin mx) e^{-my} \quad \text{—} (2)$$

Cond^n (2) $\Rightarrow$ $u(0,y) = 0 \Rightarrow C_5 e^{-my} = 0$

$\Rightarrow C_5 = 0.$

$$\Rightarrow u(x,y) = C_6 \sin mx\, e^{-my} \quad \text{—} (3).$$

Using cond^n (3) $\Rightarrow$ $u(1,y) = 0$

$\Rightarrow 0 = C_6 \sin m e^{-my}$

$\Rightarrow \sin m = 0$

$m = n\pi \quad n = 1, 2, \ldots$

$\therefore$ (3) becomes

$$u(x,y) = C_6 \sin n\pi x\, e^{-n\pi y} \quad \text{—} (I). \quad n = 1, 2, \ldots$$

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin n\pi x\, e^{-n\pi y}$$

Using cond^n (4)

$u(x,0) = x(1-x)$

(II) $x(1-x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$ is half range F. sine series

in which $b_n = \frac{2}{1} \int_0^1 x(1-x) \sin \frac{n\pi x}{1}\, dx.$

6/10/22

Unit - 3

1) Laplace Transforms

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$= F(s)$$

time → frequency

$$L^{-1}[F(s)] = f(t).$$

Defn: If $f(t)$ is a fn of time (t)
then $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt. = F(s)$ where
s = parameter.

Ex: $L[1] = \int_0^\infty e^{-st}(1)\, dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty$

$$= \frac{-1}{s}[0-1]$$

$$L[1] = \frac{1}{s}$$

$$L[K] = \frac{K}{s} \quad \text{if } s \geq 0$$

$$L[0] = 0.$$

2) $L[e^{at}] = \int_0^\infty e^{-st} e^{at}\, dt$

$$= \int_0^\infty e^{t(a-s)}\, dt$$

$$= \left[\frac{e^{-t(s-a)}}{-(s-a)}\right]_0^\infty$$

$$= \frac{-1}{(s-a)}[0-1]$$

$$\boxed{L[e^{at}] = \frac{1}{s-a}} \quad \text{if } s > a.$$

$$L\left[e^{(-3/2)t}\right] = \frac{1}{s+3/2}$$

3) $$L[\sin(at)] = \int_0^\infty e^{-st} \sin at \, dt$$

$$= \left[\frac{e^{-st}}{s^2+a^2}(-s \sin at - a\cos at)\right]_0^\infty$$

$$= 0 - \frac{1}{s^2+a^2}(0-a)$$

$$\boxed{L[\sin(at)] = \frac{a}{s^2+a^2}}$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

$$2\cos^2\theta = 1+\cos 2\theta$$

$$2\sin^2\theta = 1-\cos 2\theta$$

4) $$L[\cos(at)] = \int_0^\infty e^{-st}\cos at\,dt$$

$$= \left[\frac{e^{-st}}{s^2+a^2}(-s\cos at + a\sin at)\right]_0^\infty$$

$$= 0 - \frac{1}{s^2+a^2}(-s)$$

$$\boxed{L[\cos(at)] = \frac{s}{s^2+a^2}}$$

Q. Find $L[\cos^2 t] = L\left[\frac{1+\cos 2t}{2}\right]$

$$= L\left[\frac{1}{2}\right] + L\left[\frac{\cos 2t}{2}\right]$$

$$L[\cos^2 t] = \frac{1}{2s} + \frac{s}{2(s^2+4)}$$

5) $$L[\sinh at] = L\left[\frac{e^{x}-e^{-x}}{2}\right]$$

$$\sinh x = \frac{e^{x}-e^{-x}}{2}$$

$$\cosh x = \frac{e^{x}+e^{-x}}{2}$$

$$= L\left[\frac{e^{x}}{2}\right] - L\left[\frac{e^{-x}}{2}\right]$$

$$= \frac{1}{2(s-1)} - \frac{1}{2(s+1)}$$

$$= \int_0^\infty e^{-st}\left(\frac{e^{at} - e^{-at}}{2}\right) dt$$

$$= \frac{1}{2}\left[\int_0^\infty e^{-(s-a)t} dt - \int_0^\infty e^{-(s+a)t} dt\right]$$

$$= \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)} - \frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{(s+a)}\right]$$

$$\boxed{L[\sinh at] = \frac{a}{s^2 - a^2}}$$

6) $L[\cosh at] = \int_0^\infty e^{-st}\left(\frac{e^{at} + e^{-at}}{2}\right) dt$

$$= \frac{1}{2}\left[\int_0^\infty e^{-(s-a)t} dt - \int_0^\infty e^{-(s+a)t} dt\right]$$

$$\boxed{L[\cosh at] = \frac{s}{s^2 - a^2}}$$

7) $L[t^n] = \int_0^\infty e^{-st} t^n \, dt$

put $st = u$ as $t \to 0,\ u \to 0$

$s\,dt = du$ $\quad t \to \infty,\ u \to \infty$

$dt = \frac{du}{s}$

$$\therefore L[t^n] = \int_0^\infty e^{-u}\left(\frac{u}{s}\right)^n \frac{du}{s}$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-u} u^n \, du$$

$$\int_0^\infty e^{-x} x^{n-1} dx = \sqrt{n}$$

$$= \frac{1}{s^{n+1}} \sqrt{n+1}$$

but $\Gamma(n+1) = n!$ for n = +ve int.

$\therefore L[t^n] = \dfrac{\Gamma(n+1)}{s^{n+1}}$

$$\boxed{L[t^n] = \frac{n!}{s^{n+1}}}$$

Ex: Find $L\left[t^3 + \sin^2 2t + e^{t/2}\right]$

Sol^n: $L[t^3] + L[\sin^2 2t] + L[e^{t/2}]$

$= \dfrac{3!}{s^4} + L\left[\dfrac{1-\cos 4t}{2}\right] + L[e^{t/2}]$

$= \dfrac{6}{s^4} + L\left[\dfrac{1}{2}\right] - L\left[\dfrac{\cos 4t}{2}\right] + L[e^{t/2}]$

$= \dfrac{6}{s^4} + \dfrac{1}{2s} - \dfrac{s}{2(s^2+16)} + \dfrac{1}{s-1/2}$

*) Existence cond^n: [Sufficient cond^n]
If $f(t)$ is piecewise continuous f^n & is of exponential order, then its L.T. exists.

Piecewise continuous $\Rightarrow$ f^n is continuous in the parts of the interval.

$f(t)$ is of exp. order
if $|f(t)| < Me^{\alpha t}$ where α, M are constants.
$\Rightarrow e^{-\alpha t}|f(t)|$ is bounded $\to$ finite.

Ex: $f(t) = t^2$ is of exp. order $\Rightarrow |t^2| < e^{3t}$.
of 3.

$f(t) = e^{t^2} \Rightarrow |e^{t^2}| > Me^{\alpha t}$. not of exp. order.
$\therefore$ L.T. does not exists.

*) **Properties:**

① Linearity $\Rightarrow L[af(t)+bg(t)] = aL[f(t)] + bL[g(t)]$.

② if $L[f(t)] = F(s)$ then $L[e^{-at}f(t)] = F(s+a)$: 1st shifting prop.

Consider $L[e^{-at}f(t)] = \int_0^\infty e^{-st}[e^{-at}f(t)]dt$

$$= \int_0^\infty e^{-(s+a)t} f(t)\,dt$$

$$= F(s+a).$$

Ex: $L[e^{-3t}\sin 2t]$

$a=3$, $f(t) = 2t$.

$L[\sin 2t] = \frac{2}{s^2+4} = F(s)$

$\therefore$ Put $s = s+3$

$L[e^{-3t}\sin 2t] = F(s+3)$

$$= \frac{2}{(s+3)^2+4}$$

Q. Find L.T. of $L[e^{2t}t^2]$

Sol^n: $a = -2$ $f(t) = t^2$

$L[t^2] = \frac{2!}{s^3}$

$$= \frac{2}{s^3}$$

Put $s = s-2$

$L[e^{-(-2)t}t^2] = \frac{2}{(s-2)^3}$

Q. L.T. of $L[e^{-t}\sin^2 t]$ $L[k] = k/s$

Sol^n: $a=1$ $f(t) = \sin^2 t$

$$L[\sin^2 t] = L\left[\frac{1-\cos 2t}{2}\right]$$

$$= L\left[\frac{1}{2}\right] - L\left[\frac{\cos 2t}{2}\right]$$

$$L[\sin^2 t] = \frac{1}{2s} - \frac{s}{2(s^2+4)}$$

Put $s = s+1$

$$L[e^{-t}\sin^2 t] = \frac{1}{2(s+1)} - \frac{s+1}{2((s+1)^2+4)}$$

Q. $L[e^{-t/2}\sqrt{t}]$

Sol^n: $a = \frac{1}{2}$ $f(t) = \sqrt{t}$ $\Gamma(n+1) = n\Gamma n$

$\Gamma\frac{1}{2} = \sqrt{\pi}$

$$L[\sqrt{t}] = \frac{\Gamma(\frac{1}{2}+1)}{s^{\frac{1}{2}+1}}$$

$$= \frac{\frac{1}{2}\Gamma\frac{1}{2}}{s^{3/2}}$$

$$= \frac{\frac{1}{2}\sqrt{\pi}}{s^{3/2}}$$

Put. $s = s + \frac{1}{2}$

$$L[e^{-t/2}\sqrt{t}] = \frac{\sqrt{\pi}/2}{(s+\frac{1}{2})^{3/2}}.$$

③ change of scale → if $L[f(t)] = F(s)$ then $L[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$

Ex: $L[\sin(3t)] = \frac{1}{3}F\left(\frac{s}{3}\right)$ where $F(s) = L[\sin t]$

$= \frac{1}{3}\frac{1}{\left(\frac{s}{3}\right)^2+1} = \frac{3}{s^2+9}$ $= \frac{1}{s^2+1}$

④ If $L[f(t)] = F(s)$
then $L[F(t)] = e^{-as}f(s)$.
when $F(t) = f(t-a)$ $t>a$
$= 0$ $t<a$
(2nd shifting).

Ex: Find $L[F(t)]$ where
$F(t) = (t-1)^2$ $t>1$
$= 0$ $t<1$ — by 2nd shifting property

Here $L[f(t)] = e^{-as}F(s)$
$a=1$ $F(s) = L[f(t)]$
here $f(t-a) = f(t-1) = (t-1)^2$
$f(t) = t^2$
$L[t^2] = \frac{2!}{s^3}$

$L[t^2] = \frac{2}{s^3}$

$\therefore L[F(t)] = e^{-s}\frac{2}{s^3}$

⑤ Multiplication by t
if $L[f(t)] = F(s)$
then $L[t f(t)] = (-1)\frac{d}{ds}F(s)$

|||'ly $L[t^2 f(t)] = (-1)^2 \frac{d^2}{ds^2}F(s)$

$\vdots$ $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}F(s)$.

Ex. Find $L[t\sin 3t]$
Solⁿ: here $f(t) = \sin 3t$
$= \frac{3}{s^2+9}$

$\therefore \quad L[t f(t)] = (-1)\frac{d}{ds} F(s)$

$L[t\sin 3t] = (-1)\frac{d}{ds}\left(\frac{3}{s^2+9}\right)$

$= (-1)\left(\frac{-3(2s)}{(s^2+9)^2}\right)$

$= \frac{6s}{(s^2+9)^2}$

Ex: Find $L[t^2 e^{-3t}]$

Solⁿ. $f(t) = e^{-3t}$

$L[f(t)] = \frac{1}{s-a}$

$F(s) = \frac{1}{s+3}$

$L[t^2 e^{-3t}] = (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s+3}\right)$

$= \frac{d}{ds}\left(\frac{-1}{(s+3)^2}\right)$

$= -\frac{d}{ds}(s+3)^{-2}$

$= 2.(s+3)^{-3}$

$L[t^2 e^{-3t}] = \frac{2}{(s+3)^3}$

⑥ Division by t

If $L[f(t)] = F(s)$

then $L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds$

Ex: Find $L\left[\frac{1-\cos t}{t}\right]$

Here $f(t) = 1-\cos t$

$F(s) = L[1-\cos t]$

$F(s) = \frac{1}{s} - \frac{s}{s^2+1}$

$\therefore L\left[\frac{1-\cos t}{t}\right] = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+1}\right) ds$

$= \left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty$

$= \left[\log \frac{s}{\sqrt{s^2+1}}\right]_s^\infty$

$= \left[\log\left[\frac{s}{\sqrt{s^2(1+\frac{1}{s^2})}}\right]\right]_s^\infty$

$= \left[\log \frac{s}{s\sqrt{1+\frac{1}{s^2}}}\right]_s^\infty$

$= \log(1) - \log \frac{\cancel{s}}{\sqrt{\cancel{s}1+\frac{1}{s^2}}}$

$= \log \frac{\sqrt{s^2+1}}{s}$

(7) L.T. of derivative

If $L[f(t)] = F(s)$

then $L[f'(t)] = sF(s) - f(0)$

where $f(0) = \lim_{t\to 0} f(t)$

IIIly $L[f''(t)] = s^2F(s) - sf(0) - f'(0)$

$L[f^n(t)] = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) \ldots f^{n-1}(0)$.

Ex: Given that

$4f''(t) + f(t) = 0$

where $f(0)=0$, $f'(0)=2$

Show that $L[f(t)] = \frac{8}{4s^2+1}$

Soln: Let $L[4f''(t)] + L[f(t)] = L[0]$.

$4[s^2F(s) - sf(0) - f'(0)] + F(s) = 0$

$\therefore 4[s^2F(s) - 2] + F(s) = 0$

$F(s)[4s^2+1] = 8$

$\therefore F(s) = \frac{8}{4s^2+1}$

(8) L.T. of integral

If $L[f(t)] = F(s)$

then $L\left[\int_0^t f(u)\,du\right] = \frac{F(s)}{s}$

Ex: Evaluate

$L\left[\int_0^t e^u u^3 du\right]$

*) Properties of L.T.

If $L[f(t)] = F(s)$

1) linearity : $L[af(t) + bg(t)] = aL[f(t)] + bL[g(t)]$

2) change of scale : $L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$

3) Multiplication by t : $L[t f(t)] = (-1)\frac{d}{ds} F(s)$

$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s)$

4) Division by t : $\int_s^\infty F(s)\,ds$

5) $L[e^{-at} f(t)] = F(s+a)$.

6) 2^{nd} shifting : $L[F(t)] = e^{-as}F(s)$ where $F(s) = L[f(t)]$.
where $F(t) = f(t-a) \quad t>a$
$= 0 \quad t<a$.

7) ~~Deriv~~ Derivative : $L[f'(t)] = sF(s) - f(0)$ where $f(0) = \lim_{t\to 0} f(t)$.
$\Rightarrow L[f''(t)] = s^2F(s) - sf(0) - f'(0)$
$\therefore L[f^n(t)] = s^nF(s) - s^{n-1}f(0) - \dots f^{n-1}(0)$.

8) Integral : $L\left[\int_0^t f(u)\,du\right] = \frac{F(s)}{s}$ where $F(s) = L[f(t)]$

Ex: Find L.T. of $f(t) = \sin(2t+3) + e^{-t/2} + t^{-1/2}$

Soln: $L[f(t)] = L[\sin(2t+3)] + L[e^{-t/2}] + L[t^{-1/2}]$
$= L[\sin 2t\cos 3 + \cos 2t \sin 3] + \frac{1}{s+1/2} + \frac{\Gamma(-\frac{1}{2}+1)}{s^{-\frac{1}{2}+1}}$

$= \cos 3\, L[\sin 2t] + \sin 3\, L[\cos 2t] + \frac{1}{s+1/2} + \frac{\Gamma\frac{1}{2}}{s^{1/2}}$

$L[f(t)] = \cos 3 \frac{2}{s^2+4} + \sin 3 \frac{s}{s^2+4} + \frac{1}{s+1/2} + \frac{\sqrt{\pi}}{s^{1/2}}$

Ex: Find L.T. of $f(t) = \sin 2t \quad 0<t<\pi$
$= 0 \quad t>\pi$

Soln: $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

$= \int_0^\pi e^{-st}\sin 2t + \int_\pi^\infty 0$

$= \left[\frac{e^{-st}}{s^2+4}(-s\sin 2t - 2\cos 2t)\right]_0^\pi$

$= \frac{e^{-s\pi}}{s^2+4} - 2 - \frac{1}{s^2+4}(-2)$

$= \frac{-2}{s^2+4}\left[e^{-s\pi} - 1\right]$.

Q. Find L.T. of $f(t) = \sin^2 4t + (e^{3t}+5)^2$

Soln: $L[f(t)] = L[\sin^2 4t] + L[(e^{3t}+5)^2]$

$$= L\left[\frac{1-\cos 8t}{2}\right] + L\left[e^{6t} + 25 + 10e^{3t}\right]$$

$$= L\left[\frac{1}{2}\right] - L\left[\frac{\cos 8t}{2}\right] + L[e^{6t}] + L[10e^{3t}] + L[25]$$

$$L[f(t)] = \frac{1}{2s} - \frac{1}{2}\frac{s}{s^2+64} + \frac{1}{s-6} + \frac{10}{s-3} + \frac{25}{s}$$

Ex. Find $L[e^{t/2}\sin^2 2t]$

Soln: using 1st shifting

$$L[e^{at}f(t)] = F(s-a)$$

$$F(s) = L[f(t)]$$

$$f(t) = \frac{1-\cos 4t}{2}$$

$$L[f(t)] = \frac{1}{2s} - \frac{s}{2(s^2+16)}$$

$$L[e^{t/2}\sin^2 2t] = \frac{1}{2(s-\frac{1}{2})} - \frac{s-\frac{1}{2}}{2\left((s-\frac{1}{2})^2+16\right)}$$

Ex. Find $L\left[\int_0^t \sin u\cos 2u\, du\right]$

Soln: $f(t) = \cancel{\sin u\cos 2u} \quad \sin t\cos 2t$

$$= \cancel{\sin 3u\cos t} \quad \cancel{co} \quad \frac{\sin 3t + \sin(-t)}{2}$$

$$f(t) = \frac{\sin 3t}{2} - \frac{\sin t}{2}$$

$$L[f(t)] = \frac{1}{2}\frac{3}{(s^2+9)} - \frac{1}{2(s^2+1)}$$

$$L\left[\int_0^t \sin u\cos 2u\,du\right] = \frac{1}{s}\left[\frac{3}{2(s^2+9)} - \frac{1}{2(s^2+1)}\right]$$

7) Inverse laplace transform:

if $L[f(t)] = F(s)$.

$L^{-1}[F(s)] = f(t)$

1) $L^{-1}\left[\frac{K}{s}\right] = K$ or $L^{-1}\left[\frac{1}{s}\right] = 1$

2) $L\left[e^{at}\right] = \frac{1}{s-a} \Rightarrow L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

3) $L^{-1}\left[\frac{a}{s^2+a^2}\right] = \sin at$ $\quad L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{\sin at}{a}$

4) $L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$

5) $L^{-1}\left[\frac{a}{s^2-a^2}\right] = \sinh at$ $\quad L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{\sinh at}{a}$

6) $L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$

7) $L^{-1}\left[\frac{\Gamma n+1}{s^{n+1}}\right] = t^n$ $\quad L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{\Gamma n+1}$ or $\frac{t^n}{n!}$

$L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{\Gamma n}$ or $\frac{t^{n-1}}{(n-1)!}$

Find

1) $L^{-1}\left[\frac{5}{s}\right]$

2) $L^{-1}\left[\frac{3}{s+3}\right]$

3) $L^{-1}\left[\frac{s^3+3s^2+4}{s^5}\right]$

4) $L^{-1}\left[\frac{s+3}{s^2+4}\right]$

Soln: 1) $L^{-1}\left[\frac{5}{s}\right] = 5$

2) $L^{-1}\left[\frac{s+3-3}{s+3}\right] = L^{-1}\left[\frac{s+3}{s+3}\right] - L^{-1}\left[\frac{3}{s+3}\right]$

3) $L^{-1}\left[\frac{s^3}{s^5} + \frac{3s^2}{s^5} + \frac{4}{s^5}\right]$

$\Rightarrow L^{-1}\left[\frac{1}{s^2} + \frac{3}{s^3} + \frac{4}{s^5}\right]$

$\Rightarrow L^{-1}\left[\frac{1}{s^2}\right] + L^{-1}\left[\frac{3}{s^3}\right] + L^{-1}\left[\frac{4}{s^5}\right]$

$= t + \frac{3t^2}{2!} + \frac{4t^4}{4!}$

$= t + \frac{3t^2}{2} + \frac{4t^4}{24}$

4) $L^{-1}\left[\frac{s+3}{s^2+4}\right]$

$L^{-1}\left[\frac{s}{s^2+4}\right] + L^{-1}\left[\frac{3}{s^2+4}\right]$

$= \cos 2t + \frac{\sin 2t}{2} \cdot 3$

Properties:

1) 1^{st} shifting $\rightarrow L^{-1}[F(s-a)] = e^{at} f(t)$

2) 2^{nd} shifting $\rightarrow L^{-1}[e^{-as}F(s)] = F(t) = f(t-a)$ for $t<a$
$= 0$ $\quad t>a$

3) change of scale $\rightarrow L^{-1}\left[F\left(\frac{s}{a}\right)\right] = a f(at)$

4) Multiplication by $t \Rightarrow L^{-1}[F'(s)] = -t f(t)$

5) Division by $t \rightarrow L^{-1}\left[\int_s^{\infty} f(s)\,ds\right] = \frac{f(t)}{t}$

6) Integral $\rightarrow$ $L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(u)\,du.$

Ex: $L^{-1}\left[\frac{1}{s-3}\right] = e^{3t}$

Ex: $L^{-1}\left[\frac{1}{(s-3)^2}\right]$

$L^{-1}\left[F(s-3)\right] \Rightarrow F(s-3) = \frac{1}{(s-3)^2}$

$F(s) = \frac{1}{s^2}$

$L^{-1}\left[\frac{1}{s^2}\right] = t = f(t)$

$\therefore L^{-1}\left[\frac{1}{(s-3)^2}\right] = e^{3t}t.$

Ex: $L^{-1}\left[\frac{1}{(s+2)^5}\right]$

$= e^{-2t}L^{-1}\left[\frac{1}{s^5}\right]$

$= \frac{e^{-2t}t^4}{4!}$

Ex: $L^{-1}\left[\frac{3}{(s+1)^2+4}\right]$

$= e^{-t}L^{-1}\left[\frac{3}{s^2+4}\right]$

$= \frac{e^{-t}3\sin 2t}{2}$

Q. Find $L^{-1}\left[\frac{s+5}{(s+1)^2+9}\right]$

Soln: $L^{-1}\left[\frac{(s+1)+4}{(s+1)^2+9}\right]$

$L^{-1}\left[\frac{s+1}{(s+1)^2+9}\right] + L^{-1}\left[\frac{4}{(s+1)^2+9}\right]$

$= e^{-t}L^{-1}\left[\frac{s}{s^2+9}\right] + e^{-t}\,4\,\frac{\sin 3t}{3}$

$= e^{-t}\cos 3t + 4e^{-t}\frac{\sin 3t}{3}$

Q. Find $L^{-1}\left[\frac{1}{\sqrt{2s+5}}\right]$

Soln: $L^{-1}\left[\frac{1}{\sqrt{2(s+\frac{5}{2})}}\right]$

$=\frac{1}{\sqrt{2}}L^{-1}\left[\frac{1}{\sqrt{s+\frac{5}{2}}}\right]$

$=\frac{1}{\sqrt{2}}e^{-5/2t}L^{-1}\left[\frac{1}{s^{1/2}}\right]$

$=\frac{1}{\sqrt{2}}e^{-5/2t}\frac{t^{-1/2}}{\Gamma\frac{1}{2}}$

$=\frac{1}{\sqrt{2\pi}}e^{-5/2t}t^{-1/2}$

Q. Find $L^{-1}\left[\frac{s+12}{s^2+8s+16}\right]$

Soln: $L^{-1}\left[\frac{s+12}{(s+4)^2}\right]$

$L^{-1}\left[\frac{s+4+8}{(s+4)^2}\right]$

$L^{-1}\left[\frac{1}{s+4}\right] + L^{-1}\left[\frac{8}{(s+4)^2}\right]$

$= e^{-4t} + 8e^{-4t}t$

Q. Find $L^{-1}\left[\log\frac{s+a}{s+b}\right]$

Sol^n: $L^{-1}\left[\log(s+a) - \log(s+b)\right]$

let $F(s) = \log(s+a) - \log(s+b)$

$\frac{d}{ds}F(s) = F'(s) = \frac{1}{s+a} - \frac{1}{s+b}$

$\cancel{L^{-1}[F'(s)] = -t(e^{-at} - e^{-bt})}$

$L^{-1}\left[\frac{1}{s+a} - \frac{1}{s+b}\right] = -t\,f(t)$

$e^{-at} - e^{-bt} = -t\,f(t)$

$\therefore\ f(t) = \frac{(e^{-bt} - e^{-at})}{t}$

Q. Find $L^{-1}\left[\cot^{-1}\left(\frac{s+3}{2}\right)\right]$

Sol^n: Using $L^{-1}[F'(s)] = -t\,f(t)$

let $F(s) = \cot^{-1}\left(\frac{s+3}{2}\right)$

$\frac{d}{ds}F(s) = \frac{-1}{1+\left(\frac{s+3}{2}\right)^2}$

$L^{-1}[F'(s)] = -t\,f(t)$

$L^{-1}\left[\frac{-1}{1+\left(\frac{s+3}{2}\right)^2}\right]\frac{1}{2} = -t\,f(t)$

$L^{-1}\left[\frac{-4}{4+(s+3)^2}\right]\frac{1}{2} = -t\,f(t)$

$e^{-3t}\ L^{-1}\left[\frac{-2}{4+(s+3)^2}\right] = -t\,f(t)$

$$-2.e^{-3t}\frac{\sin 2t}{2} = -t.f(t)$$

$$t.f(t) = e^{-3t}\sin 2t.$$

$$f(t) = \frac{e^{-3t}\sin 2t}{t}.$$

Q. Find $L^{-1}\left[\int_s^\infty\left(\frac{1}{u}-\frac{1}{u+1}\right)du\right]$

Sol^n: Using $L^{-1}\left[\int_s^\infty f(u)du\right] = \frac{f(t)}{t}$

here $f(u) = \frac{1}{u}-\frac{1}{u+1}$

$$F(s) = \frac{1}{s}-\frac{1}{s+1}$$

$$L^{-1}\left[\frac{1}{s}\right]-L^{-1}\left[\frac{1}{s+1}\right]$$

$$= 1-e^{-t}$$

$$L^{-1}\left[\int_s^\infty\left(\frac{1}{s}-\frac{1}{s+1}\right)ds\right] = \frac{1-e^{-t}}{t}.$$

H.W. Find

① $L^{-1}\left[\frac{3s+2}{4s^2+12s+9}\right]$

② $L^{-1}\left[\frac{s}{(s+1)^5}\right]$

③ $L^{-1}\left[\int_s^\infty \tan^{-1}\left(\frac{2}{s^2}\right)ds\right]$

*) Inverse L.T. by Partial fraction:

$$\frac{N^r}{(ax+b)(cx+d)} = \frac{A}{ax+b}+\frac{B}{cx+d}$$ (non-repeated linear factor in D^r)

Equate N^{rs} of both sides $\Rightarrow N^r = A(cx+d)+B(ax+b).$

Any linear factor. $\dfrac{N^r}{(ax+b)^2(cx+d)} = \dfrac{A}{(ax+b)} + \dfrac{B}{(ax+b)^2} + \dfrac{C}{(cx+d)}$

$N^r = A(ax+b)(cx+d) + B(cx+d) + C(ax+b)^2.$

$$\frac{N^r}{(ax^2+bx+c)(dx+e)} = \frac{Ax+B}{ax^2+bx+c} + \frac{C}{dx+e}$$

$\Rightarrow \quad N^r = (Ax+B)(dx+e) + C(ax^2+bx+c).$

or Equate coeff. of x, x^2, const from both sides $\Rightarrow$ Eq^{ns} in A, B, C.

Partial fraction form:

when N^r's degree is less than the $D^r \rightarrow$ use Partial fraction (Partwise fraction)

Q. Find $L^{-1}\left[\dfrac{1}{s^2-3s+2}\right]$

Sol^n. $L^{-1}\left[\dfrac{1}{(s-2)(s-1)}\right]$

$$\frac{1}{(s-2)(s-1)} = \frac{A}{(s-2)} + \frac{B}{s-1}$$

$1 = A(s-1) + B(s-2)$

$s = 2 \qquad s = 1$

$\boxed{A = 1} \qquad \boxed{B = -1}$

$$L^{-1}\left[\frac{1}{s-2} - \frac{1}{s-1}\right]$$

$= e^{2t} - e^{t}.$

Q. Find $L^{-1}\left[\dfrac{s+3}{s^3-7s-6}\right]$

Sol^n:

$$\begin{array}{r|l} & s^2 - s \\ s+1 & s^3 - 7s - 6 \\ & \pm s^3 \pm s^2 \\ \hline & -s^2 - 7s - 6 \end{array}$$

$s^2 - s - 6$

$s^2 - 3s + 2s \mp 6$

$s(s-3) + 2(s-3)$

$L^{-1}\left[\frac{s+3}{(s+1)(s+2)(s-3)}\right]$

$$\begin{array}{r} s^2 - s - 6 \\ s+1 \overline{) \cancel{s^3} - 7s - 6} \\ \pm \cancel{s^3} \quad \pm s^2 \\ \hline -7s - 6 - \cancel{s^2} \\ \mp s \quad \mp \cancel{s^2} \\ \hline -6s - 6 \\ \mp 6s \mp 6 \\ \hline 0 \end{array}$$

$$\frac{1}{(s+1)(s+2)(s-3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s-3}$$

$$s+3 = A(s+2)(s-3) + B(s+1)(s-3) + C(s+1)(s+2)$$

$s = 3$ $\qquad$ $s = 1$

$6 = C(20)$ $\qquad$ $4 = -6A - 4B + \frac{3}{10} \times 6$

$C = \frac{\cancel{6}^{3}}{\cancel{20}\,10}$

$C = \frac{3}{10}$

$s = 2$

$5 = -2A - 3B + 12 \times \frac{3}{10}$

$A = \frac{-1}{2}$ $\quad$ $B = \frac{1}{5}$

$$\frac{s+3}{(s+1)(s+2)(s-3)} = \frac{-1/2}{s+1} + \frac{1/5}{s+2} + \frac{3/10}{s-3}$$

$$L^{-1}[\;''\;] = L^{-1}\left[\frac{-1/2}{s+1}\right] + L^{-1}\left[\frac{1/5}{s+2}\right] + L^{-1}\left[\frac{3/10}{s-3}\right]$$

$$= \frac{-1}{2}e^{-t} + \frac{1}{5}e^{-2t} + \frac{3}{10}e^{3t}$$

Q. Find $L^{-1}\left[\frac{1}{(s^2+1)(s^2+2)}\right]$ with partial fraction.

Soln:

$$\frac{1}{(s^2+1)(s^2+2)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2}$$

$$1 = (As+B)(s^2+2) + (Cs+D)(s^2+1).$$

$$1 = As^3 + 2As + Bs^2 + 2B + Cs^3 + Cs + Ds^2 + D.$$

$$1 = (A+C)s^3 + s^2(B+D) + s(2A+C) + 2B+D$$

$0 = A+C$

$0 = B+D$

$0 = 2A+C.$

$0 = -2C+C$

$-C = 0$

$C = 0$

$A = 0$

$2B+D = 1$

$-2D+D = 1$

$-D = 1$

$D = -1$

$B = 1$

$$\frac{1}{(s^2+1)(s^2+2)} = \frac{1}{s^2+1} - \frac{1}{s^2+2}$$

$$\therefore L^{-1}\left[\frac{1}{s^2+1} - \frac{1}{s^2+2}\right] = \sin t - \frac{\sin\sqrt{2}t}{\sqrt{2}}$$

Q. Find $L^{-1}\left[\frac{s}{s^2+6s+25}\right]$

Soln:

$$s^2+6s+25 = (s^2+6s+9)+16$$

$$= (s+3)^2+4^2$$

$$L^{-1}\left[\frac{s}{(s+3)^2+4^2}\right] = L^{-1}\left[\frac{s+3-3}{(s+3)^2+4^2}\right]$$

$$= L^{-1}\left[\frac{1}{(s+3)^2+4^2}\right] - 3L^{-1}\left[\frac{1}{(s+3)^2+4^2}\right]$$

$$= e^{-3t} - 3e^{-3t}\frac{\sin 4t}{4}$$

$$= e^{-3t}\cos 4t + \frac{3e^{-3t}\sin 4t}{4}.$$

Q. $L^{-1}\left[\frac{3s-12}{s^2+8}\right]$

Sol^n: $L^{-1}\left[\frac{3s}{s^2+8}\right] - 12\,L^{-1}\left[\frac{1}{s^2+8}\right]$

$$= 3.\cos\sqrt{8}t - \frac{12\sin\sqrt{8}t}{\sqrt{8}}$$

$$= 3\cos 2\sqrt{2}t - \frac{12\sin 2\sqrt{2}t}{2\sqrt{2}}$$

Q Find $L^{-1}\left[\frac{1}{s(s^2+1)}\right]$

Sol^n: $\frac{1}{s(s^2+1)} = \frac{As+B}{s^2+1} + \frac{C}{s}$

$1 = (As+B)(s) + C(s^2+1)$

$1 = As^2 + Bs + Cs^2 + C$

$1 = s^2(A+C) + Bs + C$

$B = 0 \qquad A+C=0$

$C = 1 \qquad A = -1$

$$\frac{1}{s(s^2+1)} = \frac{-s}{s^2+1} + \frac{1}{s}$$

$L^{-1}\left[\frac{1}{s(s^2+1)}\right] =$ ~~-sint~~ $-\cos t + 1$

*) Convolution Property:

If $f(t)$ & $g(t)$ are f^{ns} of t.

then $f(t) * g(t) = \int_0^t f(u)\,g(t-u)\,du.$ or $\int_0^t g(u)\,f(t-u)\,du.$

if $L[f(t)] = F(s)$ & $L[g(t)] = G(s)$

then $L^{-1}[F(s)\,G(s)] = f(t) * g(t) = \int_0^t f(u)\,g(t-u)\,du$

or $\int_0^t g(u)\,f(t-u)\,du$.

Ex: $L^{-1}\left[\frac{1}{s(s^2+1)}\right]$

Let $F(s) = \frac{1}{s}$ $\quad G(s) = \frac{1}{s^2+1}$

$f(t) = L^{-1}\left[\frac{1}{s}\right]$ $\quad g(t) = L^{-1}\left[\frac{1}{s^2+1}\right]$

$= 1$ $\quad g(t) = \sin t$.

$= \int_0^t \cdot f(t-u) = 1$ $\quad g(u) = \sin u$.

$\int_0^t \sin u \cdot (1)\,du$

$= \left[-\cos u\right]_0^t = -[\cos t - 1]$

$= 1 - \cos t$.

Q. Find $L^{-1}\left[\frac{1}{s^2(s+1)}\right]$

Soln: $F(s) = \frac{1}{s^2}$ $\quad G(s) = \frac{1}{s+1}$

$f(t) = L^{-1}\left[\frac{1}{s^2}\right]$ $\quad g(t) = L^{-1}\left[\frac{1}{s+1}\right]$

$f(t) = t$ $\quad g(t) = e^{-t}$

$\cancel{f(t-a) = a}$

$f(t-u) = t-u$. $\quad g(u) = e^{-u}$

OR $f(u) = u$. $\quad$ or $g(t-u) = e^{-(t-u)}$

$\therefore L^{-1}\left[\frac{1}{s^2(s+1)}\right] = f(t) * g(t)$

$$= \int_0^t u\, e^{-(t-u)}\, du.$$

$$= \int_0^t u\, e^{-t} e^{u}\, du.$$

$$= e^{-t} \int_0^t u \cdot e^{u}\, du.$$

$$= e^{-t}\left[u \int e^{u} - 1 \iint e^{u}\right]_0^t.$$

$$= e^{-t}\left[u e^{u} - e^{u}\right]_0^t$$

$$= e^{-t}\left[t e^{t} - e^{t} + 1\right].$$

$$= t + e^{-t} - 1.$$

Q. Solve by conv. property

① $L^{-1}\left[\dfrac{1}{s^2(s^2+1)}\right]$

② $L^{-1}\left[\dfrac{16}{(s-2)(s+2)^2}\right].$

Q. $y'' + y = e^{-2t}\sin t$

$y(0) = y'(0) = 0$

Soln: Consider

$$L[y''] + L[y] = L[e^{-2t}\sin t]$$

$$\Rightarrow [s^2 y(s) - s y(0) - y'(0)] + y(s) = \cancel{\frac{1}{s^2+1}}\ \frac{1}{(s+2)^2+1}$$

$$y(s)[s^2+1] = \frac{1}{(s+2)^2+1} = \frac{1}{s^2+4s+5}$$

$$y(s) = \frac{1}{(s^2+1)(s^2+4s+5)} \quad \text{— ①}$$

Consider inverse L.T. of ①

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$$L^{-1}[y(s)] = L^{-1}\left[\frac{1}{(s^2+1)(s^2+4s+5)}\right]$$

$$y(t) = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4s+5}$$

$$1 = (As+B)(s^2+4s+5) + (Cs+D)(s^2+1)$$

$$1 = As^3 + 4As^2 + 5As + Bs^2 + 4Bs + 5B + Cs^3 + Cs + Ds^2 + D$$

$$1 = s^3(A+C) + s^2(4A+B+D) + s(5A+4B+C) + 5B+D$$

$A+C=0$ $4A+B+D=0$ $5A+4B+C=0$

$5B+D=1$ $-4C+C+D=0$ $-5C+4B+C=0$

$-5C+D=0$ $-3C+D=0$ $-4C+4B=0$

$5\cdot\frac{D}{5}+D=0$ $\cancel{-3C=-D}$ $B=C$

$D=0.$ $C=\frac{D}{5}$

$$A=\frac{-1}{8} \quad B=\frac{1}{8} \quad C=\frac{1}{8} \quad D=\frac{3}{8}$$

$$\frac{1}{(s^2+1)(s^2+4s+5)} = \frac{-\frac{1}{8}s+\frac{1}{8}}{s^2+1} + \frac{\frac{1}{8}s+\frac{3}{8}}{s^2+4s+5}$$

$$L^{-1}\left[\frac{1}{(s^2+1)(s^2+4s+5)}\right] = L^{-1}\left[\frac{-\frac{1}{8}s+\frac{1}{8}}{s^2+1}\right] + L^{-1}\left[\frac{\frac{1}{8}s+\frac{3}{8}}{s^2+4s+5}\right]$$

$$= \frac{-1}{8}L^{-1}\left[\frac{s}{s^2+1}\right] + \frac{1}{8}L^{-1}\left[\frac{1}{s^2+1}\right] + \frac{1}{8}L^{-1}\left[\frac{s}{s^2+4s+5}\right] + \frac{3}{8}L^{-1}\left[\frac{1}{s^2+4s+5}\right]$$

$$y(t) = \frac{-1}{8}\cos t + \frac{1}{8}\sin t + \frac{1}{8}e^{-2t}\cos t - \frac{2}{8}e^{-2t}\sin t + \frac{3}{8}\sin t.$$

*) Special functions :

1. Unit step funcⁿ

$$u(t) = 0 \qquad t < 0$$
$$\quad = 1 \qquad t \geq 0$$

$$L[u(t)] = \int_0^\infty e^{-st} u(t)\, dt$$
$$= \int_0^\infty e^{-st}\, dt$$
$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty$$
$$L[u(t)] = \frac{1}{s}$$

2. Displaced unit step funcⁿ

$$u(t-a) = 0 \qquad t < a$$
$$\quad = 1 \qquad t \geq a$$

$$\therefore L[u(t-a)] = \int_0^\infty e^{-st} u(t-a)\, dt .$$
$$= \int_0^a 0 + \int_a^\infty e^{-st}\, dt$$
$$= \left[\frac{e^{-st}}{-s}\right]_a^\infty$$
$$L[u(t-a)] = \frac{e^{-as}}{s}$$

$$\Rightarrow L^{-1}\left[\frac{e^{-as}}{s}\right] = u(t-a).$$

Ex/ $$L^{-1}\left[\frac{e^{-\pi s}}{s}\right] = u(t-\pi) = 0 \qquad t < \pi$$
$$\qquad = 1 \qquad t \geq \pi$$

3. Dirac Delta func^n (Unit Impulse).

If $F(t) = 0 \quad t < a$

$= \frac{1}{\epsilon} \quad a \leq t \leq a+\epsilon$

$= 0 \quad t > a+\epsilon$

then $\lim_{\epsilon \to 0} F(t) = \delta(t)$ is called unit impulse f^n (dirac-delta).

where $\int_{-\infty}^{\infty} F(t)dt = 1$

4. Periodic f^n

$f(t+T) = f(t)$ with period T.

∴ L.T. of periodic f^n $f(t)$

$$L[f(t)] = \frac{\int_0^T e^{-st} f(t)dt}{1 - e^{-sT}}$$

*) ~~Fourier Transforms:~~

~~$f(x) = \frac{a_0}{2}$~~

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Fourier Transforms

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right], \quad f(x) \in [-L, L]$$

where $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx.$$

to express func^n in signal.

Any func^n $f(x)$ not necessarily periodic, can be expressed as a fourier series defined in the period $[-L, L]$ as a signal (waveform).

Using $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$ $\quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{L}}$$

where $C_n = \frac{1}{2L}\int_{-\infty}^{\infty} f(u)\, e^{-in\pi u/L}$

Exponential form:

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{L}} \quad (2) \text{ where } C_n = \frac{1}{2L}\int_{-L}^{L} f(u)\, e^{-in\pi u/L}\,du$$

Combing C_n & eq^n (2) we get in the limiting from $L \to \infty$ & $\frac{n\pi}{L} = \lambda$

Fourier Integral respresent$^{a^n}$:

$$f(x) = \frac{1}{2\pi}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u}\right] e^{i\lambda x}\,d\lambda$$

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$$F(\lambda) = \int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u}\, du = \text{F. Transform of } f(x) \quad \text{—(A)}$$

& $\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda) e^{i\lambda x} d\lambda = f(x)$ is called inv. Fourier transform.
↳ (B)

F.T. & I.F.T together is integral respresentation.

if $f(-x) = f(x) \rightarrow$ even
$= -f(x) \rightarrow$ odd.

$$\boxed{\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{-i\theta} &= \cos\theta - i\sin\theta \end{aligned}}$$

$$\int_{-a}^{a} f(x)dx = 0 \quad \text{if } f(x) = \text{odd}$$
$$= 2\int_{0}^{a} f(x)\,dx = \text{even}$$

As per even & odd f^n $f(x)$
we have

f. cosine trans $\rightarrow$ $F_c(\lambda) = \int_0^{\infty} f(u) \cos\lambda u\, du$ — (C)

Inv. Cosine trans $\Rightarrow$ $f(x) = \frac{2}{\pi}\int_0^{\infty} F_c(\lambda)\cos\lambda x\, d\lambda$ — (D)

& F. sine trans $\Rightarrow$ $F_s(\lambda) = \int_{u=0}^{\infty} f(u) \sin\lambda u\, du$ — (E)

& Inv. sine trans $\Rightarrow$ $f(x) = \frac{2}{\pi}\int_0^{\infty} F_s(\lambda) \sin\lambda x\, d\lambda$ — (F)

1) $\int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

2) $\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}\cancel{(a\cos bx + b\sin bx)}\,(a\sin bx - b\cos bx)$

3) $\int uv\, dx$

4) Beta - Gamma rel^n $\rightarrow$ $\beta(m,n) = \cancel{\Gamma a \Gamma b \Gamma a+b}$ $\Gamma m \Gamma n \Gamma m+n$

5) $\int_0^{\infty} \frac{\sin ax}{x} dx = \frac{\pi}{2}$ if $a>0$, $\frac{-\pi}{2}$ if $a<0$.

*) $f(x)$ is defined in $[-\infty, \infty]$ then

F.T. of $f(x) = F(\lambda) = \int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u} du.$ — (A)

∴ Inverse F.T.

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\lambda)\, e^{+i\lambda x} d\lambda \quad \text{— (B)}$$

*) If $f(x)$ is even f^n in $[-\infty, \infty]$

then $F_c(\lambda) = \int_0^{\infty} f(u) \cos\lambda u\, du.$ — (C)

Inverse F. cosine Trans.

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_c(\lambda) \cos\lambda x\, d\lambda \quad \text{— (D)}$$

If $f(x)$ is odd f^n in $[-\infty, \infty]$ then

F. sine trans of $f(x)$

$$F_s(\lambda) = \int_0^{\infty} f(u) \sin\lambda u\, du \quad \text{— (E).}$$

& Inverse sine trans

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_s(\lambda) \sin\lambda x\, d\lambda \quad \text{— (F)}$$

∴ Fourier Integral respresentation of $f(x)$.

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u)\, e^{-i\lambda(u-x)} du\, d\lambda.$$

or

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\lambda)\, e^{i\lambda x} d\lambda.$$

Fourier cosine integral respresen^n

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \int_0^{\infty} f(u) \cos\lambda u \cos\lambda x\, du\, d\lambda.$$

$\therefore$ Fourier sine integral representa^n

$$f(x) = \frac{2}{\pi}\int_0^\infty \int_0^\infty f(u) \sin \lambda u \sin \lambda x \, du \, d\lambda.$$

Q. Find F. sine & cosine integral represen^n for $f(x) = e^{-mx}$ $(m>0)$.

Soln: let F. cosine int. represen^n

$$\Rightarrow f(x) = \frac{2}{\pi}\int_0^\infty \int_0^\infty f(u) \cos \lambda u \cos \lambda x \, du \, d\lambda. \quad \text{--- (1)}$$

Consider, $F_c(\lambda) = \int_0^\infty f(u) \cos \lambda u \, du$

$$= \int_0^\infty e^{-mu} \cos \lambda u \, du$$

$$= \left[\frac{e^{-mu}}{m^2+\lambda^2} (-m \cos \lambda u + \lambda \sin \lambda u) \right]_0^\infty$$

$$= \frac{m}{m^2+\lambda^2} \quad \text{--- (2)}$$

Put (2) in eq^n (1)

$$\therefore f(x) = \frac{2}{\pi}\int_0^\infty \frac{m}{m^2+\lambda^2} \cos \lambda x \, d\lambda$$

let F. sine int. represen^n

$$F_s(\lambda) = \int_0^\infty f(u) \sin \lambda u \, du$$

$$= \int_0^\infty e^{-mu} \sin \lambda u \, du$$

$$= \left[\frac{e^{-mu}}{m^2+\lambda^2} (-m \sin \lambda u - \lambda \cos \lambda u) \right]_0^\infty$$

$$= \frac{\lambda}{m^2+\lambda^2}$$

$$\therefore f(x) = \frac{2}{\pi}\int_0^\infty \frac{\lambda}{m^2+\lambda^2} \sin \lambda x \, d\lambda.$$

OR P.T. $\int_0^\infty \frac{\lambda \sin\lambda x}{m^2+\lambda^2} = \frac{\pi}{2} e^{-mx}$.

Q. Find F.T. of $f(x) = 1$ for ~~odd~~ $|x| \le 1$
$= 0$ for $|x| > 1$
& hence evaluate $\int_0^\infty \frac{\sin x}{x} dx$.

Soln: Using (A)
consider $F(\lambda) = \int_{-\infty}^{\infty} f(u) e^{-i\lambda u} du$.

$= \int_{-\infty}^{-1} 0 + \int_{-1}^{1} 1 \cdot e^{-i\lambda u} du + \int_1^\infty 0.$

$= \left[\frac{e^{-i\lambda u}}{-i\lambda}\right]_{-1}^{1}$

$= \frac{e^{-i\lambda}}{-i\lambda} + \frac{e^{i\lambda}}{i\lambda}$

$F(\lambda) = \frac{1}{i\lambda}\left[e^{i\lambda} - e^{-i\lambda}\right]$

$\boxed{\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}}$

$F(\lambda) = \frac{2\sin\lambda}{\lambda}$

Now to evaluate $\int_0^\infty \frac{\sin x}{x} dx$

consider Inv. F.T. of $f(x)$

$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda) e^{i\lambda x} d\lambda$

$\because F(\lambda) = \frac{2\sin\lambda}{\lambda}$

$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin\lambda}{\lambda} e^{i\lambda x} d\lambda.$

Put $x = 0 \Rightarrow e^{i\lambda x} = 1$

$f(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin\lambda}{\lambda} d\lambda.$

classmate

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$\because f(0) = 1$ from def^n. of $f(x)$

$\therefore \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2 \sin\lambda}{\lambda} d\lambda = 1$

$\therefore \int_{-\infty}^{\infty} \frac{\sin\lambda}{\lambda} d\lambda = \pi$

$\int_{-a}^{a} f(x) dx = 2\int_{0}^{a} f(x) dx$ if $f(x)$ is even

$= 0$ if $f(x)$ is odd

$= 2\int_{0}^{\infty} \frac{\sin\lambda}{\lambda} d\lambda = \pi$

$= \int_{0}^{\infty} \frac{\sin\lambda}{\lambda} d\lambda = \frac{\pi}{2}$

Ex: S.T. Fourier Trans. of $f(x) = e^{-|x|}$ is $\frac{2}{1+\lambda^2}$

Soln: Given $f^n \rightarrow f(x) = e^{-|x|} \Rightarrow f(-x) = e^{-|x|} = f(x)$

$\Rightarrow f^n$ is even f^n $\therefore$ using F. cosine trans of $f(x)$.

$$F_c(\lambda) = 2\int_{0}^{\infty} f(u) \cos\lambda u \, du.$$

$$= 2\int_{0}^{\infty} e^{-|u|} \cos\lambda u \, du$$

$$= 2\int_{0}^{\infty} e^{-u} \cos\lambda u \, du$$

$$= 2\left[\frac{e^{-u}}{1+\lambda^2}(-\cos\lambda u + \lambda \sin\lambda u)\right]_0^{\infty}$$

$$= 2\left[\frac{1}{1+\lambda^2}\right]$$

$$= \frac{2}{1+\lambda^2}$$

//

Q. Find F. sine trans of $f(x) = 1 \quad 0 \le x \le 1$
$= 0 \quad x > 1$

& hence S.T. $\int_0^\infty \frac{\sin^3 x}{x} dx = \frac{\pi}{4}$

soln: Using formula (E)

$$F_s(\lambda) = \cancel{\int_{-\infty}^{0} 0} + \int_0^1 1 \sin \lambda u \, du + \int_1^\infty 0$$

$$= \int_0^1 \sin \lambda u \, du.$$

$$= \left[\frac{-\cos \lambda u}{\lambda} \right]_0^1$$

$$F_s(\lambda) = \frac{-\cos\lambda}{\lambda} + \frac{1}{\lambda}$$

$$= \frac{1}{\lambda}\left[1 - \cos\lambda\right]$$

$$F_s(\lambda) = \frac{1}{\lambda}\left[2\sin^2\frac{\lambda}{2}\right]$$

Consider inverse F. sine trans

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(\lambda) \sin \lambda x \, d\lambda$$

$0 < x \le 1 \quad 1$
$x > 1 \quad 0$

$$= \frac{2}{\pi}\int_0^\infty \frac{2\sin^2 \lambda/2}{\lambda} \sin \lambda x \, d\lambda.$$

Substituting $x = \frac{\pi}{\cancel{2}}$ 1/2

$$1 = \frac{2}{\pi}\int_0^\infty \frac{2\sin^2 \lambda/2}{\lambda} (\sin \lambda/2) \, d\lambda$$

$$\frac{\pi}{4} = \int_0^\infty \frac{\sin^3 \lambda/2}{\lambda} \cancel{2\sin\frac{\lambda}{2}\cos\frac{\lambda}{2}} \, d\lambda$$

$$\frac{\pi}{4} = \int_0^\infty \frac{\sin^3 \lambda/2}{\lambda/2} (\cancel{\cos \lambda/2}) \, d\lambda.$$

Q. Solve the integral eqⁿ.

$$\int_0^\infty f(x)\sin\lambda x\,dx = 1-\lambda \quad 0\le\lambda\le1$$
$$= 0 \quad \lambda>1$$

$\Rightarrow$ To find $f(x)$ by inverse.
Inv. F-sine trans.

Solⁿ: Consider F

$$f(x) = \frac{2}{\pi}\int_0^\infty F_S(\lambda)\sin\lambda x\,dx.$$

$$F_S(\lambda) = 1-\lambda \quad 0\le\lambda\le1$$
$$= 0 \quad \lambda>1$$

$$= \frac{2}{\pi}\int_0^1 (1-\lambda)\sin\lambda x\,d\lambda$$

$$= \frac{2}{\pi}\int_0^1 [\sin\lambda x - \lambda\sin\lambda x]\,d\lambda$$

$$= \frac{2}{\pi}\left[\int_0^1 \sin\lambda x\,d\lambda - \int_0^1 \lambda\sin\lambda x\,d\lambda\right]$$

$$= \frac{2}{\pi}\left[\left[\frac{-\cos\lambda x}{x}\right]_0^1 - \left[\frac{-\lambda\cos\lambda x}{x} - \frac{\sin\lambda x}{x^2}\right]_0^1\right]$$

$$= \frac{2}{\pi}\left[\left[\frac{-\cos x}{x}+\frac{1}{x}\right] - \left[\frac{-\cos x}{x} - \frac{\sin x}{x^2}\right]\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{x} \mp \frac{\sin x}{x^2}\right].$$

Q. Find F. sine trans of $f(x) = 1 \quad 0\le x\le1$
$= 0 \quad x>1$

& hence evaluate $\int_0^\infty \frac{\sin^3 x}{x}\,dx = \frac{\pi}{4}$

Solⁿ:

Q. Find $f(x)$ if $F_c(\lambda) = \frac{\sin a\lambda}{\lambda}$

Sol^n: Using formula ⓓ

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(\lambda)\cos\lambda x \, d\lambda.$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\sin a\lambda}{\lambda}\cos\lambda x \, d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\sin(a\lambda+\lambda x) + \sin(a\lambda-\lambda x)}{2\lambda} \, d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\sin\lambda(a+x) + \sin\lambda(a-x)}{\lambda} \, d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\sin\lambda(a+x)}{\lambda} d\lambda + \frac{1}{\pi}\int_0^\infty \frac{\sin\lambda(a-x)}{\lambda} \, d\lambda$$

if $a+x > 0$ & $(a-x) > 0$

$$f(x) = \frac{1}{\pi}\left\{\left[\frac{\pi}{2}\right] + \left[\frac{\pi}{2}\right]\right\}$$

$$f(x) = 1$$

if $a+x > 0$ $(a-x) < 0$

$$f(x) = \frac{1}{\pi}\left[\frac{\pi}{2} - \frac{\pi}{2}\right]$$

$$f(x) = 0.$$

Q. If $f(x) = x^{m-1}$. Find F. sine & cosine trans. of $f(x)$ using gamma f^n $\Gamma m = \int_0^\infty e^{-x} x^{m-1} dx$. by putting $x = i\lambda u$.

Sol^n: let $\Gamma m = \int_0^\infty e^{-x} x^{m-1} dx$ put $x = i\lambda u$

$dx = i\lambda du$.

$x \to 0, u \to 0$

$x \to \infty, u \to \infty$

$$\therefore \Gamma m = \int_0^\infty e^{-i\lambda u} (i\lambda u)^{m-1} i\lambda du.$$

$$= \int_0^\infty e^{-i\lambda u} u^{m-1} (i\lambda)^m du$$

$$= (i\lambda)^m \int_0^\infty e^{-i\lambda u} u^{m-1} du.$$

$$= (e^{i\pi/2}\lambda)^m \int_0^\infty (\cos\lambda u - i\sin\lambda u) u^{m-1} du.$$

$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

$i = e^{i\pi/2}$

$e^{-i\theta} = \cos\theta - i\sin\theta.$

$$\frac{\sqrt{m}\, e^{-\frac{i\pi}{2}m}}{\lambda^m} = \int_0^\infty \cos\lambda u\, u^{m-1} - i\int_0^\infty \sin\lambda u\, u^{m-1} du$$

$$\theta = \frac{\pi m}{2}$$

$$\frac{\sqrt{m}}{\lambda^m}\left[\cos\frac{m\pi}{2} - i\sin\frac{m\pi}{2}\right] = \int_0^\infty u^{m-1}\cos\lambda u\, du - i\int_0^\infty u^{m-1}\sin\lambda u\, du$$

Equating real & imag. parts from both sides.

$$\frac{\sqrt{m}}{\lambda^m}\cos\frac{m\pi}{2} = \int_0^\infty u^{m-1}\cos\lambda u\, du = F_c(\lambda)$$

$$\& \ \frac{\sqrt{m}}{\lambda^m}\sin\frac{m\pi}{2} = \int_0^\infty u^{m-1}\sin\lambda u\, du = F_s(\lambda).$$

Q. S.T. F. cosine trans. of $e^{-x^2/2}$ is $\sqrt{\frac{\pi}{2}}\, e^{-\lambda^2/2}$ using differentiation under integral sign (DUIS) formula.

$I(\alpha) = \int_a^b f(x,\alpha)\, dx$ (α = parameter)

↓ if this is not integrable with any method

$\Rightarrow$ Diff. both sides w.r.t α

$$\frac{dI}{d\alpha} = \int_a^b \left[\frac{\partial}{\partial\alpha} f(x,\alpha)\right]_{x=\text{const}} dx.$$

Diff. eq^n in $I(\alpha)$ → solving $\Rightarrow I(\alpha) = ?$

Q. Consider F. cosine trans. of $e^{-x^2/2}$

Soln:
$$F_c(\lambda) = \int_0^\infty f(u)\cos\lambda u\,du.$$

$$= \int_0^\infty e^{-u^2/2}\cos\lambda u\,du.$$

λ is parameter

Diff. both sides w.r.t λ (DUIS)

$$\frac{dF}{d\lambda} = \int_0^\infty \left(\frac{\partial}{\partial\lambda} e^{-u^2/2}\cos\lambda u\right)_{u=\text{const}} du$$

$$= \int_0^\infty e^{-u^2/2}(-\sin\lambda u).\,u\,du$$

$$\frac{dF}{d\lambda} = \int_0^\infty \underbrace{(\sin\lambda u)}_{\text{1st fn}}\underbrace{(-ue^{-u^2/2})}_{\text{2nd}}\,du. \quad —①$$

Using int. by parts.

let $\int -ue^{-u^2/2}\,du =$

Put $\frac{-u^2}{2} = t$

$$\frac{-2u\,du}{2} = dt.$$

$$-u\,du = dt.$$

$$= \int e^t\,dt$$

$$= e^t$$

$$= e^{-u^2/2}$$

$$\frac{dF}{d\lambda} = \left[\sin\lambda u\left(e^{-u^2/2}\right)\right]_0^\infty - \int_0^\infty e^{-u^2/2}\lambda\cos\lambda u\,du.$$

$$\frac{dF}{d\lambda} = [0] - \lambda\int_0^\infty e^{-u^2/2}\cos\lambda u\,du.$$

$$\frac{dF}{d\lambda} = -\lambda F_c(\lambda).$$

$$\int\frac{dF}{F} = \int\lambda\,d\lambda$$

$$\log F_c(\lambda) = \frac{-\lambda^2}{2} + c. \quad — ②$$

$$\therefore F_c(\lambda) = e^{-\lambda^2/2} \cdot e^{c}$$

$$e^{c} = C_1$$

$$F_c(\lambda) = e^{-\lambda^2/2} C_1 \quad — ③$$

③ can not be the sol^n of $e^{-x^2/2}$ as it contains C_1.

$\therefore$ To evaluate for C_1.

Put $\lambda = 0$ in ③

$$F_c(0) = e^{0} C_1 \Rightarrow C_1 = F_c(0) = \int_0^\infty e^{-u^2/2} \cos(0)\, du$$

Put $u^2/2 = t$

$$u\,du = dt$$

$$du = \frac{dt}{u}$$

$$= \frac{dt}{\sqrt{2t}}$$

$$\therefore C_1 = \int_0^\infty e^{-t} \frac{dt}{\sqrt{2t}}.$$

$$= \frac{1}{\sqrt{2}} \int_0^\infty e^{-t} t^{-1/2} dt$$

$$\because \Gamma n = \int_0^\infty e^{-x} x^{n-1} dx.$$

$$= \frac{1}{\sqrt{2}} \Gamma \frac{1}{2}$$

$$= \frac{\sqrt{\pi}}{\sqrt{2}}.$$

Put in eq^n ③

$$F_c(\lambda) = e^{-\lambda^2/2} \sqrt{\frac{\pi}{2}}.$$

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Z - transforms

*) Z- Transforms are mainly used in the discrete func^{ns} for digital systems.

Def^n: Sequence

$f(k) = \{ \dots -4, 7, \underset{\uparrow}{10}, 1, 3, 5 \dots \}$

$f(0) \rightarrow$ zeroth position

$\therefore$ $f(0) = 10$ $\quad$ $f(-1) = 7$

$f(1) = 1$ $\quad$ $f(-2) = -4.$

$f(2) = 3$

$f(3) = 5$

In general

$f(k) = \{2^k\}_{k=-\infty}^{\infty} = \{ \dots 2^{-3}, 2^{-2}, 2^{-1}, 2^{0}, 2^{1}, 2^{2}, 2^{3}, \dots \}.$

$k = -\infty$ to ∞

Note: If $f(k) = 0$ for $k < 0$
then $f(k)$ is called causal seq. $(k \geq 0)$.

$f(k) = \{0, 0 \dots, 0, 3, 5, 100 \dots \}$

*) Property:

1. Add^n $\rightarrow \{f(k)\} + \{g(k)\} = \{f(k) + g(k)\}$ resp. positions.

2. Scaling $\rightarrow \{af(k)\} = a\{f(k)\}$. $\quad a =$ scalar.

3. Linearity $\rightarrow \{af(k) + bg(k)\} = a\{f(k)\} + b\{g(k)\}$.

*) Standard series (convergent)

① Binomial $\rightarrow (a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \dots$

$(-1)(-y)$

$\therefore$ $(1+y)^n = 1+ny + \frac{n(n-1)}{2!}y^2 + \frac{n(n-1)(n-2)}{3!}y^3 + \ldots$

is the convergent series if $|y|<1$.

(2) $(1+y)^{-1} = \frac{1}{1+y} = 1-y+y^2-y^3+y^4-\ldots$

(3) $(1-y)^{-1} = 1+y+y^2+y^3+y^4+\ldots$

(4) $e^y = 1+y+\frac{y^2}{2!}+\frac{y^3}{3!}+\ldots$

(5) $a+ar+ar^2+ar^3+\ldots = G.S.$

$\Rightarrow S_\infty = \frac{a}{1-r}$

$a = 1^{st}$ term

r = common ratio.

Is convergent if $|r|<1$.

(6). In z-transforms $\to$ z = complex no. $= x+iy$.

$\therefore |z| = 1 \Rightarrow \sqrt{x^2+y^2} = 1$.

$x^2+y^2 = 1 \longrightarrow$ circle (unit).

$|z|<1 \Rightarrow$ pts. inside circle $x^2+y^2=1$

$|z|>1 \Rightarrow$ pts. outside circle.

Ex: Identify the seq.

$f(k) = 0 \quad k<0$

$= \frac{1}{4^k} \quad k \geq 0$

Sol^n: $f(k) = \left\{ \frac{1}{4}, \frac{1}{4^2}, \frac{1}{4^3}, \ldots \right\}$.

$\therefore$ Z- transform of $f(k)$.

$$\boxed{Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}}$$

$$= \{ \cdots f(-3)z^{3}, f(-2)z^{2}, f(-1)z, f(0), f(1)z^{-1}, f(2)z^{-2}, \}$$

Ex: $f(k) = 2^k \quad k \geq 0$. Find $Z\{f(k)\}$.

Soln:

$$Z\{f(k)\} = \sum_{k=0}^{\infty} 2^k z^{-k}$$

$$= \sum_{k=0}^{\infty} (2z^{-1})^k$$

$$= \{1 + 2z^{-1} + 2^2 z^{-2} + 2^3 z^{-3} + \cdots\}.$$

$\therefore$ Using $y = 2z^{-1}$ for $|y| < 1$

$$= \frac{1}{1-(2z^{-1})} \quad \text{where } |2z^{-1}| < 1 \text{ is convergent.}$$

$$Z\{f(k)\} = \frac{z}{z-2} \quad \text{where } |2z^{-1}| < 1$$

$$|2| < |z|$$

$$|z| > |2|$$

*) Standard seq. of Z- transforms

① If $f(k) = a^k \quad k \geq 0$

then $Z\{a^k\}_{k \geq 0} = \dfrac{z}{z-a}$ for $|z| > |a|$.
is called region of convergence.

~~ise~~ ∞

② If $f(k) = a^k$ for $k < 0$

Let $Z\{a^k\}_{k<0} = \sum_{k=-\infty}^{-1} a^k z^{-k}$

Put $k = -r$.

$k = -\infty$, $r = \infty$

$k = -1$, $r = 1$

$$= \sum_{r=1}^{\infty} a^{-r} z^{r}$$

$$= \sum_{r=1}^{\infty} (a^{-1}z)^{r}$$

$$= [a^{-1}z + a^{-2}z^2 + a^{-3}z^3 + \dots]$$

$$= a^{-1}z[1 + a^{-1}z + a^{-2}z^2 + \dots]$$

$$\therefore \quad Z\{a^k\} = a^{-1}z\left[\frac{1}{1-a^{-1}z}\right] \text{ where } |a^{-1}z| < 1.$$

$$= \frac{z/a}{1 - z/a} \text{ when } \left|\frac{z}{a}\right| < 1 \Rightarrow |z| < |a|.$$

$$Z\{a^k\}_{k<0} = \frac{z}{a-z}. \quad ROC \Rightarrow |z| < |a|.$$

③ If $f(k) = a^{|k|}$ for all values of k.

By def^n $Z\{f(k)\} = \sum_{k=-\infty}^{\infty} a^{|k|} z^{-k} = \sum_{-\infty}^{-1} a^{|k|} \cdot z^{-k} + \sum_{0}^{\infty} a^{|k|} z^{-k}$

Let $k = -r$

$k = -\infty \to r = \infty$

$k = -1 \to r = 1$

$$= \sum_{r=1}^{\infty} a^{|-r|} z^{r} + \sum_{0}^{\infty} a^{|k|} z^{-k}$$

$$= \sum_{r=1}^{\infty} a^{r} z^{r} + \sum_{0}^{\infty} a^{k} z^{-k}$$

$$= [az + a^2z^2 + \dots] + [1 + az^{-1} + a^2z^{-2} + \dots] \quad ①$$

Both are in ∞ G.P.

$\because \; a + ar + ar^2 + \dots = \dfrac{a}{1-r}$ for $|r| < 1$

From ①

$$= \frac{az}{1-az} + \frac{1}{1-az^{-1}} \quad \text{for } |az|<1 \text{ for } |az^{-1}| < 1.$$

$$Z\{a^{|k|}\} = \frac{az}{1-az} + \frac{z}{z-a} \quad |z| < \frac{1}{|a|} \ \& \ |z| > |a|.$$

④ Unit Impulse f^n (discrete)

$$\delta(k) = 1 \quad k=0$$
$$= 0 \quad k \neq 0$$

then $Z\{\delta(k)\} = \sum_{k=-\infty}^{\infty} \delta(k) z^{-k} = 1$

⑤ Unit step f^n

$$u(k) = 0 \quad k<0$$
$$= 1 \quad k \geq 0$$

$$Z\{u(k)\} = \sum_{k=-\infty}^{\infty} u(k) z^{-k}$$

$$= \sum_{k=-\infty}^{0} u(k) z^{-k} + \sum_{0}^{\infty} u(k) z^{-k}$$

$$= \sum_{0}^{\infty} z^{-k}$$

$$= [1 + z^{-1} + z^{-2} + \cdots]$$

$$a + ar + ar^2 + \cdots = \frac{1}{1-\frac{1}{z}} \quad \text{for } \left|\frac{1}{z}\right| < 1.$$

$$Z\{u(k)\} = \frac{z}{z-1} \quad \text{for } |z| > 1.$$

⑥ If $f(k) = \{\cos\alpha k\}$ $k \geq 0$

let $Z\{f(k)\} = \sum_{k=0}^{\infty} (\cos\alpha k) z^{-k}$

$$\boxed{\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}}$$

$$= \sum_{k=0}^{\infty} \left(\frac{e^{i\alpha k} + e^{-i\alpha k}}{2} \right) z^{-k}$$

$$\therefore \cos\alpha k = \frac{e^{i\alpha k} + e^{-i\alpha k}}{2}$$

$$= \frac{1}{2}\left[\sum_{0}^{\infty} e^{i\alpha k} z^{-k} + \sum_{0}^{\infty} e^{-i\alpha k} z^{-k}\right]$$

$$= \frac{1}{2}\left(1 + e^{i\alpha} z^{-1} + e^{2i\alpha} z^{-2} + \dots\right) + \left(1 + e^{-i\alpha} z^{-1} + e^{-2i\alpha} z^{-2} + \dots\right)\Big]$$

$$= \frac{1}{2}\left[\frac{1}{1 - e^{i\alpha} z^{-1}} + \frac{1}{1 - e^{-i\alpha} z^{-1}}\right]$$

$$= \frac{1}{2}\left[\frac{1 - e^{-i\alpha} z^{-1} + 1 - e^{i\alpha} z^{-1}}{1 - e^{i\alpha} z^{-1} - e^{-i\alpha} z^{-1} + z^{-2}}\right]$$

$$= \frac{1}{2}\left[\frac{2 - z^{-1}\left(e^{i\alpha} + e^{-i\alpha}\right)}{1 - z^{-1}\left(e^{i\alpha} + e^{-i\alpha}\right) + z^{-2}}\right]$$

$$= \frac{1}{2}\left[\frac{2 - 2\cos\alpha\, z^{-1}}{1 - 2z^{-1}\cos\alpha + z^{-2}}\right]$$

$$= \frac{1 - \frac{\cos\alpha}{z}}{1 - \frac{2\cos\alpha}{z} + \frac{1}{z^2}}$$

$$= \frac{\frac{z - \cos\alpha}{z}}{\frac{z^2 - 2z\cos\alpha + 1}{z^2}}$$

$$Z\{\cos\alpha k\} = \frac{z(z - \cos\alpha)}{z^2 - 2z\cos\alpha + 1} \quad \text{for } |z| > 1$$

for ROC → where $|e^{i\alpha} z^{-1}| < 1$ $\qquad |e^{i\alpha}| = 1$

$$|z| > |e^{i\alpha}|$$

$$|z| > 1$$

$$|e^{-i\alpha} z^{-1}| < 1$$

$$|z| > |e^{-i\alpha}|$$

Q. Find $Z\{\cos \frac{k\pi}{2}\}$

Sol^n $$Z\left\{\cos \frac{k\pi}{2}\right\} = \frac{z\left(z - \cos\pi/2\right)}{z^2 - 2z\cos\frac{\pi}{2} + 1}$$

$$= \frac{z^2}{z^2+1} \quad |z|>1$$

⑦ If $f(k) = \{\sin \alpha k\}_{k\geq 0}$

Show that $z\{\sin\alpha k\}_{k\geq 0} = \dfrac{z\sin\alpha}{z^2-2z\cos\alpha+1} \quad |z|>1$

$$\boxed{\text{using } \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}}$$

⑧ If $f(k) = \{\cosh\alpha k\}_{k\geq 0}$
then $z\{\cosh\alpha k\} = \dfrac{z(z-\cosh\alpha)}{z^2-2z\cosh\alpha+1}$ for $|z| > \max$ of $|e^{\alpha}|$ or $|e^{-\alpha}|$.

⑨ If $f(k) = \{\sinh\alpha k\}_{k\geq 0}$
then $z\{\sinh\alpha k\} = \dfrac{z\sinh\alpha}{z^2-2z\cosh\alpha+1}$ for $|z|$ max of $|e^{\alpha}|$ or $|e^{-\alpha}|$.

Table of standard sequences & transform.

	$f(k)$	$F(z) = z\{f(k)\}$
①	$\{a^k\}_{k\geq 0}$	$\dfrac{z}{z-a}$ for $\lvert z\rvert > \lvert a\rvert$
②	$\{a^k\}_{k<0}$	$\dfrac{z}{a-z}$ for $\lvert z\rvert < \lvert a\rvert$.
③	$\{a^{\lvert k\rvert}\}$	$\dfrac{az}{1-az} + \dfrac{z}{z-a} \quad \lvert a\rvert < \lvert z\rvert < \dfrac{1}{\lvert a\rvert}$
④	$\{\delta(k)\}$	1
⑤	$\{u(k)\}$	$\dfrac{z}{z-1} \quad \lvert z\rvert > 1$

(6) $\{\cos\alpha k\}_{k\geq 0}$ $\qquad \dfrac{z(z-\cos\alpha)}{z^2-2z\cos\alpha+1}$ for $|z|>1$

(7) $\{\sin\alpha k\}_{k\geq 0}$ $\qquad \dfrac{z\sin\alpha}{z^2-2z\cos\alpha+1}$ for $|z|>1$

(8) $\{\cosh\alpha k\}_{k\geq 0}$ $\qquad \dfrac{z(z-\cosh\alpha)}{z^2-2z\cosh\alpha+1}$ for $|z|>\max(|e^{\alpha}|$ or $|e^{-\alpha}|)$

(9) $\{\sinh\alpha k\}_{k\geq 0}$ $\qquad \dfrac{z\sinh\alpha}{z^2-2z\cosh\alpha+1}$ for " "

Ex: Find $z\left\{\dfrac{a^k}{k!}\right\}_{k\geq 0}$

Sol^n: Let $z\left\{\dfrac{a^k}{k!}\right\} = \sum_{k=0}^{\infty} \dfrac{a^k}{k!} z^{-k} = 1 + \dfrac{a}{1!z} + \dfrac{a^2}{2!z^2} + \dfrac{a^3}{3!z^3} + \dots$

$$= 1 + \left(\frac{a}{z}\right) + \frac{1}{2!}\left(\frac{a}{z}\right)^2 + \frac{1}{3!}\left(\frac{a}{z}\right)^3 + \dots$$

$\because e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dots$

$\therefore z\left\{\dfrac{a^k}{k!}\right\} = e^{a/z}$

Q. Find if $f(k) = 3^k \quad k<0$
$\qquad\qquad = 2^k \quad k\geq 0$

$z\{f(k)\}$

Sol^n: $z\{f(k)\}_{k<0} = \dfrac{z}{3-z} + \dfrac{z}{z-2}$

$$\sum_{k=-\infty}^{\infty} f(k)z^{-k} = \sum_{-\infty}^{-1} 3^k z^{-1} + \sum_{0}^{\infty} 2^k z^{-1}$$

$$= \frac{z}{3-z} + \frac{z}{z-2} \qquad |z|<|3| \ \& \ |z|>|2|.$$

Q. Find $z\left\{\frac{1}{2}\right\}^{|k|}$ for all k

Sol^n: $z\left\{\frac{1}{2}\right\}^{|k|} = \sum_{-\infty}^{\infty}\left(\frac{1}{2}\right)^{|k|} z^{-k}$

$$= \frac{\frac{z}{2}}{1-\frac{z}{2}} + \frac{z}{z-\frac{1}{2}}$$

$$= \frac{z}{2-z} + \frac{2z}{2z-1} \qquad \left|\tfrac{1}{2}\right| < |z| < |2|.$$

Q. Find $z\{\sin(2k+3)\}$

Sol^n: $\therefore z\{\sin(2k+3)\} = z\{\sin(2k)\cos 3 + \cos(2k)\sin 3\}$

$$= \cos 3\, z\{\sin(2k)\} + \sin 3\, z\{\cos 2k\}$$

$$= \cos 3 \cdot \frac{z\sin 2}{z^2-2z\cos 2+1} + \sin 3 \cdot \frac{z(z-\cos 2)}{z^2-2z\cos 2+1}$$

for $|z|>1$.

4) Properties:

If $z\{f(k)\} = F(z)$

① Change of scale: $z\{a^k f(k)\} = F\left(\frac{z}{a}\right)$

Ex: Find $z\{2^k U(k)\}$

$$z\{U(k)\} = \frac{z}{z-1} \quad \text{for } |z|>1.$$

$$= F(z)$$

$$z\{2^k U(k)\} = F\left(\frac{z}{2}\right)$$

$$= \frac{\frac{z}{2}}{\frac{z}{2}-1}$$

$$= \frac{z}{z-2} \qquad |z|>|2|.$$

Q. Find $z\left\{\left(\frac{1}{3}\right)^k \sin 2k\right\}$

Sol^n: $z\{\sin 2k\} = \dfrac{z \sin 2}{z^2 - 2z\cos 2 + 1}$ for $|z|>1$

$z\left\{\left(\frac{1}{3}\right)^k \sin 2k\right\} = \dfrac{3z \sin 2}{(3z)^2 - 2 \cdot 3z \cos 2 + 1}$ for $|z| > \left|\frac{1}{3}\right|$.

$= \dfrac{3z \sin 2}{(3z)^2 - 6z\cos 2 + 1}$ for $|z| > \left|\frac{1}{3}\right|$.

② If $z\{f(k)\} = F(z)$
then $z\{e^{-\alpha k} f(k)\} = F(e^{\alpha} z)$.

Ex: Find $z\left\{e^{-2k}\left(\frac{1}{4}\right)^k\right\}$

Sol^n: $z\left\{\left(\frac{1}{4}\right)^k\right\} \Rightarrow F(z) = \dfrac{z}{z - \frac{1}{4}}$ for $|z| > \left|\frac{1}{4}\right|$

$\therefore z\left\{e^{-2k}\left(\frac{1}{4}\right)^k\right\} = \dfrac{e^2 z}{e^2 z - \frac{1}{4}}$ for $|e^2 z| > \left|\frac{1}{4}\right|$

Q. Find $z\{e^{-3k} \cos 2k\}$

Sol^n: $z\{\cos 2k\} = \dfrac{z(z - \cos 2)}{z^2 - 2z\cos 2 + 1}$ $|z| > 1$

$z\{e^{-3k} \cos 2k\} = \dfrac{e^3 z(e^3 z - \cos 2)}{(e^3 z)^2 - 2e^3 z \cos 2 + 1}$ for $|z|$

③ Multiplication by K
If $z\{f(k)\} = F(z)$
then $z\{k f(k)\} = \left(-z\dfrac{d}{dz}\right) F(z)$

$z\{k^n f(k)\} = \left(-z\dfrac{d}{dz}\right)^n F(z)$.

Ex. Find $z\{k2^k\}_{k\geq 0}$

$\because z\{2^k\}_{k\geq 0} = \frac{z}{z-2}$ for $|z|>2$

$= F(z)$

$\therefore z\{k2^k\} = \left(-z\frac{d}{dz}\right)\left(\frac{z}{z-2}\right)$

$= -z\left[\frac{(z-2)-z}{(z-2)^2}\right]$

$= \frac{2z}{(z-2)^2}$

Q. Find $z\{k^2\cos\frac{\pi}{2}k\}$

Soln.: $f(k) = \cos\frac{\pi}{2}k$

$F(z) = z\{\cos\frac{\pi}{2}k\} = \frac{z(z-\cos\pi/2)}{z^2-2z\cos\pi/2+1}$

$= \frac{z^2}{z^2+1}$ $|z|>1$

$z\{k^2\cos\frac{\pi}{2}k\} = \left(-z\frac{d}{dz}\right)^2\left(\frac{z^2}{z^2+1}\right)$

$= z^2\frac{d}{dz}\left(\frac{(z^2+1)(2z)-z^2(2z)}{(z^2+1)^2}\right)$

$= z^2\frac{d}{dz}\left(\frac{2z^3+2z-2z^3}{(z^2+1)^2}\right)$

$= z^2\frac{d}{dz}\left(\frac{2z}{(z^2+1)^2}\right)$

$= z^2\left(\frac{(z^2+1)^2\cdot 2 - 2z(2(z^2+1)\cdot 2z)}{(z^2+1)^4}\right)$

$= z^2\left(\frac{2(z^2+1)^2 - 8z(z^2+1)}{(z^2+1)^4}\right)$

$= z^2\left(\frac{2(z^4+2z^2+1) - 8z^3-8z}{(z^2+1)^4}\right)$

$$= z^2\left(\frac{2z^4+4z^2+2-8z^3-8z}{(z^2+1)^4}\right)$$

$$= \frac{2z^6+4z^4-8z^5-8z^3+2z^2}{(z^2+1)^4}$$

④ Division by k

If $z\{f(k)\} = F(z)$

then $z\left\{\frac{f(k)}{k}\right\} = \int_z^{\infty} z^{-1}F(z)\,dz$

Ex. Find $z\left\{\frac{\sin ak}{k}\right\}_{k>0}$

Sol^n: here $z\{\sin ak\}_{k\geq 0} = \frac{z\sin a}{z^2-2z\cos a+1} \quad |z|>1$

$$\therefore z\left\{\frac{\sin ak}{k}\right\} = \int_z^{\infty} z^{-1}\left(\frac{z\sin a}{z^2-2z\cos a+1}\right)dz$$

$$= \sin a\int_z^{\infty}\frac{dz}{z^2-2z\cos a+1}$$

$$= \sin a\int_z^{\infty}\frac{dz}{z^2-2z\cos a+\sin^2 a+\cos^2 a}$$

$$= \sin a\int_z^{\infty}\frac{dz}{(z-\cos a)^2+\sin^2 a} \qquad \boxed{\int\frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}}$$

$$= \sin a\cdot\frac{1}{\sin a}\left[\tan^{-1}\left(\frac{z-\cos a}{\sin a}\right)\right]_z^{\infty}$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{z-\cos a}{\sin a}\right).$$

$$= \cot^{-1}\left(\frac{z-\cos a}{\sin a}\right).$$

Q. Find $Z\left\{\frac{2^k}{k}\right\}_{k\geq 1}$

<u>Soln:</u> $f(k) = \{2^k\}_{k\geq 1}$

$$Z\{2^k\}_{k\geq 1} = \sum_{k=1}^{\infty} 2^k z^{-k}$$

$$= 2z^{-1} + 2^2 z^{-2} + 2^3 z^{-3} + \cdots$$

$$= \frac{2z^{-1}}{1-2z^{-1}}$$

$$= \frac{2/z}{1-2/z}$$

$$= \frac{2}{z-2} \quad \text{where } |2z^{-1}| < 1$$

$$|2| < |z|.$$

$$Z\left\{\frac{2^k}{k}\right\} = \int_z^{\infty} z^{-1}\left(\frac{2}{z-2}\right) dz$$

$$= \int_z^{\infty} \frac{2}{z^2-2z} dz$$

$$= \int_z^{\infty} \frac{2}{z(z-2)} dz.$$

$$\frac{1}{z(z-2)} = \frac{A}{z} + \frac{B}{z-2}$$

$$1 = A(z-2) + B(z)$$

$$1 = Az - 2A + Bz$$

$$A+B=0 \qquad -2A=1$$

$$B = 1/2 \qquad A = -1/2$$

$$\frac{1}{z(z-2)} = \frac{-1}{2z} + \frac{1}{2(z-2)}$$

$$= 2\int_z^{\infty} \frac{-1}{2z} dz + 2\int_z^{\infty} \frac{1}{2(z-2)} dz$$

$$= \left(-\log z\right)_z^{\infty} + \left(\log(z-2)\right)_z^{\infty}$$

$$= \left[\log\frac{z-2}{z}\right]_z^{\infty}$$

$$\log \frac{z\left(1-\frac{2}{z}\right)}{z} - \log\left(\frac{z-2}{z}\right)$$

$$= \log\left(\frac{z-2}{z}\right)$$

⑤ Convolution thm:

$$f(k) * g(k) = f(m)g(k-m)$$

or

$$g(m)f(k-m)$$

$$z\{f(k) * g(k)\} = F(z) - G(z)$$

$$F(z) = z\{f(k)\}$$

$$G(z) = z\{g(k)\}$$

Ex: Find $z\left\{\frac{1}{1^k} * \frac{1}{2^k} * \frac{1}{3^k}\right\}_{k \geq 0}$

Soln:

$$z\left\{\frac{1}{1^k}\right\} = F(z) = \frac{z}{z-1}$$

$$z\left\{\frac{1}{2^k}\right\} = G(z) = \frac{z}{z-1/2}$$

$$z\left\{\frac{1}{3^k}\right\} = H(z) = \frac{z}{z-1/3}$$

$$z\left\{\frac{1}{1^k} * \frac{1}{2^k} * \frac{1}{3^k}\right\} = \left[\frac{z}{z-1}\right]\left[\frac{2z}{2z-1}\right]\left[\frac{3z}{3z-1}\right].$$

Inverse z-transform

$F(z)$	$f(k)\ k \geq 0$	$f(k)\ k < 0$
$\left[\frac{z}{z-a}\right]_{\substack{\lvert z\rvert > a \\ \lvert z\rvert < a}}$	a^k	$-a^k$