Tutorial - 4, 5

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Q1) $\frac{du}{dt} = \frac{d^2u}{dx^2}$ with ① $u(0,t) = 0$
② $u(l,t) = 0$
③ $u(x,0) = x$

$\Rightarrow$ Let $u(x,t) = (A\cos mx + B\sin mx)e^{-m^2t}$ — ①

$\therefore$ From condition ① $u(0,t) = 0$

$0 = Ae^{-m^2t}$

$\boxed{A = 0}$

$\therefore$ ① becomes

$u(x,t) = B\sin mx\, e^{-m^2t}$ — ②

from condn ② $u(l,t) = 0$

$\Rightarrow 0 = B\sin ml\, e^{-m^2t}$

$\sin ml = 0$

$\sin ml = \sin n\pi$

$ml = n\pi$

$m = \frac{n\pi}{l}$

$\therefore$ ② becomes -

$u(x,t)\ B\sin \frac{n\pi}{l}x\, e^{-\frac{n^2\pi^2 t}{l^2}}$ — ③ $n = 1,2,3\cdots$

~~$x = \sum_{n=1}$~~

From condition ③ $\Rightarrow u(x,0) = x$

Put $t = 0$ in eqn ③

$x = B\sin \frac{n\pi}{l}x$

~~$x = B\sin$~~

$\rightarrow$

$$x = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \quad - (4)$$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{l} \int_0^l x \sin \frac{n\pi x}{l} \, dx$$

$$\Rightarrow \frac{2}{l} \left[x \cdot \left(\frac{-\cos n\pi x/l}{n\pi/l} \right) - (1) \left(\frac{-\sin \pi x/l}{n^2\pi^2/l^2} \right) \right]_0^l$$

$$\Rightarrow \frac{2}{l} \left[\frac{x\,l}{n\pi} \left(-\cos(n\pi x)/l\right) + \frac{l^2}{n^2\pi^2} \sin \frac{n\pi x}{l} \right]_0^l$$

$$b_n = \frac{2}{l} \left[\frac{l^2}{n\pi} (-\cos \pi) \right]$$

$$b_n = \frac{-2l(-1)^n}{n\pi} \quad - (5)$$

from (3), (4) & (5)

$$\boxed{u(x,t) = \frac{2l}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi}{l} e^{\frac{-n^2\pi^2 t}{l^2}}}$$

Q2) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

① $u(x,0) = 0$
② $u(x,1) = 0$
③ $u(\infty, y) = 0$
④ $(0,y) = 50°C$

$\Rightarrow$ condition ③ $\Rightarrow u(\infty, y) = 0$

$\therefore$ appropriate soln is

$u(x,y) = (c_1 e^{mx} + c_2 e^{-mx})(c_3 \cos my + c_4 \sin my)$ — ①

$\because u(\infty, y) = 0$ temp. should remain finite for $x \to \infty$

$\Rightarrow c_1 = 0$

$\therefore u(x,y) = e^{-mx}[c_3 \cos my + c_4 \sin my]$

$u(x,y) = e^{-mx}[c_5 \cos my + c_6 \sin my]$

From condition ① $\Rightarrow u(x,0) = 0$

$\Rightarrow 0 = c_5 e^{-mx}$

$\Rightarrow \boxed{c_5 = 0}$

$\therefore$ ② becomes

$u(x,y) = 0$

$0 = c_6 \sin m e^{-mx}$

$\Rightarrow \sin m = 0$

$\sin m = \sin n\pi$

$\therefore$ $m = n\pi$ $\because n = 1, 2, 3 \dots$

$\rightarrow$

$\because$ ③ becomes

$$u(x,y) = C_6 \sin(n\pi y) e^{-n\pi x} \quad \text{— ④}$$

condition ④ $\Rightarrow u(0,y) = 50$

Put $x=0$ in eqn ④

$$50 = \sum_{n=1}^{\infty} b_n \sin(n\pi y) \quad \text{— ⑤}$$

is half range fourier sine series when

$$b_n = \frac{2}{1} \int_0^1 50 \sin \frac{n\pi y}{1} \, dy$$

$$b_n = 100 \left[-\frac{\cos n\pi y}{n\pi} \right]_0^1$$

$$b_n = 100 \left[-\frac{\cos n\pi}{n\pi} - \frac{1}{n\pi} \right]$$

$$b_n = \frac{100}{n\pi} (-1)^n$$

Putting in eqn ④ $\Rightarrow$

$$u(x,y) = \sum_{n=1}^{\infty} \frac{100}{n\pi} (-1)^n \sin(n\pi y) e^{-n\pi x}$$

$$u(x,y) = \frac{100}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(n\pi y) e^{-n\pi x}$$