Z- Transforms

Sequence - An ordered set of real or complex numbers is called a sequence. It is denoted by $\{f(k)\}$ or $\{f_k\}$. The sequence is represented in two ways.

① $\{f(k)\} = \{15, 13, 10, \underset{\uparrow}{8}, 5, 2, 0, 3\}$

This vertical arrow represents the 0^{th} position in the sequence i.e, $f(0)=8$, $f(1)=5$, $f(2)=2$, $f(3)=0$, $f(4)=3$, $f(-1)=10$, $f(-2)=13$, $f(-3)=15$.

If vertical arrow is not given then the starting term of the sequence denotes the position corresponding to $k=0$.

② The second way of specifying the sequence is to define the general term of the sequence (if possible) as a function of position i.e, k.

eg) $\{f(k)\} = \frac{1}{4^k}$, $k \in \mathbb{Z}$.

then $\{f(k)\} = \left\{ \frac{1}{4^{-8}}, \frac{1}{4^{-7}}, \cdots, \frac{1}{4^{-1}}, 1, \frac{1}{4}, \frac{1}{4^2}, \cdots \right\}$

Here $f(0)=1$, $f(1)=\frac{1}{4}$.

Causal Sequence :- Any sequence whose terms corresponding to $k<0$ are all zero is called as causal sequence.

eg) $\{f(k)\} = \{0, 0, \cdots, 0, \underset{\substack{\uparrow \\ k=0}}{1}, 2, 4, 6, \cdots\}$

Basic operations on Sequences

1] Addition: $\rightarrow$ $\{f(k)\} + \{g(k)\} = \{f(k) + g(k)\}$

2] Scaling: $\rightarrow$ If a is any scalar, $a\{f(k)\} = \{a f(k)\}$

3] Linearity :→

$\{a f(K) + b\, g(K)\} = a\{f(K)\} + b\{g(K)\}$

Important Results

1] $\frac{1}{1+y} = 1 - y + y^2 - y^3 + y^4 - \cdots$, $|y| < 1$

2] $\frac{1}{1-y} = 1 + y + y^2 + y^3 + \cdots$, $|y| < 1$

3] $(1+y)^n = 1 + ny + \frac{n(n-1)}{2!} y^2 + \frac{n(n-1)(n-2)}{3!} y^3 + \cdots$, $|y| < 1$

$= \sum_{r=0}^{n} {}^nC_r\, y^r$

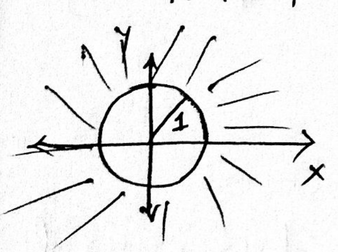
4] $e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots$

5] $a + ar + ar^2 + \cdots = \frac{a}{1-r}$ and it is convergent if $|r| < 1$

where a = first term and r = common ratio.

6] If $z = x + iy$ then $|z| > 1$ represents.

→ $|z| > 1 \Rightarrow |x + iy| > 1$



$\Rightarrow \sqrt{x^2 + y^2} > 1 \Rightarrow x^2 + y^2 > 1$

⇒ collection of points which lie outside the circle $x^2 + y^2 = 1$ i.e, centre $(0, 0)$ & radius 1.

Simillarly $|z| < 1 \Rightarrow$ represents points lies inside the circle $x^2 + y^2 = 1$.

Definition of z-transform

The z-transform of sequence $\{f(k)\}$ defined as.

$$Z\{f(k)\} = F(z) = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

where $z = x + iy$ is a complex number.

'Z' is a z-transform operator.

'$F(z)$' is z-transform of $\{f(k)\}$.

Note — For Causal sequence

$$\{f(k)\} = \{0, 0, \ldots, 0, f(0), f(1), f(2) \ldots\}$$

$$Z\{f(k)\} = F(z) = \sum_{k=0}^{\infty} f(k)\, z^{-k}$$

eg) Find z-transform of $\{f(k)\} = \{2, \underset{\uparrow}{3}, 4, 5, 6\}$

→ $$F(z) = Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

Here $f(0) = 3$, $f(1) = 4$, $f(2) = 5$, $f(3) = 6$, $f(-1) = 2$.

$$\therefore \; = \sum_{k=-1}^{3} f(k)\, z^{-k} = f(-1)\, z^{1} + f(0)\, z^{0} + f(1)\, z^{-1} + f(2)\, z^{-2} + f(3)\, z^{-3}$$

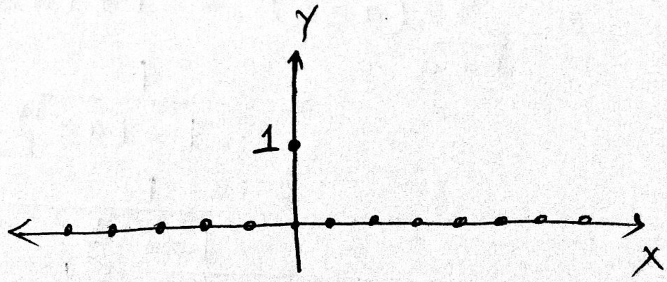
$$= 2z + 3 + \frac{4}{z} + \frac{5}{z^2} + \frac{6}{z^3}$$

Z-transform of some standard sequences

1] **Unit Impulse:-** $\delta(k) = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$

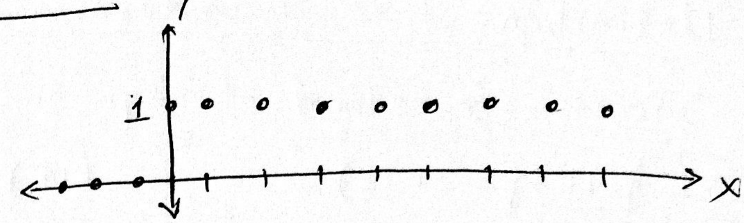
$$Z\{\delta(k)\} = \sum_{k=-\infty}^{\infty} \delta(k)\, z^{-k} = 0 + \cdots + 0 + 1\, z^{0} + 0 + \cdots + 0$$
$$= 1.$$

$$\therefore \boxed{Z\{\delta(k)\} = 1}$$



2] Discrete Unit step function :-

$$U(K) = \begin{cases} 0, & K < 0 \\ 1, & K \geq 0 \end{cases}$$



$$Z\{U(K)\} = \sum_{K=-\infty}^{\infty} f(K)\, z^{-K}$$

$$= \sum_{K=-\infty}^{-1} f(K)\, z^{-K} + \sum_{K=0}^{\infty} f(K)\, z^{-K}$$

$$= 0 + \sum_{K=0}^{\infty} 1\, z^{-K}$$

$$= 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5} + \cdots$$

$$= 1 + (z^{-1})^1 + (z^{-1})^2 + (z^{-1})^3 + (z^{-1})^4 + (z^{-1})^5 + \cdots$$

$$= \frac{1}{1-(z^{-1})}, \quad |z^{-1}| < 1$$

$$= \frac{1}{1-\frac{1}{z}}, \quad \left|\frac{1}{z}\right| < 1$$

$$= \frac{z}{z-1}, \quad |z| > 1 \qquad \text{Here } |z|>1 \text{ is ROC}$$

ROC :- Region of convergence.

$$\Rightarrow \boxed{Z\{U(K)\} = \frac{z}{z-1}, \quad |z| > 1}$$

3] $f(K) = a^K, \; K \geq 0$.

$$Z\{f(K)\} = \sum_{K=0}^{\infty} f(K)\, z^{-K} = \sum_{K=0}^{\infty} a^K z^{-K}$$

$$= \sum_{K=0}^{\infty} (a z^{-1})^K = 1 + (a z^{-1}) + (a z^{-1})^2 + (a z^{-1})^2 + \cdots$$

$$= \frac{1}{1-(a z^{-1})}, \quad |a z^{-1}| < 1$$

$$= \frac{1}{1-\frac{a}{z}}, \quad \left|\frac{a}{z}\right| < 1$$

$$\boxed{Z\{a^K\}_{K \geq 0} = \frac{z}{z-a}, \quad |z| > |a|}$$

3] $f(k) = a^k,\ k<0$

$$Z\{f(k)\} = \sum_{k=-\infty}^{-1} f(k) z^{-k} = \sum_{k=-\infty}^{-1} a^k z^{-k}$$

Put $k=-r$

k	$-\infty$	-1
r	∞	1

$$= \sum_{k=1}^{\infty} a^{-r} z^r = \sum_{r=1}^{\infty} (a^{-1}z)^r$$

$$= (a^{-1}z) + (a^{-1}z)^2 + (a^{-1}z)^3 + \cdots$$

$$= (a^{-1}z)\left[1 + (a^{-1}z) + (a^{-1}z)^2 + \cdots\right]$$

$$= (a^{-1}z)\cdot\left[\frac{1}{1-(a^{-1}z)}\right], \quad |a^{-1}z|<1$$

$$= (a^{-1}z)\left(\frac{1}{1-\frac{z}{a}}\right), \quad \left|\frac{z}{a}\right|<1$$

$$= \frac{z}{\cancel{a}}\left(\frac{\cancel{a}}{a-z}\right), \quad |z|<|a|$$

$$= \frac{z}{a-z}, \quad |z|<|a|$$

$$\therefore \boxed{Z\{a^k\}_{k<0} = \frac{z}{a-z}, \quad |z|<|a|}$$

4] $f(k) = \{a^{|k|}\},\ \forall k.$

$$\rightarrow Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k} = \sum_{-\infty}^{-1} f(k) z^{-k} + \sum_{k=0}^{\infty} f(k) z^{-k}$$

$$= \sum_{k=-\infty}^{-1} a^{|k|} z^{-k} + \sum_{k=0}^{\infty} a^{|k|} z^{-k} = \sum_{k=-\infty}^{-1} a^{-k} z^{-k} + \sum_{k=0}^{\infty} a^k z^{-k}$$

Put $k=-r$ in the first summation

$$= \left[az + (az)^2 + \cdots\right] + \left[1 + (az^{-1}) + (az^{-1})^2 + \cdots\right]$$

$$= \frac{az}{1-az} + \frac{1}{1-az^{-1}}, \quad |az|<1 \ \& \ |az^{-1}|<1$$

$$= \frac{az}{1-az} + \frac{z}{z-a}, \quad |z|<\frac{1}{|a|} \ \& \ \left|\frac{a}{z}\right|<1$$

i.e, $|a| < |z| < \frac{1}{|a|}$

6] $f(k) = \cos\alpha k, \quad k \geq 0.$

$\rightarrow$ Euler's formula, $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

$$\therefore Z\{\cos\alpha k\} = \sum_{k=0}^{\infty} \left(\frac{e^{i\alpha k} + e^{-i\alpha k}}{2} \right) z^{-k}$$

$$= \frac{1}{2}\left\{\left[\sum_{k=0}^{\infty} e^{i\alpha k} z^{-k} + \sum_{k=0}^{\infty} e^{-i\alpha k} z^{-k} \right]\right.$$

$$= \frac{1}{2}\left[\sum_{k=0}^{\infty} (e^{i\alpha} z^{-1})^k + \sum_{k=0}^{\infty} (e^{-i\alpha} z^{-1})^k \right]$$

$$= \frac{1}{2}\left[\left[1 + (e^{i\alpha} z^{-1}) + (e^{i\alpha} z^{-1})^2 + (e^{i\alpha} z^{-1})^3 + \cdots\right] + \left[1 + (e^{-i\alpha} z^{-1}) + (e^{-i\alpha} z^{-1})^2 + (e^{-i\alpha} z^{-1})^3 + \cdots\right]\right].$$

$$= \frac{1}{2}\left[\frac{1}{1 - e^{i\alpha} z^{-1}} + \frac{1}{1 - (e^{-i\alpha} z^{-1})} \right], \quad |e^{i\alpha} z^{-1}| < 1 \ \& \ |e^{-i\alpha} z^{-1}| < 1$$

$$= \frac{1}{2}\left[\frac{z}{z - e^{i\alpha}} + \frac{(e^{i\alpha} z)}{e^{i\alpha} z - 1} \right], \quad |e^{i\alpha}| < |z| \ \& \ |z| > |e^{-i\alpha}|$$

$$= \frac{1}{2}\left[\frac{z}{z - e^{i\alpha}} + \frac{z}{z - e^{-i\alpha}} \right], \quad |z| > |e^{i\alpha}| \ \& \ |z| > |e^{-i\alpha}|$$

$\Downarrow$

since $e^{i\alpha} = \cos\alpha + i\sin\alpha$

$|e^{i\alpha}| = \sqrt{\cos^2\alpha + \sin^2\alpha}$

$= 1$

$\therefore |z| > 1$

$$= \frac{1}{2}\left[\frac{z(z - e^{-i\alpha}) + z(z - e^{i\alpha})}{(z - e^{i\alpha})(z - e^{-i\alpha})} \right]$$

$$= \frac{1}{2}\left[\frac{z^2 - z e^{-i\alpha} + z^2 - z e^{i\alpha}}{(z^2 - z e^{-i\alpha} - z e^{i\alpha} + e^{i\alpha} e^{-i\alpha})} \right]$$

$$= \frac{1}{2}\left[\frac{2z^2 - 2z\left(\frac{e^{i\alpha} + e^{-i\alpha}}{2}\right)}{z^2 - 2z\left(\frac{e^{i\alpha} + e^{-i\alpha}}{2}\right) + 1} \right], \quad |z| > 1$$

$$= \frac{z^2 - z\cos\alpha}{z^2 - 2z\cos\alpha + 1} = \frac{z(z - \cos\alpha)}{z^2 - 2z\cos\alpha + 1}, \quad |z| > 1$$

$$\therefore \boxed{Z\{\cos\alpha k\} = \frac{z(z-\cos\alpha)}{z^2-2z\cos\alpha+1}, \quad |z|>1}$$

7] $$Z\underset{k\geq 0}{[\sin\alpha k]} = \frac{z\sin\alpha}{z^2-2z\cos\alpha+1}, \quad |z|>1$$

8] $$Z\underset{k\geq 0}{\{\cosh\alpha k\}} = \frac{z(z-\cosh\alpha)}{z^2-2z\cosh\alpha+1}, \quad |z|>\max(|e^{\alpha}|,|e^{-\alpha}|)$$

9] $$Z\underset{k\geq 0}{\{\sinh\alpha k\}} = \frac{z\sinh\alpha}{z^2-2z\cosh\alpha+1}, \quad |z|>\max(|e^{\alpha}|,|e^{-\alpha}|)$$

10] $f(k)=\{{}^nC_k\}, (0\leq k\leq n)$

$$Z[{}^nC_k] = \sum_{k=0}^{\infty} {}^nC_k\, z^{-k} \qquad \text{since } {}^nC_k=0 \text{ for } k>n$$

$$= {}^nC_0 + {}^nC_1 z^{-1} + {}^nC_2 z^{-2} + {}^nC_3 z^{-3} + \cdots + {}^nC_n z^{-n}$$

$$= (1+z^{-1})^n, \quad |z|>0$$

11] $$Z\underset{k\geq n}{[{}^kC_n]} = z^{-n}(1-z^{-1})^{-(n+1)}, \quad |z|>1$$

12] $$Z\left[\frac{a^k}{k!}\right], k\geq 0 = e^{a/z}$$

$$\rightarrow \quad Z\left[\frac{a^k}{k!}\right] = \sum_{k=0}^{\infty} \frac{a^k}{k!} z^{-k} = \sum_{k=0}^{\infty} \frac{(az^{-1})^k}{k!}$$

$$= 1 + \frac{(az^{-1})}{1!} + \frac{(az^{-1})^2}{2!} + \frac{(az^{-1})^3}{3!} + \cdots$$

$$= e^{az^{-1}} = e^{a/z}$$

Find z-transform of

① $\left(\frac{1}{5}\right)^k,\ k \geq 0$.

$\rightarrow\ z[f(k)] = \sum_{k=0}^{\infty} \left(\frac{1}{5}\right)^k z^{-k} = \sum_{k=0}^{\infty} \left(\frac{z^{-1}}{5}\right)^k$

$= 1 + \left(\frac{z^{-1}}{5}\right) + \left(\frac{z^{-1}}{5}\right)^2 + \left(\frac{z^{-1}}{5}\right)^3 + \cdots$

$= \frac{1}{1-\left(\frac{z^{-1}}{5}\right)},\ \left|\frac{z^{-1}}{5}\right| < 1$

$= \frac{5z}{5z-1},\ \left|\frac{1}{5z}\right| < 1$ i.e, $|z| > \frac{1}{5}$

② $3^k,\ k < 0$

$\rightarrow\ z[f(k)] = \sum_{k=-\infty}^{-1} f(k) z^{-k}$

$= \sum_{k=-\infty}^{-1} 3^k z^{-k}$ Put $k = -r$

k	$-\infty$	-1
r	∞	1

$= \sum_{r=1}^{\infty} 3^{-r} z^{r} = \sum_{r=1}^{\infty} (3^{-1}z)^r$

$= (3^{-1}z) + (3^{-1}z)^2 + (3^{-1}z)^3 + \cdots$

$= (3^{-1}z)\left[1 + (3^{-1}z) + (3^{-1}z)^2 + \cdots\right]$

$= (3^{-1}z)\left[\frac{1}{1-(3^{-1}z)}\right],\ |3^{-1}z| < 1$

$= \frac{z}{\cancel{3}}\left(\frac{\cancel{3}}{3-z}\right),\ \left|\frac{z}{3}\right| < 1$

$= \frac{z}{3-z},\ |z| < 3$

③ $f(k) = 4^k + 5^k,\ k \geq 0$

$\rightarrow\ z[f(k)] = \sum_{k=0}^{\infty} f(k) z^{-k}$

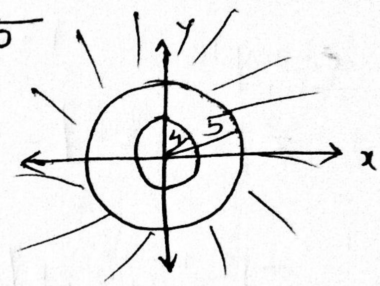
$$= \sum_{K=0}^{\infty} (4^K + 5^K) z^{-K}$$

$$= \sum_{K=0}^{\infty} 4^K z^{-K} + \sum_{K=0}^{\infty} 5^K z^{-K} = \sum_{K=0}^{\infty} (4z^{-1})^K + \sum_{K=0}^{\infty} (5z^{-1})^K$$

$$= \left[1 + (4z^{-1}) + (4z^{-1})^2 + \cdots\right] + \left[1 + (5z^{-1}) + (5z^{-1})^2 + \cdots\right]$$

$$= \frac{1}{1-4z^{-1}} + \frac{1}{1-5z^{-1}}, \quad |4z^{-1}| < 1 \ \& \ |5z^{-1}| < 1$$

$$= \frac{z}{z-4} + \frac{z}{z-5}, \quad |z| > 4 \ \& \ |z| > 5$$



$$= \frac{z}{z-4} + \frac{z}{z-5}, \quad |z| > 5$$

(4) $f(K) = \underset{K<0}{4^K} + \underset{K\geq 0}{3^K}$

$$\rightarrow \ z[f(K)] = \sum_{K=-\infty}^{\infty} f(K) z^{-K}$$

$$= \sum_{K=-\infty}^{-1} 4^K z^{-K} + \sum_{K=0}^{\infty} 3^K z^{-K}$$

$$= \sum_{K=-\infty}^{-1} (4^K z^{-K}) + \sum_{K=0}^{\infty} (3z^{-1})^K$$

Put $K = -r$

$$= \sum_{r=1}^{\infty} 4^{-r} z^{r} + \sum_{K=0}^{\infty} (3z^{-1})^K = \sum_{r=1}^{\infty} (4^{-1}z)^r + \sum_{K=0}^{\infty} (3z^{-1})^K$$

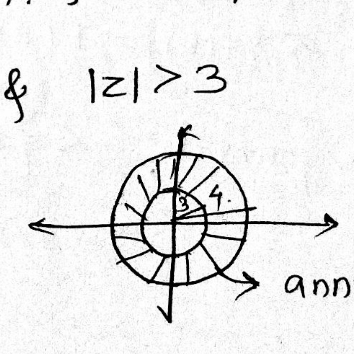
$$= \left[(4^{-1}z) + (4^{-1}z)^2 + (4^{-1}z)^3 + \cdots\right] + \left[1 + (3z^{-1}) + (3z^{-1})^2 + \cdots\right]$$

$$= 4^{-1}z\left[1 + (4^{-1}z) + (4^{-1}z)^2 + \cdots\right] + \frac{1}{1-3z^{-1}}, \quad |3z^{-1}| < 1$$

$$= 4^{-1}z\left[\frac{1}{1-4^{-1}z}\right] + \frac{1}{1-3z^{-1}}, \quad |4^{-1}z| < 1 \ \& \ |3z^{-1}| < 1$$

$$= \frac{z}{\cancel{4}}\left[\frac{\cancel{4}}{4-z}\right] + \frac{z}{z-3}, \quad |z| < 4 \ \& \ |z| > 3$$

$$= \frac{z}{4-z} + \frac{z}{z-3}, \quad 3 < |z| < 4$$



annulus.

Properties of z-transform

*1] **Linearity**:- If $\{f(k)\}$ and $\{g(k)\}$ are such that they can be added and a & b are constants then,

$$z\{af(k)+b\,g(k)\} = a\,z\{f(k)\} + b\,z\{g(k)\}$$

*2] **Change of scale** :- If $z\{f(k)\} = F(z)$ then

$$z^{-1}\{a^k f(k)\} = F\left(\frac{z}{a}\right)$$

example :- Find z-transform of $f(k) = 3^k \cos 2k,\ k \geq 0$

$$z[\cos 2k] = \frac{z(z-\cos 2)}{z^2 - 2z\cos 2 + 1}, \quad |z|>1$$

$$z[3^k \cos 2k] = \frac{(z/3)\left((z/3)-\cos 2\right)}{(z/3)^2 - 2(z/3)\cos 2 + 1}, \quad |z/3|>1$$

↳ Using change of scale property.

*3] If $z\{f(k)\} = F(z)$ then $z\{e^{-ak} f(k)\} = F(e^{a} z)$

ex] Find $z[e^{-3k}\cos 5k]$

→ $$z[\cos 5k] = \frac{z(z-\cos 5)}{z^2 - 2z\cos 5 + 1}, \quad |z|>1$$

Now using property ③

$$z[e^{-3k}\cos 5k] = \frac{e^3 z\,(e^3 z - \cos 5)}{(e^3 z)^2 - 2(e^3 z)\cos 5 + 1}, \quad |e^3 z|>1$$

*4] **Multiplication by k**

If $z\{f(k)\} = F(z)$ then $z\{k f(k)\} = -z \frac{d}{dz} F(z)$

In general, $z\{k^n f(k)\} = \left(-z\frac{d}{dz}\right)^n F(z)$

example ① Find $z\{k\,5^k\}$, $k \geq 0$.

$\rightarrow$ $z\{5^k\}_{k\geq 0} = \frac{z}{z-5}$

$z\{k\,5^k\} = -z\frac{d}{dz}\left(\frac{z}{z-5}\right)$ ... using multipli^n by 'k' rule

$= -z\left[\frac{(z-5)-(1)z}{(z-5)^2}\right] = -z\left[\frac{\cancel{z}-5-\cancel{z}}{(z-5)^2}\right]$

$= \frac{5z}{(z-5)^2}$

② $z\{k\}$, $k \geq 0$

$\rightarrow$ $z\{1\}$ OR $z\{U(k)\}$ $k \geq 0$ --- Very Imp step.

$z\{U(k)\} = \frac{z}{z-1}$

Now using multiplication by k

$z\{k\} = -z\frac{d}{dz}\left(\frac{z}{z-1}\right)$

$= \frac{z}{(z-1)^2}$

5] Division by k

If $z\{f(k)\} = F(z)$ then $z\left\{\frac{f(k)}{k}\right\} = -\int_z^{*} z^{-1}F(z)\,dz$.

ex] $z\left\{\frac{\sin 3k}{k}\right\}$, $k > 0$.

$\rightarrow$ $z\{\sin 3k\} = \frac{z\sin 3}{z^2 - 2z\cos 3 + 1}$

$z\left\{\frac{\sin 3k}{k}\right\} = \int_z^{\infty} z^{-1}\left(\frac{z\sin 3}{z^2-2z\cos 3+1}\right)dz = \sin 3\int_z^{\infty}\frac{1}{z^2-2z\cos 3+1}\,dz$

$= \sin 3\int_z^{\infty}\frac{dz}{(z-\cos 3)^2+\sin^2 3} = \cancel{\sin 3}\left[\frac{1}{\cancel{\sin 3}}\left(\tan^{-1}\left(\frac{z-\cos 3}{\sin 3}\right)\right)\right]_z^{\infty}$

$= \tan^{-1}\infty - \tan^{-1}\left(\frac{z-\cos 3}{\sin 3}\right) = \frac{\pi}{2} - \tan^{-1}\left(\frac{z-\cos 3}{\sin 3}\right)$

$= \cot^{-1}\left(\frac{z-\cos 3}{\sin 3}\right)$

6] Shifting Property

(a) If $Z\{f(k)\} = F(z)$ then

$$Z\{f(k+n)\} = z^n F(z) \quad \& \quad Z\{f(k-n)\} = z^{-n} F(z)$$

(b) For one sided z-transform. $(k \geq 0)$

$$Z\{f(k+n)\} = z^n F(z) - \sum_{r=0}^{n-1} f(r) z^{n-r} \quad , k \geq 0.$$

$$Z\{f(k-n)\} = z^{-n} F(z) + \sum_{r=-n}^{-1} f(r) z^{-(n+r)} \quad , k < 0$$

(c) For Causal sequence

$$Z\{f(k+n)\} = z^n F(z) - \sum_{r=0}^{n-1} f(r) z^{n-r}$$

$$Z\{f(k-n)\} = z^{-n} F(z)$$

7] Convolution

$$Z\{\{f(k)\} * \{g(k)\}\} = F(z) \cdot G(z)$$

where

$$\{f(k)\} * \{g(k)\} = \sum_{m=-\infty}^{\infty} f(m)\, g(k-m)$$

Ex– Verify convolution theorem for $f_1(k) = k$, $f_2(k) = k$, $k \geq 0$

$\rightarrow$ $$Z\{k\} = Z\{k \cdot 1\} = -z \frac{d}{dz}\left(\frac{z}{z-1}\right)$$

$$= \frac{z}{(z-1)^2}$$

$\therefore$ RHS $\Rightarrow$ $$F(z)\, G(z) = \frac{z}{(z-1)^2} \frac{z}{(z-1)^2} = \frac{z^2}{(z-1)^4}$$

Now LHS $\Rightarrow$ $$\{f_1(k) * f_2(k)\} = \sum_{m=-\infty}^{\infty} f(m)\, g(k-m)$$

$$= \sum_{m=0}^{\infty} f_1(m)\, f_2(k-m) = \sum_{m=0}^{\infty} m(k-m)$$

$$= k \sum_{m=0}^{\infty} m - \sum_{m=0}^{\infty} m^2$$

Now we know

$$1+2+3+\cdots = \frac{n(n+1)}{2}$$

$$1^2+2^2+3^2+\cdots = \frac{n(n+1)(2n+1)}{6}$$

using these results we get,

$$= k\sum m - \sum m^2$$

$$= k\left(\frac{k(k+1)}{2}\right) - \frac{k(k+1)(2k+1)}{6}$$

$$= \frac{k(k+1)}{6}(3k-2k-1) = \frac{k}{6}(k+1)(k-1) = \frac{k}{6}(k^2-1)$$

Now apply z-transform.

$$Z\{f_1(k) * f_2(k)\} = Z\left\{\frac{k}{6}(k^2-1)\right\}$$

$$= \frac{1}{6}Z\{\cancel{z}k^3 - k\} = \frac{1}{6}\left[Z\{k^3\} - Z\{k\}\right]$$

$$= \frac{1}{6}\left[\left(-z\frac{d}{dz}\right)^3\left(\frac{z}{z-1}\right) - \left(-z\frac{d}{dz}\right)\left(\frac{z}{z-1}\right)\right]$$

$$= \frac{1}{6}\left[\frac{z(z^2+4z+1)}{(z-1)^4} - \frac{z}{(z-1)^2}\right]$$

$$= z\left[\frac{\cancel{z^2}+4z+\cancel{1}-\cancel{z^2}+2z-\cancel{1}}{6(z-1)^4}\right] = \frac{z}{6}\left[\frac{\cancel{6}z}{(z-1)^4}\right]$$

$$= \frac{z^2}{(z-1)^4} = RHS.$$

$\therefore$ LHS = RHS

Hence convolution theorem is verified

Ex:- $Z[\cos(5k+7)]$, $k \geq 0$.

$\rightarrow$ $\cos(5k+7) = \cos 5k \cos 7 - \sin 5k \sin 7$

$$Z[\cos(5k+7)] = Z[\cos 5k \cos 7] - Z[\sin 5k \sin 7]$$

$$= \cos 7 \, Z[\cos 5k] - \sin 7 \, Z[\sin 5k]$$

$$= \cos 7\left(\frac{z(z-\cos 5)}{z^2 - 2z\cos 5 + 1}\right) - \sin 7\left(\frac{z\sin 5}{z^2 - 2z\cos 5 + 1}\right)$$

$$= \frac{\cos 7(z^2 - z\cos 5) - z\sin 5 \sin 7}{z^2 - 2z\cos 5 + 1}$$

Questions (HW)

I Find z-transform of the following.

1. $f(k) = 3^k, k \geq 0$
2. $f(k) = 2, k \geq 0$.
3. $f(k) = \left(\frac{1}{3}\right)^k, k \geq 0$
4. $\left(\frac{1}{2}\right)^{|k|}$, ~~k≥~~ for all k
5. $f(k) = 2^k + \left(\frac{1}{2}\right)^k, k \geq 0$
6. $f(k) = \frac{3^k}{k!}, k \geq 0$.
7. $f(k) = \cos\left(\frac{k\pi}{8} + \alpha\right), k \geq 0$
8. $f(k) = \sinh\left(\frac{k\pi}{2}\right), k \geq 0$.
9. $f(k) = 2^k \cos(3k+2)$
10. $f(k) = e^{-3k} \sin 4k, k \geq 0$
11. $f(k) = k e^{-5k}, k \geq 0$
12. $f(k) = k^2, k \geq 0$.

Ex:- Find z-transform of $f(k) = k\, a^{k-1}\, U(k-1)$, $k \geq 0$.

$\rightarrow$ $Z\{U(k)\} = \frac{z}{z-1}$

$$Z\{a^k U(k)\} = \frac{(z/a)}{(z/a)-1} = \frac{z}{z-a}$$ ... Using change of scale

Now using shifting property (since it is Causal sequence)

$$Z\{a^{k-1} U(k-1)\} = z^{-1}\frac{z}{z-a} = \frac{1}{z-a}$$

Now $$Z\{k\, a^{k-1} U(k-1)\} = -z\frac{d}{dz}\left(\frac{1}{z-a}\right) = -z\left(-\frac{1}{(z-a)^2}\right) = \frac{z}{(z-a)^2}$$

Ans