2/8/22 Books

B.S. Grewal

CCA - 60
End T. - 40

1) $\int x^n dx = \frac{x^{n+1}}{n+1}$

4) $\int \sin x \, dx = -\cos x$

2) $\int e^{ax} dx = \frac{e^{ax}}{a}$

5) $\int \cos x \, dx = \sin x$

3) $\int a^x dx = \frac{a^x}{\log a}$

6) $\int \tan x \, dx = \log(\sec x)$

12) $\int \sec^2 x \, dx = \tan x + c$

7) $\int \frac{dx}{x} = \log x$

13) $\int \operatorname{cosec}^2 x \, dx = -\cot x + c$

17) $\int \cot x \, dx = \log(\sin x) + c$

8) $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$

9) $\int u.v \, dv = u.\int v \, dx - \int \frac{du}{dx}\left[\int v \, dx\right] dx$

10) $\int_a^b f(x) \, dx = [g(x)]_a^b = g(b) - g(a)$

11) $\int a \, dx = ax + c$

18. $\int \sec x \, dx = \log|\sec x + \tan x| + c$

14) $\int \frac{1}{\sqrt{1 - x^2}} dx = \sin^{-1} x + c$

19. $\int \operatorname{cosec} x \, dx = \log|\operatorname{cosec} x - \cot x| + c$

15) $\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$

16) $\int \frac{1}{|x|\sqrt{x^2-1}} dx = \sec^{-1} x + c$

# $1^{st}$ Order Diff. Equations

1) $x\,dy + y\,dx = 0$

$x\,dy = -y\,dx$

$\int \frac{1}{y}\,dy = \int -\frac{1}{x}\,dx$

$\log y = -\log x + \log c$

$\boxed{xy = c}$

2) $\frac{dy}{dx} + y = 5$

$dy + y\,dx = 5\,dx$

$dy = (5-y)\,dx$

$\int \frac{dy}{(5-y)} = \int dx$

$\boxed{-\log(5-y) = x + c}$

* $\frac{dy}{dx} + y = 5$

$\Rightarrow \frac{dy}{dx} + Py = Q$

$I.F. = e^{\int P dx} = e^{\int dx} = e^{x}$

Sol^n :-

$y \times I.F. = \int I.F. \times Q\,dx + c$

$$y \cdot e^{x} = \int e^{x} 5\, dx$$

$$y \cdot e^{x} = 5e^{x} + c$$

$$e^{x}(y-5) = c$$

$$\Rightarrow -\log(y-5) = x + c$$

Formulae :- ..

20) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c$

21) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c$

22) $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$

23) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log \left| x + \sqrt{x^2 - a^2} \right| + c$

24) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + c$

25) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log \left| x + \sqrt{x^2 + a^2} \right| + c$

26) $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log \left| x + \sqrt{x^2 - a^2} \right| + c$

27) $\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log \left| x + \sqrt{x^2 + a^2} \right| + c$

28) $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + c$

22/08/22

# Linear Differential eqⁿs:

* $n^{th}$ order :-

- Prequisite :- Roots of polynomial eqⁿ.

1) $1^{st}$ order → linear

$D+1=0 \Rightarrow D=-1$ is root.

2) $2^{nd}$ order → quadratic

1) $D^2-4D+4=0 \Rightarrow$

$D(D-2)-2(D-2)=0$

$D_1=2, D_2=2, -$

2)

$D^2+D+1=0$

$d=b^2-4ac$

$=1^2-4\times1\times1$

$=1-4$

$=-3$

$Root = \dfrac{-b\pm\sqrt{D}}{2a} = \dfrac{-1\pm\sqrt{-3}}{2\times1}$

$= \dfrac{-1\pm\sqrt{3}i}{2}$

3) Cubic eqⁿ :- $3^{rd}$ order

$(a^3+b^3) = (a+b)(a^2-ab+b^2)$

$(a^3-b^3) = (a-b)(a^2+ab+b^2)$

ex. : Find roots for -

$$D^3 - 1 = (D-1)(D^2+D+1)$$

$\therefore D = 1$ or $D = \dfrac{-1+\sqrt{3}i}{2}$, $D = \dfrac{-1-\sqrt{3}i}{2}$

Synthetic division $\longrightarrow$ for roots

E.g. : $D^3 - 7D - 6 = 0$

If $D = -1 \Rightarrow f(D) = 0$

$\Rightarrow (D+1)$ is factor of $D^3 - 7D - 6 = 0$

$$\begin{array}{c|cccc} D=-1 & 1 & 0 & -7 & -6 \\ & & -1 & 1 & 6 \\ \hline & 1 & -1 & -6 & \boxed{0} \end{array} \Rightarrow (D^2 - D - 6)$$

$\therefore (D+1)(D^2 - D - 6) = 0$

$(D+1)(D-3)(D+2) = 0$

2) $D^3 - 2D + 4 = 0$

$D = -2$ $\qquad (D+2)$ is one factor

$$\begin{array}{c|cccc} D=-2 & 1 & 0 & -2 & 4 \\ & & -2 & 4 & -4 \\ \hline & 1 & -2 & 2 & \boxed{0} \end{array}$$

$(D+2)(D^2 - 2D + 2) = 0$

$(D+2)$

$D^2 - 2D + 2 = 0$

$$d = \frac{-2 \pm \sqrt{4 - 4 \times 1 \times 2}}{2}$$

$= -1 \pm i$

$\therefore (D+2)(-1+i)(-1-i) = C$

24/08/22

# Linear Differential Eq$^{rs}$ ($n^{th}$ order)

$\left(\frac{dy}{dx}\right) + P(y) = Q \longrightarrow 1^{st}$ order L.D.E.

where $P, Q \to$ consts. or $f^r$ of '$x$'.

$I.F. = e^{\int P dx} = e^{mx}$

Solve: $\boxed{y \times I.F. = \int I.F. \times Q\, dx + C}$

$ye^{mx} = \int e^{mx} Q\, dx + C$

$ye^{mx} = f(x) + C$

Sol$^n$. :— $\boxed{y = e^{-mx} f(x) + ce^{-mx}}$

$2^{nd}$ order $\rightarrow$

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + qy = 0$$

$n^{th}$ order $\rightarrow$

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_n y = 0 \text{ or } f(x)$$

where $a_0, a_1, a_2, \ldots, a_n \rightarrow$ constant co-efficients

Use diff. operator $D = \frac{d}{dx}$

$\Rightarrow$

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n) y = 0 \text{ or } f(x)$$

$\downarrow$

Polynomial in D

$$\phi(D) y = 0 \text{ or } f(x) \quad \text{—— (3)}$$

To find the solution -

Consider $\phi(D) = 0$ is Auxilliary eq$^n$ (A.E.)

$\hookrightarrow$ algebraic eq$^n$ in D

$\therefore$ find roots of $\phi(D) = 0$

e.g.

$$\frac{dy}{dx} + m.y = 0$$

$$(D+m)y = 0$$

$\Rightarrow$ $D + m = 0$

$D = -m$ $\therefore$ Solution is $y = ce^{-mx}$

Hence for $(D-m_1)y = 0 \to$ 1st order L.D.E.

Sol$^n$ $\Rightarrow$ $y = c_1 e^{m_1 x}$

$(D-m_2)y = 0 \to$ 2nd order L.D.E.

$\Rightarrow y = c_2 e^{m_2 x}$

$\therefore$ From (3) sol$^n$ $\to$ G.S. $\Rightarrow$

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots$$

$\Rightarrow$ Depending on nature of roots of $\phi(D)=0$

I] roots are real and different

Say $D = m_1, D = m_2, \dots, D = m_n$

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$$

II] if $\phi(D) = 0$ has real and equal roots

Say $D = m_1 = m_2, m_3, m_4, \dots, m_n$

Then

$$y = (c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$$

III] if $\phi(D) = 0$ has imaginary and diff. roots

$D = \alpha + i\beta$, $D = \alpha - i\beta = m_2$

$\therefore$ Sol$^n \Rightarrow y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x}$

$y = c_1 e^{\alpha x} \cdot e^{i\beta x} + c_2 e^{\alpha x} \cdot e^{-i\beta x}$

$= e^{\alpha x}\left[c_1 e^{i\beta x} + e^{-i\beta x}\right]$ —— (4)

$e^{i\theta} = \cos\theta + i\sin\theta$ [Euler's form]

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

So in eq$^n$ (4)

$$y = e^{\alpha x}\left[c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)\right]$$

$$y = e^{\alpha x}\left[\cos\beta x . \underbrace{(c_1 + c_2)}_{A} + \sin\beta x . \underbrace{(ic_1 - ic_2)}_{B}\right]$$

$$\therefore \quad y = \left[A\cos\beta x + B\sin\beta x\right] e^{\alpha x}$$

IV] if $\phi(D) = 0$ has <u>imaginary</u> and <u>equal</u> roots

Let $D = m_1 = \alpha + i\beta = m_2$

$D = m_3 = \alpha - i\beta = m_4$

Using case II,

G.S. $y = (c_1 + c_2 x)e^{m_1 x} + (c_3 + c_4 x)e^{m_3 x}$

or

$$y = e^{\alpha x}\left[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x\right]$$

E.g. Find solⁿ of -

1)
$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

$$[D^2 - 5D + 6]y = 0$$

$\therefore$ A.E. $\Rightarrow [D^2 - 5D + 6] = 0$

$(D-2)(D-3) = 0$

$\therefore$ $D = 2$ and $D = 3$ are the real and distinct roots

Solⁿ :- $\boxed{y = c_1 e^{2x} + c_2 e^{3x}}$

$\Rightarrow$ complementary fⁿ (C.F.)

2)
$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$$

$\Rightarrow (D^2 - 4D + 4)y = 0$

$\therefore$ A.E. $\Rightarrow [D^2 - 4D + 4] = 0$

$(D-2)^2 = 0$

$\therefore$ $D = 2, 2$ real and equal

Solⁿ :- $\boxed{y = (c_1 + c_2 x)e^{2x}}$

3) $(D^2+2D+2)y = 0$

A.E. $\Rightarrow [D^2+2D+2] = 0$

$$D = \frac{-2 \pm \sqrt{4-4(1)(2)}}{2(1)}$$

$$= \frac{-2 \pm 2i}{2}$$

$\therefore D = -1+i$ or $D = -1-i$

imaginary and distinct

Using case III,

$\alpha = -1, \beta = 1$

$$y = e^{-x}[A\cos x + B\sin x]$$

4) $(D^2+D+1)^2 y = 0$

A.E. $\Rightarrow (D^2+D+1)^2 = 0$

$\therefore (D^2+D+1) = 0$

$$D = \frac{-1 \pm \sqrt{1-4(1)(1)}}{2(1)}$$

$$= \frac{-1 \pm \sqrt{3}i}{2}$$

$\therefore D = \frac{-1+\sqrt{3}i}{2}$ & $D = \frac{-1-\sqrt{3}i}{2}$

$= m_1 = m_2$ $\qquad = m_3 = m_4$

$\alpha = -1/2$ & $\beta = \sqrt{3}/2$

Using IV,

$$y = e^{-x/2}\left[(c_1+c_2x)\cos\frac{\sqrt{3}}{2}x + (c_3+c_4x)\sin\frac{\sqrt{3}}{2}x\right]$$

# DETT Tut 1

1) $2D^2 - D - 10 = 0$

$$\boxed{\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

Roots $= \dfrac{-(-1) \pm \sqrt{(1)^2 - 4 \times (-10) \times 2}}{2 \times 2}$

$$= \frac{1 \pm \sqrt{1 + 80}}{4}$$

$$= \frac{1 \pm \sqrt{81}}{4} = \frac{1 \pm 9}{4} =$$

$x = \dfrac{1+9}{4}$, $\quad y = \dfrac{1-9}{4} =$

$= \dfrac{10}{4}$, $\quad y = \dfrac{-8}{4} = -2$

$= \dfrac{5}{2}$

2) $D^3 - 6D^2 + 11D - 6 = 0$

$D = 1 \Rightarrow (D - 1) = 0$

$D = 1$

| | | | |
|---|---|---|---|
| 1 | -6 | 11 | -6 |
| | 1 | -5 | 6 |
| 1 | -5 | 6 | 0 |

$(D-1)(D^2 - 5D + 6) = 0$

$(D-1)(D^2 - 3D - 2D + 6) = 0$

$(D-1)(D-3)(D-2) = 0$

Roots :- $D_1 = 3$, $D_2 = 1$, $D_3 = 2$

3) $D^3 - 3D^2 + 3D - 1 = 0$

$D = 1$, $(D-1)$

| $D=1$ | 1 | -3 | 3 | -1 |
| --- | --- | --- | --- | --- |
| | | 1 | -2 | 1 |
| | 1 | -2 | 1 | 0 |

$(D-1)(D^2 - 2D + 1) = 0$

$(D-1)$. taking quadratic method to solve

roots are :- $\dfrac{2 \pm \sqrt{4-4}}{2}$

$= \dfrac{2 \pm \sqrt{0}}{2}$

$= \pm\dfrac{2}{2} \Rightarrow +1$ or $-1$

roots are $D_1 = 1$, $D_2 = 1$, and $D_3 = 1$

(4) $(D^4+1)=0$

$(D^2)^2+(1)^2$

$(D^2)^2+2D^2+1-2D^2$

$(D^2+1)^2-(\sqrt{2}D)^2$

$A \qquad B$

$$\left((D^2+1)-(\sqrt{2}D)\right)\left((D^2+1)+(\sqrt{2}D)\right)=0$$

$(D^2+1)=\sqrt{2}D$ → $D^2-\sqrt{2}D+1=0$

$$\frac{\sqrt{2}\pm\sqrt{2-4}}{2}$$

$$=\frac{\sqrt{2}\pm\sqrt{-2}}{2}$$

$$=\frac{1\pm i}{\sqrt{2}}$$

or

$\Rightarrow \frac{1+i}{2}$ and $\frac{1-i}{2}$

$D^2+1=-\sqrt{2}D$

$D^2+\sqrt{2}D+1=0$

→ & $-\frac{1+i}{2}$ and $-\frac{1-i}{2}$.

$(a+b)^2$

$a^2+2ab+b^2$

1) $\int e^{x} e^{e^{x}} \, dx$

assuming $e^{x} = t$

$e^{x} dx = dt$

$\int e^{t} \, dt$

$e^{t} + C$

$\boxed{I.F = e^{e^{x}} + C}$

(ii) $I = \int \underset{u}{x^2} \underset{v}{\sin 3x} \, dx$

$= u \int v \, dx - \int \left( \frac{du}{dx} \int v \, dx \right) dx$

$= -\frac{x^2 \cos 3x}{3} - \int \frac{2x(-\cos 3x)}{3} \, dx$

$= -\frac{x^2 \cos 3x}{3} + \frac{2}{3} \int x \cos 3x \, dx$

$+ \frac{2}{3} \int I_1 \, dx$

$I_1 = \int x \cos 3x \, dx$

$= \frac{x \sin 3x}{3} - \int 1 \cdot \frac{\sin 3x}{3} \, dx \Rightarrow \frac{x \sin 3x}{3} + \frac{1}{3} \cos 3x$

$= -\frac{x^2 \cos 3x}{3} + \frac{2}{3} \left[ \frac{x \sin 3x + \cos 3x}{3} \right]$

$= -\frac{x^2 \cos 3x}{3} + \frac{2}{9} x \sin 3x$

$+ \frac{2}{9} \cos 3x$

$+ C$

29/08/22

Find the sol$^n$ of:-

1) $(D^4 - 16)y = 0$

$\therefore$ A.E. $\Rightarrow D^4 - 16 = 0$

$\Rightarrow (D^2 - 4)(D^2 + 4) = 0$

$(D+2)(D-2)(D^2+4) = 0$ $\alpha = 0, \beta = 2$

$\Rightarrow D = -2, D = 2$, $D = 2i, D = -2i$

Case I Case II

G.S. :-

C.F. $\Rightarrow y = c_1 e^{2x} + c_2 e^{-2x} + (c_3 \cos 2x + c_4 \sin 2x)e^{0x}$

2) $(D^4 - 2D^3 + D^2)y = 0$

$\therefore$ A.E. $\Rightarrow D^4 - 2D^3 + D^2 = 0$

$D^2(D^2 - 2D + 1) = 0$

$D^2(D-1)(D-1) = 0$

$\therefore D = 0, 0, 1, 1$

G.S. :- Using case I and II,

C.F. $y = (c_1 + c_2 x)e^{0x} + (c_3 + c_4 x)e^{x}$

3) $(D^3 + D^2 - 2D + 12)y = 0$

A.E. $\Rightarrow D^3 + D^2 - 2D + 12 = 0$

By trial and error,

$D = -3$ case I

By synthetic division,

| | | | | |
|---|---|---|---|---|
| $D = -3$ | 1 | 1 | $-2$ | 12 |
| | | $-3$ | 6 | $-12$ |
| | 1 | $-2$ | 4 | $\boxed{0}$ |

$\Rightarrow (D^2 - 2D + 4) = 0$

$$D^2 - 2D + 4 = 0$$

$$\Rightarrow D = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(4)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{4-16}}{2} = \frac{2 \pm 2\sqrt{-3}}{2}$$

$$= \frac{1 \pm i\sqrt{3}}{1}$$

$$\therefore \quad D = 1 + i\sqrt{3} \quad \text{or} \quad D = 1 - i\sqrt{3}$$

G.S. :— Case III

$\therefore$ C.F. $\Rightarrow$

$$y = c_1 e^{-3x} + e^{x}\left[c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x\right]$$

---

L.D.E. $\quad \phi(D)y = f(x)$

# G.S. $\Rightarrow$ G.S. = C.F. + Particular integral (P.I.)

C.F. $\downarrow$ arbitrary constant

Particular integral $\downarrow$ free from arbitrary constant

To find P.I.

Consider $\quad y_p = \left[\dfrac{1}{\phi(D)}\right] f(x)$

$\hookrightarrow$ inverse operator of $\phi(D)$

$\phi(D) \rightarrow$ differential operator

$1/\phi(D) \rightarrow$ integral operator

$$D \rightarrow \frac{d}{dx} \Rightarrow \frac{1}{D} \rightarrow \int$$

$$D^2 \rightarrow \frac{d^2}{dx^2} \Rightarrow \frac{1}{D^2} \rightarrow \iint$$

**Note:—** $(D-m_1)$ is a factor of $\phi(D)$ then $\frac{1}{(D-m_1)}$ is inverse operator

If, $y_p = \frac{1}{\phi(D)} f(x)$

Following methods are used for P.I.

1) General method
2) Short cut method
3) Method of variation of parameters.

1) **General method:—** consider $y_p = \frac{1}{\phi(D)} f(x)$

If $\phi(D) = (D-m_1) \Rightarrow$

$$\boxed{y_p = \frac{1}{(D-m_1)} f(x) = e^{m_1 x} \int e^{-m_1 x} f(x)\, dx}$$

If $\phi(D) = (D+m_1)$

then

$$\boxed{y_p = \frac{1}{(D+m_1)} f(x) = e^{-m_1 x} \int e^{m_1 x} f(x)\, dx}$$

For real factors $\Rightarrow \phi(D) = (D-m_1)(D-m_2)$

$$y_p = \frac{1}{(D-m_1)(D-m_2)} f(x)$$

$$= \frac{1}{D-m_1} \left[ e^{m_2 x} \int e^{-m_2 x} f(x)\, dx \right] \rightarrow g(x)$$

$$\boxed{y_p = \frac{1}{D-m_1} \cdot g(x) = e^{m_1 x} \int e^{-m_1 x} \cdot g(x)\, dx}$$

TUT

# DETT

Tut 2 , Name: Aditya Vishwaraj.

Q 3 $(D^3+8)y=0$

AE: $D^3+8=0$

$D=-2$ (hit & trial method)

| | | | | |
|---|---|---|---|---|
| $D=-2$ | 1 | 0 | 0 | 8 |
| | | -2 | 4 | -8 |
| | 1 | -2 | 4 | 0 |

$D^2-2D+4=0$

roots $\Rightarrow \dfrac{2\pm\sqrt{4-4\times 4}}{2}$

$\Rightarrow \dfrac{2\pm\sqrt{12}}{2} = \dfrac{2\pm 2\sqrt{3}i}{2}$

$\Rightarrow$ $\Rightarrow 1\pm\sqrt{3}i$

Using Case III

$cf = c_1e^{-2x}+e^{x}[c_2\cos\sqrt{3}x+c_3\sin\sqrt{3}x]$

$\boxed{\text{opt (a)}}$

T-2

Q. 2 $(D^3 - 6D^2 + 12D - 8)y = e^{2x} + 3^x$

AE → $D^3 - 6D^2 + 12D - 8 = 0$

$\boxed{D = 2}$ (by hit & trial method)

| | | | | |
|---|---|---|---|---|
| $D=2$ | 1 | $-6$ | 12 | $-8$ |
| | | 2 | $-8$ | 8 |
| | 1 | $-4$ | 4 | 0 |

$D^2 - 4D + 4 = 0$

$$\frac{4 \pm \sqrt{16 - 4 \times 4 \times 1}}{2}$$

$$= \frac{4 \pm \sqrt{16 - 16}}{2} = \frac{4 \pm 0}{2} = 2, 2,$$

roots are → 2, 2, 2

Case II

$CF = (c_1 + c_2 x + c_3 x^2) e^{2x}$

$\boxed{\text{opt } c}$

T-3

① $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$

$D^3 - 6D^2 + 11D - 6 = 0$

by hit & trial method :-

↳ $(D-1) = 0$

$D = 1$

| $D=1$ | 1 | $-6$ | 11 | $-6$ |
|---|---|---|---|---|
| | | 1 | $-5$ | 6 |
| | 1 | $-5$ | 6 | 0 |

$D^2 - 5D + 6$

$D^2 - 3D - 2D + 6 = 0$

$D(D-3) - 2(D-3) = 0$

$D = 2, D = 3$

Three roots are, $D = 1, D = 2, D = 3$

$CF = C_1e^{x} + C_2e^{2x} + C_3e^{3x}$

opt ⓐ

T-4

(4) $(D^4 - 25)y = 0$

AE :- $((D^2) - (5)^2) = 0$

$(D^2 - 5)(D^2 + 5) = 0$

$D^2 - 5 = 0$ $\qquad D^2 + 5 = 0$

$D^2 = 5$ $\qquad \rightarrow \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$D = \pm\sqrt{5}$

$\rightarrow \dfrac{-0 \pm \sqrt{0 - 4 \times 5}}{2}$

$= \pm \dfrac{\sqrt{-20}}{2}$ $\qquad 4 \times 5$

$= \dfrac{\pm \cancel{2}\sqrt{5}\, i}{\cancel{2}} = \pm\sqrt{5}\, i$

Case (1) & (111)

$cf = (c_1 \cos\sqrt{5}x + c_2 \sin\sqrt{5}x)e^{0x} + c_3 e^{-\sqrt{5}x} + c_4 e^{\sqrt{5}x}$

$= (c_1 \cos\sqrt{5}x + c_2 \sin\sqrt{5}x) + c_3 e^{-\sqrt{5}x} + c_4 e^{\sqrt{5}x}$

option (C)

T- 5

(5) $(D^4 - 9D^2 + 20)y = 0$

AE

$$D^4 - 9D^2 + 20 = 0$$

$$D^4 - 5D^2 - 4D^2 + 20 = 0$$

$$D^2(D^2 - 5) - 4(D^2 - 5) = 0$$

$$(D^2 - 4)(D^2 - 5) = 0$$

$D^2 - 4 = 0$, $D^2 - 5 = 0$

$D^2 = 4$, $D^2 = 5$

$D = \pm\sqrt{4}$, $D = \pm\sqrt{5}$

$= \pm 2$, $D = \pm\sqrt{5}$

4 | 20
| 1

Using case (1)

$$CF: c_1 e^{2x} + c_2 e^{-2x} + c_3 e^{\sqrt{5}x} + c_4 e^{-\sqrt{5}x}$$

Also: $y = c_1 \cosh\sqrt{5}x + c_2 \sinh\sqrt{5}x + c_3 \cosh 2x + c_4 \sinh 2x$

$\because$ $\boxed{\cosh x = \frac{e^x + e^{-x}}{2}}$ & $\boxed{\sinh x = \frac{e^x - e^{-x}}{2}}$

$$\therefore y = c_1\left[\frac{e^{\sqrt{5}x} + e^{-\sqrt{5}x}}{2}\right] + c_2\left[\frac{e^{\sqrt{5}x} - e^{-\sqrt{5}x}}{2}\right] + c_3\left[\frac{e^{2x} + e^{-2x}}{2}\right] + c_4\left[\frac{e^{2x} - e^{-2x}}{2}\right]$$

$$\Rightarrow e^{\sqrt{5}x}\underbrace{\left[\frac{c_1 + c_2}{2}\right]}_{A} + e^{-\sqrt{5}x}\underbrace{\left[\frac{c_1 - c_2}{2}\right]}_{B} + e^{2x}\underbrace{\left[\frac{c_3 + c_4}{2}\right]}_{C} + e^{-2x}\underbrace{\left[\frac{c_3 - c_4}{2}\right]}_{D}$$

$$\Rightarrow \left(A e^{\sqrt{5}x} + B e^{-\sqrt{5}x} + C e^{2x} + D e^{-2x}\right)$$

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Theory

# for method ①. we have one option:-

Partial fraction:-

$$\frac{1}{(D-m_1)(D-m_2)} = \frac{A}{(D-m_1)} + \frac{B}{(D-m_2)}$$

$$= \frac{A}{(D-m_1)} f(x) + \frac{B}{(D-m_2)} f(x)$$

Ex:-

$(D-2)y = x$. find CF and P.I.

→ AE = $D-2=0 \Rightarrow D=2 \Rightarrow$ Cf. $= Ce^{2x}$ — ①

PI :- $\frac{1}{(D-2)}(x) = e^{2x}\int e^{-2x} \cdot x \, dx$

$$= e^{2x}\left[x\left(\frac{e^{-2x}}{-2}\right) - \int \frac{e^{-2x}}{2}(1)\,dx\right]$$

$$= e^{2x}\left[\frac{xe^{-2x}}{-2} + \frac{e^{-2x}}{-4}\right] \text{ — ⑪}$$

GS = CF + PI

$$GS: Ce^{2x} + e^{2x}\left[\frac{xe^{-2x}}{2} + \frac{e^{-2x}}{-4}\right]$$

$$\boxed{GS: Ce^{2x} + \left[\frac{x}{2} + \frac{1}{4}\right]}$$

(Ex) $(D^2+3D+2)y = e^{e^x}$

$CF = D^2+3D+2$

$D^2+2D+D+2$

$= D(D+2)+1(D+2)$

$\Rightarrow D=-1,\ D=-2.$

$$\boxed{CF = c_1 e^{-x} + c_2 e^{-2x}} \quad \text{— (I)}$$

$$\underbrace{(D+2)(D+1)}_{\phi(D)}y = \underbrace{e^{e^x}}_{f(x)}$$

$$P.I = \frac{1}{(D+1)(D+2)} e^{e^x}$$

$$= \frac{1}{D+2}\left[\frac{1}{D+1} e^{e^x}\right]$$

$$= \frac{1}{D+2}\left[e^{-x}\int e^x \cdot e^{e^x}\,dx\right]$$

put $e^x = t$

$e^x\,dx = dt$

$$= \frac{1}{D+2}\left[e^{-x}\int e^t\,dt\right] = \frac{1}{D+2}\left[\underbrace{e^{-x}\,e^{e^x}}_{g(x)}\right]$$

$$= e^{-2x}\int\left(e^{2x}\cdot e^{-x}\cdot e^{e^x}\right)dx$$

$$= e^{-2x}\int\left(e^x \cdot e^{e^x}\right)dx \Rightarrow e^{-2x}\cdot e^{e^x} \quad \text{— (II)}$$

$GS = CF + PI$

$= c_1 e^{-x} + c_2 e^{-2x} + e^{-2x} \cdot e^{e^x}$.

Q. $(D^2+D)y = \dfrac{1}{1+e^x}$

$AE \Rightarrow D^2 + D = 0$

$D^2 = -D$

$D = -1, 0$

$CF = c_1 e^{-x} + c_2$

$PI = \dfrac{1}{D(D+1)} \cdot \left(\dfrac{1}{1+e^x}\right)$

$= \dfrac{1}{D^2+D}\left(\dfrac{1}{1+e^x}\right)$

$= \dfrac{1}{D(D+1)}\left(\dfrac{1}{1+e^x}\right)$ — ①

Consider =

$$\frac{1}{D(D+1)} = \frac{A}{D} + \frac{B}{D+1}$$

$\Rightarrow N^r = 1 = A(D+1) + B(D)$

1, $D = 0$, $A = 1$ ; $D = -1$, $B = -1$.

$\therefore \dfrac{1}{D(D+1)} = \dfrac{1}{D} - \dfrac{1}{D+1}$

This eqⁿ becomes

$$PI = \frac{1}{D}\left(\frac{1}{1+e^x}\right) - \frac{1}{D+1}\left(\frac{1}{1+e^x}\right)$$

$$= \int \frac{1\,dx}{1+e^x} - e^{-x}\int e^x \frac{1\,dx}{1+e^x}.$$

divide num & den by $e^x$

$$= \int \frac{e^{-x}\,dx}{e^{-x}+1} - e^{-x}\int \frac{e^x}{1+e^x}dx$$

put $1+e^{-x}=t$

$-e^{-x}dx = dt$

put $1+e^x = u$

$e^x dx = du$

$$= \int -\frac{dt}{t} - e^{-x}\int \frac{du}{u}$$

$$= -\log(1+e^{-x}) - e^{-x}[\log(1+e^x)]$$

* Short cut method :-

**Case I :-** $f(x) = e^{ax}$

$\Rightarrow$ $PI = \dfrac{1}{\phi(D)} e^{ax}$ . put $D = a$

$= \dfrac{1}{\phi(a)} e^{ax}$

$\rightarrow$ If $\phi(a) = 0 \Rightarrow a$ is root of $\phi(D)$
$\Rightarrow (D-a)$ is factor

$\Rightarrow$ $PI = \dfrac{1}{\phi'(D)} \cdot x e^{ax}$ then put $D = a$

$= \dfrac{1}{\phi'(a)} x e^{ax}$ if $\phi'(a) \neq 0$.

# If $\phi'(a) = 0$; then double differentiation of D & then substitute $\boxed{D = a}$

$= \dfrac{1}{\phi''(a)} x^2 e^{ax}$ . . . . continue till non zero answer in denominator

# NOTE :-

(i) $I = \dfrac{1}{(D-a)} e^{ax} = \dfrac{x}{1!} e^{ax}$

$= \dfrac{1}{(D-a)^2} e^{ax} = \dfrac{x}{2!} e^{ax}$

$= \dfrac{1}{(D-a)^r} e^{ax} = \dfrac{x}{r!} e^{ax}$.

(ii) for constant. $k = e^{ax} \Rightarrow$ put $D = 0$

(iii) for $f(x) = a^x \Rightarrow e^{x \log a} \Rightarrow$ put $D = \log a$.

(iv) for $f(x) = a^{-x} = e^{-x \log a} \Rightarrow$ put $D = -\log a$

Example :- 1) Solve $(D^2 + 3D + 2)y = e^{2x}$

$$CF = C_1 e^{-2x} + C_2 e^{-x}$$

$$PI = \frac{1}{(D+1)(D+2)} \cdot e^{2x} \quad \to (a)$$

put $D = a = 2$.

$$PI = \frac{1}{12} e^{2x}.$$

2) $(D^2 + 3D + 2)y = e^{-2x}$

$$CF = C_1 e^{-2x} + C_2 e^{-x}$$

$$PI = \frac{1}{D^2+3D+2} \cdot e^{-2x}, \quad \text{if } D = -2, \ \phi(D) = 0$$

→ differentiation of Denominator & multiply by $x$.

$$\phi'(D) = \frac{(x)}{2D+3} \cdot e^{-2x}, \quad a = -2 = D.$$

$$= \frac{x}{-1} \cdot e^{-2x} \Rightarrow \boxed{PI = -xe^{-2x}}$$

3. $(D^2+3D+2)y = e^{-2x} + 2^x + \frac{3}{2}$.

$CF = C_1e^{-x} + C_2e^{-2x}$

$PI = \frac{1}{D^2+3D+2} e^{-2x} + \frac{1}{D^2+3D+2} 2^x + \frac{1}{D^2+3D+2}\left(\frac{3}{2}\right)$.

$= \frac{1}{D^2+3D+2} e^{-2x} + \frac{1}{D^2+3D+2} e^{x\log 2} + \frac{1}{D^2+3D+2} \cdot \frac{3}{2} \cdot e^{0x}$

$D = -2$ $\qquad$ $D = \log 2$ $\qquad$ $D = 0$.

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Eg. $(D^2 - 4D + 4)y = e^{2x} + 3$

$$P.I. = \frac{1}{(D-2)^2} e^{2x} + \frac{1}{(D-2)^2} \cdot 3 \cdot e^{0x}$$

Put $D = 2$ ... Put $D = 0$

$$= \frac{x^2}{2!} e^{2x} + \frac{3}{4}$$

$$\therefore \boxed{P.I. = \frac{x^2}{2!} e^{2x} + \frac{3}{4}}$$

#1 Case II :— If $f(x) = \sin(ax+b)$ or $\cos(ax+b)$ then replace $D^2$ of $\phi(D)$ as $(-a^2)$

$$P.I. = \frac{1}{\phi(D)} \sin(ax+b) \text{ or } \cos(ax+b)$$

$D^2 = -a^2$

$$= \frac{1}{\phi(-a^2)} \sin(ax+b) \text{ or } \cos(ax+b)$$

if $\phi(-a^2) \neq 0$

if $\phi(-a^2) = 0 \Rightarrow$ diff. $D^r$ and multiply by $x$.

Note –

i) $\sin^2 x = \frac{1 - \cos 2x}{2}$

ii) $\cos^2 x = \frac{1 + \cos 2x}{2}$

iii) $\sin 2x = 2 \sin x \cos x$

iv) $\sin A \cos B = \dfrac{\sin(A+B) + \sin(A-B)}{2}$

v) $\cos A \cos B = \dfrac{\cos(A+B) + \cos(A-B)}{2}$

vi) $\sin A \sin B = \dfrac{\cos(A-B) - \cos(A+B)}{2}$

Solve :-

1) $(D^2 + 2D + 1)y = \sin x$

$C.F. \Rightarrow (c_1 + c_2 x)e^{-x}$

$P.I. = \dfrac{1}{D^2 + 2D + 1} \sin x \longrightarrow a = 1$

Put $D^2 = -1$

$= \dfrac{1}{-1 + 2D + 1} \sin x$

$= \dfrac{1}{2D} \sin x$

$P.I. = \dfrac{1}{2} \int \sin x \, dx = \dfrac{-\cos x}{2}$

$g.s. = C.F. + P.I.$

$$\boxed{g.s. = (c_1 + c_2 x)e^{-x} - \frac{\cos x}{2}}$$

Note :- If $D^r$ contains $\dfrac{1}{mD + n}$ term in $\phi(D)$

then bring $D^2$ term by multiplying $N^r$ and $D^r$ by conjugate $\rightarrow mD - n$

2) $(D^3 + 4D)y = \cos 2x$

C.F. $\Rightarrow D = 0, 2i, -2i$

$= c_1 + (c_2 \cos 2x + c_3 \sin 2x)$

P.I. $= \dfrac{1}{D^3 + 4D} \cos 2x \longrightarrow a = 2$

Put $D^2 = -2^2 = -4$

$= \dfrac{1}{-4D + 4D \longrightarrow 0} \cos 2x$ X

$\therefore$ Diff. $D^r$ and multiply by $x$,

$= x \cdot \dfrac{1}{3D^2 + 4} \cos 2x$

Put $D = -4$

$= x \cdot \dfrac{1}{-8} \cos 2x$

$\therefore$ P.I. $= y_p = -\dfrac{x \cos 2x}{8}$

$$\therefore \boxed{G.S. = c_1 + (c_2 \cos 2x + c_3 \sin 2x) - \frac{x \cos 2x}{8}}$$

3) $(D^2+1)y = \sin 2x \cos x$

$C.F. \Rightarrow D = \pm i$

$C.F. = (c_1 \cos x + c_2 \sin x)$

$$P.I. = \frac{1}{D^2+1}(\sin 2x \cos x)$$

$$= \frac{1}{D^2+1}\left[\frac{\sin(2x+x)+\sin(2x-x)}{2}\right]$$

$$= \frac{1}{D^2+1}\left[\frac{\sin 3x}{2}+\frac{\sin x}{2}\right]$$

$$= \frac{1}{D^2+1}\frac{\sin 3x}{2} + \frac{1}{D^2+1}\frac{\sin x}{2}$$

$$= \frac{1}{2}\left[\frac{1}{D^2+1}\sin 3x + \frac{1}{D^2+1}\sin x\right]$$

Put $D^2 = -9$ Put $D^2 = -1$

$$= \frac{1}{2}\left[\frac{\sin 3x}{-8} + x \cdot \frac{1}{2D}\sin x\right]$$

$$= \frac{\sin 3x}{-16} + \frac{x}{4}\int \sin x$$

$$P.I. = \frac{\sin 3x}{-16} - \frac{x}{4}\cos x$$

$$\boxed{G.S. = c_1\cos x + c_2 \sin x - \frac{\sin 3x}{16} - \frac{x\cos x}{4}}$$

Case III :— when $f(x) = \cosh(ax+b)$ or $\sinh(ax+b)$

Put $D^2 = a^2$ in $\phi(D)$

or $\sinh x = \dfrac{e^x - e^{-x}}{2}$

$\cosh x = \dfrac{e^x + e^{-x}}{2}$

E.g. $(D^2+1)y = \sinh(3x)$

$P.I. = \dfrac{1}{D^2+1} \sinh 3x$

$= \dfrac{1}{10} \sinh 3x$

H.W.

1) $(D^2+3D+2)y = \sin^2 x$

2) $(D^2-1)y = \cos x \cos 2x$

1) $(D^2+3D+2)y = \sin^2 x$

$y = \dfrac{1}{D^2+3D+2} \times \sin^2 x$

$\sin^2 x = \dfrac{1-\cos 2x}{2}$

$y = \dfrac{1}{D^2+3D+2} \left(\dfrac{1-\cos 2x}{2}\right)$

$= \dfrac{1}{2}\left(\dfrac{1}{D^2+3D+2}\right) - \dfrac{1}{2}\left(\dfrac{1}{D^2+3D+2} \cos 2x\right)$

$= \dfrac{1}{2}\left(\dfrac{1}{D^2+3D+2}\right)e^{0x} - \dfrac{1}{2}\left(\dfrac{1}{D^2+3D+2} \cos 2x\right)$

$D = 0$ $\qquad D^2 = -4$

$\rightarrow P.I. = \frac{1}{\cancel{2y}} = \frac{1}{2}\left(\frac{1}{3D-2}\right)\cos 2x$

Multiply Num & Den by Conjugate of $(3D-2)$ i.e $(3D+2)$

2. $(D^2-1)y = \cos x \cdot \cos 2x$

$$(D^2-1)y = \frac{\cos 3x}{2} + \frac{\cos(-x)}{2}$$

$$= \frac{\cos 3x}{2} - \frac{\cos x}{2}.$$

$$y = \frac{1}{D^2-1} \times \frac{\cos 3x}{2} - \frac{1}{D^2-1} \times \frac{\cos x}{2}$$

$D^2 = -9$ $\qquad$ $D^2 = -1$

$$= \frac{1}{2}\left(\frac{\cos 3x}{-10}\right) - \frac{1}{2}\left(\frac{\cos x}{-2}\right)$$

$$PI = -\frac{\cos 3x}{20} + \frac{\cos x}{4}$$

$CF \Rightarrow D^2 - 1 = 0$

$D^2 = 1$

$D = \pm 1$

$CF = C_1 e^{x} + C_2 e^{-x}.$

$$GS = CF + PI = C_1 e^{x} + C_2 e^{-x} + \frac{\cos x}{4} - \frac{\cos 3x}{20}$$

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Case IV :— If $f(x) = x^m$ in $\phi(D)y = f(x)$

$$P.I. = \frac{1}{\phi(D)} x^m$$

$$= [\phi(D)]^{-1} x$$

$= [$expanding by binomial expression$]\, x^m$

adjust →

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \dots$$

where $x$ can be in form of $D = \frac{d}{dx}$

# Depending upon $x^m$ → $m$ derivatives are possible

E.g. $(D^2-1)y = x^3$

$$P.I. = \frac{1}{D^2-1} x^3$$

$$= \frac{-1}{[1-D^2]} x^3$$

$$= -[1-D^2]^{-1} x^3$$

Using $(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$

$$= -\left[1 + D^2 + D^4 + \dots\right] x^3$$

Neglecting terms from $D^4$,

$= -[1+D^2]x^3$

$= -[x^3+D^2x^3]$

$D = \frac{d}{dx}$

$D(x^3) = 3x^2$

$D^2(x^3) = 6x$

$D^3(x^3) = 6$

$D^4(x^3) = 0$

$\boxed{P.I. = -[x^3+6x]}$

2) $(D^2-D+1)y = x^3-3x^2+1$

$\Rightarrow$

$$P.I. = \frac{1}{D^2-D+1}(x^3-3x^2+1)$$

$$= [1+(D^2-D)]^{-1}(x^3-3x^2+1)$$

Use $(1+x)^{-1} = 1-x+x^2-x^3+\ldots$

$$P.I. = [1-(D^2-D)+(D^2-D)^2-\ldots\ldots](x^3-3x^2+1)$$

Since maximum derivatives possible for $(x^3-3x^2+1)$ are 3; we discard the terms from $D^4$.

$$\therefore P.I. = [1-D^2+D+D^4-2D^3+D^2](x^3-3x^2+1)$$

$$= [1+D-D^3](x^3-3x^2+1)$$

$$= x^3-3x^2+1+D(x^3-3x^2+1) - D^3(x^3-3x^2+1)$$

$\therefore \boxed{P.I. = (x^3 - 3x^2 + 1) + (3x^2 - 6x) - 6}$

<u>Case V</u> :— If $f(x) = e^{ax}.V$ (where V is any other $f^n$ of $x$)

$$P.I. = \frac{1}{\phi(D)} e^{ax}.V$$

replace $D = D + a$

$$P.I. = e^{ax}. \frac{1}{\phi(D+a)} V$$

then using any of the previous cases depending upon v find P.I.

Q. $(D^2 - 4D + 3)y = x^3 e^{2x}$

here $V = x^3$

$a = 2$

$Df(x) = 2D + 4$

$D^2 f(x) = 2.$

$$P.I. = \frac{1}{D^2 + 4D + 3} e^{2x}.x^3$$

Put $D = D + 2$

$$P.I. = e^{2x} \frac{1}{(D+2)^2 - 4(D+2) + 3} x^3$$

$$= e^{2x} \frac{1}{D^2 + 4D + 4 - 4D - 8 + 3} x^3$$

$$= e^{2x}. \frac{1}{D^2 - 1} x^3$$

$$= -e^{2x} (1 - D^2)^{-1} x^3$$

$$= -e^{2x} [1 + D^2] x^3$$

$$= -e^{2x} [1 + 2] x^3$$

$$= -3 e^{2x}. x^3$$

2) $(D^4 - 4D + 4)y = e^x \cos 2x$

$$P.I. = \frac{1}{D^4 - 4D + 4} e^x \cos 2x$$

$a = 1$

$V = \cos 2x$

Put $D = D + 1$

$$\therefore P.I. = e^x \frac{1}{(D+1)^2 - 4(D+1) + 4} \cos 2x$$

$$= e^x \frac{1}{D^2 - 2D + 1} \cos 2x$$

Now, using case II,

Replace $D^2 = -2^2 = 4$

$$\therefore P.I. = e^x \frac{1}{-4 - 2D + 1} \cos 2x$$

$$= e^x \frac{1}{-(2D+3)} \cos 2x$$

$$= e^x \frac{(2D-3)}{-(2D+3)(2D-3)} \cdot \cos 2x$$

$$= -e^x \frac{(2D-3)}{4D^2 - 9} \cos 2x$$

Put $D^2 = -4$,

$$P.I. = \frac{-e^x (2D-3)}{-25} \cos 2x$$

$$= \frac{e^x}{25} \left[2D(\cos 2x) - 3\cos 2x\right]$$

$$\boxed{P.I. = \frac{e^x}{25} \left[2(-2\sin 2x) - 3\cos 2x\right]}$$

Case VI :- If $f(x) = x \cdot V$

$$P.I. = \frac{1}{\phi(D)} x(V)$$

$$= \left[ x - \frac{\phi'(D)}{\phi(D)} \right] \frac{1}{\phi(D)} V$$

... using the case for P.I. as per V $f^n$

# TUT-2

Name: Aditya Vishwaraj            Roll No: 57

1. $(D^2 - 9D + 18)y = e^{e^{-3x}}$

CF = $D^2 - 9D + 18 = 0$

$D^2 - 6D - 3D + 18 = 0$

$(D-6)(D-3)$

$\Rightarrow D = 6, D = 3$

$$\boxed{CF = c_1 e^{6x} + c_2 e^{3x}}$$

PI = $y = \dfrac{1}{(D^2 - 9D + 18)} e^{e^{-3x}}$

$\Rightarrow y = \dfrac{1}{(D-6)(D-3)} e^{e^{-3x}}$

$\Rightarrow y = \dfrac{1}{D-6}\left(\dfrac{1}{D-3} e^{e^{-3x}}\right)$

$\Rightarrow y = \dfrac{1}{D-6}\left(e^{3x} \int e^{-3x} \cdot e^{e^{-3x}} dx\right)$

put $e^{-3x} = t$

$e^{-3x} dx = \dfrac{dt}{3}$

$= \dfrac{1}{D-6}\left(e^{3x} \int e^{-3x} \cdot e^{e^{-3x}} dx\right)$

$= \dfrac{1}{D-6}\left(e^{3x} \cdot \left(-\dfrac{1}{3}\right) \int e^{t} dt\right) = \dfrac{1}{D-6}\left[e^{3x} \cdot -\dfrac{1}{3}\left(e^{t}\right)\right]$

$$= \frac{1}{D-6}\left[-\frac{1}{3} e^{3x}\left(e^{e^{3x}}\right)\right]$$

$$\Rightarrow \frac{1}{3} e^{6x} \int \left[e^{-6x} \cdot e^{3x} \cdot e^{e^{-3x}} dx\right]$$

$$e^{-3x} = u$$
$$e^{-3x} dx = -\frac{du}{3}$$

$$\Rightarrow -\frac{1}{3} e^{6x} \int \left(e^{u} du\right]$$

$$= \frac{1}{9} e^{6x} \left[e^{u}\right]$$

$$= \frac{1}{9} e^{6x} \left[e^{e^{-3x}}\right]$$

$$GS = CF + PI$$

$$= c_1 e^{6x} + c_2 e^{3x} + \frac{1}{9} e^{6x}\left[e^{e^{-3x}}\right]$$

2. $(D^3+D)y = \cos x$

AE = $D^3+D=0$

$D^3 = -D$

$D^2 = -1$

$D = \pm i$

$CF = C_1 \cos x + C_2 \sin x$

$PI = \frac{1}{D^3+D} \cos x$ , $a = 1$

$D^2 = -1$.

$= \frac{1}{D(D^2+1)} = \frac{1}{D(-1+1)}$ it's coming out to be zero.

Hence diff $D^r$ and multiply by $x$.

$= \frac{x}{3D^2+1} \cos x$

$= \frac{x}{3(-1)+1} \cos x$

$= \frac{x \cos x}{-2}$

$GS = CF + PI = C_1 \cos x + C_2 \sin x - \frac{x \cos x}{2}$.

3. $(D^3 - 5D^2 + 8D - 4)y = e^{2x} + 2^{-x} + 3.$

$AE = D^3 - 5D^2 + 8D - 4 = 0$

$D = 1$, $(D-1)$ is a factor

| $D=1$ | 1 | -5 | 8 | -4 |
|---|---|---|---|---|
| | | 1 | -4 | 4 |
| | 1 | -4 | 4 | 0 |

$D^2 - 4D + 4 = 0$

$D^2 - 2D - 2D + 4 = 0$

$(D-2) - 2(D-2) = 0$

$D = 2, D = 2.$

$e^{-x\log 2}$

$CF: (C_1 + C_2 x)e^{2x} + C_3 e^{x}.$

$PI = \frac{1}{\quad} \cdot e^{2x} + 2^{-x} + 3.$

$= \frac{1}{D^3 - 5D^2 + 8D - 4} e^{2x} + \frac{1}{D^3 - 5D^2 + 8D - 4} e^{-x\log 2}$

$D = 2$

$D = -\log 2$

$+ \frac{1}{(D-1)(D+2)(D+2)} 3 \cdot e^{0x}$

$D^r$ coming out to be zero.

So; diff $D^r$ and multiply by $x$

$D = 0$

$= \frac{x}{3D^2 - 10D + 8} e^{2x} + \frac{1}{(-\log 2)^3 - 5(-\log 2)^2 + 8(-\log 2) - 4}$

$D = 2.$

$+ \frac{3}{4}$

$\Rightarrow$ again it coming out to be zero;

diff again and multiply by $x$.

$$PI = \frac{x^2}{6D-10} e^{2x} + \frac{2^{-x}}{(-\log 2)^3 - 5(-\log 2)^2 + 8(-\log 2) - 4} - \frac{3}{4}$$

$$GS = c_1 e^x + (c_2 + c_3 x)e^{2x} + PI.$$

4. $(D^3+8)y = x^4 + 2x + 1.$

$C.F. = 0$

$\because D^3 + (2)^3$

$= (a+b)(a^2 - ab + b^2)$

$= (D+2)(D^2 - 2D + (2)^2)$

$\Rightarrow D^3 + 8 = 0$

$D^3 = -8$ $\Rightarrow D = -1 \pm i\sqrt{3}, -2$

$\alpha = 1, \quad \beta = \sqrt{3}$

$$CF = e^{-x}(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x) + c_3 e^{-2x}.$$

$$PI = \frac{1}{D^3+8}(x^4 + 2x + 1)$$

$$= 8\left(1 + \left(\frac{D}{2}\right)^3\right)^{-1}[x^4 + 2x + 1]$$

$$= 8\left(1 - \left(\frac{D}{2}\right)^3 + \left(\left(\frac{D}{2}\right)^3\right)^2 + \ldots\right)(x^4 + 2x + 1)$$

$$= 8\left(1 - \frac{D^3}{8}\right) * (x^4 + 2x + 1)$$

$D(f(x)) = 4x^3 + 2$

$D''(f(x)) = 12x^2$

$D'''(f(x)) = 24x$

$D^{iv}(f(x)) = 24.$

Since the $f(x)$ can be differentiate 4 times Hence we will discard $D^5$ onwards.

$PI = 1 - 24x\,(x^4+2x-1)$

$= x^4+2x-1 - 24x^5 - 48x^2 + 24x$

$= -24x^5 + x^4 - 48x^2 + 26x - 1$

$GS = CF + PI$

$= e^{-x}(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x) + c_3 e^{-2x}$

$-24x^5 + x^4 - 48x^2 + 26x + 1$.

5. $(D^2+2D+1)y = \dfrac{e^{-x}}{x+2}$

$AE = D^2+2D+1$

roots are :- $\dfrac{-2 \pm \sqrt{4-4}}{2a}$

$= -1$ and $-1$

$CF = (c_1 + c_2 x)e^{-x}$.

$1 \times 1$

$D^2+D+D+1$

$D(D+1)+1(D+1)$

$PI = \dfrac{1}{D^2+2D+1}\left(\dfrac{e^{-x}}{x+2}\right)$

put $D = D-1$

$= e^{-x} \dfrac{1}{(D-1)^2+2(D-1)+1} \cdot \dfrac{1}{x+2}$

$= e^{-x} \dfrac{1}{D^2 - \cancel{2D} + \cancel{1} + \cancel{2D} - \cancel{2} + \cancel{1}} \cdot \dfrac{1}{x+2}$

$= e^{-x} \dfrac{1}{D^2} \dfrac{1}{x+2}$

$= e^{-x} \frac{1}{D^2} \cdot \frac{1}{x+2}$

$= e^{-x} \frac{1}{D} \int \frac{1}{x+2}$

$= e^{-x} \frac{1}{D} \log|x+2| + c_3$

$= e^{-x} \int \log(x+2) + c_3$

$PI = e^{-x} \left[ x \log(x+2) - 2x + \log(x+2) \right] + c_4$

$GS = PI + CF$

$= (c_1 + c_2 x) e^{-x} + e^{-x} \left[ x \log(x+2) - 2x + \log(x+2) \right] + c_4$

6. $(D^2 + 3D + 2) y = x \sin 2x.$

$\Rightarrow AE \Rightarrow \quad D^2 + 2D + D + 2 = 0$

$\Rightarrow D(D+2) + 1(D+2) = 0$

$\Rightarrow (D+1)(D+2) = 0$

$D = -1, \ D = -2$

$CF = c_1 e^{-x} + c_2 e^{-2x}$

$PI \Rightarrow y = \frac{1}{D^2 + 3D + 2} \cdot x \sin 2x$

$= \left[ x - \frac{2D+3}{D^2+3D+2} \right] \cdot \frac{1}{D^2+3D+2} \cdot \sin 2x.$

$D^2 = -4.$

$$= \left[x - \frac{2D+3}{D^2+3D+2}\right] \frac{1}{-4+3D+2} \sin 2x$$

$$= \left[x - \frac{2D+3}{D^2+3D+2}\right] \frac{1}{3D-2} \sin 2x.$$

Multiply $N^r$ & $D^r$ by its conjugate

$$= \left[x - \frac{2D+3}{D^2+3D+2}\right] \frac{3D+2}{9D^2-4} \sin 2x$$

$$= \left[x - \frac{2D+3}{D^2+3D+2}\right] \frac{3D+2}{-40} \sin 2x$$

$$= \left[x - \frac{2D+3}{D^2+3D+2}\right] \frac{2\times 3 \times \cos 2x + 2\sin 2x}{-40}$$

$$= \left[\frac{6x\cos 2x + 2x\sin 2x}{-40}\right] - \left[\frac{2D+3}{D^2+3D+2}\right]\left(\frac{6x\cos 2x}{-40}\right)$$

$$- \left(\frac{2D+3}{D^2+3D+2}\right)\left(\frac{2\sin 2x}{-40}\right)$$

$\Downarrow$

$$PI = \left(\frac{7-30x}{200}\right)\cos 2x + \left(\frac{12-5x}{100}\right)\sin 2x$$

Tut.

## (iii) Method of variation of parameters :-
for P.I.

Procedure :-

If C.F of the LDE ($2^{nd}$ order) is in form of $Ay_1 + By_2$ where A & B are arbitary constants; or will say parameter.

By varying the parameter A & B ;

Let P.I $= uy_1 + vy_2$

where

$$u = \int \frac{-y_2 f(x)}{W} dx \quad \& \quad v = \int \frac{y_1 f(x)}{W} dx$$

where

W = Wronskian

$$= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

Ex: (i) $(D^2+4)y = \sec 2x$ $\to f(x)$

AE: $D^2 + 4 = 0$

$D^2 = -4$

$D = \pm 2i$

CF: $A\cos 2x + B\sin 2x$

Let's take PI as,

$$PI = \underset{y_1}{u\cos 2x} + \underset{y_2}{v\sin 2x}$$

$$u = \int \frac{-y_2 f(x)}{w}$$

$W = y_1 y_2' - y_2 y_1'$

$= 2\cos 2x \cdot \cos 2x$

$+ 2\sin 2x \cdot \sin 2x$

$= 2\cos^2 2x + 2\sin^2 2x$

$= 2(\cos^2 2x + \sin^2 2x)$

$= 2$

$$= -\int \frac{\sin 2x \cdot \sec 2x}{2}\, dx$$

$$= -\int \frac{\sin 2x \cdot \frac{1}{\cos 2x}}{2}\, dx$$

$$= -\frac{1}{2}\int \tan 2x\, dx$$

$$u = -\frac{1}{4}\underline{|\log(\sec 2x)|} + c_1$$

$$v = \int \frac{y_1 f(x)}{w} = \int \frac{\cos 2x \cdot \sec 2x}{2} = \frac{1}{2}\int dx = \frac{x}{2}.$$

$PI = uy_1 + vy_2$

$= -\frac{1}{4} \log|\sec 2x| \cos 2x + \frac{x}{2} \sin 2x.$

(ii) $(D^2 - 6D + 9)y = \frac{e^{3x}}{x^2}$

$AE = D^2 - 6D + 9 = 0$

$D = 3, 3.$

$CF = (c_1 + c_2 x)e^{3x} = c_1 e^{3x} + c_2 x e^{3x}$

$= \underbrace{A}_{u}\underbrace{e^{3x}}_{y_1} + \underbrace{B}_{v}\underbrace{xe^{3x}}_{y_2}$

$PI = uy_1 + vy_2$

$y_1 = e^{3x}$

$y_1' = 3e^{3x}$

$y_2 = xe^{3x}$

$y_2' = 3xe^{3x} + e^{3x}$

$u = -\int \frac{y_2 f(x)}{w}$

$= -\int \frac{xe^{3x} \cdot \frac{e^{3x}}{x^2}}{e^{6x}}$

$= -\int \frac{1}{x} dx$

$= -\log|x| + c.$

$w = y_1 y_2' - y_2 y_1'$

$= e^{3x}(3xe^{3x} + e^{3x}) - xe^{3x} \cdot 3e^{3x}$

$= \cancel{3xe^{6x}} - e^{6x} - \cancel{3xe^{6x}}$

$= e^{6x}$

$v = \int \frac{\cancel{e^{3x}}}{\cancel{e^{6x}}} \cdot \frac{\cancel{e^{3x}}}{x^2} dx$

$= \int \frac{1}{x^2} dx \Rightarrow \Rightarrow -\frac{1}{x}$

HW :-

1) $(D^2+1)y = \operatorname{cosec} x$

2) $(D^2-2D+2)y = e^x \tan x$

Ans: $(D^2+1)y = \operatorname{cosec} x$

AE. $D^2+1=0$

$D^2 = -1$

$D = \pm i$

$cf = A\cos x + B\sin x$

$PI = uy_1 + vy_2$

$$u = \int \frac{-y_2 f(x)}{W}$$

$W = y_1 y_2' - y_2 y_1'$

$= \cos x \cdot \cos x - \sin x(-\sin x)$

$= \cos^2 x + \sin^2 x = 1$

$u = -\int y_2 f(x)$

$= -\int \sin x \operatorname{cosec} x \, dx$

$= -\int \sin x \cdot \frac{1}{\sin x} dx \Rightarrow -\int dx$

$= -x + c_1$

$v = \int y_1 f(x) = \int \cos x \cdot \operatorname{cosec} x$

$= \int \cot x = -\log|\operatorname{cosec} x| + c_2$

$PI = uy_1 + vy_2$

$= -x\cos x - \log|\operatorname{cosec} x| \cdot \sin x$

$GS = CF + PI$

(2) $(D^2 - 2D + 2)y = e^x \tan x.$

$AE \rightarrow D^2 - 2D + 2 = 0$

roots $\Rightarrow \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \dfrac{2 \pm \sqrt{4 - 4 \times 2 \times 1}}{2}$

$= \dfrac{2 \pm \sqrt{-4}}{2} = 1 \pm i$

$CF = (A\cos x + B\sin x)e^x = \underset{u}{A}\,\underset{y_1}{\cos x e^x} + \underset{v}{B}\,\underset{y_2}{\sin x e^x}$

$PI = uy_1 + vy_2$

$y_2' = \sin x e^x + e^x \cos x$
$= e^x(\sin x + \cos x)$

$u = \int \dfrac{-y_2 f(x)}{w}$

$y_1' = \cos x e^x - e^x \sin x$
$= e^x(\cos x - \sin x)$

$= -\int \dfrac{\sin x \cancel{e^x} \cdot \cancel{e^x} \cdot \tan x}{\cancel{e^{2x}}}$

$W = y_1 y_2' - y_2 y_1'$

$= \cos x e^x \cdot e^x(\sin x + \cos x)$

$- \sin x e^x e^x(\cos x - \sin x)$

$= -\int \sin x \cdot \dfrac{\sin x}{\cos x}$

$= \cancel{\sin x \cos x e^{2x}} + \cos^2 x e^{2x}$

$- \cancel{\sin x \cos x e^{2x}}$

$+ \sin^2 x e^{2x}$

$= -\int \dfrac{\sin^2 x}{\cos x}$

$= e^{2x}.$

$$u = -\int \frac{\sin^2 x}{\cos x} = -\int \frac{1-\cos^2 x}{\cos x}$$

$$= -\int \frac{1}{\cos x} + \int \cos x$$

$$v = \int \frac{y_1 \, f(x)}{w}$$

$$= \int \frac{\cancel{\cos x} \cdot \cancel{e^{x}} \cdot \cancel{e^{x}} \frac{\sin x}{\cancel{\cos x}}}{\cancel{e^{2x}}}$$

$$= \int \sin x = -\cos x$$

$$PI = u y_1 + v y_2$$

* LDE with variable co-eff.:

(Cauchy's & Legendre's form)

→ constant coefficient of LDE :→

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} \cdots + a_n y = f(x)$$

where $a_0, a_1 \ldots a_n \to$ constt co-efficient

→ GS = CF + P.I.

→ if LDE is of form:-

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = f(x) \quad \text{— (1)}$$

Cauchy's form.

$$\Rightarrow (a_0 x^n D^n + a_1 x^{n-1} D^{n-1} + \cdots + a_n) y = f x$$

to reduce this eqn to constt co-eff ; where $D = \frac{d}{dx}$

put $x = e^z \Rightarrow z = \log x$. $\frac{dz}{dx} = \frac{1}{x}$

$$\therefore \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \cdot \frac{dy}{dz}$$

$$\therefore x \frac{dy}{dx} = \frac{dy}{dz} \quad \text{if } \frac{d}{dz} = D$$

$$\boxed{\therefore x \frac{dy}{dx} = Dy}$$

Similarily

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dz}\left(\frac{dy}{dx}\right) \cdot \frac{dz}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dz}\left(\frac{1}{x} \cdot \frac{dy}{dz}\right) \cdot \frac{1}{x}$$

$$\frac{x\,d^2y}{dx^2} = \frac{1}{x}\left[\frac{d^2y}{dz^2}\right] + \frac{dy}{dz}\left[-\frac{1}{x\cancel{x}}\right] \cdot \cancel{x}$$

$$\therefore x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

$$x^2\frac{d^2y}{dx^2} = D^2y - Dy \qquad D = \frac{dy}{dz}$$

$$= D(D-1)y$$

Similarily

$$x^3\frac{d^3y}{dx^3} = D(D-1)(D-2).y$$

$\therefore$ substituting $x\frac{dy}{dx}$, $x^2\frac{d^2y}{dx^2}$ in (1)

it reduces to constt: co-efficient form whose solution is $\boxed{GS = CF + PI}$ interms of $z$.

→ replace the original variably in $x$.

Ex. $$x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = \cos(\log x) + x\sin(\log x). \quad \text{---} \ (1)$$

Ans

put $x = e^z \Rightarrow x\frac{dy}{dx} = Dy$

$\hookrightarrow z = \log x$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

$$\frac{2 + \sqrt{\cancel{4}-16}}{2}$$
$$= \frac{2 \pm 2\sqrt{3}\,i}{2}$$
$$= 1 \pm \sqrt{3}\,i$$

eqn (1) becomes:-

$$D(D-1)y - Dy + 4y = \cos z + e^z \sin z$$

$$(D^2 - 2D - 4)y = \cos z + e^z \sin z \quad \text{---} \ (11)$$

AE : $D^2 - 2D - 4 = 0$

$$CF = e^z\left[c_1 \cos\sqrt{3}z + c_2 \sin\sqrt{3}z\right]$$

now $$PI = \frac{1}{D^2 - 2D + 4}[\cos z] + \frac{1}{D^2 - 2D + 4}[e^z \sin z]$$

$\hookrightarrow$ put $D^2 = -1$ $\qquad$ $\hookrightarrow$ put $D = (D+1)$

$$= \frac{1}{-1 - 2D + 4}[\cos z] + e^z \frac{1}{(D+1)^2 - 2(D+1) + 4}\sin z$$

$$= \frac{1}{3 - 2D}\cos z + e^z \frac{1}{D^2 + 3}\sin z$$

$$= \frac{3 + 2D}{9 - 4D^2}\cos z + e^z \frac{1}{D^2 + 3}\sin z$$

$$= \frac{3 + 2D}{9 - 4(-1)}\cos z + e^z \frac{1}{-1 + 3}\sin z$$

$$PI = \frac{3\cos z - 2\sin z}{13} + \frac{\sin z}{2} \cdot e^{z}$$

GS = CF + PI

## * Legendre's form:-

$$a_0(ax+b)^n \frac{d^n y}{dx^n} + a_1(ax+b)^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_n y = f(x) \quad \text{—— (1)}$$

put $ax+b = e^z \Rightarrow z = \log(ax+b)$

$$\therefore \frac{dz}{dx} = \frac{a}{a+b}$$

$$\therefore (ax+b)\frac{dy}{dx} = aDy \qquad \rightarrow D = \frac{d}{dz}$$

$$\therefore (ax+b)^2 \frac{d^2y}{dx^2} = a^2 D(D-1)y$$

$\therefore$ (1) reduces to constt co-efficient form.

GS = CF + P.I.

* Solve,

$$(2x+1)^2 \frac{d^2y}{dx^2} - 2(2x+1)\frac{dy}{dx} - 12y = 6x \quad \text{— (1)}$$

→ $2x+1 = e^z \Rightarrow z = \log(2x+1)$

$\Rightarrow (2x+1)\frac{dy}{dx} = 2Dy$

$\Rightarrow (2x+1)^2 \frac{d^2y}{dx^2} = (2)^2 \cdot D(D-1)y$

$= 4[D(D-1)y]$

eq^n (1) becomes

$$4(D(D-1)y) - 2(2Dy) - 12y = 6\left[\frac{e^z-1}{2}\right]$$

$= 4D(D-1)y - 4Dy - 12y = 3e^z - 3$

$\Rightarrow y(4D^2 - 4D - 4D - 12) = 3e^z - 3$

$\Rightarrow (4D^2 - 8D - 12)y = 3e^z - 3$

AE = $4D^2 - 8D - 12 = 0$

$D^2 - 2D - 3 = 0$

$D^2 - 3D + D - 3 = 0$

$D(D-3) + 1(D-3) = 0$

$D = -1, \quad D = 3.$

CF = $C_1 e^{-x} + C_2 e^{3x}$

$$PI = \frac{3}{4}\left(\frac{1}{(D^2-2D-3)}e^{2}\Big|_{D=1} - \frac{1}{(D^2-2D-3)}e^{0z}\Big|_{D=0}\right)$$

$$= \frac{3}{4}\left(\frac{1}{1-2-3}e^2 - \frac{1}{0-0-3}\right)$$

$$= \frac{3}{4}\left(\frac{1}{-4}e^2 + \frac{1}{3}\right)$$

$$= \frac{3}{4}\left(\frac{e^2}{-4} + \frac{1}{3}\right)$$

GS = C.F + P.I.

$$\Rightarrow y = c_1 e^{3z} + c_2 e^{-z} + \frac{3}{4}\left(\frac{e^2}{-4} + \frac{1}{3}\right)$$

$$\boxed{GS = c_1(2x+1)^3 + c_2(2x+1)^{-1} + \frac{3}{4}\left[\frac{1}{3} - \frac{2x+1}{4}\right].}$$

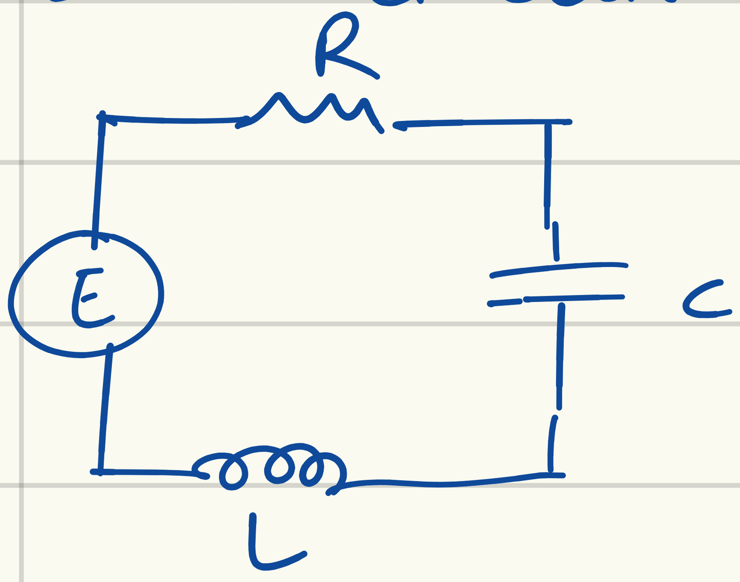
H.W

1) $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 5y = x^2 \sin(\log x)$

2) $(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 2\sin\log(1+x)$

## * Application to electric circuit.

→ LRC circuit :- To form LDE → Kirchoff's law




↳ Potential diff across $R \to Ri$

" " " $L \to L\frac{di}{dt}$

" " " $C \to Q/C$

Constant / $f^n$ of (t).

→ LDE ⇒ $L\frac{di}{dt} + Ri + \frac{Q}{C} = E(t)$

$$\boxed{\frac{dQ}{dt} = i}$$

$$\Rightarrow L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E(t) \quad \text{—— (i)}$$

∴ Eq$^n$ (i) is 2$^{nd}$ order LDE in Q ⇒ sol$^n$ ⇒ $Q(t) = f^n$ of t.

**Ex:-** 1) A ckt → $L = 0.05\,H$, $R = 5\,\Omega$ & $C = 4\times10^{-4}\,F$
if $Q = 0$; $i = 0$ at $t = 0$.
Find $Q(t)$ & $i(t)$ when if there is constant emf of 110 V app.

→ Let LDE → $L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E(t)$

$$\Rightarrow 0.05\frac{d^2Q}{dt^2} + 5\frac{dQ}{dt} + \frac{Q}{4\times10^{-4}} = 110$$

divide eq$^n$ by 0.05

$$\Rightarrow \frac{d^2Q}{dt^2} + 100\frac{dQ}{dt} + 50000\,Q = 2200 \quad \text{—— (1)}$$

LDE with const coefficient

To find CF

$AE = D^2 + 100D + 50000 = 0$

$\Rightarrow$ roots:

$$D = \frac{-100 \pm \sqrt{(100)^2 - 4(50{,}000)}}{2}$$

$$\Rightarrow D = -50 + 50\sqrt{19}\, i$$

$$CF = e^{-50t}\left(A \cos 50\sqrt{19}t + B \sin 50\sqrt{19}t\right)$$

$$PI = \frac{1}{D^2 + 100D + 50000}(2200)e^{0t}$$

put $D = 0$

$$= \frac{2200}{50000} = 0.044.$$

Sol$^n$ :) $Q(t) = CF + PI$

$$Q(t) = e^{-50t}\left[A \cos 50\sqrt{19}t + B \sin 50\sqrt{19}t\right] + 0.044 \quad \text{--- (ii)}$$

initially when $t = 0$; $i = 0$, $Q = 0$

put $t = 0$, $Q = 0$.

$$0 = A + 0.044$$

$$\boxed{A = -0.044}$$

Diff eq$^n$ (ii) to get $i(t)$.
w.r.t to 't'

$$\Rightarrow \frac{dQ}{dt} = i(t) \Rightarrow \frac{dQ}{dt} = -50e^{-50t}\left[A \cos \omega t + B \sin \omega t\right]$$

$$+ e^{-50t}\left[-A\omega \sin \omega t + B\omega \cos \omega t\right]$$

$$i(t) = A\left[-50e^{-50t}\cos\omega t - e^{-50t}\omega\sin\omega t\right]$$
$$+ B\left[-50e^{-50t}\sin\omega t + e^{-50t}\omega\cos\omega t\right]$$

— (ii)

at $t = 0$; $i = 0$

$$0 = -50A + B\omega$$

$$50 \times 0.044) = B \cdot 50\sqrt{19}$$

$$B = \frac{50\times(0.044)}{217.94}$$

$$= -0.010.$$

put A & B in (ii) & (iii)

eq^n (ii) becomes

$$q(t) = e^{-50t}\left[-0.044\cos 50\sqrt{19}t - 0.010\sin 50\sqrt{19}t\right]$$
$$+ 0.044$$

eq^n (iii) becomes

$$q(t) = -0.044\left[-50e^{-50t}\cos 50\sqrt{19}t - e^{-50t}\omega\sin 50\sqrt{19}t\right]$$
$$-0.010\left[-50e^{-50t}\sin 50\sqrt{19}t + e^{-50t}\cos 50\sqrt{19}t\right]$$

2) Determine $q(t)$ & $i(t)$.

with LRC, $L = 0.5H$

$R = 6\Omega$

$C = 0.02f$

& $E(t) = 24 \sin \omega t$, when $t = 0$, $q = 0$, $i = 0$

also find steady state cond.

Ans :- LDE.

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E(t)$$

$$\Rightarrow 0.5D^2 + 6D + \frac{q}{0.02} = 24 \sin 10t \quad \text{— (1)}$$

$$\Rightarrow D^2 + 12D + 100q = 48 \sin 10t$$

CF →

At $\Rightarrow$ $D^2 + 12D + 100 = 0$

roots $= -6 \pm 8i$

$$CF = e^{-6x}(A \cos 8t + B \sin 8t)$$

$$PI: \frac{1}{D^2 + 12D + 100} 48 \sin 10t.$$

$D^2 = -100$

$$= 48 \frac{1}{\cancel{-100} + 12D + \cancel{100}} \sin 10t$$

$$= 48 \frac{1}{12D} \sin 10t$$

$$= \frac{48}{12} \int \sin 10t \Rightarrow 4 \int \sin 10t = -\frac{4}{10} \cos 10t$$

$$\boxed{PI = -0.4 \cos 10t}$$

HW →

put A & B in (ii) & (iii)

(ii) becomes

$$Q(t) = e^{-6t}\left[0.4\cos 8t + 0.3\sin 8t\right] - 0.4\cos 10t$$

(iii) becomes

&

$$I(t) = 0.4e^{-6t}\left[-6\cos 8t - 8\sin 8t\right] + 0.3e^{-6t}\left[-6\sin 8t + 8\cos 8t\right] + 4\sin 10t.$$

→ for steady state condition

$$\boxed{t \to \infty, \quad e^{-\infty} = 0}$$

$\therefore$ $Q(t) \rightarrow$ is function of cosine $\not\rightarrow$ $f^n \rightarrow t \rightarrow \infty$

↓

Cannot be determined

HW:

Find Q & I in LRC ckt if

$L = 2;\ R = 4;\ C = 0.05$

& $E = 100v$, initially $t = 0$, $Q = 0$, $I = 0$

& find steady state cond.

$sol^n :\rightarrow$

$$Q(t) = CF + PI$$

$$Q(t) = e^{-6t}\left[A\cos 8t + B\sin 8t\right] - 0.4\cos 10t \quad \text{--- (i)}$$

Since, $Q = 0, t = 0.$

$$0 = 1[A] - 0.4$$

$$\boxed{A = 0.4}$$

for $i(t)$, D. $Q(t)$ wrt $t$.

$$i(t) = -6e^{-6t}\left[A\cos 8t + B\sin 8t\right] + e^{-6t}\left[-8A\sin 8t + 8B\cos 8t\right] + 0.4\times 10\sin 10t$$

$$= Ae^{-6t}\left[-6\cos 8t - 8\sin 8t\right] + Be^{-6t}\left[-6\sin 8t + 8\cos 8t\right) + 4\sin 10t \quad \text{--- (ii)}$$

Sin $i = 0, t = 0$

$$0 = A[-6] + B[8] + 0$$

$$\therefore 0 = -6A + 8B,$$

$$6A = 8B$$

$$B = \frac{6A}{8} = \frac{6\times 0.4}{8} = 0.3$$

# UNIT 2 | Application of L.D.E.

$z = ax + by$ (1) [a,b -> arb. const.]

Diff. partially w.r.t. x,

$$\left(\frac{\partial z}{\partial x}\right)_{y = const.} = a$$

$$\left(\frac{\partial z}{\partial y}\right)_{x = const.} = b$$

From (1),

$$\boxed{z = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}}$$ in P.D.E. of 1st order

## Standard P.D.E.

1) Wave equation

$$\frac{\partial^2 x}{\partial x^2} = c^2 \frac{\partial^2 x}{\partial y^2}$$

2) Heat e uation (1-dim)

$$\frac{\partial u}{dt} = c^2 \frac{\partial^2 u}{\partial x^2}$$

3) Laplace equation (2-dim)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Formation of heat equation → through semi infinite bar (1-dim heat flow)

# Laws :- i) Heat flows from higher to lower temp.
ii) The quantity of heat flowing through the surface is proportional to the area and rate of change of temperature w.r.t. the dist. normal to the area.

$A \rightarrow$ cross-sectional area
$x \rightarrow$ dist. normal to $A$

$\therefore$ quantity of heat entering into the slab $= Q_1 \propto A \dfrac{du}{dx}$

where $\mu(x,t)$ is temp. depending upon $x$ and $t$ = time

Similarly, if $Q_2$ is quantity of heat flowing out of surface then $Q_2 = -kA\left(\dfrac{\partial u}{\partial x}\right)$ —— (2)

Total heat retained in the slab
$\Rightarrow Q_1 - Q_2 = kA\left[\left(\dfrac{\partial u}{\partial}\right.\right.$

Equating (3) & (4),

$\Rightarrow$

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* $$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{——— (i)}$$ [Heat flow equation]

solution = $u(x,t) = ?$

Solⁿ by method of seperation of variables.

Let $u = F(x) \cdot G(t)$ be the solution ①.

Diff ①. w.r.t. partially & Diff (1) partially wrt $x$ (2 times)

$$\frac{\partial u}{\partial t} = F(x) \cdot G'(t) \qquad \frac{\partial^2 u}{\partial x^2} = F''(x) \cdot G(t)$$

from ① $F(x) \cdot G'(t) = c^2 F''(x) G(t)$

$$\Rightarrow \boxed{\frac{G'}{c^2 G} = \frac{F''}{F} = k \text{ (say)}}$$

$$\underbrace{\boxed{\frac{G'}{c^2 G} = k} \quad \& \quad \boxed{\frac{F''}{F} = k}} \text{ ——— ②}$$

→ Depending upon $k$

→ Case I :- $k = 0$

$\Rightarrow G'(t) = 0$ & $F''(x) = 0$

$\Rightarrow \boxed{G(t) = c_1}$ $\boxed{F(x) = c_2 x + c_3}$

• Solution is

$$u(x,t) = F(x) \cdot G(t)$$
$$= c_1 (c_2 x + c_3)$$
$$\boxed{u(x,t) = Ax + B}$$

$A = c_1 c_2$, $B = c_3 c_1$

→ $\boxed{\text{Case - II}}$ :

When $k>0$ : Let $k = m^2$

$\therefore$ (2) $\Rightarrow \dfrac{G'}{c^2 G} = m^2$ & $\dfrac{F''}{F} = m^2$

$\Rightarrow G' - c^2 m^2 G = 0$ & $F'' - m^2 F = 0$.

ordinary D'E

$\Rightarrow (D - c^2 m^2) G(t) = 0$ & $(D^2 - m^2) F(x) = 0$

$\Rightarrow G(t) = C_1 e^{c^2 m^2 t}$ & $F(x) = c_2 e^{mx} + c_3 e^{-mx}$

from ① solution;

$$u(x,t) = F(x) \cdot G(t)$$

$$u(x,t) = c_1 e^{c^2 m^2 t}\left[c_2 e^{mx} + c_3 e^{-mx}\right]$$

$$\boxed{u(x,t) = \left[A e^{mx} + B e^{-mx}\right] e^{c^2 m^2 t}}$$

→ $\boxed{\text{Case - III}}$ → If $k<0$. say $k = -m^2$

$\therefore$ 2

$$\frac{G'}{c^2 G} = -m^2 \quad , \& \quad \frac{F''}{F} = -m^2$$

$\Rightarrow G' + c^2 m^2 G = 0$ & $F'' + m^2 F = 0$

$\Rightarrow \boxed{G(t) = C_1 e^{-c^2 m^2 t}}$ & $(D^2 + m^2) F(x) = 0$

$D = \pm mi$

$\Rightarrow \boxed{F(x) = C_2 \cos mx + C_1 \sin mx}$

$\therefore$ Solⁿ $u(x,t) = F(x) \cdot G(t)$

$= C_1 e^{-c^2 m^2 t} (C_2 \cos mx + C_3 \sin mx)$

$$\boxed{u(x,t) = \left[A \cos mx + B \sin mx\right] e^{-c^2 m^2 t}}$$ → most appropriate

Note # as the problem is of heat conduction →
as the time increases, temp. should decrease
⇒ Solution obtained for $k>0$; is an appropriate solution for heat eqⁿ

$$\Rightarrow \boxed{u(x,t) = [A\cos mx + B\sin mx]e^{-c^2m^2t}}$$

* $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ with boundary condition

(1) $u(0,t) = 0 \quad \forall t$ → this sign represents for "all"

(2) $u(l,t) = 0 \quad \forall t$

(3) $u(x,0) = 50 \quad 0 \le x \le l$

Solⁿ: Let the solⁿ be $u(x,t) = [A\cos mx + B\sin mx]e^{-c^2m^2t}$ —— ①

from ①.

$u(0,t) = 0 \Rightarrow$

$0 = Ae^{-c^2m^2t}$ → This eqⁿ can't be zero else there will no solⁿ & $\boxed{eq^n\ ① = 0}$

$\Rightarrow \boxed{A = 0}$

then ① becomes

$u(x,t) = [B\sin mx]e^{-c^2m^2t}$ —— ①'

new from codⁿ ②

$u(l,t) = 0 \Rightarrow 0 = B\sin m e^{-c^2mt}$ → They can't be zero else ① will be zero

$\Rightarrow \sin m = 0$

$\Rightarrow \boxed{m = n\pi}$ when $n = 0,1,2,3,\ldots$

then (II) becomes

$$u(x,t) = B\sin(n\pi)x\, e^{-c^2 n^2 \pi^2 t} \quad \text{--- (3)}$$

$\because$ Condition (3) $\Rightarrow$ $u(x,0) = 50$

$\therefore$ put $t=0$ in (3)

$$u(x,0) = 50.$$

$$\Rightarrow \boxed{B\sin(n\pi x) = 50} \qquad n = \cancel{0}, 1, 2, 3, \ldots$$

OR

$$\boxed{\underbrace{50}_{f(x)} = \sum_{n=1}^{\infty} b_n \sin(n\pi x)} \quad \text{--- (4)}$$

↳ equivalent to half range fourier sine series in $0 \le x \le l$ where $l$ = length of rod.

$$\text{where } b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$\therefore b_n = 2\int_0^1 50 \sin n\pi x\, dx$$

$$= 100\left[\frac{-\cos n\pi x}{n\pi}\right]_0^1$$

$$\therefore b_n = \frac{-100}{n\pi}\left[\cos n\pi - \cos 0\right]$$

$$= \frac{-100}{n\pi}\left[(-1)^n - 1\right]$$

$$\therefore b_n = 100\left[\frac{1-(-1)^n}{n\pi}\right] \quad \text{——} \quad (5)$$

from (3), (4) and (5)

$$u(x,t) = \sum_{n=1}^{\infty} 100\left[\frac{1-(-1)^n}{n\pi}\right]\sin(n\pi x)\, e^{-c^2n^2\pi^2 t}$$

$$\boxed{u(x,t) = \frac{100}{\pi}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n}\sin(n\pi x)\, e^{-c^2n^2\pi^2 t}}$$

temp at any time 't'.

→ if temp required at half distance ⇒ put $x = \frac{1}{2}$ mt.

$$u\left(\frac{1}{2}, t\right) = \frac{100}{\pi}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n}\sin\left(\frac{n\pi}{2}\right) e^{-c^2n^2\pi^2 t}$$

→ 0 for $n$ = even

$(-1)^n$ for $n$ = odd

* Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ with

(1) $u(0,t) = 0 \quad \forall t$

(2) $u(\pi, t) = 0 \quad \forall t$

(3) $u(x,0) = \pi x - x^2 \quad 0 \le x \le \pi$

Solⁿ:- Let $u(x,t) = (A\cos mx + B\sin mx)e^{-c^2m^2t}$ —— ①

from condⁿ ①

$u(0,t) = 0$

$\Rightarrow \quad 0 = Ae^{-c^2m^2t}$

$\boxed{A = 0}$

① becomes

$\boxed{u(x,t) = B\sin mx\, e^{-c^2m^2t}}$ —— ②

given $c=1 \Rightarrow c^2 = 1$

from condⁿ ⓘⓘ, eqⁿ ② becomes

$u(\pi, t) = 0$

$0 = B\sin m\pi \cdot e^{-m^2t}$

$\sin m\pi = 0$

$\sin m\pi = \sin n\pi \qquad n = 0, 1, 2, -$

$\boxed{m = n}$ where $n = 0, 1, 2, -$

Hence eqⁿ ② becomes

$\boxed{u(x,t) = B\sin(nx)\, e^{-n^2t}}$ —— ③

From condⁿ (iii) . eqⁿ (3) becomes

$$u(x,0) = \pi x - x^2$$

put $t=0$ in eqⁿ (3)

$$u(x,0) = B\sin(n)x.$$

$$\pi x - x^2 = B\sin(nx)$$

$$\boxed{\underbrace{\pi x - x^2}_{f(x)} = \sum_{n=1}^{\infty} b_n \sin(nx)} \quad \longrightarrow (4)$$

$$b_n = \frac{2}{l}\int_0^l f(x) \cdot \sin\frac{n\pi x}{l}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2) \sin\left(\frac{n\cancel{\pi} x}{\cancel{\pi}}\right)$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2) \cdot \sin(nx).$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(-\frac{\cos nx}{n}\right) - \int_0^{\pi}\left[\sin nx \cdot (\pi - 2x)\right]\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(-\frac{\cos nx}{n}\right) - \pi\int_0^{\pi} \sin nx + \int_0^{\pi} \sin nx \cdot 2x\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{-\cos nx}{n}\right)\right]_0^{\pi} - \pi\left(\frac{-\cos nx}{n}\right)_0^{\pi} + I_3 \quad \text{———} (5)$$

$$I_3 = \int_0^{\pi} \sin nx \cdot 2x$$

$$= 2x \cdot \frac{-\cos nx}{n} - \int_0^{\pi} \sin nx \cdot 2$$

$$: \left[2x \cdot \frac{\cos nx}{n} - 2 \frac{-\cos nx}{n}\right]_0^{\pi}$$

$$= \left[- \frac{2x \cos nx}{n} + \frac{2 \cos nx}{n}\right]_0^{\pi}$$

putting $I_3$ in (9)

$$b_n = \frac{2}{\pi}\left[\pi x - x^2 \left(\frac{-\cos nx}{n}\right) + \frac{\pi \cos nx}{n} - \frac{2x \cos nx}{n} + \frac{2\cos nx}{n}\right]_0^{\pi}$$

using generalized rule :-

$$b_n = \frac{2}{\pi}\Bigg[(\pi x - x^2)\left(-\frac{\cos nx}{n}\right) - (\pi - 2x)\left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\Bigg]_0^{\pi}$$

$$b_n = \frac{2}{\pi}\Bigg[\pi^2 - \pi^2\left(\frac{-\cos n\pi}{n}\right) - (\pi - 2\pi)\left(\frac{-\sin n\pi}{n^2}\right) + (-2)\left(\frac{\cos n\pi}{n^3}\right)$$

$$- (0 - 0^2)\left(\frac{-\cos n0^{\circ}}{n}\right) -$$

# Generalised rule for int by parts.

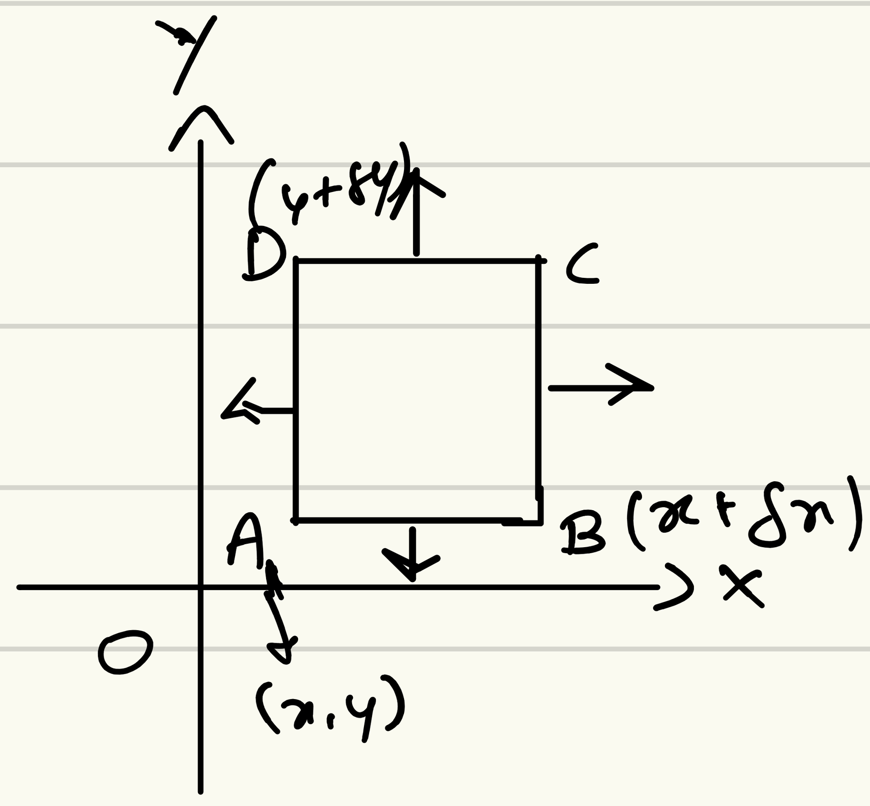
$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 + \ldots$$

$u', u'' \rightarrow$ derivative of $1^{st}$ fn

$v_1, v_2 \rightarrow$ integrals of $2^{nd}$ fn

* 2-Dim Heat flow (Laplace eqⁿ)

In steady state condition (not depending upon time t)




Heat flows through metal plate ABCD entering through AB & AD and flowing out from CD & BC.

∴ As per law of heat condition ⇒ quantity of heat flowing is proportional to the area & rate of change of temp. wrt distance normal to surface

⇒ sides $\delta x, \delta y$ ⇒ $\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$

'u' is temp at any time

is Laplace eqⁿ ⇒ 'u' is temp depending upon x & y

→ Solution of Laplace eqⁿ :-

[Method of seperation of variable]

Let $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ ——— (1)

Let $u = X \cdot Y$ be solⁿ of (1) where X is fⁿ of x alone & Y is fⁿ of y alone

$\therefore \frac{\partial^2 u}{\partial x^2} = X''Y$ & $\frac{\partial^2 u}{\partial y^2} = XY''$

from (1)

$\Rightarrow \boxed{X''Y + Y''X = 0} \Rightarrow \boxed{\frac{X''}{X} = -\frac{Y''}{Y} = K}$

Considering $\frac{X''}{X} = k$ & $\frac{-Y''}{Y} = -k$

$\hookrightarrow$ (2) $\qquad\qquad$ $\hookrightarrow$ (3)

$\therefore$ (2) $\Rightarrow$ $\boxed{X'' - kX = 0}$ ; (3) $\Rightarrow$ $\boxed{Y'' + kY = 0}$

which are ordinary diff eqn.

→ Case (I) when $k = 0$

$\Rightarrow D^2 x = 0$ & $D^2 y = 0$

$\Rightarrow X = c_1 + c_2 x$ & $Y = c_3 + c_4 y$

$\therefore u = X \cdot Y$

$$\boxed{u(x,y) = (c_1 + c_2 x) + (c_3 + c_4 y)}$$

solution for $k=0$

→ Case (II) when $k > 0$, say $k = m^2$

$\therefore$ (2) & (3) becomes

$(D^2 - m^2) X = 0$ & $(D^2 + m^2) Y = 0$

$\Rightarrow X = c_1 e^{mx} + c_2 e^{-mx}$ & $Y = c_3 \cos my + c_4 \sin my$

$\therefore u(x,y) = X \cdot Y$

$$\boxed{u(x,y) = (c_1 e^{mx} + c_2 e^{-mx})(c_3 \cos my + c_4 \sin my)}$$

solution for $k > 0$

* Case-III, when $k<0$, say; $k=-m^2$

$\therefore$ ② & ③ becomes

$(D^2+m^2)x=0$ ; & $(D^2-m^2)y=0$

$\therefore$ solⁿ is

$X=(c_1\cos mx + c_2\sin mx)$ & $Y=(c_3 e^{my}+c_4 e^{-my})$

$$\boxed{u(x,y)=(c_1\cos mx + c_2\sin mx)(c_3e^{my}+c_4e^{-my})}$$

solⁿ for $k<0$

Note- As per the boundary condⁿ appropriate solⁿ is selected

1) if $u(x,\infty)=0$, $\forall x\in(0,l)$ $\Rightarrow$ Select solⁿ for $k<0$.

$\Rightarrow \boxed{u(x,y)=(c_1\cos mx + c_2\sin mx)(c_3e^{my}+c_4e^{-my})}$

2) if $u(\infty,y)=0$, $\forall x\in(0,l)$ $\Rightarrow$ select solⁿ for $k>0$

$\Rightarrow u(x,y)=(c_1e^{mx}+c_2e^{-mx})(c_3\cos my + c_4\sin my)$

→ Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

using b.c.

1) $u(x, \infty) = 0$
2) $u(0, y) = 0$
3) $u(l, y) = 0$
4) $u(x, 0) = x(l - x)$
$\forall \; 0 < x < l$

Using solⁿ $(k<0)$

$u(x,y) = (c_1 \cos mx + c_2 \sin mx)(c_3 e^{my} + c_4 e^{-my})$

1) $c_3 = 0$;

$u(x,y) = (c_1 \cos mx + c_2 \sin mx)(c_4 e^{-my})$

$= (c_1 c_4 \cos mx + c_2 c_4 \sin mx) e^{-my}$

$$\boxed{u(x,y) = (A \cos mx + B \sin mx) e^{-my}.} \quad \text{— (2)}$$

2) $u(0,y) = 0 \Rightarrow$ eqⁿ (2) becomes

$0 = A e^{-my} \Rightarrow \boxed{A = 0}$

$$\Rightarrow \boxed{u(x,y) = B \sin mx \, e^{-my}} \quad \text{— (3)}$$

3) Using condⁿ (3)

$u(l, y) = 0$

$\Rightarrow 0 = C_6 \sin m e^{-my} \Rightarrow \sin m = 0.$

$m = n\pi, \quad n = 1, 2, 3 \ldots$

③ becomes

$u(x,y) = C_6 \sin n\pi x \, e^{-n\pi y}$

$\Downarrow$ $\quad n = 1, 2, 3, \ldots$

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin n\pi x \, e^{-n\pi y} \quad \text{——— } ①$$

(iv) Using condⁿ ④
$u(x,0) = x(1-x)$

① becomes

$$x(1-x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$$

is half f. sine series in which

$$b_n = \frac{2}{1} \int_0^1 x(1-x) \sin \frac{n\pi x}{1} \, dx.$$

# Laplace Transforms

* $\left.\begin{array}{l}ODE\\PDE\end{array}\right\} \xrightarrow{L.T} A.E$

time $\rightarrow$ frequency

$f(t) \rightarrow F(s)$

Parameter

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$= F(s)$$

- Analog system $\rightarrow$ continuous $\rightarrow$ L.T, F.T

Digital system $\rightarrow$ Discrete — Z.T

---

* **Defn :-**

If $f(t)$ is function of time 't'.
then

$$\underbrace{L[f(t)]}_{\text{time}} = \int_0^\infty e^{-st} f(t)\,dt = \underbrace{F(s)}_{\text{frequency}}$$

$s \rightarrow$ parameter

Ex : 1 :-

$$L[1] = \int_{t=0}^{\infty} e^{-st} (1)\, dt$$

$$= \left[ \frac{e^{-st}}{-s} \right]_0^{\infty} = -\frac{1}{s} \left[ e^{-\infty} - e^{0} \right] \qquad \boxed{e^{-\infty} \to 0}$$

$$= -\frac{1}{s} (0 - 1)$$

$$= \frac{1}{s}$$

$$L[k] = \int_0^{\infty} e^{-st} (k)\, dt = k \left[ \frac{e^{-st}}{s} \right]_0^{\infty} = \frac{k}{s}$$

2) $$L[e^{at}] = \int_0^{\infty} e^{-st} \cdot e^{at}\, dt$$

$$= \int_0^{\infty} e^{-(s-a)t}\, dt = \left[ \frac{e^{-(s-a)t}}{-(s-a)} \right]_0^{\infty}$$

$$= \frac{-1}{s-a} \left[ e^{-\infty} - e^{0} \right]$$

$$= \frac{1}{s-a} \quad \text{if } s > a.$$

Ex → $L\left[e^{(-3/2)t}\right] = \dfrac{1}{s+\frac{3}{2}}$ (from last exa)

(3) $L[\sin(at)] = \int_0^\infty e^{-st}\sin(at)\,dt$

$\rightarrow \left\{\dfrac{e^{-st}}{s^2+a^2}\left[-s\sin at - a\cos at\right]\right\}_0^\infty$

$= 0 - \dfrac{1}{s^2+a^2}(0-a)$

$$\boxed{L[\sin(at)] = \frac{a}{s^2+a^2} = F(s)}$$

Formullas

$\int e^{ax}\sin bx\,dx = \dfrac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

&

$\int e^{ax}\cos bx\,dx = \dfrac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

(4) $L[\cos(at)] = \int_0^\infty e^{-st}\cos(at)\,dt$

$= \left\{\dfrac{e^{-st}}{s^2+a^2}\left[-s\cos at + a\sin at\right]\right\}_0^\infty$

$= 0 - \dfrac{1}{s^2+a^2}[-s+0]$

$$\boxed{L[\cos(at)] = \frac{s}{s^2+a^2}}$$

Formula :- $2\cos^2\theta = 1+\cos 2\theta$

$2\sin^2\theta = 1-\cos 2\theta$

5) $L[\cos^2 t] = L\left[\frac{1+\cos 2t}{2}\right]$

$= L\left[\frac{1}{2}\right] + L\left[\frac{\cos 2t}{2}\right]$

$= \frac{1}{2s} + \frac{1}{2}\left(\frac{s}{s^2+4}\right)$

6) $L[\sinh(at)] = \int_0^\infty e^{-st}\left(\frac{e^{at}-e^{-at}}{2}\right)dt$

$\sinh ax = \frac{e^{ax}-e^{-ax}}{2}$

$\cosh ax = \frac{e^{ax}+e^{-ax}}{2}$

$= \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)} - \frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty$

$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{(s+a)}\right]$

$= \frac{1}{2}\left[\frac{2a}{s^2-a^2}\right] = \frac{a}{s^2-a^2}$

7) $L[\cosh(at)] = \int_0^\infty e^{-st}\left(\frac{e^{at}+e^{-at}}{2}\right)dt = \frac{s}{s^2-a^2}$

8) $L[t^n] = \int_0^\infty e^{-st}\, t^n\, dt$ , put $st = u$ as $t\to 0,\ u\to 0$

$\Rightarrow s\,dt = du$ as $t\to\infty,\ u\to\infty$

$dt = \frac{du}{s}$

$\boxed{t = \frac{u}{s}}$

$$\rightarrow L[t^n] = \int_0^{\infty} e^{-u}\left(\frac{u}{s}\right)^n \frac{du}{s}$$

$$= \frac{1}{s^{n+1}} \int_0^{\infty} e^{-u} \cdot u^n \, du$$

$$= \frac{1}{s^{n+1}} \Gamma(n+1)$$

but $n! = \Gamma(n+1)$ ($n$ = +ve int)

$$\boxed{L[t^n] = \frac{1}{s^{n+1}} \, n!}$$

9. Find; $L\left[t^3 + \sin^2 2t + e^{\sqrt{t}}\right]$

$$= L[t^3] + L[\sin^2 2t] + L[e^{t/2}]$$

$$= \frac{3!}{s^{3+1}} + L\left[\frac{1 - \cos 4t}{2}\right] + L\left[e^{\frac{1}{2}t}\right]$$

$$= \frac{6}{s^4} + \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+16}\right] + \frac{1}{s - \frac{1}{2}} = F(s)$$

using $L[t^n]$

$L[\cos at]$

$L[e^{at}]$

| | $f(s)$ | $f(t)$ |
|---|---|---|
| 1. | $1$ | $1/s \quad s>0$ |
| 2. | $e^{at}$ | $1/(s-a) \quad s>a$ |
| 3. | $\sin at$ | $\dfrac{a}{s^2+a^2}$ |
| 4. | $\cos at$ | $\dfrac{s}{s^2+a^2}$ |
| 5. | $\sinh at$ | $\dfrac{a}{s^2-a^2}$ |
| 6. | $\cosh at$ | $\dfrac{s}{s^2-a^2}$ |
| 7. | $t^n$ | $\dfrac{\overline{\lvert n+1}}{s^{n+1}}$ or $\dfrac{n!}{s^{n+1}}$ |

* Existence condⁿ [sufficient condition]

If $f(t)$ is piecewise continuation $f^n$ & is of exponential order then its L.T exists.

laplace transformation.

Exponential order means. $|f(t)| < Me^{at}$

– Piecewise continuous means $f^n$ is continuous in the part of the interval.

Ex:-

$$f(t) = 0 \quad 0 < t < 1$$
$$= 1 \quad 1 < t < 2$$
$$= 0 \quad t > 2$$

from defⁿ

$$L[f(t)] = \int_0^2 e^{-st} f(t)\, dt$$

$$= \int_0^1 0 + \int_1^2 1\, e^{-st} dt + \int_2^\infty 0$$

$$= \int_1^2 t e^{-st} dt$$

$$= t\left(\frac{e^{-st}}{-s}\right) - \int \frac{e^{-st}}{s} dt$$

$\therefore$ it will come in terms of $s$ = $\underline{F(s)}$.

# $f(t)$ is of exponential order :-

if $|f(t)| < Me^{\alpha t}$ where $\alpha, M$ are constants.

$\Rightarrow e^{-\alpha t} |f(t)|$ is bounded $\Rightarrow$ finite.

Ex:- 1) $f(t) = t^2$ is of exp order $= |t^2| < e^{3t}$

2) $f(t) = e^{t^2} \Rightarrow |e^{t^2}| > Me^{\alpha t}$ → not of exp order

$\therefore$ LT doesn't exist.

3) $f(t) = \tan t = \int e^{-st} \tan t \, dt$

→ not exists

→ LT doesnot exist.

* Properties :-

P (i) Linearity ⇒ $L[a f(t) + b g(t)] = a L[f(t)] + b L[g(t)]$

P (ii) If $L[f(t)] = F(s)$ then $L[e^{-at} f(t)] = F(s+a)$ → 1st shifting property

Consider $L[e^{-at} f(t)] = \int_0^\infty e^{-st} [e^{at} f(t)] = \int_0^\infty e^{-(s+a)t} f(t) \, dt$

$\Rightarrow F(s+a)$

Eg:-4) $L[e^{-3t} \sin 2t]$, $a = 3$, $f(t) = \sin 2t$

$\therefore L[\sin 2t] = \frac{2}{s^2+4} \Rightarrow F(s) = \frac{2}{(s+3)^2+4}$

$\therefore$ put $s = s+3$ ⇒

$L[e^{-3t} \sin 2t] = F(s+3).$

eg 5) $L[e^{2t} t^2]$ → find L.T.

$a = -2, \quad f(t) = t^2$

$\therefore \; L[f(t)] = L[t^2] = \frac{2!}{s^{2+1}} = \frac{2}{s^3} = F(s)$ — ①

$\therefore \; L[e^{-(-2t)} t^2] = F(s-2)$ → put $s = s-2$ in ①

$= \frac{2}{(s-2)^3}$

6.) $L[e^{-t} \sin^2 t]$

$a = 1, \quad f(t) = \sin^2 t = \frac{1 - \cos 2t}{2}$

$L\left[e^{-t} \frac{(1-\cos 2t)}{2}\right] = L\left[\frac{e^{-t}}{2}\right] - L\left[\frac{e^{-t}\cos 2t}{2}\right]$

$= \frac{1}{2} L[e^{-t}] - \frac{1}{2} L[e^{-t} \cos 2t]$

$\hookrightarrow L[\cos 2t] = \frac{s}{s^2+4} = F(s)$

$= \frac{1}{2}\left(\frac{1}{s+1}\right) - \frac{1}{2} F(s+1)$

$= \frac{1}{2}\left(\frac{1}{s+1}\right) - \frac{1}{2}\left(\frac{s+1}{(s+1)^2+4}\right)$

7) $L\left[e^{-t/2} \cdot \sqrt{t}\right]$

$a = \frac{1}{2}$. $\quad f(t) = \sqrt{t} = t^{1/2}$

$\Gamma n+1 = n\Gamma n$

$\Gamma\frac{1}{2} = \sqrt{\pi}$

$\Rightarrow F(s) = L[\sqrt{t}]$

$= L[t^{1/2}]$

$= \frac{\Gamma \frac{1}{2}+1}{s^{\frac{1}{2}+1}}$

$= \frac{\frac{1}{2}\Gamma\frac{1}{2}}{s^{3/2}}$

$= \frac{\sqrt{\pi}/2}{s^{3/2}} = F(s)$

$\because a = \frac{1}{2}$

$\therefore$ changing $s = (s + \frac{1}{2})$

$$L\left[e^{-t/2} \cdot \sqrt{t}\right] = \frac{\sqrt{\pi}}{2\left(s+\frac{1}{2}\right)^{3/2}}$$

P-(iii) change of scale $\rightarrow$ if $L[f(t)] = F(s)$

then

$$L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$$

Ex:-

$$L\left[\sin(3t)\right] = \frac{1}{3} F\left(\frac{s}{3}\right)$$

where $F(s) = L[\sin t] = \frac{1}{s^2+1}$

$$= \frac{1}{3}\left[\frac{1}{\left(\frac{s}{3}\right)^2+1}\right]$$

$$= \left(\frac{3}{s^2+9}\right)$$

* Properties of L.T. (continue...)

P(iv) If $L[f(t)] = F(s)$

then

$$L[F(t)] = e^{-as} f(s)$$

where

$$\left\{\begin{array}{ll} F(t) = f(t-a) & t > a \\ \quad = 0 & t < a \end{array}\right.$$

Ex:- $F(t) = (t-1)^2 \quad t > 1$

$\quad = 0 \quad t < 1$ by $2^{nd}$ shifting

here $L[F(t)] = e^{-as} F(s)$

here

$$a = 1 \quad \& \quad F(s) = L[f(t)]$$

here

$$f(t-a) = f(t-1) = (t-1)^2$$

$\therefore f(t) = t^2 \Rightarrow L[t^2] = \frac{2!}{s^{2+1}} = \frac{2}{s^3} = F(s)$

$\therefore L[F(t)] = e^{-s}\left[\frac{2}{s^3}\right]$

P (V) Multiplication by t.

if $L[f(t)] = F(s)$

then

$$L[t\,F(t)] = (-1)\frac{d}{ds}F(s)$$

III'y: $L[t^2 F(t)] = (-1)^2 \frac{d^2}{ds^2} F(s)$

$\vdots$

$$L[t^n F(t)] = (-1)^n \frac{d^n}{ds^n} F(s)$$

Ex: 1) find $L[t\sin 3t]$

here

$$f(t) = \sin 3t \Rightarrow F(s) = \frac{3}{s^2+9}$$

$$\Rightarrow L[t\sin 3t] = (-1)\frac{d}{ds}\left(\frac{3}{s^2+9}\right)$$

$$= (-1)\,3\left[\frac{-1}{(s^2+9)^2}\cdot 2s\right]$$

$$= \frac{6s}{(s^2+9)^2}$$

2) find $L[t^2 e^{-3t}]$

$f(t) = e^{-3t}$ using $L[e^{at}] = \frac{1}{s-a}$

here; $a = -3$

$$F(s) = \frac{1}{s+3}.$$

$\therefore \; L[t^2 f(t)] = (-1)^2 \dfrac{d^2}{ds^2} F(s)$

$= \dfrac{d^2}{ds^2}\left[\dfrac{1}{s+3}\right]$

$= \dfrac{d}{ds}\left[\dfrac{-1}{(s+3)^2}\right] = -\dfrac{d}{ds}\left[(s+3)^{-2}\right]$

using $\dfrac{dx^n}{dx} = nx^{n-1}$

$= +2(s+3)^{-3}$

$= \dfrac{2}{(s+3)^3}$

A

P (vi):- division by t

If $L[f(t)] = F(s)$,
then
$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds$$

Ex:- Find $L\left[\dfrac{1-\cos t}{t}\right]$

here

$f(t) = 1 - \cos t$

$F(s) = L[1-\cos t]$

$= L[1] - L[\cos t]$

$F(s) = \dfrac{1}{s} - \dfrac{s}{s^2+1}$

$$\therefore\ L\left[\frac{1-\cos t}{t}\right] = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+1}\right) ds$$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{2s}{s^2+1}\right) \frac{ds}{2}$$

$$= \left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty$$

$$= \left[\log\left(\frac{s}{\sqrt{s^2+1}}\right)\right]_s^\infty$$

$$= \left[\log\left(\frac{s}{\sqrt{s^2\left(1+\frac{1}{s^2}\right)}}\right)\right]_s^\infty$$

$$= \left[\log\left(\frac{\cancel{s}}{\cancel{s}\sqrt{1+\frac{1}{s^2}}}\right)\right]_s^\infty$$

$$= \left[\log\left(\frac{1}{\sqrt{1+\frac{1}{s^2}}}\right)\right]_s^\infty$$

$$= \log 1 - \log\left(\frac{s}{\sqrt{s^2+1}}\right)$$

$$= \log\left(\frac{\sqrt{s^2+1}}{s}\right)$$

$$\boxed{\int \frac{f'(x)}{f(x)} = \log f(x)]}$$

P (vii) L.T of derivatives

If $L[f(t)] = F(s)$

then

$$L[f'(t)] = sF(s) - f(0)$$

where

$$f(0) = \lim_{t \to 0} f(t)$$

$|||^{ly}$

$$L[f''(t)] = s^2 F(s) - s f(0) - f'(0)$$

$$\vdots$$

$$L[f^n(t)] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{n-1}(0) \ldots$$

Ex:- Given that $4f''(t) + f(t) = 0$

where $f(0) = 0$, $f'(0) = 2$

show that

$$L[f(t)] = \frac{8}{4s^2+1}$$

Sol$^n$:- Let

$$L[4f''(t)] + L[f(t)] = L[0]$$

$$\Rightarrow 4[s^2 F(s) - s\underset{0}{f(0)} - \underset{2}{f'(0)}] + F(s) = 0$$

$$= 4[s^2 F(s) - 2] + F(s) = 0$$

$$\Rightarrow F(s)[4s^2+1] = 8 \quad \therefore F(s) = \frac{8}{4s^2+1}$$

P(viii)., L.T. of integral :-

if $L[f(t)] = F(s)$

then

$$L\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$$

HW

Ex. Evaluate :-

$$L\left[\int_0^t e^u u^3\,du\right]$$