Unit I

LDE

* Integral formulae

① $\int x^n dx = \frac{x^{n+1}}{n+1} + C$

② $\int e^{ax} dx = \frac{e^{ax}}{a}$

$\int \frac{f'(x)}{f(x)} dx = \log(f(x))$

③ $\int a^x dx = \frac{a^x}{\log a}$

④ $\int \sin x \, dx = \cancel{\sin x} -\cos x$

⑤ $\int \cos x \, dx = \sin x$

⑥ $\int \tan x \, dx = \log(\sec x)$

⑦ $\int \frac{dx}{x} = \log x$

⑧ $\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1}\frac{x}{a}$

⑨ Integration by parts.

$$\int \underbrace{\underset{\text{1st}}{u} \cdot \underset{\text{2nd}}{v}}_{\text{LIATE}} \, dx = u\int v dx - \int \left(\int v dx\right) \frac{du}{dx} dx$$

⑩ $\int_{x=a}^{b} f(x) dx = [g(x)]_a^b = g(b) - g(a)$

Eg -

① $\frac{dy}{dx} + 1 = 0 \longrightarrow y = ?$

$\Rightarrow \frac{dy}{dx} = -1$

$\Rightarrow \int dy = -\int dx$ → variable separable form

$= y = -x + C$

$= \underline{y + x = C}$

② $x\,dy + y\,dx = 0$

$\Rightarrow \frac{dy}{y} + \frac{dx}{x} = 0$

$\Rightarrow \int \frac{dy}{y} = -\int \frac{dx}{x}$

$\Rightarrow \log y = -\log x + \log C$

$\Rightarrow \log y + \log x = \log C$

$\Rightarrow \log(xy) = \log C$

$\Rightarrow xy = C$

③ $\frac{dy}{dx} + y = 5$

$dy = (5 - y)\,dx$

$1/dx = \frac{5 - y}{dy}$

$\int dx/1 = \int dy/5 - y$

$x = -\log(5 - y) + C$

$\underline{x + \log(5 - y) = C}$

Linear Diff Eqn (nth order)

* 1st order L.D.E in y

$$\boxed{y = f(x)e^{-mx} + ce^{-mx}}$$

* General solution -

$$\boxed{y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots c_n e^{m_n x}}$$

for liner Diff Eq of n^{th} order.

* Depending upon nature of roots of $\phi(D) = 0$

case I) $\boxed{y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots c_n e^{m_n x}}$

↳ if roots are real & distinct

case II) If $\phi(D) = 0$ has real & Equal roots.

i.e. $D = m_1 = m_2 \cdots m_n$

$$\therefore \boxed{\begin{array}{c} y = (c_1 c_2 x)e^{m_1 x} + c_3 e^{m_3 x} + \cdots c_n e^{m_n x} \\ \text{or} \\ y = (c_1 \cancel{x} + c_2 x)e^{m_1 x} \end{array}}$$

case III) If $\phi(D) = 0$ has imaginary & diff roots.

i.e. $D = \alpha + i\beta = m_1$

$D = \alpha - i\beta = m_2$

$$\Rightarrow \boxed{y = e^{\alpha x}[A\cos\beta x + B\sin\beta x]}$$

i.e. $\Rightarrow e^{\alpha x}[\cos\beta x \underbrace{(c_1 + c_2)}_{A} + \sin\beta x \underbrace{(ic_1 - ic_2)}_{B}]$

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caseIV ⟹ If $\phi(D)=0$ has imaginary & equal roots.

i.e. $D = m_1 = \alpha + i\beta = m_2$

$D = m_3 = \alpha - i\beta = m_4$

$$\therefore \boxed{y = e^{\alpha x}\left[(C_1 + C_2 x)\cos\beta x + (C_3 + C_4)\sin\beta x\right]}$$

Examples -

① $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

case ①

operator form (O.F) ⟹ $D = \frac{d}{dx}$

$\therefore [D^2 - 5D + 6]y = 0$

$\therefore$ auxillary eqn becomes -

$D^2 - 5D + 6 = 0$

$(D-2)(D-3) = 0$

6: +3, −2

⟹ $D = 2$ & $D = 3$

Roots are real & distin't

$\therefore$ from Case-I

$$y = C_1 e^{2x} + C_2 e^{3x}$$

$$D = \frac{-b \pm \sqrt{(b)^2 - 4ac}}{2a}$$

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(2) $\frac{d^2y}{dx^2} - \frac{4dy}{dx} + 4y = 0$

case 2 O.F $\Rightarrow$ $D = d/dx$

$D^2 - 4D + 4 = 0$ $\quad 4 \to -2 \; -2$

$\therefore$ A.E $= D^2 - 4D + 4 = 0$

$(D-2)^2 = 0$

$D = 2, 2 \to$ equal roots.

$\therefore$ soln: C.F $\Rightarrow$ $y = (C_1 + C_2 x)e^{2x}$

from case II $\to$

(3) $(D^2 + 2D + 2)y = 0$

A.E $\Rightarrow$ $D^2 + 2D + 2 = 0$

case 3 $D = \frac{-2 \pm \sqrt{4 - 4(2)}}{2(1)}$

$D = \frac{-2 \pm \sqrt{-4}}{2}$

$D = \frac{-2 \pm \sqrt{4}\sqrt{-1}}{2} = \frac{-2 \pm 2i}{2}$

$D = -1 \pm i$

$\therefore D = -1 + i \; \& \; D = -1 - i$

$\therefore$ Roots are immaginary & distinct

Using case III $\Rightarrow$

$\alpha = 1$

$y = e^{-x}[A\cos x + B\sin x]$

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(4) $(D^2+D+1)^2 y = 0$

A.E $\Rightarrow (D^2+D+1)^2 = 0$

Case 4

$\therefore D = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot 1}}{2(1)}$

$= \frac{-1 \pm \sqrt{1-4}}{2}$

$= \frac{-1 \pm \sqrt{3}\, i}{2}$

$\therefore D = \frac{-1+\sqrt{3}\, i}{2}$ & $D = \frac{-1-\sqrt{3}\, i}{2}$

$= m_1 = m_2$ $\qquad = m_3 = m_4$

$\therefore$ Using case IV

$\alpha = -1/2 \qquad \beta = \sqrt{3}/2$

$\therefore y = e^{-x/2}\left[(c_1 + c_2 x)\cos\frac{\sqrt{3}}{2}x + (c_3 + c_4 x)\sin\frac{\sqrt{3}}{2}x\right]$

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* Multiple cases problems -

① $(D^4 - 16)y = 0$

$D = \frac{d}{dx}$

A.E $\Rightarrow D^4 - 16 = 0$

$(D^2)^2 - (4)^2 = 0$

$\downarrow \quad \downarrow$

$A \quad B$

$\Rightarrow (D^2 - 4)(D^2 + 4) = 0$

$\Rightarrow (D+2)(D-2)(D^2+4) = 0$

$\Rightarrow D = -2$

$D = 2$

$D^2 + 4 = 0$

$\Rightarrow D^2 = -4$

$D = \sqrt{-4}$

$D = \pm 2i$

$\therefore$ The roots are

$\underbrace{D = 2,\ D = -2}_{\text{case 1}},\ \underbrace{D = 2i,\ D = -2i}_{\text{case 3}}$ $\alpha = 0$

$\beta = 2$

$\therefore C.F \Rightarrow y = c_1 e^{2x} + c_2 e^{-2x} + (c_3 \cos 2x + c_4 \sin 2x)$ e^{0x}

$C.F \Rightarrow y = c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos 2x + c_4 \sin 2x$

$\Rightarrow$

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② $(D^4 - 2D^3 + D^2)y = 0$

A.E $\Rightarrow D^4 - 2D^3 + D^2 = 0$

$D^2(D^2 - 2D + 1) = 0$

$D^2 = 0$ & $D^2 - 2D + 1 = 0$

$\therefore \underbrace{D = 0, D = 0}$ $\quad (D-1)^2 = 0$

$\underbrace{D = 1, D = 1}$

using case ① & ⑪

$C.F \Rightarrow y = (c_1 + c_2 x)e^{0x} + (c_3 + c_4 x)e^{x}$

③ $(D^3 + D^2 - 2D + 12)y = 0$

A.E $\Rightarrow D^3 + D^2 - 2D + 12 = 0$

put $D = -3$ in

$$\begin{array}{c|cccc} -3 & 1 & +1 & -2 & \cancel{+} \\ & & -3 & +6 & \cancel{-} \\ \hline & 1 & -2 & 4 & \boxed{0} \end{array}$$

$(D+3)(D^2 - 2D + 4) = 0$

$D = \frac{(-2) \pm \sqrt{4 - 4(4)}}{2}$

$\therefore$ Roots are

$D = -3$

$1 + i\sqrt{3}$

$1 - i\sqrt{3}$

$D = \frac{2 \pm \sqrt{12}}{2}$

$D = \frac{2 \pm 2\sqrt{3}}{2}$

$\therefore \alpha = 1$

$\beta = \sqrt{3}$

$D = 1 \pm \sqrt{3}$

using case ① & ⑪

$y = c_1 e^{-3x} + e^{x}[c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x]$

* L.D.E -

$\phi(D)y = f(x)$

G.S = C.F + Particular integral (PI)

General soln = Complementary function + Particular integral

$\underbrace{\text{Complementary function}}_{\text{arbitrary constat}}$ $\underbrace{\text{Particular integral}}_{\text{free from arbitary constant.}}$

- To find P.I consider $y_p = \frac{1}{\phi(D)} f(x)$

$D = \frac{d}{dx} \Rightarrow \frac{1}{D} = \text{int} = \int$

$D^2 = \frac{d^2}{x^2} \Rightarrow \frac{1}{D^2} = \iint$

Note - $(D-m_1)$ is a factor of $\phi(D)$ then $\frac{1}{D-m_1}$ is the inverse op of ϕD

$$\boxed{P.I \Rightarrow y_p = \frac{1}{\phi(D)} f(x)}$$

Methods used for P.I -

1. General method
2. shortcut method
3. method of variation of parameters

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1) General Method -

consider $Y_p = \frac{1}{\phi(D)} f(x)$

- If $\phi(D) = (D-m_1)$

$$\Rightarrow \boxed{Y_p = \frac{1}{(D-m_1)} f(x) = e^{m_1 x} \int e^{-m_1 x} f(x)\,dx}$$

- If $\phi(D) = (D+m_1)$

then $\Rightarrow$ $$\boxed{Y_p = \frac{1}{(D+m_1)} f(x) = e^{-m_1 x} \int e^{m_1 x} f(x)\,dx}$$

- For real factors $\Rightarrow$ $\phi(D) = (D-m_1)(D-m_2)$

$$Y_p = \frac{1}{(D-m_1)(D-m_2)} f(x)$$

$$= \frac{1}{D-m_1}\left[\underbrace{e^{m_2 x}\int e^{-m_2 x} f(x)\,dx}_{g(x)}\right]$$

$$\therefore \boxed{Y_p = \frac{1}{D-m_1} \cdot g(x) = e^{m_1 x}\int e^{-m_1 x} \cdot g(x)\,dx}$$

$$\boxed{\frac{1}{D} f(x) = \int f(x)\,dx}$$

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* General Method for P.I

① $(D^2+D)y = \frac{1}{1+e^x}$

$A.E \Rightarrow D^2+D=0$

$D^2=0$ $\qquad C.F = C_1e^{-x}+C_2$

$D=-1,0$

$$P.I = \frac{1}{D^2+D}\left(\frac{1}{1+e^x}\right)$$

$$= \frac{1}{D(D+1)}\,\frac{1}{1+e^x} \quad -①$$

Consider

$$\frac{1}{D(D+1)} = \frac{A}{D} + \frac{B}{D+1} \qquad \text{P.F form}$$

$N^r \Rightarrow 1 = A(D+1) + B(D)$

put $D=0$ $\qquad D=-1$

$1=A$ $\qquad B=-1$

The eqn becomes. $\qquad \frac{1}{D(D+1)} = \frac{1}{D} - \frac{1}{D+1}$

$$P.I = \frac{1}{D}\left(\frac{1}{1+e^x}\right) - \frac{1}{D+1}\left(\frac{1}{1+e^x}\right)$$

$$P.I = \int \frac{dx}{1+e^x} - e^{-x}\int e^x \frac{1}{1+e^x}\,dx$$

$\div$ N/D by e^x

$$\int \frac{e^{-x}dx}{e^{-x}+1} - e^{-x}\int \frac{e^x}{1+e^x}\,dx$$

Put $1+e^{-x}=t$ | $1+e^{x}=u$

$\Rightarrow -e^{x}\,dx = dt$ | $\Rightarrow e^{x}\,dx = du$

$$\Rightarrow \int -\frac{dt}{t} - e^{-x}\int \frac{du}{u}$$

$$= -\log(1+e^{-x}) - e^{-x}(\log(1+e^{x}))$$

Q) Solve: $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^{x}}$

step1 find $y_c = ?$

A.E $\phi(D)=0$

$$D^2y + 3Dy + 2y = 0$$

$$(D^2+3D\cancel{+}2y = 0$$

$$(D+2)(D+1) = 0$$

$$D=-2,\ D=-1$$

$$\boxed{y = c_1e^{-x} + c_2e^{-2x}} \quad -(1)$$

step2 find $y_p = ?$

$$P.I = y_p = \frac{1}{\phi(D)} f(x)$$

$$= \frac{1}{(D+2)(D+1)}(e^{e^{x}})$$

$$y_p = \frac{1}{(D+2)}\left[\underbrace{\frac{1}{(D+1)}e^{e^{x}}}\right]$$

using formula:

$$1/D+m = f(x) = e^{-mx}\int e^{mx}f(x)dx\}$$

$$y_p = \frac{1}{(D+2)}\left[e^{-x}\int e^{x}e^{e^{x}}dx\right]$$

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$$y_p = \frac{1}{(D+2)}\left[e^{-x}\int e^{x}\cdot e^{e^x}dx\right]$$

put $e^x = t$

$e^x dx = dt$

$$y_p = \frac{1}{(D+2)}\left[e^{-x}\int e^{t}dt\right]$$

$$y_p = \frac{1}{(D+2)}\left[e^{-x}\cdot e^{e^x}\right]$$

using formula.

$$y_p = e^{-2x}\int e^{2x}\cdot e^{-x}\cdot e^{e^x}dx$$

$$= e^{-2x}\int e^{x}\cdot e^{e^x}dx$$

put $e^x = t$

$e^x dx = dt$

$$y_p = e^{-2x}\int e^{t}dt$$

$$\boxed{y_p = e^{-2x}e^{e^x}}$$

$\therefore$ General soln is –

$G.S = C.F + y_p$

$$G.S = C_1e^{-x} + C_2e^{-2x} + e^{-2x}\cdot e^{e^x}$$

* Shortcut Method-

Case I) If $f(x) = e^{ax}$

$$P.I = \frac{1}{\phi(D)} e^{ax}$$ replace D with a

$$= \frac{1}{\phi a} e^{ax} \ldots \cancel{\text{if } f(a)=0}$$

- If $\phi(a) = 0 \Rightarrow a$ is root of $\phi(D)$
  $\Rightarrow D-a$ is factor of $\phi(D)$

$$P.I = \frac{1}{\phi(D)} x e^{ax}$$ then put $D=a$

$$= \frac{1}{\phi(a)} x e^{ax}$$ if $\phi'(a) \neq 0$

if $\phi'(a) = 0$ then double diff of D
then sub $D=a$

x with x
& dif ϕD

$$= \frac{1}{\phi''(a)} x^2 e^{ax} \ldots$$ continue till non zero As.

Note -> (1)

$$P.I = \frac{1}{(D-a)} e^{ax} = \frac{x}{1!} e^{ax}$$

$$= \frac{1}{(D-a)^2} e^{ax} = \frac{x}{2!} e^{ax}$$

$$= \frac{1}{(D-a)^r} e^{ax} = \frac{x}{r!} e^{ax}$$

$$\boxed{\begin{aligned} \text{case I)}\ f(x) &= e^{ax} \ \cdots\ D = a \\ &= a^{x} \ \cdots\ D = \log a \\ &= k \ \cdots\ D = 0 \\ &= a^{-x} \ \cdots\ D = -\log a \end{aligned}}$$

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(2) for constant

$$k = e^{ax} \Rightarrow \text{put } D = 0$$

(3) for $f(x) = a^{x}$

$$\Rightarrow e^{x\log a} = e^{\log a^{x}} \quad \text{put } D = \log a$$

(4) for $f(x) = a^{-x} = e^{-x\log a}$

put $D = -\log a$

Example -

(1) $(D^2 + 3D + 2)y = e^{2x}$

$$C.F = c_1 e^{-2x} + c_2 e^{-x}$$

$$P.I = \frac{1}{(D+1)(D+2)} \cdot e^{2x} \quad \text{Put } D = 2$$

$$P.I = \frac{1}{12} e^{2x}$$

(2) $(D^2 + 3D + 2y)y = e^{-2x}$

$$C.F = c_1 e^{-x} + c_2 e^{-2x}$$

$$P.I = \frac{1}{D^2 + 3D + 2} \cdot e^{-2x} \quad \text{if } D = -2,\ \phi(D) = 0$$

$\rightarrow$ differentiation of denom & multiply by x

$\Rightarrow$

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$$\phi(D) = \frac{x}{2D+3} \cdot e^{-2x} \quad a = 2 = D$$

$$P.I = \frac{1}{-e} e^{-2x} = -e^{-2x}$$

~~(3) $(D^2+3D+2)y$~~
~~$(D+1)(D+2)$~~
~~$D=-1,\ D=-2$~~
~~$\therefore\ y_c = c_1 e^{-x} + c_2 e^{-2x}$~~
~~step II > $y_p = \frac{1}{\phi(D)} f(x)$~~

(2) $(D+3)y = 5^x$

$\rightarrow y = \frac{1}{D+3} 5^x$

$a^x \Rightarrow D = \log a = \log 5$

$$y_p = \frac{1}{(\log 5 + 3)} 5^x$$

(3) $(D^2 - 2D + 3)y = \frac{3}{2}$

$$y = \frac{1}{(D^2 - 2D + 3)} \frac{3}{2}$$

$D = 0$

$$y_p = \frac{1}{(0 - 0 + \cancel{3})} \frac{\cancel{3}}{2}$$

$$= \frac{1}{3}$$

(5) $(D^2 - 9)y = e^{3x}$

$$y = \frac{1}{(D^2-9)} e^{3x}$$

$D = a = 3 = 0$

case of failure

$$y = \frac{x}{(2D - 0)} e^{3x}$$

$D = 3$

$$\boxed{y = \frac{x e^{3x}}{6}}$$

(4) $(D^2-4D+4)y = e^{2x}+3$

$$P.I = \frac{1}{(D-2)^2} e^{2x} + \frac{1}{(D-2)^2} 3 \cdot e^{0x}$$

put $D=2$

$$P.I = \frac{x^2}{2!} e^{2x} + \frac{3}{4}.$$

(5) $D^2+5D+6 = e^{2x}$

step I) $y_c = ?$ $D^2+5D+6 = 0$

$D=-2$, $D=-3$

$\therefore$ $y_c = c_1 e^{-2x} + c_2 e^{-3x}$

step II) $y_p = ?$ $y_p = \frac{1}{\phi(D)} f(x)$

$$y_p = \frac{1}{D^2+5D+6} e^{2x}$$

Put $D=2$

$$y_p = \frac{1}{(2)^2+5(2)+6} e^{2x}$$

$$y_p = \frac{1}{4+10+6} e^{2x}$$

$$\boxed{y_p = \frac{1}{20} e^{2x}}$$

$$G.S = c_1 e^{-2x} + c_2 e^{-3x} + \frac{1}{20} e^{2x}$$

Case II)

If $f(x) = \sin(ax+b)$ or $\cos(ax+b)$
Then replace D^2 of $\phi(D)$ as $(-a^2)$

$$\boxed{P.I = \frac{1}{\phi(D)} \sin(ax+b) \text{ or } \cos(ax+b) \qquad \phi(D) \longrightarrow \text{put } D^2 = -(a)^2}$$

$$= \frac{1}{\phi(-a^2)} \sin(ax+b) \text{ or } \cos(ax+b)$$
if $\phi(-a)^2 \neq 0$

if $\phi(-a^2) = 0 \Rightarrow$ diff D^r & multiply by x

Note -

① $\sin^2 x = \dfrac{1-\cos 2x}{2}$

② $\cos^2 x = \dfrac{1+\cos 2x}{2}$

③ $\sin 2x = 2\sin x \cos x$

④ $\sin A \cos B = \dfrac{\sin(A+B) + \sin(A-B)}{2}$

⑤ $\cos A \cos B = \dfrac{\cos(A+B) + \cos(A-B)}{2}$

⑥ $\sin A \sin B = \dfrac{\cos(A-B) - \cos(A+B)}{2}$

examples -

① $(D^2+2D+1)y = \sin x$

$C.F \Rightarrow (c_1 + c_2 x)e^{-x}$

$$P.I = \frac{1}{D^2+2D+1}\sin x$$

put $D^2 = -1$

$$= \frac{1}{\cancel{-1}+2D+\cancel{1}}\sin x$$

$$= \frac{1}{2D}\sin x$$

$$P.I = \frac{1}{2}\int \sin x\, dx = \frac{-\cos x}{2}$$

$G.S = C.F + P.I$

$$= (c_1 + c_2 x)e^{-x} - \frac{\cos x}{2}$$

Note - if D^r contains

$\frac{1}{aD+n}$ term in $\phi(D)$ then bring D^2 term by multiplying N^r & D^r by conjugate $\Rightarrow mD-n$

$\rightarrow$

(2) $(D^3+4D)y = \cos 2x$

$C.F \Rightarrow D = 0, 2i, -2i$

$= c_1 + (c_2 \cos 2x + c_3 \sin 2x)$

$P.I = \dfrac{1}{D^3+4D} \cos 2x \Rightarrow a = 2$

put $D^2 = -2^2 = -4$

$= \dfrac{1}{-4D+4D} \cos 2x$ $= 0$

$\therefore$ Diff D^r & multiply by x

$= x \dfrac{1}{3D^2+4} \cos 2x$

Put $D = -4$

$= x \cdot \dfrac{1}{-8} \cos 2x$

$\therefore P.I = y_p = -\dfrac{x \cos 2x}{8}$

$$\boxed{G.S = c_1 + (c_2 \cos 2x + c_3 \sin 2x) - \dfrac{x \cos 2x}{8}}$$

③ $(D^2+1)y = \sin 2x \cos x$

$C.F \Rightarrow c_1 \cos x + c_2 \sin x$

$P.I = \frac{1}{D^2+1} \sin 2x \cos x$

$= \frac{1}{D^2+1} \left(\frac{\sin 3x + \sin x}{2} \right)$

$= \frac{1}{2} \left[\frac{1}{D^2+1} \sin 3x + \frac{1}{D^2+1} \sin x \right]$

put $D^2 = -9$ put $D^2 = -1$

$= \frac{1}{2} \left[\frac{\sin 3x}{-8} + x \frac{1}{2D} \sin x \right]$

$= \frac{\sin 3x}{-16} + \frac{x}{4} \left(\int \sin x \, dx \right)$

$P.I = \frac{\sin 3x}{-16} - \frac{x \cos x}{4}$

$G.S = C.F + P.I$

$y = c_1 \cos x + c_2 \sin x - \frac{\sin 3x}{16} - \frac{x \cos x}{4}$

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~~4 Structure of Atom~~

Case III

hyperbolic trigo function

$$\boxed{\begin{array}{l}\text{when } f(x) = \cosh(ax+b) \\ \qquad\qquad \text{or } \sinh(ax+b) \\ \text{put } D^2 = a^2 \text{ in } \phi(D)\end{array}}$$

Eg —

1. $(D^2+1)y = \sinh(3x)$

$\sinh x = \frac{e^x - e^{-x}}{2}$

$\cosh x = \frac{e^x + e^{-x}}{2}$

$$P.I = \frac{1}{D^2+1} \sinh 3x \quad \text{put } D^2 = 3^2 = 9$$

$$= \frac{1}{10} \sinh 3x$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

2. $(D^2+3D+2)y = \sin^2 x$

$$y = \frac{1}{D^2+3D+2} \times \sin^2 x \qquad \sin^2 x = \frac{1-\cos 2x}{2}$$

$$y = \frac{1}{D^2+3D+2}\left(\frac{1-\cos 2x}{2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{D^2+3D+2}\right) - \frac{1}{2}\left(\frac{1}{D^2+3D+2}\cos 2x\right)$$

$$= \frac{1}{2}\left(\frac{1}{D^2+3D+2}\right)e^{0x} - \frac{1}{2}\left(\frac{1}{D^2+3D+2}\cos 2x\right)$$

$$D = 0 \qquad\qquad D^2 = -4$$

solve again

② $(D^2-1)y = \cos x \cdot \cos 2x$

$$(D^2-1)\,y = \frac{\cos 3x}{2} + \frac{\cos(-x)}{x}$$

$$= \frac{\cos 3x}{2} - \frac{\cos x}{2}$$

$$y = \frac{1}{D^2-1} \times \frac{\cos 3x}{2} - \frac{1}{D^2-1} \cdot \frac{\cos x}{2}$$

$$D^2 = -9 \qquad\qquad D^2 = -1$$

$$= \frac{1}{2}\left(\frac{\cos 3x}{-10}\right) \frac{-1}{2}\left(\frac{\cos x}{-2}\right)$$

$$P.I = -\frac{\cos 3x}{20} + \frac{\cos x}{4}$$

$$C.F \Rightarrow D^2-1 = 0$$

$$D^2 = 1$$

$$D = \pm 1$$

$$C.F = C_1e^{x} + C_2e^{-x}$$

$$GS = C.F + P.I$$

$$= C_1e^{x} + C_2e^{x} + \cos x - \cos 3x$$

Case IV) If $f(x) = x^m$ in $\phi(D)y = f(x)$

$$P.I = \frac{1}{\phi(D)} x^m$$

$$= [\phi(D)]^{-1} x^m$$

(expanding by Binomial exp). x^m

adjust

for binomial

→ $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$

→ $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$

$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots$

where x can be in form of $D = \frac{d}{dx}$

Note: Depending upon $x^m \Rightarrow m$ derivatives are possible

$\Rightarrow$

* Example -

① $(D^2-1)y = x^3$

$$P.I = \frac{1}{(D^2-1) \to \phi(D)} x^3$$

$$= \frac{-1}{[1-D^2)} x^3$$

$$= -[1-D^2]^{-1} x^3$$

Using $(1-x)^{-1} = 1 + x + x^2 + x^3 \cdots$

$$= -[1 + D^2 + (D^2)^2 + \cdots] x^3$$

$$= -[1 + D^2 + D^4 + \cdots] x^3$$

$$\therefore D = \frac{d}{dx}$$

$$D(x^3) = 3x^2$$

$$D^2(x^3) = 6x$$

$$D^3(x^3) = 6$$

$$D^4(x^3) = 0$$

$\therefore$ Neglecting the terms from $D^4 \cdots$

$$\therefore = -[1 + D^2] x^3$$

$$= -[x^3 + D^2(x^3)]$$

$$\underline{P.I = -[x^3 + 6x]}$$

$\Rightarrow$

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(2) $(D^2 - D + 1)y = x^3 - 3x^2 + 1$ use $(1+x)^{-1}$
$= 1 - x + x^2 - x^3 \ldots$

$\Rightarrow$ $P.I = \frac{1}{D^2 - D + 1}(x^3 - 3x^2 + 1)$

$= [1 + (D^2 - D)]^{-1}(x^3 - 3x^2 + 1)$

$\therefore$ expanding

$= [1 - (D^2 - D) + (D^2 - D)^2 \ldots](x^3 - 3x^2 + 1)$

$\therefore$ max derivatives possible for $(x^3 - 3x^2 + 1)$ are 3

$\therefore$ Discarding the terms from $D^4 \ldots$

$= [1 - D^2 + D + (D^4 - 2D^3 + D^2) \ldots](x^3 - 3x^2 + 1)$

$= [1 + D - 2D^3](x^3 - 3x^2 + 1)$

$= x^3 - 3x^2 + 1 + D(x^3 - 3x^2 + 1) - D^2(x^3 - 3x^2 + 1)$

$P.I = (x^3 - 3x^2 + 1) + (3x^2 - 6x) - (6)$

Case V)

If $f(x) = e^{ax} \cdot V$ (where V is any other function of x)

$$P.I = \frac{1}{\phi(D)} e^{ax} V$$

replace $D = D + a$

$$P.I = e^{ax} \frac{1}{\phi(D+a)} V$$

Then using any of the previous cases depending upon V find P.I

Example –

① $(D^2 - 4D + 3) y = x^3 e^{2x}$

$$P.I = \frac{1}{D^2 - 4D + 3} \cdot e^{2x} x^3$$

Put $D = D + 2$

$$= e^{2x} \frac{1}{(D+2)^2 - 4(D+2) + 3} x^3$$

$$= e^{2x} \frac{1}{D^2 + \cancel{4D} + 4 - \cancel{4D} - 8 + 3}$$

$$= e^{2x} \frac{1}{D^2 - 1} x^3$$ using case IV for P.I

$$= -e^{2x} (1 - D^2)^{-1} x^3$$

$$= -e^{2x} [x^3 + 6x]$$

(2) Find P.I $(D^2-4D+4)y = e^x \cos 2x$

$$P.I = \frac{1}{D^2-4D+4} e^x \cos 2x \rightarrow a=1, \quad V=\cos 2x$$

put $D = D+1$

$$= e^x \frac{1}{(D+1)^2-4(D+1)+4} \cos 2x$$

$$= e^x \frac{1}{(D^2-2)D+1} \cos 2x$$

Now using case II) for P.I

replace $D^2 - 2^2 = -4$

$$= e^x \cdot \frac{1}{-4-2D+1} \cos 2x$$

$$= e^x \cdot \frac{1}{-(2D+3)\times(2D-3)} \cos 2x \quad \text{i.e. } \frac{n}{u} \text{ by } (2D-3)$$

$$= \frac{-e^x(2D-3)}{4D^2-9} \cos 2x$$

put $D^2 = -4$

$$= \frac{e^x(2D-3)}{-25} \cos 2x$$

$$= \frac{e^x}{25}(2D-3)\cos 2x$$

$$= \frac{e^x}{25}[2D(\cos 2x) - 3\cos 2x]$$

$$= \frac{e^x}{25}[2(-2\sin 2x) - 3\cos 2x]$$

Case VI

If $f(x) = x \cdot V$

$$P.I = \frac{1}{\phi(D)} x \cdot V$$

$$= \left[x - \frac{\phi'(D)}{\phi(D)} \right] \frac{1}{\phi(D)} V$$

∴ Using the case for P.I as per V f^n

* Method of variation of Parameters

Procedure!
If C.F of the LDE (2^{nd} order) is in form of $Ay_1 + By_2$ where A & B are arbitary constants or will say parameter.

By varying the parameter A & B;

$$\text{Let PI} = uy_1 + vy_2$$

where, $u = \int \frac{-y_2 f(x)}{w} dx$ & $v = \int \frac{(y_1) f(x)}{w} dx$

where $w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$

$\rightarrow$

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(1) $(D^2+4)y = \sec 2x$ $\quad D^2+4=0$
$\quad\quad\quad\quad \to f(x)$ $\quad D^2=-4$
$\quad D=\pm 2i$

$C.F \Rightarrow A\cos 2x + B\sin 2x$

Let $P.I = u\cos 2x + v\sin 2x$

$\therefore y_1 = \cos 2x$ & $y_2 = \sin 2x$

where $u = \int \frac{-y_2 f(x)}{W} dx$

$v = \int \frac{y_1 f(x)}{W} dx$

$\therefore W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$\Rightarrow W = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin x & 2\cos x \end{vmatrix}$

$W = 2$

$\Rightarrow u = -\int \frac{\sin 2x}{2} \sec 2x \, dx$

$-u = -\frac{1}{2}\int \frac{\sin 2x}{\cos 2x} dx = \frac{-1}{2}\int \tan 2x \, dx$

$u = -\frac{1}{2} \frac{\log(\sec 2x)}{2}$

$v = \int \frac{y_1 f(x)}{W} dx = \int \frac{\cos 2x}{2} \cdot \sec 2x \, dx$

$= \int dx/2 = x/2$

From (1)

$P.I = uy_1 + vy_2$
$= \frac{-\log(\sec 2x)}{4} \cos 2x + \frac{x}{2} \sin 2x$

② $(D^2-6D+9)y = \frac{e^{3x}}{x^2}$

$C.F \Rightarrow C_1 e^{3x} + C_2 x e^{3x}$

$\Rightarrow (C_1 + C_2 x) e^{3x}$ $\quad w = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & 3^x \end{vmatrix}$

$\Rightarrow \underset{\underset{u}{\downarrow}}{A} \underbrace{e^{3x}}_{y_1} + \underset{\underset{v}{\downarrow}}{B} \underbrace{xe^{3x}}_{y_2}$

$P.I = uy_1 + vy_2$ — ①

where $u = \int \frac{-y_2 f(x)}{w} dx$

$v = \int \frac{y_1 f(x)}{w} dx$

$w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & 3xe^{3x} + e^{3x} \end{vmatrix}$

$w = y_1 y_2' - y_2 y_1'$

$= e^{3x}(3xe^{3x} + e^{3x}) - xe^{3x}(3e^{3x})$

$= \cancel{3xe^{6x}} + e^{6x} - \cancel{3xe^{6x}}$

$\boxed{w = e^{6x}}$

$\Rightarrow u = \int - \frac{xe^{3x}}{e^{6x}} \cdot \frac{e^{3x}}{x^2} dx = \int \frac{-1}{x} dx = -\log x$

$v = \int \frac{e^{3x}}{e^{6x}} \cdot \frac{e^{3x}}{x^2} dx = \int \frac{1}{x^2} dx = -\frac{1}{x}$

$P.I = -\log x e^{3x} - \frac{1}{\cancel{x}} \cancel{x} e^{3x}$

* Caueshy's & Legendre's form

* Linear Differential Eqn with Variable Coefficients

- Constant coeff- LDE

$\Rightarrow a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} \cdots a_n y = f(x)$

where $a_0, a_1, \cdots a_n \Rightarrow$ const coeff

$$\underline{G.S = C.F + P.I}$$

if LDE is of form

$$\boxed{a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1}}{dx^{n-1}} y + \cdots a_n y = fx \quad -(1)}$$

is caushey's form.

(I) $a_0 x^n, a_1 x^{n-1} D^{n-1} + \cdots a_n) y = f(x)$

where $D = \frac{d}{dx}$

to reduce this eqn to constant coeff form

put $x = e^z \Rightarrow z = \log x \Rightarrow \frac{dz}{dx} = \frac{1}{x}$

$\therefore \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \frac{dy}{dz}$

$\therefore x \frac{dy}{dx} = \frac{dy}{dz}$ if $\frac{d}{dz} = D$

$\therefore x \frac{dy}{dx} = Dy$

* eg -

① $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = \cos(\log x) + x\sin(\log x)$ — ①

→ Put $x = e^z \Rightarrow x\frac{dy}{dx} = Dy$, $x^2\frac{d^2y}{dx^2} = D(D-1)y$

$\hookrightarrow z = \log x$

① becomes -

$D(D-1)y - Dy + 4y = \cos z + e^z \sin z$ — ②

$\Rightarrow (D^2 - 2D + 4)y = \cos z + e^z \sin z$

with const coeff

A.E $\Rightarrow D^2 - 2D + 4 = 0$

$\Rightarrow D = \frac{2 \pm \sqrt{4-16}}{2}$

$= \frac{2 \pm 2\sqrt{3}}{2} = 1 \pm \sqrt{3}i$

C.F $\Rightarrow e^z[C_1 \cos\sqrt{3}z + C_2 \sin\sqrt{3}z]$

now

P.I $= \frac{1}{D^2-2D+4}[\cos z] + \frac{1}{D^2-2D+4} e^z \sin z$

$\hookrightarrow$ put $D^2 = -1$ $\hookrightarrow$ as e.v put $D = D+a$

$= \frac{1}{3-2D}\cos z + e^z \frac{1}{(D+1)^2 - 2(D+1) + 4}\sin z$

$= \frac{(3+2D)}{9-4D^2}\cos z + e^z \frac{1}{D^2+3}\sin z$

$D^2 = -1$ $D^2 = -1$

$= \boxed{\frac{3\cos z - 2\sin z}{13} + \frac{\sin z\, e^z}{2}}$

→

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$G.S = C.F + P.I$

$\Rightarrow y = e^{z}\left[c_1 \cos\sqrt{3}z + c_2 \sin\sqrt{3}z\right] + \frac{3\cos z - 2\sin z}{13} + \frac{e^{z}\sin z}{2}$

put $z = \log x \quad e^{z} = x$

$\therefore y = x\left[c_1 \cos(\sqrt{3}\log x) + c_2 \sin(\sqrt{3}\log x)\right] + \frac{3\cos\log x - 2\sin\log x}{13} + \frac{x\sin\log x}{2}$

---

* Legendre's form

$a_0(ax+b)^n \frac{d^n y}{dx^n} + a_1(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots a_n y = f(x)$

Put $ax+b = e^{z} \Rightarrow z = \log(ax+b)$

$\therefore \frac{dz}{dx} = \frac{a}{ax+b}$

$\therefore (ax+b)\frac{dy}{dx} = aDy \rightarrow D = \frac{d}{dz}$

$(ax+b)\frac{d^2y}{dx^2} = a^2 D(D-1)y$

$\therefore$ (1) reduces to const coeff form

$G.S = C.F + P.I$

eg -

① $(2x+1)^2 \dfrac{d^2y}{dx^2} - 2(2x+1)\dfrac{dy}{dx} - 12y = 6x$ — ①

$\Big[$ $a=2$

put $2x+1 = e^z \Rightarrow z = \log(2x+1)$

$\therefore (2x+1)\dfrac{dy}{dx} = 2Dy \quad ; \quad D = \dfrac{d}{dz}$

& $(2x+1)^2 \dfrac{d^2y}{dx^2} = (2)^2 D(D-1)y$

$\therefore$ ① becomes

$4D(D-1)y - 2(2Dy - 12y = \cancel{6}^{3}\left[\dfrac{e^z-1}{2}\right]$

$\Rightarrow (4D^2 - 8D - 12)y = 3(e^z-1)$ — ②

$\therefore$ A.E $\Rightarrow 4D^2 - 8D - 12 = 0$

$\Rightarrow (D^2 - 2D - 3) = 0 \qquad D = 3, -1$

C.F $\Rightarrow c_1e^{3z} + c_2e^{-z}$

&

P.I $= \dfrac{1}{4(D^2-2D-3)} \cdot 3(e^z-1)$

$= \dfrac{3}{4}\left[\dfrac{1}{D^2-2D-3}e^z - \dfrac{1}{D^2-2D-3}e^0\right]$

put $D=1$ put $D=0$

P.I $= \dfrac{3}{4}\left[\dfrac{e^z}{4} + \dfrac{1}{3}\right]$

$\therefore$ G.S $=$ C.F $+$ P.I

$y = c_1e^{3z} + c_2e^{-z} + \dfrac{3}{4}\left[\dfrac{1}{3} - \dfrac{e^z}{4}\right]$

replace $z = \log(2x+1)$

$\Rightarrow y = c_1(2x+1)^3 + c_2(2x+1)^{-1} + \dfrac{3}{4}\left[\dfrac{1}{3} - \dfrac{(2x+1)}{4}\right]$

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Unit II

App of PDE

$z = ax + by \Rightarrow z = f(x, y)$

a & b are arbitary constants
Differentiate partially w.r.t x

$$\left(\frac{\partial z}{\partial x}\right)_{y = const} = a$$

ex $\left(\frac{\partial z}{\partial y}\right)_{x = const} = b$

sub a & b in $z = ax + by$

$$\therefore z = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} \rightarrow$$ is PDE of 1st order
it's soln is
$z = ax + by$

- Standard PDE

① wave eqn $\Rightarrow \frac{\partial^2 u}{\partial x^2} = c^2\frac{\partial^2 u}{\partial y^2}$

② Heat eqn (1-dim) $\Rightarrow \frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$

③ Laplace eqn (2-dim) $\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Unit 3
Laplace Transform

ODE, PDE → L.T. → A.E	Analog SYS → Laplace, Fourier } constant fn
time → freq	Digital SYS → Z-Tf } discrete fn.

Objective -
To solve the differential eqns in the areas of electric ckt, Beams of string problems. For diff eqn in case of discrete fn.

$$L[f(t)] = \int_0^\infty e^{-st} = f(t)\,dt$$

(st → parameter)

$$= F(s) \quad L^{-1}[F(s)] = f(t)$$

Defn: -
if f(t) is fn of time (t)
then $\boxed{L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = F(s)}$

where

time domain → s is parameter ← Frequency domain

① $L[1] \Rightarrow \int_{t=0}^{\infty} e^{-st}(1)\,dt = \left[\frac{e^{-st}}{-s}\right]_0^{\infty}$

$= -\frac{1}{s}\left[e^{-\infty} - e^{0}\right]$ $\quad \because \left\{\begin{matrix} e^{\infty} = \infty \\ e^{-\infty} = 0 \end{matrix}\right\}$

$= \frac{-1}{s}[0-1]$

$= \frac{1}{s}$

$\therefore \boxed{L[1] = \frac{1}{s}} \quad s > 0$

② $L[e^{at}] = \int_0^{\infty} e^{-st}\, e^{at}\,dt = \int e^{-(s-a)t}\,dt$

$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^{\infty}$

$= -\frac{1}{(s-a)}\left[e^{-\infty} - e^{0}\right]$

$= -\frac{1}{s-a}[0-1]$

$\boxed{L[e^{at}] = \frac{1}{s-a} \text{ if } s > a}$

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③ $L[\sin(at)] = \int_0^\infty e^{-st} \sin(at)\, dt$

$= \left[\frac{e^{-st}}{s^2+a^2}[-s\sin at - a\cos at]\right]_0^\infty$

$\qquad\qquad\qquad\qquad \Downarrow \qquad\quad \Downarrow$

$\qquad\qquad\qquad\qquad 0 \qquad\quad 1$

$= 0 - \frac{1}{s^2+a^2}(0-a) \cdots t=0$

$\boxed{= \frac{a}{s^2+a^2}}$

$= F(s)$

$\therefore \boxed{L[\sin(at)] = F(s)}$

Formulae

$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

& $\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

④ $L[\cos(at)] = \int_0^\infty e^{-st} \cos(at)\, dt$

$= \left\{\frac{e^{-st}}{s^2+a^2}[-s\cos at + a\sin at]\right\}_0^\infty$

$L[\cos(at)] = \frac{s}{s^2+a^2} = F(s) \cdots \int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

$\therefore \boxed{L[\cos(at)] = \frac{s}{s^2+a^2}}$

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(5) $L[\sinh at] = \int_0^\infty e^{-st}\left(\frac{e^{at} - e^{at}}{2}\right) dt$

$= \frac{1}{2}\left[\int_0^\infty e^{-(s-a)t} dt - \int_0^\infty e^{-(s+a)t} dt\right]$

$= \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)} - \frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty$

$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{-s+a}\right]$

$\boxed{L[\sinh at] = \frac{a}{s^2 - a^2}}$

$= F(s)$

(6) $L[\cosh at] = \frac{s}{s^2 - a^2}$

$\Rightarrow = L[\cosh(at\} = L\left[\frac{e^{at} + e^{-at}}{2}\right]$

$= \frac{1}{2} L\{e^{at} + e^{-at}\}$

$= \frac{1}{2}\left[L\{e^{at}\} + L\{e^{-at}\}\right]$

$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s-(-a)}\right]$

$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$

$\boxed{L[\cosh at] = \frac{s}{s^2 - a^2}}$

$= \frac{1}{\cancel{2}}\left[\frac{\cancel{2}s}{s^2 - a^2}\right]$

$a^2 - b^2 = (a-b)(a+b)$

$\boxed{L(\cosh(at)) = \frac{s}{s^2 - a^2}}$

(7) $L[t^n] = \int_0^\infty e^{-st} t^n dt$

put $st = u$ as $t \to 0, u \to 0$

$\Rightarrow s\,dt = du$

$dt = \frac{du}{s}$ $t \to \infty$ $u \to \infty$

$\therefore L[t^n] = \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^n \frac{du}{s}$ $t = \frac{u}{s}$

$= \frac{1}{s^{n+1}} \int_0^\infty e^{-u} u^n du$

$= \frac{1}{s^{n+1}} \Gamma(n+1)$

but $\Gamma(n+1) = n!$ (n = +ve int)

$$\therefore \boxed{L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}} \text{ or } \frac{n!}{s^{n+1}}}$$

IMP

	L(f(t)) f(s)	f(t)
(1)	1	$1/s$ $s>0$
(2)	e^{at}	$1/(s-a)$ $s>a$
(3)	$\sin at$	$\frac{a}{s^2+a^2}$
(4)	$\cos at$	$\frac{s}{s^2+a^2}$
(5)	$\sinh at$	$\frac{a}{s^2-a^2}$
(6)	$\cosh at$	$\frac{s}{s^2-a^2}$
(7)	t^n	$\frac{\Gamma(n+1)}{s^{n+1}}$ or $\frac{n!}{s^{n+1}}$

Standard Function

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eg -

$L[t^3 + \sin^2 2t + e^{\sqrt{t}}]$

$= L[t^3] + L[\sin^2 2t] + L[e^{\sqrt{t}}]$

$t^n \qquad \frac{1-\cos 4t}{2} \qquad e^{1/2} \quad a = 1/2$

$= \frac{3!}{s^{3+1}} + L\left[\frac{1-\cos 4t}{2}\right] + L\left[e^{1/t}\right]$

$= \frac{6}{s^4} + \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+16}\right] + \frac{1}{s-1/2}$

using

$L[t^n] \qquad L[\cos at] \qquad L[e^{at}]$

eg) $L[\sin^2(3t)] = ?$

$\therefore L[\sin^2(3t)] = L\left[\frac{1-\cos 2\times 3t}{2}\right]$

$= L\left[\frac{1-\cos 6t}{2}\right]$

$= \frac{1}{2} L[1-\cos 6t]$

$= \frac{1}{2} L[1] - L[\cos 6t]$

$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+a^2}\right]$

* Existence $cond^n$ [sufficient condn]

If $f(t)$ is piecewise continous fn & is of experimental order then it's L.T exists.

→ laplace transformation.

exponential order means $|f(t)| < Me^{\alpha t}$

- Piecewise continuous means fn is continuous in the part of the interval

---

Q) $L[\cos^3 t] = ?$ $\qquad \cos^3 x = \frac{3\cos x + \cos 3x}{4}$

$= \frac{1}{4} L[3\cos t + \cos 3t]$

$= \frac{1}{4} L[3\cos t] + L[\cos 3t]$

$= \frac{1}{4} \quad 3 \times \frac{s}{s^2+1} + \frac{s}{s^2+9}$

$= \frac{1}{4}\left[\frac{3s}{s^2+1} + \frac{s}{s^2+9}\right]$

$= \frac{1}{4}\left[\frac{3s(s^2+9) + s(s^2+1)}{(s^2+1)(s^2+9)}\right]$

$= \frac{1}{4} \quad \frac{3s^3 + 27s + s^3 + s}{(s^2+1)(s^2+9)}$

$= \frac{1}{4} \quad \frac{4s^3 + 28s}{(s^2+1)(s^2+9)}$

$= \frac{1}{\cancel{4}}\left[\frac{\cancel{4}(s^3 + 7s)}{(s^2+1)(s^2+9)}\right]$

* Properties of L.T

$$\text{If } L[f(t)] = F(s)$$

① Linearity: $L[af(t) + bg(t)]$
$= aL[f(t)] + bL[g(t)]$

② Change of scale:
$$L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$$

③ Multiplication by t:

$$L[tf(t)] = (-1) \frac{d}{ds} F(s)$$

$$\Rightarrow L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s)$$

④ Division by t:

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds$$

⑤ 1st Shifting

$$L[e^{-at} f(t)] = F(s+a)$$

or

$$L[e^{at} f(t)] = F(s-a)$$

⑥ 2nd Shifting:

$$L[F(t)] = e^{-as} F(s)$$

where $(F(s) = L[f(t)]$

where $F(t) = F(t-a) \quad t>a$
$= 0 \quad t<a$

⑦ Derivative

$$L[f'(t)] = sF(s) - f(0)$$

where $f(0) = \lim_{t\to 0} f(t)$

$$\Rightarrow L[f''(t)] = s^2 F(s) - sf(0) - f'(0)$$

$$\cdots L[f^n(t)] = s^n F(s) - s^{n-1} f(0) \cdots f^{n-1}(0)$$

⑧ Integral

$$L\left[\int_0^t f(u)du\right] = \frac{1}{s} F(s)$$

where $F(s) = L[f(t)]$

I shifting - $L[e^{at}F(t)] = F(s-a)$
or
$L[e^{-at}f(t)] = F(s+a)$

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① $L[e^{-3t}\sin 2t]$

here $a=-3$, $f(t)=\sin 2t$

$L[\sin 2t] = \frac{2}{s^2+4} = F(s)$

$\therefore$ put $s = s+a = s+3$

$\Rightarrow L[e^{-3t}\sin 2t] = F(s+3)$

$= \frac{2}{(s+3)^2+4}$

② $L[e^{-t}\sin^2 t]$ $\quad \because f(t) = \sin^2 t = \frac{1-\cos 2t}{2}$

$L[e^{-t}(\frac{1-\cos 2t}{2})] = L[\frac{e^{-t}}{2}] - \underbrace{L[\frac{e^{-t}\cos 2t}{2}]}$

$= \frac{1}{2}L[e^{-t}]$ $\quad L[e^{-at}f(t)] = F(s+a)$

$a=1$, $f(t)=\cos 2t$

$\therefore L[\cos 2t] = \frac{s}{s^2+a}$

$= F(s)$

$= \frac{1}{2}\frac{1}{s+1} - \frac{1}{2}F(s+1)$

$= \frac{1}{2}\left(\frac{s+1}{(s+1)^2+4}\right)$

③ $L[e^{2t}t^2]$

here $a=2$, $f(t)=t^2$

$\therefore L[f(t)] = L[t^2]$

$= \frac{2!}{s^{2+1}} = \frac{2}{s^3} = F(s)$

$\therefore L[e^{2t}t^2] = F(s-a)$ $\quad s = s-2$

$= \frac{2}{(s-2)^3}$

Change of scale
If $L[f(t)] = F(s)$
then $L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$

① $L[\sin(3t)]$
using formula
$L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$

$\therefore L[\sin(3t)] = \frac{1}{3} F\left(\frac{s}{3}\right)$

where $F(s) = L[\sin t]$
$= \frac{1}{s^2+1}$

$= \frac{1}{3}\left[\frac{1}{(s/3)^2+1}\right]$

$= \frac{1}{3} \frac{\cancel{9}\,3}{s^2+9}$

Multiplication by t

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s)$$

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Q1) $\Rightarrow L[t\sin 3t]$

here $f(t) = \sin 3t$

$\therefore L[f(t)] = L[\sin 3t]$

$= \frac{3}{s^2+9} = F(s)$

using formula.

$\therefore L[t f(t)] = -1 \frac{d}{ds} F(s)$

$\Rightarrow L[t\sin 3t] = -1 \frac{d}{ds}\left(\frac{3}{s^2+9}\right)$

$= (-1)\cdot 3 \left[\frac{-1}{(s^2+9)^2} \cdot 2s\right]$

$= \frac{6s}{(s^2+9)^2}$

Q2) $L[t^2 e^{-3t}]$

here $f(t) = e^{-3t}$

$\therefore L[f(t)] = L[e^{-3t}]$

$= \frac{1}{s-(-3)} = \frac{1}{s+3} = F(s)$

using formula

$\therefore L[t^n f(t)] = -1^n \frac{d^n}{ds^n} F(s)$

$\Rightarrow L[t^2 e^{-3t}] = -1^2 \frac{d^2}{ds^2}\left(\frac{1}{s+3}\right)$

$= \frac{d^2}{ds^2}\left(\frac{1}{s+3}\right)$

$= \frac{d}{ds}\left[\frac{-1}{(s+3)^2}\right]$

$= \frac{-d}{ds}\left[(s+3)^{-2}\right]$

$= -2(s+3)^{-3}$

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(Q3) $\frac{t \sinh at}{2a}$

here $f(t) = \frac{\sinh at}{2a}$

$$L[f(t)] = L\left[\frac{\sinh at}{2a}\right]$$

$$= \frac{1}{2a}\left[\frac{1}{s^2-a^2}\right]$$

Now $L\left[\frac{t \sinh at}{2a}\right] = (-1)\frac{d}{ds}\left\{\frac{1}{2}\left[\frac{1}{s^2-a^2}\right]\right\}$

$$= (-1)\frac{1}{2}\frac{1}{(s^2-a^2)^2} -2s$$

$$= \frac{s^2-a^2}{(s^2+a^2)^2}$$

(Q4) $t \sin^3 t$

$$L[\sin^3 t] = L\left[\frac{3}{4}\sin t - \frac{1}{4}\sin 3t\right]$$

$$= \frac{3}{4}L[\sin t] - \frac{1}{4}L[\sin 3t]$$

$$= 3/4\left(\frac{1}{s^2+1} - \frac{1}{s^2+9}\right)$$

$$L[t\sin^3 t] = (-1)\frac{d}{ds}\left[\frac{3}{4}\left(\frac{1}{s^2+1} - \frac{1}{s^2+9}\right)\right]$$

$$= \frac{-3}{4}\left[\frac{(-2s)}{(s^2+1)^2} - \frac{(-2s)}{(s^2+9)^2}\right]$$

$$= \frac{3s}{4}\left[\frac{1}{(s^2+1)^2} - \frac{1}{(s^2+9)^2}\right]$$

Division t

$$L\left[\frac{F(t)}{t}\right] = \int_s^\infty F(s)\,ds$$

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Q) $L\left[\frac{\sin^2 t}{t}\right]$

here $f(t) = (\sin^2 t)$

$\therefore L[f(t)] = L[\sin^2 t]$

$= L\left[\frac{1-\cos 2t}{2}\right]$ $\because \sin^2\theta = \frac{1-\cos 2\theta}{2}$

$= \frac{1}{2} L[1] - L[\cos 2t]$

$L[\sin^2 t] = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$

$\Rightarrow L\left[\frac{\sin^2 t}{t}\right] = \frac{1}{2}\int_0^\infty \left[\frac{1}{s} - \frac{s}{s^2+4}\right] ds$

$= \frac{1}{2}\int_0^\infty \left[\frac{1}{s} - \frac{1}{2}\times\frac{2s}{s^2+4}\right] ds$

$\boxed{\int \frac{1}{x}\,dx = \log x}$

$= \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty$

$= \frac{1}{4}\left[2\log s - \log(s^2+4)\right]_s^\infty$

$= \frac{1}{4}\left[\log s^2 - \log(s^2+4)\right]_s^\infty$

$\boxed{\begin{array}{l}\log a^b = b\log a \\ \log\frac{a}{b} = \log a - \log b\end{array}}$

$= \frac{1}{4}\left[\log\frac{s^2}{(s^2+4)}\right]_0^\infty$

$= \frac{1}{4}\left[\log\frac{s^2}{s^2(1+4/s^2)}\right]_s^\infty$

$= \frac{1}{4}\left[\log\frac{1}{1+4/s^2}\right]_s^\infty$

Ans $= \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)$

$\log\infty = \infty$

$\frac{4}{\infty} = 0$

$\frac{a}{0} = \infty$ $\quad \frac{a}{\infty} = 0$

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Q2) $L\left[\frac{1-\cos t}{t}\right]$

Here $f(t) = 1-\cos t$

$$L[f(t)] = L[1-\cos t]$$
$$= L[1] - L[\cos t]$$
$$= \frac{1}{s} - \frac{s}{s^2+1}$$

Now

$$L\left[\frac{1-\cos t}{t}\right] = \int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+1}\right] ds$$

$$= \int_s^\infty \frac{1}{s} ds - \frac{1}{2}\int_s^\infty \frac{2s}{s^2+1} ds$$

$\boxed{\frac{f'(x)}{f(x)} = \log[f(x)]}$

$$= \left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty$$

$$= \left(\log \frac{s}{\sqrt{s^2+1}}\right)_s^\infty$$

$$= \log \left.\frac{s}{\sqrt{s^2\left(1+\frac{1}{s^2}\right)}}\right]_s^\infty$$

$$= \log\left[\frac{\cancel{s}}{\cancel{s}\sqrt{1+\frac{1}{s^2}}}\right]_s^\infty$$

$$= \log(1) - \log\frac{s}{\sqrt{s^2+1}}$$

$$= \log\frac{\sqrt{s^2+1}}{s}$$

2nd shifting

If $L[f(t)] = F(s)$ then $L[F(t)] = e^{-as} F(s)$

where $F(t) = f(t-a) \quad t > a$

$= 0 \quad t < a$

Q1) $L[F(t)$ where $F(t) = (t-1)^2 \quad t > 1$

$= 0 \quad t < 1$

here, by second shifting property –

$L[F(t)] = e^{-as} F(s)$

$\therefore a = 1 \quad F(s) = L[f(t)]$

same

$f(t-a) = f(t-1) = (t-1)^2$

$\therefore f(t) = t^2 \Rightarrow L[t^2] = \frac{2!}{s^{2+1}} = \frac{2}{s^3} = F(s)$

$\therefore L[F(t)] = e^{-s}\left[\frac{2}{s^3}\right]$

Q2) $L[F(t)]$ if $F(t) = (t-1)^2 \quad t > 1$

$= 0 \quad 0 < 1$

Here $f(t-a) = (t-1)^2$ where $a = 1$

same

$\therefore f(t) = t^2$ & $F(s) = \frac{2}{s^3}$

by second shifting th, with $a = 1$,

$L[F(t)] = e^{-as} F(s) = e^{-s}\left(\frac{2}{s^3}\right)$

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(Q) $f(t) = 0 \quad 0 < t < 1$
$= t \quad 1 < t < 4$
$= 0 \quad t > 4$

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$\therefore L[f(t)] = \int_0^1 e^{-st} f(t) dt + \int_1^4 e^{-st} f(t) dt + \int_4^\infty e^{-st} (0)\, dt$$

$$= \int_0^1 e^{-st}(0) dt + \int_1^4 e^{-st}(t) dt + \int_4^\infty e^{-st}(0) dt$$

$$= \int_1^4 e^{-st}(t)\, dt + 0 + 0$$

$$= \int_1^4 e^{-st}\, t\, dt$$

$$= t \int_1^4 e^{-st} dt + \int_1^4 (1) \frac{e^{-st}}{s} dt$$

$$= -1 \left[\frac{e^{-st}}{s} t\right]_1^4 - \frac{1}{s}\left[\frac{e^{-st}}{s}\right]_1^4$$

$$= -\frac{1}{s}(4e^{-4s} - e^{-s}) - \frac{1}{s^2}(e^{-4s} - e^{-s})$$

$$\therefore F(s) = \frac{e^{-s}}{s} - \frac{4e^{-4s}}{s} + \frac{e^{-s}}{s^2} - \frac{e^{-4s}}{s^2}$$

Derivative

if $L[f(t)] = F(s)$

then $L[f'(t)] = sF(s) - f(0)$

where $f(0) = \lim_{t \to 0} f(t)$

sim. $L[f''(t)] = s^2F(s) - sf(0) - f'(0)$

$L[f^n(t)] = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) \cdots f^{n-1}(0)$

Q) Given: $4f''(t) + f(t) = 0$

where $f(0) = 0$, $f'(0) = 2$

show that $L[f(t)] = \frac{8}{4s^2+1}$

Let

$L[4f''(t)] + L[f(t)] = L[0]$

$\Rightarrow 4[s^2F(s) - s\underbrace{f(0)}_{0} - \underbrace{f'(0)}_{2}] + F(s) = 0$

$\Rightarrow 4[s^2F(s) - 0 - 2] + F(s) = 0$

$\Rightarrow 4[s^2F(s) - 2] + F(s) = 0$

$[4s^2F(s) - 8] + F(s)$

$\Rightarrow F(s)[4s^2+1] = 8$

$\therefore F(s) = \frac{8}{4s^2+1}$

Use derivative property –

(I) $L[e^{at}] = \frac{1}{s-a}$

→ Let $f(t) = e^{at}$. Then $f'(t) = a e^{at}$, $f(0) = 1$

Now

$$L[f'(t)] = s\,L[e^{at}] - f(0)$$

$$L[e^{at}] = s\,L[e^{at}] - 1$$

$$a\,L[e^{at}] = s\,L[e^{at}] - 1$$

$$L[e^{at}] = \frac{1}{s-a}$$

(II) $L[\sin at] = \frac{a}{(s^2+a^2)}$

Let $f(t) = \sin at$ $\qquad f(0) = 0$

Then $f'(t) = a\cos at$ $\qquad f'(0) = a$

$f''(t) = -a^2 \sin at$

Now $L[f''(t)] = s^2 L[f(t)] - s f(0) - f'(0)$

$$L[-a^2 \sin at] = s^2 L[\sin at] - s(0) - a$$

or

$$-a^2 L[\sin at] = s^2 L[\sin at] - a$$

$$L[\sin at] = \frac{a}{a^2+a^2}$$

Integral

If $L[f(t)] = F(s)$

then $L\left[\int_0^t f(u)\,du\right] = \frac{F(s)}{s}$

where $F(s) = L[f(t)]$

Inverse Laplace Transform

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$F(s)$	$f(t) = L^{-1}[F(s)]$
① $\frac{1}{s}$	1
② $\frac{1}{s-a}$	e^{at}
③ $\frac{1}{s^2+a^2}$	$\frac{\sin at}{a}$
④ $\frac{s}{s^2+a^2}$	$\cos at$
⑤ $\frac{1}{s^2-a^2}$	$\frac{\sinh at}{a}$
⑥ $\frac{s}{s^2-a^2}$	$\cosh at$
⑦ $\frac{1}{s^{n+1}}$	$\frac{t^n}{\Gamma{n+1}}$
⑧ $\frac{1}{s^{n+1}}$	$\frac{t^n}{n!}$

if n is a positive int.

$$L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{\Gamma n} \text{ or } \frac{t^{n-1}}{(n-1)!}$$

$$\Gamma n = (n-1)!$$

$$L^{-1}[F(s)] = f(t)$$

Properties -

(1) 1st shifting -

$$L^{-1}[F(s-a)] = e^{at} f(t)$$

(2) 2nd shifting

$$L^{-1}[e^{-as} F(s)] = F(t) = f(t-a) \quad \text{for } t<a$$

$$= 0 \quad t>a$$

(3) change of scale

$$L^{-1}\left[F\left(\frac{s}{a}\right)\right] = af(at)$$

(4) Moltiplication by t -

$$L^{-1}[F'(s)] = -tf(t)$$

(5) Division by t -

$$L^{-1}\left[\int_s^\infty F(s)\, ds\right]$$

$$= \frac{f(t)}{t}$$

(6) Integral → $L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(u)\, du$

extra: $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{2a^3}(\sin at - at\cos at)$

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{t}{2a}\sin at$$

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① $L^{-1}\left[\frac{1}{s-3}\right] = e^{3t}$ using formula

② $L^{-1}\left[\frac{1}{s+3}\right] = e^{-3t}$

③ $L^{-1}\left[\frac{s}{s^2+25}\right] = \cos 5t$

④ $L^{-1}\left[\frac{s}{s^2-25}\right] = \cosh(5t)$

⑤ $L^{-1}\left[\frac{1}{s^2+25}\right] = \frac{\sin 5t}{5}$

⑥ $L^{-1}\left[\frac{1}{s^2-25}\right] = \frac{\sinh 5t}{}$

⑦ $L^{-1}\left[\frac{1}{s^2+1}\right] = \frac{\sin 1t}{1} = \sin t$

⑧ $L^{-1}\left[\frac{3}{s^2+4}\right] = 3L^{-1}\left[\frac{1}{s^2+4}\right] = \frac{3 \times \sin 2t}{2}$

using property.

(1) $L^{-1}\left[\frac{1}{(s-3)^2}\right] = \frac{t^{2-1}}{(2-1)!}\cdot e^{3t}$
$= te^{3t}$

(2) $L^{-1}\left[\frac{1}{(s+3)^2}\right] = \frac{t^{2-1}}{(2-1)!}e^{3t}$
$= te^{-3t}$

L^{-1} using 1st shifting.
&
$L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}$

(3) $L^{-1}\left[\frac{(s-2)}{(s-2)^2+25}\right] = \cos 5t \cdot e^{2t}$

(4) $L^{-1}\left[\frac{(s-3)}{(s-3)^2-25}\right] = \cosh(5t)\cdot e^{3t}$

(5) $L^{-1}\left[\frac{1}{(s-1)^2+25}\right] = \frac{\sin 5t}{5} \times e^{t}$

(6) $L^{-1}\left[\frac{1}{(s-2)^2-25}\right] = \frac{\sinh 5t}{5}\ e^{2t}$

(4) $L^{-1}\left[\frac{3}{(s-2)^2+4}\right]$

$= e^{2t} + L^{-1}\left[\frac{3}{s^2+4}\right] \quad \because L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{\sin at}{a}$

$= e^{2t} \cdot 3\left(\frac{\sin 2t}{2}\right)$

(5) $L^{-1}\left[\frac{s+5}{(s+1)^2+9}\right]$

split terms-

$\Rightarrow L^{-1}\left[\frac{s+1}{(s+1)^2+9}\right] + 4L^{-1}\left[\frac{1}{(s+1)^2+9}\right]$

$= e^{-t} L^{-1}\left[\frac{s}{s^2+9}\right] + 4e^{-t}\left[\frac{1}{s^2+9}\right]$

$= e^{-t}\cos 3t + \frac{4e^{-t}\sin 3t}{3}$

(6) $L^{-1}\left[\frac{s+12}{s^2+8s+16}\right]$

$= L^{-1}\left[\frac{s+12}{(s+4)^2}\right]$

$= L^{-1}\left[\frac{(s+4)+8}{(s+4)^2}\right]$

$= L^{-1}\left[\frac{\cancel{s+4}}{(s+4)^2}\right] + 8L^{-1}\left[\frac{1}{(s+4)^2}\right]$

$$= L^{-1}\left[\frac{1}{s+4}\right] + 8L^{-1}\left[\frac{1}{(s+4)^2}\right]$$

$$= e^{-4t} + 8e^{-4t} L^{-1}\left[\frac{1}{s^2}\right]$$

$$= e^{-4t} + 8e^{-4t} t$$

(7) $L^{-1}\left[\log\frac{s+a}{s+b}\right] = L^{-1}\left[\log(s+a) - \log(s+b)\right]$

Let $F(s) = \log(s+a) - \log(s+b)$

$\Rightarrow \frac{d}{ds}F(s) = F'(s) = \left(\frac{1}{s+a} - \frac{1}{s+b}\right)$

using

$L^{-1}[F'(s)] = -tf(t) \Rightarrow f(t)$

$= -\frac{1}{t} L^{-1}[F'(s)]$

$\therefore L^{-1}\left[\frac{1}{s+a} - \frac{1}{s+b}\right] = -tf(t)$

$\Rightarrow e^{-at} - e^{-bt} = -tf(t)$

$\Rightarrow f(t) = \frac{e^{-bt} - e^{-at}}{t}$

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* Inverse L.T by Partial fraction.

When Nr's degree is less than the Dr - use partial fraction (part wise fraction)

(1) $\dfrac{Nr}{(ax+b)(cx+d)}$ ⇒ Non-repeated linear factors in D^r

then equate Nr from b sides

$= \dfrac{A}{ax+b} + \dfrac{B}{cx+d} \Rightarrow N^r = A(cx+d) + B(ax+b)$

put 2 least values of x to get A & B

(2) If any linear factor is repeated.

$\Rightarrow \dfrac{Nr}{(ax+b)^2(cx+d)} = \dfrac{A}{ax+b} + \dfrac{B}{(ax+b)^2} + \dfrac{C}{cx+d}$

→ equate Nr & find A, B, C

(3) For quadratic factor in Dr

$\dfrac{Nr}{(ax^2+bx+c)(dx+c)} = \dfrac{Ax+B}{ax^2+bx+c} + \dfrac{C}{dx+C}$

equate Nrs ⇒ $N^r = (Ax+B)(dx+c) + C(ax^2+bx+c)$

put 3 values of x & find A, B, C

or

Equate co-eff of x, x^2, const from both sides

⇒ eqns in A, B, C ⇒ solving ⇒ A = ?
B = ?
C = ?

Q) $L^{-1}\left[\frac{1}{s^2-3s+2}\right] = L^{-1}\left[\frac{1}{(s-2)(s-1)}\right]$

$D^r = 2$

Using P.F

$$\frac{1}{(s-2)(s-1)} = \frac{A}{s-2} + \frac{B}{s-1} = \frac{1}{s-2} - \frac{1}{s-1}$$

$\Rightarrow N^r = L.H.S = 1 = A(s-1) + B(s-2)$

put $s = 1$ | $s = 2$

$\Rightarrow B = -1$ | $A = 1$

Q) Find $L^{-1}\left[\frac{s+3}{s^3-7s-6}\right]$

put $s = -1 \Rightarrow$ is factor

$\therefore (s+1)$ is factor of s^3-7s-6

$\Rightarrow L^{-1}\left[\frac{s+3}{(s+1)(s+2)(s-3)}\right]$

$$\frac{s+3}{(s+1)(s+2)(s-3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s-3}$$

Put

$\therefore s+3 = A(s+2)(s-3) + B(s+1)(s-3) + C(s+1)(s+2)$

put $s = -1$ | $s = -2$ | $s = 3$

$2 = -4A$ | $1 = 5B$ | $6 = 20C$

$A = -1/2$ | $B = 1/5$ | $C = 3/10$

$$\therefore \frac{s+3}{(s+1)(s+2)(s-3)} = \frac{-1/2}{s+1} + \frac{1/5}{s+2} + \frac{3/10}{s-3}$$

$$\therefore L^{-1}[\,11\,] = L^{-1}\left[\frac{-1/2}{s+1}\right] + \frac{1}{5}L^{-1}\left[\frac{1}{s+2}\right] + \frac{3}{10}L^{-1}\left[\frac{1}{s-3}\right]$$

$$= -\tfrac{1}{2}e^{-t} + \frac{1}{5}e^{-2t} + \frac{3}{10}e^{3t}$$

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Q> Find $L^{-1}\left[\frac{1}{(s^2+1)(s^2+2)}\right]$ with partial fraction

Let $\frac{1}{(s^2+1)(s^2+2)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2}$

$N^r \Rightarrow 1 = (As+B)(s^2+2) + (Cs+D)(s^2+1)$
$= As^3 + 2As + Bs^2 + 2B + Cs^3 + Cs + Ds^2 + D$

$1 = (A+C)s^3 + s^2(B+D) + s(2A+C) + (2B+D)$

equating the coeff of s^3, s^2, s & constants from sides-

$0 = A+C$ — ①
$0 = B+D$ — ②
$0 = 2A+C$ — ③
$1 = 2B+D$ — ④

Solving ② & ④

$B+D=0$
$2B+D=1$
$-2B-D=-1$
$-B=-1 \Rightarrow B=1$
$\Rightarrow D=(-1)$

$\frac{1}{(s^2+1)(s^2+2)} = \frac{1}{s^2+1} - \frac{1}{s^2+2}$

$\Rightarrow L^{-1}\left[\frac{1}{(s^2+1)(s^2+2)}\right] = L^{-1}\left[\frac{1}{s^2+1}\right] - L^{-1}\left[\frac{1}{s^2+\sqrt{2}^2}\right]$

$= \sin t - \frac{1}{\sqrt{2}}\sin(\sqrt{2}t)$

$L\left[\int_0^t f(u)\,g(t-u)\,du\right] = F(s)G(s)$

inverse

* Convolution prop for inverse transform.

If $f(t)$ & $g(t)$ are fn of t
then

$$f(t) * g(t) = \int_0^t f(u)\,g(t-u)\,du$$

$$\text{or } \int_0^t g(u)\,f(t-u)\,du$$

~~inverse~~ if $L[f(t)] = F(s)$ & $L[g(t)] = G(s)$

then $\boxed{L^{-1}[F(s)\,G(s)] = f(t) * g(t)}$

$$= \int_0^t f(u)\,g(t-u)\,du$$

$$\text{or } \int_0^t g(u) \cdot f(t-u)\,du$$

~~ex-) $L^{-1}\left[\frac{1}{s(s^2+9)}\right]$ by convolution th~~

Laplace.

$$L[f(t)] = \bar{f}(s)$$
$$L[g(t)] = g(s)$$

Laplace $L\left[\int_0^t f(u)\,g(t-u)\,du\right] = f(s)\,g(s)$

$$L^{-1}[f(s)] = f(t)$$
$$L^{-1}[g(s)] = g(t)$$

Inverse. $L^{-1}[f(s)\,g(s)] = \int_0^t f(u)\,g(t-u)\,du$

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Q) $L^{-1}\left[\frac{1}{s(s^2+9)}\right]$ by convolution th.

$\Rightarrow L^{-1}\left[\frac{1}{s(s^2+9)}\right] = L^{-1}\left[\frac{1}{s} \times \frac{1}{s^2+9}\right]$

$\frac{1}{s} \to f(s)$, $\frac{1}{s^2+9} \to g(s)$

where.

$f(s) = 1/s$

$g(s) = 1/s^2+9$

$L[f(s)] = \frac{1}{s}$, $L[g(s)] = \frac{1}{s^2+9}$

$f(s) = L^{-1}\left[\frac{1}{s}\right]$, $g(s) = L^{-1}\left[\frac{1}{s^2+9}\right]$

$f(s) = 1$ $\qquad g(s) = \frac{\sin 3t}{3}$

$\therefore$ by convolution thm
we have.

$t = u$

$L^{-1}[f(s)\,g(s)] = \int_0^t f(u)\,g(t-u)\,du$

$t = t-u$

$L^{-1}\left[\frac{1}{s} \times \frac{1}{s^2+9}\right] = \int_0^t 1 \cdot \frac{\sin 3(t-u)}{3}\,du$

$L^{-1}\left[\frac{1}{s(s^2+9)}\right] = \frac{1}{3}\int_0^t \sin 3t - 3u\,du$

$= \frac{1}{3}\left[\frac{-\cos(3t-3u)}{-3}\right]_0^t$

$= \frac{1}{3}\left[\frac{\cos(3t-3t)}{3} - \frac{\cos(3t-0)}{3}\right]$

$= 1/9\,[\cos 0 - \cos 3t]$

$= 1/9\,(1 - \cos 3t)$

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Q) $L^{-1}\left[\dfrac{1}{s(s^2+1)}\right]$ by convo prop

Let $F(s) = \dfrac{1}{s}$ $g(s) = \dfrac{1}{s^2+1}$

$L[F(s)] = \dfrac{1}{s}$ $L[g(s)] = \dfrac{1}{s^2+1)}$

$F(s) = L^{-1}\left[\dfrac{1}{s}\right]$, $g(s) = L^{-1}\left[\dfrac{1}{s^2+1}\right]$

$F(s) = 1$ $g(s) = \dfrac{\sin \cancel{3} t}{1}$

by convolution th.

$$L^{-1}[f(t)g(t)] = \int_0^t f(u)\, g(t-u)\, du$$

$$L^{-1}\left[\frac{1}{s(s^2+1)}\right] = \int_0^t (1) \sin(t-u)\, du$$

$$= \cancel{\int} [-\cos u]_0^t$$

$$= -[\cos t - 1]$$

$$= 1 - \cos t.$$

Q) find $L^{-1}\left[\frac{1}{(s^2+1)^2}\right]$

by co

$$L^{-1}\left[\frac{1}{(s^2+1)^2}\right] = L^{-1}\left[\underbrace{\frac{1}{s^2+1}}_{F(s)} \times \underbrace{\frac{1}{s^2+1}}_{g(s)}\right]$$

where.

$$F(s) = \frac{1}{s^2+1}, \quad g(s) = \frac{1}{s^2+1}$$

$$L^{-1}[F(s)] = \frac{1}{s^2+1}, \quad L[g(s)] = \frac{1}{s^2+1}$$

$$[F(s)] = L^{-1}\left[\frac{1}{s^2+1}\right], \quad [g(s)] = L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$F(s) = \sin t \qquad g(s) = \sin t$$

by convo th

$$L^{-1}[F(s)\,g(s)] = \int_0^t f(u)\,g(t-u)\,du$$

$$L^{-1}\left[\frac{1}{(s^2+1)^2}\right] = \int_0^t \sin u \cdot \sin(t-u)\,du$$

$$= \int \frac{1}{2}\left[\cos(2u-t) - \cos t\right] du$$

$$= \frac{1}{2}\left[\frac{\sin(2u-t)}{2} - u\cos t\right]_0^t \qquad \because \sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

$$= \frac{1}{2}\left[\frac{\sin(2t-t)}{2} - t\cos t - \left(\frac{\sin(0-t)}{2} - 0\right)\right]$$

$$= \frac{1}{2}\left[\frac{\sin t}{2} - t\cos t + \frac{\sin t}{2}\right]$$

$$= \frac{1}{2}[\sin t - t\cos t]$$

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* Applications of LT

to solve ODE

$$L\frac{d^2y}{dt^2} - \frac{3dy}{dt} + 5y = 0$$

$$y'' \quad 3y' \quad + 5y = 0$$

$$\Rightarrow \underline{L[y'(t)] = sy(s) - y(0)}$$

Q) $\frac{d^2y}{dt^2} - \frac{2dy}{dt} - 8y = 0 \quad -(1)$

if $y(0) = 3$ & $y'(0) = 6$

tak L.T of eqn (1)

$$L[y''(t)] - 2L[y'(t)] - 8L[y(t)] = 0$$

$$[s^2y(s) - sy(0) - y'(0)] - 2[sy(s) - y(0)] - 8y(s) = 0$$

$$\Rightarrow y(s)[s^2 - 2s - 8] - 3s \cancel{-6} \cancel{+6} = 0$$

$$\therefore y(s) = \frac{3s}{s^2 - 2s - 8} \quad -(2)$$

consider inverse L.T of (2)

$$L^{-1}[y(s)] = L^{-1}\left[\frac{3s}{s^2 - 2s - 8}\right]$$

$$= L^{-1}\left[\frac{3s}{(s-4)(s+2)}\right]$$

$$\therefore \text{let } \frac{3s}{(s-4)(s+2)} = \frac{A}{s-4} + \frac{B}{s+2} = \frac{2}{s-4} + \frac{1}{s+2}$$

on solving

$$y(t) = 2e^{4t} + e^{-2t}.$$

Q) $y'' + 9y = 18t$, $y(0) = 0$ & $y(\pi/2) = 0$

given $y(0) = 0$, $y\left(\frac{\pi}{2}\right) = 0$

$\Rightarrow y(t)$ at $t = \pi/2$

$\hookrightarrow y(t)$ at $t = 0$

$\rightarrow$ consider L.T of ①

$L[y''] + 9L[y] = 18L[t]$

$\Rightarrow \left[s^2 y(s) - \underbrace{s y(0)}_{\to 0} - y'(0)\right] + 9y(s)$

using $L[t^n] = \frac{n!}{s^{n+1}}$

$\because y'(0)$ is not given

Let $y'(0) = k$ (assume)

$\Rightarrow y(s)[s^2+9] - k = 18/s^2$

$\Rightarrow y(s) = \left[\frac{18}{s^2} + k\right]\frac{1}{s^2+9}$

$= \frac{18}{s^2(s^2+9)} + \frac{k}{s^2+9}$ — ②

$\therefore L^{-1}[y(s)] = L^{-1}\left[\frac{18}{s^2(s^2+9)}\right] + L^{-1}\left[\frac{k}{s^2+9}\right]$

Let $\frac{18}{s^2(s^2+9)} = \frac{As+B}{s^2} + \frac{Cs+D}{s^2+9}$

$\Rightarrow 18 = (As+B)(s^2+9) + (Cs+D)(s^2)$

$\rightarrow$

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$\Rightarrow 0 = A + C \Rightarrow C = 0$

$0 = B + D \Rightarrow D = -2$

$0 = 9A \Rightarrow A = 0$

$18 = 9B \Rightarrow B = 2$

$= \frac{2}{s^2} - \frac{2}{(s^2+9)}$

$\because$ from (2)

$y(t) = L^{-1}\left[\frac{2}{s^2}\right] - L^{-1}\left[\frac{2}{s^2+9^2}\right] + L^{-1}\left[\frac{K}{s^2+9}\right]$

$y(t) = 2t + \frac{2}{3}\sin 3t + \frac{K}{3}\sin 3t$ — (3)

$\because y(\pi/2) = 0$

$\therefore$ put $t = \frac{\pi}{2}$ in (3)

$\therefore y(\pi/2) = 0$

$0 = \cancel{2}\left(\frac{\pi}{\cancel{2}}\right) + \frac{2}{3}(-1) - K/3$

$\therefore K = 3\pi - 2$

* Special fonctions

$f(t)$	$L[f(t)] = F(s)$
① $u(t)$	$\frac{1}{s}$
② $u(t-a)$	$\frac{e^{-as}}{s}$
③ $f(t)\, u(t-a)$	$e^{-as} L[f(t+a)]$
⑤ $f(t-a)\, u(t-a)$	$e^{-as} F(s) = e^{-as} L[f(t)]$
⑤ $\delta(t)$	1
⑥ $\delta(t-a)$	e^{-as}
⑦ $f(t)\, \delta(t-a)$	$e^{-as} f(a)$
⑧ Periodic fn $f(t)$	$\frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$ $T =$ period

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(1) Find $L[t^2 u(t-2)]$

$\because a=2, \quad f(t)=t^2$

using (3)

i.e. $L[f(t)u(t-a)] = e^{-as} L[f(t+a)]$

$= e^{-2s} L[f(t+2)]$

$= e^{-2s} L[(t+2)^2]$

$= e^{-2s} [t^2+4t+4]$

$= e^{-2s} [L(t)^2 + 4L(t) + 4L(1)]$

$= e^{-2s} \left[\frac{2}{s^3} + 4\left(\frac{1}{s^2}\right) + \frac{4}{s}\right]$

(2) $L[\sin t\, u(t-4)]$

$\because a=4 \quad f(t)=\sin t$

using (3)

$L[f(t)u(t-a)] = e^{-as} L[f(t+a)]$

$= e^{-4s} L[\sin(t-4)]$

$\left(\sin A+B = \sin A \cos B + \cos A \sin b\right)$

$= e^{-4s} L[\sin t \cos 4 + \cos t \sin 4]$

$e^{-4s} [\cos 4\, L\sin t + \sin 4\, L\cos t]$

$= e^{-4s} \left[\frac{\cos 4}{1+s^2} + \frac{\sin 4 s}{s^2+1}\right]$

(3) $L[\sin 2t\, \delta(t-2)]$

using (7)

$$L[f(t)\,\delta(t-a)] = e^{-as} f(a)$$

$$\because a=2 \quad f(t) = \sin 2t$$

$$\Rightarrow f(2) = \sin 4$$

$$\therefore L[\sin 2t\, \delta(t-2)] = e^{-2s} \sin 4$$

(4) $L[t\,u(t-4) - t^3\,\delta(t-2)]$

using (3) & (7)

$$\because f(t)\,u(t-a) = e^{-as} L[f(t+a)]$$

$$f(t)\,\delta(t-a) = e^{-as} f(a)$$

$$\Rightarrow L[t\,u(t-4)] = e^{-4s} L[f(t+4)]$$

$$= e^{-4s} L[(t+4)]$$

$$= e^{-4s}\{L[t] + 4L[1]\}$$

$$= e^{-4s}\{1/s^2 + 4/s\}$$

$$L[t^3\,\delta(t-2)] = e^{-2s} f(2)$$

$$= e^{-2s}\, t$$

Unit - 4

Fourier Transforms

The function F(S), defined by.

$$F(S) = \int_{-\infty}^{\infty} f(x) \cdot e^{isx} \, dx$$

is called fourier transform of f(x)

Also the function f(x) is defined by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(S) \cdot e^{-isx} \, ds$$

is called Inverse fourier transform of F(S)

---

if $f(-x) = f(x) \rightarrow$ even

$= -f(x) \rightarrow$ odd

$$\Rightarrow \int_{-a}^{a} f(x)\,dx = 0 \quad \text{if } f(x) = \text{odd}$$

$$= 2\int_{0}^{a} f(x)\,dx. \quad = \text{even}$$

As per even & odd f^n $f(x)$
we have

1) * fourier cosine transform

$$F_C(\lambda) = \int_0^\infty f(u) \cos \lambda u \, du \quad -$$

* Inverse Cosine transform.

$$f(x) = \frac{2}{\pi} \int_0^\infty F_C(\lambda) \cos(\lambda) x \, dx$$

2) Fsine transform.

$$F_S(\lambda) = \int_{u=0}^\infty f(u) \sin \lambda u \, du$$

Inverse sine transform

$$f(x) = \frac{2}{\pi} \int_{\lambda=0}^\infty F_S(\lambda) \sin \lambda x \, dx$$

Fourier Transform.

① $f(x)$ defined in $[-\infty,\infty]$ then

F.T of $f(x) = F(\lambda) = \int_{u=-\infty}^{\infty} f(u) e^{-i\lambda u} du$

Inverse ⇒
$$f(x) = \frac{1}{2\pi} \int_{\lambda=-\infty}^{\infty} F(\lambda) e^{i\lambda u} dx$$

② If $f(x)$ is even f^n in $[-\infty,\infty]$
then cosine trans –
$$F_c(\lambda) = \int_{u=0}^{\infty} f(u) \cos\lambda u \, du$$

Inverse ⇒ Cosine Trans
$$f_c(x) = \frac{2}{\pi} \int_{\lambda=0}^{\infty} F_c(\lambda) \cos\lambda x \, dx$$

③ If $f(x)$ is odd f^n in $[-\infty,\infty]$ then
fourier Sine trans of $f(x)$
$$F_s(\lambda) = \int_{u=0}^{\infty} f(u) \sin\lambda u \, du$$

Inverse ⇒ Sine trans.
$$F(x) = \frac{2}{\pi} \int_{\lambda=0}^{\infty} F_s(\lambda) \sin\lambda x \, d\lambda$$

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(4) Fourier integral representation of $f(x)$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) e^{-i\lambda(u-x)} \, du \, d\lambda$$

(5) Fourier cosine integral representation

$$f(x) = \frac{2}{\pi} \int_{0}^{\infty} \int_{0}^{\infty} f(u) \cos\lambda u \cos\lambda x \, du \, d\lambda$$

(6) Fourier sine integral representation

$$f(x) = \frac{2}{\pi} \int_{0}^{\infty} \int_{0}^{\infty} f(u) \sin\lambda u \sin\lambda x \, du \, d\lambda$$

IMP Results -

(1) $$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} [a \sin bx - b \cos bx]$$

(2) $$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \cos bx + b \sin bx)$$

(3) $$\int uv = uv_1 - u'v_2 + u''v_3 - u'''v_4 + u^{iv}v_5 \cdots$$

Here dash $\rightarrow$ derivative
suffix $\rightarrow$ Integration

This rule is known as integration by parts.

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Q1) Find the fourier Transform of

$$f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

hence, evaluate $\int_0^\infty \frac{\sin x}{x}\, dx$

$\Rightarrow$ The fourier transform of $f(x)$ is given by

$$F\{f(x)\} = F(s) = \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$x < 1$

$-x < 1 \Rightarrow x > -1$

$x > 1$

$x > 1 = x < -1$

$-\infty \quad -1 \quad 0 \quad 1 \quad 2 \quad \infty$

$$= \cancel{\int_{-\infty}^{-1} f(x)\, e^{isx}\, dx}_{\,0} + \int_{-1}^{1} f(x)\, e^{isx}\, dx + \cancel{\int_{1}^{\infty} f(x)\, e^{isx}\, dx}_{\,0}$$

$$= \int_{-1}^{1} f(x)\, e^{isx}\, dx$$

$$= \int_{-1}^{1} 1 \cdot e^{isx}\, dx$$

$$= \left| \frac{e^{isx}}{is} \right|_{-1}^{1} = \frac{e^{is} - e^{-is}}{is}$$

$$= \frac{2\sin s}{s} \qquad \left[\sin x = \frac{e^{ix} - e^{-ix}}{2i} \right.$$

$$\Rightarrow F\{f(x)\} = F(s) = \frac{2\sin s}{s} \quad \text{--- (1)}$$

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Now by inverse fourier transform

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s) \cdot e^{-isx} ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin s}{s} e^{-isx} ds \quad \text{.. from (1)}$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin s}{s} \cdot e^{-isx} ds$$

put $x = 0$,

$$1 = f(0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin s}{s} ds$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin s}{s} ds = \pi$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{\sin s}{s} ds = \pi$$

X

$$\text{Or, } \int_{0}^{\infty} \frac{\sin s}{s} ds = \frac{\pi}{2}$$

For $f(s) = \sin s$
Put $s = -s$
$f(-s) = \sin(-s)$
$= -\sin s = -f(s)$
Also for $f(s) = s$
$f(-s) = -s = -f(s)$
both are odd fn
but $\frac{odd}{odd}$ = even

$$\lim_{a \to \infty} \int_{-a}^{a} \frac{\sin s}{s} ds = \pi$$

$$\Rightarrow \lim_{a \to \infty} 2\int_{0}^{a} \frac{\sin s}{s} ds = \pi$$

$$\Rightarrow \lim_{a \to \infty} \int_{0}^{a} \frac{\sin s}{s} ds = \frac{\pi}{2}$$

$$\text{or } \boxed{\int_{0}^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}}$$

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Z transform

The z-transform of the sequence $\{f(k)\}_{k=-\infty}^{\infty}$ is defined as

$$\boxed{Z[\{f(k)\}] = F(z) = \sum_{k=-\infty}^{\infty} f(k) z^{-k}}$$

The inverse z-transform is written as

$$\boxed{Z^{-1}[F(z)] = \{f(k)\}}$$

where,
z is a complex number
Z is an operator of z transform
$F(z)$ is the z transform of $\{f(k)\}$

* Property

① Addn $\rightarrow \{f(k)\} + \{g(k)\} = \{f(k) + g(k)\}$ respositions

② Scaling $\rightarrow \{a f(k)\} = a\{f(k)\}$ a = scalar

③ Linearity $\rightarrow \{a f(k) + b g(k)\} = a\{f(k)\} + b\{g(k)\}$

Unit V

Z-transforms.

Analog < Laplace T / Fourier T.

Digital → Z-T. Discrete fn.

Defn: The z-transform of a sequence u_n is denoted as $z(u_n)$ is defined a

$$\boxed{z[u_n] = \sum_{n=-\infty}^{\infty} u_n z^{-n} = \bar{u}(z)} \quad (1)$$

where $\bar{u}(z)$ is the z-transform of u_n & z is a complex number

Z-transform exists only when the infinite series in (1) is convergent

The sequence u_n is called the inverse Z-Transform of $\bar{u}(z)$ & is written as

$$z^{-1}[\bar{u}(z)] = u_n$$

eg - show that $z[a^n] = \frac{z}{z-a}$, $n \geq 0$

$$z(u_n) = \sum_{n=0}^{\infty} u_n z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$

$$= 1 + \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^1 + \cdots$$

$$= \frac{1}{1-a/z} = \frac{z}{z-a}$$

Standard series –

① Binomial $\rightarrow$ $(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \cdots$

$\therefore (1+y)^n = 1 + ny + \frac{n(n-1)}{2!}y^2 + \frac{(n)(n-1)(n-2)}{3!}y^3 + \cdots$ is convergent if $|y|<1$

② $(1+y)^{-1} = \frac{1}{1+y} = 1 - y + y^2 - y^3 + y^4 \cdots$

③ $\frac{1}{1-y} = (1-y)^{-1} = 1 + y + y^2 + y^3 + y^4 \cdots$

④ $e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots$

⑤ $a + ar + ar^2 + ar^3 + \cdots$

G.S $\Rightarrow$ $S_\infty = \frac{a}{1-r}$

is convergent if $|r|<1$

a = 1st term

r = common ratio

⑥ In z-transforms $\rightarrow$ z = complex no $= x + iy$

$\therefore |z| = 1 \Rightarrow \sqrt{x^2+y^2} = 1$

$\Rightarrow \underline{x^2+y^2=1}$ circle

$\therefore |z|<1 \Rightarrow$ points inside the circle

$\therefore |z|>1 \Rightarrow$ points outside the circle.

Standard sequences & transform

$f(K)$ $\qquad$ $F(z) = Z\{f(K)\}$

① $\{a^K\}_{K\geq 0}$ $\qquad$ $\frac{z}{z-a}$ for $|z| > |a|$

② $\{a^K\}_{K<0}$ $\qquad$ $\frac{z}{a-z}$ for $|z| < |a|$

③ $\{a^{|K|}\}_{\forall K}$ $\qquad$ $\frac{az}{1-az} + \frac{z}{z-a}$ $\quad |a| < |z| < \frac{1}{|a|}$

④ $\{\delta(K)\}$ $\qquad$ 1

⑤ $\{U(K)\}$ $\qquad$ $\frac{z}{z-1}$ for $|z| > 1$

⑥ $\{\cos \alpha K\}_{K\geq 0}$ $\qquad$ $\frac{z(z-\cos\alpha)}{z^2 - 2z\cos\alpha + 1}$ for $|z| > 1$

⑦ $\{\sin \alpha K\}_{K\geq 0}$ $\qquad$ $\frac{z\sin\alpha}{z^2 - 2z\cos\alpha + 1}$ for $|z| > 1$

⑧ $\{\cosh \alpha K\}_{K\geq 0}$ $\qquad$ $\frac{z(z-\cosh\alpha)}{z^2 - 2z\cosh\alpha + 1}$ $\left\{\text{for } |z| > \max |e^{a}| \text{ or } |e^{-a}|\right.$

⑨ $\{\sinh \alpha K\}_{K\geq 0}$ $\qquad$ $\frac{z\sinh\alpha}{z^2 - 2z\cosh\alpha + 1}$ for $|z| > \max |e^{a}|$ or $|e^{-a}|$

⑩ $Z[{}^{K}C_n]_{K\geq n} = z^{-n}(1-z^{-1})^{-(n+1)}$ $\qquad |z| > 1$

⑪ $Z\left[\frac{a^K}{K!}\right], K\geq 0 = e^{a/z}$

Q1) Find z transform of the sequence
$\left\{ \frac{1}{2^k} \right\}$, $-5 \le k \le 6$

→ By defn of z- transform, we have

$$F(z) = \sum_{k=-5}^{6} \frac{1}{2^k} z^{-k}$$

$$\Rightarrow F(z) = \frac{1}{2^{-5}} z^{-(-5)} + \frac{1}{2^{-4}} z^{-(-4)} \ldots \frac{1}{2^6} z^{-6}$$

$$= (2)^5 z^5 + (2)^4 z^4 + \cdots + \frac{1}{(2^6) z^6}$$

$$= 32z^5 + 16z^4 + 8z^3 + \cdots \frac{1}{64z^6}$$

Q2) Find the z transfom of $\{a^k\}$, $k \ge 0$

⇒ By def of z- transform, we have

$$F(z) = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$

$$\Rightarrow F(z) = \sum_{k=0}^{\infty} a^k z^{-k}$$

$$\Rightarrow F(z) = a^0 z^{-0} + a^1 z^{-1} + a^2 z^{-2} + \cdots\cdots$$

$$F(z) = \cancel{\frac{a}{z}}\ 1 + \frac{a}{z} + \frac{a^2}{z^2} + \cdots$$

which is a GP series whose sum, $S = \frac{a}{1-r}$
where $a = 1$ $r = a/z$

$$\Rightarrow F(z) = \frac{1}{1 - a/z} = \frac{1}{\frac{z-a}{z}} = \frac{z}{z-a}$$

(3) Find z-transform of

(1) $\left(\frac{1}{5}\right)^k, k \geq 0$

by rule $\rightarrow$ ~~F(z)~~

$$z[f(k)] = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$

$$\therefore z[f(k)] = \sum_{k=0}^{\infty} \left(\frac{1}{5}\right)^k z^{-k}$$

$$= \sum_{k=0}^{\infty} \left(\frac{z^{-1}}{5}\right)^k$$

$$= 1 + \left(\frac{z^{-1}}{5}\right) + \left(\frac{z^{-1}}{5}\right)^2 + \left(\frac{z^{-1}}{5}\right)^3 + \cdots$$

$$= \frac{1}{1 - \left(\frac{z^{-1}}{5}\right)}, \left|\frac{z^{-1}}{5}\right| < 1$$

$$= \frac{5z}{5z-1}, \left|\frac{1}{5z}\right| < 1 \text{ i.e., } |z| > \frac{1}{5}$$

(2) $3^k, k < 0$

$$\Rightarrow z[f(k)] = \sum_{k=-\infty} f(k) z^{-k}$$

$$= \sum_{k=-\infty}^{-1} 3^k z^{-k} \qquad \text{Put } k = -r \quad \begin{array}{c|c|c} k & -\infty & -1 \\ \hline r & \infty & 1 \end{array}$$

$$= \sum_{r=1}^{\infty} 3^{-r} z^r = \sum_{r=1}^{\infty} (3^{-1} z)^r$$

$$= (3^{-1}z) + (3^{-1}z)^2 + (3^{-1}z)^3 + \cdots$$

$$= (3^{-1}z)\left[1 + (3^{-1}z) + (3^{-1}z)^2 + \cdots\right]$$

$$= \frac{z}{\cancel{3}}\left(\frac{\cancel{3}}{3-z}\right), \left|\frac{z}{3}\right| < 1$$

$$= \frac{z}{3-z}, |z| < 3$$

(3) $f(k) = 4^k + 5^k, \; k \geq 0$

$\rightarrow Z[f(k)] = \sum_{k=0}^{\infty} f(k) z^{-k}$

$= \sum_{k=0}^{\infty} (4^k + 5^k) z^{-k}$

$= \sum_{k=0}^{\infty} 4^k z^{-k} + \sum_{k=0}^{\infty} 5^k z^{-k}$

$= \sum_{k=0}^{\infty} (4z^{-1})^k + \sum_{k=0}^{\infty} (5z^{-1})^k$

$= [1 + (4z^{-1}) + (4z^{-1})^2 + \cdots] + [1 + (5z^{-1}) + (5z^{-1})^2 + \cdots]$

$= \frac{1}{1-4z^{-1}} + \frac{1}{1-5z^{-1}}, \; |4z^{-1}| < 1 \; \& \; |5z^{-1}| < 1$

$= \frac{z}{z-4} + \frac{z}{z-5}, \; |z| > 4 \; \& \; |z| > 5$

$= \frac{z}{z-4} + \frac{z}{z-5}, \; |z| > 5$

(4) If $f(k) = 3^k \quad k < 0$

$\quad = 2^k \quad k \geq 0$

find $Z\{f(k)\} = \{3^k\}_{k<0} \; \& \; \{2^k\}_{k \geq 0}$

$= \sum_{k=-\infty}^{\infty} f(k) z^{-k}$

$= \sum_{-\infty}^{-1} 3^k z^{-k} + \sum_{0}^{\infty} 2^k z^{-k}$

use formulae (1) & (2) in table.

$= \frac{z}{3-z} + \frac{z}{z-2}$ for $|z| < |3|$ & $|z| > |2|$

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* Properties of z-transform.

① * Linearity: If $\{f(k)\}$ & $\{g(k)\}$ are such that they can be added and a & b are constants then,

$$z\{af(k) + bg(k)\} = az\{f(k)\} + bz\{g(k)\}$$

② * change of scale – If $z\{f(k)\} = F(z)$ then $z^{-1}\{a^k f(k)\} = F(z/a)$

eg) Find z-transform of $f(k) = 3^k \cos 2k,\ k \geq 0$

$$z[\cos 2k] = \frac{z(z-\cos 2)}{z^2 - 2z\cos 2 + 1},\ |z| > 1$$

$$z[3^k \cos 2k] = \frac{(z/3)((z/3) - \cos 2)}{(z/3)^2 - 2(z/3)\cos 2 + 1},\ |z/3| > 1$$

↳ using change of scale prop

③ * If $z\{f(k)\} = F(z)$ then $z[e^{-ak} f(k)\} = F(e^a z)$

eg) find $z[e^{-3k} \cos 5k]$

$$\Rightarrow z[\cos 5k] = \frac{z(z-\cos 5)}{z^2 - 2z\cos 5 + 1},\ |z| > 1$$

Now using prop ③

$$z[e^{-3k} \cos 5k] = \frac{e^3 z(e^3 z - \cos 5)}{(e^3 z)^2 - 2(e^3 z)\cos 5 + 1},\ |e^3 z| > 1$$

① Multiplication by K

If $z\{f(k)\} = F(z)$ then $z\{k f(k)\} = -z\frac{dF(z)}{dz}$

in general, $z\{k^n f(k)\} = \left(-z\frac{d}{dz}\right)^n F(z)$

eg- ① find $z[k5^k], k \geq 0$

$\rightarrow z[5^k]_{k\geq 0} = \frac{z}{z-5}$

$z\{k5^k\} = -z\frac{d}{dz}\left(\frac{z}{z-5}\right)$ ... using multipl. by 'k' rule.

$= -z\left[\frac{(z-5)-(1)z}{(z-5)^2}\right]$

$= -z\left[\frac{z-5-z}{(z-5)^2}\right]$

$= \frac{5z}{(z-5)^2}$

② $z\{k\}, k \geq 0$

$\rightarrow z\{1\}$ or $z\{U(k)\}$ $k \geq 0$ ... very imp step

$z\{U(k)\} = \frac{z}{z-1}$

Now using multiplication by K

$z\{k\} = -z\frac{d}{dz}\left(\frac{z}{z-1}\right)$

$\rightarrow$

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(5) Division by K

If $z\{f(k)\} = F(z)$ then $z\left\{\frac{f(k)}{k}\right\} = -\int_z z^{-1}\frac{F(z)}{dz}$

ex) $z\left\{\frac{\sin 3k}{k}\right\}, \ k > 0$

$\rightarrow z\{\sin 3k\} = \frac{z\sin 3}{z^2 - 2z\cos 3 + 1}$

$$z\left\{\frac{\sin 3k}{k}\right\} = \int_z^{\infty} z^{-1}\left(\frac{z\sin 3}{z^2 - 2z\cos 3 + 1}\right) dz$$

$$= \sin 3 \int_z^{\infty} \frac{1}{z^2 - 2z\cos 3 + 1} dz$$

$$= \sin 3 \int_z^{\infty} \frac{dz}{(z-\cos 3)^2 + \sin^2 3}$$

$$= \cancel{\sin 3}\left[\frac{1}{\cancel{\sin 3}}\left(\tan^{-1}\left(\frac{z-\cos 3}{\sin 3}\right)\right)\right]_z^{\infty}$$

$$= \tan^{-1}\infty - \tan^{-1}\left(\frac{z-\cos 3}{\sin 3}\right)$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{z-\cos 3}{\sin 3}\right)$$

$$= \cot^{-1}\left(\frac{z-\cos 3}{\sin 3}\right)$$

6) Shifting Property -

ⓐ If $z\{f(k)\} = F(z)$ then

$$z\{f(k+n)\} = z^n F(z) \ \& \ z\{f(k-n)\} = z^{-n} F(z)$$

ⓑ For one sided z-transform $k \geq 0$)

$$z\{f(k+n)\} = z^n F(z) - \sum_{r=0}^{n-1} f(r) z^{n-r}, \ k \geq 0$$

$$z\{f(k-n)\} = z^{-n} F(z) + \sum_{r=-n}^{-1} f(r) z^{-(n+r)}, k < 0$$

ⓒ For causal sequence

$$z\{f(k+n)\} = z^n F(z) - \sum_{r=0}^{n-1} f(r) z^{n-r}$$

$$z\{f(k-n)\} = z^{-n} F(z)$$

7) Convolution.

$$z\{f(k)\} \times \{g(k)\} = F(z) \cdot G(z)$$

Where $\{f(k)\} * \{g(k)\} = \sum_{m=-\infty}^{\infty} f(m) g(k-m)$

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(7) Ex $\rightarrow$ verify convolution Th for $f_1(k)=k$, $f_2(k)=k$, $k\geq 0$

$\rightarrow$ $$z\{k\} = z\{k\cdot 1\} = -z\frac{d}{dz}\left(\frac{z}{z-1}\right)$$

$$= \frac{z}{(z-1)^2}$$

$\therefore$ RHS $\Rightarrow F(z)\,G(z)$

$$= \frac{z}{(z-1)^2}\,\frac{z}{(z-1)^2} = \frac{z^2}{(z-1)^3}$$

Now,

LHS $\Rightarrow$ $$\{f_1(k) * f_2(k)\} = \sum_{m=-\infty}^{\infty} f(m)\,g(k-m)$$

$$= \sum_{m=0}^{\infty} f_1(m)\,f_2(k-m) = \sum_{m=0}^{\infty} m(k-m)$$

$$= k\sum_{m=0}^{\infty} m - \sum_{m=0}^{\infty} m^2$$

Now we know

$$1+2+3+\cdots = \frac{n(n+1)}{2}$$

$$1^2+2^2+3^2+\cdots = \frac{n(n+1)(2n+1)}{6}$$

Using these results we get:

$$= k\sum m - \sum m^2$$

$$= k\left(\frac{k(k+1)}{2}\right) - \frac{k(k+1)(2k+1)}{6}$$

$$= \frac{k(k+1)}{6}(3k-2k-1)$$

$$= \frac{k}{6}(k+1)(k-1).$$

$$= \frac{k}{6}(k^2-1)$$

halfonly

eg) $Z[\cos(5k+7)]$, $k \geq 0$

$\rightarrow \cos(5k+7) = \cos 5k \cos 7 - \sin 5k \sin 7$

$$Z[\cos(5k+7)] = Z[\cos 5k \cos 7] - Z[\sin 5k \sin 7]$$

$$= \cos 7 \, Z[\cos 5k] - \sin 7 \, Z[\sin 5k]$$

$$= \cos 7 \left(\frac{z(z-\cos 5)}{z^2 - 2z\cos 5 + 1} \right) - \sin 7 \left(\frac{z\sin 5}{z^2 - 2z\cos 5 + 1} \right)$$

$$= \frac{\cos 7 \, (z^2 - z\cos 5) - z\sin 5 \sin 7}{z^2 - 2z\cos 5 + 1}$$

Inverse 2-Trans.

$$z^{-1}\left[F(z)\right]_{|z|} = \{f(k)\}$$

$F(z)$	$f(k)$ $k \geq 0$, $\lvert z\rvert > \lvert a\rvert$	$f(k)$ $k < 0$, $\lvert z\rvert < \lvert a\rvert$
① $\left[\dfrac{z}{z-a}\right]_{\lvert z\rvert > a,\ \lvert z\rvert < a}$	$a^k u(k)$	$-a^k$
② $\dfrac{1}{z-a}$	$a^{k-1} u(k-1)$	
③ $\dfrac{z}{z-1}$	$u(k)$	
④ $\dfrac{z(z-\cos\alpha)}{z^2-(z\cos\alpha+1)}$	$\cos\alpha k$	
⑤ $\dfrac{z\sin\alpha}{z^2(2z\cos\alpha+1)}$	$\sin\alpha k$	
⑥ $\dfrac{z^2}{(z-a)^2}$	$(k+1)a^k$	$-(k+1)a^k$