CHAPTER 1

Basic Probability

RANDOM EXPERIMENTS

We are all familiar with the importance of experiments in science and engineering. Experimentation is useful to us because we can assume that if we perform certain experiments under very nearly identical conditions, we will arrive at results that are essentially the same. In these circumstances, we are able to control the value of the variables that affect the outcome of the experiment.

However, in some experiments, we are not able to ascertain or control the value of certain variables so that the results will vary from one performance of the experiment to the next even though most of the conditions are the same. These experiments are described as *random*. The following are some examples.

EXAMPLE 1.1. If we toss a coin, the result of the experiment is that it will either come up "tails," symbolized by T (or 0), or "heads," symbolized by H (or 1), i.e., one of the elements of the set $\{H, T\}$ (or $\{0, 1\}$).

EXAMPLE 1.2. If we toss a die, the result of the experiment is that it will come up with one of the numbers in the set $\{1, 2, 3, 4, 5, 6\}$.

EXAMPLE 1.3. If we toss a coin twice, there are four results possible, as indicated by $\{HH, HT, TH, TT\}$, i.e., both heads, head on first and tail on second, etc.

EXAMPLE 1.4. If we are making bolts with a machine, the result of the experiment is that some may be defective. Thus when a bolt is made, it will be a member of the set {defective, nondefective}.

EXAMPLE 1.5. If an experiment consists of measuring "lifetimes" of electric light bulbs produced by a company, then the result of the experiment is a time t in hours that lies in some interval—say, $0 \leqq t \leqq 4000$—where we assume that no bulb lasts more than 4000 hours.

SAMPLE SPACES

A set S that consists of all possible outcomes of a random experiment is called a *sample space*, and each outcome is called a *sample point*. Often there will be more than one sample space that can describe outcomes of an experiment, but there is usually only one that will provide the most information.

EXAMPLE 1.6. If we toss a die, one sample space, or set of all possible outcomes, is given by {1, 2, 3, 4, 5, 6} while another is {odd, even}. It is clear, however, that the latter would not be adequate to determine, for example, whether an outcome is divisible by 3.

It is often useful to portray a sample space graphically. In such cases it is desirable to use numbers in place of letters whenever possible.

EXAMPLE 1.7. If we toss a coin twice and use 0 to represent tails and 1 to represent heads, the sample space (see Example 1.3) can be portrayed by points as in Fig. 1-1 where, for example, (0, 1) represents tails on first toss and heads on second toss, i.e., *TH*.

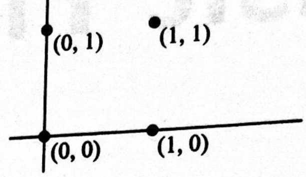


Fig. 1-1

If a sample space has a finite number of points, as in Example 1.7, it is called a *finite sample space*. If it has as many points as there are natural numbers 1, 2, 3, . . . , it is called a *countably infinite sample space*. If it has as many points as there are in some interval on the x axis, such as $0 \leqq x \leqq 1$, it is called a *noncountably infinite sample space*. A sample space that is finite or countably infinite is often called a *discrete sample space*, while one that is noncountably infinite is called a *nondiscrete sample space*.

EVENTS

An *event* is a subset A of the sample space S, i.e., it is a set of possible outcomes. If the outcome of an experiment is an element of A, we say that the event A *has occurred*. An event consisting of a single point of S is often called a *simple* or *elementary event*.

EXAMPLE 1.8. If we toss a coin twice, the event that only one head comes up is the subset of the sample space that consists of points (0, 1) and (1, 0), as indicated in Fig. 1-2.

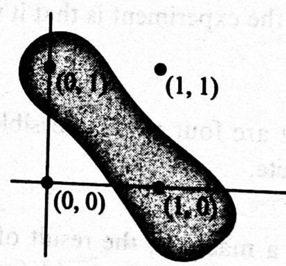


Fig. 1-2

As particular events, we have S itself, which is the *sure* or *certain event* since an element of S must occur, and the empty set $\varnothing$, which is called the *impossible event* because an element of $\varnothing$ cannot occur.

By using set operations on events in S, we can obtain other events in S. For example, if A and B are events, then

1. $A \cup B$ is the event "either A or B or both." $A \cup B$ is called the *union* of A and B.
2. $A \cap B$ is the event "both A and B." $A \cap B$ is called the *intersection* of A and B.
3. A' is the event "not A." A' is called the *complement* of A.
4. $A - B = A \cap B'$ is the event "A but not B." In particular, $A' = S - A$.

If the sets corresponding to events A and B are disjoint, i.e., $A \cap B = \varnothing$, we often say that the events are *mutually exclusive*. This means that they cannot both occur. We say that a collection of events $A_1, A_2, \ldots, A_n$ is mutually exclusive if every pair in the collection is mutually exclusive.

EXAMPLE 1.9. Referring to the experiment of tossing a coin twice, let A be the event "at least one head occurs" and B the event "the second toss results in a tail." Then $A = \{HT, TH, HH\}$, $B = \{HT, TT\}$, and so we have

$$A \cup B = \{HT, TH, HH, TT\} = S \qquad A \cap B = \{HT\}$$

$$A' = \{TT\} \qquad A - B = \{TH, HH\}$$

THE CONCEPT OF PROBABILITY

In any random experiment there is always uncertainty as to whether a particular event will or will not occur. As a measure of the *chance*, or *probability*, with which we can expect the event to occur, it is convenient to assign a number between 0 and 1. If we are sure or certain that the event will occur, we say that its probability is 100% or 1, but if we are sure that the event will not occur, we say that its probability is zero. If, for example, the probability is $\frac{1}{4}$, we would say that there is a 25% chance it will occur and a 75% chance that it will not occur. Equivalently, we can say that the *odds* against its occurrence are 75% to 25%, or 3 to 1.

There are two important procedures by means of which we can estimate the probability of an event.

1. **CLASSICAL APPROACH.** If an event can occur in h different ways out of a total number of n possible ways, all of which are equally likely, then the probability of the event is h/n.

EXAMPLE 1.10. Suppose we want to know the probability that a head will turn up in a single toss of a coin. Since there are two equally likely ways in which the coin can come up—namely, heads and tails (assuming it does not roll away or stand on its edge)—and of these two ways a head can arise in only one way, we reason that the required probability is 1/2. In arriving at this, we assume that the coin is *fair*, i.e., not *loaded* in any way.

2. **FREQUENCY APPROACH.** If after n repetitions of an experiment, where n is very large, an event is observed to occur in h of these, then the probability of the event is h/n. This is also called the *empirical probability* of the event.

EXAMPLE 1.11. If we toss a coin 1000 times and find that it comes up heads 532 times, we estimate the probability of a head coming up to be $532/1000 = 0.532$.

Both the classical and frequency approaches have serious drawbacks, the first because the words "equally likely" are vague and the second because the "large number" involved is vague. Because of these difficulties, mathematicians have been led to an *axiomatic approach* to probability.

THE AXIOMS OF PROBABILITY

Suppose we have a sample space S. If S is discrete, all subsets correspond to events and conversely, but if S is nondiscrete, only special subsets (called *measurable*) correspond to events. To each event A in the class C of events, we associate a real number $P(A)$. Then P is called a *probability function*, and $P(A)$ the *probability* of the event A, if the following axioms are satisfied.

Axiom 1. For every event A in the class C,

$$P(A) \geq 0 \tag{1}$$

Axiom 2. For the sure or certain event S in the class C,

$$P(S) = 1 \tag{2}$$

Axiom 3. For any number of mutually exclusive events $A_1, A_2, \ldots$, in the class C,

$$P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots \tag{3}$$

In particular, for two mutually exclusive events A_1, A_2,

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) \tag{4}$$

SOME IMPORTANT THEOREMS ON PROBABILITY

From the above axioms we can now prove various theorems on probability that are important in further work.

Theorem 1-1: If $A_1 \subset A_2$, then $P(A_1) \leqq P(A_2)$ and $P(A_2 - A_1) = P(A_2) - P(A_1)$.

Theorem 1-2: For every event A,

$$0 \leqq P(A) \leqq 1, \tag{5}$$

i.e., a probability is between 0 and 1.

Theorem 1-3:

$$P(\varnothing) = 0 \tag{6}$$

i.e., the impossible event has probability zero.

Theorem 1-4: If A' is the complement of A, then

$$P(A') = 1 - P(A) \tag{7}$$

Theorem 1-5: If $A = A_1 \cup A_2 \cup \cdots \cup A_n$, where $A_1, A_2, \ldots, A_n$ are mutually exclusive events, then

$$P(A) = P(A_1) + P(A_2) + \cdots + P(A_n) \tag{8}$$

In particular, if $A = S$, the sample space, then

$$P(A_1) + P(A_2) + \cdots + P(A_n) = 1 \tag{9}$$

Theorem 1-6: If A and B are any two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \tag{10}$$

More generally, if A_1, A_2, A_3 are any three events, then

$$\begin{aligned} P(A_1 \cup A_2 \cup A_3) = {} & P(A_1) + P(A_2) + P(A_3) \\ & - P(A_1 \cap A_2) - P(A_2 \cap A_3) - P(A_3 \cap A_1) \\ & + P(A_1 \cap A_2 \cap A_3) \end{aligned} \tag{11}$$

Generalizations to n events can also be made.

Theorem 1-7: For any events A and B,

$$P(A) = P(A \cap B) + P(A \cap B') \tag{12}$$

Theorem 1-8: If an event A must result in the occurrence of one of the mutually exclusive events $A_1, A_2, \ldots, A_n$, then

$$P(A) = P(A \cap A_1) + P(A \cap A_2) + \cdots + P(A \cap A_n) \tag{13}$$

ASSIGNMENT OF PROBABILITIES

If a sample space S consists of a finite number of outcomes $a_1, a_2, \ldots, a_n$, then by Theorem 1-5,

$$P(A_1) + P(A_2) + \cdots + P(A_n) = 1 \tag{14}$$

where $A_1, A_2, \ldots, A_n$ are elementary events given by $A_i = \{a_i\}$.

It follows that we can arbitrarily choose any nonnegative numbers for the probabilities of these simple events as long as *(14)* is satisfied. In particular, if we assume *equal probabilities* for all simple events, then

$$P(A_k) = \frac{1}{n}, \qquad k = 1, 2, \ldots, n \tag{15}$$

and if A is any event made up of h such simple events, we have

$$P(A) = \frac{h}{n} \tag{16}$$

This is equivalent to the classical approach to probability given on page 5. We could of course use other procedures for assigning probabilities, such as the frequency approach of page 5.

Assigning probabilities provides a *mathematical model*, the success of which must be tested by experiment in much the same manner that theories in physics or other sciences must be tested by experiment.

EXAMPLE 1.12. A single die is tossed once. Find the probability of a 2 or 5 turning up.

The sample space is $S = \{1, 2, 3, 4, 5, 6\}$. If we assign equal probabilities to the sample points, i.e., if we assume that the die is fair, then

$$P(1) = P(2) = \cdots = P(6) = \frac{1}{6}$$

The event that either 2 or 5 turns up is indicated by $2 \cup 5$. Therefore,

$$P(2 \cup 5) = P(2) + P(5) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

CONDITIONAL PROBABILITY

Let A and B be two events (Fig. 1-3) such that $P(A) > 0$. Denote by $P(B \mid A)$ the probability of B *given that* A has occurred. Since A is known to have occurred, it becomes the new sample space replacing the original S. From this we are led to the definition

$$P(B \mid A) \equiv \frac{P(A \cap B)}{P(A)} \tag{17}$$

or

$$P(A \cap B) \equiv P(A)P(B \mid A) \tag{18}$$

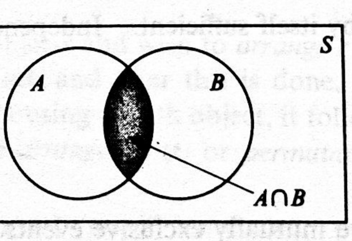


Fig. 1-3

In words, *(18)* says that the probability that both A and B occur is equal to the probability that A occurs times the probability that B occurs given that A has occurred. We call $P(B \mid A)$ the *conditional probability* of B given A, i.e., the probability that B will occur given that A has occurred. It is easy to show that conditional probability satisfies the axioms on page 5.

EXAMPLE 1.13. Find the probability that a single toss of a die will result in a number less than 4 if *(a)* no other information is given and *(b)* it is given that the toss resulted in an odd number.

(*a*) Let B denote the event {less than 4}. Since B is the union of the events 1, 2, or 3 turning up, we see by Theorem 1-5 that

$$P(B) = P(1) + P(2) + P(3) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$$

assuming equal probabilities for the sample points.

(*b*) Letting A be the event {odd number}, we see that $P(A) = \frac{3}{6} = \frac{1}{2}$. Also $P(A \cap B) = \frac{2}{6} = \frac{1}{3}$. Then

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{1/3}{1/2} = \frac{2}{3}$$

Hence, the added knowledge that the toss results in an odd number raises the probability from 1/2 to 2/3.

THEOREMS ON CONDITIONAL PROBABILITY

Theorem 1-9: For any three events A_1, A_2, A_3, we have

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_1 \cap A_2) \tag{19}$$

In words, the probability that A_1 and A_2 and A_3 all occur is equal to the probability that A_1 occurs times the probability that A_2 occurs given that A_1 has occurred times the probability that A_3 occurs given that both A_1 and A_2 have occurred. The result is easily generalized to n events.

Theorem 1-10: If an event A must result in one of the mutually exclusive events $A_1, A_2, \ldots, A_n$, then

$$P(A) = P(A_1)P(A \mid A_1) + P(A_2)P(A \mid A_2) + \cdots + P(A_n)P(A \mid A_n) \tag{20}$$

INDEPENDENT EVENTS

If $P(B \mid A) = P(B)$, i.e., the probability of B occurring is not affected by the occurrence or non-occurrence of A, then we say that A and B are *independent events*. This is equivalent to

$$P(A \cap B) = P(A)P(B) \tag{21}$$

as seen from (*18*). Conversely, if (*21*) holds, then A and B are independent.

We say that three events A_1, A_2, A_3 are *independent* if they are pairwise independent:

$$P(A_j \cap A_k) = P(A_j)P(A_k) \qquad j \neq k \qquad \text{where } j, k = 1, 2, 3 \tag{22}$$

and

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3) \tag{23}$$

Note that neither (*22*) nor (*23*) is by itself sufficient. Independence of more than three events is easily defined.

BAYES' THEOREM OR RULE

Suppose that $A_1, A_2, \ldots, A_n$ are mutually exclusive events whose union is the sample space S, i.e., one of the events must occur. Then if A is any event, we have the following important theorem:

***Theorem 1-11* (Bayes' Rule):**

$$P(A_k \mid A) = \frac{P(A_k)P(A \mid A_k)}{\sum_{j=1}^{n} P(A_j)P(A \mid A_j)} \tag{24}$$

This enables us to find the probabilities of the various events $A_1, A_2, \ldots, A_n$ that can *cause* A to occur. For this reason Bayes' theorem is often referred to as a *theorem on the probability of causes*.

1.7. A ball is drawn at random from a box containing 6 red balls, 4 white balls, and 5 blue balls. Determine the probability that it is (*a*) red, (*b*) white, (*c*) blue, (*d*) not red, (*e*) red or white.

(*a*) **Method 1.**

Let R, W, and B denote the events of drawing a red ball, white ball, and blue ball, respectively. Then

$$P(R) = \frac{\text{ways of choosing a red ball}}{\text{total ways of choosing a ball}} = \frac{6}{6+4+5} = \frac{6}{15} = \frac{2}{5}$$

Method 2.

Our sample space consists of $6 + 4 + 5 = 15$ sample points. Then if we assign equal probabilities $1/15$ to each sample point, we see that $P(R) = 6/15 = 2/5$, since there are 6 sample points corresponding to "red ball."

(*b*) $P(W) = \dfrac{4}{6+4+5} = \dfrac{4}{15}$

(*c*) $P(B) = \dfrac{5}{6+4+5} = \dfrac{5}{15} = \dfrac{1}{3}$

(*d*) $P(\text{not red}) = P(R') = 1 - P(R) = 1 - \dfrac{2}{5} = \dfrac{3}{5}$ by part (*a*).

(*e*) **Method 1.**

$$P(\text{red or white}) = P(R \cup W) = \frac{\text{ways of choosing a red or white ball}}{\text{total ways of choosing a ball}}$$

$$= \frac{6+4}{6+4+5} = \frac{10}{15} = \frac{2}{3}$$

This can also be worked using the sample space as in part (*a*).

Method 2.

$$P(R \cup W) = P(B') = 1 - P(B) = 1 - \frac{1}{3} = \frac{2}{3} \quad \text{by part } (c).$$

1.15. Box I contains 3 red and 2 blue marbles while Box II contains 2 red and 8 blue marbles. A fair coin is tossed. If the coin turns up heads, a marble is chosen from Box I; if it turns up tails, a marble is chosen from Box II. Find the probability that a red marble is chosen.

Let R denote the event "a red marble is chosen" while I and II denote the events that Box I and Box II are chosen, respectively. Since a red marble can result by choosing either Box I or II, we can use the results of Problem 1.14 with $A = R$, $A_1 = I$, $A_2 = II$. Therefore, the probability of choosing a red marble is

$$P(R) = P(I)P(R \mid I) + P(II)P(R \mid II) = \left(\frac{1}{2}\right)\left(\frac{3}{3+2}\right) + \left(\frac{1}{2}\right)\left(\frac{2}{2+8}\right) = \frac{2}{5}$$

1.17. Suppose in Problem 1.15 that the one who tosses the coin does not reveal whether it has turned up heads or tails (so that the box from which a marble was chosen is not revealed) but does reveal that a red marble was chosen. What is the probability that Box I was chosen (i.e., the coin turned up heads)?

Let us use the same terminology as in Problem 1.15, i.e., $A = R$, $A_1 = I$, $A_2 = II$. We seek the probability that Box I was chosen given that a red marble is known to have been chosen. Using Bayes' rule with $n = 2$, this probability is given by

$$P(I \mid R) = \frac{P(I)P(R \mid I)}{P(I)P(R \mid I) + P(II)P(R \mid II)} = \frac{\left(\frac{1}{2}\right)\left(\frac{3}{3+2}\right)}{\left(\frac{1}{2}\right)\left(\frac{3}{3+2}\right) + \left(\frac{1}{2}\right)\left(\frac{2}{2+8}\right)} = \frac{3}{4}$$

Random Variables and Probability Distributions

RANDOM VARIABLES

Suppose that to each point of a sample space we assign a number. We then have a *function* defined on the sample space. This function is called a *random variable* (or *stochastic variable*) or more precisely a *random function* (*stochastic function*). It is usually denoted by a capital letter such as X or Y. In general, a random variable has some specified physical, geometrical, or other significance.

EXAMPLE 2.1. Suppose that a coin is tossed twice so that the sample space is $S = \{HH, HT, TH, TT\}$. Let X represent the number of heads that can come up. With each sample point we can associate a number for X as shown in Table 2-1. Thus, for example, in the case of HH (i.e., 2 heads), $X = 2$ while for TH (1 head), $X = 1$. It follows that X is a random variable.

Table 2-1

Sample Point	HH	HT	TH	TT
X	2	1	1	0

It should be noted that many other random variables could also be defined on this sample space, for example, the square of the number of heads or the number of heads minus the number of tails.

A random variable that takes on a finite or countably infinite number of values (see page 4) is called a *discrete random variable* while one which takes on a noncountably infinite number of values is called a *nondiscrete random variable*.

DISCRETE PROBABILITY DISTRIBUTIONS

Let X be a discrete random variable, and suppose that the possible values that it can assume are given by $x_1, x_2, x_3, \ldots$, arranged in some order. Suppose also that these values are assumed with probabilities given by

$$P(X = x_k) = f(x_k) \qquad k = 1, 2, \ldots \tag{1}$$

It is convenient to introduce the *probability function*, also referred to as *probability distribution*, given by

$$P(X = x) = f(x) \tag{2}$$

For $x = x_k$, this reduces to (*1*) while for other values of x, $f(x) = 0$.

In general, $f(x)$ is a probability function if

1. $f(x) \geq 0$
2. $\sum_x f(x) = 1$

where the sum in 2 is taken over all possible values of x.

EXAMPLE 2.2. Find the probability function corresponding to the random variable X of Example 2.1.

Assuming that the coin is fair, we have

$$P(HH) = \frac{1}{4} \qquad P(HT) = \frac{1}{4} \qquad P(TH) = \frac{1}{4} \qquad P(TT) = \frac{1}{4}$$

Then

$$P(X = 0) = P(TT) = \frac{1}{4}$$

$$P(X = 1) = P(HT \cup TH) = P(HT) + P(TH) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$P(X = 2) = P(HH) = \frac{1}{4}$$

The probability function is thus given by Table 2-2.

Table 2-2

x	0	1	2
$f(x)$	1/4	1/2	1/4

DISTRIBUTION FUNCTIONS FOR RANDOM VARIABLES

The *cumulative distribution function*, or briefly the *distribution function*, for a random variable X is defined by

$$F(x) = P(X \leq x) \tag{3}$$

where x is any real number, i.e., $-\infty < x < \infty$.

The distribution function $F(x)$ has the following properties:

1. $F(x)$ is nondecreasing [i.e., $F(x) \leq F(y)$ if $x \leq y$].
2. $\lim_{x \to -\infty} F(x) = 0$; $\lim_{x \to \infty} F(x) = 1$.
3. $F(x)$ is continuous from the right [i.e., $\lim_{h \to 0^+} F(x + h) = F(x)$ for all x].

DISTRIBUTION FUNCTIONS FOR DISCRETE RANDOM VARIABLES

The distribution function for a discrete random variable X can be obtained from its probability function by noting that, for all x in $(-\infty, \infty)$,

$$F(x) = P(X \leq x) = \sum_{u \leq x} f(u) \tag{4}$$

where the sum is taken over all values u taken on by X for which $u \leq x$.

If X takes on only a finite number of values $x_1, x_2, \ldots, x_n$, then the distribution function is given by

$$F(x) = \begin{cases} 0 & -\infty < x < x_1 \\ f(x_1) & x_1 \leqq x < x_2 \\ f(x_1) + f(x_2) & x_2 \leqq x < x_3 \\ \vdots & \vdots \\ f(x_1) + \cdots + f(x_n) & x_n \leqq x < \infty \end{cases} \tag{5}$$

EXAMPLE 2.3. (*a*) Find the distribution function for the random variable X of Example 2.2. (*b*) Obtain its graph.

(*a*) The distribution function is

$$F(x) = \begin{cases} 0 & -\infty < x < 0 \\ \frac{1}{4} & 0 \leqq x < 1 \\ \frac{3}{4} & 1 \leqq x < 2 \\ 1 & 2 \leqq x < \infty \end{cases}$$

(*b*) The graph of $F(x)$ is shown in Fig. 2-1.

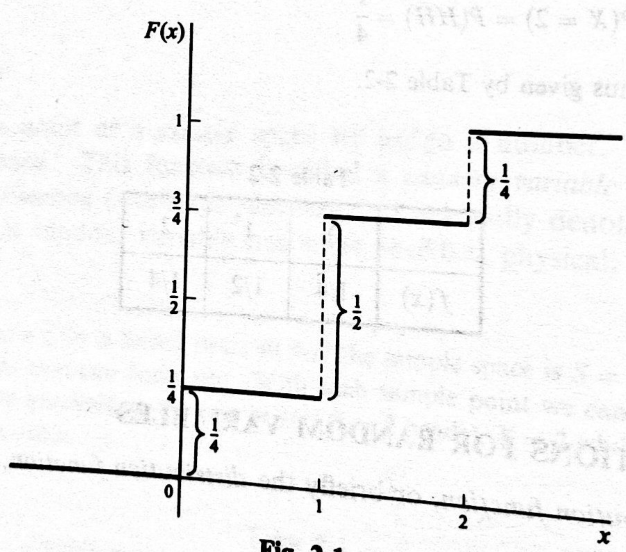


Fig. 2-1

The following things about the above distribution function, which are true in general, should be noted.

1. The magnitudes of the jumps at 0, 1, 2 are $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$ which are precisely the probabilities in Table 2-2. This fact enables one to obtain the probability function from the distribution function.
2. Because of the appearance of the graph of Fig. 2-1, it is often called a *staircase function* or *step function*. The value of the function at an integer is obtained from the higher step; thus the value at 1 is $\frac{3}{4}$ and not $\frac{1}{4}$. This is expressed mathematically by stating that the distribution function is *continuous from the right* at 0, 1, 2.

3. As we proceed from left to right (i.e. going *upstairs*), the distribution function either remains the same or increases, taking on values from 0 to 1. Because of this, it is said to be a *monotonically increasing function*.

It is clear from the above remarks and the properties of distribution functions that the probability function of a discrete random variable can be obtained from the distribution function by noting that

$$f(x) = F(x) - \lim_{u \to x^-} F(u). \tag{6}$$

CONTINUOUS RANDOM VARIABLES

A nondiscrete random variable X is said to be *absolutely continuous*, or simply *continuous*, if its distribution function may be represented as

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(u)\,du \qquad (-\infty < x < \infty) \tag{7}$$

where the function $f(x)$ has the properties

1. $f(x) \geq 0$
2. $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$

It follows from the above that if X is a continuous random variable, then the probability that X takes on any one particular value is zero, whereas the *interval probability* that X lies *between two different values*, say, a and b, is given by

$$P(a < X < b) = \int_{a}^{b} f(x)\,dx \tag{8}$$

EXAMPLE 2.4. If an individual is selected at random from a large group of adult males, the probability that his height X is precisely 68 inches (i.e., 68.000... inches) would be zero. However, there is a probability greater than zero that X is between 67.000... inches and 68.500... inches, for example.

A function $f(x)$ that satisfies the above requirements is called a *probability function* or *probability distribution* for a continuous random variable, but it is more often called a *probability density function* or simply *density function*. Any function $f(x)$ satisfying Properties 1 and 2 above will automatically be a density function, and required probabilities can then be obtained from (*8*).

EXAMPLE 2.5. (*a*) Find the constant c such that the function

$$f(x) = \begin{cases} cx^2 & 0 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

is a density function, and (*b*) compute $P(1 < X < 2)$.

(*a*) Since $f(x)$ satisfies Property 1 if $c \geqq 0$, it must satisfy Property 2 in order to be a density function. Now

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{3} cx^2\,dx = \left.\frac{cx^3}{3}\right|_0^3 = 9c$$

and since this must equal 1, we have $c = 1/9$.

(*b*)
$$P(1 < X < 2) = \int_{1}^{2} \frac{1}{9} x^2\,dx = \left.\frac{x^3}{27}\right|_1^2 = \frac{8}{27} - \frac{1}{27} = \frac{7}{27}$$

In case $f(x)$ is continuous, which we shall assume unless otherwise stated, the probability that X is equal to any particular value is zero. In such case we can replace either or both of the signs $<$ in (*8*) by $\leqq$. Thus, in Example 2.5,

$$P(1 \leqq X \leqq 2) = P(1 \leqq X < 2) = P(1 < X \leqq 2) = P(1 < X < 2) = \frac{7}{27}$$

EXAMPLE 2.6. (*a*) Find the distribution function for the random variable of Example 2.5. (*b*) Use the result of (*a*) to find $P(1 < x \leqq 2)$.

(*a*) We have

$$F(x) = P(X \leqq x) = \int_{-\infty}^{x} f(u)\,du$$

If $x < 0$, then $F(x) = 0$. If $0 \leqq x < 3$, then

$$F(x) = \int_0^x f(u)\,du = \int_0^x \frac{1}{9}u^2\,du = \frac{x^3}{27}$$

If $x \geqq 3$, then

$$F(x) = \int_0^3 f(u)\,du + \int_3^x f(u)\,du = \int_0^3 \frac{1}{9}u^2\,du + \int_3^x 0\,du = 1$$

Thus the required distribution function is

$$F(x) = \begin{cases} 0 & x < 0 \\ x^3/27 & 0 \leqq x < 3 \\ 1 & x \geqq 3 \end{cases}$$

Note that $F(x)$ increases monotonically from 0 to 1 as is required for a distribution function. It should also be noted that $F(x)$ in this case is continuous.

(*b*) We have

$$\begin{aligned} P(1 < X \leqq 2) &= P(X \leqq 2) - P(X \leqq 1) \\ &= F(2) - F(1) \\ &= \frac{2^3}{27} - \frac{1^3}{27} = \frac{7}{27} \end{aligned}$$

as in Example 2.5.

Solved Problems

DISCRETE RANDOM VARIABLES AND PROBABILITY DISTRIBUTIONS

2.1. Suppose that a pair of fair dice are to be tossed, and let the random variable X denote the sum of the points. Obtain the probability distribution for X.

The sample points for tosses of a pair of dice are given in Fig. 1-9, page 15. The random variable X is the sum of the coordinates for each point. Thus for $(3, 2)$ we have $X = 5$. Using the fact that all 36 sample points are equally probable, so that each sample point has probability 1/36, we obtain Table 2-4. For example, corresponding to $X = 5$, we have the sample points $(1, 4)$, $(2, 3)$, $(3, 2)$, $(4, 1)$, so that the associated probability is 4/36.

Table 2-4

x	2	3	4	5	6	7	8	9	10	11	12
$f(x)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

2.2. Find the probability distribution of boys and girls in families with 3 children, assuming equal probabilities for boys and girls.

Problem 1.37 treated the case of n mutually independent trials, where each trial had just two possible outcomes, A and A', with respective probabilities p and $q = 1 - p$. It was found that the probability of getting exactly x A's in the n trials is ${}_nC_x p^x q^{n-x}$. This result applies to the present problem, under the assumption that successive births (the "trials") are independent as far as the sex of the child is concerned. Thus, with A being the event "a boy," $n = 3$, and $p = q = \frac{1}{2}$, we have

$$P(\text{exactly } x \text{ boys}) = P(X = x) = {}_3C_x\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{3-x} = {}_3C_x\left(\frac{1}{2}\right)^3$$

where the random variable X represents the number of boys in the family. (Note that X is defined on the sample space of 3 trials.) The probability function for X,

$$f(x) = {}_3C_x\left(\frac{1}{2}\right)^3$$

is displayed in Table 2-5.

Table 2-5

x	0	1	2	3
$f(x)$	1/8	3/8	3/8	1/8

2.5. A random variable X has the density function $f(x) = c/(x^2+1)$, where $-\infty < x < \infty$. *(a)* Find the value of the constant c. *(b)* Find the probability that X^2 lies between 1/3 and 1.

(a) We must have $\int_{-\infty}^{\infty} f(x)\,dx = 1$, i.e.,

$$\int_{-\infty}^{\infty} \frac{c\,dx}{x^2+1} = c\tan^{-1} x \Big|_{-\infty}^{\infty} = c\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = 1$$

so that $c = 1/\pi$.

(b) If $\frac{1}{3} \leqq X^2 \leqq 1$, then either $\frac{\sqrt{3}}{3} \leqq X \leqq 1$ or $-1 \leqq X \leqq -\frac{\sqrt{3}}{3}$. Thus the required probability is

$$\frac{1}{\pi}\int_{-1}^{-\sqrt{3}/3} \frac{dx}{x^2+1} + \frac{1}{\pi}\int_{\sqrt{3}/3}^{1} \frac{dx}{x^2+1} = \frac{2}{\pi}\int_{\sqrt{3}/3}^{1} \frac{dx}{x^2+1}$$

$$= \frac{2}{\pi}\left[\tan^{-1}(1) - \tan^{-1}\left(\frac{\sqrt{3}}{3}\right)\right]$$

$$= \frac{2}{\pi}\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \frac{1}{6}$$

2.6. Find the distribution function corresponding to the density function of Problem 2.5.

$$F(x) = \int_{-\infty}^{x} f(u)\,du = \frac{1}{\pi}\int_{-\infty}^{x} \frac{du}{u^2+1} = \frac{1}{\pi}\left[\tan^{-1} u \Big|_{-\infty}^{x}\right]$$

$$= \frac{1}{\pi}[\tan^{-1} x - \tan^{-1}(-\infty)] = \frac{1}{\pi}\left[\tan^{-1} x + \frac{\pi}{2}\right]$$

$$= \frac{1}{2} + \frac{1}{\pi}\tan^{-1} x$$

2.7. The distribution function for a random variable X is

$$F(x) = \begin{cases} 1 - e^{-2x} & x \geqq 0 \\ 0 & x < 0 \end{cases}$$

Find *(a)* the density function, *(b)* the probability that $X > 2$, and *(c)* the probability that $-3 < X \leqq 4$.

(a)
$$f(x) = \frac{d}{dx}F(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & x < 0 \end{cases}$$

(b)
$$P(X > 2) = \int_{2}^{\infty} 2e^{-2u}\,du = -e^{-2u}\Big|_{2}^{\infty} = e^{-4}$$

Another method.

By definition, $P(X \leqq 2) = F(2) = 1 - e^{-4}$. Hence,

$$P(X > 2) = 1 - (1 - e^{-4}) = e^{-4}$$

(c)
$$P(-3 < X \leqq 4) = \int_{-3}^{4} f(u)\,du = \int_{-3}^{0} 0\,du + \int_{0}^{4} 2e^{-2u}\,du$$

$$= -e^{-2u}\Big|_{0}^{4} = 1 - e^{-8}$$

DEFINITION OF MATHEMATICAL EXPECTATION

A very important concept in probability and statistics is that of the *mathematical expectation*, *expected value*, or briefly the *expectation*, of a random variable. For a discrete random variable X having the possible values $x_1, \ldots, x_n$, the expectation of X is defined as

$$E(X) = x_1 P(X = x_1) + \cdots + x_n P(X = x_n) = \sum_{j=1}^{n} x_j P(X = x_j) \tag{1}$$

or equivalently, if $P(X = x_j) = f(x_j)$,

$$E(X) = x_1 f(x_1) + \cdots + x_n f(x_n) = \sum_{j=1}^{n} x_j f(x_j) = \sum x f(x) \tag{2}$$

where the last summation is taken over all appropriate values of x. As a special case of (2), where the probabilities are all equal, we have

$$E(X) = \frac{x_1 + x_2 + \cdots + x_n}{n} \tag{3}$$

which is called the *arithmetic mean*, or simply the *mean*, of $x_1, x_2, \ldots, x_n$.

If X takes on an infinite number of values $x_1, x_2, \ldots$, then $E(X) = \sum_{j=1}^{\infty} x_j f(x_j)$ provided that the infinite series converges absolutely.

For a continuous random variable X having density function $f(x)$, the expectation of X is defined as

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx \tag{4}$$

provided that the integral converges absolutely.

The expectation of X is very often called the *mean* of X and is denoted by μ_X, or simply μ, when the particular random variable is understood.

EXAMPLE 3.1. Suppose that a game is to be played with a single die assumed fair. In this game a player wins \$20 if a 2 turns up, \$40 if a 4 turns up; loses \$30 if a 6 turns up; while the player neither wins nor loses if any other face turns up. Find the expected sum of money to be won.

Let X be the random variable giving the amount of money won on any toss. The possible amounts won when the die turns up $1, 2, \ldots, 6$ are $x_1, x_2, \ldots, x_6$, respectively, while the probabilities of these are $f(x_1), f(x_2), \ldots, f(x_6)$. The probability function for X is displayed in Table 3-1. Therefore, the expected value or expectation is

$$E(X) = (0)\left(\frac{1}{6}\right) + (20)\left(\frac{1}{6}\right) + (0)\left(\frac{1}{6}\right) + (40)\left(\frac{1}{6}\right) + (0)\left(\frac{1}{6}\right) + (-30)\left(\frac{1}{6}\right) = 5$$

Table 3-1

x_j	0	+20	0	+40	0	−30
$f(x_j)$	1/6	1/6	1/6	1/6	1/6	1/6

It follows that the player can expect to win \$5. In a fair game, therefore, the player should be expected to pay \$5 in order to play the game.

EXAMPLE 3.2. The density function of a random variable X is given by

$$f(x) = \begin{cases} \frac{1}{2}x & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

The expected value of X is then

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^2 x\left(\frac{1}{2}x\right)dx = \int_0^2 \frac{x^2}{2}\,dx = \frac{x^3}{6}\bigg|_0^2 = \frac{4}{3}$$

BINOMIAL DISTRIBUTION

63.1 BINOMIAL DISTRIBUTION $P(r) = {}^nC_r\, p^r \cdot q^{n-r}$

To find the probability of the happening of an event once, twice, thrice, ... r times ...... exactly in n trials.

Let the probability of the happening of an event A in one trial be p and its probability of not happening be $1 - p = q$.

We assume that there are n trials and the happening of the event A is r times and its not happening is $n - r$ times.

$$\underbrace{A\,A \ldots\ldots A}_{r \text{ times}} \qquad \underbrace{\bar{A}.\bar{A} \ldots\ldots \bar{A}}_{n-r \text{ times}} \qquad \ldots(1)$$

A indicates its happening, $\bar{A}$ its failure and $P(A) = p$ and $P(\bar{A}) = q$.

We see that (1) has the probability

$$\underbrace{pp \ldots\, p}_{r \text{ times}}\ \underbrace{q \cdot q \ldots\, q}_{n-r \text{ times}} = p^r \cdot q^{n-r} = p^r q^{n-r} \qquad \ldots(2)$$

Clearly (1) is merely one order of arranging r A's.

The probability of (1) $= p^r q^{n-r} \times$ Number of different arrangements of r A's and $(n-r)$ $\bar{A}$'s.

The number of different arrangements of r A's and $(n-r)$ $\bar{A}$'s $= {}^nC_r$

$\therefore$ Probability of the happening of an event r times $= {}^nC_r\, p^r \cdot q^{n-r}$.

$$P(r) = {}^nC_r\, p^r \cdot q^{n-r} \quad (r = 0, 1, 2, \ldots, n).$$
$$= (r+1)\text{th term of } (q+p)^n$$

If $r = 0$, probability of happening of an event 0 times $= {}^nC_0\, q^n p^0 = q^n$

If $r = 1$, probability of happening of an event 1 time $= {}^nC_1\, q^{n-1} p$

If $r = 2$, probability of happening of an event 2 times $= {}^nC_2\, q^{n-2} p^2$

If $r = 3$, probability of happening of an event 3 times $= {}^nC_3\, q^{n-3} p^3$ and so on.

These terms are clearly the successive terms in the expansion of $(q+p)^n$.

Hence it is called Binomial Distribution.

Example 1. *Find the probability of getting 4 heads in 6 tosses of a fair coin.*

Solution. $p = \dfrac{1}{2}, q = \dfrac{1}{2}, n = 6, r = 4.$

We know that $\quad P(r) = {}^nC_r\, q^{n-r} p^r$

$$P(4) = {}^6C_4\, q^{6-4} p^4 = \frac{6 \times 5}{1 \times 2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^4 = 15 \times \left(\frac{1}{2}\right)^6 = \frac{15}{64} \quad \textbf{Ans.}$$

Example 2. *If on an average one ship in every ten is wrecked, find the probability that out of 5 ships expected to arrive, 4 at least will arrive safely.*

Solution. Out of 10 ships, one ship is wrecked.

i.e., Nine ships out of ten ships are safe.

$$p\,(\text{safety}) = \frac{9}{10}$$

P (At least 4 ships out of 5 are safe) $= P\,(4 \text{ or } 5) = P\,(4) + P\,(5)$

$$= {}^5C_4\, p^4 q^{5-4} + {}^5C_5\, p^5 q^0 = 5\left(\frac{9}{10}\right)^4\left(\frac{1}{10}\right) + \left(\frac{9}{10}\right)^5 = \left(\frac{9}{10}\right)^4\left(\frac{5}{10} + \frac{9}{10}\right) = \frac{7}{5}\left(\frac{9}{10}\right)^4 \quad \textbf{Ans.}$$

Example 3. *The overall percentage of failures in a certain examination is 20. If six candidates appear in the examination, what is the probability that at least five pass the examination?*

Solution. Probability of failures $= 20\% = \frac{20}{100} = \frac{1}{5}$

Probability of pass $(P) = 1 - \frac{1}{5} = \frac{4}{5}$

Probability of at least five pass $= P\,(5 \text{ or } 6)$

$$= P\,(5) + P(6) = {}^6C_5 P^5 q + {}^6C_6 P^6 q^0 = 6\left(\frac{4}{5}\right)^5\left(\frac{1}{5}\right) + \left(\frac{4}{5}\right)^6 = \left(\frac{4}{5}\right)^5\left[\frac{6}{5} + \frac{4}{5}\right] = 2\left(\frac{4}{5}\right)^5 = \frac{2048}{3125}$$

$= 0.65536$ **Ans.**

Example 4. *Ten percent of screws produced in a certain factory turn out to be defective. Find the probability that in a sample of 10 screws chosen at random, exactly two will be defective.*

Solution. $P = \frac{1}{10}, \quad q = \frac{9}{10}, \ n = 10, \ r = 2$

$$P\,(r) = {}^nC_r\, P^r q^{n-r}$$

$$P(2) = {}^{10}C_2\left(\frac{1}{10}\right)^2\left(\frac{9}{10}\right)^{10-2} = \frac{10\times 9}{1\times 2}\left(\frac{1}{10}\right)^2\left(\frac{9}{10}\right)^8 = \frac{1}{2}\cdot\left(\frac{9}{10}\right)^9 = 0.1937 \quad \textbf{Ans.}$$

Example 5. *The probability that a man aged 60 will live to be 70 is 0.65. What is the probability that out of 10 men, now 60, at least 7 will live to be 70 ?*

Solution. The probability that a man aged 60 will live to be 70 $= p = 0.65$

$$q = 1 - p = 1 - 0.65 = 0.35$$

Number of men $= n = 10$

Probability that at least 7 men (7 or 8 or 9 or 10) will live to 70

$$= P\,(7) + P\,(8) + P\,(9) + P\,(10) = {}^{10}C_7\, q^3 p^7 + {}^{10}C_8\, q^2 p^8 + {}^{10}C_9\, q\, p^9 + p^{10}$$

$$= \frac{10\times 9\times 8}{1\times 2\times 3}(0.35)^3(0.65)^7 + \frac{10\times 9}{1\times 2}(0.35)^2(0.65)^8 + 10(0.35)(0.65)^9 + (0.65)^{10}$$

$$= (0.65)^7\,[\,120\,(0.35)^3 + 45\,(0.35)^2\,(0.65) + 10\,(0.35)\,(0.65)^2 + (0.65)^3]$$

$$= (0.65)^7 \times 125\,[120 \times (0.07)^3 + 45 \times (0.07)^2\,(0.13) + 10\,(0.07)\,(0.13)^2 + (0.13)^3]$$

$$= 0.04902 \times 125\,[0.04 + 0.028665 + 0.011830 + 0.002197]$$

$$= 6.1275 \times 0.082692 = 0.5067$$

Ans.

Mean and Variance of Binomial Distribution :

We shall first obtain moments of the binomial distribution about r = 0 (about origin).

$$\mu_1' = \sum_{r=0}^{n} r \cdot {}^nC_r\, p^r q^{n-r}$$

$$= 0 \cdot q^n + 1 \cdot {}^nC_1 p^1 q^{n-1} + 2 \cdot {}^nC_2\, p^2 q^{n-2} + 3 \cdot {}^nC_3 p^3 q^{n-3} \ldots np^n$$

$$= npq^{n-1} + 2 \cdot \frac{n(n-1)}{2!}\, p^2 q^{n-2} + 3 \cdot \frac{n(n-1)(n-2)}{2!}\, p^3 q^{n-3} \ldots np^n$$

$$= np \times \left[q^{n-1} + (n-1)\, pq^{n-2} + \frac{(n-1)(n-2)}{2!}\, p^2 q^{n-3} \ldots p^{n-1} \right]$$

$$= np\,[(q+p)^{n-1}] \qquad \boxed{\mu' = np}$$

But $\mu_1' = \mu_1 = \text{Mean}$

Hence Mean of the Binomial distribution or which is also the expectation of variable r is np.

Now consider

$$\mu_2' = \sum_{r=0}^{n} r^2\, {}^nC_r\, p^r q^{n-r}$$

$$= \sum_{r=0}^{n} \{r(r-1) + r\}\, {}^nC_r\, p^r q^{n-r} = \sum_{r=2}^{n} r(r-1)\, {}^nC_r\, p^r q^{n-r} + \sum_{r=0}^{n} r\, {}^nC_r\, p^r q^{n-r}$$

$$= \sum_{r=2}^{n} r(r-1)\, {}^nC_r\, p^r q^{n-r} + np$$

$$= 1.2\,{}^nC_2\,p^2 q^{n-2} + 2.3\,{}^nC_3\,p^3 q^{n-3} + 3.4\,{}^nC_4\,p^4 q^{n-4} \ldots n\,(n-1)\,p^n + np$$

$$= 1.2\,\frac{n\,(n-1)}{2!}\,p^2 q^{n-2} + 2.3\,\frac{n\,(n-1)\,(n-2)}{3!}\,p^3 q^{n-3}$$

$$+ 3.4\,\frac{n\,(n-1)\,(n-2)\,(n-3)}{4!}\,p^4 q^{n-4} \ldots n\,(n-1)\,p^n + np$$

$$= n\,(n-1) \times p^2 \left[q^{n-2} + (n-2)\,p\,q^{n-3} + \frac{(n-2)\,(n-3)}{2!}\,p^2 q^{n-4} \ldots p^{n-2} \right] + np$$

$$= n\,(n-1)\,p^2\,\{q+p\}^{n-2} + np = n\,(n-1)\,p^2 + np$$

$$= n^2 p^2 - np^2 + np = n^2 p^2 + np\,\{1-p\} = n^2 p^2 + npq$$

$$\mu_2 = \text{Variance} = \mu_2' - \mu_1'^2 = n^2 p^2 + npq - (np)^2$$

$$\boxed{\sigma^2 = npq}$$

$$\boxed{\sigma = \text{Standard deviation} = \sqrt{npq}}$$

ILLUSTRATIONS

Ex. 1 : *Point out the fallacy of the statement 'The Mean of Binomial distribution is 3 and variance 5'.*

Sol. : Given Mean $= np = 3$, Variance $= npq = 5$

$$\therefore \quad q = \frac{npq}{np} = \frac{5}{3} > 1$$

which is not possible since probability cannot exceed unity.

Ex. 2 : *The Mean and Variance of Binomial distribution are 6 and 2 respectively. Find $p\,(r \geq 1)$.* **(Dec. 2012)**

Sol. : Here r denotes the number of successes in n trials. Given that

Mean $= np = 6$ and Variance $= npq = 2$

$$\therefore \quad q = \frac{npq}{np} = \frac{2}{6} = \frac{1}{3}$$

$$p = 1 - q = 1 - \frac{1}{3} = \frac{2}{3}$$

$$np = 6 \quad \therefore \; n = \frac{6}{2/3} = 9$$

$$p\,(r \geq 1) = 1 - p\,(r = 0) = 1 - q^n = 1 - \left(\frac{1}{3}\right)^9 = 0.999949$$

6.9 POISSON DISTRIBUTION

When 'p' be the probability of success is very small and n the number of trials is very large and np is finite then we get another distribution called Poisson distribution. It is considered as limiting case of Binomial distribution with $n \to \infty$, $p \to 0$ and np remaining finite.

Consider the Binomial distribution

$$B(n, p, r) = {}^nC_r\, p^r q^{n-r}$$

$$= \frac{n(n-1)(n-2)\ \ldots\ (n-(r-1))}{r!}\, p^r (1-p)^{n-r}$$

Let $z = np \quad \therefore \quad p = \dfrac{z}{n}$

$$B(n, p, r) = \frac{np\,(np-p)\,(np-2p)\ \ldots\ (np-(r-1)\,p)}{r!} \times \frac{(1-p)^n}{(1-p)^r}$$

$$= \frac{z\left(z-\frac{z}{n}\right)\left(z-\frac{2z}{n}\right)\ldots\left[z-(r-1)\frac{z}{n}\right]}{r!} \times \frac{\left(1-\frac{z}{n}\right)^n}{\left(1-\frac{z}{n}\right)^r}$$

Now taking the limit as $n \to \infty$ and $np = z$, $p \to 0$

$$\lim B(n, p, r) = \frac{z^r e^{-z}}{r!} \left[\lim_{n \to \infty} \left(1-\frac{z}{n}\right)^n = e^{-z} \text{ and } \lim_{n \to \infty} \left(1-\frac{z}{n}\right)^r = 1\right]$$

This is called Poisson distribution which may be denoted by p (r).

Thus the probability of r successes in a series of large number of trials n with p the probability of success at each trial, a small number is given by,

$$p(r) = \frac{z^r e^{-z}}{r!}$$

Here Mean of the Poisson distribution is given by

$$\text{Mean} = \text{Lim } np = z$$

$$\text{Variance} = \text{Lim } npq = z \text{ [Lim } q = 1 \text{ as } p \to 0]$$

$$\text{Standard deviation} = \sqrt{z}$$

ILLUSTRATIONS

Ex. 1 : *A manufacturer of cotter pins knows that 2% of his product is defective. If he sells cotter pins in boxes of 100 pins and guarantees that not more than 5 pins will be defective in a box, find the approximate probability that a box will fail to meet the guaranteed quality.* **(May 2010)**

Sol. : Here, n = 100.

p the probability of defective pins $= \frac{2}{100} = 0.02$

$$z = \text{mean number of defective pins in a box}$$

$$z = np = 100 \times 0.02 = 2$$

Since p is small, we can use Poisson distribution.

$$p(r) = \frac{e^{-z} z^r}{r!} = \frac{e^{-2} 2^r}{r!}$$

Probability that a box will fail to meet the guaranteed quality is

$$p(r > 5) = 1 - p(r \le 5)$$

$$= 1 - \sum_{r=0}^{5} \frac{e^{-2} 2^r}{r!} = 1 - e^{-2} \sum_{r=0}^{5} \frac{2^r}{r!} = 0.0165$$

Ex. 2 : *In a certain factory turning out razor blades, there is a small chance of 1/500 for any blade to be defective. The blades are supplied in a packet of 10. Use Poisson distribution to calculate the approximate number of packets containing no defective and two defective blades, in a consignment of 10,000 packets.*

Sol. : Here $p = 0.002$, $n = 10$, $z = np = 0.02$

$$p(\text{no defective}) = p(r = 0) = \frac{e^{-0.02} (0.02)^0}{0!} = \frac{1}{e^{0.02}}$$

$$p(2 \text{ defectives}) = p(r = 2) = \frac{e^{-0.02} (0.02)^2}{2}$$

Number of packets containing no defective blades in a consignment of 10,000 packets

$$= 10{,}000 \times \frac{1}{e^{0.02}} = 9802$$

Number of packets containing 2 defective blades

$$= 10{,}000 \times \frac{(0.02)^2}{2 \times e^{0.02}} = 2$$

Ex. 3 : *In a Poisson distribution if p (r = 1) = 2p (r = 2), find p (r = 3).*

Sol. :

$$p\,(r) = \frac{e^{-z}\,z^r}{r!}$$

$$p\,(r = 1) = \frac{e^{-z}\,z}{1}, \quad p\,(r = 2) = \frac{e^{-z}\,z^2}{2}$$

$$\therefore \quad ze^{-z} = 2 \times \frac{e^{-z}\,z^2}{2} \quad \text{which gives } z = 1$$

$$p\,(r = 3) = \frac{e^{-1}\,(1)}{3!} = e^{-1}\,\frac{1}{6} = \frac{1}{6e} = 0.0613$$

Ex. 4 : *The accidents per shift in a factory are given by the table :*

Accidents x per shift	*0*	*1*	*2*	*3*	*4*	*5*
Frequency f	*142*	*158*	*67*	*27*	*5*	*1*

Fit a Poisson distribution to the above table and calculate theoretical frequencies.

Sol. : z = The mean number of accidents

$$= \frac{0 \times 142 + 1 \times 158 + 2 \times 67 + 3 \times 27 + 4 \times 5 + 5 \times 1}{142 + 158 + 67 + 27 + 5 + 1}$$

$$= \frac{158 + 134 + 81 + 20 + 5}{400} = \frac{398}{400} = 0.995$$

$$p\,(r) = \frac{e^{-0.995}\,(0.995)^r}{r!} \qquad p\,(0) = e^{-0.995} = 0.3697$$

$$p\,(1) = 0.36785 \qquad p\,(2) = 0.813$$

$$p\,(3) = 0.0607 \qquad p\,(4) = 0.0151$$

$$p\,(5) = 0.003$$

Theoretical frequencies are

$$p\,(0) \times 400 = 0.3697 \times 400 = 148$$

$$p\,(1) \times 400 = 0.36785 \times 400 = 147$$

$$p\,(2) \times 400 = 0.183 \times 400 = 73$$

$$p\,(3) \times 400 = 0.0607 \times 400 = 24$$

$$p\,(4) \times 400 = 0.0151 \times 400 = 6$$

$$p\,(5) \times 400 = 0.003 \times 400 = 1$$

Ex. 6 : *The average number of misprints per page of a book is 1.5. Assuming the distribution of number of misprints to be Poisson, find*

(i) the probability that a particular book is free from misprints.

(ii) number of pages containing more than one misprint if the book contains 900 pages.

Sol. : Let X : Number of misprints on a page in the book.

Given : $X \rightarrow P(Z = 1.5)$ $\quad E(r) = Z = 1.5$

Here the p.m.f. is given by,

$$P(r) = \frac{e^{-z} Z^r}{r!}$$

$$P(r) = \frac{e^{-1.5} (1.5)^r}{r!}$$

(i) $$P(r = 0) = \frac{e^{-1.5} (1.5)}{0!} = e^{-1.5} = 0.223130$$ (From statistical tables)

Note : The Poisson probabilities for m = 0.1, 0.2, 0.3 ... 15.0 are given in the statistical tables.

(ii) $$P[r > 1] = 1 - P[r \leq 1]$$

$$= 1 - \{P(R = 0) + P(r = 1)\}$$

$$= 1 - \left\{e^{-1.5} + \frac{e^{-1.5}(1.5)^1}{1!}\right\}$$

$$= 1 - \{0.223130 + 0.334695\} \quad \left[\begin{array}{l}\text{From statistical}\\ \text{tables}\end{array}\right]$$

$$= 0.442175$$

$\therefore$ Number of pages in the book containing more than one misprint.

$$= (900)\,P[r > 1] = (900)\ (0.442175)$$

$$= 397.9575 \approx 398$$

Ex. 7 : *Number of road accidents on a high way during a month follows a Poisson stribution with mean 5. Find the probability that in a certain month number of cidents on the highway will be*

(i) Less than 3

(ii) Between 3 and 5

(iii) More than 3.

Sol. : Let X : number of road accidents on a highway during a month.

Given : $X \rightarrow P(Z = 5)$

$\therefore$ The p.m.f is given by,

$$P[r] = \frac{e^{-z} Z^{-r}}{r!} \ ; \ r = 0, 1, 2 \ldots\ldots$$

$$P[r] = \frac{e^{-5} 5^r}{r!}$$

(i) $$P[r < 3] = P[r \leq 2] = P[r = 0] + P[r = 1] + P[r = 2]$$

$$= \frac{e^{-5} 5^0}{0!} + \frac{e^{-5} 5^1}{1!} + \frac{e^{-5} 5^2}{2!}$$

$$= 0.006738 + 0.033690 + 0.084224$$

(From statistical tables)

$$= 0.124652$$

(ii) $$P[3 \leq r \leq] = P(3) + P(4) + P(5)$$

$$= 0.140374 + 0.175467 + 0.175467$$

$$= 0.491308$$

(iii) $$P[r > 3] = 1 - P[r \leq 3]$$

$$= 1 - [P(0) + P(1) + P(2) + P(3)]$$

$$= 0.734974$$

6.10 NORMAL DISTRIBUTION

Normal distribution is obtained as a limiting form of Binomial distribution when n the number of trials is very large and neither p nor q is very small. Most of the modern statistical methods have been based on this distribution.

Normal distribution curve is given by the equation

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \qquad \dots (1)$$

Its shape is as shown in Fig. 6.9.

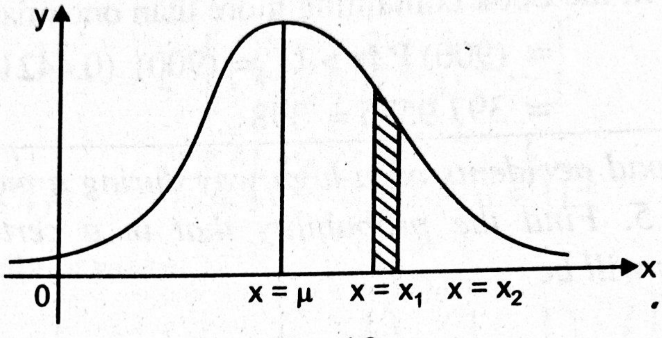


Fig. 6.9

The area under the curve from $x = x_1$ to $x = x_2$ gives the probability of the variable x lying between the values $x = x_1$ and $x = x_2$. The total area under the curve (which is symmetrical about $x = \mu$) is given by $\int_{-\infty}^{\infty} y\, dx = 1$.

Numbers μ and σ occurring in equation (1) are respectively the mean and the standard deviation of the distribution.

If the origin is shifted to $(\mu, 0)$, the equation of curve becomes

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2} \qquad \dots (2)$$

We shall now obtain Normal distribution as a limiting case of Binomial distribution as $n \to \infty$.

The probability function of the Binomial distribution with parameters n and p is given by

$$p(x) = {}^nC_x\, p^x q^{n-x} = \frac{n!}{x!\,(n-x)!} p^x q^{n-x}, \quad x = 0, 1, 2, \dots n \qquad \dots (3)$$

Let us now consider the standard Binomial variate.

$$z = \frac{x - np}{\sqrt{npq}}\ ; \qquad x = 0, 1, 2 \dots n \qquad \dots (4)$$

When $\quad x = 0, \quad z = \frac{-np}{\sqrt{npq}} = \sqrt{\frac{np}{q}}$

and when $\quad x = n, \quad z = \frac{n - np}{\sqrt{npq}} = \sqrt{\frac{nq}{p}}$

Thus in the limit as $n \to \infty$, z takes the values from $-\infty$ to ∞.

Hence the distribution of x will be a continuous distribution over the range $-\infty$ to ∞.

Using Stirling's approximation to n ! for large n i.e.

$$\lim_{n \to \infty} n! = \sqrt{2\pi}\, e^{-n}\, n^{n+1/2}$$

Now considering $\lim n \to \infty$ and hence $x \to \infty$

$$\lim p(x) = \lim \left[\frac{\sqrt{2\pi}\, e^{-n}\, n^{n+1/2}\, p^x\, q^{n-x}}{\sqrt{2\pi}\, e^{-x}\, x^{x+1/2} \sqrt{2\pi}\, e^{-(n-x)}\, (n-x)^{n-x+1/2}}\right]$$

$$= \lim \left[\frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{npq}}\, \frac{(np)^{x+1/2}\,(nq)^{n-x+1/2}}{x^{x+1/2}\,(n-x)^{n-x+1/2}}\right]$$

$$= \lim \left[\frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{npq}} \cdot \left(\frac{np}{x}\right)^{x+1/2} \left(\frac{nq}{n-x}\right)^{n-x+1/2}\right] \quad \ldots (5)$$

From (4), we have $\quad x = np + z\sqrt{npq}$

$$\therefore \quad \frac{x}{np} = 1 + z\sqrt{\frac{q}{np}}$$

Also $\quad n - x = n - np - z\sqrt{npq} = nq - z\sqrt{npq}$

$$\therefore \quad \frac{n-x}{nq} = 1 - z\sqrt{\frac{p}{nq}}$$

Also $\quad dz = \dfrac{1}{\sqrt{npq}}\, dx$

Hence the probability differential of the distribution of z in the limit is given from (5) by

$$d\,G(z) = g(z)\, dz = \lim_{n \to \infty} \left[\frac{1}{\sqrt{2\pi}} \times \frac{1}{N}\right] dz \quad \ldots (6)$$

where $\quad N = \left[\dfrac{x}{np}\right]^{x+1/2} \left[\dfrac{n-x}{nq}\right]^{n-x+1/2}$

Taking log on both the sides,

$$\log N = \left(x + \frac{1}{2}\right) \log(x/np) + \left(n - x + \frac{1}{2}\right) \log\left(\frac{n-x}{nq}\right)$$

$$= \left(np + z\sqrt{npq} + \frac{1}{2}\right) \log\left[1 + z\sqrt{q/np}\right] + \left(nq - z\sqrt{npq}\ \frac{1}{2}\right) \log\left[1 - z\sqrt{p/nq}\right]$$

$$= \left(np + z\sqrt{npq} + \frac{1}{2}\right)\left[z\sqrt{\frac{q}{np}} - \frac{1}{2} z^2\left(\frac{q}{np}\right) + \frac{1}{3} z^2 \left(\frac{q}{np}\right)^{3/2} \ldots + \ldots\right]$$

$$+\left(nq - z\sqrt{npq} + \frac{1}{2}\right)\left[-z\sqrt{\frac{p}{nq}} - \frac{1}{2} z^2\left(\frac{p}{nq}\right) - \frac{1}{3} z^3 \left(\frac{p}{nq}\right)^{3/2} \ldots + \ldots\right]$$

$$= \left\{ z\sqrt{npq} - \frac{1}{2} qz^2 + \frac{1}{3} z^3 \frac{q^{3/2}}{\sqrt{np}} + z^2 q - \frac{1}{2} z^3 \frac{q^{3/2}}{\sqrt{np}} + \frac{1}{2} z \sqrt{\frac{q}{np}} \right.$$

$$\left. - \frac{1}{4} z^2 \frac{q}{np} + \frac{+1}{6} z^3 \left(\frac{q}{np}\right)^{3/2} \ldots \right\}$$

$$+\left[-z\sqrt{npq}-\frac{1}{2}z^2p-\frac{1}{3}z^3\frac{p^{3/2}}{\sqrt{nq}}+z^2p+\frac{1}{2}z^3\frac{p^{3/2}}{\sqrt{nq}}-\frac{1}{2}z\sqrt{\frac{p}{nq}}-\frac{1}{4}z^2\left(\frac{p}{nq}\right)-\frac{1}{6}z^3\left(\frac{p}{nq}\right)^{3/2}\ldots\right]$$

(terms higher than fourth power of z are neglected.)

Now, we can write

$$\log N = \left[\frac{z}{2\sqrt{n}}\left\{\sqrt{\frac{q}{p}}-\sqrt{\frac{p}{q}}\right\}-\frac{1}{2}z(p+q)+z^2(p+q)\right]$$

+ terms containing power of n in denominator

Taking the limit as $n \to \infty$

$$\lim_{n\to\infty}\log N = \frac{z^2}{2} \text{ or } \lim_{n\to\infty} N = e^{-z^2/2}$$

Putting in (6), we get $dG(z) = g(z)\,dz = \frac{1}{\sqrt{2\pi}}e^{-z^2/2}\,dz$ for $-\infty < z < \infty$.

This is the probability density function of the normal distribution with mean 0 and unit variance.

If X is a normal variate with mean μ and s.d. σ then $Z = (X-\mu)/\sigma$ is a standard normal variate. The probability density function of a normal variate with mean μ and variance σ^2 is given by

$$p(X) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}, \quad -\infty < X < \infty$$

6.10.1 Mean Deviation from the Mean

$$\text{M.D.} = \int_{-\infty}^{\infty}|x-\mu|\,p(x)\,dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}|x-\mu|\,e^{-(x-\mu)^2/2\sigma^2}\,dx$$

Put $\frac{x-\mu}{\sigma} = z$; $dx = \sigma\,dz$ $= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}\sigma^2|z|\,e^{-z^2/2}\,dz = \frac{\sigma}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|z|\,e^{-z^2/2}\,dz$

$$= \frac{\sigma}{\sqrt{2\pi}}\cdot 2\int_0^{\infty}|z|\,e^{-z^2/2}\,dz$$

[$f(z) = |z| = e^{-z^2/2}$ is even function of z]

$$\text{M.D.} = \sqrt{\frac{2}{\pi}}\,\sigma\int_0^{\infty}ze^{-z^2/2}\,dz \quad [|z| = z \text{ for } 0 < z < \infty]$$

$$= \sqrt{\frac{2}{\pi}}\,\sigma\int_0^{\infty}\frac{1}{2}e^{-z^2/2}\,d(z^2) = \sqrt{\frac{2}{\pi}}\cdot\frac{\sigma}{2}\left[\frac{e^{-z^2/2}}{-1.2}\right]_0^{\infty}$$

$$= \sqrt{\frac{2}{\pi}}\,\sigma[1] = \frac{4}{5}\sigma \text{ (approximately)}$$

10.2 Area Property (Normal Probability Integral)

The probability of random value x lying between $x = \mu$ and $x = x_1$ is given by

Table 6.1

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5259
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7793	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8906	0.8925	0.8943	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9494	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9871	0.9678	0.9686	0.9693	0.9699	0.9708
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9783	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9634	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.98686	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9809	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9988	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9998	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9898

0.9306
−0.5
0.4306

In each row and each column 0.5 to be subtracted.

$$P(\mu < x < x_1) = \int_{\mu}^{x_1} P(x)\,dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{\mu}^{x_1} e^{-(x-\mu)^2/2\sigma^2}\,dx$$

Put $\frac{x-\mu}{\sigma} = z$ i.e. $x - \mu = \sigma z$

when $x = \mu$, $z = 0$

$x = x_1$, $z = z_1$ (say)

$$P(\mu < x < x_1) = P(0 < z < z_1)$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_0^{z_1} e^{-z^2/2} \cdot \sigma\,dz = \frac{1}{\sqrt{2\pi}} \int_0^{z_1} e^{-z^2/2} \cdot dz = \int_0^{z_1} f(z)\,dz$$

The definite integral $\int_0^{z_1} f(z)$ is known as normal probability integral and gives the area under standard normal curve between $z = 0$ and $z = z_1$.

ILLUSTRATIONS

Ex. 1 : *The mean weight of 500 students is 63 kgs and the standard deviation is 8 kgs. Assuming that the weights are normally distributed, find how many students weigh 52 kgs ? The weights are recorded to the nearest kg.*

Sol. : The frequency curve for the given distribution is

$$y = \frac{500}{8\sqrt{2\pi}} \text{Exp}\left(-\frac{1}{2}\left(\frac{x-63}{8}\right)^2\right) \quad \ldots (1)$$

Since the weights are recorded to the nearest kg, the students weighing 52 kgs have their actual weights between $x = 51.5$ and 52.5 kg. So the area under the curve (1) from $x = 51.5$ to $x = 52.5$ is to be obtained.

$$z = \frac{x-\mu}{\sigma} \Rightarrow z_1 = \frac{51.5-63}{8} = -1.4375 = -1.44 \text{ (appx)}$$

$$z_2 = \frac{52.5-63}{8} = -1.3125 = -1.31 \text{ (appx)}$$

The number of students weighing 52 kg

$$= 500 \int_{51.5}^{52.5} p(x)\,dx = \frac{500}{\sqrt{2\pi}} \int_{-1.4375}^{-1.3125} e^{-z^2/2}\,dz$$

$$= 500\,(A_1 - A_2)$$

$$= 500\,(0.4251 - 0.4049) = 10 \text{ students approximately.}$$

where, $A_1 = 0.4251$ is the area for $z_1 = 1.44$,

and $A_2 = 0.4049$ is the area for $z_2 = 1.31$.

Ex. 2 : *For a normal distribution when mean $\overline{x}$ = 1, S.D. = 3, find the probabilities for the intervals :*

(i) $3.43 \le x \le 6.19$; (ii) $-1.43 \le x \le 6.19$

Sol. : (i) $z_1 = \dfrac{3.43-1}{3} = 0.81, \quad z_2 = \dfrac{6.19-1}{3} = 1.73$

Required probability $= A_1 - A_2$

where, A_1 is area corresponding to $z_1 = 1.73$

A_2 is area corresponding to $z_2 = 0.81$

$= (0.4582 - 0.2910) = 0.1672$

(ii) $z_1 = \dfrac{-1.43-1}{3} = -0.81, \quad z_2 = \dfrac{6.19-1}{3} = 1.73$

Required probability $= A_1 + A_2 = 0.2910 + 0.4582 = 0.7492$

Ex. 3 : *Assuming that the diameters of 1000 brass plugs taken consecutively from machine form a normal distribution with mean 0.7515 cm and standard deviation 0.0020 cm. How many of the plugs are likely to be approved if the acceptable diameter is 0.752 ± 0.004 cm ?*

Sol. :

$$\sigma = 0.0020, \quad \mu = 0.7515$$

$$x_1 = 0.752 + 0.004 = 0.756$$

$$x_2 = 0.752 - 0.004 = 0.748$$

$$z_1 = \frac{x_1-\mu}{\sigma} = \frac{0.756-0.7515}{0.0020} = 2.25$$

$$z_2 = \frac{x_2-\mu}{\sigma} = \frac{0.748-0.7515}{0.0020} = -1.75$$

A_1 corresponding to z_1 = 2.25 (Refer table 6.1)

= 0.4878

A_2 corresponding to $z_2 = 1.75 = 0.4599$

$p(0.748 < x < 0.756) = 0.4878 + 0.4599 = 0.9477$

Number of plugs likely to be approved $= 1000 \times 0.9477 = 948$ approximately.

Ex. 4 : *In a certain examination test, 2000 students appeared in a subject of statistics. Average marks obtained were 50% with standard deviation 5%. How many students do you expect to obtain more than 60% of marks, supposing that marks are distributed normally ?* **(May 2010, Dec. 2012)**

Sol. :

$$\mu = 0.5, \quad \sigma = 0.05$$

$$x_1 = 0.6, \quad z_1 = \frac{0.6-0.5}{0.05} = 2$$

A corresponding to z = 2 is 0.4772

$p(x \ge 6) = 0.5 - 0.4772 = 0.0228$

Number of students expected to get more than 60% marks

$= 0.0228 \times 200 = 46$ students approximately.

Ex. 5 : *In a distribution, exactly normal, 7% of the items are under 35 and 89% a... under 63. Find the mean and standard deviation of the distribution.*

Sol. : From Fig. 6.10, it is clear that 7% of items are under 35 means area under 35 ... 0.07. Similarly area for $x > 63$ is 0.11.

$$p(x < 35) = 0.07 \text{ and } p(x > 63) = 0.11$$

$x = 35$, $x = 63$ are located as shown in Fig. 6.10.

When $x = 35$, $z = \dfrac{35 - \mu}{\sigma} = -z_1$ (say), (– ve sign because $x = 35$ to the left of $x = \mu$)

When $x = 63$, $z = \dfrac{63 - \mu}{\sigma} = z_2$ (say), (+ve sign for $x = 63$ lies to the right of $x = \mu$)

$\therefore$ From Table 6.1, we get

Area $A_1 = p(0 < z < z_1) = 0.43$ corresponds to $z_1 = 1.48$ (appx)

& Area $A_2 = p(0 < z < z_2) = 0.39$ corresponds to $z_2 = 1.23$ (appx)

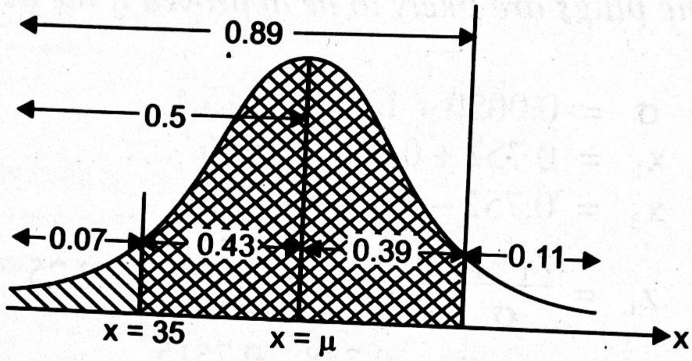


Fig. 6.10

Thus, we get two simultaneous equations

$$\frac{35 - \mu}{\sigma} = -z_1 = -1.48 \quad \ldots (1)$$

$$\frac{63 - \mu}{\sigma} = z_2 = 1.23 \quad \ldots (2)$$

Subtracting (1) from (2),

$$\frac{28}{\sigma} = 2.71 \Rightarrow \sigma = 10.33 \text{ (approximately)}$$

and (2) $\Rightarrow$

$$\mu = 63 - \sigma \times 1.23$$

$$= 63 - 10.33 \times 1.23 = 50.3 \text{ (approximately)}$$

Ex. 6 : *Let $x \to N(4, 16)$. Find (i) $P(x > 5)$, (ii) $P(x < 2)$, (iii) $P(x > 0)$, (iv) $P(6 < x < 8)$, (v) $P(|x| > 6)$.*

Sol. : Let $x \to N(4, 16) = N(\mu, \sigma^2)$, hence $\mu = 4$, and $\sigma^2 = 16 \Rightarrow \sigma = 4$.

(i) $$P(x > 5) = P\left(z = \frac{x-\mu}{\sigma} > \frac{5-4}{4}\right)$$

$$= P(z > 1/4)$$

$\therefore$ From normal probability integral table we get area of shaded region as,

$$p(z > 1/4) = 0.40129$$

(ii) $$p(x < 2) = P\left(\frac{x-\mu}{\sigma} > \frac{2-4}{4}\right)$$

$$= P\left(z < \frac{-2}{4}\right)$$

$$= P(z < -0.5)$$

$$= P(z > 0.5) \quad \text{(Due to symmetry)}$$

$$= 0.30854 \quad \text{(From the table)}$$

(iii) $$p(x > 0) = P\left(\frac{x-\mu}{\sigma} > \frac{0-4}{4}\right)$$

$$= P(z > -1) = B$$

Since, only tail area is given in the table we use the fact that $A + B = 1$.

$\therefore$ $$p(z > -1) = 1 - A = 1 - p(z < -1)$$

$$= 1 - p(z > 1) \quad \text{(Due to symmetry)}$$

$$= 1 - 0.15866$$

$$= 0.84134$$

(iv) $$p(6 < x < 8) = P\left(\frac{6-4}{4} < \frac{x-\mu}{\sigma} < \frac{8-4}{\sigma}\right)$$

$$= P\left(\frac{2}{4} < z < 1\right)$$

$$= P(0.5 < z < 1) = A$$

$$= (A + B) - B$$

$$= P(z > 0.3) - P(z > 1)$$

$$= 0.30854 - 0.15866$$

$$= 0.14988$$

(v) $$p(|x| > 6) = p(x > 6) + P(x < -6)$$

$$= p\left(\frac{x-\mu}{6} > \frac{6-4}{4}\right) + P\left(\frac{x-\mu}{\sigma} > \frac{-6-4}{4}\right)$$

$$= p(z > 0.5) + p(z < -2.5)$$

$$= p(z > 0.5) + p(z > 2.5)$$

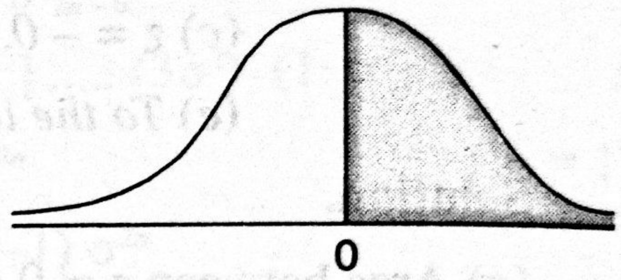
$$= 0.30854 + 0.0062097)$$

$$= 0.31475$$

Example 10. *Students of a class were given an aptitude test Their marks were found to be normally distributed with mean 60 and standard deviation 5. What percentage of students scored more than 60 marks ?*

Solution. $x = 60,\ \bar{x} = 60,\ \sigma = 5$

$$z = \frac{x - \bar{x}}{\sigma} = \frac{60 - 60}{5} = 0$$



if $x > 60$ then $z > 0$

Area lying to the right of $z = 0$ is 0.5.

The percentage of students getting more than 60 marks = 50 % **Ans.**

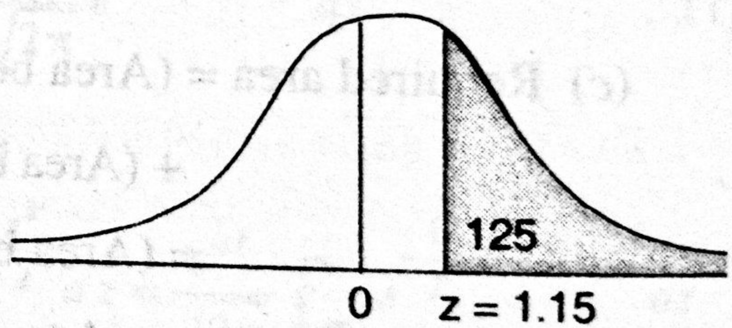
Example 11. *Assume mean height of soldiers to be 68.22 inches with a variance of 10.8 inches square. How many soldiers in a regiment of 1,000 would you expect to be over 6 feet tall, given that the area under the standard normal curve between $x = 0$ and $x = 0.35$ is 0.1368 and between $x = 0$ and $x = 1.15$ is 0.3746.*

(U.P. III Semester Dec. 2001)

Solution. Mean $= \bar{x} = 68.22$ inch

variance $= \sigma^2 = 10.8$ inches squares

If $x = 72$ inches then $z = \frac{x - \mu}{\sigma} = \frac{72 - 68.22}{\sqrt{10.8}} = 1.15$

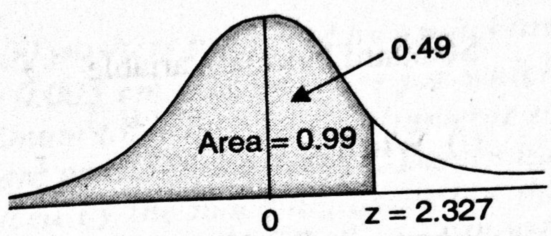


$$P(x > 72) = P(z > 1.15)$$

$$= 0.5 - P(0 \le z \le 1.15) = 0.5 - 0.3746 = 0.1254$$

Number of soldiers $= 1000 \times 0.1254 = 125.4 \approx 125$ (app.) **Ans.**

$$z = \frac{x-\mu}{\sigma} \quad \Rightarrow \quad z = \frac{x-64.5}{3.3} \qquad \ldots (1)$$



Area between 0 and $z = 0.99 - 0.5 = 0.49$

From the table, z for area 0.49 is 2.327.

Putting the value of z in (1), we get

$$\Rightarrow \quad \frac{x-64.5}{3.3} = 2.327 \quad \Rightarrow \quad x - 64.5 = 3.3 \times 2.327$$

$$x - 64.5 = 7.68$$

$$\Rightarrow \qquad x = 7.68 + 64.5 = 72.18 \text{ inches}$$

Ans.

Hence 99% students are of height less than 72.18 inches.

Example 12. *A sample of 100 dry battery cells tested to find the length of life produced the following results:*

$\bar{x}$ = 12 hours, σ = 3 hours

Assuming the data to be normally distributed, what percentage of battery cells are expected to have life

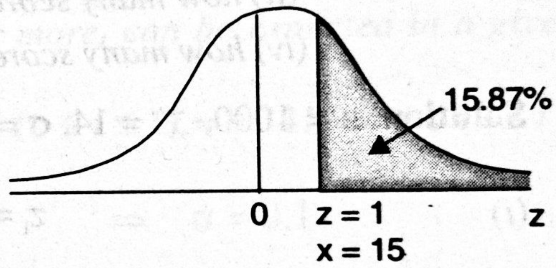
(i) more than 15 hours   *(ii) less than 6 hours*

(iii) between 10 and 14 hours ?   *(U.P. III Semester Dec. 2003)*

Solution. Here, Mean = $\bar{x}$ = 12 hours

and Standard deviation = σ = 3 hours

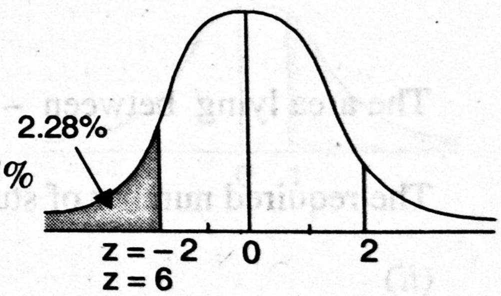
x denotes the length of life of dry battery cells.



$$z = \frac{x-\bar{x}}{\sigma}$$

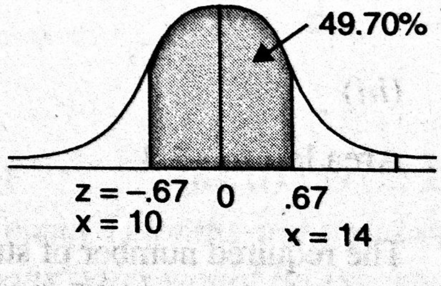
(*i*) When $x = 15$, then $\qquad z = \frac{15-12}{3} = 1$

$$\therefore \qquad P(x > 15) = P(z > 1)$$
$$= P(0 < z < \infty) - P(0 < z < 1)$$
$$= 0.5 - 0.3413 = 0.1587 = 15.87\%$$



(*ii*) When x = 6, then $\qquad z = \frac{6-12}{3} = \frac{-6}{3} = -2$

$$P(x < 6) = P(z < -2)$$
$$= P(z > 2) = 0.5 - P(0 < z < 2)$$
$$= 0.5 - 0.4772 = 0.0228 = 2.28\%$$



(*iii*) When $x = 10$, then $\qquad z = \frac{10-12}{3} = \frac{-2}{3} = -0.67$

When $x = 14$, then $\qquad z = \frac{14-12}{3} = \frac{2}{3} = 0.67$

$$P(10 < x < 14) = P(-0.67 < z < 0.67)$$
$$= 2\,P(0 < z < 0.67) = 2 \times 0.2485 = 0.4970 = 49.70\%$$

Ans.

= 0.1587

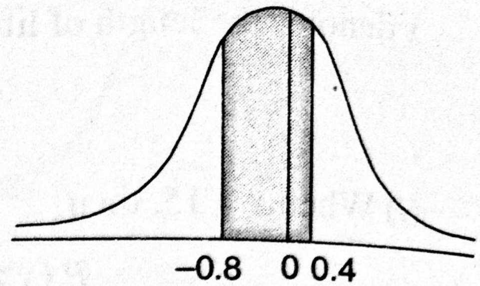
Example 14. *In a sample of 1000 cases, the mean of a certain test is 14 and standard deviation is 2.5. Assuming the distribution to be normal, find*

(i) how many students score between 12 and 15 ?

(ii) how many score above 18 ? *(iii) how many score below 8 ?*

(iv) how many score 16 ?

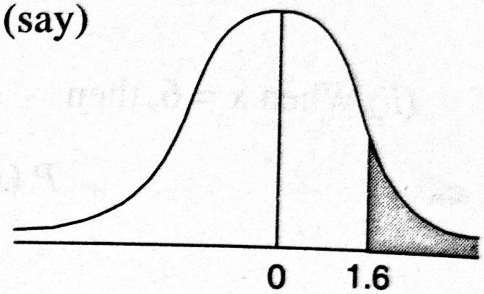
Solution. $n = 1000, \ \bar{x} = 14, \sigma = 2.5$



(*i*)
$$z_1 = \frac{x - \bar{x}}{\sigma} = \frac{12-14}{2.5} = -0.8$$
$$z_2 = \frac{15-14}{2.5} = \frac{1}{2.5} = 0.4$$

The area lying between -0.8 to 0.4 = Area from 0 to -0.8 + area from 0 to 0.4
$$= 0.2881 + 0.1554 = 0.4435$$

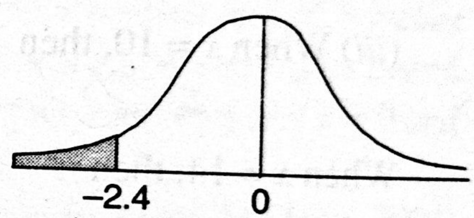
The required number of students = $1000 \times 0.4435 = 443.5 = 444$ (say)



(*ii*)
$$z_1 = \frac{18-14}{2.5} = \frac{4}{2.5} = 1.6$$

Area right to 1.6
$$= 0.5 - \text{Area between 0 and 1.6}$$
$$= 0.5 - 0.4452 = 0.0548$$

The required number of students
$$= 1000 \times 0.0548 = 54.8 = 55 \text{ (say)}$$

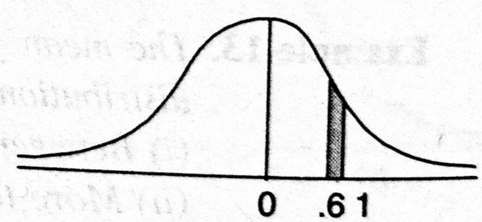


(*iii*)
$$z = \frac{8-14}{2.5} = -\frac{6}{2.5} = -2.4$$

Area left to -2.4
$$= 0.5 - \text{area between 0 and } -2.4$$
$$= 0.5 - 0.4918 = 0.0082$$

The required number of students = $1000 \times 0.0082 = 8.2 = 8$ (say)

(*iv*) Area between 15.5 and 16.5



$$z_1 = \frac{15.5-14}{2.5} = 0.6$$
$$z_2 = \frac{16.5-14}{2.5} = 1$$

Area between 0.6 and 1 $= 0.3413 - 0.2257 = 0.1156$

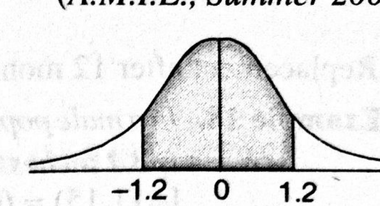
The required number of students = $0.1156 \times 1000 = 115.6 = 116$ say

Ans.

Example 15. *The mean inside diameter of a sample of 200 washers produced by a machine is 0.502 cm and the standard deviation is 0.005 cm. The purpose for which these washers are intended allows a maximum tolerance in the diameter of 0.496 to 0.508 cm, otherwise the washers are considered defective. Determine the percentage of defective washers produced by the machine, assuming the diameters are normally distributed* (*A.M.I.E., Summer 2001*)

Solution. $$z_1 = \frac{x-\bar{x}}{\sigma} = \frac{0.496-0.502}{0.005} = -1.2$$

$$z_2 = \frac{x-\bar{x}}{\sigma} = \frac{0.508-0.502}{0.005} = +1.2$$



Area for non-defective washers = Area between $z = -1.2$ and $z = +1.2$

$= 2$ Area between $z = 0$ and $z = 1.2.$

$= 2 \times (0.3849) = 0.7698 = 76.98\%$

Percentage of defective washers = $100 - 76.98$

$= 23.02\%$ **Ans.**

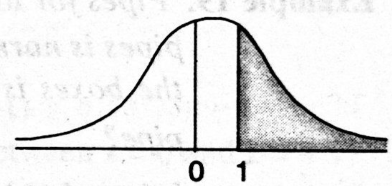
Example 16. *A manufacturer of envelopes knows that the weight of the envelopes is normally distributed with mean 1.9 gm and variance 0.01 gm. Find how many envelopes weighing (i) 2 gm or more, (ii) 2.1 gm or more, can be expected in a given packet of1000 envelopes.*

[Given : if t is the normal variable, then $\phi\,(o \le t \le 1) = 0.3413$ and $\phi\,(0 \le t \le 2) = 0.4772$]

Solution. $\mu = 1.9$ gm, Variance $= 0.01$ gm $\Rightarrow \sigma = 0.1$

(*i*) $x = 2$ gms or more

$$z = \frac{x-\mu}{\sigma} = \frac{2-1.9}{0.1} = \frac{0.1}{0.1} = 1$$



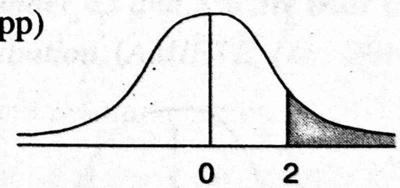
$P\,(z > 1) =$ Area right to $z = 1$

$= 0.5 - 0.3413 = 0.1587$

Number of envelopes heavier than 2 gm in a lot of 1000

$= 1000 \times 0.1587 = 158.7 = 159$ (app)

(*ii*) $$z = \frac{2.1-1.9}{0.1} = \frac{0.2}{0.1} = 2$$



$P\,(z > 2) =$ Area right to $z = 2$

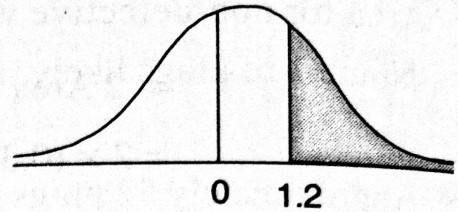
$= 0.5 - 0.4772 = 0.0228$

Number of envelopes heavier than 2.1 gm in a lot of 1000

$= 1000 \times 0.0228 = 22.8 = 23$ (app) **Ans.** (*i*) 159 (*ii*) 23

Example 18. *In a male population of 1000, the mean height is 68.16 inches and standard deviation is 3.2 inches. How many men may be more than 6 feet (72 inches) ?*
[ϕ *(1.15) = 0.8749,* ϕ (1.2) = 0.8849, ϕ (1.25) = *0.8944*]
where the argument is the standard normal variable.

Solution. Male population = 1000
Mean height = 68.16 inches
Standard deviation = 3.2 inches



Men more than 72 inches = ?

ϕ (1.15) = 0.8749, ϕ (1.2) = 0.8849

ϕ (1.25) = 0.8944

$$z = \frac{x - \bar{x}}{\sigma} = \frac{72 - 68.16}{3.2} = 1.2$$

ϕ (1.2) = 0.8949

ϕ for more than 1.2 = 1 – 0.8849 = 0.1151

Men more than 72 inches = $1000 \times 0.1151 = 115.1 = 115$ (say) **Ans.**

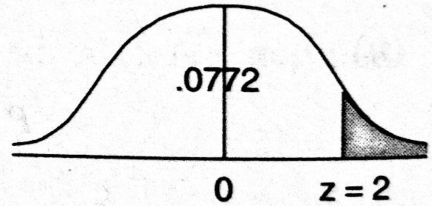
Example 19. *Pipes for tobacco are being packed in fancy plastic boxes. The length of the pipes is normally distributed with* $\mu = 5''$ *and* $a = 0.1''$*. The internal length of the boxes is 5.2". What is the probability that the box would be small for the pipe?*

[*given that* ϕ *(1.8)* = *0.9641,* ϕ*(2)* = *0.9772,* ϕ (2.5) = 0.9938]

Solution. $\mu = 5''$, $\sigma = 0.1''$, $x = 5.2''$

ϕ (1.8) = 0.9641, ϕ (2) = 0.9722, ϕ (2.5) = 0.9938

$$z = \frac{x - \mu}{\sigma} = \frac{5.2 - 5}{0.1} = 2$$



ϕ (2) = 0.9772

ϕ (z > 2) = 1– 0.9772 = 0.0228

The box will be small if the length of the pipe is more than 5.2" ($z = 2$).
Hence the probability is 0.0228

Ans.

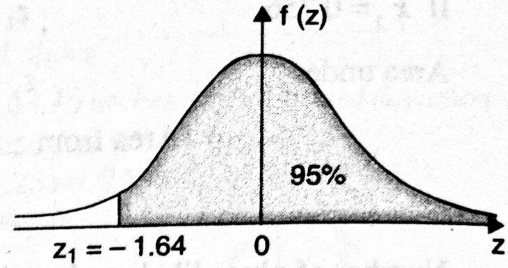
Example 23. *The income of a group of 10,000 persons was found to be normally distributed with mean Rs. 750 p.m. and standard deviation of Rs. 50. Show that, of this group, about 95% had income exceeding Rs. 668 and only 5% had income exceeding Rs. 832. Also find the lowest income among the richest 100.*

(U.P. III Semester Dec. 2004)

Solution. Mean $= \mu = 750$

Standard deviation $= \sigma = 50$

and $z = \dfrac{x-\mu}{\sigma}$

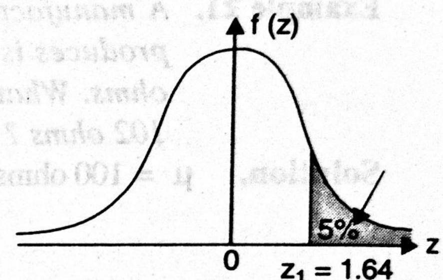


(i) If $x_1 = 668$, then $z_1 = \dfrac{668-750}{50} = -1.64$

$P(x_1 > 668) = P(z_1 < -1.64)$

$= 0.5 + P(-1.64 \le z \le 0) = 0.5 + P(0 \le z \le 1.64) = 0.5 + 0.4495 = 0.9495$

$\therefore$ Percentage of persons having income exceeding Rs. 668 = 94.95% $\approx$ 95% (aaprox.)



(ii) If $x = 832$, then $z = \dfrac{832-750}{50} = 1.64$

$P(x_2 > 832) = P(z_2 > 1.64)$
$= 0.5 - 0.4495$
$= 0.0505$

$\therefore$ Percentage of persons having income exceeding Rs. 832 = 5.05% = 5% (approx.)

(iii) Let x be the lowest income among the richest 100 persons.

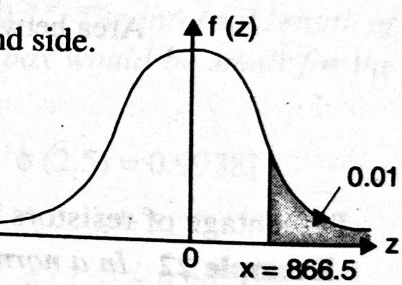
100 persons = 1% of 10,000

100 persons represents 1% area under the curve on the right hand side.

Thus the area between 0 and z

$= 0.5 - 0.01 = 0.49$

From the table z for area 0.49 is 2.33



$z = \dfrac{x-\mu}{\sigma}$

$\Rightarrow \quad 2.33 = \dfrac{x-750}{50} \quad \Rightarrow \quad x - 750 = 50 \times 2.33$

$\Rightarrow \quad x - 750 = 116.5 \quad \Rightarrow \quad x = 866.5$

Hence, the minimum income among the 100 richest persons is equal to Rs. 866.5. **Ans.**

Example 24. *Fit a normal curve to the following data :*

Length of line (in cm)	8.60	8.59	8.58	8.57	8.56	8.55	8.54	8.53	8.52
Frequency	2	3	4	9	10	8	4	1	1

Solution. Let assumed mean = 8.56 cm

x_i	f_i	$x_i - 8.56$	$f_i(x_i - 8.56)$	$f_i(x_i - 8.56)^2$
8.60	2	.04	.08	.0032
8.59	3	.03	.09	.0027
8.58	4	.02	.08	.0016
8.57	9	.01	.09	.0009

8.56	10	0	0	0
8.55	8	− .01	− .08	.0008
8.54	4	− .02	− .08	.0016
8.53	1	− .03	− .03	.0009
8.52	1	− .04	− .04	.0016
	$\Sigma f_i = 42$		$\Sigma f_i (x_i - 8.56) = 0.11$	$\Sigma f_i (x_i - 8.56)^2 = 0.0133$

$$\text{Mean} = a + \frac{\Sigma f_i (x_i - 8.56)}{\Sigma f_i} = 8.56 + \frac{0.11}{42} = 8.56 + 0.00262 = 8.56262 \quad \textbf{Ans.}$$

$$\text{Standard deviation} = \sqrt{\frac{\Sigma f_i (x_i - 8.56)^2}{\Sigma f_i} - \left(\frac{\Sigma f_i (x_i - 8.56)}{\Sigma f_i}\right)^2} = \sqrt{\frac{0.0133}{42} - \left(\frac{0.11}{42}\right)^2}$$

$$= \sqrt{0.000316666 - 0.000006859} = \sqrt{0.00030980} = 0.0176$$

Hence, the equation of the normal curve fitted to the given data is

$$P(x) = \frac{N}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma}}$$

where, $\mu = 8.56262$

$\sigma = 0.0176$ and $N = 42$ **Ans.**

EXERCISE 65.2

1. In a regiment of 1000, the mean height of the soldiers is 68.12 units and the standard deviation is 3.374 units. Assuming a normal distribution, how many soldiers could be expected to be more than 72 units? It is given that
   $P(z = 1.00) = 0.3413$, $P(z = 1.15) = 0.3749$ and
   $P(z = 1.25) = 0.3944$, where z is the standard normal variable. **Ans. 125**

2. The lifetime of radio tubes manufactured in a factory is known to have an average value of 10 years. Find the probability that the lifetime of a tube taken randomly (*i*) exceeds 15 years, (*ii*) is less than 5 years, assuming that the exponential probability law is followed. **Ans. (*i*) 0.2231, (*ii*) 0.3935.**

3. The breaking strength X of a cotton fabric is normally distributed with $E(x) = 16$ and $\sigma(x) = 1$. The fabric is said to be good if $X \geq 14$. What is the probability that a fabric chosen at random is good. Given that $\phi(2) = 0.9772$ **Ans. 0.9772**

4. A manufacturer knows from experience that the resistance of resistors he produces is normal with mean it $\mu = 140\ \Omega$ and standard deviation $\sigma = 5\Omega$. Find the percentage of the resistors that will have resistance between $138\ \Omega$ and $142\ \Omega$. (given $\phi(0.4)$ 0.6554, where z is the standard normal variate). **Ans. 31.08%**

5. A manufacturing company packs pencils in fancy plastic boxes. The length of the pencils is normally distributed with $\mu = 6''$ and $\sigma = 0.2''$. The internal length of the boxes is 6.4". What is the probability that the box would be too small for the pencils (Given that a value of the standardized normal distribution function is $\phi(2) = 0.9772$). **Ans. 0.0228.**

6. A manufacturer produces airmail envelopes, whose weight is normal with mean $\mu = 1.95$ gm and standard deviation $\sigma = 0.05$ gm. The envelopes are sold in lots of 1000. How many envelopes in a lot will be heavier than 2 gm? Use the fact that $\frac{1}{\sqrt{2\pi}}\int_0^1 \exp\left(\frac{-x^2}{2}\right) dx = 0.3413$ **Ans. 159**

7. The mean height of 500 students is 151 cm and the standard deviation is 15 cm. Assuming that the heights are normally distributed, find how many student's height lie between 120 and 155 cm. **Ans. 294**