Inverse z-transform

The operation of obtaining the sequence $\{f(k)\}$ from $F(z)$ is defined as inverse z-transform & is defined as

$$z^{-1}[F(z)] = \{f(k)\}$$

Where 'z^{-1}' is the inverse z-transform operator.

Table :-

$F(z)$	Inverse z-transform $f(k)$ if $\lvert z\rvert > \lvert a\rvert, k>0$	Inverse z-transform if $\lvert z\rvert < \lvert a\rvert$, $k<0$
$\frac{z}{z-a}$	$a^k U(k)$ OR a^k	$-a^k$
$\frac{z^2}{(z-a)^2}$	$(k+1)a^k$	$-(k+1)a^k$
$\frac{z^3}{(z-a)^3}$	$\frac{1}{2!}(k+1)(k+2)a^k U(k)$	$-\frac{1}{2!}(k+1)(k+2)a^k U(-k+2)$
$\frac{z^n}{(z-a)^n}$	$\frac{1}{(n-1)!}(k+1)\cdots(k+n-1)a^k U(k)$	$-\frac{1}{(n-1)!}(k+1)\cdots(k+n-1)a^k U(-k+n-1)$
$\frac{1}{z-a}$	$a^{k-1}U(k-1)$	$-a^{k-1}U(-k)$
$\frac{1}{(z-a)^2}$	$(k-1)a^{k-2}U(k-2)$	$-(k-1)a^{k-2}U(-k+1)$
$\frac{z}{z-1}$	$U(k)$	—
$\frac{z(z-\cos\alpha)}{z^2-2z\cos\alpha+1}, \lvert z\rvert>1$	$\cos\alpha k$	—
$\frac{z\sin\alpha}{z^2-2z\cos\alpha+1}, \lvert z\rvert>1$	$\sin\alpha k$	—

There are three methods to find inverse z-transform

1] Power Series Method

2] Partial fraction Method

3] Inversion integral Method.

Partial fraction Method

To apply this, it is necessary that the degree of numerator must be less than degree of denominator. if not, carry out actual division till numerator satisfies the above condition. then write it as

$$F(z) = P(z) + \frac{Q(z)}{R(z)}$$

Then use partial fraction method for $\frac{Q(z)}{R(z)}$

*Note :- obtain partial fraction of $\frac{F(z)}{z}$ and not that of $F(z)$.

Ex – ① Find $z^{-1}\left(\frac{z}{(z-1)(z-2)}\right)$, if $|z| \geq 2$.

Here $F(z) = \frac{z}{(z-1)(z-2)}$

$$\frac{F(z)}{z} = \frac{1}{(z-1)(z-2)}$$

Now use partial fraction method.

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$1 = A(z-2) + B(z-1)$$

Put $z=1$ $\Rightarrow$ $1 = -A$ $\Rightarrow$ $\boxed{A=-1}$

Put $z=2$ $\Rightarrow$ $1 = B$ $\Rightarrow$ $\boxed{B=1}$

$$\therefore \frac{F(z)}{z} = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$F(z) = -\frac{z}{z-1} + \frac{z}{z-2}$$

$$F(z) = -\left(\frac{z}{z-1}\right) + \left(\frac{z}{z-2}\right)$$

Apply inverse z-transforms on both sides,

$$z^{-1}[F(z)] = -z^{-1}\left[\frac{z}{z-1}\right] + z^{-1}\left[\frac{z}{z-2}\right], \quad |z| \geq 2 \text{ i.e, } |z| \geq 1$$

$$\therefore \{f(k)\} = -(1)^k + (2)^k, \quad k \geq 0.$$

$$= 2^k - 1, \quad k \geq 0.$$

2] Find $z^{-1}\left[\frac{z^2}{(z-\frac{1}{2})(z-\frac{1}{3})}\right]$, if $\frac{1}{3} < |z| < \frac{1}{2}$

→ $$F(z) = \frac{z^2}{(z-\frac{1}{2})(z-\frac{1}{3})}$$

$$\frac{F(z)}{z} = \frac{z}{(z-\frac{1}{2})(z-\frac{1}{3})}$$

$$\frac{F(z)}{z} = \frac{3}{z-\frac{1}{2}} + \frac{(-2)}{(z-\frac{1}{3})}$$

$$F(z) = 3\left(\frac{z}{z-\frac{1}{2}}\right) - 2\left(\frac{z}{z-\frac{1}{3}}\right)$$

Apply inverse z-transform on both sides.

$$z^{-1}[F(z)] = 3z^{-1}\left(\frac{z}{z-\frac{1}{2}}\right) - 2z^{-1}\left(\frac{z}{z-\frac{1}{3}}\right)$$

$$\frac{1}{3} < |z| < \frac{1}{2} \Rightarrow |z| < \frac{1}{2} \ \& \ |z| > \frac{1}{3}$$

$$\{f(k)\} = \underset{k<0}{3\left[-\left(\frac{1}{2}\right)^k\right]} - \underset{k\geq 0}{2\left(\frac{1}{3}\right)^k}.$$

$$\{f(k)\} = \underset{k<0}{-3\left(\frac{1}{2}\right)^k} - \underset{k\geq 0}{2\left(\frac{1}{3}\right)^k}$$

Ex. 4 : *Find* $Z^{-1}\dfrac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)},\quad |z| > \dfrac{1}{2}.$ **(Dec. 0**

Sol. :

$$F(z) = \frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}$$

$$\frac{F(z)}{z} = \frac{z}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)} = \frac{3}{z-\frac{1}{2}} - \frac{2}{z-\frac{1}{3}}$$

$$F(z) = 3\cdot\frac{z}{z-\frac{1}{2}} - 2\cdot\frac{z}{z-\frac{1}{3}}.$$

$$\therefore \quad Z^{-1}[F(z)] = 3\cdot Z^{-1}\left(\frac{z}{z-\frac{1}{2}}\right) - 2\cdot Z^{-1}\left(\frac{z}{z-\frac{1}{3}}\right)$$

$$|z| > \frac{1}{2} \Rightarrow |z| > \frac{1}{3}$$

$$\{f(k)\} = 3\cdot\left(\frac{1}{2}\right)^k - 2\left(\frac{1}{3}\right)^k,\quad k \geq 0$$

Ex. 6 : Find $Z^{-1}\left[\dfrac{z^3}{(z-1)\left(z-\frac{1}{2}\right)^2}\right]$, $|z| > 1$. **(Dec. 04, 12; May 05)**

Sol. :

$$F(z) = \frac{z^3}{(z-1)\left(z-\frac{1}{2}\right)^2}$$

$$\frac{F(z)}{z} = \frac{z^2}{(z-1)\left(z-\frac{1}{2}\right)^2}$$

$$\frac{z^2}{(z-1)\left(z-\frac{1}{2}\right)^2} = \frac{A}{z-1} + \frac{B}{z-\frac{1}{2}} + \frac{C}{\left(z-\frac{1}{2}\right)^2}$$

$$z^2 = A\left(z-\frac{1}{2}\right)^2 + B\left(z-\frac{1}{2}\right)(z-1) + C(z-1)$$

$$z = 1 \Rightarrow A = 4; \quad z = \frac{1}{2} \Rightarrow C = -\frac{1}{2}$$

$$z = 0 \Rightarrow B = -3$$

$$\frac{F(z)}{z} = \frac{4}{z-1} - \frac{3}{z-\frac{1}{2}} - \frac{\frac{1}{2}}{\left(z-\frac{1}{2}\right)^2}$$

$$F(z) = 4\cdot\frac{z}{z-1} - 3\cdot\frac{z}{z-\frac{1}{2}} - \frac{1}{2}\,\frac{z}{\left(z-\frac{1}{2}\right)^2}, \quad |z| > 1$$

$$\{f(k)\} = 4(1)^k - 3\cdot\left(\frac{1}{2}\right)^k - \frac{1}{2}\cdot k\left(\frac{1}{2}\right)^{k-1}, \quad k \geq 0$$

$$= 4 - 3\left(\frac{1}{2}\right)^k - k\left(\frac{1}{2}\right)^k, \; k \geq 0, \; |z| > 1$$

(May 200

Ex. 7 : *Find* $Z^{-1}\left(\dfrac{z(z+1)}{z^2-2z+1}\right)$, $|z| > 1$.

Sol. :

$$\frac{F(z)}{z} = \frac{z+1}{z^2-2z+1} = \frac{z+1}{(z-1)^2}$$

$$= \frac{(z-1)+2}{(z-1)^2} = \frac{1}{z-1} + \frac{2}{(z-1)^2}$$

$$F(z) = \frac{z}{z-1} + 2\cdot\frac{z}{(z-1)^2},\quad |z| > 1$$

$$\therefore \quad \{f(k)\} = (1)^k + 2\,.\,k\,(1)^{k-1},\ k \geq 0,\quad |z| > 1$$

$$\{f(k)\} = 1 + 2k,\quad k \geq 0,\ |z| > 1.$$

Ex. 8 : *Find* $Z^{-1}\left(\dfrac{z^3}{(z-1)\left(z-\frac{1}{2}\right)^2}\right)$, $|z| > \dfrac{1}{2}$.

Sol. :

$$F(z) = \frac{z^3}{(z-1)\left(z-\frac{1}{2}\right)^2}$$

$$\frac{F(z)}{z} = \frac{z^2}{(z-1)\left(z-\frac{1}{2}\right)^2} = \frac{4}{z-1} - \frac{3}{z-\frac{1}{2}} - \frac{1/2}{\left(z-\frac{1}{2}\right)^2}$$

$$F(z) = 4\cdot\frac{z}{z-1} - 3\cdot\frac{z}{z-\frac{1}{2}} - \frac{1}{2}\,\frac{z}{\left(z-\frac{1}{2}\right)^2}$$

$$\therefore \quad \{f(k)\} = 4\,(1)^k - 3\cdot\left(\frac{1}{2}\right)^k - \frac{1}{2}\cdot k\cdot\left(\frac{1}{2}\right)^{k-1},\quad k \geq 0$$

$$= 4 - (3+k)\left(\frac{1}{2}\right)^k,\quad k \geq 0.$$

Ex. 13 : *Show that* $Z^{-1}\left[\dfrac{z^2}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)}\right] = \{x_k\}$ *for* $|z| < \frac{1}{5}$,

where $x_k = 4\left(\frac{1}{5}\right)^k - 5\left(\frac{1}{4}\right)^k,\ k < 0.$

Sol. :

$$\frac{X(z)}{z} = \frac{z}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)} = \frac{5}{z-\frac{1}{4}} - \frac{4}{z-\frac{1}{5}}$$

$$X(z) = 5 \cdot \frac{z}{z-\frac{1}{4}} - 4 \cdot \frac{z}{z-\frac{1}{5}}$$

$$|z| < \frac{1}{5} \Rightarrow |z| < \frac{1}{4}$$

$$\therefore \quad \{x_k\} = 5\left[-\left(\frac{1}{4}\right)^k\right] - 4\left[-\left(\frac{1}{5}\right)^k\right],\ k < 0$$

$$= -5\left(\frac{1}{4}\right)^k + 4\left(\frac{1}{5}\right)^k,\quad k < 0.$$

Ex. 15 : *Show that* $Z^{-1}\left\{\dfrac{z^2}{z^2+1}\right\} = \{x_k\}$ *for* $|z| > 1$, *where* $x_k = \cos\dfrac{k\pi}{2}$, $k \geq 0$.

(May 2006, Dec. 2006)

Sol. :

$$X(z) = \frac{z^2}{z^2+1} = \frac{z\left(z-\cos\frac{\pi}{2}\right)}{z^2 - 2z\cos\frac{\pi}{2}+1} \qquad \text{(Note this adjustment)}$$

$$\{x_k\} = Z^{-1}\left(\frac{z\left(z-\cos\frac{\pi}{2}\right)}{z^2 - 2z\cos\frac{\pi}{2}+1}\right)$$

$$\{x_k\} = \cos\frac{k\pi}{2},\ k \geq 0.$$

Find f(k) if :

1. $\frac{1}{z-a}$, $|z| < |a|$, $|z| > |a|$

Ans. : $-a^{k-1}$, $k \le 0$; a^{k-1}, $k \ge 1$

2. $\frac{1}{z+a}$, $|z| > a$

Ans. : $(-a)^{k-1}$, $k \ge 1$

3. $\frac{1}{(z-a)^2}$, $|z| < |a|$, $|z| > |a|$

Ans. : $\frac{-k+1}{a^{-k+2}}$, $k \le 0$; $(k-1)\, a^{k-2}$, $k \ge 2$

4. $\frac{1}{(z-5)^3}$, $|z| > 5$, $|z| < 5$

Ans. : $\frac{(k-2)(k-1)}{2} 5^{k-3}$, $k \ge 3$;

$\frac{-(-k+1)(-k+2)}{2} \frac{1}{5^{-k+3}}$, $k \le 0$

5. $\frac{1}{(z-3)(z-2)}$ **(Dec. 2011)**

if (i) $|z| < 2$, (ii) $2 < |z| < 3$, (iii) $|z| > 3$

Ans. : (i) $-3^{k-1} + 2^{k-1}$, $k \le 0$

(ii) $f(k) = -3^{k-1}$, $k \le 0$

$= -2^{k-1}$, $k \ge 1$

(iii) $f(k) = 3^{k-1} - 2^{k-1}$, $k \ge 1$

$= 0$, $k \le 0$

6. $\frac{z+2}{z^2 - 2z + 1}$, $|z| > 1$

Ans. : $3k - 2$, $k \ge 1$

7. $\frac{2z^2 - 10z + 13}{(z-3)^2 (z-2)}$ $2 \le |z| < 3$

Ans. : $f(k) = 2^{k-1}$, $k \ge 1$

$= \frac{-k-2}{3^{-k+2}}$, $k < 0$

8. $\frac{z^2}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)}$ **(Dec. 2008, 2012)**

if (i) $\frac{1}{5} < |z| < \frac{1}{4}$, (ii) $|z| < \frac{1}{5}$

Ans. : (i) $f(k) = -\frac{1}{5}\left(\frac{1}{4}\right)^k$, $k < 0$

$= -4\left(\frac{1}{5}\right)^k$, $k \ge 0$

(ii) $f(k) = 4\,(5)^{-k}$, $k < 0$

$= -5\,.\,(4)^{-k}$, $k < 0$

9. $\frac{3z^2 + 2z}{z^2 + 3z + 2}$, $1 < |z| < 2$

Ans. : $f(k) = -5$, $k \ge 0$

$= -8\,(2)^k$, $k < 0$

10. $\frac{z}{(z-2)(z-3)}$,

if (i) $|z| < 2$, (ii) $2 < |z| < 3$, (iii) $|z| < 3$

Ans. : (i) $2^k - 3^k$, $k \le 0$

(ii) $f(k) = -2^k$, $k > 0$

$= -3^k$, $k \le 0$

(iii) $3^k - 2^k$, $k \ge 0$

11. $\frac{z^3}{(z-1)(z-2)^2}$, $|z| > 2$ **(May 07)**

Ans. : $1 + k\,.\,2^{k+1}$, $k \ge 0$

12. $\frac{z^3}{(z-3)(z-2)^2}$, $|z| > 3$ **(Dec. 08)**

Ans. : $3^{k+2} - 2^{k+2} - k\,.\,2^{k+1}$, $k \ge 0$

13. $\frac{z^2}{z^2 + a^2}$, $|z| > |a|$ **(Dec. 2008)**

Ans. : $a^k \cos\frac{k\pi}{2}$

14. $\frac{z}{(z-1)(z-2)}$, $|z| > 2$

Ans. : $2^k - 1$, $k \ge 0$

III. Inversion Integral Method

We have, $Z[\{f(k)\}] = F(z) = \sum_{k=0}^{\infty} f(k)\, z^{-k}$

$$F(z) = f(0) + f(1)\, z^{-1} + f(2)\, z^{-2} + \ldots\ldots + f(k)\, z^{-k} + \ldots\ldots$$

By multiplying both sides of this last equation by z^{k-1}, we obtain

$$F(z) \cdot z^{k-1} = f(0) \cdot z^{k-1} + f(1)\, z^{k-2} + f(2)\, z^{k-3} + \ldots\ldots + f(k)\, z^{-1} + \ldots$$

Now we integrate both sides of the above equation along a circle such that all the poles (i.e. values of z such that F (z) is infinite) of F (z) lie within a circle C, in anticlockwise direction, we get

$$\oint_C F(z)\, z^{k-1}\, dz = \oint_C f(0)\, z^{k-1}\, dz + \oint_C f(1)\, z^{k-2}\, dz + \ldots + \oint_C f(k)\, z^{-1}\, dz + \ldots$$

Applying Cauchy's theorem of complex integration, we see that all terms of R.H.S. of above equation are zero except the term

$$\oint_C f(k)\, z^{-1}\, dz = (2\pi i)\, f(k)$$

$$\therefore \quad \oint_C F(z)\, z^{k-1}\, dz = \oint_C f(k)\, z^{-1}\, dz = (2\pi i)\, f(k)$$

$$\therefore \quad \boxed{f(k) = \frac{1}{2\pi i} \oint_C F(z)\, z^{k-1}\, dz} \qquad \ldots \text{(I)}$$

Equation (I) is known as the inversion integral for inverse of Z-transform and equation (I) is equivalent to stating that

$$\boxed{f(k) = \sum \text{[Residues of } F(z)\, z^{k-1} \text{ at the poles of } F(z)]}$$

We have from the theory of complex variables,

(i) Residue for simple pole $z = a$ is $= [(z-a)\, z^{k-1}\, F(z)]_{z=a}$

(ii) Residue for r times repeated poles at z = a is

$$= \frac{1}{(r-1)!} \cdot \frac{d^{r-1}}{dz^{r-1}} \left[(z-a)^r z^{k-1} F(z)\right]_{z=a}$$

The method of inversion integral is most convenient method than earlier methods, in determining inverse of Z-transform.

Additional Results :

1. **Pole of F (z) :** Pole of F (z) is the value (or values) of z for which F (z) is infinite.

e.g. $$F(z) = \frac{z}{(z-a)(z-b)}$$

Here z = a and z = b are the poles. These are also called as simple poles of F (z).

2. **Multiple pole of F (z) :** If a pole is repeated more than once, it is called a multiple pole.

e.g. $$F(z) = \frac{z^3}{(z-1)(z-2)^2}$$

Here z = 1 is a simple pole and

z = 2 is called a double pole.

The following examples will illustrate the inversion integral method.

ILLUSTRATIONS

Ex. 1 : Find $Z^{-1}\left[\frac{1}{(z-2)(z-3)}\right]$ by inversion integral method.

Sol. : $$F(z) = \frac{1}{(z-2)(z-3)}$$

The poles of F (z) are simple poles at z = 2, z = 3.

Consider $$F(z)\, z^{k-1} = \frac{z^{k-1}}{(z-2)(z-3)}$$

Residue of $z^{k-1} \cdot F(z)$ at z = 2 is

$$= [z^{k-1} F(z) . (z-2)]_{z=2} = \left[\frac{z^{k-1}}{(z-2)(z-3)} \cdot (z-2)\right]_{z=2}$$

$$= \left[\frac{z^{k-1}}{z-3}\right]_{z=2} = \frac{2^{k-1}}{-1} = -2^{k-1} \qquad \ldots (i)$$

Residue of z^{k-1} F (z) at z = 3 is

$$= \left[z^{k-1} F(z)\ (z-3)\right]_{z=3} = \left[\frac{z^{k-1}}{(z-2)(z-3)}\ (z-3)\right]_{z=3}$$

$$= \left[\frac{z^{k-1}}{z-2}\right]_{z=3} = \frac{3^{k-1}}{1} = 3^{k-1} \quad \cdots \text{(ii)}$$

From (i) and (ii),

$\therefore$ f (k) = algebraic sum of all the residues of z^{k-1} F (z)

$$= 3^{k-1} - 2^{k-1}, \quad k \geq 1, \ |z| > 3.$$

Ex. 2 : *Find* $Z^{-1}\left[\dfrac{z^3}{(z-1)\left(z-\frac{1}{2}\right)^2}\right]$ *by using inversion integral method.*

Sol. :
$$F(z) = \frac{z^3}{(z-1)\left(z-\frac{1}{2}\right)^2}$$

The poles of F (z) are simple poles at z = 1 and double pole at $z = \frac{1}{2}$.

Consider
$$F(z)\ z^{k-1} = \frac{z^{k+2}}{(z-1)\left(z-\frac{1}{2}\right)^2}$$

Residue of F (z) . z^{k-1} at z = 1 is

$$= \left[z^{k-1} F(z) . (z-1)\right]_{z=1}$$

$$= \left[\frac{z^{k+2}}{(z-1)\left(z-\frac{1}{2}\right)^2}\ (z-1)\right]_{z=1}$$

$$= \left[\frac{z^{k+2}}{\left(z-\frac{1}{2}\right)^2}\right]_{z=1} = \frac{1}{\frac{1}{4}}$$

Residue of z^{k-1} F (z) for 2 times repeated pole at $z = \frac{1}{2}$ is

$$= \frac{1}{(2-1)!}\ \frac{d^{2-1}}{dz^{2-1}}\left[\left(z-\frac{1}{2}\right)^2 . \ z^{k-1}\ F(z)\right]_{z=\frac{1}{2}}$$

$$= \frac{1}{1!}\ \frac{d}{dz}\left[\left(z-\frac{1}{2}\right)^2 \frac{z^{k+2}}{(z-1)\left(z-\frac{1}{2}\right)^2}\right]_{z=\frac{1}{2}}$$

$$= \frac{d}{dz}\left[\frac{z^{k+2}}{(z-1)}\right]_{z=\frac{1}{2}}$$

$$= \left[\frac{(k+2)\,z^{k+1}}{z-1} - \frac{z^{k+2}}{(z-1)^2}\right]_{z=\frac{1}{2}}$$

$$= \frac{(k+2)\left(\frac{1}{2}\right)^{k+1}}{\left(-\frac{1}{2}\right)} - \frac{\left(\frac{1}{2}\right)^{k+2}}{\left(\frac{1}{2}\right)^2}$$

$$= -(k+2)\left(\frac{1}{2}\right)^k - \left(\frac{1}{2}\right)^k = -(k+3)\left(\frac{1}{2}\right)^k$$

$$f(k) = 4 - (k+3)\left(\frac{1}{2}\right)^k, \quad k \geq 0, \quad |z| > 1.$$

Ex. 3 : *Obtain $\{f(k)\}$ by use of the inversion integral when $F(z)$ is given by*

$$F(z) = \frac{10z}{(z-1)(z-2)}$$ **(Dec. 10, 12; May 11)**

Sol. : The poles of F (z) are simple poles at $z = 1$, $z = 2$.

Consider $F(z)\,z^{k-1} = \dfrac{10 \cdot z^k}{(z-1)(z-2)}$

Residue of $z^{k-1} F(z)$ at $z = 1$ is

$$= [z^{k-1}\,F(z)\,(z-1)]_{z=1}$$

$$= \left[\frac{10\,z^k}{(z-1)(z-2)}\,(z-1)\right]_{z=1}$$

$$= \left[\frac{10\,z^k}{(z-2)}\right]_{z=1} = \frac{10}{-1} = -10$$

Residue of $z^{k-1}\,F(z)$ at $z = 2$ is

$$= [z^{k-1}\,F(z)\,(z-2)]_{z=2}$$

$$= \left[\frac{10\,z^k}{(z-1)(z-2)}\,(z-2)\right]_{z=2}$$

$$= \left[\frac{10\,z^k}{(z-1)}\right]_{z=2} = \frac{10\,.\,(2)^k}{1} = 10 \cdot (2)^k$$

$f(k)$ = algebraic sum of all the residues of $z^{k-1}\,F(z)$.

$$= 10\,[2^k - 1], \quad k \geq 0.$$

Ex. 5 : Find $Z^{-1}\left(\frac{z^2}{z^2+1}\right)$ *by using inversion integral method.*

(Dec. 2005, May 2008, Dec. 2012)

Sol. :

$$F(z) = \frac{z^2}{z^2+1}$$

$$z^{k-1}F(z) = \frac{z^{k+1}}{(z+i)(z-i)}$$

which has poles at $z = i,\ z = -i$

Residue of $F(z)\ z^{k-1}$ at $z = i$

$$= \left[z^{k-1}F(z)\ (z-i)\right]_{z=i} = \left[\frac{z^{k+1}}{(z+i)}\right]_{z=i} = \frac{(i)^{k+1}}{2i} = \frac{(i)^k}{2}$$

Residue of $F(z)\ z^{k-1}$ at $z = -i$ is

$$= \left[z^{k-1}F(z)\ (z+i)\right]_{z=-i}$$

$$= \left(\frac{z^{k+1}}{z-i}\right)_{z=-i} = \frac{(-i)^{k+1}}{-2i}$$

$$= \frac{(-i)^k}{2}$$

$$\therefore \quad f(k) = \frac{(i)^k}{2} + \frac{(-i)^k}{2} = \frac{(i)^k + (-i)^k}{2}$$

But $\quad i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\frac{\pi}{2}}$

$$(i)^k = e^{i\frac{k\pi}{2}}$$

Similarly, $\quad (-i)^k = e^{-i\frac{k\pi}{2}}$

$$\frac{(i)^k + (-i)^k}{2} = \frac{e^{ik\pi/2} + e^{-ik\pi/2}}{2} = \cos k\frac{\pi}{2}$$

$$f(k) = \cos k\frac{\pi}{2},\ k \geq 0,\ |z| > 1.$$

Find inverse Z-transforms by inversion integral method :

1. $\dfrac{z(z+1)}{(z-1)(z^2+z+1)}$ **Ans.** $\dfrac{2}{3}\left(1-\cos\dfrac{2\pi k}{3}\right)$

2. $\dfrac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}$ **(May 2007) Ans.** $3\left(\dfrac{1}{2}\right)^k - 2\left(\dfrac{1}{3}\right)^k$

3. $\dfrac{z^3}{\left(z-\frac{1}{4}\right)^2 (z-1)}$ **Ans.** $\dfrac{16}{9} - \dfrac{4}{9}\left(\dfrac{1}{4}\right)^k - \dfrac{1}{3}(k+1)\left(\dfrac{1}{4}\right)^k$

4. $\dfrac{2z^2+3z}{z^2+z+1}$ **Ans.** $2\cos\dfrac{2\pi k}{3} + \dfrac{4}{\sqrt{3}}\sin\dfrac{2\pi k}{3}$

5. $\dfrac{z}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)}$ **(Dec. 2010) Ans.** $20\left[\left(\dfrac{1}{4}\right)^k - \left(\dfrac{1}{5}\right)^k\right]$

4.12 SOLUTIONS OF DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS USING Z-TRANSFORM

A relation between f (k) and f (k + 1), f (k + 2), f (k + 3), ... is called *difference equation* and an expression for f (k) in terms of k which satisfies the equation is called its solution.

A Laplace transform, transforms a differential equation to algebraic equation, the Z-transform, transforms a difference equation to algebraic equation in z and initial data is automatically included in algebraic equation. We take Z- transform of the entire equation to solve a difference equation and write F (z). The inverse Z-transform of F (z) gives the required solution.

Additional Results :

$$Z\{f(k)\} = F(z)$$
$$Z\{f(k+1)\} = z\,F(z) - z\,f(0)$$
$$Z\{f(k+2)\} = z^2\,F(z) - z^2 f(0) - z\,f(1)$$
$$Z f(k-1) = z^{-1}\,F(z)$$
$$Z f(k-2) = z^{-2}\,F(z).$$

Note : f (k) is considered causal sequence.

Ex. 1 : *Obtain $f(k)$ given that $f(k+1) + \frac{1}{2} f(k) = \left(\frac{1}{2}\right)^k$, $k \geq 0$, $f(0) = 0$.*

(May 2005, 2008, 2011; Dec. 2010)

Sol. : Taking Z-transform of both sides, we get

$$Z\{f(k+1)\} + \frac{1}{2} Z\{f(k)\} = Z\left(\frac{1}{2}\right)^k$$

$$[z F(z) - z f(0)] + \frac{1}{2} F(z) = \frac{z}{z - \frac{1}{2}}, \quad |z| > \frac{1}{2}$$

$$\left(z + \frac{1}{2}\right) F(z) = \frac{z}{z - \frac{1}{2}}$$

$$F(z) = \frac{z}{\left(z - \frac{1}{2}\right)\left(z + \frac{1}{2}\right)}$$

$$\frac{F(z)}{z} = \frac{1}{\left(z - \frac{1}{2}\right)\left(z + \frac{1}{2}\right)} = \frac{1}{z - \frac{1}{2}} - \frac{1}{z + \frac{1}{2}}$$

$$F(z) = \frac{z}{z - \frac{1}{2}} - \frac{z}{z + \frac{1}{2}}$$

$$= \left(\frac{1}{2}\right)^k - \left(-\frac{1}{2}\right)^k, \quad k \geq 0.$$

Ex. 2 : *Obtain $f(k)$, given that $12 f(k+2) - 7 f(k+1) + f(k) = 0$, $k \geq 0$, $f(0) = 0$, $f(1) = 3$.* **(Dec. 2006, 2007, 2012)**

Sol. : Taking Z-transform of both sides, we get

$$12 . Z\{f(k+2)\} - 7 . Z\{f(k+1)\} + Z f(k) = 0$$

$$12 . [z^2 F(z) - z^2 f(0) - z f(1)] - 7 [z F(z) - z f(0)] + F(z) = 0$$

$$12 . [z^2 F(z) - 3z] - 7z F(z) + F(z) = 0$$

$$(12 z^2 - 7z + 1) F(z) = 36 z$$

$$F(z) = \frac{36 z}{(4z - 1)(3z - 1)}$$

$$\frac{F(z)}{z} = \frac{36}{(4z-1)(3z-1)} = 36 \cdot \left\{\frac{-4}{4z-1} + \frac{3}{3z-1}\right\}$$

$$F(z) = 36 \cdot \left[\frac{3z}{3z-1} - \frac{4z}{4z-1}\right]$$

$$F(z) = 36 \left[\frac{z}{z-\frac{1}{3}} - \frac{z}{z-\frac{1}{4}}\right]$$

$$\{f(k)\} = 36 \cdot \left[\left(\frac{1}{3}\right)^k - \left(\frac{1}{4}\right)^k\right], \; k \geq 0.$$

Ex. 3 : *From the equation $y_k - 3y_{k-1} + 2y_{k-2} = 1$, $k \geq 0$ and $y_{-1} = y_{-2} = 2$, show that the unilateral transform $Y(z)$ of the sequence $\{y_k\}$, using the given initial conditions, is $\dfrac{z(3z^2-6z+4)}{(z-1)^2\,(z-2)}$.*

Sol. Taking Z-transform of both sides, we get

$$Z\{y_k\} - 3Z\{y_{k-1}\} + 2 \cdot Z\{y_{k-2}\} = Z\{1\}$$

$$Y(z) - 3 \cdot [z^{-1}Y(z) + y_{-1}z^0] + 2[z^{-2}Y(z) + y_{-1}z^{-1} + y_{-2}z^0] = \frac{z}{z-1}$$

$$Y(z) - 3 \cdot [z^{-1}Y(z) + 2] + 2[z^{-2}Y(z) + 2z^{-1} + 2] = \frac{z}{z-1}.$$

$$(1 - 3z^{-1} + 2z^{-2})\,Y(z) = \frac{z}{z-1} + 2 - \frac{4}{z}$$

$$\left(\frac{z^2-3z+2}{z^2}\right) Y(z) = \frac{z^2+2z^2-2z-4z+4}{z(z-1)}$$

$$Y(z) = \frac{(3z^2-6z+4)\,z}{(z-1)(z^2-3z+2)} = \frac{z(3z^2-6z+4)}{(z-1)(z-1)(z-2)}$$

$$Y(z) = \frac{z(3z^2-6z+4)}{(z-1)^2(z-2)}$$

[**Note :** $Z\{y_{k-1}\} = z^{-1}\,Y(z) + y_{-1}\,z^0$
$Z\{y_{k-2}\} = z^{-2}\,Y(z) + y_{-1}\,z^{-1} + y_{-2}\,z^0$]

Ex. 4 : *Obtain the output of the system, where input is U_k and the system is given by $y_k - 4y_{k-2} = U_k$, where*

$$U_k = \left(\frac{1}{2}\right)^k, \quad k \geq 0$$
$$= 0, \quad k < 0$$

Sol. : We have, $Z\{y_k\} = Y(z)$ **(May 2006)**

$$Z\{y_{k-2}\} = z^{-2}\,Y(z)$$

Since $\{y_k\}$ is considered causal sequence.

$\therefore$ y_{-1}, y_{-2} are zero.

$$\therefore \quad Z\{y_k\} - 4Z\{y_{k-2}\} = Z\left\{\left(\frac{1}{2}\right)^k\right\}$$

$$Y(z) - 4z^{-2}\,Y(z) = \frac{z}{z-\frac{1}{2}}$$

$$\left(\frac{z^2-4}{z^2}\right) Y(z) = \frac{z}{\left(z-\frac{1}{2}\right)}$$

$$Y(z) = \frac{z^3}{\left(z-\frac{1}{2}\right)(z^2-4)}$$

$$\frac{Y(z)}{z} = \frac{z^2}{(z-2)(z+2)\left(z-\frac{1}{2}\right)}$$

$$Y(z) = \frac{2}{3}\,\frac{z}{z-2} + \frac{2}{5}\,\frac{z}{z+2} - \frac{1}{15}\cdot\frac{z}{z-\frac{1}{2}}$$

$$\{y_k\} = \frac{2}{3}\cdot 2^k + \frac{2}{5}(-2)^k - \frac{1}{15}\left(\frac{1}{2}\right)^k, \; k \geq 0.$$

Ex. 5 : *Solve* $y_k - \frac{5}{6}\,y_{k-1} + \frac{1}{6}\,y_{k-2} = \left(\frac{1}{2}\right)^k,\; k \geq 0.$ **(Dec. 2004)**

Sol. : $Z\{y_k\} - \frac{5}{6}\,Z\{y_{k-1}\} + \frac{1}{6}\,Z\{y_{k-2}\} = Z\left(\frac{1}{2}\right)^k$

y_{-1}, y_{-2} are zero, since y_k is considered as causal sequence.

$$Y(z) - \frac{5}{6}\cdot z^{-1}\,Y(z) + \frac{1}{6}\,z^{-2}\,Y(z) = \frac{z}{z-\frac{1}{2}}$$

$$\left(1-\frac{5}{6}\,z^{-1}+\frac{1}{6}\,z^{-2}\right) Y(z) = \frac{z}{z-\frac{1}{2}}$$

$$\left(\frac{z^2-\frac{5}{6}z+\frac{1}{6}}{z^2}\right) Y(z) = \frac{z}{z-\frac{1}{2}}$$

$$Y(z) = \frac{z^3}{\left(z-\frac{1}{2}\right)\left(z^2-\frac{5}{6}z+\frac{1}{6}\right)}$$

$$\frac{Y(z)}{z} = \frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}$$

$$= \frac{z^2}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{2}\right)^2} = \frac{4}{z-\frac{1}{3}} - \frac{3}{z-\frac{1}{2}} + \frac{\frac{3}{2}}{\left(z-\frac{1}{2}\right)^2}$$

$$Y(z) = 4 \cdot \frac{z}{z-\frac{1}{3}} - 3 \cdot \frac{z}{z-\frac{1}{2}} + \frac{3}{2} \cdot \frac{z}{\left(z-\frac{1}{2}\right)^2}$$

$$\{y_k\} = 4 \cdot \left(\frac{1}{3}\right)^k - 3 \cdot \left(\frac{1}{2}\right)^k + \frac{3}{2} \cdot k \cdot \left(\frac{1}{2}\right)^{k-1}$$

$$= 4 \cdot \left(\frac{1}{3}\right)^k - 3 \cdot \left(\frac{1}{2}\right)^k + 3 \cdot k \cdot \left(\frac{1}{2}\right)^k, \quad k \geq 0.$$

EXERCISE 4.4

Solve the following difference equations :

1. $f(k+2) + 3f(k+1) + 2f(k) = 0, \; f(0) = 0, \; f(1) = 1$

(Dec. 2005, May 2007, 2009)

Ans. : $f(k) = (-1)^k - (-2)^k, \; k \geq 0, \; |z| > 2$

2. $f(k+2) - 3f(k+1) + 2f(k) = U(k)$

where $f(k) = 0$ for $k \leq 0$ and $U(0) = 1$

and $U(k) = 0$ for $k < 0$ and $k > 0$.

Ans. : $f(k) = 2^{k-1} - 1, \; k > 0, \; |z| > 2$

3. $4f(k) + f(k-2) = 4\left(\frac{1}{2}\right)^k \sin\frac{k\pi}{2}, \; k \geq 0$

Ans. : $f(k) = (k+1)\left(\frac{1}{2}\right)^k \sin\frac{k\pi}{2}, \; k \geq 0, \; |z| > \frac{1}{2}$

4. $u_{n+2} + u_{n+1} + u_n = 0, \; u_o = 1, \; u_1 = 1$

Ans. : $u_n = \cos\frac{2n\pi}{3} + \sqrt{3}\sin\frac{2n\pi}{3}, \; n \geq 0$