INVERSE LAPLACE TRANSFORMS

11.1 INTRODUCTION

We have so far discussed how to find Laplace transform of given function f(t). However from application point of view this will not be very useful unless we obtain inverse transform f(t) of a given function F(s).

In this chapter, we shall consider the inverse problem of finding f(t) for a given F(s) i.e. given a function F(s), to find a function f(t) of which F(s) is Laplace transform. Applications of Laplace transform to differential equations are also discussed in this chapter.

11.2 DEFINITION

If the Laplace transform of f(t) is F(s), i.e. L [f(t)] = F(s), then f(t) is called the inverse Laplace transform of F(s) and we write symbolically

$$\boxed{L^{-1}[F(s)] = f(t)} \quad \ldots (1)$$

where L^{-1} is called inverse Laplace transform operator.

11.3 LINEARITY PROPERTY

Theorem : If c_1 and c_2 are any constants and $F_1(s)$ and $F_2(s)$ are the Laplace transform of $f_1(t)$ and $f_2(t)$ respectively, then

$$\begin{aligned} L^{-1}[c_1 F_1(s) + c_2 F_2(s)] &= c_1 L^{-1}[F_1(s)] + c_2 L^{-1}[F_2(s)] \\ &= c_1 f_1(t) + c_2 f_2(t) \end{aligned}$$

Proof : To prove this, we have given

$$F_1(s) = L[f_1(t)] \quad \text{and} \quad F_2(s) = L[f_2(t)]$$

We know by the Linearity property of Laplace transform (see article 10.5)

$$\begin{aligned} L[c_1 f_1(t) + c_2 f_2(t)] &= c_1 L[f_1(t)] + c_2 L[f_2(t)] \\ &= c_1 F_1(s) + c_2 F_2(s) \end{aligned}$$

Therefore by definition (1) above, we have

$$\begin{aligned} L^{-1}[c_1 F_1(s) + c_2 F_2(s)] &= c_1 f_1(t) + c_2 f_2(t) \\ &= c_1 L^{-1}[F_1(s)] + c_2 L^{-1}[F_2(s)]. \end{aligned}$$

The result is easily extended to more than two functions.

Note : The property of Laplace transformation expressed in this theorem is, of course, the property of Linearity. In other words, the inverse Laplace transform is Linear transform.

Table of Inverse Laplace Transforms :

Sr. No.	**F(s)**	**$f(t) = L^{-1}[F(s)]$**
1	$\frac{1}{s}$	1
2	$\frac{1}{s-a}$	e^{at}
3	$\frac{1}{s^2+a^2}$	$\frac{\sin at}{a}$
4	$\frac{s}{s^2+a^2}$	$\cos at$
5	$\frac{1}{s^2-a^2}$	$\frac{\sinh at}{a}$
6	$\frac{s}{s^2-a^2}$	$\cosh at$
7	$\frac{1}{s^{n+1}}$	$\frac{t^n}{\left\lvert\overline{n+1}\right.}$
8	$\frac{1}{s^{n+1}}$ if n a positive integer	$\frac{t^n}{n!}$

Ex. 2 : Obtain the inverse Laplace transform of each of the following functions :

(i) $\dfrac{2}{s^2+16}$ (ii) $\dfrac{4s}{s^2-16}$ (iii) $\dfrac{2s-5}{4s^2+25}$ (iv) $\dfrac{3s-12}{s^2+8}$ (v) $\dfrac{s-4}{s^2-4}$ ✓ (vi) $\dfrac{s\cos\alpha+\omega\sin\alpha}{s^2+\omega^2}$

Sol. : (i)
$$L^{-1}\left[\frac{2}{s^2+16}\right] = 2\,L^{-1}\left[\frac{1}{s^2+16}\right] = 2\,L^{-1}\left[\frac{1}{s^2+4^2}\right]$$

$$= 2\left(\frac{\sin 4t}{4}\right) = \frac{\sin 4t}{2}$$

$\left\{\because\right.$ From Table of I.L.T. $L^{-1}\left[\dfrac{1}{s^2+a^2}\right] = \dfrac{\sin at}{a}$

(ii)
$$L^{-1}\left[\frac{4s}{s^2-16}\right] = 4\,L^{-1}\left[\frac{s}{s^2-16}\right] = 4\,L^{-1}\left[\frac{s}{s^2-4^2}\right]$$

$$= 4\cosh 4t.$$

$\left\{\right.$From Table of I.L.T $L^{-1}\left[\dfrac{s}{s^2-a^2}\right] = \cosh a$

(iii)
$$L^{-1}\left[\frac{2s-5}{4s^2+25}\right] = \frac{1}{4}\,L^{-1}\left[\frac{2s-5}{s^2+25/4}\right] = \frac{1}{4}\,L^{-1}\left[\frac{2s-5}{s^2+(5/2)^2}\right]$$

$$= \frac{1}{2}\,L^{-1}\left[\frac{s}{s^2+(5/2)^2}\right] - \frac{5}{4}\,L^{-1}\left[\frac{1}{s^2+(5/2)^2}\right]$$

[By Linearity property

$$= \frac{1}{2}\cos\frac{5}{2}t - \frac{5}{4}\cdot\frac{2}{5}\sin\frac{5}{2}t = \frac{1}{2}\left(\cos\frac{5t}{2} - \sin\frac{5t}{2}\right).$$

(iv)
$$L^{-1}\left[\frac{3s-12}{s^2+8}\right] = 3\,L^{-1}\left[\frac{s}{s^2+8}\right] - 12\,L^{-1}\left[\frac{1}{s^2+8}\right]$$

[By Linearity propert

$$= 3\,L^{-1}\left[\frac{s}{s^2+\left(2\sqrt{2}\right)^2}\right] - 12\,L^{-1}\left[\frac{1}{s^2+\left(2\sqrt{2}\right)^2}\right]$$

$$= 3\cos\left(2\sqrt{2}\right)t - 12\frac{1}{2\sqrt{2}}\sin\left(2\sqrt{2}\right)t$$

$$= 3\cos\left(2\sqrt{2}\right)t - 3\sqrt{2}\sin\left(2\sqrt{2}\right)t.$$

(v)
$$L^{-1}\left[\frac{s-4}{s^2-4}\right] = L^{-1}\left[\frac{s}{s^2-4}\right] - 4\,L^{-1}\left[\frac{1}{s^2-4}\right]$$

[By Linearity propert

$$= \cosh 2t - 4\left(\frac{\sinh 2t}{2}\right)$$

$$= \cosh 2t - 2\sinh 2t.$$

$$L^{-1}\left[\frac{s}{s^2+6s+25}\right] = L^{-1}\left[\frac{s+3-3}{s^2+6s+9+16}\right] = L^{-1}\left[\frac{(s+3)-3}{(s+3)^2+4^2}\right]$$

$$= L^{-1}\left[\frac{s+3}{(s+3)^2+4^2}\right] - 3\,L^{-1}\left[\frac{1}{(s+3)^2+4^2}\right]$$

$$= e^{-3t}\,L^{-1}\left[\frac{s}{s^2+4^2}\right] - 3e^{-3t}\,L^{-1}\left[\frac{1}{s^2+4^2}\right]$$

$$= e^{-3t}\ (\cos 4t) - 3e^{-3t}\left(\frac{\sin 4t}{4}\right)$$

$$= e^{-3t}\left(\cos 4t - \frac{3}{4}\sin 4t\right)$$

(ii) Let $$L^{-1}\left[\frac{2s+1}{(s^2+s+1)^2}\right] = f(t)$$

∴ $$L^{-1}\left[\int_s^{\infty}\frac{2s+1}{(s^2+s+1)^2}\,ds\right] = \frac{f(t)}{t}$$

∴ $$L^{-1}\left[\left\{-\frac{1}{(s^2+s+1)}\right\}_s^{\infty}\right] = \frac{f(t)}{t}$$

∴ $$L^{-1}\left[\frac{1}{s^2+s+1}\right] = \frac{f(t)}{t}$$

or $$L^{-1}\left[\frac{1}{(s+1/2)^2+(\sqrt{3}/2)^2}\right] = \frac{f(t)}{t}$$

∴ $$e^{-t/2}\,L^{-1}\left[\frac{1}{s^2+\left(\sqrt{3}/2\right)^2}\right] = \frac{f(t)}{t}$$ [By the First

∴ $$e^{-t/2}\,\frac{2}{\sqrt{3}}\,\sin\frac{\sqrt{3}}{2}t = \frac{f(t)}{t}$$

or $$\frac{2t}{\sqrt{3}}\,e^{-t/2}\sin\frac{\sqrt{3}}{2}t = f(t)$$

Hence $$L^{-1}\left[\frac{2s+1}{(s^2+s+1)^2}\right] = f(t) = \frac{2t}{\sqrt{3}}\;e^{-t/2}\sin\frac{\sqrt{3}}{2}\,t$$

Table of Theorems of Inverse Laplace Transforms

	If $L^{-1}[F(s)] = f(t)$, then
	$L^{-1}[F(s+a)] = e^{-at} f(t)$
	$L^{-1}[e^{-as} F(s)] = f(t-a)\,U(t-a) = \begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases}$
	$L^{-1}[F(ks)] = \frac{1}{k} f\left(\frac{t}{k}\right)$
D	$L^{-1}\left[\frac{d}{ds} F(s)\right] = -t\,f(t)$
E	$L^{-1}\left[\int_{s}^{\infty} F(s)\,ds\right] = \frac{f(t)}{t}$
F	$L^{-1}[s\,F(s)] = f'(t)$, if $f(0) = 0$
G	$L^{-1}\left[\frac{F(s)}{s}\right] = \int_{0}^{t} f(u)\,du$
H	$L^{-1}[F(s)\,G(s)] = \int_{0}^{t} f(u)\,g(t-u)\,du = f(t) * g(t)$

(ii) Here the denominator has non-repeated linear factors.

Let $$\frac{11s^2 - 2s + 5}{(s-2)(2s-1)(s+1)} = \frac{A}{s-2} + \frac{B}{(2s-1)} + \frac{C}{s+1} \qquad \ldots \text{(i)}$$

Multiplying both sides of (i) by $(s-2)(2s-1)(s+1)$, we obtain

$$11s^2 - 2s + 5 = A(2s-1)(s+1) + B(s-2)(s+1) + C(s-2)(2s-1)$$

Putting s = 2, we get $\quad 11(4) - 2(2) + 5 = A(4-1)(2+1) \quad \therefore \quad A = 5$

Putting s = 1/2, we get $\quad 11(1/4) - 2(1/2) + 5 = B(1/2-2)(1/2+1) \quad \therefore \quad B = -3$

Putting s = – 1, we get $\quad 11(-1) - 2(-1) + 5 = C(-3)(-3) \quad \therefore \quad C = 2$

Hence $$L^{-1}\left[\frac{11s^2 - 2s + 5}{(s-2)(2s-1)(s+1)}\right] = L^{-1}\left[\frac{5}{s-2} + \frac{-3}{2s-1} + \frac{2}{s+1}\right]$$

$$= 5L^{-1}\left[\frac{1}{s-2}\right] - \frac{3}{2}L^{-1}\left[\frac{1}{s-1/2}\right] + 2L^{-1}\left[\frac{1}{s+1}\right]$$

$$= 5e^{2t} - \frac{3}{2}e^{t/2} + 2e^{-t}$$

(iii) Here the denominator has repeated linear factors.

Let $$\frac{21s - 33}{(s+1)(s-2)^3} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{(s-2)^2} + \frac{D}{(s-2)^3} \qquad \ldots ($$

Multiplying both sides of (i) by $(s+1)(s-2)^3$, we obtain

$$21s - 33 = A(s-2)^3 + B(s+1)(s-2)^2 + C(s+1)(s-2) + D(s+1) \qquad \ldots ($$

Putting s = – 1, we get $\quad 21(-1) - 33 = A(-27) \quad \therefore \quad A = 2$

Putting s = 2, we get $\quad 21(2) - 33 = D(3) \quad \therefore \quad D = 3$

The method fails to determine B and C. However, since we know A and D, we determine and C from (ii) as

Equating coefficients of s^3 on both sides of (ii), we have

$$0 = A + B \qquad \therefore \qquad B = -A = -2$$

To obtain C we can substitute s = 0 on both sides of (ii), we have

$$-33 = A(-8) + B(4) + C(-2) + D \text{ or } -33 = -16 - 8 - 2C + D \quad \therefore \quad C = 6$$

Thus $$L^{-1}\left[\frac{21s-33}{(s+1)(s-2)^3}\right] = L^{-1}\left[\frac{2}{s+1} + \frac{-2}{s-2} + \frac{6}{(s-2)^2} + \frac{3}{(s-2)^3}\right]$$

$$= 2L^{-1}\left[\frac{1}{s+1}\right] - 2L^{-1}\left[\frac{1}{s-2}\right] + 6L^{-1}\left[\frac{1}{(s-2)^2}\right] + 3L^{-1}\left[\frac{1}{(s-2)^3}\right]$$

$$= 2e^{-t} - 2e^{2t} + e^{2t}t + 3e^{2t}\frac{t^2}{2}.$$

$\left\{\text{By First Shifting The} \right.$

$\left.\text{and } L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{n!}\right.$

EXERCISE 11.2

1. Find the inverse Laplace transforms of each of the following fu

(i) $\frac{s}{(s+a)^2}$ (ii) $\frac{1}{(s+4)^{3/2}}$ (iii) $\frac{1}{s^2+2s+2}$ (iv) $\frac{6s-4}{s^2-4s+20}$

(vi) $\frac{2s+5}{s^2-2s-3}$ (vii) $\frac{1}{\sqrt{2s+3}}$ (viii) $\frac{1}{\sqrt[3]{8s-27}}$

Ans. (i) $e^{-at}(1-at)$ (ii) $2e^{-4t}\frac{\sqrt{t}}{\sqrt{\pi}}$ (iii) $e^{-t}(\cos t + 6 \sin t)$ (iv) $2e^2$

2. Obtain the inverse Laplace transforms of the following functions :

(i) $\dfrac{e^{-5s}}{(s-2)^4}$ (ii) $\dfrac{e^{4-3s}}{(s+4)^{5/2}}$ (iii) $\dfrac{e^{-3s}}{s^2-9}$ (iv) $\dfrac{e^{-s}}{\sqrt{s+1}}$ (v) $\dfrac{s\,e^{-s/2}+\pi e^{-s}}{s^2+\pi^2}$

(vi) $\dfrac{s\,e^{-\pi s}}{s^2-4s+29}$ (vii) $\dfrac{(s+1)\,e^{-\pi s}}{s^2+s+1}$ (viii) $\dfrac{e^{-s}+e^{-2s}}{s^2-3s+2}$ (ix) $\dfrac{\left(1-\sqrt{s}\right)e^{-s}}{s^{3/2}}$

(x) $\dfrac{\left(1-\sqrt{s}\right)^2 e^{-s}}{s^3}$ (xi) $\dfrac{e^{-s}\,(1-e^{-s})}{s\,(s^2+1)}$ (Dec. 91, 92, May 91, 95)

Ans. (i) $\dfrac{1}{6}(t-5)^3\,e^{2\,(t-5)}\,U(t-5)$ (ii) $\dfrac{4\,(t-3)^{3/2}\,e^{-4\,(t-4)}}{3\sqrt{\pi}}\,U(t-3)$

(iii) $\dfrac{1}{3}\sinh 3\,(t-3)\,U(t-3)$ (iv) $\dfrac{e^{-(t-1)}}{\sqrt{\pi\,(t-1)}}\,U(t-1)$

(v) $\sin \pi t\,[U(t-1/2)-U(t-1)]$ (vi) $e^{2\,(t-\pi)}\left\{\cos 5\,(t-\pi)+\dfrac{2}{5}\sin 5\,(t-\pi)\right\}U(t-\pi)$

(vii) $\dfrac{e^{-(t-\pi)/2}}{\sqrt{3}}\left\{\sqrt{3}\cos\dfrac{\sqrt{3}}{2}(t-\pi)+\sin\dfrac{\sqrt{3}}{2}(t-\pi)\right\}U(t-\pi)$

(viii) $\{e^{2\,(t-1)}-e^{(t-1)}\}\,U(t-1)+\{e^{2\,(t-2)}-e^{(t-2)}\}\,U(t-2)$

(ix) $\left\{\dfrac{2\sqrt{t-1}}{\sqrt{\pi}}-1\right\}U(t-1)$ (x) $\left\{\dfrac{(t-1)^2}{2}-\dfrac{8}{3}\,\dfrac{(t-1)^{3/2}}{\sqrt{\pi}}+(t-1)\right\}U(t-1)$

(xi) $\{1-\cos(t-1)\}\,U(t-1)-\{1-\cos(t-2)\}\,U(t-2).$

3. Find the inverse Laplace transform of the following :

(i) $\tan^{-1}\dfrac{1}{s}$ (ii) $\tan^{-1}(s+1)$ (iii) $\log\left(\dfrac{s+2}{s+1}\right)$ (iv) $\dfrac{1}{2}\log\dfrac{s-1}{s+1}$ (v) $\log\left(\dfrac{1+s}{s}\right)$

(vi) $\log\left(\dfrac{s}{s-1}\right)$ (vii) $\dfrac{1}{2}\log\left(\dfrac{s^2-a^2}{s^2}\right)$ (viii) $\dfrac{1}{2}\log\dfrac{s^2+b^2}{(s-a)^2}$ (ix) $s\log\dfrac{s}{\sqrt{s^2+1}}+\cot^{-1}s$

(May 91, 96, Dec. 92, 9

Ans. (i) $\dfrac{\sin t}{t}$ (ii) $-e^{-t}\dfrac{\sin t}{t}$ (iii) $\dfrac{e^{-t}-e^{-2t}}{t}$ (iv) $\dfrac{\sinh t}{t}$ (v) $\dfrac{1-e^{-t}}{t}$ (vi) $\dfrac{e^{t}-1}{t}$

(vii) $\dfrac{1-\cosh at}{t}$ (viii) $\dfrac{e^{-at}-\cos bt}{t}$ (ix) $\dfrac{1-\cos t}{t^2}$

inverse Laplace transform of derivative, to find

11. Using partial fractions, find the inverse Laplace transforms of the following

(i) $\dfrac{5s+3}{(s-1)(s^2+2s+5)}$ (ii) $\dfrac{s^2-3}{(s+2)(s-3)(s^2+2s+5)}$ (iii) $\dfrac{27s-12s}{(s+4)(s^2+9)}$

(iv) $\dfrac{s^2+2s-4}{(s^2+2s+5)(s^2+2s+2)}$ (v) $\dfrac{s}{(s^2+1)(s^2+2)}$ (vi) $\dfrac{1}{(s^2+2s+5)^2}$

(vii) $\dfrac{s^3+3s^2-s-3}{(s^2+2s+5)^2}$ (viii) $\dfrac{s}{s^4+4}$ (ix) $\dfrac{a(s^2-2a^2)}{s^4+4a^4}$ (x) $\dfrac{1}{s^3-a^3}$ (xi) $\dfrac{s^3}{s^4+64}$

(xii) $\dfrac{s^3+16s-24}{s^4+20s^2+64}$ **(Dec. 90, 96,**

Ans. (i) $e^t - e^{-t}\cos 2t + \frac{3}{2}e^{-t}\sin 2t$ (ii) $\frac{3}{50}e^{3t} - \frac{1}{25}e^{-2t} - \frac{1}{50}e^{-t}(\cos 2t - 18 \sin 2$

(iii) $3(e^{-4t} - \cos 3t)$ (iv) $\frac{3}{2}e^{-t}\sin 2t - 2e^{-t}\sin t$ (v) $\frac{1}{2}\sin t - \frac{1}{2}te^{-t}$

(vi) $\frac{1}{16}e^{-t}(\sin 2t - 2t\cos 2t)$ (vii) $e^{-t}(\cos 2t - 2t\sin 2t)$ (viii) $\frac{1}{2}\sinh t \sin$

(ix) $\cos at \sinh at$ (x) $\frac{1}{3a^2}\left[e^{at} - e^{-at/2}\left\{\cos\frac{\sqrt{3}}{2}at + \sqrt{3}\sin\frac{\sqrt{3}}{2}at\right\}\right]$

(xi) $\cosh 2t \cos 2t$ (xii) $\frac{1}{2}\sin 4t + \cos 2t - \sin 2t$

11.10 APPLICATIONS TO DIFFERENTIAL EQUATIONS

The Laplace transform is useful in solving differential equations and corresp initial and boundary value problems. The solution of differential equations in functions of an impulsive type can also be solved by the use of Laplace transform in efficient manner. The general process of solution consists three main steps :

1. The given differential equation is transformed into an simple algebraic e (called subsidiary equation).
2. The subsidiary equation is solved by pure algebraic manipulations.
3. The solution of the subsidiary equation is then transformed back to obtain the of the given differential equation.

In this way the Laplace transform method reduces the problem of solving diff equation to an algebraic problem. Another advantage of this method over the classical is that it solves initial value problem directly without first finding general solution (c solution) and then evaluating the arbitrary constants. We shall now illustrate this m the following applications.

Note :

(i) $$L\left[\frac{dy}{dt}\right] = L\,[y'] = s\,Y(s) - y(0)$$

(ii) $$L\left[\frac{d^2y}{dt^2}\right] = L\,[y''] = s^2\,Y(s) - s\,y(0) - y'(0)$$

(iii) $$L\left[\frac{d^3y}{dt^3}\right] = L\,[y'''] = s^3\,Y(s) - s^2\,y(0) - s\,y'(0) - y''(0)$$

(iv) $$L\left[\frac{d^4y}{dt^4}\right] = L\,[y^{iv}] = s^4\,Y(s) - s^3\,y(0) - s^2\,y'(0) - s\,y''(0) - y'''(0)$$

Ex. 2 : Solve each of the following by using Laplace transforms :

(i) $\dfrac{d^2x}{dt^2} + 9x(t) = 18\,t,$ $\quad x(0) = 0, \quad x(\pi/2) = 0.$

(ii) $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + 5y = e^{-t}\sin t,$ $\quad y(0) = 0, \quad y'(0) = 1.$

(iii) $y'' + 4y' + 13y = \dfrac{1}{3}\; e^{-2t}\sin 3t,$ $\quad y(0) = 1, \quad y'(0) = -2.$

(iv) $(D^2 + n^2)\,x = a\sin(nt + \alpha),$ $\quad x(0) = x'(0) = 0.$ **(June 92, 96, Dec. 93, 95)**

Sol. : (i) Taking Laplace transform of both sides, we get

$$L\left[\frac{d^2x}{dt^2}\right] + 9\,L\,[x(t)] = 18\,L\,[t]$$

$$\{s^2\,X(s) - s\,x(0) - x'(0)\} + 9\,X(s) = \frac{18}{s^2}$$

Since x'(0) is not known, let x'(0) = A. Then

$$\{s^2\,X(s) - s(0) - A\} + 9\,X(s) = \frac{18}{s^2} \qquad \{\because\ x(0) = 0\}$$

or

$$(s^2 + 9)\,X(s) = A + \frac{18}{s^2}$$

$$X(s) = \frac{A}{s^2+9} + \frac{18}{s^2\,(s^2+9)}$$

$$= \frac{A}{s^2+9} + \frac{18 + 2s^2 - 2s^2}{s^2\,(s^2+9)}$$

$$= \frac{A}{s^2+9} - \frac{2}{s^2+9} + \frac{2\,(s^2+9)}{s^2\,(s^2+9)}$$

$$= \frac{(A-2)}{s^2+9} + \frac{2}{s^2}$$

Thus

$$x(t) = \left(\frac{A-2}{3}\right)\sin 3t + 2t$$

To determine A, we put t = π/2 and obtain

$$x\left(\frac{\pi}{2}\right) = \left(\frac{A-2}{3}\right)\sin 3\pi/2 + 2\pi/2$$

$$0 = \left(\frac{A-2}{3}\right)(-1) + \pi \quad \text{or} \quad \left(\frac{A-2}{3}\right) = \pi$$

$$x = \pi\sin 3t + 2t$$

(ii) Taking Laplace transform of both sides, we get

$$L\left[\frac{d^2y}{dt^2}\right] + 2\,L\left[\frac{dy}{dt}\right] + 5\,L\,[y(t)] = L\,[e^{-t}\sin t]$$

$$Y(s) - s\,y(0) - y'(0)\} + 2\,\{s\,Y(s) - y(0)\} + 5\,Y(s) = \frac{1}{(s+1)^2 + 1}$$

$$\{s^2\,Y(s) - s(0) - 1\} + 2\,\{s\,Y(s) - 0\} + 5\,Y(s) = \frac{1}{s^2 + 2s + 2} \qquad \{\because\ y(0) = 0,\ y'(0) = 1\}$$

$$\therefore \quad \{s^2\,Y(s) - s(0) - 1\} + 2\,\{s\,Y(s) - 0\} + 5\,Y(s) = \frac{1}{s^2 + 2s + 2}$$

$$\therefore \quad (s^2 + 2s + 5)\,Y(s) - 1 = \frac{1}{s^2 + 2s + 2}$$

$$\therefore \quad Y(s) = \frac{1}{(s^2 + 2s + 5)} + \frac{1}{(s^2 + 2s + 2)\,(s^2 + 2s + 5)}$$

$$= \frac{s^2 + 2s + 3}{(s^2 + 2s + 2)\,(s^2 + 2s + 5)}$$

$$\therefore \quad y(t) = \frac{1}{3}\,e^{-t}\,(\sin t + \sin 2t)$$ [Refer solved example 14 (iv), see 11.9]

(iii) Taking Laplace transform of each sides, we have

$$L\,[y''] + 4\,L\,[y'] + 13\,L\,[y(t)] = \frac{1}{3}\,L\,[e^{-2t}\sin 3t]$$

$$\therefore \quad \{s^2\,Y(s) - s\,y(0) - y'(0)\} + 4\,\{s\,Y(s) - y(0)\} + 13\,Y(s) = \frac{1}{3}\cdot\frac{3}{(s+2)^2 + 9}$$

$$\therefore \quad \{s^2\,Y(s) - s(1) - (-2)\} + 4\,\{s\,Y(s) - 1\} + 13\,Y(s) = \frac{1}{(s+2)^2 + 9} \quad \{\because y(0) = 1,\ y'(0) = -2\}$$

or

$$(s^2 + 4s + 13)\,Y(s) - s - 2 = \frac{1}{(s+2)^2 + 9}$$

or

$$Y(s) = \frac{s+2}{s^2 + 4s + 13} + \frac{1}{(s^2 + 4s + 13)\,[(s+2)^2 + 9]}$$

$$= \frac{s+2}{(s+2)^2 + 9} + \frac{1}{[(s+2)^2 + 9]^2}$$

Taking inverse Laplace transform, we get

$$y(t) = L^{-1}\left[\frac{s+2}{(s+2)^2 + 9}\right] + L^{-1}\left[\frac{1}{\{(s+2)^2 + 9\}^2}\right]$$

$$= e^{-2t}\left[\frac{s}{s^2 + 9}\right] + e^{-2t}\,L^{-1}\left[\frac{1}{(s^2 + 9)^2}\right]$$

$$= e^{-2t}\cos 3t + e^{-2t}\,L^{-1}\left[\frac{1}{s^2 + 9}\cdot\frac{1}{s^2 + 9}\right]$$

$$= e^{-2t}\cos 3t + e^{-2t}\left\{\frac{\sin 3t}{3} * \frac{\sin 3t}{3}\right\}$$ [By the Convolution Theorem]

$$= e^{-2t}\cos 3t + \frac{e^{-2t}}{9}\int_0^t \sin 3u\,\sin 3\,(t - u)\,du$$

$$= e^{-2t}\cos 3t + \frac{e^{-2t}}{9}\int_0^t \frac{\cos(6u - 3t) - \cos 3t}{2}\,du$$

$$= e^{-2t}\cos 3t + \frac{e^{-2t}}{18}\left[\frac{\sin(6u - 3t)}{6} - u\cos 3t\right]^{t}$$

$$= e^{-2t}\cos 3t + \frac{e^{-2t}}{18}\left(\frac{\sin 3t}{3} - t\cos 3t\right)$$

$$= e^{-2t}\cos 3t + \frac{e^{-2t}}{54}\,(\sin 3t - 3t\cos 3t).$$

(iv) Given equation is $x''(t) + n^2 x(t) = a \sin nt \cos \alpha + a \cos nt \sin \alpha$.

Taking Laplace transform of the equation, we have

$$L[x''(t)] + n^2 L[x(t)] = a\cos\alpha\; L[\sin nt] + a\sin\alpha\, L[\cos nt]$$

$$\{s^2 X(s) - s\,x(0) - x'(0)\} + n^2 X(s) = a\cos\alpha\left(\frac{n}{s^2+n^2}\right) + a\sin\alpha\left(\frac{s}{s^2+n^2}\right)$$

Using the given conditions $x(0) = x'(0) = 1$, we get

$$(s^2+n^2)\,X(s) = a\cos\alpha\left(\frac{n}{s^2+n^2}\right) + a\sin\alpha\left(\frac{s}{s^2+n^2}\right)$$

$$\therefore \qquad X(s) = a\cos\alpha\left[\frac{n}{(s^2+n^2)^2}\right] + a\sin\alpha\left[\frac{s}{(s^2+n^2)^2}\right]$$

Thus
$$x(t) = (a\cos\alpha)\,L^{-1}\left[\frac{n}{(s^2+n^2)^2}\right] + (a\sin\alpha)\,L^{-1}\left[\frac{s}{(s^2+n^2)^2}\right]$$

$$= (a\cos\alpha)\left\{\frac{n}{2n^3}(\sin nt - nt\cos nt)\right\} + (a\sin\alpha)\left\{\frac{t\sin nt}{2n}\right\}$$

$$= \frac{a\cos\alpha}{2n^2}(\sin nt - nt\cos nt) + \frac{a\sin\alpha}{2n}(t\sin nt)$$

[**Note :** (i) $\therefore\; L^{-1}\left[\frac{1}{s^2+n^2}\right] = \frac{\sin nt}{n} \qquad \therefore\; L^{-1}\left[-\frac{d}{ds}\,\frac{1}{s^2+n^2}\right] = \frac{t\sin nt}{n}$

$$\therefore \qquad \left[\frac{2s}{(s^2+n^2)^2}\right] = \frac{t\sin nt}{n} \quad \text{or} \quad L^{-1}\left[\frac{s}{(s^2+n^2)^2}\right] = \frac{t\sin nt}{2n}$$

(ii) $$L^{-1}\left[\frac{1}{s}\cdot\frac{s}{(s^2+n^2)^2}\right] = \int_0^t \frac{t\sin nt}{2n}\,dt = \frac{1}{2n^3}\,(\sin nt - n\,t\cos nt)$$]

Ex. 3 : Solve the differential equation

$$y'' + 4y = f(t), \quad y(0) = 0, \quad y'(0) = 1.$$

Sol. : Taking Laplace transform of both sides, we have

$$L[y''] + 4\,L[y(t)] = L[f(t)]$$

$$\{s^2 Y(s) - s\,y(0) - y'(0)\} + 4\,Y(s) = F(s)$$

Using the given conditions $y(0) = 0$, $y'(0) = 1$, we get

$$\{s^2 Y(s) - s(0) - 1\} + 4\,Y(s) = F(s)$$

or
$$(s^2+4)\,Y(s) - 1 = F(s)$$

$$\therefore \qquad Y(s) = \frac{1}{s^2+4} + \frac{F(s)}{s^2+4}$$

Then using the convolution theorem, we have

$$y(t) = \sin t + f(t) * \frac{\sin 2t}{2} = \sin t + \frac{1}{2}\int_0^t f(u)\sin 2(t-u)\,du.$$

[Note that in this case,actual Laplace transform of f(t) does not enter into final solution.]

Ex. 4 : Solve Example 3 if :

(i) $f(t) = U(t-2)$ [Heaviside's unit step function]

(ii) $f(t) = \delta(t)$ [Dirac delta function]

(iii) $f(t) = \delta(t-2)$

(iv) $f(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & t > 1 \end{cases}$ **(May 93, 96, Dec. 92, 94)**

Sol. : (i) Here $f(t) = U(t-2)$ $\therefore$ $F(s) = \dfrac{e^{-2s}}{s}$ $\left\{\because L[U(t-a)] = \dfrac{e^{-as}}{s}\right.$

Now from result (i) in Example 3 above, we have

$$Y(s) = \frac{1}{s^2+4} + \frac{e^{-2s}}{s(s^2+4)}$$

or

$$Y(s) = \frac{1}{(s^2+4)} + \frac{e^{-2s}}{4}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

Taking inverse Laplace transform, we get

$$y(t) = \frac{1}{2}\sin 2t + \frac{1}{4}L^{-1}\left[e^{-2s}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)\right]$$

$$\therefore \quad y(t) = \frac{1}{2}\sin 2t + \frac{1}{4}\{1-\cos 2(t-2)\}\,U(t-2)$$

[By the Second Shifting Theorem]

$\{\because L[\delta(t)] = 1\}$

(ii) Here $f(t) = \delta(t)$ $\therefore$ $F(s) = 1$

Now from result (i) in Example 3 above, we have

$$Y(s) = \frac{1}{s^2+4} + \frac{1}{s^2+4} = \frac{2}{s^2+4}$$

$$y(t) = \sin 2t$$

$\sqrt{s^2 + 1}$

10.10 LAPLACE TRANSFORMS OF SPECIAL FUNCTIONS

In discussion of certain types of physical and engineering problems, number of situation such as

(i) a force acting on the part of the system or voltage acting for finite interval of time.

(ii) a large force acting, for very short time (impulsive force) or over a very short area. For examples, heavy excitation is introduced by putting on and off switch in one action, concentrated load acting on a beam.

(iii) a periodic function or periodic voltage.

The analytical representation of such functions and the nature of their Laplace transforms are of great practical importance. Hence in the following articles, such functions and their Laplace transforms are discussed.

10.11 UNIT STEP FUNCTION

The unit step function also called *Heaviside's unit step function U(t),* is defined as

$$\boxed{U(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}}$$ ... (2

This unit step function is a curve which has value zero at all points to the left of the origin and is equal to 1 (unity) on the right of the origin (see Fig. 10.4).

The displaced unit step function U(t – a) is defined as

$$\boxed{U(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}}$$ ... (2

where a ≥ 0.

The displaced unit step function U(t – a) represents curve U(t) which is displace (translated) a distance a units to the right along t-axis (see Fig. 10.5).

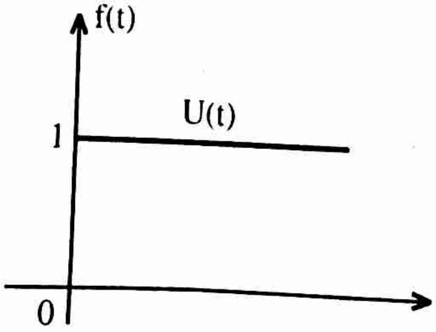


Fig. 10.4 : Unit step function U(t)

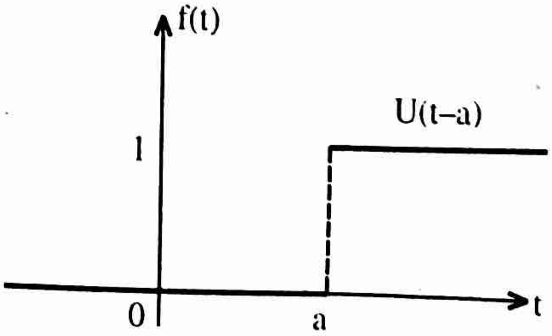


Fig. 10.5 : Displaced unit step function U(t – a)

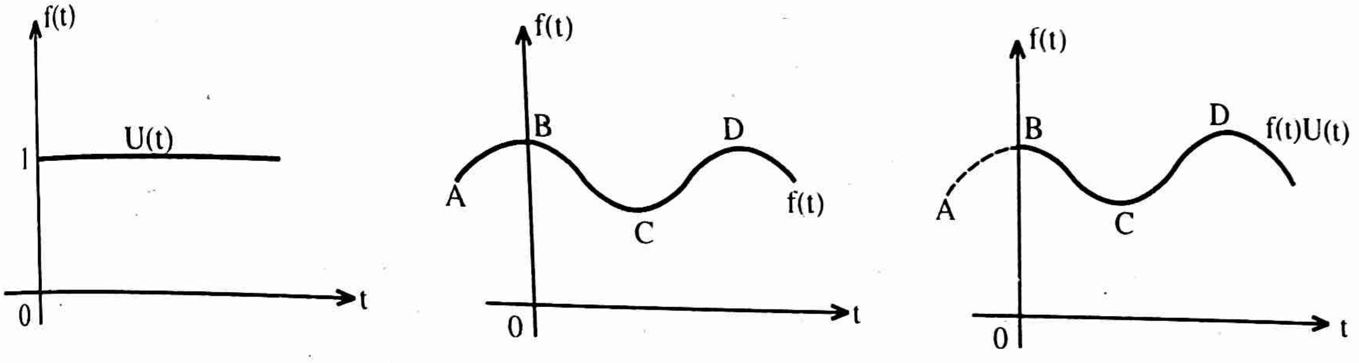
Note 1 : Observe that at t = a, a step of unit height is formed, hence the name unit step *function*. Other notations to represent unit step functions are H(t), H(t – a), $u_0(t)$, $u_a(t)$.

Note 2 : Unit step functions U(t) and U(t – a) are extensively used to represent a portion of *the curve* of the function f(t) as explained in the following cases.

Case I : f(t) U(t) :

When the function f(t) is multiplied by unit step function U(t), the resultant function f(t) U(t) will represent the part of the function f(t) on the right of the origin, the part of f(t) on the left of the origin being cut off (i.e. vanishes for t < 0) (see Fig. 10.6) i.e.

$$f(t)\,U(t) = \begin{cases} 0 & t < 0 \\ f(t) & t \geq 0 \end{cases}$$



(a) (b) (c)

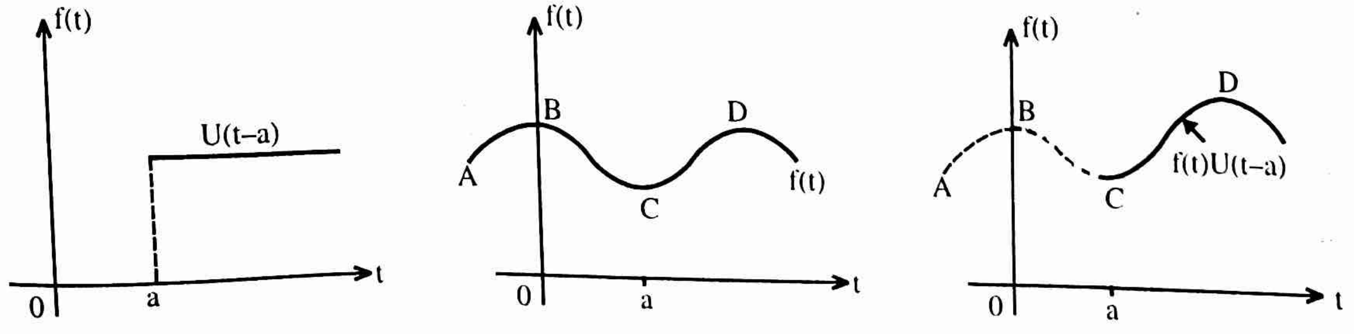
Fig. 10.6

In Fig. 10.6 (c), the part AB of the curve f(t) is cut off.

Case II : f(t) U(t – a) :

When the function f(t) is multiplied by displaced unit step function U(t – a), the resultant function f(t) U(t – a) will represent the part of the function f(t) on the right of t = a, the part of the function f(t) on the left of t = a is cut off (see Fig. 10.7) i.e.

$$f(t)\,U(t-a) = \begin{cases} 0 & t < a \\ f(t) & t \geq a \end{cases}$$



(a) (b) (c)

Fig. 10.7

In Fig. 10.7 (c), the part ABC of the function f(t) is cut off.

10.12 TRANSFORMS USING UNIT STEP FUNCTIONS

1. Laplace transform of Unit step function U(t) :

By definition of Laplace transform

$$L[U(t)] = \int_0^\infty e^{-st} U(t)\, dt = \int_0^\infty e^{-st}(1)\, dt \qquad \left[\because U(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}\right]$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s} \qquad \text{where } s > 0$$

Hence

$$\boxed{L[U(t)] = \frac{1}{s}} \qquad \dots (30)$$

Laplace transform of Displaced unit step function U(t – a) :

By definition of Laplace transform

$$L[U(t-a)] = \int_0^\infty e^{-st} U(t-a)\, dt$$

$$= \int_0^a e^{-st}(0)\, dt + \int_a^\infty e^{-st}(1)\, dt \qquad \left[\because U(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}\right]$$

$$= 0 + \left[\frac{e^{-st}}{-s}\right]_a^\infty = \frac{e^{-as}}{s} \qquad \text{where } s > 0$$

Hence

$$\boxed{L[U(t-a)] = \frac{e^{-as}}{s}} \qquad \dots (31)$$

(May 97)

10.14 DIRAC DELTA FUNCTION OR UNIT IMPULSE FUNCTION

Consider the function

$$F(t) = \begin{cases} t & t < a \\ 1/\varepsilon & a \le t \le a + \varepsilon \\ 0 & t > a + \varepsilon \end{cases} \qquad \ldots (37)$$

where $\varepsilon > 0$ and whose graph is shown in the Fig. 10.17.

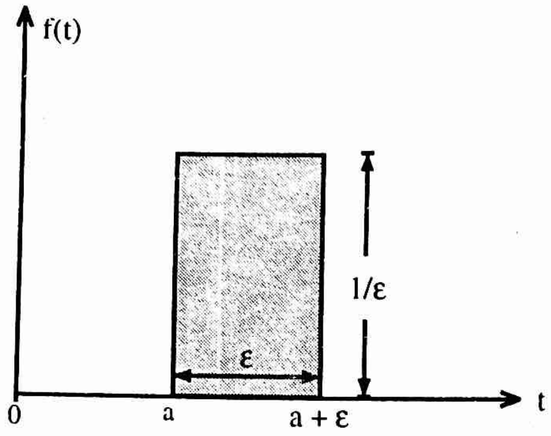


Fig. 10.17

The area enclosed by F(t) and t-axis is obtain by integrating F(t) between the limits $-\infty$ and ∞. Thus

$$\int_{-\infty}^{\infty} F(t)\,dt = \int_{-\infty}^{a} F(t)\,dt + \int_{a}^{a+\varepsilon} F(t)\,dt + \int_{a+\varepsilon}^{\infty} F(t)\,dt$$

$$= 0 + \int_{a}^{a+\varepsilon} \frac{1}{\varepsilon}\,dt + 0$$

$$= \frac{1}{\varepsilon}\,[t]_{a}^{a+\varepsilon} = \frac{1}{\varepsilon}\,[a + \varepsilon - a] = 1$$

As $\varepsilon \to 0$, the function F(t) tends to infinity at $t = a$ and zero else where, with the property that its integral is unity. This is concise manner of expressing a function of arbitrary large magnitude acting for an infinitesimal time such that the product of duration and intensity remains unity. The resulting function is called *unit impulse* or the δ function at $t = a$.

Note : Geometrically, as $\varepsilon \to 0$ the height of the rectangular shaded region increases indefinitely and width decreases in such a way that the area is always equal to 1

i.e. $$\int_{-\infty}^{\infty} F(t)\,dt = 1.$$

Definition : The limiting form of the function F(t) in (37) (as $\varepsilon \to 0$) with the property that its integral is equal to 1 is called *unit impulse function* or *Dirac delta function* denoted by $\delta(t - a)$.

Hence

$$\boxed{\delta(t-a) = \lim_{\epsilon\to 0} F(t)} \qquad \ldots (38)$$

where f(t) is given by (37).

In particular, if a = 0, the unit impulse function at t = 0 is given by

$$\boxed{\delta(t) = \lim_{\varepsilon\to 0} F(t)} \qquad \ldots (39)$$

where

$$F(t) = \begin{cases} 0 & t < 0 \\ 1/\varepsilon & 0 < t < \varepsilon \\ 0 & t > \varepsilon \end{cases}$$

10.15 RELATION BETWEEN UNIT STEP FUNCTION AND DIRAC DELTA FUNCTION

The function F(t) defined in (37), can be expressed in terms of unit step functions as

$$F(t) = \frac{1}{\varepsilon}\,[U(t-a) - U(t-a-\varepsilon)]$$

Taking limit as $\varepsilon \to 0$, we have

$$\lim_{\varepsilon\to 0} F(t) = \lim_{\varepsilon\to 0}\left[\frac{U(t-a) - U(t-a-\varepsilon)}{\varepsilon}\right]$$

$$= \frac{d}{dt}\,U(t-a) = U'(t-a)$$

Hence

$$\boxed{\delta(t-a) = U'(t-a)} \qquad \ldots (40)$$

10.16 LAPLACE TRANSFORM OF DIRAC DELTA FUNCTION

We shall first find the Laplace transform of F(t) defined in (37) and then proceed to the limit as $\varepsilon \to 0$.

By definition of Laplace transform

$$L[F(t)] = \int_0^\infty e^{-st} F(t)\,dt = \int_a^{a+\varepsilon} e^{-st}\frac{1}{\varepsilon}\,dt = \frac{1}{\varepsilon}\left[\frac{e^{-st}}{-s}\right]_a^{a+\varepsilon}$$

$$= \frac{1}{s\varepsilon}\,[e^{-as} - e^{-s(a+\varepsilon)}] = \frac{e^{-as}}{s}\left[\frac{1-e^{-s\varepsilon}}{\varepsilon}\right]$$

$$\therefore \quad L[\delta(t-a)] = \lim_{\varepsilon\to 0} L[F(t)] = \lim_{\varepsilon\to 0}\left[\frac{e^{-as}}{s}\left(\frac{1-e^{-s\varepsilon}}{\varepsilon}\right)\right]$$

$$= \frac{e^{-as}}{s}\lim_{\varepsilon\to 0}\left(\frac{1-e^{-s\varepsilon}}{\varepsilon}\right) \qquad \left(\text{Indeterminate Form } \frac{0}{0}\right)$$

$$= \frac{e^{-as}}{s}\lim_{\varepsilon\to 0}\left(\frac{se^{-s\varepsilon}}{1}\right) \qquad \text{[By L'Hospital's Rule]}$$

$$= \frac{e^{-as}}{s}\,(s) = e^{-as}$$

Hence $\boxed{L\,[\delta\,(t-a)] \;=\; e^{-as}}$

In particular, if a = 0, we have

$$\boxed{L\,[\delta(t)] \;=\; 1}$$

10.17 PROPERTIES OF DIRAC DELTA FUNCTION

By definitions of U(t – a) and δ(t – a), we have

$$[U(t-a)]_{t=0} \;=\; \text{and } [U'(t-a)]_{t=0} = [\delta'(t-a)]_{t=0} = 0$$

(A) $$L\,[H'(t-a)] \;=\; L\,[\delta(t-a)]$$

$$L\,[H'(t-a)] \;=\; s\,L\,[H(t-a)] - [H(t-a)]_{t=0}$$ [By

$$= s\cdot\frac{e^{-as}}{s} - 0$$ [By

$$= e^{-as} \;=\; L\,[\delta(t-a)]$$

(B) $$L\,[H''(t-a)] \;=\; L\,[\delta'(t-a)]$$

$$L\,[H''(t-a)] \;=\; s^2\,L\,[H(t-a)] - s\,[H(t-a)]_{t=0} - [H'(t-a)]_{t=0}$$

[B

$$= s^2\cdot\frac{e^{-as}}{s} - 0 - 0$$ [B

$$= s\,e^{-as} = L\,[\delta'(t-a)]$$

$-\infty$

10.19 PERIODIC FUNCTIONS

Definition : A function f(t) defined for t > 0 is said to be periodic with period T (> 0) if

$$f(t + T) = f(t) \quad \text{for all } t > 0$$

From the above definition it follows that, if n is any integer, then

$$f(t + nT) = f(t) \quad \text{for all } t > 0$$

Hence 2T, 3T, 4T, ... are also periods of f(t).

The Laplace transform of periodic function is discussed in the following theorem.

Theorem : If a function f(t) is periodic with period T, then

$$L\,[f(t)] = \frac{\int_0^T e^{-st}\, f(t)\, dt}{1 - e^{-sT}}$$

Proof : By definition

Ex. 3 : Show that the function f(t) whose graph is the triangular wave shown in the Fig. 10.20 has the Laplace transform $\frac{1}{as^2} \tanh \frac{as}{2}$. **(May 94, Dec. 96)**

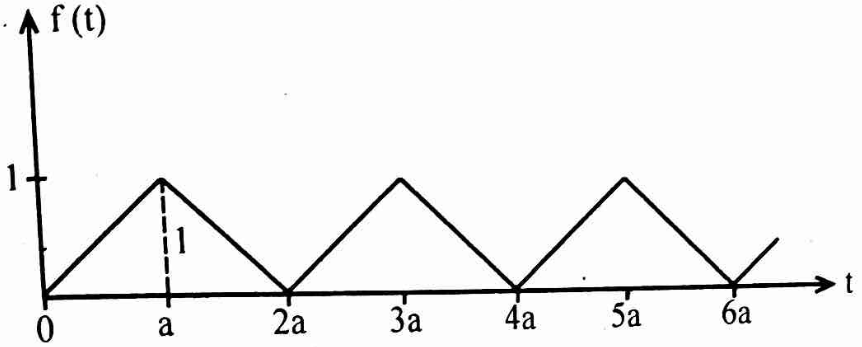


Fig. 10.20 : Periodic triangular wave

Sol. : The given function is defined as

$$f(t) = \begin{cases} t/a & 0 < t < a \\ \frac{1}{a}(2a - t) & a < t < 2a \end{cases} \quad \text{and } f(t + 2a) = f(t) \qquad \left(\text{Using } \frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}\right)$$

Here period T = 2a, hence using result (46), we have

$$L\,[f(t)] = \frac{\int_0^T e^{-st} f(t)\,dt}{1 - e^{-sT}} = \frac{1}{1 - e^{-2as}} \int_0^{2a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1 - e^{-2as}} \left\{ \int_0^a e^{-st} (t/a)\,dt + \int_a^{2a} e^{-st} \frac{1}{a} (2a - t)\,dt \right\}$$

$$= \frac{1}{a\,(1 - e^{-2as})} \left\{ \left[t \left(\frac{e^{-st}}{-s}\right) - (1) \left(\frac{e^{-st}}{s^2}\right) \right]_0^a + \left[(2a - t) \left(\frac{e^{-st}}{-s}\right) - (-1) \left(\frac{e^{-st}}{s^2}\right) \right]_a^{2a} \right\}$$

$$= \frac{1}{a\,(1 - e^{-2as})} \left\{ \left(- \frac{a\,e^{-as}}{s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2} \right) + \left(\frac{e^{-2as}}{s^2} + \frac{a\,e^{-as}}{s} - \frac{e^{-as}}{s^2} \right) \right\}$$

$$= \frac{1}{a\,(1 - e^{-2as})} \left\{ \frac{1 - 2e^{-as} + e^{-2as}}{s^2} \right\} = \frac{(1 - e^{-as})^2}{as^2\,(1 - e^{-as})\,(1 + e^{-as})}$$

$$= \frac{1}{as^2} \, \frac{1 - e^{-as}}{1 + e^{-as}} = \frac{1}{as^2} \tanh \frac{as}{2}$$

Table of Laplace Transform of Special Functions

Sr. No.	f(t)	L [f(t)] = F(s)
1.	$U(t)$	$\frac{1}{s}$
2.	$U(t-a)$	$\frac{e^{-as}}{s}$
3.	$f(t-a)\,U(t-a)$	$e^{-as}\,L[f(t)] = e^{-as}\,F(s)$
4.	$f(t)\,U(t)$	$F(s)$
5.	$\delta(t)$	1
6.	$\delta(t-a)$	e^{-as}
7.	$f(t)\,\delta(t-a)$	$e^{-as}\,f(a)$
8.	Periodic function $f(t)$	$\dfrac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-st}}$, T is period
9.	$\operatorname{erf}\left(\sqrt{t}\right)$	$\frac{1}{s\sqrt{s+1}}$
10.	$\operatorname{erf}(t)$	$\frac{1}{s}\, e^{s^4/4}\ \operatorname{erfc}(s/2)$
11.	$S_i(t)$	$\frac{1}{s}\tan^{-1}\frac{1}{s}$
12.	$C_i(t)$	$\frac{1}{2s}\log(s^2+1)$
13.	$E_i(t)$	$\frac{1}{s}\log(s+1)$