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CHAPTER

4

RELATIONS AND FUNCTIONS

SYLLABUS

Relations : Definition of binary relations, Equivalence relations and partitions, Partial ordering relations and lattices, Chains and anti chains.

Functions : Definitions, Domain, Range, One-to-one and on-to, Inverse and composition, Pigeonhole Infinite sets and countability, Principle, Discrete numeric functions and generating functions, Job scheduling problem.

Recurrence relations : Recurrence relation, Linear recurrence relations with constant coefficients, Homogeneous solutions, Total solutions, solutions by the method of generating functions.

OBJECTIVES :

- To study relations among objects of sets.
- To study function as input-output relation.

UTILITY :

- In preparing relational database.
- Recursive and generating functions are important tools in software development.

KEY CONCEPTS :

- Binary relation
- Equivalence relation
- Partial order
- Partition
- Bijective
- Pigeon hole principle
- Generating functions
- Recurrence relations.

4.0 INTRODUCTION

This chapter is concerned with two themes, relations and functions. In the preceding chapter we dealt with sets, elements and general properties of sets. Now we progress further and study the various relationships that may exist between elements of a set. We study various properties of a relation, including its matrix and graphical representations.

The concept of relation is of primary importance in computer science, especially in the study of data structure such as linked list, array, relational models etc. Relations are also important in the analysis of algorithms, information system etc.

Function is a special type of relation. It is basically an input–output relation. Many concepts in computer science can be conveniently stated in the language of functions. Recursive and generating functions are of especial importance in software development.

In this chapter, we discuss infinite sets and their cardinalities, where the concept of bijective function is used. We also briefly discuss an important principle called as the Pigeon Hole Principle, and use it to solve some problems related to counting.

4.1 RELATIONS

A common notion of relation is a type of association that exists between two or more objects. Consider the following examples :

(i) x is the father of y.

(ii) x was born in the city y in the year z.

(iii) The number x is greater than the number y.

(iv) Prof x teaches the subject y to the class z in classroom u.

In general, one can have relation among n objects, (where n is a positive integer). In describing a relation, it is necessary not only to specify the objects, but also the order in which they appear. In the example " x is the father of y ", the respective positions of x and y matter; x should precede y, and not vice-versa. Hence in the following definition, we introduce the concept which gives the necessary ordering of the objects.

4.1.1 Definition

An **ordered n-tuple,** for $n > 0$, is a sequence of objects or elements, denoted by $(a_1, a_2, ..., a_n)$.

If $n = 2$, the ordered n-tuple is called an **ordered pair.**

If $n = 3$, the ordered n-tuple is called an ordered triple ; and so on.

As pre-requisite to study relations, we consider sets whose elements are ordered n-tuples and study their properties.

4.2 PRODUCT SETS

Definition : Let A and B be non-empty sets. We define the **product set** or the **cartesian product** $A \times B$ is defined as

$$A \times B = \left\{ (a, b) \,\middle|\, a \in A \text{ and } b \in B \right\}.$$

If $A = \phi$ or $B = \phi$, then $A \times B = \phi$.

Examples

(i) Let $A = \{a, b, c\}$, $B = \{1, 2\}$.

Then $A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$

(ii) Let A = set of students = {Shilpa, Ramesh, Aparna}

and B = set of marks in DSGT = {65, 56, 72},

then A × B = {(Shilpa, 65), (Shilpa, 56), (Shilpa, 72), (Ramesh, 65), (Ramesh, 56), (Ramesh, 72), (Aparna, 65), (Aparna, 56), (Aparna, 72)}.

(iii) If **R** denotes the set of all real numbers, then $\mathbf{R} \times \mathbf{R}$ denotes the set of all points in the co-ordinate plane.

(iv) We know that a complex number $x + iy$ can be considered as an ordered pair (x, y). Hence if C denotes the set of all complex number then C is the cartesian product $\mathbf{R} \times \mathbf{R}$.

From the above examples, it is clear that product of sets is non-commutative,

i.e. $A \times B \neq B \times A$.

The following theorem, establishes certain important properties of the product operation.

4.2.1 Theorem

If A, B, C are sets, then

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(iii) $(A \cup B) \times C = (A \times C) \cup (B \times C)$

(iv) $(A \cap B) \times C = (A \times C) \cap (B \times C)$

Proof : (i) We shall show that every element (x, y) of $A \times (B \cup C)$ is an element of $(A \times B) \cup (A \times C)$ and vice-versa. $(x, y) \in A \times (B \cup C) \leftrightarrow x \in A$ and $y \in (B \cup C) \leftrightarrow x \in A$ and $(y \in B$ or $y \in C)$. Since 'and' distributes over 'or', this implies $(x \in A$ and $y \in B)$ or $(x \in A$ and $y \in C)$

$\leftrightarrow (x, y) \in (A \times B)$ or $(x, y) \in (A \times C)$

$\leftrightarrow x \in (A \times B) \cup (A \times C)$.

Thus (i) is proved.

(ii) $(x, y) \in A \times (B \cap C) \leftrightarrow x \in A$ and $y \in (B \cap C) \leftrightarrow x \in A$ and ($y \in B$ and $y \in C$)

$\leftrightarrow$ ($x \in A$ and $y \in B$) and ($x \in A$ and $y \in C$)

$\leftrightarrow (x, y) \in (A \times B)$ and $(x, y) \in (A \times C)$

$\leftrightarrow (x, y) \in (A \times B) \cap (A \times C)$

(iii) $(x, y) \in (A \cup B) \times C \leftrightarrow x \in (A \cup B)$ and $y \in C \leftrightarrow$ ($x \in A$ or $x \in B$) and $y \in C \leftrightarrow$ ($x \in A$ and $y \in C$) or ($x \in B$ and $y \in C$) $\leftrightarrow (x, y) \in (A \times C)$ or $(x, y) \in (B \times C) \leftrightarrow (x, y) \in (A \times C) \cup (B \times C)$.

(iv) $(x, y) \in (A \cap B) \times C \leftrightarrow x \in (A \cap B)$ and $y \in C \leftrightarrow$ ($x \in A$ and $x \in B$) and $y \in C$

$\leftrightarrow$ ($x \in A$ and $y \in C$) and ($x \in B$ and $y \in C$)

$\leftrightarrow (x, y) \in (A \times C)$ and $(x, y) \in (B \times C)$

$\leftrightarrow (x, y) \in (A \times C) \cap (B \times C)$.

The next theorem gives an important result pertaining to the cardinality of the product set.

4.2.2 Theorem

If A and B are finite sets with cardinalities m, n respectively then $|A \times B| = m \cdot n$.

Proof : Since $|A| = m$,

$$\text{Let } A = \{a_1, a_2, \ldots, a_m\}.$$

$$\text{Similarly, } B = \{b_1, b_2, \ldots, b_n\}, \text{ as } |B| = n.$$

$$\text{Now } A \times B = \{(a_i, b_j) \mid 1 \le i \le m,\ 1 \le j \le n\}$$

Since for each element a_i in A, there exists a corresponding element $b_j \in B$ in the ordered pair (a_i, b_j), the set $A \times B$ consists of exactly $m \cdot n$ elements. Hence $|A \times B| = m \cdot n$.

Example :

$$\text{If } A = \{n \in N \mid 1 \le n \le 100\}$$

$$\text{and } B = \{n \in N \mid 1 \le n \le 50\}$$

$$\text{then } |A \times B| = 100 \times 50 = 5000$$

The definition of product of two sets is generalised for a finite collection of sets $\{A_1, A_2, \ldots, A_n\}$ by defining $A_1 \times A_2 \times \ldots\ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i, 1 \le i \le n\}$.

If all $A_i = A$, then $A_1 \times A_2 \times \ldots \times A_n = A^n$.

If A, B, C are non-empty sets ;

$$\text{then } A \times B \times C = \{(a, b, c) \mid a \in A, b \in B, c \in C\}$$

This set is clearly different from

$$A \times (B \times C) = \{(a, (b, c)) \mid a \in A, (b, c) \in B \times C\}$$

and also different from the set

$$(A \times B) \times C = \{((a, b), c) \mid (a, b) \in A \times B, c \in C\}$$

Distinguishing the three types is quite a problem sometimes, though normally by product of sets A, B, C, we mean $A \times B \times C$.

SOLVED EXAMPLES

1. If $A = \{1\}$, $B = \{a, b\}$, $C = \{2, 3\}$, find $A \times B \times C$, A^2, $B^2 \times A$, C^3.

Solution :
$$A \times B \times C = \{(1, a, 2), (1, b, 2), (1, a, 3), (1, b, 3)\}$$
$$A^2 = \{(1, 1)\}$$
$$B^2 = \{(a, a), (a, b), (b, a), (b, b)\}$$
$$B^2 \times A = \{((a, a), 1), ((a, b), 1), ((b, a), 1), ((b, b), 1)\}$$
$$C^3 = C \times C \times C = \{(2, 2, 2), (2, 2, 3), (2, 3, 3), (2, 3, 2), (3, 2, 2), (3, 3, 2), (3, 2, 3), (3, 3, 3)\}$$

2. If $A \subseteq C$ and $B \subseteq D$, prove that $A \times B \subseteq C \times D$.

Solution : Let $(a, b) \in A \times B$.

This implies $a \in A$ and $b \in B$. Since $A \subseteq C$ and $B \subseteq D$, $a \in C$ and $b \in D$ so that $(a, b) \in C \times D$. Hence $A \times B \subseteq C \times D$.

3. Show that $A \times B = B \times A \leftrightarrow A = \phi$ or $B = \phi$ or $A = B$.

Solution : Let $A \times B = B \times A$. Then $(a, b) \in A \times B \leftrightarrow (a, b) \in B \times A \leftrightarrow a \in A$ iff $a \in B$ and $b \in B$ iff $b \in A \leftrightarrow A = B$ iff $A \times B \neq \phi$. If $A \times B = \phi$, then $A = \phi$ or $B = \phi$. Hence the result.

4. If $A = \{1, 2\}$, $B = \{1, 2, 3\}$, find $(A \times B) \cap (B \times A)$

Solution :
$$A \times B = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\}$$
$$B \times A = [(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 2)\}.$$
Hence $(A \times B) \cap (B \times A) = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$.

EXERCISES – I

1. Let $A = \{a, b\}$ and $B = \{4, 5, 6\}$. List the elements in
   (a) $A \times B$, (b) $B \times A$, (c) $A \times A$, (d) $B \times B$, (e) $(A \times B) \times A$, (f) $A \times A \times B$, (g) $A \times (B \times A)$.
2. If $A = \{a, b, c\}$, $B = \{1, 2\}$ and $C = \{\#, *\}$ list all the elements in $A \times B \times C$.
3. If $A = \{1, 2\}$, construct the set $P(A) \times A$.
4. If $A \times B \subseteq C \times D$, does it necessarily follow that $A \subseteq C$ and $B \subseteq D$?
5. Is it possible $A \subseteq A \times A$, for some set A ?
6. Prove or disprove
   (i) $(A \cup B) \times (C \cup D) = (A \times C) \cup (B \times D)$
   (ii) $(A - B) \times (C - D) = (A \times C) - (B \times D)$
   (iii) $(A \oplus B) \times (C \oplus D) = (A \times C) \oplus (B \times D)$
   (iv) $(A \oplus B) \times C = (A \times C) \oplus (B \times C)$
7. Prove that $A \times B = B \times A$ iff $A = B$.
8. Consider the ordered triple $\{1, 3, 5\}$.
   (a) Represent the ordered triple as an ordered pair.
   (b) Represent the ordered pair in (a) as a set.

9. Let $A = \{ a \mid 1 \le a \le 2 \}$ and

$B = \{ b \mid 0 \le b \le 1 \}$.

Find

(a) $A \times B$
(b) $B \times A$
(c) Represent $A \times B$ and $B \times A$ graphically.

10. Is $(A \times B) \times C = A \times (B \times C)$? Justify.

4.3 BASIC CONCEPTS OF RELATION

4.3.1 Definition

Let, $\{A_1, A_2, \ldots, A_n\}$ be a finite collection of sets. A subset R of $A_1 \times A_2 \times \ldots \times A_n$ is called an **n-ary relation** on $A_1, A_2, \ldots, A_n$.

If $R = \phi$, then R is called **void** or empty relation.

If $R = A_1 \times A_2 \times \ldots \times A_n$, then R is called the **universal** relation.

If $A_i = A$ for all i, then R is called an n-ary relation on A.

If n = 1, 2 or 3, then R is called a **unary, binary** or **ternary** relation respectively.

Examples

(i) Let **Z** be the set of all integers. Then the property " x is an even integer ", can be characterised as a relation which is unary. In this case, the relation

$$R = \{ x \in Z \mid x \text{ is even} \}.$$

(ii) Let $A = \{1, 2, 5, 6\}$ and let R be the relation characterised by the property " x is less than y ". Then $R = \{(1, 2), (1,5), (1, 6), (2, 5), (2, 6), (5, 6)\}$.

where R is binary.

(iii) Let $A = \{1, 2, 3\}$ and let R be the relation characterised by the property " x + y is less than z ". Then $R = \{1, 1, 3\}$, which is a ternary.

(iv) Let $A = \{2, 3, 4\}$ and let R be the relation characterised by the property " x + y is divisible by z ". Then $R = \{(2, 2, 2), (2, 2, 4), (2, 4, 2), (2, 4, 3), (3, 3, 2), (3, 3, 3), (4, 2, 2), (4, 2, 3), (4, 4, 2), (4, 4, 4)\}$ where R is a ternary relation.

Among the relations, binary relations are the most important, being widely used in various applications. Hence in what follows, we will discuss binary relations and their properties in detail.

4.3.2 Binary Relation

Let A and B be non-empty sets. Then a binary relation R from A to B is a subset of $A \times B$, i.e. $R \subseteq A \times B$. The **domain** of R, denoted by D(R), is the set of elements in A that are related to some element in B, i.e.

$$D(R) = \{ a \in A \mid \text{for some } b \in B, (a, b) \in R \}$$

The range of R denoted by Rn (R) is the set of elements in B, that are related to some element in A, i.e.

$$\text{Rn}(R) = \{ b \in B \mid \text{for some } a \in A, (a, b) \in R \}$$

Clearly $D(R) \subseteq A$ and $\text{Rn}(R) \subseteq B$.

Example : Let A = {2, 3, 4, 5} and let R be the relation on A defined as aRb iff a < b. Find D(R) and Rn(R).

Solution : $R = \{(2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}$

Then $D(R) = \{2, 3, 4\}$ and $\text{Rn}(R) = \{3, 4, 5\}$

As R is basically a set, all the rules of set operations are applicable to R. Hence if A, B are sets with binary relations R and S, then

$$R \cup S = \{ (a, b) \mid (a, b) \in R \vee (a, b) \in S \}$$

$$R \cap S = \{ (a, b) \mid (a, b) \in R \wedge (a, b) \in S \}$$

The set $A \times B$ is the **universal** relation and the empty set ϕ is the **void** relation.

4.3.3 Complement of a Relation

A relation as a set has its complement, which is defined below.

The complement of a relation R, denoted by $\bar{R}$ is as the set

$$\bar{R} = \{ (a, b) \mid (a, b) \notin R \}, \text{ i.e.}$$

$$a \bar{R} b \text{ iff } a \not{R} b.$$

Example :

1. Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c\}$.

Let $R = \{(1, a), (1, b), (2, c), (3, a), (4, b)\}$

and $S = \{(1, b), (1, c), (2, a), (3, b), (4, b)\}$

Find (i) $\bar{R}$ and $\bar{S}$ (ii) Verify De Morgan's laws for R and S.

Solution :

(i) $A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c), (4, a), (4, b), (4, c)\}$.

$\therefore$ $\bar{R} = \{(1; c), (2, a), (2, b), (3, b), (3, c), (4, a), (4, c)\}$

$\bar{S} = \{(1, a), (2, b), (2, c), (3, a), (3, c), (4, a), (4, c)\}$.

(ii) De Morgan's laws states that

$$\overline{R \cup S} = \bar{R} \cap \bar{S} \text{ and } \overline{R \cap S} = \bar{R} \cup \bar{S}$$

$R \cup S = \{(1, a), (1, b), (1, c), (2, a), (2, c), (3, a), (3, b), (4, b)\}$

$$\overline{R \cup S} = \{(2, b), (3, c), (4, a), (4, c)\}$$

$$\bar{R} \cap \bar{S} = \{(2, b), (3, c), (4, a), (4, c)\}$$

Hence $\overline{R \cup S} = \bar{R} \cap \bar{S}$

$$R \cap S = \{(1, b), (4, b)\}$$

$$\therefore \quad \overline{R \cap S} = \{(1, a), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c), (4, a), (4, c)\}$$

$$\bar{R} \cup \bar{S} = \{(1, a), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c), (4, a), (4, c)\}$$

Hence $\overline{R \cap S} = \bar{R} \cup \bar{S}$

Thus De Morgan's laws are verified for relations R and S.

4.3.4 Converse of a relation

Given a relation from A to B, one may define a relation from B to A as follows.

Let R be a relation from A to B. Then the **converse** of R, denoted by R^c is the relation from to A, defined as

$$R^c = \left\{ (b, a) \mid (a, b) \in R \right\}$$

Clearly $R^c \subseteq B \times A$

For example, if A = N the set of natural numbers and R is relation <, then R^c is the realtion >. The converse relation is also called as the **inverse** relation and is denoted by R^{-1}.

The following theorem gives important properties of the converse relation.

4.3.5 Theorem

Let, R, S be relations from A to B. Then

(i) $(R^c)^c = R$

(ii) $(R \cup S)^c = R^c \cup S^c$

(iii) $(R \cap S)^c = R^c \cap S^c$

Proof : (i) is immediate, by the definition of the converse.

(ii)
$$\begin{aligned}(R \cup S)^c &= \left\{ (b, a) \mid (a, b) \in R \text{ or } (a, b) \in S \right\} \\ &= \left\{ (b, a) \mid (b, a) \in R^c \vee (b, a) \in S^c \right\} \\ &= R^c \cup S^c\end{aligned}$$

(iii)
$$\begin{aligned}(R \cap S)^c &= \left\{ (b, a) \mid (a, b) \in R \wedge (a, b) \in S \right\} \\ &= \left\{ (b, a) \mid (b, a) \in R^c \wedge (b, a) \in S^c \right\} \\ &= R^c \cap S^c\end{aligned}$$

Examples

1. Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c\}$

Let $R = \{(1, a), (3, a), (3, c)\}$.

Find (i) R^c, (ii) $D(R^c)$, (ii) $Rn(R^c)$.

Solution : (i) $R^c = \{(a, 1), (a, 3), (c, 3)\}$

(ii) $D(R^c) = \{a, c\} = Rn(R)$

(iii) $Rn(R^c) = \{1, 3\} = D(R)$

2. Let $A = \{2, 3, 4, 6\}$. Let R and S be relations on A such that

$$R = \left\{ (a, b) \;\middle|\; a = b + 1 \text{ or } b = 2a \right\}$$

and $S = \left\{ (a, b) \;\middle|\; a \text{ divides } b \right\}$, find $(R \cap S)^c$.

Solution : $R = \{(3, 2), (4, 3), (2, 4), (3, 6)\}$

and $S = \{(2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (6, 6)\}$

$\therefore$ $R \cap S = \{(2, 4), (3, 6)\}$

$\therefore$ $(R \cap S)^c = \{(4, 2), (6, 3)\}$.

4.3.6 Composition of Binary Relations

We shall now discuss relations that are formed from an existing **sequence** of relations. These are called as composite relations. Real life abounds with such relations. Consider for example, the relationship of grand father who is father's (or mother's) father.

The concept of composite relations plays an important role in the execution of programs, where a sequence of data conversions takes place, from decimal to binary and from binary to floating point.

Let us now formally define composite relation.

Definition : Let R_1 be a relation from A to B and R_2 a relation from B to C. The **composite relation** from A to C, denoted by $R_1 \bullet R_2$ (or $R_1 R_2$) is defined as

$$R_1 \bullet R_2 = \left\{ (a, c) \;\middle|\; a \in A \wedge c \in C \wedge \exists\, b\, [b \in B \wedge (a, b) \in R_1 \wedge (b, c) \in R_2] \right\}$$

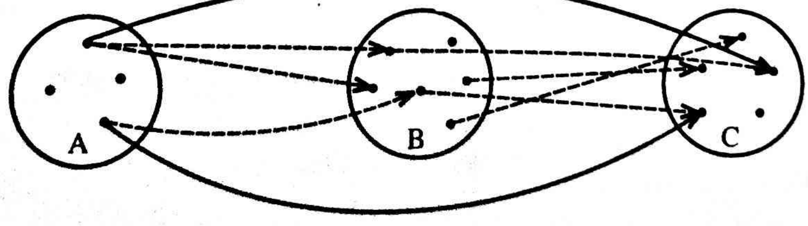


Fig. 4.1

Note that if R_1 is a relation from A to B and R_2 from C to D, $R_1 \bullet R_2$ is not defined unless $B = C$. In general if $\{A_1, A_2, \ldots, A_{n+1}\}$ is a finite collection of sets where R_1 is a relation from A_1

to A_2, R_2 from A_2 to A_3, ... R_n from A_n to A_{n+1}, then $R_1 \bullet R_2 \bullet \ldots \bullet R_n$ is a relation from A_1 to A_{n+1}. We shall denote this relation simply as $R_1\ R_2\ R_3 \ldots R_n$.

In particular if $A_1 = A_2 = \ldots = A_{n+1} = A$ and $R_1 = R_2 = \ldots = R_n = R$, then we denote $R_1\ R_2 \ldots R_n$ by R^n which is a relation on A. Hence given R, one can compute R^2, R^3 and so on.

The operation of composition is clearly not commutative; i.e. $R_1\ R_2 \neq R_2\ R_1$. In fact $R_2\ R_1$ may not be defined, even though $R_1\ R_2$ is. However, the operation is associative, as established in the following theorem.

4.3.7 Theorem

Let R_1, R_2 and R_3 be relations from A to B, B to C and C to D. Then $(R_1\ R_2)\ R_3 = R_1\ (R_2\ R_3)$.

Proof : Since we have to prove essentially the equality of two sets, we shall show that

$$(R_1\ R_2)\ R_3 \subseteq R_1\ (R_2\ R_3) \text{ and conversely.}$$

Let $(a, d) \in (R_1\ R_2)\ R_3$, where $a \in A$, $d \in D$. Note that $R_1\ R_2$ is a relation from A to C. Then this means that there exists an element $c \in C$ such that $(a, c) \in R_1\ R_2$ and $(c, d) \in R_3$. Now $(a, c) \in R_1\ R_2$ implies there exists an element $b \in B$ such that $(a, b) \in R_1$ and $(b, c) \in R_2$. Since $(b, c) \in R_2$ and $(c, d) \in R_3$, if follows that $(b, d) \in R_2\ R_3$. Again since $(a, b) \in R_1$ and $(b, d) \in R_2\ R_3$, $(a, d) \in R_1\ (R_2\ R_3)$.

Similarly, we can prove $R_1\ (R_2\ R_3) \subseteq (R_1\ R_2)\ R_3$. This proves the associativity property of relations. The next result deals with the converse of the composition.

4.3.8 Theorem

Let R_1 be a relation from A to B and R_2 from B to C. Then $(R_1\ R_2)^c = R_2^c\ R_1^c$.

Proof: R_1^c is a relation from B to A and R_2^c from C to B. Hence anyway both $(R_1\ R_2)^c$ and $R_2^c R_1^c$ are relations from C to A. We shall now prove that these relations are equal. Let $(c, a) \in (R_1\ R_2)^c$. This implies that $(a, c) \in R_1\ R_2$. Hence there exists an element $b \in B$ such that $(a, b) \in R_1$ and $(b, c) \in R_2$. It follows that $(b, a) \in R_1^c$ and $(c, b) \in R_2^c$, so that $(c, a) \in R_2^c\ R_1^c$. Hence $(R_1\ R_2)^c \subseteq R_2^c\ R_1^c$. Similarly, we can prove $R_2^c\ R_1^c \subseteq (R_1\ R_2)^c$. Hence the equality is proved.

Examples

1. Let $A = \{a, b, c, d\}$ where

$$R_1 = \{(a, a), (a, b), (b, d)\} \text{ and}$$

$$R_2 = \{(a, d), (b, c), (b, d), (c, b)\}$$

Find $R_1\ R_2$, $R_2\ R_1$, R_2^2, R_2^3.

Solution :

$$R_1\ R_2 = \{(a, d), (a, c)\}$$

$$R_2\ R_1 = \{(c, d)\}.$$

$$R_1^2 = \{(a, a), (a, b), (a, d)\}$$

$$R_2^2 = \{(b, b), (c, c), (c, d)\}$$

$$R_2^3 = R_2\ R_2^2 = \{(b, c), (c, b), (b, d)\}$$

2. Let $A = \{2, 3, 4, 5, 6\}$ and let R_1, R_2 be relations on A such that

$$R_1 = \left\{ (a, b) \mid a - b = 2 \right\} \text{ and}$$

$$R_2 = \left\{ (a, b) \mid a + 1 = b \text{ or } a = 2b \right\}.$$

Find the composite relations.

(i) $R_1 R_2$, (ii) $R_2 R_1$ (iii) $R_1 R_2 R_1$ (iv) R_1^2 (v) $R_1 R_2^2$.

Solution:

$$R_1 = \{(4, 2), (5, 3), (6, 4)\}$$

$$R_2 = \{(2, 3), (3, 4), (4, 5), (5, 6), (4, 2), (6, 3)\}$$

(i) $R_1 R_2 = \{(4, 3), (5, 4), (6, 2), (6, 5)\}$

(ii) $R_2 R_1 = \{(3, 2), (5, 4), (4, 3)\}$

(iii) $R_1 R_2 R_1 = R_1 (R_2 R_1)$

$$= \{(5, 2), (6, 3)\}$$

3. Let $A = \{1, 2, 3, 4\}$,

Let R_1 be the relation on A defined as

$$R_1 = \left\{ (x, y) \mid x + y = 5 \right\} \text{ and } R_2 \text{ defined as}$$

$$R_2 = \left\{ (x, y) \mid y - x = 2 \right\}.$$

Verify that $(R_1 R_2)^c = R_2^c R_1^c$.

Solution:

$$R_1 = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$$

$$R_2 = \{(1, 3), (2, 4)\}$$

$$R_1 R_2 = \{(3, 4), (4, 3)\}$$

$\therefore$ $(R_1 R_2)^c = \{(4, 3), (3, 4)\}$

$$R_1^c = \{(4, 1), (3, 2), (2, 3), (1, 4)\}$$

$$R_2^c = \{(3, 1), (4, 2)\}$$

$$R_2^c R_1^c = \{(3, 4), (4, 3)\} = (R_1 R_2)^c$$

4.4 MATRIX REPRESENTATION OF A RELATION

Let $A = \{a_1, a_2, \dots, a_n\}$ and

$B = \{b_1, b_2, \dots, b_n)$ be finite sets containing respectively m and n elements.

Let R be a relation from A to B. By definition $R \subseteq A \times B$. Hence we can represent R by a $m \times n$ matrix $M_R = [m_{ij}]$, which is defined as follows :

$$m_{ij} = 1 \quad \text{if } (a_i, b_j) \in R$$
$$= 0 \quad \text{if } (a_i, b_j) \notin R.$$

The matrix M_R is called as the matrix of R.

The matrix representation of R is useful in verifying certain properties of R.

Examples

1. Let $A = \{a, b, c, d\}$ and $B = \{1, 2, 3\}$.

   Let $R = \{(a, 1), (a, 2), (b, 1), (c, 2), (d, 1)\}$

   Find the relation matrix.

Solution : M_R will have 4 rows and 3 columns.

$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

2. Let $A = \{1, 2, 3, 4, 8\}, B = \{1, 4, 6, 9\}$.

   Let $a R b$ iff $a \mid b$ (a divides b).

   Find the relation matrix.

Solution : $R = \{(1, 1), (1, 4), (1, 6), (1, 9), (2, 4), (2, 6), (3, 6), (3, 9), (4, 4)\}$

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 8 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 4 & 6 & 9 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

3. Let $A = \{1, 2, 3, 4, 8\} = B$; $a R b$ iff $a + b \leq 9$. Find its relation matrix.

Solution : $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 8), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4), (8, 1)\}$

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4. Let $A = \{a, b, c, d\}$ and let

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

   Find R.

Solution : $R = \{(a, a), (a, b), (b, c), (b, d), (c, c), (c, d), (d, a), (d, c)\}$

4.4.1 Relation Matrix Operations

A relation matrix has entries which are either one or zero. Such a matrix is called a Boolean matrix. Let us see how to add or multiply two such matrices.

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ Boolean matrix. We define

$$A + B = [c_{ij}] \text{ where}$$

$$c_{ij} = 1 \quad \text{if } a_{ij} = 1 \text{ or } b_{ij} = 1$$

$$= 0 \quad \text{if } a_{ij} \text{ and } b_{ij} \text{ are both zero.}$$

Similarly, if $A = [a_{ij}]$ is an $m \times n$ Boolean matrix and $B = [b_{ij}]$ is an $n \times r$ Boolean matrix, then $A \cdot B = [d_{ij}]$ an $m \times r$ matrix

where
$$d_{ij} = 1 \quad \text{if } a_{ij} = b_{ij} = 1$$
$$= 0 \quad \text{if } a_{ij} = 0 \text{ or } b_{ij} = 0$$

Example : Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ then

$$A + B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } A \cdot B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

4.4.2 Properties of Relation Matrix

Let R_1 be a relation from A to B, R_2 from B to C. Then the relation matrices satisfy the following properties :

1. $M_{R_1 \cdot R_2} = M_{R_1} \cdot M_{R_2}$
2. $M_{R^c} = \text{transpose of } M_R$ (for $R = R_1$ or $R = R_2$)
3. $M_{(R_1 \cdot R_2)^c} = M_{R_2{}^c R_1{}^c} = M_{R_2{}^c} \cdot M_{R_1{}^c}$

The proofs are left as exercises.

Examples :

1. Let $A = \{1, 2, 3, 4\}$, and let

$$R_1 = \{(1, 1), (1, 2), (2, 3), (2, 4), (3, 4), (4, 1), (4, 2)\}$$

and $R_2 = \{(3, 1), (4, 4), (2, 3), (2, 4), (1, 1), (1,4)\}$

Verify (i) $M_{R_1 R_2} = M_{R_1} \; M_{R_2}$

(ii) $M_{R_1{}^c} = \text{transpose of } M_{R_1}$

(iii) $M_{(R_1 R_2)^c} = M_{R_2{}^c} \cdot M_{R_1{}^c}$

Solution : (i)
$$M_{R_1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \quad M_{R_2} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_1 R_2 = \{(1, 1), (1, 4), (1, 3), (2, 1), (2, 4), (3, 4), (4, 4), (4, 1), (4, 3)\}$$

$$\therefore \quad M_{R_1 R_2} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

$$M_{R_1} \cdot M_{R_2} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

(ii)
$$R_1^c = \{(1, 1), (2, 1), (3, 2), (4, 2), (4, 3), (1, 4), (2, 4)\}$$

$$M_{R_1^c} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \text{transpose of } M_{R_1}$$

(iii)
$$(R_1 R_2)^c = \{(1, 1), (4, 1), (3, 1), (1, 2), (4, 2), (4, 3), (4, 4), (1, 4), (3, 4)\}$$

$$\therefore \quad M_{(R_1 R_2)^c} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Now
$$M_{R_2^c} \cdot M_{R_1^c} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = M_{R_2^c} \cdot M_{R_1^c}$$

4.5 GRAPHICAL REPRESENTATION OF A RELATION

If A is a finite set and R is a relation on A, it is possible to represent R pictorially by means of a graph. The elements of A are represented by points or circles, called as **nodes** or **vertices**. If aRb, this is indicated by drawing an arc from a to b with an arrow head pointing in the direction a → b. If aRa, this is shown by drawing a loop around a. These arcs (or loops) are called as **edges** of the graph. The resulting graph is called a **directed graph** or **digraph** of R.

The various types are illustrated in the following figures.

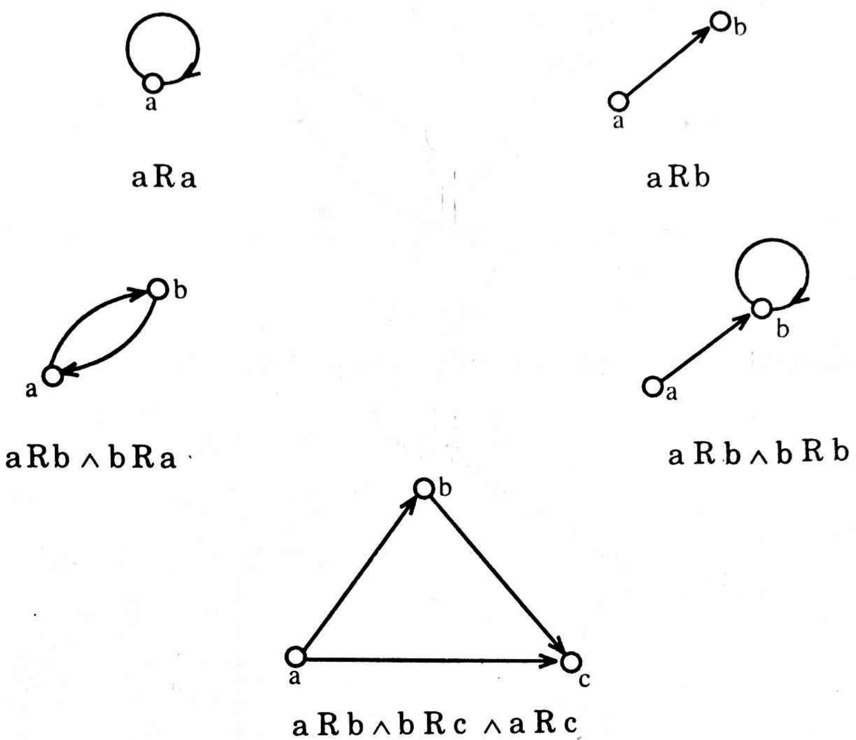


Fig. 4.2

Examples :

1. Let A = {2, 3, 4, 5} and let

R = {(2, 3), (3, 2), (3, 4), (3, 5), (4, 3), (4, 4), (4, 5)}

Draw its digraph.

Solution :

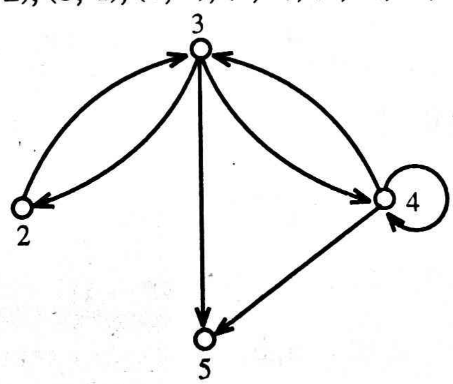


Fig. 4.3

2. Let A = {a, b, c, d} and

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Draw the digraph of R.

Solution : R = {(a, a), (a, b), (a, d), (b, b), (b, c), (c, c), (c, d), (d, a)}

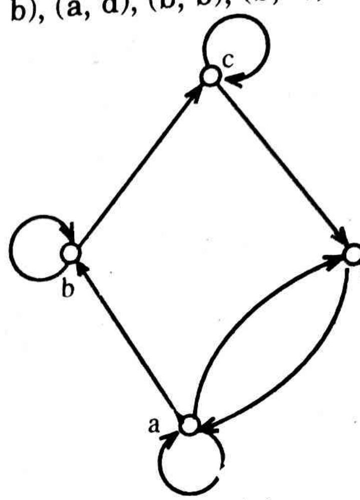


Fig. 4.4

3. Find the relation determined by the digraph and give its matrix.

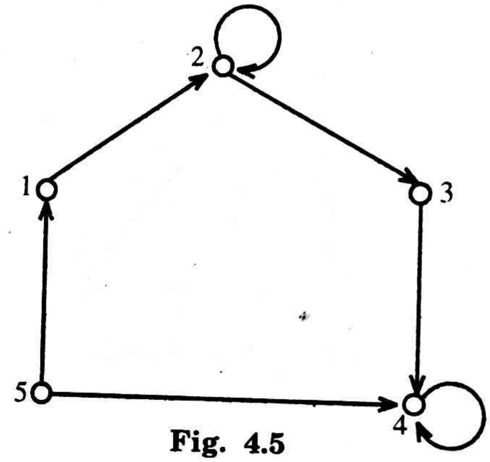


Fig. 4.5

Solution : A = {1, 2, 3, 4, 5} R = {(1, 2), (2, 2), (2, 3), (3, 4), (4, 4), (5, 1), (5, 4)}

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

EXERCISE – II

1. Let A be the product set {1, 2, 3} × {a, b}. How many relations are there on A ?
2. A = {1, 2, 3, 4}, B = {1, 4, 6, 8, 9}; aRb if and only if $b = a^2$. Find the domain, range of R. Find also its relation matrix and draw its digraph.
3. Let A = **R**, set of real numbers. Consider the following relation on A ; (a, b) ∈ R iff 2a + 3b = 6. Find domain of R and also its range.
4. Let A = {1, 2, 3, 4, 5} and let R = {(1, 1,), (1, 2), (2, 1), (1, 3), (1, 4), (4, 5), (5, 1), (1, 5), (4, 1)}. Draw the digraph of R.

5. For a set A = {1, 2, 3, 4, 5}, the relation matrix is

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Draw its digraph.

6. Let A = {1, 2, 3, 4}.

If $R = \{ (a, b) \mid (a - b) \text{ is an integral non-zero multiple of } 2 \}$

and $S = \{ (a, b) \mid (a - b) \text{ is an integral non-zero multiple of } 3 \}$

Find $R \cup S$ and $R \cap S$.

7. For a set A = {1, 2, 3, 4, 5} relations R_1 and R_2 are given by

$R_1 = \{(1, 2), (3, 4), (2, 2)\}$ and $R_2 = \{(4, 2), (2, 5), (3, 1), (1, 3)\}$

Find (a) $R_1 R_2$, (b) $R_2 R_1$, (c) $R_1 (R_2 R_1)$, (d) $(R_1 R) R_1$, (e) R_1^3, (f) R_2^2.

8. If A = B = {1, 2, 3}, $R_1 = \{(1, 1), (1, 2), (2, 3), 93, 1)\}$

and $R_2 = \{(2, 1), (3, 1), (3, 2), (3, 3)\}$

Compute

(a) Complement of R_1,

(b) Converse of R_2

(c) $R_1 \oplus R_2$.

9. Let A = B = {1, 2, 3, 4}, R = {(1, 1), (1, 3), (2, 3), (3, 1), (4, 2), (4, 4)} and

S = {(1, 2), (2, 3), (3, 1), (3, 2), (4, 3)}

Compute

(a) $M_{R \cap S}$ (b) $M_{R \cup S}$ (c) M_{R^c} (d) $M_{\overline{S}}$

10. Let A be set of workers and B be a set of jobs. Let R_1 be a binary relation from A to B such that (a, b) is in R_1 if worker a is assigned to job b. (We assume that a worker might be assigned to more than one job and more than one worker might be assigned to the same job.) Let R_2 be a binary relation on A such that (a_1, a_2) is in R_2 if a_1, a_2 can get along with each other if they are assigned to the same job. State a condition in terms of R_1, R_2 and (possibly) binary relations derived from R_1 and R_2 such that an assignment of the workers to the jobs according to R_1 will not put workers that cannot get along with one another on the same job.

4.6 SPECIAL PROPERTIES OF BINARY RELATIONS

In many applications to computer science, we deal with relations on a set A, rather then relations from A to B. These relations have certain properties which are useful in storing data, more efficiently, on the computer. Let R be a relation on a set A.

(a) **Reflexive relation :** R is reflexive if for **every** element a ∈ A, a R a i.e (a, a) ∈ R.

R is not a reflexive relation if for **some** element a ∈ A, a $\not R$ a, i.e. (a, a) ∉ R.

Examples :

(i) Let A = {a, b} and let R = {(a, a), (a, b), (b, b)}.

Then R is reflexive.

(ii) Let A = {1, 2} and let R = {(1, 1), (1, 2)}.

R is not reflexive since (2, 2) ∉ R.

(b) Irreflexive relation R is said to be **irreflexive** if for every element a ∈ A, a $\not R$ a, i.e. (a, a) ∉ R.

Examples

(i) Let A = {1, 2} and let R = {(1, 2), (2, 1)}. Then R is irreflexive since (1, 1), (2, 2) ∉ R.

(ii) Let A = {1, 2} and let R = {(1, 2), (2, 2). Then R is not irreflexive since (2, 2) ∈ R.

Note that R is not reflexive either; since (1, 1) ∉ R.

If R is reflexive, the corresponding relation matrix M_R will have its diagonal entries as one. If R is irreflexive, the diagonal elements will be zeros.

Digraph of a reflexive relation.

Fig. 4.6

(c) **Symmetric relation :** R is said to be **symmetric** if whenever a R b, then b R a. It then follows that, R is not symmetric if for some a and b ∈ A, a R b but b $\not R$ a.

The relation matrix corresponding to a symmetric relation is a symmetric matrix. If its ij-th entry is 1, its ji-th entry is also 1. If its ji-th entry is 0, its ij-th entry is also 0 (for i ≠ j).

Examples :

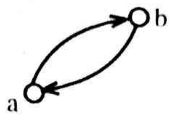
(i) Let A be set of people. Let a R b if a is a friend of b. Then obviously b is related to a. Hence the relation of being " friend " is a symmetric relation.

(ii) Let A be set of lines in a plane. For lines l_1, l_2 ∈ A, let l_1 R l_2 if l_1 is parallel to l_2. Then l_2 R l_1 since the relation of being " parallel to." is a symmetric relation.

(iii) Let A be set of people and let a R b if a is brother of b. Then this is not a symmetric relation since b can be the sister of a. This relation will be symmetric only if A is the set of males.

(iv) Let $A = \{1, 2\}$ and let $R = \{(1, 1), (2, 2)\}$. This is an example of a symmetric relation which is also reflexive.

Digraph of a symmetric relation



or

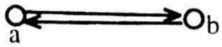


Fig. 4.7

(d) Asymmetric relation R is said to be **asymmetric** if whenever a R b, then $b \not R a$. Hence R is not asymmetric if for some a and $b \in A$, we have both a R b and b R a.

Examples

(i) Let $A = \mathbf{R}$ the set of real numbers and let R be the relation ' < '. Then $a < b \rightarrow b \not< a$. Hence < is asymmetric.

(ii) Let $A = \{2, 4, 5\}$ and let R be the relation " is a divisor of ".

Then $R = \{(2, 2), (2, 4), (4, 4), (5, 5)\}$.

R is not asymmetric since (2, 2) (also (4, 4), (5, 5)) $\in$ R.

(e) Antisymmetric relation R is antisymmetric if whenever a R b and b R a then a = b. It follows that R is not antisymmetric if we have elements $a, b \in A$ such that $a \neq b$ but both a R b and b R a.

An equivalent definition of antisymmetric relation R is : If $a \neq b$, then either $a \not R b$ or $b \not R a$.

This definition is sometimes useful to verify whether a given relation is antisymmetric.

Examples

(i) Let $A = \mathbf{R}$ and let R be the relation ' ≤ '. Then $a \leq b$ and $b \leq a \rightarrow a = b$. Hence ' ≤ ' is an antisymmetric relation.

(ii) Let $A = \{1, 2, 3\}$ and let $R = \{(1, 2), (2, 1), (2, 3)\}$.

R is not antisymmetric since (1, 2) and (2, 1) $\in$ R.

R is not symmetric either since (2, 3) $\in$ R but (3, 2) $\notin$ R.

R is also not asymmetric since both (1, 2) and (2, 1) $\in$ R.

(f) Transitive relation R is said to be **transitive** if whenever a R b and b R c, then a R c. It follows that a relation R is not transitive if there exist elements $a, b, c \in A$ such that a R b and b R c, but $a \not R c$. If such elements a, b, c do not exist, then R is transitive.

Examples

(i) Let A = **R** and let R be the relation '≤'. Then R is clearly transitive.

(ii) Let A = set of triangles and let R be the relation of being congruent. Then for triangles a, b, c ∈ A, a R b and b R c → a R c. Hence R is transitive.

(iii) Let A be set of people and let R be the relation of being " brother of ". Then a is brother of b and b is brother of c implies a is brother of c. Hence R is transitive.

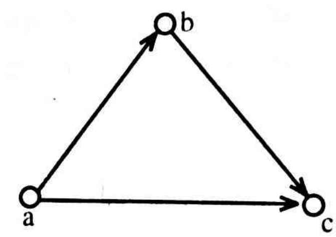
(iv) Let A = N the set of natural numbers, and let

$$R = \{ (a, b)\ a, b \in N \mid a + b \text{ is an odd number} \}$$

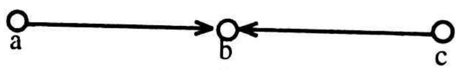
Then R is not transitive since (1, 2) and (2, 1) ∈ R, but (1, 1) ∉ R.

Digraphs of transitive relation.

(i)



(ii)



(a R b but b $\not R$ c)

Fig. 4.8

4.7 EQUIVALENCE RELATION

A binary relation R on a set A is called as equivalence relation if it is reflexive, symmetric and transitive.

The following are some of the common but important examples of equivalence relations.

Examples

(i) Let A = **R** and R be ' equality ' of numbers.

(ii) Consider all subsets of a universal set and R be the relation, " equality " of sets.

(iii) A is the set of triangles and R is ' similarity ' of triangles.

(iv) A is a set of students and R is the relation of being in "the same class or division."

(v) Let A be set of statement forms and R be the relation of " logical equivalence ".

(vi) A is set of lines in a plane and R is the relation of lines being "parallel."

The digraph of an equivalence relation will have the following characteristics. Every vertex will have a loop; if there is an arc from a to b, there should be an arc from b to a; if there is an arc from a to b and one from b to c, there should be an arc from a to c. In short, the following is a typical digraph of an equivalence relation.

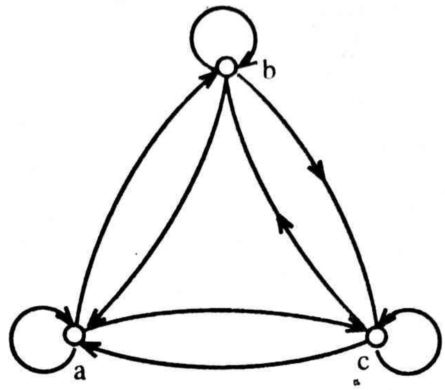


Fig. 4.9

Examples

1. Let A = {a, b, c, d}, R = {(a, a), (b, a), (b, b), (c, c), (d, d), (d, c).

   Determine whether R is an equivalence relation.

Solution : R is reflexive since (a, a), (b, b), (c, c) and (d, d) $\in$ R.

But R is not symmetric since (b, a) $\in$ R but (a, b) $\notin$ R. Hence R is not an equivalence relation.

2. Let A = {a, b, c} and let

$$M_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$ Determine whether R is an equivalence relation.

Solution : R = {(a, a), (b, b), (b, c), (c, b), (c, c)}.

R is reflexive since (a, a), (b, b), (c, c) $\in$ R.

R is symmetric since (b, c) $\in$ R $\rightarrow$ (c, b) $\in$ R,

R is transitive since

(b, b) and (b, c) $\in$ R implies (b, c) $\in$ R,

(b, c) and (c, b) $\in$ R implies (b, b) $\in$ R,

(c, c) and (c, b) $\in$ R implies (c, b) $\in$ R,

(c, b) and (b, b) $\in$ R implies (c, b) $\in$ R

(c, b) and (b, c) $\in$ R implies (c, c) $\in$ R

(b, c) and (c, c) $\in$ R implies (b, c) $\in$ R.

Hence R is an equivalence relation.

3. Determine whether the relation R whose digraph is given below is an equivalence relation.

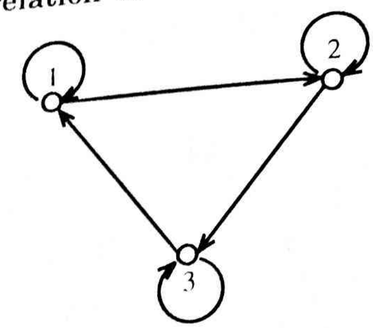


Fig. 4.10

Solution : R is reflexive since there is a loop around each vertex. But R is not symmetric, since $(1, 2) \in R$ but $(2, 1) \notin R$. Hence R is not an equivalence relation.

4. Determine whether the relation R whose digraph is given below is an equivalence relation.

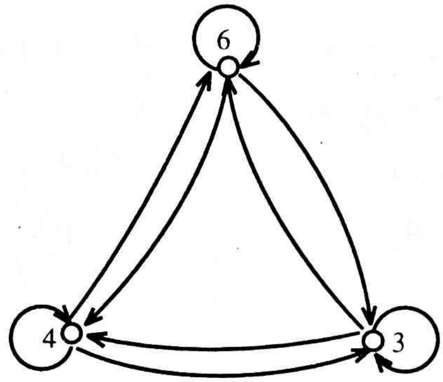


Fig. 4.11

Solution : R is reflexive since each vertex has a loop. R is also symmetric since every pair of distinct vertices is connected by a pair of double arcs, each pointing in the opposite direction.

In fact $R = \{(3, 3), (3, 4), (3, 6), (4, 3), (4, 4), (4, 6), (6, 3), (6, 4), (6, 6)\}$

Hence it is clear that R is also transitive. Therefore R is an equivalence relation.

4.7.1 Some important properties of equivalence relations

1. If R_1 and R_2 are equivalence relations on a set A, then $R_1 \cap R_2$ is an equivalence relations.

Proof : For each $a \in A$, $(a, a) \in R_1$ and $(a, a) \in R_2$; hence $(a, a) \in R_1 \cap R_2$. Therefore $R_1 \cap R_2$ is reflexive. Let $(a, b) \in R_1 \cap R_2$; then $(a, b) \in R_1$ and $(a, b) \in R_2$. Since R_1 and R_2 are both symmetric this implies $(b, a) \in R_1$ and $(b, a) \in R_2$, so that $(b, a) \in R_1 \cap R_2$. Hence $R_1 \cap R_2$ is symmetric. Let (a, b), and $(b, c) \in R_1 \cap R_2$. Then $(a, b), (b, c) \in R_1$ and $(a, b), (b, c) \in R_2$. But R_1 and R_2 are transitive; therefore $(a, c) \in R_1$ and $(a, c) \in R_2$, so that $(a, c) \in R_1 \cap R_2$. Hence $R_1 \cap R_2$ is transitive. This proves that $R_1 \cap R_2$ is an equivalence relation.

2. If R_1 and R_2 are equivalence relations, it is not necessary that $R_1 \cup R_2$ is also an equivalence relation.

Counter-example :

$$
\begin{aligned}
\text{Let } A &= \{a, b, c\} \\
R_1 &= \{(a, a), (b, b), (c, c), (a, b), (b, a)\} \\
\text{and } R_2 &= \{(a, a), (b, b), (c, c), (a, c), (c, a)\}
\end{aligned}
$$

Both R_1 and R_2 are equivalence relations.

$$R_1 \cup R_2 = \{(a, a), (b, b), (c, c), (a, b), (b, a), (a, c), (c, a)\}$$

is not an equivalence relation since it is not transitive (b, a) and $(a, c) \in R_1 \cup R_2$ but $(b, c) \notin R_1 \cup R_2$.

In general, if a relation R is not transitive, we can find a relation containing R which is transitive and is the smallest set with this property.

This set is called as the **transitive closure** of R. This notion is discussed in Art 4.11.

4.7.2 Equivalence Classes

Let R be an equivalence relation on a set A. For every $a \in A$, let $[a]_R$ denote the set $\{ x \in A \mid x \, R \, a \}$. Then $[a]_R$ is called as the **equivalence class of a with respect to R.**

$[a]_R \neq \phi$ since $a \in [a]_R$.

The **rank** of R is the number of distinct equivalence classes of R if the number of classes is finite; otherwise the rank is said to be infinite.

In what follows, we will drop the suffix R, and denote the equivalence class of a simply as [a].

The following theorem gives an important characterisation of the equivalence classes.

Theorem 1 : Let R be an equivalence relation on a set A. Then the following hold :

(i) For all $a, b \in A$, either $[a] = [b]$ or $[a] \cap [b] = \phi$.

(ii) $A = \bigcup\limits_{a \in A} [a]$

$a \in A$

Proof : (i) If $A = \phi$, there is nothing to prove. Hence assume $A \neq \phi$. If $A = \{a\}$, singleton set, the result is trivially true. Therefore consider elements $a, b \in A$. Suppose $[a] \neq [b]$. Then we have to show that $[a] \cap [b] = \phi$. Suppose this is not true. Let $c \in [a] \cap [b]$ then c R a and c R b. Since R is symmetric it follows that a R c and c R b. But R is transitive as well. Hence we have a R b, i.e. $b \in [a]$ and $a \in [b]$, which means that $[a] = [b]$, a contradiction. Hence $[a] \cap [b] = \phi$.

(ii) Clearly for each $a \in A$, $[a] \subseteq A$.

Hence $\bigcup\limits_{a \in A} [a] \subseteq A$. Conversely, let $x \in A$. Then $x \in [a]$ for some $a \in A$. This implies that x R a, i.e. a R x. Hence $a \in [x]$ which means that $[a] = [x]$. Therefore $A \subseteq \bigcup\limits_{a \in A} [a]$.

Hence it follows that $A = \bigcup\limits_{a \in A} [a]$.

Examples :

1. Let $A = \{a, b, c\}$ and let $R = \{(a, a), (b, b), (c, c), (a, b), (b, a)\}$

where R is clearly an equivalence relation.

The equivalence classes of the elements of A are :

$$[a] = \{a, b\}$$

$$[b] = \{b, a\} = [a]$$

$$[c] = \{c\}.$$

The rank of R is 2.

2. Let $A = \{1, 2, 3, 4\}$ and let $R = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1), (2, 3), (3, 2), (3, 3), (4, 4)\}$. Show that R is an equivalence relation and determine the equivalence classes, and hence find the rank of R.

Solution : R is reflexive since $(1, 1), (2, 2), (3, 3), (4, 4) \in R$.

R is symmetric since both $(1, 2), (2, 1) \in R$.

Similarly $(2, 3), (3, 2), (1, 3), (3, 1) \in R$.

R is transitive since $(1, 2)$ and $(2, 1) \in R$ implies $(1, 1) \in R$.

Similarly

$$(1, 3), (3, 1) \in R \to (1, 1) \in R,$$

$$(2, 3), (3, 2) \in R \to (2, 2) \in R$$

$$(3, 1), (1, 3) \in R \to (3, 3) \in R$$

$$(3, 2), (2, 1) \in R \to (3, 1) \in R.$$

Hence R is an equivalence relation. The equivalence classes of A are :

$$[1] = \{1, 2, 3\}$$

$$[2] = \{1, 2, 3\} = [1]$$

$$[3] = \{3, 1, 2\} = [1]$$

$$[4] = \{4\}.$$

Hence there two distinct equivalence classes. Hence rank of R is 2.

The following is an important example of an equivalence relation and equivalence classes.

3. (Residue classes modulo a positive integer)

Let Z denote the set of integers. Let n be a positive integer and define a relation R on Z by setting a R b iff $n \mid (a - b)$. Show that R is an equivalence relation and determine its equivalence classes.

Solution : R is reflexive since $n \mid (a - a)$ i.e. n divides zero.

R is symmetric, since a R b $\rightarrow$ $n \mid a - b$ which implies $n \mid (b - a)$, i.e. b R a. Let a R b and b R c. Then $n \mid (a - b)$ and $n \mid (b - c)$ which implies $n \mid [(a - b) + (b - c)] \rightarrow n \mid (a - c)$, i.e. a R c. Hence R is transitive. R is therefore an equivalence relation.

The equivalence classes are [0], [1], [2], ..., [n – 1]. This is because [n] = [0], [n + 1] = [1] and so on. Also note that for any integer m, [–m] = [m].

We denote the set of these equivalence classes by Z_n.

$$Z_1 = \{[0]\} = \{\ldots -1, 0, 1, 2 \ldots\}$$
$$= Z$$
$$Z_2 = \{[0], [1]\}$$
$$Z_3 = \{[0], [1], [2]\}$$
$$Z_4 = \{[0], [1], [2], [3]\} \text{ and so on.}$$

The relation R is known as the congruence relation.

4.7.3 Partitions

We shall now discuss the concept of partition which is closely related to that of equivalence relation.

Definition : A partition of a non-empty set A is a collection of sets $\{A_1, A_2, \ldots, A_n\}$ such that

(i) $A = \bigcup_{i=1}^{n} A_i$

(ii) $A_i \cap A_j = \phi$, for $i \neq j$ (i.e. the sets A_i are mutually disjoint).

We denote a partition of A by the symbol π. An element of a partition is called a **block.** The **rank** of π is the number of blocks of π.

For a given non-empty set, its partition is not unique; we can have different partitions of the same set.

Examples

1. Let $A = \{1, 2, 3\}$

   Then $\pi_1 = \{\{1, 2\}, \{3\}\}$ is a partition of A.

   Similarly, $\pi_2 = \{\{1, 3\}, \{2\}\}$ is another partition of A. A third partition is

   $\pi_3 = \{\{1\}, \{2\}, \{3\}\}$ and so on .

2. Let Z = set of all integers,

   E = set of all even integers

   O = set of all odd integers.

   Then {E, O} is a partition of Z.

3. The rooms (flats) in a building block form a partition.

4. The main memory of a multi-programmed computer system is partitioned and a separate program is stored in each block of the partition.

The following diagram represents a partition of a set.

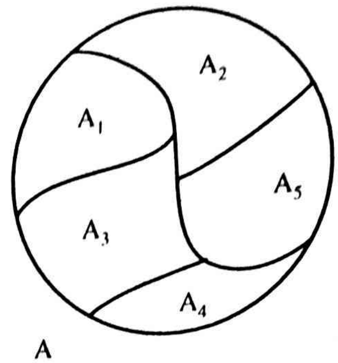


Fig. 4.12

The following theorem establishes the fact that equivalence relations and partitions are different descriptions of the same concept.

Theorem 1 : Let A be a non-empty set and R an equivalence relation on A, then the set of equivalence classes $\left\{ [a]_R \,\middle|\, a \in A \right\}$ constitutes a partition of A.

Proof : This theorem is actually a corollary to Theorem 1 of Art 4.9.2 in which we have shown that

(i) $A = \bigcup_{a \in A} [a]$

(ii) $[a] \cap [b] = \phi$ if $[a] \neq [b]$.

The above conditions are the same as that required for a partition of A. Thus the theorem is proved.

In the above theorem, we have shown that an equivalence relation induces a partition on A. The converse is also true, as proved in the following theorem.

Theorem 2 : Let A be a non-empty set, and let π be a partition of A. Then π induces an equivalence relation on A.

Proof : Let π be a partition of the set A, and define the binary relation R on A as a R b iff there exists a set A_i in π such that a, b $\in A_i$. We shall show that R is an equivalence relation.

(i) R is reflexive since a R a. This is because π being a partition of A, a $\in A_i$ for some i.

(ii) R is symmetric by definition of R.

(iii) R is transitive. Let a R b and b R c. This implies that for some i, a, b $\in A_i$, and for some j, b, c $\in A_j$. Therefore b $\in A_i \cap A_j$. But since π is a partition, this is possible only if $A_i = A_j$. Hence a, c $\in A_i$ which means that a R c. This proves that R is transitive.

The above theorems, thus, establish a natural correspondence between partition of a set and equivalence relation on the set.

Examples

1. Let $A = \{a, b, c, d\}$, $\pi = \{\{a, b\}, \{c\}, \{d\}\}$. Find the equivalence relation induced by π and construct is digraph.

Solution : $R = \{(a, a), (a, b), (b, a), (b, b), (c, c), (d, d)\}$.

The digraph of R is :

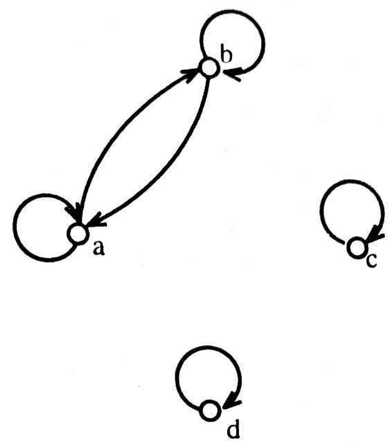


Fig. 4.13

2. Let $A = \{1, 2, 3, 5\}$ and $\pi = \{\{1, 2\}, \{3\}, \{4, 5\}\}$. Find the equivalence relation determined by π and draw its digraph.

Solution : $R = \{(1, 1), (1\ 2), (2, 1), (2, 2), (3, 3), (4, 4), (4, 5), (5, 4), (5, 5)\}$.

Digraph of R

Fig. 4.14

Let R be an equivalence relation on A. We denote by A/R the partition induced by R. Hence a partition of A is called as a **quotient** set of A.

Examples :

1. Let $A = \{1, 2, 3\}$ and let $R = \{(1, 1), (2, 2), (1, 3), (3, 1), (3, 3)\}$. Find A/R.

Solution : A/R is the partition of A induced by R.

Hence $A/R = \{\{1, 3\}, \{2\}\}$.

2. Let Z be the set of integers. Define a relation R on Z as a R b iff $6 \mid (a - b)$, show that R is an equivalence relation and find Z/R.

Solution : Since $6 \mid (a - a)$, a R a. Hence R is reflexive.

If $6 \mid (a - b)$, then $6 \mid (b - a)$, which shows that R is symmetric.

If $6 \mid (a-b)$ and $6 \mid (b-c)$ then obviously $6 \mid ((a-b)+(b-c))$, i.e. $6 \mid (a-c)$. Hence R is also transitive. R is therefore an equivalence relation.

$$Z/R = \{[0], [1], [2], [3], [4], [5]\}$$

where

$$[0] = \{\ldots -12, -6, 0, 6, 12, 18 \ldots\}$$

$$[1] = \{, -11, -5, 1, 7, 13, \ldots\}$$

$$[2] = \{, \ldots -10, -4, 2, 8, 14, \ldots\}$$

$$[3] = \{\ldots, \ldots, -3, 3, 9, 15, \ldots\}$$

$$[4] = \{\ldots, \ldots, -2, 4, 10, 16, \ldots\}$$

$$[5] = \{\ldots, -7, -1, 5, 11, 17, \ldots\}$$

The quotient set Z/R is denoted by Z_6 and is called as the set of congruence classes modulo 6. R is also called as a congruence relation.

3. Let $X = \{a, b, c, d, e\}$ and $C = \{\{a, b\}, \{c\}, \{d, e\}\}$. Show that the partition C defines an equivalence relation on X. **(Dec. 2002)**

Solution : C induces a relation R on X, as follows : For elements $x, y \in X$, xRy if and only if x and y are members of the same subset element of C.

For example, aRb since both $a, b \in \{a, b\}$ in C. R is reflexive since aRa, bRb, cRc, dRd, eRe. Also R is symmetric since aRb and bRa. R is also transitive since aRb, bRa, aRa, bRb, dRe, eRd, dRd, eRe. Hence R is an equivalence relation.

$\pi_1 + \pi_2 = \{\{a, b, c, ...$

4.8 COMPATIBLE RELATION

Definition : A relation R on a set A is said to be compatible if it is reflexive and symmetric.

Examples :

(i) All equivalence relations are compatible relations.

(ii) The relation of 'being friend of' is a compatible relation.

(iii)

$$\text{Let } A = \{a, b, c\}$$
$$B = \{b, c, d\}$$
$$C = \{d, e, f\}.$$

Define a relation R on these sets as : A set X is related to a set Y, i.e. X R Y iff $X \cap Y \neq \phi$.

In this example A R B, B R C, but A $\not R$ C.

R is therefore a compatible relation which is not an equivalence relation.

The graph of a compatible relation is drawn by omitting the loop at each vertex and using a single edge with no arrow between them.

If R is a compatible relation, its relation matrix is symmetric with the diagonal elements being 1.

Definition : A **covering** of a set A is a collection of subsets $\{A_1, A_2, \ldots, A_k\}$ of A, such that $\bigcup_{i=1}^{k} A_i = A$.

We have seen that an equivalence relation induces a partition on A. We shall now show that a compatible relation induces a covering on A.

Definition : Let A be a non-empty set and let R be a compatible relation on A. A subset M of A is called a **maximal compatibility block** if every element of M is compatible with every other element of M and no element of A – M is compatible with all elements of M.

For example, consider the diagram given below for the compatible relation R on the set A = {1, 2, 3, 4, 5}.

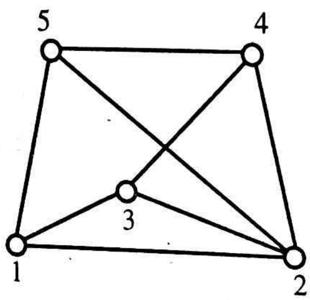


Fig. 4.15

The maximal compatibility blocks are

$$M_1 = \{1, 2, 3\},\ M_2 = \{2, 3, 4\},\ M_3 = \{1, 2, 5\},\ M_4 = \{2, 4, 5\}.$$

$A = M_1 \cup M_2 \cup M_3$ Hence $\{M_1, M_2, M_3\}$ forms a covering for A.

Similarly, $\{M_1, M_4\}$ forms a covering for A.

4.9 TRANSITIVE CLOSURE

Definition : The transitive closure of a relation R is the smallest transitive relation containing R. We denote transitive closure of R by R^*.

The following theorem gives a method to find the transitive closure.

Theorem : Let A be a set with $|A| = n$, and let R be a relation on A. Then

$$R^* = R \cup R^2 \cup R^3 \cup \ldots \cup R^n.$$

Proof : Let $(a, b) \in R^*$. Then since R^* is the transitive closure of R, there exists a sequence of elements $x_1, x_2, \ldots, x_R$ in A such that $a = x_1$, $b = x_R$ and $(x_i, x_{i+1}) \in R$ for $1 \le i \le R - 1$. This means that $(a, b) \in R^{k-1}$. Hence, $R^* \subseteq R \cup R^2 \cup R^3 \cup \ldots \cup R^n$.. Conversely $R^i \subseteq R^*$ for $1 \le i \le n$. Hence $R^* = R \cup R^2 \cup R^3 \cup \ldots \cup R^n$. Now we have only to show that R^* is the smallest transitive relation, containing R. Let S be a transitive relation containing R. Let $(a, c) \in R^2$. This implies that there exists an element $b \in A$ such that $(a, b) \in R$ and $(b, c) \in R$.

Since $R \subseteq S$ and S is transitive it follows that $(a, c) \in S$. Hence $R^2 \subseteq S$. Proceeding thus, we can show that $R^3, R^4, \ldots, R^n$ are all subsets of S. Hence $R \cup R^2 \cup \ldots \cup R^n \subseteq S$, which proves the claim.

Examples :

1. Let $A = \{1, 2, 3, 4\}$ and $R = \{(1, 2), (2, 3), (3, 4)\}$ be a relation on A. Find R^* and draw its digraph.

Solution :

$$R = \{(1, 2), (2, 3), (3, 4)\}$$
$$R^2 = \{(1, 3), (2, 4)\}$$
$$R^3 = \{(1, 4)\}$$
$$R^4 = \phi$$

Hence
$$R^* = R \cup R^2 \cup R^3$$
$$= \{(1, 2), (2, 3), (3, 4), (1, 3), (2, 4), (1, 4)\}$$

Digraph of R^*

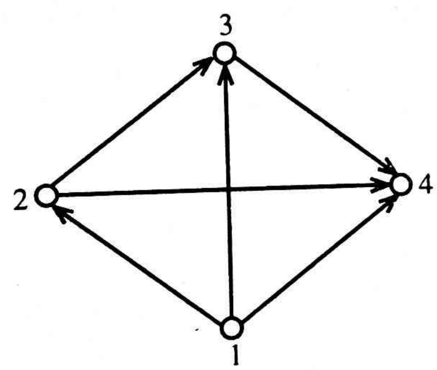


Fig. 4.16

2. Let
$$A = \{a, b, c, d\},$$
$$R_1 = \{(a, a), (b, b), (c, c), (a, b)\} \text{ and}$$
$$R_2 = \{(a, a), (b, d), (d, c)\}$$

Find $(R_1 \cup R_2)^*$ and draw its digraph.

Solution :

$$R = R_1 \cup R_2 = \{(a, a), (b, b), (c, c), (a, b), (b, d), (d, c)\}$$
$$R^2 = \{(a, b), (b, d), (a, d), (b, c), (a, a), (b, b), (c, c), (d, c)\}$$
$$R^3 = \{(a, a), (a, b), (a, c), (a, d), (b, b), (b, c), (b, d), (d, c), (c, c)\}$$
$$R^4 = \{(a, a), (a, b), (a, c), (a, d), (b, b), (b, c), (b, d), (c, c), (d, c)\}$$
$$\therefore \quad R^* = R \cup R^2 \cup R^3$$
$$= \{(a, a), (b, b), (c, c), (a, b), (b, d), (d, c), (a, d), (b, c), (a, c)\}$$

Digraph of R^*

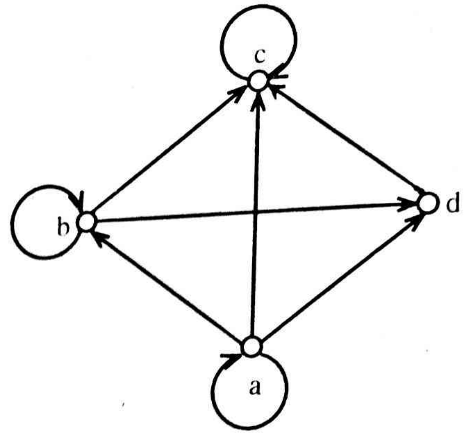


Fig. 4.17

4.9.1 Warshall's Algorithm

Finding the transitive closure of a relation, by computing various powers of R or products of the relation matrix M_R, is quite impractical for large sets and relations. Warshall's Algorithm offers an alternative but efficient method for computing the transitive closure.

Let R be a relation on a set $A = \{a_1, a_2, \ldots a_n\}$ and let R^* denote the transitive closure of R. A **path** of length m in R from a to b is a finite sequence a, $x_1, x_2, \ldots, x_{m-1}$, b, beginning with a and ending with b, such that a R x_1, x_1 R x_2, … , x_{m-1} R b. Note that a path of length m involves m + 1 elements of A, not necessarily distinct. All vertices in the path, except a and b are called as **interior vertices** of the path. For $1 \le k \le n$, define a Boolean matrix $\mathbf{W_k}$ as $\mathbf{W}_k$ has 1 in position i, j if and only if there is a path from a_i to a_j in R whose interior vertices, if any, come from the set $\{a_1, a_2, \ldots, a_k\}$.

Suppose we have a path as shown in the diagram below,

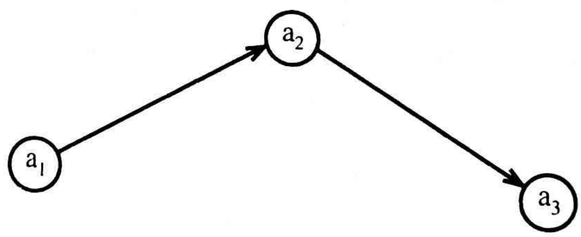


Fig. 4.18

then W_2 will have a 1 in the first row and third column.

Since any vertex must come from the set $\{a_1, a_2, \ldots a_n\}$, it follows that the matrix W_n has 1 in position i, j if and only if some path in R connects a_i and a_j. Hence $W_n = M_{R^*}$. Define $W_o = M_R$. Then we will have a sequence $\mathbf{W_o}, \mathbf{W_1}, \ldots, \mathbf{W_n}$ whose first term is M_R and last term is M_{R^*}.

Warshall's algorithm gives a procedure to compute each matrix $\mathbf{W_k}$ from the previous matrix W_{k-1}. Beginning with the matrix of R, we proceed one step at a time, until we reach the matrix of R^*, in n steps. The matrices W_k, being different from powers of the matrix M_R, result in a considerable saving of steps in the computation of the transitive closure of R.

Suppose $W_{k-1} = [u_{ij}]$ and $W_k = [v_{ij}]$. If $v_{ij} = 1$, there is a path from a_i to a_j whose interior vertices come from the set $\{a_1, a_2, \ldots a_k\}$. If a_k is not an interior vertex of this path, then all the interior vertices must come actually from $\{a_1, a_2, \ldots, a_{k-1}\}$, hence $u_{ij} = 1$. If a_k is an interior vertex of the path, then we have the situation as shown below.

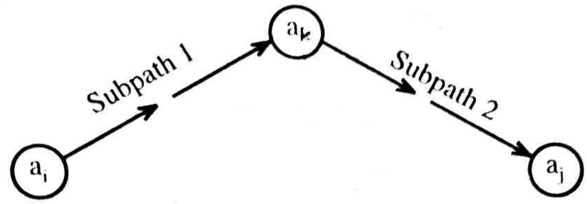


Fig. 4.19

Since there is a subpath from a_i to a_k whose interior vertices come from $\{a_1, a_2, \ldots, a_{k-1}\}$, we must have $u_{ik} = 1$. Similarly $u_{kj} = 1$.

Hence $v_{ij} = 1$ if and only if

(1) $u_{ij} = 1$

(2) $u_{ik} = 1$ and $u_{kj} = 1$.

This is the basis for Warshall's algorithm. If $\mathbf{W_{k-1}}$ has 1 in position i, j then by (1) so will $\mathbf{W_k}$. A new 1 can be added in position i, j of $\mathbf{W_k}$ if and only if column k of W_{k-1} has a 1 in position i, and row k of W_{k-1} has a 1 in position j.

Thus we have the following procedure for computing $\mathbf{W_k}$ from W_{k-1}.

Step 1 : Transfer to W_k, all the 1's in W_{k-1}.

Step 2 : List the locations $p_1, p_2, \ldots$ in column k of W_{k-1} where the entry is 1, and the locations $q_1, q_2, \ldots$ in row k of W_{k-1}, where the entry is 1.

Step 3 : Put 1's in all the positions p_i, q_j of $\mathbf{W_k}$ (if they are not already there).

The above procedure is illustrated in the following problems.

Examples :

1. et $A = \{1, 2, 3, 4\}$ and $R = \{(1, 2), (2, 4), (1, 3), (3, 2)\}$.

Find the transitive closure of R by Warshall's algorithm.

Solution : $$W_o = M_R = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For W_1, $k = 1$, $a_k = 1$ is not an interior vertex for any path in R. Hence $W_1 = W_0$, without any addition to the entries.

For W_2, $k = 2$; $a_k = 2$ is an interior vertex for the path from 3 to 4. It is also an interior vertex for the path from 1 to 4. Hence W_2 has 1 in the position (3, 4) and also 1 in the position (1, 4).

$$\text{Hence} \qquad W_2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For W_3, $k = 3$, $a_k = 3$. Although 3 is an interior vertex for the path from 1 to 2, since the entry 1 is already in the position (1, 2), there is no new addition to the entries in W_2. Hence $W_3 = W_2$.

For W_4, $k = 4$, $a_k = 4$ which is not an interior vertex of any path in R. Hence $W_4 = W_3 = W_2$.

But $\qquad M_{R^*} = W_4$

$$\text{Hence} \qquad M_{R^*} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, we obtain the transitive closure R^* as {(1, 2), (1, 3), (1, 4), (2, 4), (3, 2), (3, 4)}.

2. Let $A = \{a_1, a_2, a_3, a_4, a_5\}$ and let R be a relation on A whose matrix is

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Find M_{R^*} by Warshall's algorithm.

Solution : $\qquad R = \{(a_1, a_1), (a_1, a_4), (a_2, a_2), (a_3, a_4), (a_3, a_5), (a_4, a_1), (a_5, a_2), (a_5, a_5)\}$

$\qquad W_0 = M_R$

For $k = 1$, a_1 is an interior vertex for the path a_4 to a_4. Hence W_1 has 1 in the position (4, 4).

$$W_1 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$W_2 = W_1$ as there is already 1 in the position (5, 2). a_3 is not an interior vertex for any path in R. Hence $W_3 = W_2 = W_1$.

a_4 is an interior vertex for the path a_3 to a_1. Hence W_4 has 1 in the position (3, 1).

$$\therefore \qquad W_4 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

a_5 is an interior vertex in the path from a_3 to a_2. Hence

$$W_5 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$W_5 = M_{R^*}$$

Hence

$$M_{R^*} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

The algorithm for Warshall's method is given below :

Warshall's Algorithm

```
FOR i: = 1 TO n DO
  FOR j: = 1 TO n DO
      If a [j, i] = 1 THEN
         FOR k: = 1 TO n DO
           If a [i, k] = 1 THEN
              a [j, k] : = 1;
              END;
         END;
      END;
  END;
END;
```

4.10 PARTIAL ORDERING RELATIONS

An **order relation** is a transitive relation on a set by means of which we can compare elements of set.

Definition : A binary relation R on a non-empty set A is a **partial order** if R is reflexive, antisymmetric and transitive.

The ordered pair (A, R) is called a **partially ordered set** or **poset.**

Examples :

(i) The relation ' $\leq$ ' is a partial order relation on the set of real numbers.

(ii) The relation of ' being a subset ' is a partial order on any collection of subsets of a set A; i.e. the ordered pair (P(A), $\subseteq$) is a poset.

(iii) The lexicographic ordering on the set of alphabets is a partial order.

We will use the symbol $\leq$ to denote an arbitrary partial order. This notation should not be confused with the ' $\leq$ ' of number systems. Thus $a \leq b$ will mean a R b for an arbitrary partial order relation R, and $(A, \leq)$ will be the corresponding poset.

4.10.1 Hasse Diagrams

The posets can be depicted by digraphs. However a more economical way to describe a poset is by **Hasse diagrams**. A Hasse diagram is a simpler version of a digraph, incorporating the following rules :

(i) All arrow heads that appear on the edges are omitted.

(ii) Loops are omitted as reflexivity is implied, by definition of a partial order.

(iii) Similarly an arc (or edge) is not present in the diagram if it is implied by transitivity. There is an arc from a to b only if there is no element c such that $a \leq c$ and $c \leq b$.

(iv) An arc pointing upward is drawn from a to b if $a \neq b$ and $a \leq b$. Arrow heads are not used.

Examples :

1. Let $A = \{2, 3, 4, 6\}$ and let a R b if a divides b. Show that R is a partial order and draw its Hasse diagram.

Solution : $R = \{(2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (6, 6)\}$

R is reflexive since $(2, 2), (3, 3), (4, 4), (6, 6) \in R$. R is antisymmetric since if $a \mid b$ $b \nmid a$ unless $a = b$. R is also transitive since $a \mid b$ and $b \mid c$ implies $a \mid c$. Hence R is a partial order.

Hasse diagram for R

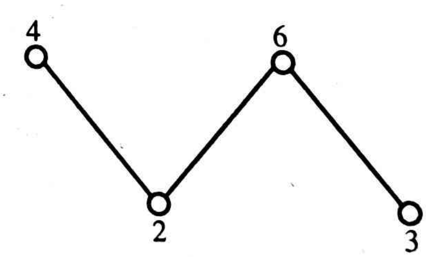


Fig. 4.20

2. If $A = \{1, 2, 3, 4\}$, $R = \{(1, 1), (1, 2), (2, 2), (2, 4), (1, 3), (3, 3), (3, 4), (1, 4), (4, 4)\}$ then show that R is a partial order and draw its Hasse diagram.

Solution : R is reflexive since $(1, 1), (2, 2), (3, 3), (4, 4) \in R$.

R is antisymmetric since $(1, 2) \in R$ but $(2, 1) \notin R$, $(2, 4) \in R$ but $(4, 2) \notin R$. Similarly $(1, 3)$, $(1, 4), (3, 4) \in R$ but $(3, 1), (4, 1), (4, 3) \notin R$. One can also similarly check that R is transitive.

Hasse diagram

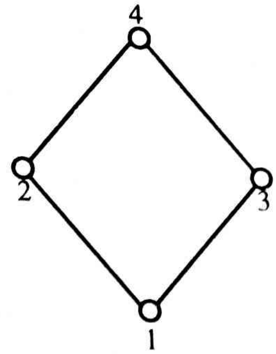


Fig. 4.21

3. Let $A = \{a, b, c\}$. Show that $(P(A), \subseteq)$ is a poset and draw its Hasse diagram.

Solution : $P(A) = \{A, \phi, \{a, b\}, \{b, c\}, \{a, c\}, \{a\}, \{b\}, \{c\}\}$

Set containment $\subseteq$ is always a partial order since for any subset B of A, $B \subseteq B$, i.e. $\subseteq$ is reflexive. If $B \subseteq C$ and $C \subseteq B$, $B = C$ (antisymmetry). If $B \subseteq C$ and $C \subseteq D$ then $B \subseteq D$ (transitivity).

Hasse diagram

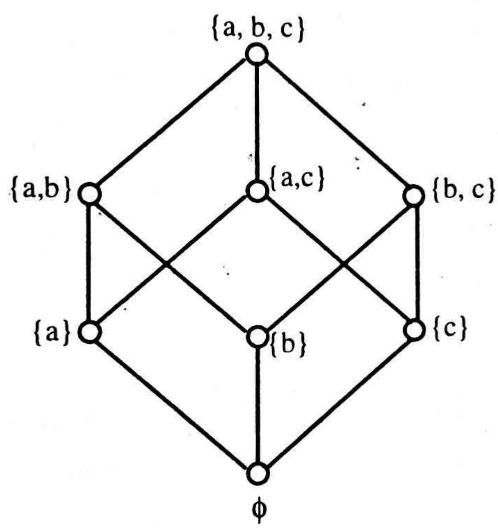


Fig. 4.22

4.11 CHAINS AND ANTICHAINS

Definition : Let $(A, \leq)$ be a poset. A subset of A is called a **chain** if every pair of elements in the subset are related.

In any chain with a finite number of elements $\{a_1, a_2, \ldots, a_k\}$ there is an element a_{i1} that is less than (i.e. related to) every element in the chain, and there is an element a_{i2} that is less than every other element except a_{i1}. Continuing in this manner, we shall have a sequence $a_{i1} \leq a_{i2} \leq a_{i3} \leq \ldots \leq a_{ik}$. The number of elements in the chain is called as the length of the chain.

If A itself is a chain, the poset $(A, \leq)$ is called a totally ordered set or linearly ordered set.

Definition : A subset of A is called an antichain if no two distinct elements in the subset are related.

Examples :

(i) Let $A = \{1, 2, 3\}$ and let the partial order $\leq$ mean " less than or equal to ". Then $(A, \leq)$ is a chain and its Hasse diagram is

Fig. 4.23

(ii) Let $A = \{a, b\}$ and consider its poset $\{P(A), \subseteq\}$

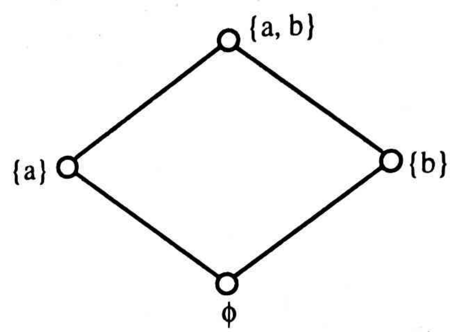


Fig. 4.24

Then the following subsets are chains

$\{\phi, \{a\}, \{a, b\}\}$, $\{\phi, \{b\}, \{a, b\}\}$, $\{\phi, \{a\}\}$, $\{\phi, \{b\}\}$, $\{\{a\}, \{a, b\}\}$, $\{\{b\}, \{a, b\}\}$.

The following subset is an antichains is $\{\{a\}, \{b\}\}$.

In the above example, the length of the longest chain is 3 and the number of elements in the antichain is 2.

We state the following theorem (without proof) that shows a close relation between chains and antichains.

Theorem 1 : Set $(A, \leq)$ be a poset. Suppose the length of the longest chain in A is n. Then the elements is A can be partitioned into n disjoint antichains.

SOLVED PROBLEMS

1. Let R_1 be a binary relation on A such that (a, b) is in R_1 if book a costs more and contains fewer pages than book b. Is R_1 reflexive ? Symmetric ? Antisymmetric ? Transitive ?

Solution : R_1 is obviously not reflexive and not symmetric. If a R b, then b $\not R$ a. Similarly if b R a, then a $\not R$ b. Hence both the conditions a R b and b R a cannot be fulfilled simultaneously. Hence by the law of contrapositive, R_1 is antisymmetric. Let a R b and b R c. This implies a costs more than b, b costs more then c, a contains fewer pages than b, b contains fewer pages than c. Hence combining all these statements a costs more than c and contains few pages than c. Hence R is a transitive relation.

2. Let R be a binary relation on the set of all positive integers such that

$$R = \{ (a, b) \mid a - b \text{ is an odd positive integer} \}.$$

Is R reflexive, symmetric, antisymmetric, transitive ?

Is R an equivalence relation ? a partial ordering relation.

Solution : R is not reflexive since $(a, a) \notin R$ as $a - a = 0$.

R is not symmetric since $a - b$ is odd implies $b - a$ is odd, but it is not positive. $(3, 2), \in R$ but $(2, 3) \notin R$. R is also not transitive, since although $a - b$ is odd and $b - c$ is odd, $a - c = (a - b) + (b - c)$ which is even. Hence $(a, c) \notin R$ whenever $(a, b) \in R$ and $(b, c) \in R$. R is antisymmetric since $a R b \rightarrow b \not R a$. None of the conditions for an equivalence relation are satisfied. Hence R is not an equivalence relation. R is also not a partial order.

3. Let R be a binary relation on the set of all strings of 0's and 1's such that

$$R = \{ (a, b) \mid a \text{ and } b \text{ are strings that have the same number of 0's} \}$$

Is R reflexive ? Symmetric ? Antisymmetric ? Transitive ? An equivalence relation ? A partial order ?

Solution : R is reflexive since $(a, a) \in R$. R is symmetric since if a and b have the same number of 0's, then b and a will have the same number of 0's. R is transitive since if a and b have the same number of 0's, b and c have the same number of 0's, then obviously a and c have the same number of 0's. R is obviously not antisymmetric. R is an equivalence relation but not a partial order.

4. Let S be the set of all points in a plane. Let R be the relation such that for any two points a and b, $(a, b) \in R$ if b is within one inch from a. Examine if R will be an equivalence relation.

Solution : R is reflexive since a is **within** one inch (i.e. 0 inch) from itself.

R is symmetric since if b is within 1 inch from a, a is also within 1 inch from b.

But R is not transitive since b is within 1 inch from a, c is within 1 inch from b need not imply that c is within 1 inch from a. Hence R is not an equivalence relation.

5. Let T be a set of triangles in a plane and define R as the set

$$R = \{ (a, b) \mid a, b \in T, \ a \text{ is congruent to } b \}.$$

Show that R is an equivalence relation.

Solution : a triangle is congruent to itself. Hence R is reflexive. If a is congruent to b, then b is congruent to a. Hence R is symmetric. If a is congruent to b, b is congruent to c, then a is congruent to c. Hence R is transitive. The relation satisfies all the three properties of an equivalence relation.

6. Consider subset as a relation on a given set. Check whether it is reflexive, symmetric, antisymmetric, equivalence or partial ordering relation.

Solution : Let A be the given set. Consider subsets B, C of A. Any set is its own subset. Hence the relation is reflexive. If $B \subseteq C$, then $C \subseteq B$ only if $B = C$. Hence the relation is not symmetric but antisymmetric.

If $B \subseteq C$ and $C \subseteq D$ then $B \subseteq D$. Hence the subset relation is transitive. The relation is therefore not an equivalence relation but a partial ordering relation.

7. From the following digraphs, write the relation a set of ordered pairs. Are the relations equivalence relations ? (May 96)

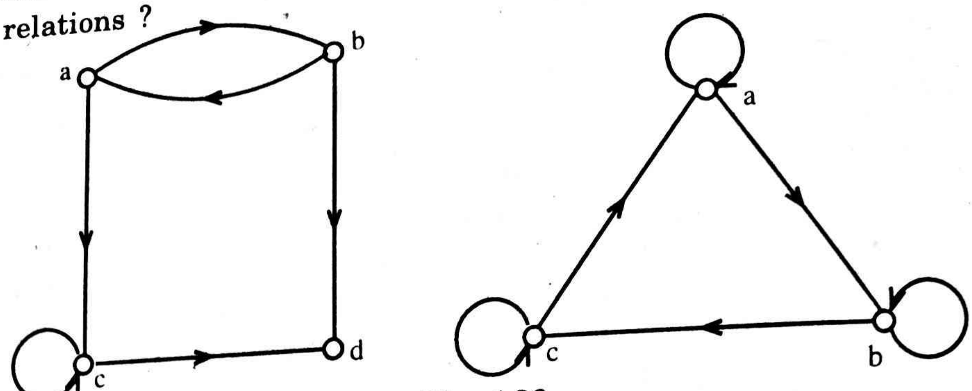


Fig. 4.26

Solution : $R_1 = \{(a, b), (b, a), (a, c), (c, d), (b, d), (c, c)\}$ and

$R_2 = \{(a, a), (b, b), (c, c), (c, a), (c, c), (a, b)\}$

R_1 is not an equivalence relation since R_1 is not reflexive. R_2 is also not an equivalence relation since R_2 is not symmetric, as $(a, b) \in R$ but $(b, a) \notin R$.

8. Consider the following relation on $\{1, 2, 3, 4, 5, 6\}$, $R = \left\{ (i, j) \mid |i - j| = 2 \right\}$

Is R reflexive, symmetric, transitive ? Draw a graph of R.

Solution : R is not reflexive. R is symmetric but not transitive.

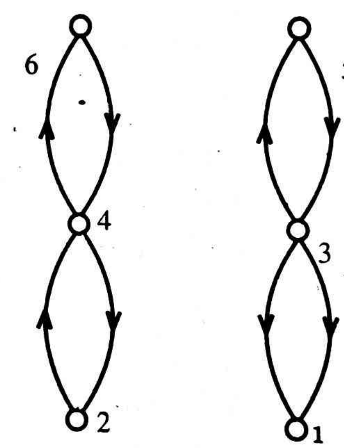
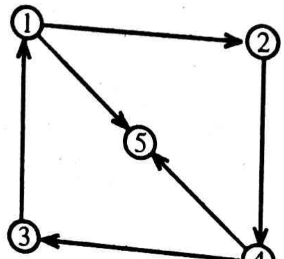
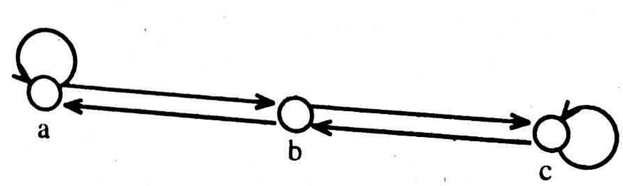


Fig. 4.27

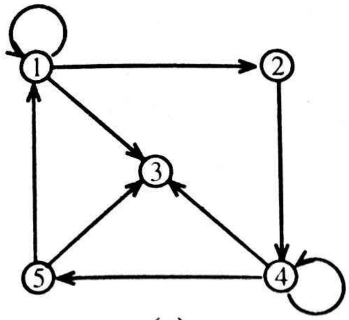
9. Consider the relations defined by the digraphs. Determine whether the given relations are reflexive, symmetric, antisymmetric or transitive. Which graphs are equivalence relations and which are partial orders ?



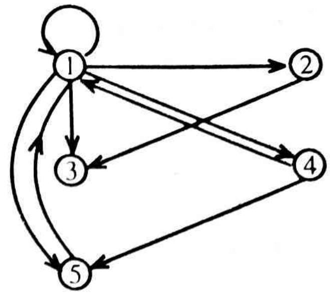
(a)



(b)



(c)



(c)

Fig. 4.28

Solution : $R_1 = \{(1, 2), (1, 5), (2, 4), (3, 1), (4, 3), (4, 5)\}$

$R_2 = \{(a, a), (a, b), (b, a), (b, c), (c, b), (c, c)\}$

$R_3 = \{(1, 1), (1, 2), (1, 3), (2, 4), (4, 3), (4, 4), (4, 5), (5, 1)\}$

$R_4 = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (4, 1), (4, 5), (5, 1)\}$

R_1 is not reflexive, not symmetric and not transitive also.

R_1 is also antisymmetric since $a \neq b \rightarrow a \not{R}\, b \vee b \not{R}\, a$.

R_2 is not reflexive, not transitive but symmetric. R_2 is not antisymmetric.

R_3 is not reflexive, not symmetric, but is transitive and antisymmetric.

R_4 is not reflexive, not symmetric and not transitive since $(4, 1) \in R$ and $(1, 3) \in R$ but $(4, 3) \notin R$. R_4 is not antisymmetric.

None of the relations are equivalence relations or partial orders.

10. The following relations R_1 and R_2 are defined over the set $A = \{1, 2, 3, 4, 5\}$. Show that they are partial order relations and draw their Hasse diagram.

R_1	1	2	3	4	5
1	✓				
2	✓	✓			
3	✓	✓	✓		
4	✓	✓		✓	✓
5	✓				✓

R_2	1	2	3	4	5
1	✓	✓	✓	✓	
2		✓	✓	✓	
3			✓	✓	
4				✓	
5			✓	✓	✓

Solution : R_1 is reflexive since (1, 1), (2, 2), (3, 3), (4, 4), (5, 5) $\in R_1$. R_1 is also antisymmetric since (1, 2) $\notin R_1$, but (2, 1) $\in R_1$ (1, 3) $\notin R_1$ but (3, 1) $\in R_1$. Likewise we can check for other elements. R_1 is transitive (4, 5) $\in R_1$, (5, 1) $\in R_1$, $\rightarrow$ (4, 1) $\in R_1$. Hence R_1 is a partial order relation.

R_2 is reflexive since ✓ appears along the diagonal of the square. R_2 is antisymmetric since whenever (a, b) $\in R_2$, (b, a) $\notin R_2$, unless a = b. R_2 is also transitive. (1, 3) and (3, 4) $\in R_2 \rightarrow$ (1, 4) $\in R_2$. (5, 3) and (3, 4) $\in R_2 \rightarrow$ (5, 4) $\in R_2$. Likewise we can check transitivity for all possible combinations. Hence R_2 is also a partial order.

Hasse diagrams

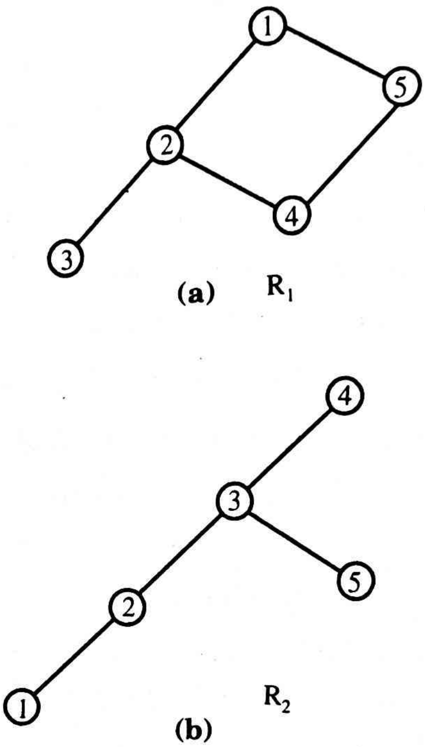


Fig. 4.29

11. Let R be a symmetric and transitive relation on a set A. Show that if for every a in A there exists b in A such that (a, b) is in R, then R is an equivalence relation.

Solution : We have only to show that R is reflexive. Let a $\in$ A, then there exists b $\in$ A, such that (a, b) $\in$ R. Since R is symmetric, this implies (b, a) $\in$ R. Now (a, b) and (b, a) $\in$ R. Hence by transitivity (a, a) $\in$ R, i.e. R is reflexive.

12. Show that the transitive closure of a symmetric relation is symmetric.

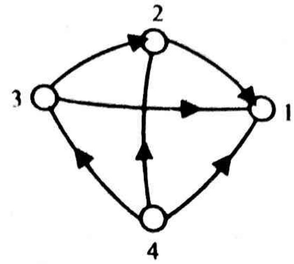
Solution : let R^* denote the transitive closure of R. $R^* = R \cup R^2 \cup$. Let (a, b) $\in R^*$. Then (a, b) $\in R^k$ for some positive integer k. Then there exists a sequence of elements a, x_1, x_2, ... x_{k-1}, b, such that (a, x_1) $\in$ R, (x_1, x_2) $\in$ R, ..., (x_{k-1}) $\in$ R.

Since R is symmetric this implies (b, x_{k-1}), (x_{k-1}, x_{k-2}), ... , (x_2, x_1), (x_1, a) $\in$ R. Since R^* is transitive this implies that (b, a) $\in R^*$. Hence R^* is transitive.

Ex. 13 : Let X = {1, 2, 3, 4} and R = {(x, y) | x > y}. Draw the graph of R and also give its matrix.

$$R = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$$

Graph of R :



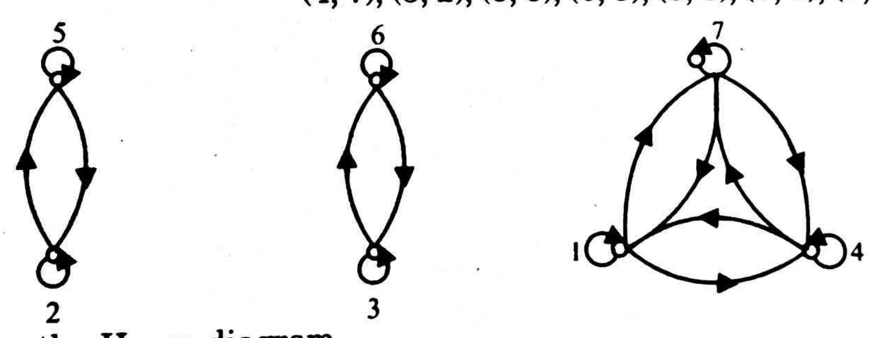
$$M_R = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Ex. 14 : Let X = {1, 2, ... 7} and R = {(x, y) | x − y is divisible by 3}. Show that R is an equivalence relation. Draw the graph of R.

Solution : R is reflexive since $\forall\, x \in X$, $x - x = 0$ is divisible by 3. Hence for $\forall\, x \in X$, $(x, x) \in R$. R is symmetric since for every $(x, y) \in R$, $(y, x) \in R$, as $y - x = -(x - y)$ is divisible by 3. Let (x, y) and $(y, z) \in R$. Then $\pm\, x - z = (x - y) + (y - z)$ is clearly divisible by 3. Hence R is an equivalence relation.

Graph of R :

$$R = \{(1, 1), (1, 4), (1, 7), (2, 2), (2, 5), (3, 3), (3, 6), (4, 1), (4, 4), (4, 7), (5, 2), (5, 5), (6, 3), (6, 6), (7, 1), (7, 4), (7, 7)\}$$



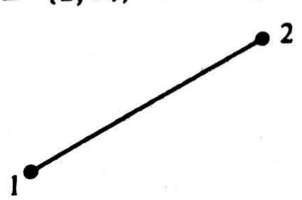
Ex. 15 : Draw the Hasse diagram.

Let A be the set of factors of a particular positive integer m and let ≤ be a relation divides, i.e.

$$\leq \; = \{(x, y) \mid x \in A \wedge y \in A \wedge (x \text{ divides } y)\}.$$

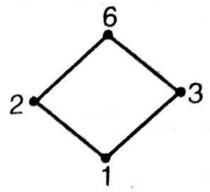
(i) m = 2, (ii) m = 6, (iii) m = 12, (iv) m = 45.

Solution : (i) m = 2, $A = \{1, 2\}$, $R = \{(1, 1), (1, 2), (2, 2)\}$

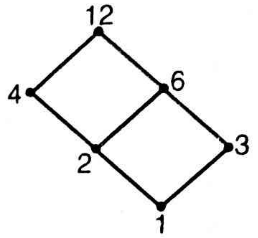


(ii) m = 6, A = {1, 2, 3, 6}

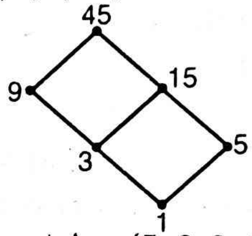
R = {(1, 1), (1, 2), (1, 3), (1, 6), (2, 2), (2, 6), (3, 3), (3, 6), (6, 6)}



(iii) m = 12, A = {1, 2, 3, 4, 5, 6, 12}



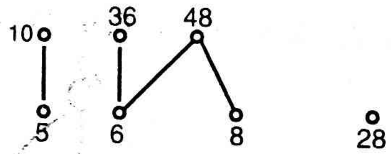
(iv) m = 45, A = {1, 3, 9, 15, 45}



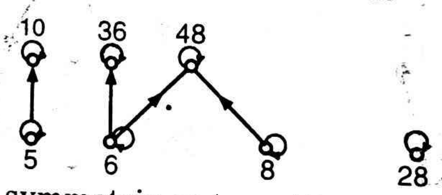
Ex. 16 : Let R be the relation on the set A = {5, 6, 8, 10, 28, 36, 48}. Let R = {(a, b) | a is a divisor of b}. Draw the Hasse diagram and compare it with digraph. Determins whether R is reflexive, transitive and symmetric.

Solution : R = {(5, 5), (5, 10), (6, 6), (6, 36), (6, 48), (8, 8), (8, 48), (10, 10), (28, 28), (36, 36), (48, 48)}.

Hasse diagram :



Digraph :



R is reflexive, but not symmetric or transitive.

Ex. 17 : Find the transitive closure of R by Warshall Algorithm where A = {1, 2, 3, 4, 5, 6} and R = {(x, y) | |x – y| = 2}.

Solution :

R = {(1, 3), (3, 1), (2, 4), (4, 2), (3, 5), (5, 3), (4, 6), (6, 4)}

$$W_R = M_R = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

By notation, let $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $a_4 = 4$, $a_5 = 5$, $a_6 = 6$. For $k = 1$, $W_k = W_1$, $a_k = 1$ is an interior vertex for the path from 3 to 1 and 1 to 3. Hence W_1 has 1 in the position (3, 3).

$$\therefore \quad W_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Similarly for $k = 2$, $a_2 = 2$ is an interior vertex for the path (4, 2) and (2, 4). $k = 3$, $a_3 = 3$ is an interior vertex for the path (1, 3) and (3, 1). $k = 4$, $a_4 = 4$ is an interior vertex for the path (2, 4) and (4, 2). $k = 5$, $a_5 = 5$ is an interior vertex for the path (3, 5) and (5, 3). $k = 6$, $a_k = 6$ is an interior vertex for the path (4, 6) and (6, 4).

Hence the final matrix becomes

$$W_6 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Hence transitive closure of R is

$$R^* = \{(1, 1), (1, 3), (1, 5), (2, 2), (2, 4), (2, 6), (3, 1), (3, 3), (3, 5), (4, 2), (4, 4), (4, 6), (5, 1), (5, 3), (5, 5), (6, 2), (6, 4), (6, 6)\}$$

Ex. 18 : Use Warshall's Algorithm to find the transitive closure of R, where

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{and } A = \{1, 2, 3\}.$$

Solution :

$$R = \{(1, 1), (1, 3), (2, 2), (3, 1), (3, 2)\}$$

$$W_O = W_R$$

For $k = 1$, $a_k = 1$ is an interior vertex for the path (3, 1) and (1, 3). Hence W_1 will have 1 as its (3, 3) entry.

For $k = 2$, $a_k = 2$ is an interior vertex for the path (3, 2) and (2, 2), but already there is 1 at (3, 2) position.

For $k = 3$, $a_k = 3$ is an interior vertex for the paths (1, 3), (3, 2) and (1, 3), (3, 1). Hence we have to include 1 at the (1, 2) position, whereas there is already 1 at the (1, 1) position.

$$W_3 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Hence transitive closure of R

$$R^* = \{(1, 1), (1, 3), (2, 1), (2, 2), (3, 1), (3, 2), (3, 3)\}$$

EXERCISE – III

1. Let $A = \{1, 2, 3, 4\}$. Consider the following relations on A.

$$R_1 = \{(1, 3), (2, 3), (4, 1)\}$$
$$R_2 = \{(1, 1), (2, 1)\}$$
$$R_3 = \{(3, 4)\}$$
$$R_4 = \{(1, 1), (2, 2), (3, 3)\}$$
$$R_5 = \{(1, 3), (2, 4)\}.$$

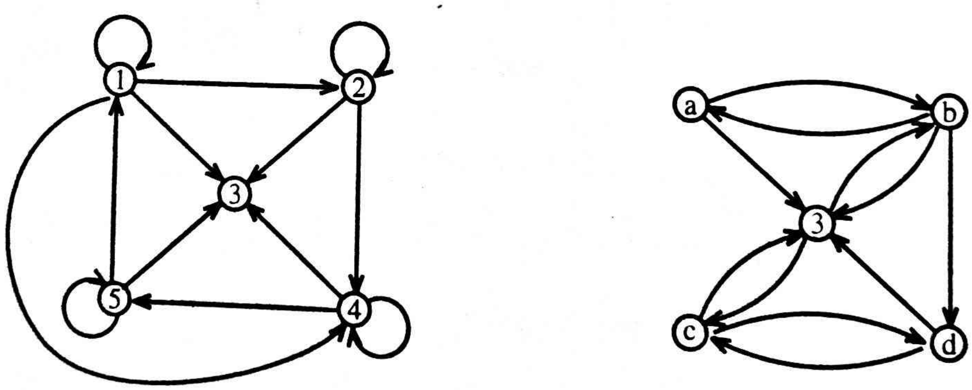
Determine which relations are (i) reflexive, (ii) symmetric, (iii) transitive, (iv) equivalence, (v) partial ordering relation ?

2. Let $A = \{a, b, c\}$. Determine whether the relation R whose matrix M_R is given is an equivalence relation.

(a) $M_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

(b) $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3. Determine whether the relation R whose digraph is given is reflexive, irreflexive, symmetric, antisymmetric or transitive ?



(a) (b)

Fig. 4.30

4. Let Z be the set of integers and let a R b iff b is a multiple of a. Determine which of the five properties are satisfied by R ?

5. Let A be a set of lines in a plane. Define the following relation on A : l_1 A l_2 if and only if l_1 is perpendicular to l_2. Determine what properties of a relation are satisfied by R ?

6. Let P be the set of all people. Let R be a binary relation on P such that a R b iff a is a brother of b. What properties of a relation are satisfied by R ?

7. Let R_1 and R_2 be relations on a set A. Prove the following :

   (a) If R_1 is symmetric, then so are R_1^c and $\bar{R}_1$.

   (b) If R_1 and R_2 are symmetric, then so are $R_1 \cap R_2$ and $R_1 \cup R_2$.

   (c) If R_1 and R_2 are transitive, then so is $R_1 \cap R_2$. Is $R_1 \cup R_2$ transitive ?

8. If R_1 and R_2 are relations on any set A, prove or disprove the following :

   (a) If R_1 and R_2 are reflexive then so is $R_1 R_2$.

   (b) If R_1 and R_2 are irreflexive, then so is $R_1 R_2$.

   (c) If R_1 and R_2 are symmetric, then so is $R_1 R_2$.

   (d) If R_1 and R_2 are antisymmetric, then so is $R_1 R_2$.

   (e) If R_1 and R_2 are transitive, then so is $R_1 R_2$.

9. If A = {1, 2, 3, 4, 5} and R = {(1, 2), (3, 4), (4, 5), (4, 1), (1, 1)}, find its transitive closure.

10. Let R be a transitive and reflexive relation on A. Let T be a relation on A such that (a, b) ∈ T iff both (a, b) and (b, a) are in R. Show that T is an equivalence relation.

11. Let R be a reflexive relation on a set A. Show that R is an equivalence relation iff (a, b) and (a, c) ∈ R implies that (b, c) ∈ R.

12. Let R be the relation defined on the integers by a R b iff a – b is even. Show that R is an equivalence relation and determine the equivalence classes.

13. Partition the set A = {1, 2, 3, 4, 5} by collection of sets {{1, 2}, {3}, {4, 5}}.

    Determine the equivalence relation induced by the partition.

14. Let A = {1, 2, 3, 4} and

    R = {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1), (2, 3), (3, 2), (3, 3), (4, 4)}

Show that R is an equivalence relation and determine the equivalence classes.

15. Let A = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

    and let A_1 = {1, 2, 3, 4}

    A_2 = {5, 6, 7}

    A_3 = {8, 9, 10}

    Form a partition

    $\pi_1 = \{A_1, A_2, A_3\}$ of A.

    Let A_4 = {4, 8, 10}, A_5 = {3, 7, 9}, A_6 = {1, 2, 5, 6} form another partition

    $\pi_2 = \{A_4, A_5, A_6\}$ of A.

    Find the sum and product of the two partitions.

16. Let A be a set of people and R be a binary relation on A such that (a, b) is in R if a is a friend of b. Show that R is a compatible relation.

17. Let R_1 and R_2 be two compatible relations on A. Is $R_1 \cap R_2$ a compatible relation ? Is $R_1 \cup R_2$ a compatible relation ?

18. Let A be a set of English words and R be a binary relation on A such that two words in A are related if they have one or more letters in common. Show that R is a compatible relation.

19. Determine whether the relation R is a partial order on the set A.

(i) $A = Z$, and a R b iff $a = 2b$

(ii) $A = Z$, and a R b iff $b^2 \mid a$.

20. Let $A = \{1, 2, 3, 4\}$ and $R = \{(1, 1), (1, 2), (2, 2), (2, 4), (1, 3), (3, 3), (3, 4), (1, 4), (4, 4)\}$. Show that R is a partial order and draw its Hasse diagram. Determine the chains and antichains.

21. Consider the poset whose Hasse diagram is given below. Find the length of the longest chain. Find also the antichains.

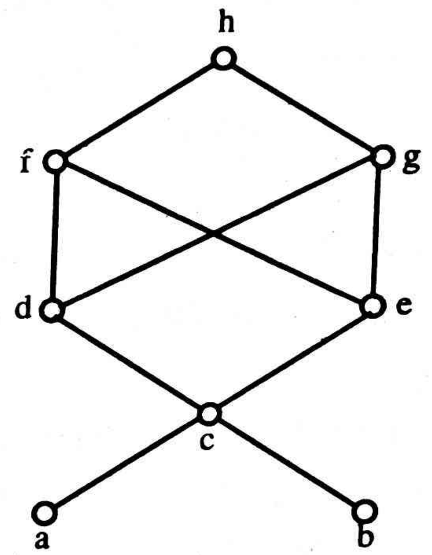


Fig. 4.31

22. Given $S = \{1, 2, 3, 4, 5\}$ and relation R on S, where $R = \{(x, y) \mid x + y = 5\}$. What are the properties of R ?

23. Let R be a relation on a set A,

$$A = \{2, 3, 4, 6, 8, 12, 38, 48\} \text{ defined by}$$

$$R = \{(a, b) \mid a \text{ is divisor of } b\}$$

Draw the diagraph and Hasse diagram.

4.13.1 Maximal and Minimal elements

An element a ∈ A is called a **maximal element** if there is no element b ∈ A such that b ≠ a and a ≤ b. An element c ∈ A is called a minimal element if there is no element d ∈ A such that d ≠ c and d ≤ a.

Example : Consider the poset whose Hasse diagram is given below.

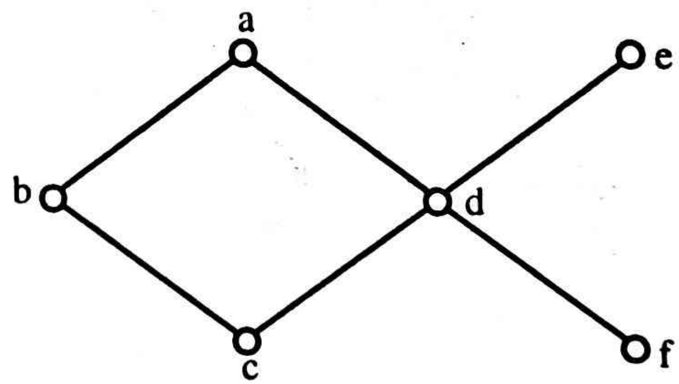


Fig. 4.32

Maximal elements are a, e. Minimal elements are c, f.

4.13.2 Upper Bound and Lower Bound

Let a, b be elements in a poset (A, ≤). An element c is said to be an **upper bound** of a and b if a ≤ c and b ≤ c. An element c is said to be a **least upper bound** (lub) of a and b if c is an upper bound of a and b and if there is no other upper bound d of a and b such that d ≤ c.

Similarly an element e is said to be a **lower bound** of a and b if e ≤ a and e ≤ b; and e is called a **greatest lower bound** (glb) of a and b if there is no other lower bound f of a, b such that e ≤ f.

Example : Consider the poset whose diagram is given below.

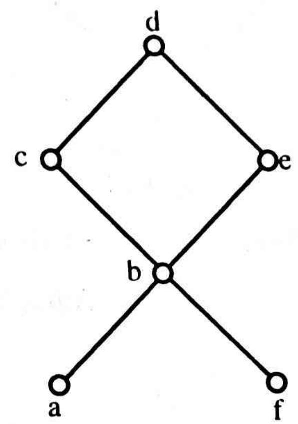


Fig. 4.33

Then

(i) upper bounds for {c, e} is d

lub {c, e} = d.

lower bounds for {c, e} are the elements b, a and f

glb {c, e} = b.

(ii) upper bounds for {a, f} are the elements b, c, e, d.

lub {a, f} = b.

lower bounds for a, f do not exist.

(iii) upper bounds for {b, d} is d.

lub {b, d} = d.

lower bounds for {b, d} are b, a and f.

glb {b, d} = b.

In this manner, one can find the bounds for various pairs of elements.

4.13.3 Lattice – Definition

A lattice is a poset (L, ≤) in which every subset {a, b} of L, has a least upper bound and greatest lower bound.

Examples :

1. For any set A, consider its power set P(A). Then (P(A), ⊆) is a poset. It is also a lattice since for any pair of subsets B, C of A, lub {B, C} = B ∪ C and glb {B, C} = B ∩ C.

2. Let N be the set of natural numbers. For a, b ∈ N, let a ≤ b if b is divisible by a. Then (N, ≤) is a lattice since for any pair of elements a, b ∈ N, lub {a, b} = $l$cm of a and b and glb {a, b} = gcd of a and b.

3. Consider the poset whose diagram is given below.

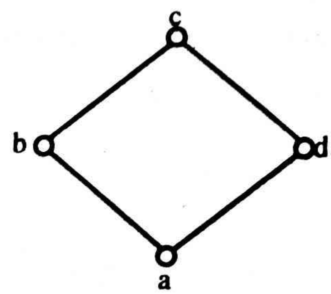


Fig. 4.34

If A = {a, b, c, d}, every pair of elements has a lub and glb. Hence (A, ≤) is a lattice.

4. Let A = {2, 3, 4, 6, 12} and define a ≤ b as a divides b.

Consider the diagram of the poset (A, ≤).

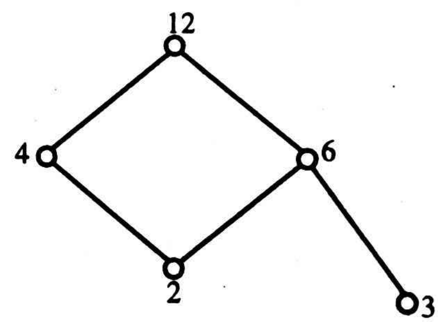


Fig. 4.35

The poset is not a lattice since the pair {2, 3} do not have a greatest lower bound. If the element 1 is included in A, then (A, ≤) will be a lattice.

4.13.4 Lattice Operators

Let (L, ≤) be a lattice. For any pair of elements a, b ∈ L, denote lub {a, b} by a ∨ b and glb {a, b} by a ∧ b. a ∨ b is called as the **join** of a and b. a ∧ b is called as the meet of a and b. The meet and join are therefore binary operators.

4.13.5 Basic properties of lattices

Theorem 1 : For any element a ∈ L, a ∨ a= a and a ∧ a = a (Idempotent property).

Proof : Since a ∨ a is an upper bound for a, a ≤ a ∨ a ... (1)

By reflexivity, a ≤ a ... (2)

Since a ∨ a = lub {a, a}, from (1) and (2) it follows that

$$a \vee a \leq a \quad \text{... (3)}$$

But ≤ is antisymmetric.

Hence

$$a \vee a = a$$

Similarly, we can prove a ∧ a = a.

Theorem 2 : For any a, b $\in$ L

$$a \vee (a \wedge b) = a$$

$$a \wedge (a \vee b) = a$$

(Absorption property of join and meet)

Proof : Since $a \vee (a \wedge b)$ is an upper bound for a and $a \wedge b$, it follows that

$$a \leq a \vee (a \wedge b) \quad \text{... (1)}$$

Now $a \wedge b$ is a lower bound for a and b, hence $a \wedge b \leq a$.

Now $a \leq a$ and $a \wedge b \leq a$. Hence a is an upper bound for the pair $\{a, a \wedge b\}$.

Since $a \vee (a \wedge b) = \text{lub } \{a, a \wedge b\}$, it follows that

$$a \vee (a \wedge b) \leq a \quad \text{... (2)}$$

From (1) and (2) by antisymmetry of $\leq$, we have

$$a = a \vee (a \wedge b)$$

Similarly, we can prove $a \wedge (a \vee b) = a$.

Theorem 3 : The meet and join operations are associative,

i.e. $\quad a \vee (a \vee c) = (a \vee b) \vee c$

and $\quad a \wedge (b \wedge c) = (a \wedge b) \wedge c$.

Proof : Proof is left as an exercise.

Theorem 4 : The meet and join operations are commutative.

Proof : Left as an easy exercise.

In general, the distributive law is not satisfied for a lattice, as the following example demonstrates.

4.13.6 Example of a non-distributive lattice

1. Consider the Hasse diagram given below of a lattice.

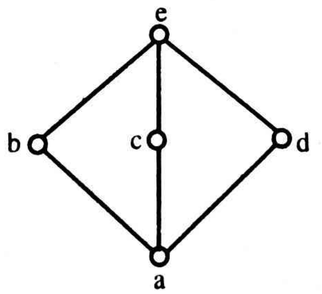


Fig. 4.36

We have $b \wedge (c \vee d) = b \wedge e = b$ ... (1)

On the other hand $(b \wedge c) \vee (b \wedge d) = a \vee a = a$.

Since $a \neq b$, the distributive law does not hold for this lattice.

4.13.7 Universal Bounds

An element in a lattice (L, ≤) is called **universal lower bound** if for every a ∈ L, $l \leq a$. Similarly an element u in L is called **universal upper bound** if for every a ∈ L, a ≤ u.

One can easily see that **these universal bounds are unique. We denote the universal lower bound by 0 and the universal upper bound by 1. 0 and 1 are merely symbols and should not** confused with the numbers 0 and 1. All lattices do not have the universal bounds. The set of real numbers with the usual order (≤) has neither the universal lower bound nor the universal upper bound.

Theorem 1 : Let (L, ≤) be a lattice with universal bounds 0 and 1. Then for every a ∈ L, $a \vee 1 = 1$, $a \wedge 1 = a$, $a \vee 0 = a$, $a \wedge 0 = 0$.

Proof : Easy exercise.

4.13.8 Complement and Complemented Lattice

Definition : Let (L, ≤) be a lattice with universal bounds 0 and 1. For an element a ∈ L, b is said to be a complement of a if $a \vee b = 1$ and $a \wedge b = 0$.

Note by the commutativity property of meet and join, if b is the complement of a, then a is the complement of b.

In a lattice an element can have more than one complement, as demonstrated in the following example.

Example : Let A = {1, 2, 3, 5, 30} and let a ≤ b iff a divides b. The Hasse diagram is

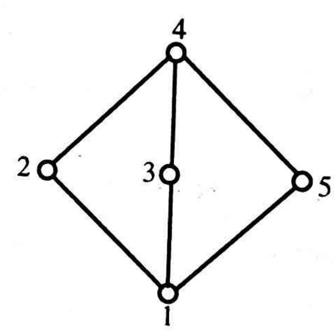


Fig. 4.37

Now

$$2 \wedge 3 = 1, \quad 2 \vee 3 = 30$$

$$2 \wedge 5 = 1, \quad 2 \vee 5 = 30$$

Hence 2 has two complements 3 and 5.

Hence complement is not unique.

However, as the following theorem will prove, there is a special type of lattice, in which every element has a unique complement.

Theorem 1 : Let (L, ≤) be a complemented distributive lattice. Then every element in L has a unique complement.

Proof : Let a ∈ L. Suppose there exist elements $a_1, a_2 \in L$.

Such that $a \wedge a_1 = 0$ and $a \vee a_1 = 1$,

and $a \wedge a_2 = 0$ and $a \vee a_2 = 1$,

then we have to prove $a_1 = a_2$.

Consider $a_1 = a_1 \wedge 1 = a_1 \wedge (a \vee a_2)$

$= (a_1 \wedge a) \vee (a_1 \wedge a_2)$ (by distributivity)

$= 0 \vee (a_1 \wedge a_2) = a_1 \wedge a_2$ ... (1) (Commutativity)

Similarly, $a_2 = a_2 \wedge 1 = a_2 \wedge (a \vee a_1)$

$= (a_2 \wedge a) \vee (a_2 \wedge a_1) = 0 \vee (a_2 \wedge a_1) = a_1 \wedge a_2$ ... (2)

From (1) and (2), a_1 and a_2 are both equal to $a_1 \wedge a_2$.

Hence $a_1 = a_2$, which proves the uniqueness of the complement.

We denote the complement of a by a'.

The following theorem proves De Morgan's laws for a complemented distributive lattice.

Theorem 2 : Let $(L, \leq)$ be a complemented distributive lattice.

Show that $(a \vee b)' = a' \wedge b'$ and $(a \wedge b)' = a' \vee b'$.

Proof : We use the uniqueness of complement.

Consider $(a \vee b) \vee (a' \wedge b')$

$= (a \vee b \vee a') \wedge (a \vee b \vee b')$

$= ((a \vee a') \vee b) \wedge (a \vee 1)$

(By associativity, commutativity and distributivity)

$= (1 \vee b) \wedge (a \vee 1) = |\wedge| = 1$

Next consider $(a \vee b) \wedge (a' \wedge b')$

$= ((a \wedge a') \wedge b') \vee (b \wedge b' \wedge a')$

(Using distributivity, commutativity and associativity)

$= (0 \wedge b) \vee (0 \wedge a')$

$= 0 \vee 0 = 0$

Hence $a' \wedge b'$ satisfies the conditions for complement of $a \vee b$. By uniqueness of complement, it follows that

$$(a \vee b)' = a' \wedge b'.$$

One can similarly prove that

$$(a \wedge b)' = a' \vee b'.$$

4.13.9 Principle of Duality

Any statement about lattices involving the join and meet operations and the relations $\leq$ = $\geq$ remains true if $\wedge$ is replaced by $\vee$ and $\vee$ by $\wedge$, $\leq$ by $\geq$ and $\geq$ by $\leq$. (= remains as =).

Hence if " $a \vee a = a$ " is true, then so is " $a \wedge a = a$".

If " $a \wedge b \leq a$ " is true, then so is " $a \vee b' \geq a$ " (or " $a \leq a \vee b$") is true.

If a lattice has universal bounds 0 and 1, then in the dual statement, 0 is replaced by 1 and 1 is replaced by 0.

This concept is known as the principle of duality. Use this principle in the following theorem.

Theorem 1 : If the meet operation is distributive over the join operation in a lattice, then the join operation is also distributive over the meet operation. The converse is also true.

Proof : It is given that $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

We obtain $(a \vee b) \wedge (a \vee c) = [(a \vee b) \wedge a] \vee [(a \vee b) \wedge c]$

$$= a \vee [(a \vee b) \wedge c]$$

$$= a \vee [(a \wedge c) \vee (b \wedge c)]$$

$$= [a \vee (a \wedge c)] \vee (b \wedge c)$$

$$= a \vee (b \wedge c)$$

Hence we have proved that if the meet operation is distributive over the join operation, the join operation is also distributive over the meet operation.

The converse is obtained by the principle of duality.

SOLVED EXAMPLES

1. Show that in a distributive lattice $(A, \leq)$ if $a \wedge x = a \wedge y$ and $a \vee x = a \vee y$ for some a, then $x = y$.

Solution : By Absorption property

$$x = x \wedge (a \vee x) = (x \wedge a) \vee (x \wedge x) \quad \text{(Distributive law)}$$

$$= (a \wedge x) \vee x \quad \text{(Commutativity and Idempotent)}$$

$$= (a \wedge y) \vee x$$

$$= (a \vee x) \wedge (y \vee x)$$

$$= (a \vee y) \wedge (x \vee y)$$

$$= (a \wedge x) \vee y = (a \wedge y) \vee y = y$$

2. Let $(A, \leq)$ be a lattice with a universal upper and lower bounds 0 and 1. For any element $a \in A$, prove

$$a \vee 1 = a, \qquad a \wedge 1 = a$$

$$a \vee 0 = a, \qquad a \wedge 0 = 0$$

Solution : $a \vee 1 = \text{lub } \{a, 1\}$

Hence by definition $1 \leq a \vee 1$, but by definition glb, $a \vee 1 \leq 1$

Hence $a \vee 1 = 1$

$a \vee 0 = \text{lub } \{a, 0\}$

Hence $a \leq a \vee 0$

Now $0 \leq a$ for any $a \in A$

and $a \leq a$.

Hence a is an upper bound for $\{a, 0\}$. But $a \vee 0$ is the least upper bound for $\{a, 0\}$.

Hence it follows that $a \vee 0 \leq a$.

By reflexivity of $\leq$, we obtain

$$a \vee 0 = a$$

$a \wedge 1 = a$ is the dual statement of

$a \vee 0 = a$, and

$a \wedge 0 = 0$ is the dual of

$$a \vee 1 = 1$$

Hence by the principle of duality, these statements are also true.

3. Let $(A, \leq)$ be a lattice. For any $a, b \in A$, prove that $a \leq b$ iff $a \wedge b = a$ iff $a \vee b = b$. Let $a \leq b$. Then a is a lower bound for $\{a, b\}$. But $a \wedge b = \text{glb}\,\{a, b\}$. Hence $a \leq a \wedge b$. By definition of $a \wedge b$, $a \wedge b \leq a$. Hence by reflexivity of $\leq$, $a \wedge b = a$.

Conservely let $a \wedge b = a$.

Hence $\quad a = \text{glb}\,\{a, b\}$

Therefore $\quad a \leq b.$

Similarly, we can prove that $a \leq b$ iff $a \vee b = b$.

4. Prove that in a complemented distributive lattice, if $b \wedge \bar{c} = 0$, then $b \leq c$.

Solution :

$$\begin{aligned} b &= b \wedge 1 = b \wedge (c \vee \bar{c}) \\ &= (b \wedge c) \vee (b \wedge \bar{c}) \\ &= (b \wedge c) \vee 0 \\ &= b \wedge c \end{aligned}$$

But $\quad b \wedge c = \text{glb}\,\{b, c\}$

Hence $\quad b \leq c.$

5. Show that the set of all divisors of 70 forms a lattice.

Solution : Let $A = \{1, 2, 5, 7, 10, 14, 35, 70\}$, and let '$\leq$' is " a divisor of ".

The join operation $\vee = \text{lcm}\,\{a, b\}$ and meet $\wedge = \text{gcd}\,\{a, b\}$.

The universal upper bound '1' is 70 and the lower bound '0' is 1.

The Hasse diagram is

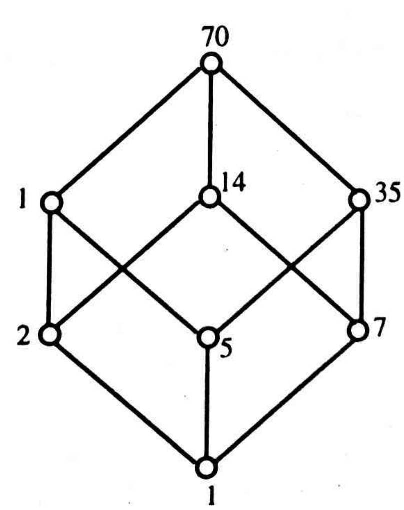


Fig. 4.38

8. For the set X = {2, 3, 6, 12, 24, 36}, a relation $\le$ is defined as $x \le y$ if x divides y. Draw the Hasse diagram for (X, $\le$). Answer the following :

(i) What are the maximal elements ?

(ii) What are the minimal elements ?

(iii) Give one example of chain and one example of antichain.

(iv) What is the maximum length of chain ?

(v) Is the poset a lattice ?

Solution :

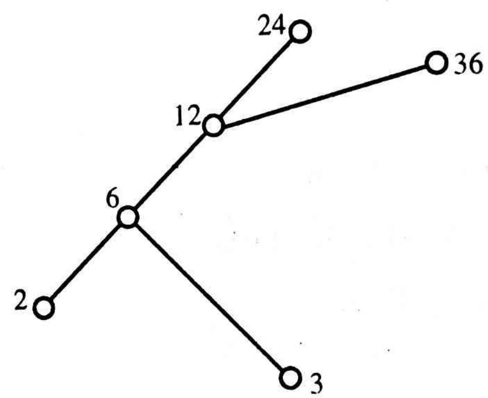


Fig. 4.25

(i) Maximal elements are 24, 36.

(ii) Minimal elements are 2, 3.

(iii) Chain : {2, 6, 12, 24}

Antichain : {2, 3} or {24, 36}.

(iv) Maximum length of chain is 3.

(v) The poset is not a lattice since the set {2, 3} has no greatest lower bound.

The set {24, 36} has no least upper bound.

Hence (A, ≤) is a lattice.

6. For any a, b, c, d in a lattice (A, ≤), if (a ≤ b) and (c ≤ d) then prove

(i) $a \vee c \leq b \vee d$, (ii) $a \wedge c \leq b \wedge d$.

Solution : (i) $a \leq b$ and $b \leq b \vee d$.

Hence $a \leq b \vee d$. (by transitivity of ≤)

Similarly $c \leq d$ and $d \leq b \vee d$

$c \leq b \vee d$.

Hence $b \vee d$ is an upper bound for {a, c}. But by definition

$$a \vee c = \text{lub}\,\{a, c\}$$

Hence $a \vee c \leq b \vee d$.

(ii) is proved on similar lines.

7. For set A = {a, b} and lattice (P(A), ⊆), construct the tables for ∨ and ∧.

$$P(A) = \{A, \phi\ \{a\}, \{b\}\}$$

$$A \equiv 1 \text{ and } \phi = 0$$

$$\vee \equiv \cup \text{ (union)}, \quad \wedge \equiv \cap \text{ (intersection)}$$

We have the following tables for ∨ and ∧ operations.

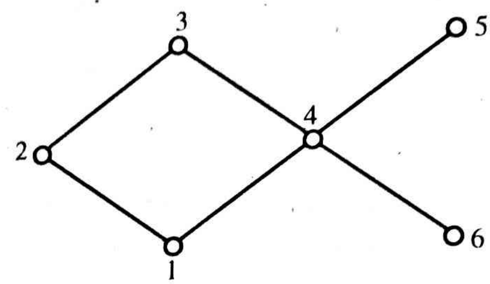
∨	0	1	{a}	{b}
0	0	1	{a}	{b}
1	1	1	1	1
{a}	{a}	1	{a}	1
{b}	{b}	1	1	{b}

∧	0	1	{a}	{b}
0	0	0	0	0
1	0	1	{a}	{b}
{a}	0	{a}	{a}	0
{b}	0	{b}	0	{b}

EXERCISE – IV

1. Determine all the maximal and minimal elements of the poset.

(a)



(b)

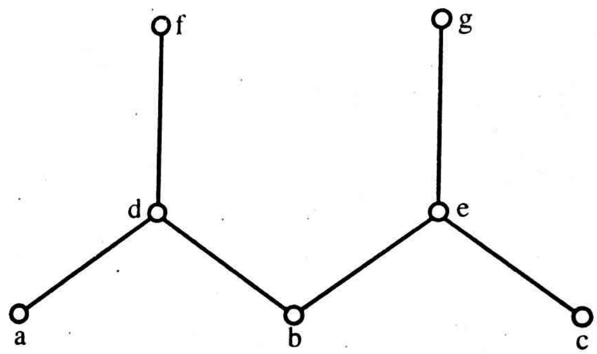


Fig. 4.39

2. In (1) above, determine, if any, least upper bound and greatest lower bound of the following sets :

(a) {2, 4, 6}, {3, 5}, {1, 6}, {2, 3, 5}.

(b) {d, e}, {f, g}, {a, d, e}, {d, g}.

3. On each of the following sets, let the partial order $\leq$ denote ' is a divisor of '. Draw the corresponding Hasse diagrams and determine which posets are lattices.

(i) A = {1, 2, 3, 5, 30}

(ii) A = {1, 2, 3, 4, 6}

(iii) A = {2, 3, 4, 16, 12, 24, 36}

(iv) A = {1, 3, 5, 9, 15, 45}

(v) A = {2, 3, 5, 7, 10, 14, 21}

4. Is the Cartesian product of two lattices always a lattice ? Prove your claim.

5. If a and b are elements in a lattice $(A, \leq)$, then show that $a \wedge b = b$ iff $a \vee b = a$.

6. Prove the associative laws for a lattice.

i.e. $a \vee (b \vee c) = (a \vee b) \vee c$ and

$a \wedge (b \wedge c) = (a \wedge b) \wedge c$

7. For elements a, b, c in a lattice $(A, \leq)$, show that if $a \leq b$ then $a \vee (b \wedge c) \leq b \wedge (a \vee c)$.

8. Show that a lattice $(A, \leq)$ is distributive iff for any element a, b, c in A,

$(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) = (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$

Hint : To show that $(A, \leq)$ is distributive, consider the elements a, $b \vee c$ and $(a \vee b) \wedge (a \vee c)$.

9. A lattice $(A, \leq)$ is called a modular lattice if for any a, b, c in A where $a \leq c$, $a \vee (b \wedge c) = (a \vee b) \wedge c$.

Show that a lattice is modular iff $a \vee (b \wedge (a \vee c)) = (a \vee b) \wedge (a \vee c)$.

10. Show that for any elements a, b, c in a modular lattice,

$$(a \vee b) \wedge c = b \wedge c \text{ implies}$$

$$(a \vee b) \wedge a \equiv b \vee a.$$

4.14 FUNCTIONS

In this section, we deal with functions which form a special class of binary relations. We shall discuss the various properties of functions and focus our attention on some special types of functions.

4.14.1 Definitions

Let A and B be non-empty sets. A **function** f from A to B, denoted as $f : A \rightarrow B$, is a relation from A to B such that for every $a \in A$, there exists a **unique** $b \in B$ such that $(a, b) \in f$.

Normally if $(a, b) \in f$, we write $f(a) = b$.

An important point to be reemphasised is that f is a relation with the following special property :

If $f(a) = b$ and $f(a) = c$ then $b = c$.

This condition implies that to each element $a \in A$, a unique element $b \in B$ should be assigned by the relation f.

Consider the following relation

$$f : R^+ \rightarrow R \text{ where}$$

$$f(x) = \sqrt{x}$$

f is obviously not a function since $f(4) = +2$ as well as $-\sqrt{2}$.

Hence in general only a many-to-one relation or a one-to-one relation is a function. A one-to-many relation is not a function.

The set A is called as the **domain** of f, denoted **D(f)**. The set B is called as the **codomain**, and the set $\{ f(a) \mid a \in A \}$, which is a subset of B, is called as the **range** of f, and denoted as **R(f)**. The element a is called an **argument** of the function f and f(a) is called the **value** of the function for the argument a.

As t is a relation we may also express f as a set of ordered pairs, i.e.

$$f = \{ (a, f(a)) \mid a \in A, f(a) \in B \}.$$

Functions are also called as **mappings** or **transformations**, since they can be thought of as rules for assigning to each element $a \in A$, the unique element $f(a) \in B$. In this context, it is customary to refer to f(a) as **image** of a and a as the **pre-image** of b which is equal to f(a).

A typical way of representing a function graphically is given below.

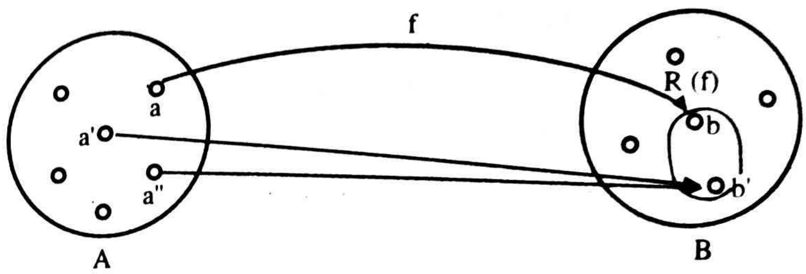


Fig. 4.40

$R(f) \subseteq B$, i.e. range of f is a proper subset of B or is equal to B.

To describe a function completely it is necessary to specify its domain, codomain and the value f(a), for each argument a.

Examples:

1. Let A be a non-empty set. Then we can always define a function $f : A \to A$ (i.e. $B = A$) as $f(a) = a$.

   f is called the **identity** function on A and is denoted by $\mathbf{1_A}$.

2. Let $f : N \to N$ be defined as $f(x) = 2x$.

   f is a function with R(f) being the set of even natural numbers.

3. Let $f : \mathbf{R} \to \mathbf{R}$ be defined as $f(x) = x^2$.

   f is a function. Geometrically, R(f) is the parabola $y = x^2$.

4. Let $A = \{1, 2, 3\}$ and $B = \{a, b, c, d\}$

   Let $f: A \to B$ be defined as

   $f(1) = a$

   $f(2) = c$

   $f(3) = a$

f is a function, which is graphically represented as

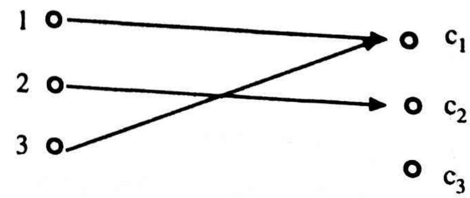


Fig. 4.41

$$R(f) = \{a, c\}.$$

5. Let $A = \{a, b, c\}$ and $B = \{e, f\}$

Let $R = \{(a, e), (b, e), (a, f), (c, e)\}$

The graph of R is

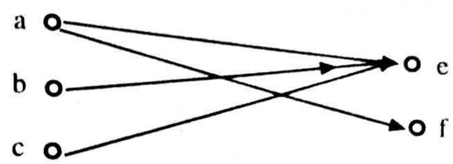


Fig. 4.42

The relation R is not a function since f(a) = e as well as f(a) = f, which violates the definition of a function.

To the computer engineer, a function is a procedure which gives a unique output for any suitable input. The next example demonstrates this aspect.

6. Let P be a computer program that accepts an integer as input and produces an integer as output. Let A = B = Z. Then P determines a relation f_P as follows : $(a, b) \in f_P$ implies that b is the output produced by program P when the input is a. f_P is clearly a function, since any particular input corresponds to a unique output.

7. Let A be a finite set and let P(A) denote its power set. Define

$$f : P(A) \rightarrow Z^+ \text{ by}$$

$$f(S) = |S|, \text{ for any } S \in P(A), \text{ (i.e. } S \subseteq A),$$

$|S|$ denotes the cardinality of S. f is clearly a function.

4.14.2 Partial functions

In actual application of functions, it is often convenient to treat the domain of a function as a subset of another set known as the **source.** (In this case, the codomain is appropriately called as the **target** set). In other words, the function has the set A as its domain but is not defined for some arguments. This leads to the following definition.

Definition : Let A and B be two sets. A partial function f with **domain A** and codomain B is any function from A' to B where $A' \subset A$. For any element $x \in A - A'$, the value of f(x) is said to be undefined.

A partial function

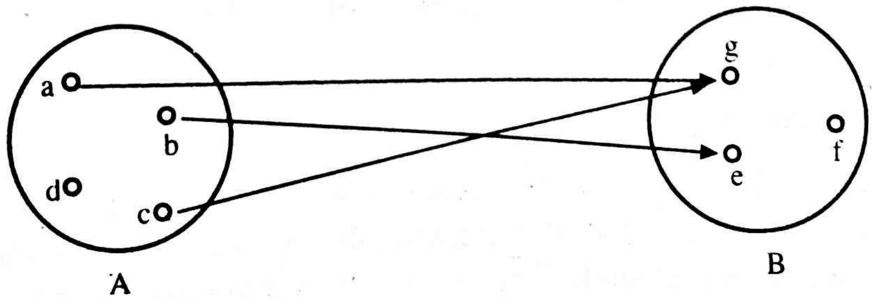


Fig. 4.43

To make the distinction more clear, the function which is not a partial function is sometimes called as a **total function**.

However, in what follows, we will use the unqualified term " function " to denote total function and the qualifier " partial " while referring to partial functions.

4.13.3 Examples

(i) The function $f : \mathbf{R} \to \mathbf{R}$ defined as $f(x) = 1/x$, is a partial function, as it is undefined for $x = 0$.

(ii) The function $f : \mathbf{R} \to \mathbf{R}$ defined as $f(x) = \sqrt{x}$ is a partial function, as $\sqrt{x}$ is not defined for $x < 0$, in **R**.

(iii) Computer programs represent partial functions. Let P be a program which has one natural number as its input, and which, for some input values will never terminate, or terminates abnormally (e.g. while attempting to divide by 0, an illegal operation). Then P is not defined for such arguments and hence can be regarded as a partial function from N to **R**.

The following function plays an important role in computer applications.

(iv) Hashing functions : A symbol table is constructed by a compiler. The identifiers in a program are read and inserted into a table (say) with 1000 spaces, labelled 0 to 999. A unique identifier is called a key. To determine to which space (location) in the table a particular key is assigned we create a hashing function from the set of keys to the set of locations. Hashing functions generally use a mod function.

$$\begin{aligned} \text{Let} \quad I &= \text{set of all possible identifiers} \\ \text{and} \quad N &= \{0, 1, \ldots, 999\}. \text{ We may define} \\ f : I &\to N \text{ as} \\ f(i) &= |i|^3 \pmod{1000} \end{aligned}$$

For example,

$$\begin{aligned} f(23) &= 23^3 \pmod{1000} \\ &= 12167 \pmod{1000} \\ &= 167 \end{aligned}$$

Hence an identifier with length 23 characters, is inserted in position 167.

4.13.4 Equivalent Functions

Definition : Let $f : A \to B$ and $g : C \to D$ be functions. Then f and g are said to be equivalent or **identical** only if $A = C$, $B = D$ and $f(a) = g(a)$ for all $a \in A$.

The function $f : Z \to Z$ given by $f(x) = x$ and $g : Z \to Z^+$ given by $g(x) = x$ are not equivalent.

4.13.5 Composite Function

Let $f : A \to B$ and $g : B \to C$ be two functions. Then the composition of f and g denoted as **gof** is a relation from A to C, where $gof(a) = g(f(a))$. $gof : A \to C$ is also a function. This is because if there exists elements $c, d \in C$ such that $gof(a) = c$ and $gof(a) = d$, for some $a \in A$, this would imply that $g(f(a)) = c$ and $g(f(a)) = d$. But f is a function, hence f(a) is unique. Then since g is also a function, it follows that c = d. Hence gof is a function from A to C. Note that gof is defined only when the range of f is a subset of the domain of g.

The diagram given below depicts a composite function.

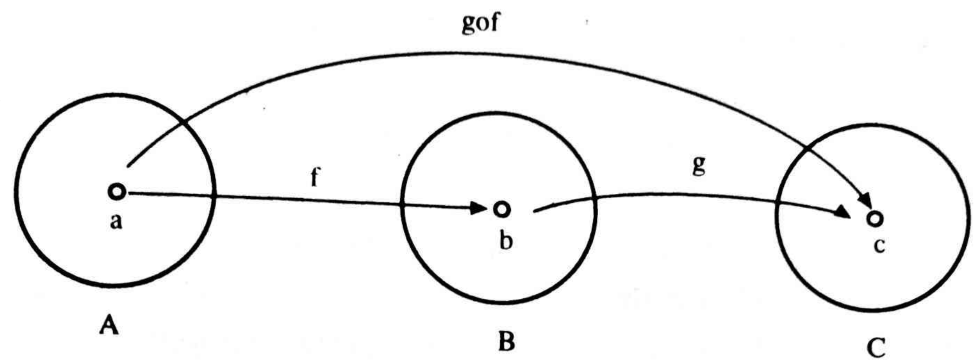


Fig. 4.44

The rule to compose two functions can be extended to a finite number of functions : $f_1 : A_1 \to A_2$, $f_2 : A_2 \to B_3, \ldots, f_n : A_n \to A_{n+1}$, where range of f_i = domain of f_{i+1}, for $1 \le i \le n$.

Thus $f_n \circ f_{n-1} \circ \ldots \circ f_1$ (denoted usually as $f_n\, f_{n-1} \ldots f_1$) is a function from A_1 to A_{n+1}.

In particular if $A_1 = A_2 = \ldots = A_{n+1} = A$ and $f_1 = f_2 = \ldots = f_n = f$ then f o f o ... o f (n times) denoted as f^n is the composite function from A to A.

Examples

1. Let $f : Z \to Z$ be defined as

$$f(x) = x^2 + 2x + 2, \text{ and}$$

$g : Z \to Z$ be defined as

$$g(x) = x - 1$$

Then $gof : Z \to Z$ is defined as

$$gof\,(x) = x^2 + 2x + 2 - 1$$
$$= (x + 1)^2$$

In this case, we also have the function $fog : Z \to Z$ which is defined

$$fog\,(x) = (x - 1)^2 + 2\,(x - 1) + 2$$
$$= x^2 + 1$$

$fof : Z \to Z$ is defined as

$$fof\,(x) = f\,(x^2 + 2x + 2)$$
$$= (x^2 + 2x + 2)^2 + 2\,(x^2 + 2x + 2) + 2$$

$gog : Z \to Z$ is defined as

$$gog\,(x) = g\,(x - 1) = x - 1 - 1 = x - 2.$$

2. Let $f : Z \to \mathbf{R}$ be defined as $f(x) = \dfrac{(x + 1)}{2}$, and

$g : \mathbf{R} \to \mathbf{R}$ be defined as

$$g(x) = x^2$$

Then $gof : Z \rightarrow \mathbf{R}$ is defined as $gof(x) = g(f(x))$

$$= g\left(\frac{x+1}{2}\right) = \frac{(x+1)^2}{4}$$

3. Let A = set of students (or their names)

B = set of their examination seat numbers,

and C = set of the students mark lists.

$f : A \rightarrow B$ is defined as

$f(s)$ = n, where n is the seat number of the students

$g : B \rightarrow C$ is defined as

$g(x) = l$, where l is the mark list corresponding to the seat number n.

Then $gof : A \rightarrow C$ is the function defined as $gof(s) = l$, l being the mark list of the students whose seat number is n.

4. Suppose a manufacturer has a list of all the parts which are supplied to him, together with the supplier's name. He has also a list of suppliers names, together with the suppliers addresses. To obtain the address from which to order a given part, he composes two functions $f : P \rightarrow S$ and $g : S \rightarrow A$ to obtain $gof : P \rightarrow A$, which gives the address a of the supplier of part p.

5. Let $f : A \rightarrow B$, $g : B \rightarrow C$ and $h : C \rightarrow D$ be defined as shown in the following graphs.

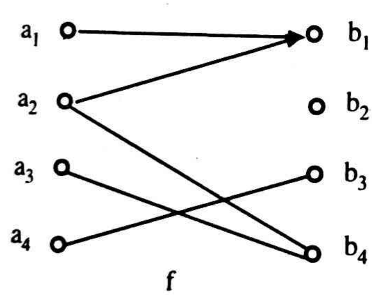


Fig. 4.45

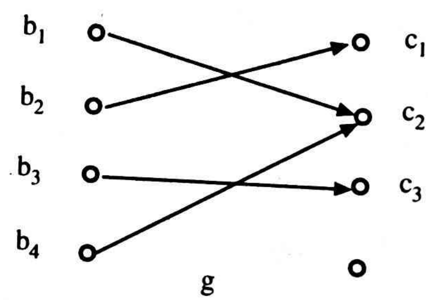


Fig. 4.46

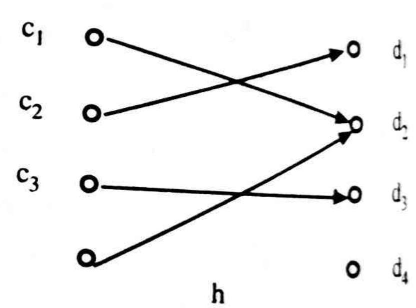


Fig. 4.47

Then we have the following composite functions

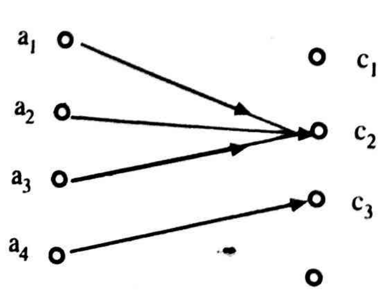


Fig. 4.48

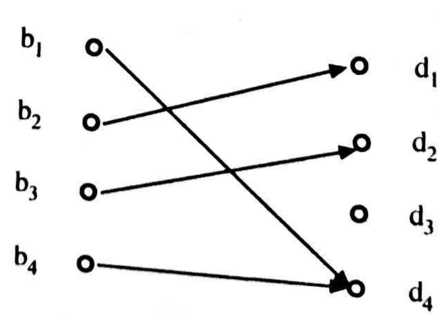


Fig. 4.49

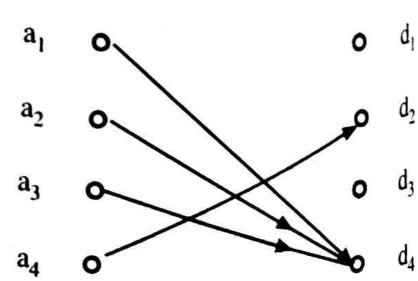


Fig. 4.50

Note that composition is associative. Hence ho (gof) = (hog) of. Hence dispersing with the brackets, we simply write hogof.

4.14.6 Special Types of Functions

Definitions : Let $f : A \rightarrow B$ be a function.

(i) f is called a **surjective** (onto) function if $f(A) = B$, i.e. range of f is equal to the codomain of f.

(ii) f is called an **injective** (one-to-one) function if for elements a, $a' \in A$, $a \neq a'$ implies $f(a) \neq f(a')$, or equivalently if $f(a) = f(a')$, then $a = a'$.

(iii) f is called bijection (one-to-one and onto) function if f is both surjective and injective.

Functions with these properties are called **surjections, injections** and **bijections** respectively.

The following diagrams represent these three types of functions.

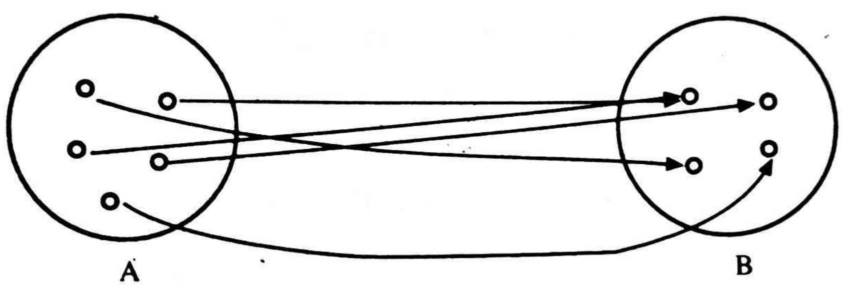


Fig. 4.51 : Surjection

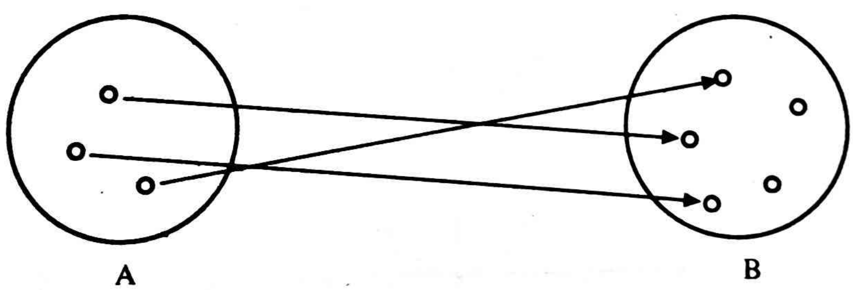


Fig. 4.52 : Injection

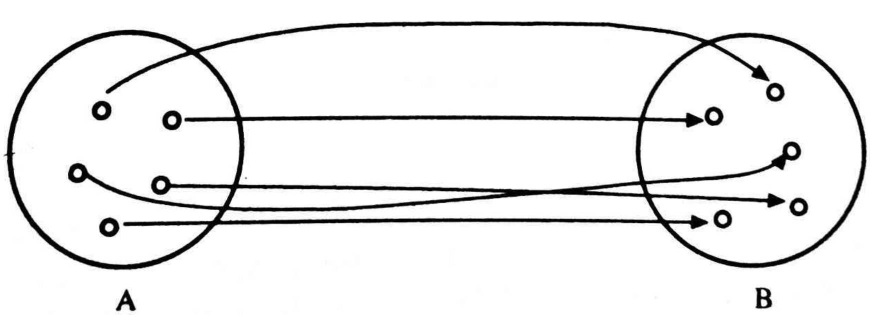


Fig. 4.53 : Bijection

Examples

(i) The identity function $1_A : A \rightarrow A$ is both surjective and injective, hence is a bijective function.

(ii) Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be defined as $f(x) = x + 1$. Then f is an injective function. We know that geometrically, this function represents a straight line.

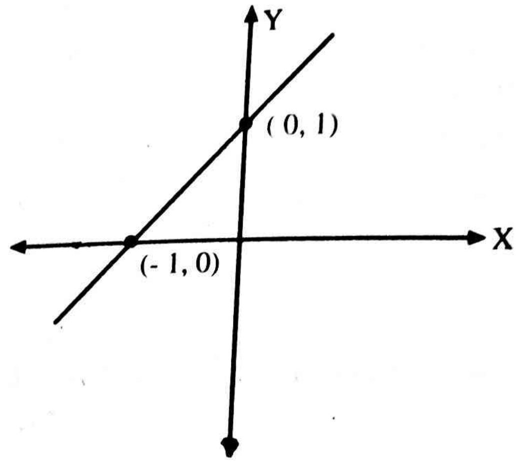


Fig. 4.54

In general, any function f defined as $f(x) = mx + c$ is an injective function.

(iii) Let $f : \mathbf{R} \rightarrow \mathbf{R}^+$ be defined as $f(x) = x^2$

This is a surjective function, since for any $y \in \mathbf{R}^+$, (i.e. $y > 0$) there exists $x \in \mathbf{R}$ such that $f(x) = y$, i.e. $x = \pm \sqrt{y}$.

Geometrically this function represents the parabola $y = x^2$, symmetric about the y-axis.

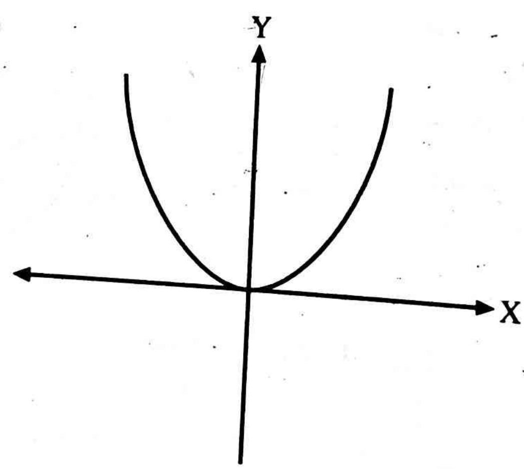


Fig. 4.55

Clearly the function is not injective.

(iv) The function in example (ii) is also a surjective function, since for any $y \in \mathbf{R}$ there exists $x \in \mathbf{R}$ such that $f(x) = y$, i.e. $x + 1 = y$ which implies $x = y - 1$. This means that for every element y in the co-domain **R**, there is a pre-image x in the domain **R**, whose image is y. Hence the function f is surjective. f is therefore a bijective function.

(v) Let E be the set of even integer and define a function $f : E \rightarrow Z$ as $f(x) = 2x$. Then f is an injective function, which is not surjective.

(vi) Let A be the set of students in a class, and B be the set of their roll numbers. Assign to a student his or her roll number. The assignment is then a bijective function.

(vii) Let A be the set of students and B be the set of their ages (in years). Assign to a student his or her age. This assignment is then a many-one function (i.e. not injective) which is surjective.

(viii) Let A be the set of students and B be the set of integers {0, 1, 2, ... 100}. Assign to a student an integer which is his marks (out of 100) in a particular subject, assuming that all the students in the set, appeared for this subject. Then this assignment is a function which is not necessarily injective or surjective.

The following theorems, give some important properties of the injective, surjective and bijective functions, for composite functions.

Theorem 1 : Let $f : A \to B$ and $g : B \to C$ be functions. Then

(i) If f and g are surjective functions, then gof is surjective.

(ii) If f and g are injective, then gof is injective.

(iii) If f and g are bijective, then gof is bijective.

Proof : (i) $gof : A \to C$. We have to show $gof(A) = C$. Let $c \in C$; then since g is surjective, there exists an element $b \in B$ such that $g(b) = c$. Since f is surjective as well, for the element $b \in B$, there exists an element $a \in A$ such that $f(a) = b$. Then $gof(a) = g(f(a)) = g(b) = c$. Hence $c \in gof(A)$, i.e. gof is surjective.

(ii) Let elements $a, a' \in A$ such that $gof(a) = gof(a')$. We have to prove $a = a'$. Let $f(a) = b$ and $f(a') = b'$, where elements $b, b' \in B$. Then $gof(a) = gof(a')$ implies $g(b) = g(b')$. But g is injective; hence $b = b'$. This implies $f(a) = f(a')$. Since f is injective, we have $a = a'$. Hence gof is injective.

(iii) Since (i) and (ii) are true, it follows that gof is bijective.

However, converse to the theorem is not true; as shown by the following examples :

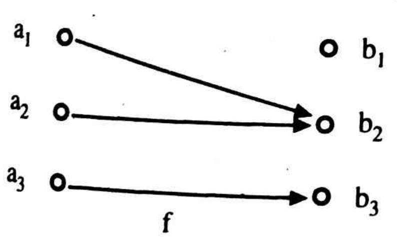


Fig. 4.56

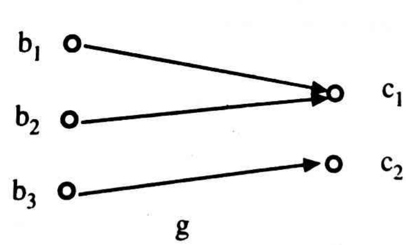


Fig. 4.57

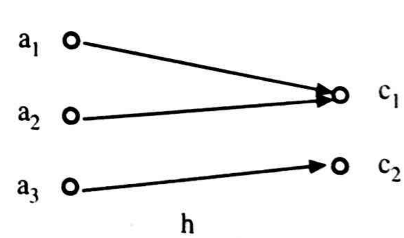


Fig. 4.58

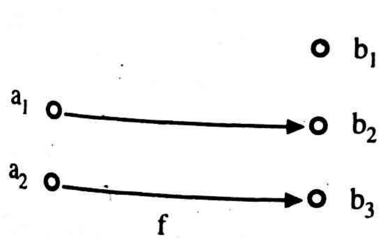


Fig. 4.59

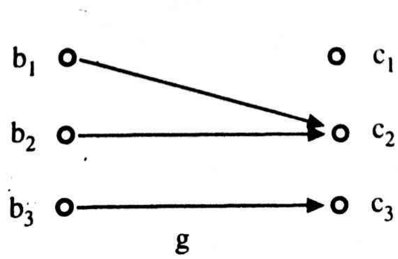


Fig. 4.60

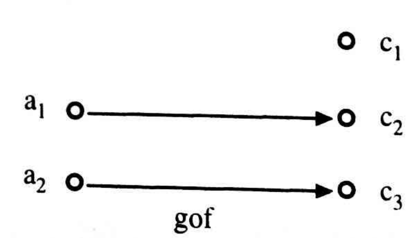


Fig. 4.61

The following theorem, however gives a " partial converse " to the above theorem.

Theorem 2 : Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be functions. Then

(i) If gof is surjective, then g is surjective.

(ii) If gof is injective, then f is injective.

(iii) If gof is bijective, then g is surjective and f is injective.

Proof : (i) We have to prove $g(B) = C$. Let $c \in C$. Since gof is surjective, there exists an element $a \in A$ such that gof $(a) = c$, i.e. $g(f(a)) = c$. Let $f(a) = b$. Then $g(b) = c$ which implies that b is the pre-image of c in B. Hence $g(B) = C$, i.e. g is surjective.

(ii) Let for elements $a, a' \in A$, $f(a) = f(a')$. We have to prove $a = a'$. Since $f(a) = f(a')$, it follows that $g(f(a)) = g(f(a'))$ i.e. gof (a) = gof (a'). Since gof is injective, it follows that $a = a'$. Hence f is injective

(iii) This statement is true, as consequence of (i) and (ii).

4.14.7 Inverse function

The concept of inverse of a function is analogous to that of the converse of a relation.

Definition : Let $f : A \rightarrow B$ be a bijection from A to B. The inverse of f denoted by f^{-1} is the function $f^{-1} : B \rightarrow A$ such that

$$f^{1} o f = 1_A \text{ and}$$
$$f o f^{-1} = 1_B$$

Example : Let $f : \{a_1, a_2, a_3\} \rightarrow \{b_1, b_2, b_3\}$ be defined as

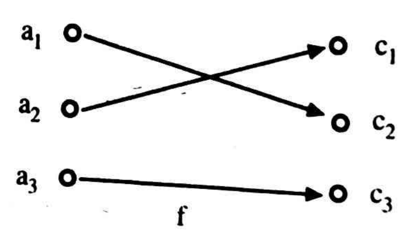


Fig. 4.62

Then f^{-1} is given by the graph

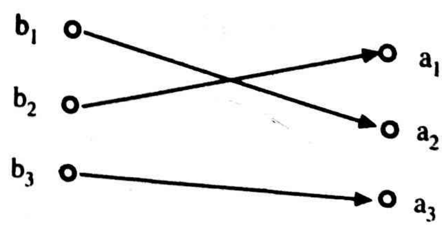


Fig. 4.63

Then $f^{-1} o f = 1_A$ and $f o f^{-1} = 1_B$.

1. **Properties of the inverse function**

(i) $(f^{-1})^{-1} = f$. (proof left as an easy exercise)

(ii) If f and g are bijective functions from A to B, and B to C respectively, then $(gof)^{-1} = f^{-1} o f^{-1}$.

Proof : First observe that both $(gof)^{-1}$ and $f^{-1} o^{-1}$ are functions from C to A. Hence domains of $(gof)^{-1}$ and $f^{-1} o g^{-1}$ are equal; and so are their codomains. We have only to prove $(gof)^{-1}(c) = f^{-1} o g^{-1}(c)$ for all $c \in C$.

Let $(gof)^{-1}(c) = a$, then

$(gof)(a) = c$ which means that $g(f(a)) = c$ since g^{-1} exists, we have $f(a) = g^{-1}(c)$. Since f^{-1} also exists, $a = f^{-1}(g^{-1}(c))$

$= f^{-1} o g^{-1}(c)$.

Hence $(gof)^{-1}(c) = f^{-1} o g^{-1}(c)$ for all $c \in C$.

Hence $(gof)^{-1} = f^{-1} o g^{-1}$

2. **One-sided Inverse Functions**

We have seen that if $f\ A \rightarrow B$ is a bijective function, then f^{-1} exists and $f^{-1} o f = 1_A$ and $f o f^{-1} = 1_B$.

We then say that f has a **left inverse** as well as a **right inverse**. Only bijections have a two sided inverse. However, there are some functions which possess one-sided inverses. The existence of these one-sided inverses are determined by whether the function is injective or surjective.

Definition : Let $f : A \rightarrow B$ and $g : B \rightarrow A$ be functions. If $gof = 1_A$ then g is a left inverse of f, while f is a right inverse of g.

We have the following theorem.

Theorem 1 : Let $f : A \rightarrow B$ be a function $(A \neq \phi, B \neq \phi)$

(i) f has a left inverse if and only if f is injective.

(ii) f has a right inverse if and only if f is surjective.

Proof : Assume first that f is injective. We have to show that f has a left inverse, i.e. we must define a function $g : B \rightarrow A$ such that $gof = 1_A$. Let $b \in B$. Then either $b \in f(A)$ or $b \in B - f(A)$. If $b \in f(A)$, then $b = f(a)$ for some $a \in A$. In that case, define $g(b) = a$. Otherwise, if $b \in B - f(A)$, choose any arbitrary element $a' \in A$ and define $g(b) = a'$. The function g is well-defined since exactly one value is specified for each $b \in B$ as f is injective. Then $gof(b) = g(b) = a$. Hence $gof = 1_A$. Conversely, let f have a left inverse. We have to show that f is injective. Let $g : B \rightarrow A$ such that $gof = 1_A$. Let $f(a) = f(a')$. We have to show $a = a'$. Now $g(f(a)) = g(f(a'))$ since g is well defined. This implies $1_A(a) = 1_A(a')$ i.e. $a = a'$. Hence f is injective.

(ii) Let f be surjective. We have to show that f has a right inverse, i.e. we must define a function $g : B \rightarrow A$ so that $fog = 1_B$. Let $b \in B$. Since f is surjective, there exists an element $a \in A$ such that $f(a) = b$. Define $g(b) = a$. Then $fog(b) = f(a) = b$. Hence $fog = 1_B$.

Conversely, let f have a right inverse $g : B \rightarrow A$ such that $fog = 1_B$. We have to show that f is surjective. Let $b \in B$ and let $g(b) = a$. Then fog **(b) = b which** implies that $f(a) = b$. Hence f is surjective, as there exists an element $a \in A$ such **that f(a) = b.**

The next theorem deals with bijective function.

Theorem 2 : If $f : A \rightarrow B$ is bijective, then the left and right inverses of f are equal.

Proof : Left as an exercise.

SOLVED EXAMPLES

1. A function $f : N \rightarrow N$, where N is the set of natural members including 0. Comment on the type of the following functions (one-one/onto etc.)

(i) $f(j) = j^2 + 2$

(ii) $f(j) = 1$ if j is odd

$= 0$ if j is even.

Solution : (i) f is one-one function since if $f(j) = f(k)$, then $j^2 + 2 = k^2 + 2$ which implies $j = k$.

But f is not onto since there is no natural number $j \in N$, such that $f(j) = 0$, since $j^2 + 2 = 0 \rightarrow j^2 = -2$. Hence 0 has no preimage. Hence f is not onto.

(ii) f is not onto since $R(f) = \{0, 1\}$. f is also not one-one since all odd numbers (even numbers) have the same image.

2. Functions f, g, h are defined on a set

$$X = \{1, 2, 3\} \text{ as}$$

$$f = \{(1, 2), (2, 3), (3, 1)\}$$

$$g = \{(1, 2), (2, 1), (3, 3)\}$$

$$h = \{(1, 1), (2, 2), (3, 1)\}.$$

(i) Find fog, gof. Are they equal ?

(ii) Find fogoh and fohog.

Solution : We may depict f, g, h graphically as

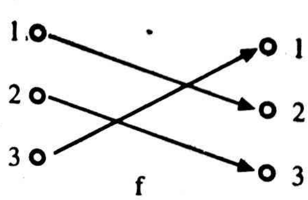


Fig. 4.64

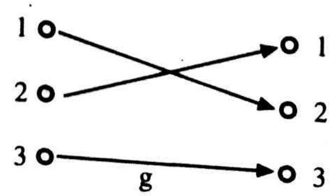


Fig. 4.65

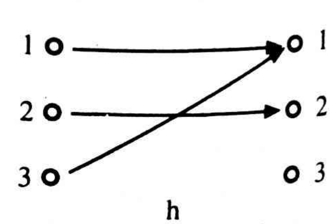


Fig. 6.66

(i) fog is depicted as

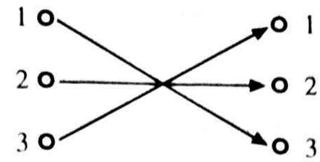


Fig. 4.67

Hence gof = {(1, 3), (3, 1), (2, 2)}

gof is depicted as

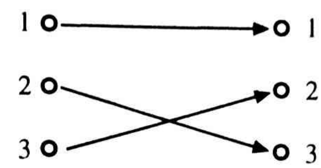


Fig. 6.68

$\therefore$ gof = {(1, 1), (2, 3), (3, 2)}

$\therefore$ fog $\neq$ gof

(ii) fogoh = (fog) oh, which can be depicted as

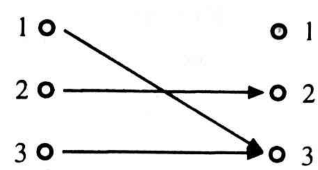


Fig. 4.69

$\therefore$ fogoh = {(1, 3), (2, 2), (3, 3)}

fohog = fo (hog)

hog can be depicted as

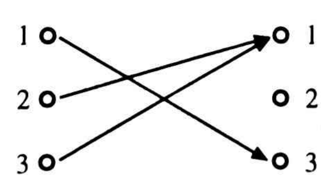


Fig. 4.70

$\therefore$ fo (hog) is

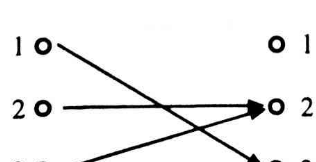


Fig. 4.71

$\therefore$ fohog = {(1, 3), (2, 2), (3, 2)}

3. Let $A = \{a, b, c, d\}$, $B = \{s, t, u\}$, $C = \{l, m, n\}$.

Obtain the composition of the following functions $f : A \to B$, $g : B \to C$

where
$$f = \{(a, s), (b, t), (c, u), (d, t)\}$$
$$g = \{(s, m), (t, l), (u, n)\}.$$

Solution :

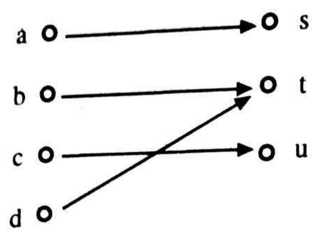


Fig. 4.72

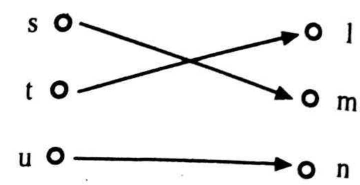


Fig. 4.73

then gof is

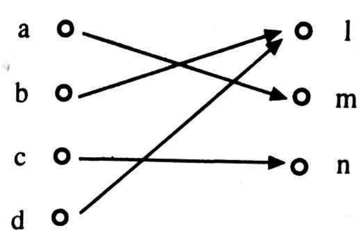


Fig. 4.74

$\therefore$ $gof = \{(a, m), (b, l), (c, n), (d, l)\}$.

4. Let $A = \{1, 2, 3, 4, 5\}$, $g : A \to A$ is as shown in the figure.

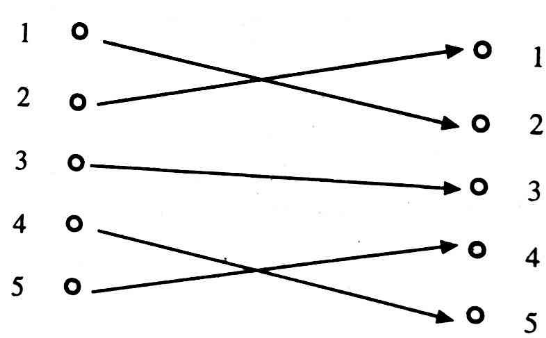


Fig. 4.75

Find the composition gog, go (gog). Determine whether each is one-to-one or onto function.

Solution : gog is

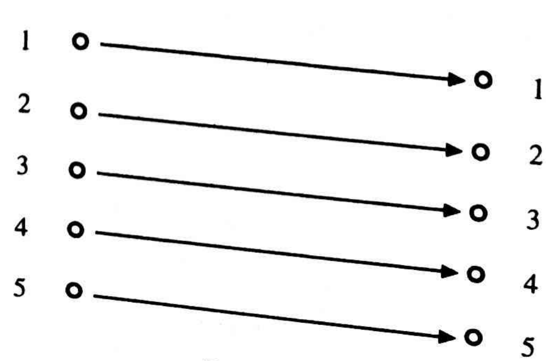


Fig. 4.76

gog is one-one and onto function.

go (gog) is

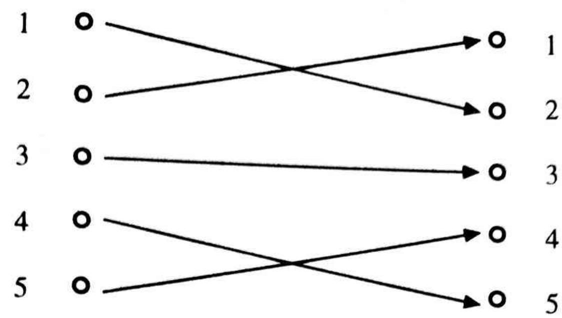


Fig. 4.77

go (gog) is also one-one and onto function.

5. The functions f : A → B, g : B → C, h : C → D are defined in the following diagram. Determine the range of each function. State which functions are into and which are onto. Draw the diagram of the composite function hogof.

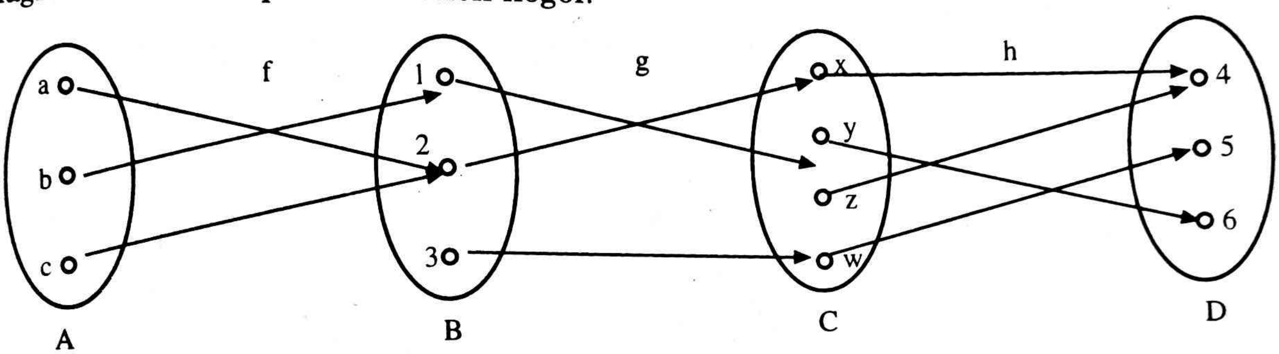


Fig. 4.78

Solution :

R(f)	=	{1, 2},	f is into
R(g)	=	{x, y, w},	g is into
R(h)	=	{4, 5, 6},	h is onto.

gof is

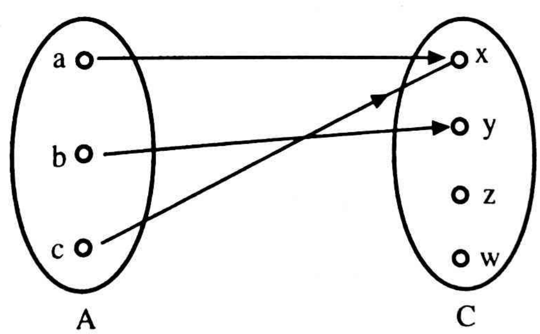


Fig. 4.79

h o (gof) is

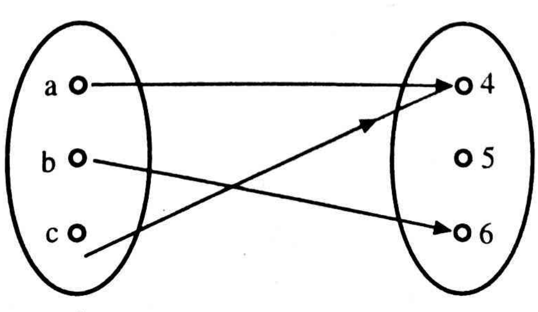


Fig. 4.80

6. State whether the following functions are one-one
   (i) To each person on the earth assign the number which corresponds to his age.
   (ii) To each country assign the number of people living in the country.
   (iii) To each book written by only one author, assign the author.
   (iv) To each country having prime minister, assign the prime minister.

Solution :

(i) Many than one person can have the same age. f is not one-one but many-one.

(ii) More than one country may not have exactly the same population. Hence function is one-one.

(iii) The same author may have written more than one book. Hence function is many-one and not one-one.

(iv) function is one-one.

Ex. 7 : Let $f(x) = x + 2$, $g(x) = x - 2$ and $h(x) = 3x$ for $x \in R$, where R = set of real numbers. Find gof, fog, fof, gog, foh, hog, hof, fohog.

Solution :

$$\begin{aligned}
gof\ (x) &= g\ (f(x)) = g\ (x + 2) \\
&= (x + 2) - 2 = x \\
fog\ (x) &= f\ (g(x)) = f\ (x - 2) \\
&= (x - 2) + 2 = x \\
fof\ (x) &= f\ (f(x)) = f\ (x + 2) \\
&= (x + 2) + 2 = x + 4 \\
gog\ (x) &= g\ (g(x)) = g\ (x - 2) = (x - 2) - 2 = x - 4 \\
foh\ (x) &= f\ (h(x)) = f\ (3x) = 3x + 2 \\
hog\ (x) &= h\ (g(x)) = h\ (x - 2) \\
&= 3\ (x - 2) = 3x - 6 \\
hof\ (x) &= h\ (f(x)) = h\ (x + 2) \\
&= 3\ (x + 2) = 3x + 6 \\
fohog\ (x) &= foh\ (g(x)) \\
&= foh\ (x - 2) = f\ (h(x - 2)) \\
&= f\ (3x - 6) = (3x - 6) + 2 \\
&= 3x - 4
\end{aligned}$$

Ex. 8 : Let $f(x) = 2x + 3$, $g(x) = 3x + 4$, $h(x) = 4x$ for $x \in R$, where R = set of real numbers. Find gof, fog, foh, hof, goh.

Solution :

$$\begin{aligned}
gof\ (x) &= g\ (f(x)) \\
&= g\ (2x + 3) = 3\ (2x + 3) + 4 \\
&= 6x + 13 \\
fog\ (x) &= f\ (g(x)) = f\ (3x + 4) = 2\ (3x + 4) + 3 = 6x + 11
\end{aligned}$$

$$\text{foh}(x) = f(h(x)) = f(4x)$$
$$= 2(4x) + 3 = 8x + 3$$
$$\text{hof}(x) = h(2x + 3) = 4(2x + 3) = 8x + 12$$
$$\text{goh}(x) = g(4x) = 3(4x) + 4 = 12x + 4.$$

Ex. 9 : If $f(x) = x^2 + 1$ and $g(x) = x + 2$ are functions from R to R, where R is the set of real numbers, find fog and gof.

Solution :

$$\text{fog}(x) = f(g(x)) = f(x + 2)$$
$$= (x + 2)^2 + 1 = x^2 + 4x + 5$$
$$\text{gof}(x) = g(f(x)) = g(x^2 + 1)$$
$$= (x^2 + 1) + 2 = x^2 + 3.$$

Ex. 10 : Let $f(x) = ax + b$ and $g(x) = cx + d$, where a, b , c, d are constants. Determine for which constants a, b, c, d it is true that fog = gof.

Solution :

$$\text{fog}(x) = f(g(x)) = f(cx + d)$$
$$= a(cx + d) + b = acx + ad + b$$
$$\text{gof}(x) = g(f(x)) = g(ax + b)$$
$$= c(ax + b) + d = acx + cb + d.$$
$$\therefore \quad \text{fog} = \text{gof} \Rightarrow acx + ad + b$$
$$= acx + cd + d$$
$$\therefore \quad ad + b = cb + d$$
$$\Rightarrow \quad d(a - 1) = b(c - 1)$$
$$\text{i.e.} \quad \frac{b}{d} = \frac{a - 1}{c - 1}$$

is the relation between the constants if fog = gof.

4.14 BIJECTION AND CARDINALITY OF FINITE SETS

Recall that cardinality of a finite set $\left(\text{denoted as } |A|\right)$ is the number of distinct elements in that set. The concept of bijection is a powerful tool to compare the cardinalities of two sets, especially for infinite sets, as we shall see later. For the present, we shall confine ourselves to finite sets. We have the following theorem.

4.14.1 Theorem

Let A and B be finite sets and suppose there is a bijection from A to B. Then $|A| = |B|$.

Proof : Let $|A| = m$ and $|B| = n$.

$$\text{Then } A = \{a_1, a_2, a_3, \ldots, a_m\}$$
$$\text{and } B = \{b_1, b_2, \ldots, b_n\}.$$

Let $f : A \to B$ be the bijection. Then since f is surjective for each $b_i \in B$ $(1 \le i \le n)$, there exists an element $a_i \in A$ such that $f(a_i) = b_i$. This means that $m \ge n$. But f is also injective ; hence for $a_i \ne a_j$, $f(a_i) \ne f(a_j)$, i.e. $b_i \ne b_j$, i.e. $m \le n$. Hence $|A| = |B|$,

Conversely if A and B are finite sets of same cardinality, we have the following theorem.

4.14.2 Theorem

If A and B are finite sets of same cardinality, and $f : A \to B$ is a function then f is injective iff f is surjective.

Proof : Let $f : A \to B$ be injective. Then $|Range\ (f)| = |A|$. As A and B have the same cardinality, and Range (f) $\subseteq$ B, it follows that Range (f) = B. This proves that f is surjective. Conversely, let f be surjective. We have to prove f is injective. Let f(a) = f(a') for elements a, a' $\in$ A. Suppose a $\neq$ a'. Then, assuming $|A| = m\ (= |B|)$, this would imply that the remaining (m – 2) elements of B. This is possible only if atleast one element is mapped onto two different elements in B, which contradicts the fact that f is a function. Hence our supposition that a $\neq$ a' is false. Hence a = a' which means that f is injective.

4.14.3 Pigeonhole Principle

In theorem 4.19.1 we have proved that if A and B are finite sets and a bijection exists from A to B, then their cardinalities are the same.

Hence if A and B are any two sets such that $|A| > |B|$, then no bijection can exist from A to B. This fact is stated as a principle, famously known as the ' Pigeon hole principle '.

This principle states that if there are n + 1 pigeons and only n pigeonholes, then two pigeons will share the same hole.

The pigeonhole principle though self-evident (and seemingly trivial) serves as a powerful tool in solving many intricate problems in counting.

Examples :

1. Show that if seven numbers from 1 to 12 are chosen, then two of them will add upto 13.

Solution : We form the six different sets, each containing two numbers that add upto 13, $A_1 = \{1, 12\}$, $A_2 = \{2, 11\}$, $A_3 = \{3, 10\}$, $A_4 = \{4, 9\}$, $A_5 = \{5, 8\}$, $A_6 = \{6, 7\}$. Each of the seven numbers chosen must belong to one of these sets. Since there are only six sets, by pigeonhole principle, two of the chosen numbers must belong to the same set ; hence their sum is 13.

2. Let S be a square whose sides have length 2 units. Show that for any five points on or inside S, there must be two points whose distance apart is almost $\sqrt{2}$ units.

Solution : Divide the square into four equal squares, as shown in Fig. 4.81. If five points are chosen in the square, we can assign each of them to a square that contains it. If a point belongs to more than one square, we assign it to one of them arbitrarily. Then the five points are assigned to four square regions, so by the pigeonhole principle atleast two points must belong to the same region. These two cannot be more than $\sqrt{2}$ units apart, as the side of each square being 1 units, the length of the diagonal is $\sqrt{2}$ units, which is the maximum distance, that the two points can be apart.

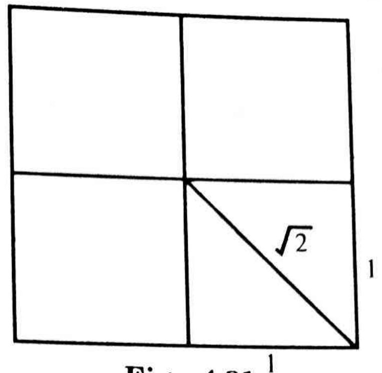


Fig. 4.81

3. Show that if any 51 numbers are chosen from the set {1, 2, ..., 100}, then one of them will be a multiple of the other.

Solution : Every positive integer n can be written as $n = 2^k m$, where m is odd and $k \geq 0$. Since the set contains only 50 odd numbers and 51 numbers are chosen, it follows from pigeonhole principle that two of the numbers chosen will have the same odd factor. Let these numbers be n_1 and n_2. Then $n_1 = 2^{k_1} m$ and $n_2 = 2^{k_2} m$, for some $k_1, k_2 \geq 0$. Then if $k_1 \geq k_2$, n_1 is a multiple of n_2 ; otherwise n_2 is a multiple of n_1.

4. Show that among n + 1 arbitrarily chosen positive integers, there are two whose difference is divisible by n.

Solution : We use Euclid's division algorithm. Given positive integers a and b, we can divide a by b and get a quotient q and remainder r, i.e.

$$a = bq + r.$$

Let $S = \{a_1, a_2, ..., a_{n+1}\}$ be the set of n + 1 arbitrarily chosen positive integers.

Define $f : S \to \{0, 1, 2, ..., n-1\}$

by $f(a_i) = r_i$, the remainder left after dividing by n.

Here $|S| = n + 1$ and cardinality of the co-domain is n. Hence by pigeonhole principle $f(a_i) = f(a_j)$ for $i \neq j$. This means that $r_i = r_j$. Hence $a_i - a_j = n(q_i - q_j)$. This means that there are two integers a_i and a_j in S whose difference is divisible by n.

5. A sports tournament consisting of 45 events is spread over 30 days. There is atleast one event per day. Prove that no matter how the events are arranged there will be a period of consecutive days during which exactly 14 events will take place.

Solution : Let a_i denote the total number of events that takes place upto and including the i-th day. Hence $a_1 \geq 1$ and $a_{30} = 45$, and we have a sequence $a_1, a_2, a_3, \ldots, a_{30} = 45$, which is strictly increasing since there is atleast one event per day. Adding 14 to each term in the sequence, we obtain $a_1 + 14, a_2 + 14, \ldots, a_{30} + 14 = 59$. Now consider the sequence $a_1, a_2, \ldots, a_{30}, a_1 + 14, a_2 + 14, \ldots, a_{30} + 14$, which consists of 60 numbers ranging from 1 to 59. Hence by pigeonhole principle, two of these numbers must be the same. Since $a_i \neq a_j$ for $i \neq j$, it follows that $a_j = a_i + 14$ for some $j > i$. Hence $a_j - a_i = 14$, which means that there is a period of consecutive days from the i-th day, during which exactly 14 games take place.

6. If a set of 16 numbers is selected from {2, ..., 50}, atleast 2 numbers will be in the set with a common divisor greater than 1.

Solution : In the set {2, ..., 50}, there are 15 prime numbers viz. {2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47}.

Suppose in any set of 16 numbers from 2 to 50, no two have a common divisor greater than one. Consider the prime factors of these numbers. By our assumption no two numbers will have common prime factor. This would mean that there should be atleast 16 different prime numbers. This contradicts the fact that there are only 15 prime numbers. Hence our assumption is wrong. Therefore, in any set of 16 numbers from 2 to 50, 2 numbers will have a common divisor greater than 1.

Ex. 7 : Show that among any n + 1 positive integers not exceeding 2n, there must be an integer that divides one of the other integers.

Solution : Let us denote the n + 1 positive integers as $a_1, a_2, \dots, a_{n+1}$. Then we can write each integer a_j as $a_j = 2^{k_j} b_j$, for $j = 1, 2, \dots, n + 1$, where k_j is a non-negative integer and b_j is odd positive integer. For example, $1 = 2^0 \cdot 1$, $2 = 2^1 \cdot 1$, $4 = 2^2 \cdot 1$, $6 = 2^1 \cdot 3$, and so on. Now $b_1, b_2, \dots, b_{n+1}$ are all odd positive integers less than 2n. Since there are only n odd positive integers which are less than 2n, it follows from pigeon-hole principle that $b_i = b_j$ for some i and j. Then $a_i = 2^{k_j} q$ and $a_j = 2^{k_j} q$, where $q = q_i = q_j$. If $k_i < k_j$, a_i divides a_j; otherwise a_j divides a_i. Hence the result.

4.15.4 The Extended Pigeonhole Principle

Let A and B be two non-empty finite sets. If cardinality of A is greater than B, the following theorem (without proof) is a stronger version of the pigeonhole principle.

Theorem : Let $f : A \to B$ be a function and let $|A| = m$, $|B| = n$, $m > n$. Let $k = \left\lfloor \frac{(m-1)}{n} \right\rfloor + 1$.

Then there exist k elements $a_1, a_2, \dots, a_k \in A$ such that $f(a_1) = f(a_2) \dots = f(a_k)$.

Examples :

1. Prove that among 100,000 people, there are two who are born at exactly the same time (hour, minute and second).

Solution : Let A be the set of people (pigeons) and B, the set of seconds (pigeonholes) of one day.

$$|A| = 100{,}000 = m, \quad |B| = 24 \times 3600 = 86400 = n.$$

Then $$k = \left\lfloor \frac{100000 - 1}{86400} \right\rfloor + 1 = 1 + 1 = 2.$$

Hence there are atleast two who are born on the same day.

2. Show that there must be atleast 90 ways to choose six numbers from 1 to 15 so that all the choices have the same sum.

Solution : $m = {}^{15}C_6 = 5005$

The lowest sum of 6 numbers chosen from 1 to 15

$$= 1 + 2 + 3 + 4 + 5 + 6 = 21$$

Highest sum $= 10 + 11 + 12 + 13 + 14 + 15 = 75$

Hence $n = 75 - 21 + 1 = 55$

Hence by the pigeonhole principle

$$k = \left[\frac{m-1}{n}\right] + 1 = \left(\frac{5004}{55}\right) + 1 = 91$$

Hence in atleast 90 ways, we can choose six numbers from 1 to 15 so that all the choices have the same sum.

Ex. 3 : There are 3000 students in a college which offers 7 distinct courses of 4 year's duration. A student who has taken a course in Discrete Mathematics learns that the largest classroom can hold only 100 students. She at once realizes there is a problem. What is the problem ?

Solution : Since there are 7 distinct classes of 4 year's duration, we have $7 \times 4 = 28$ different classes. Hence, by extended pigeon-hole principle, each classroom must hold atleast $\left[\frac{3000-1}{28}\right] + 1 = 107 + 1 = 108$ students. But since the capacity of the largest classroom is only 100, this is obviously a problem.

Ex. 4 : In a group of six people at a party, each pair of individuals consists of two mutual acquaintances or two strangers. Show that there are either three mutual acquaintances or three mutual strangers in the group.

Solution : Let A be one of the six people. Divide the remaining five into two sets, one consisting only of acquaintances of A, the other only of strangers to A. By Extended Pigeon-hole Principle, cardinality of one of the sets must be atleast $[5/2] = 3$. Hence it follows that in the group there are either 3 or more who are acquaintances of A, or there are 3 or more who are strangers to A. Let us assume the former, i.e. say B, C, D are acquaintances of A. Since any pair of individuals are either acquaintances or strangers, if say B, C are acquaintances, then together with A, A, B, C form a group of 3 mutual acquaintances. On the other hand if B, C, D are mutual strangers, they form a group of three mutual strangers.

If we assume the latter, when B, C, D are strangers to A, the proof follows in similar manner.

Ex. 5 : A man hiked for 10 hours and covered a total distance of 45 miles. It is known that he hiked 6 miles in the first hour and only 3 miles in the last hour. Show that he must have hiked atleast 9 miles within a certain period of two consecutive hours.

Solution : Let a_i, $1 \leq i \leq 10$, denote the number of miles hiked by the man during the i^{th} hour. Then $a_1 = 6$, $a_{10} = 3$. Hence $a_2 + a_3 + \ldots + a_9 = 45 - (6 + 3) = 36$ miles. Consider the set

$$A = \{(a_2, a_3), (a_4, a_5), (a_6, a_7), (a_8, a_9)\}$$

of pairs of consecutive hours.

Apply Pigeon-hole Principle to the sum $(a_2 + a_3) + (a_4 + a_5) + (a_6 + a_7) + (a_8 + a_9) = 36$.

Since sum of 4 numbers is 36, value of one number should be atleast 9.

Hence the man must have hiked atleast 9 miles within a certain period of two consecutive hours.

EXERCISE – V

1. Let A = {a, b, c, d}, B = {s, t, v}, C = {1, 2}. Define $f : A \to B$, $g : B \to C$, $h : C \to D$ as shown in the figures.

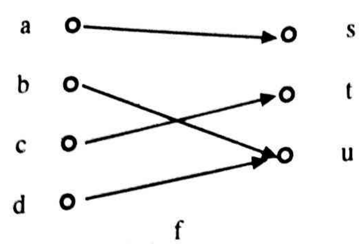


Fig. 4.82

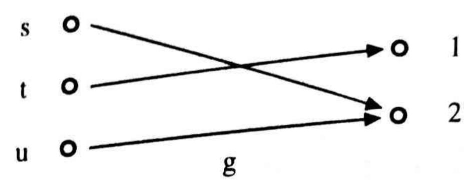


Fig. 4.83

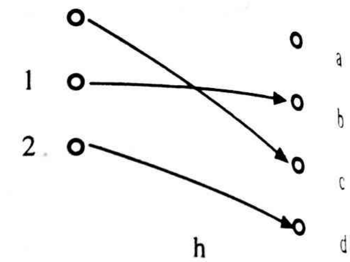


Fig. 4.84

Find the compositions

(i) goh, (ii) foh, (iii) hog, (iv) gof.

Find in each case the domain and codomain.

2. The following diagrams define functions f_1 g_1 h which map the set {1, 2, 3, 4} into itself. Find

(i) range of f, (ii) range of g, (iii) range of h, (iv) fog, (v) hof, (vi) gog.

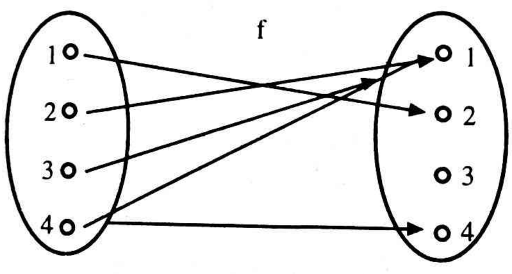


Fig. 4.85

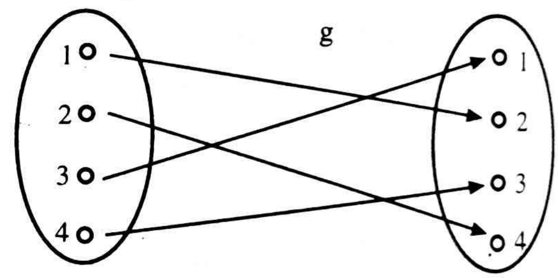


Fig. 4.86

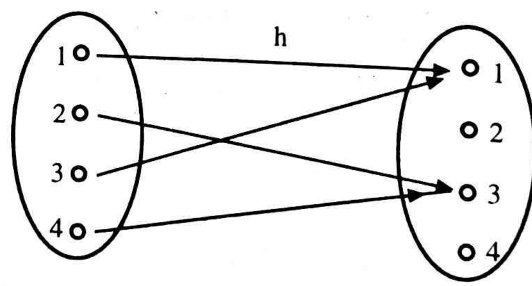


Fig. 4.87

3. Let f, g, h be defined by the following diagrams. Find which of them are injective, surjective or bijective.

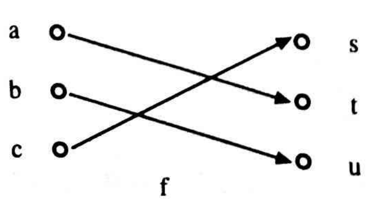


Fig. 4.88

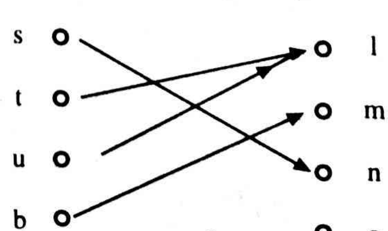


Fig. 4.89

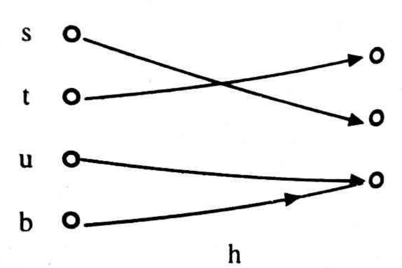


Fig. 4.90

Let $X = \{1, 2, 3\}$, f, g, h, s be functions from X to X given by

$$f = \{(1, 2), (2, 3), (3, 1)\}$$

$$g = \{(1, 2), (2, 1), (3, 3)\}$$

$$h = \{(1, 1), (2, 2), (3, 1)\}$$

$$s = \{(1, 1), (2, 2), (3, 3)$$

Find fohog, sog, fos.

5. Give example one each of the following functions :

(i) A function which is injective but not surjective.

(ii) A function which is surjective but not injective.

(iii) A function which is neither surjective nor injective.

(iv) A function which is bijective.

6. Construct an example to show that for two functions f and g from A to A, $fog \neq gof$.

7. Let f, g, h be functions from N to N where N, is the set of natural numbers including 0, such that

$$f(n) = n + 1$$

$$g(n) = 2n$$

$$h(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

Determine fof, fog, gof, goh, hog, (fog)oh.

8. Show that for any functions $f : A \to B$, $g : B \to C$ and $h : C \to D$,

$$ho\ (gof) = (hog)of$$

9. Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d\}$ and let $f : A \to B$ be given by

$$f = \{(1, a), (2, a), (3, c), (4, a)\}.$$

Determine whether f^{-1} exists. If so, find f^{-1}.

10. Let $f : \mathbf{R} \to \mathbf{R}$, $g : \mathbf{R} \to \mathbf{R}$ be defined by $f(a) = 2a + 1$, $g(b) = b/3$. Verify $(gof)^{-1} = f^{-1}og^{-1}$.

11. Let $A = \{a, b, c, d\}$ and $B = \{1, 2, 3\}$. Determine whether the following relations from A to B is a function. If it is a function, find its range.

(i) $R = \{(a, 1), (b, 2), (c, 1), (d, 2)\}$

(ii) $R = \{(a, 1), (b, 2), (a, 2), (c, 1), (d, 2)\}$

(iii) $R = \{(a, 3), (b, 2), (c, 1)\}$

(iv) $R = \{(a, 1), (b, 1), (c, 1), (d, 1)\}$

12. Determine whether the following functions are one-one, onto (or both or neither).

(i) $A = \{1, 2, 3, 4\} = B$

$$f = \{(1, 1), (2, 3), (3, 4), (4, 2)\}$$

(ii) $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$

$f = \{(1, a), (2, a), (3, c)\}$

(iii) $A = B = \{1, 2, 3, 4, 5\}$,

$f = \{(1, 3), (2, 2), (3, 4), (4, 5), (5, 1)\}$

(iv) $A = \{1, 2, 3, 4, 5\}$, $B = \{a, b, c, d\}$

$f = \{(1, a), (2, c), (3, b), (4, d)\}$

13. Show that one of any m consecutive integers is divisible by m.
14. Suppose the figures from 1 to 12 on a clock dial are reshuffled among themselves. Prove that there exists a pair of adjacent figures which add upto atleast 14.
15. Suppose that at a party, there are atleast 6 persons. Prove that either there exists 3 persons, every two of whom know each other or else there exist 3 persons no two of whom know each other.
16. Suppose 18 students in a class appear at an entrance examination. Prove that there exist two among them whose seat numbers differ by a multiple of 17.
17. If 101 integers are selected from the set {1, 2, ..., 200}, prove that among the selected integers there exist two integers such that one of them is a multiple of the other.
18. Show that if seven points are chosen a region bounded by a regular hexagon of side 1 unit each, two of the points must be no further apart than 1 unit.
19. How many friends must you have to guarantee that atleast five of them will have birthdays in the same month ?

4.16 INFINITE SETS AND COUNTABILITY

Although many problems in counting involve only finite sets, it is the infinite set which plays a more important and significant role in computability theory. It is the infinite set which provides an insight into the limitations of that which can be computed algorithmically. One can show, with the help of infinite sets that there are tasks which cannot be performed by any computer.

This section is therefore devoted to the study of some important infinite sets and their properties.

4.16.1 Infinite Sets

We have an intuitive idea of an ' infinite ' set, it is a set with an infinite number of elements, an inexhaustible storehouse of elements,elements which do not cease to appear; one will never reach the last element in the set. We are also familiar with some infinite sets; the set of natural numbers, set of integers etc. Let us now formally define an infinite set.

Definition : A set A is infinite if there exists an injection $f : A \rightarrow A$ such that f(A) is a proper subset of A. If no such injection exists, the set is finite.

Examples :

1. The set of natural numbers N is an infinite set.

   Consider $f : N \rightarrow N$, where $f(x) = 2x$.

   f(N) is the set of all positive even integers which is a proper subset of N.

2. The set of real numbers **R** is an infinite set.

Define $f : \mathbf{R} \to \mathbf{R}$ as

$$f(x) = x + 1, \qquad \text{if } x \geq 0$$
$$= x, \qquad \text{if } x < 0.$$

Clearly f is an injective function. Note that if $y \in R$ such that $y = x + 1$, then $x = y - 1$. Hence $x \geq 0$ implies $y \geq 1$. Hence range (f) = $\{ y \in \mathbf{R} \mid y < 0 \wedge y \geq 1 \}$, which is a proper subset of **R.**

4.16.2 Properties of Infinite Sets

1. Let B be a subset of A. Then if B is infinite, A is infinite.

Proof : Since B is infinite, we have an injection $f : B \to B$ such that f(B) is a proper subset of B. Define a function $g : A \to A$ as

$$g(a) = f(a), \qquad \text{if } a \in B$$
$$g(a) = a \text{ if } a \in A - B.$$

g is injective and g(A) does not include the elements in B – g(B). Hence A is infinite.

2. If A is infinite, its power set P(A) is infinite.

Proof : The mapping $f : A \to P(A)$, where $f(a) = \{a\}$, $a \in A$ is an injective function.

3. If A and B are sets where A or B is infinite then $A \cup B$ is infinite.

Proof : Define $f : A \to A \cup B$ as $f(a) = a$, $a \in A$

4. If A and B are sets where A (or B) is infinite and $B \neq \phi$ (or $A \neq \phi$) then $A \times B$ is infinite.

Proof : Choose an element $b_0 \in B$ (as $B \neq \phi$) and define $f : A \to A \times B$ as

$$f(a) = (a, b_0)$$

Clearly f is an injective function.

Examples

(i) The set of integers Z is infinite.

Solution : N, the set of natural numbers is a subset of Z, by prop.1, 4.21.2.

(ii) The set of rational numbers Q is infinite.

Solution : N (or Z) is a subset of Q, by prop.1, 4.21.2.

(iii) The set $N \times N$ is infinite.

Solution : N is infinite and product of two infinite sets is infinite.

(iv) The intersection of two infinite sets is not infinite.

Solution : (i) Let E = set of even integers

and O = set of odd integers.

Then $E \cap O = \phi$ which is a finite set

(ii) Let A = [0, 1] and B = [1, 2]

then $A \cap B = \{1\}$.

(v) Let A and B be infinite sets such that $B \subset A$. Is the set $A - B$ necessarily infinite ? Give example to support your assertion.

Solution : No, $A - B$ need not be infinite

Let $A = [0, 1]$ and $B = [0, 1]$

Then $A - B = \{1\}$.

(vi) The set of all strings in {a, b} of prime length.

Solution : The set of prime numbers is infinite. Hence the set of all strings in {a, b} of prime length is also infinite.

4.17 COUNTABILITY

If A is a finite set with cardinality n, we can describe A as $\{a_1, a_2, \ldots, a_n\}$, that is, we can list or **enumerate** the elements in the set. In other words, there exists a bijection $f : A \to \{1, 2, \ldots, n\}$, $\{1, 2, \ldots, n\}$ being a subset of N. This property enables us to " count " the elements in the set. Hence we say that a finite set is **countable.**

We extend the concept of countability to infinite sets as follows.

Definition : An infinite set A is said to be **countable** if there exists a bijection $f : N \to A$.

A countably infinite set is also called a **denumerable** set.

This definition does not mean that we can actually 'count' the elements in a denumerable set and say how many there are ! All it means is that it is theoretically possible to put the elements into an infinite list or sequence.

If A is a denumerable set then we list the elements as f(1), f(2), f(3), or $a_1, a_2, a_3, \ldots$, where $a_i = f(i)$.

One can also compare the 'sizes' of two sets (finite or infinite) by the following definition.

Definition : If A and B are sets and there exists a bijection $f : A \to B$, then A and B have the same cardinality.

We denote the cardinality of N by No (aleph nought). Hence if A is countably infinite then $|A|$ = No.

If A and b are sets and there exists a bijection $f : A \to B$, then if A is countable, so is B and vice versa.

Example : The set of integers Z is countable.

Solution : Define $f : N \to Z$ as

$$f(n) = \frac{(n+1)}{2}, n = 1, 3, 5, \ldots\ldots$$

$$= -\frac{n}{2}, n = 0, 2, 4 \ldots$$

Clearly f is a bijection and hence Z is countable.

4.18 PROPERTIES OF COUNTABLE SETS

4.18.1 Theorem

A subset of a countable set is countable.

Proof : Let B be a subset of A. If B is finite, then it is countable. Assume therefore that B is infinite, in which case A is also infinite. Since A is denumerable, there is a bijection $f : N \to A$. Hence the elements in A can be sequentially arranged as f(1), f(2), f(3) .... Let $b \in B$. Since B is a subset of A, b = f(i) for some $i \in N$. Designate this element as $f(i_1)$. Since B is an infinite set, there exists $b' \in B$, where $b' \neq b$. Since $b' \in A$, we denote this element as $f(i_2)$. Proceeding in this manner, the elements in B can be sequentially arranged as $f(i_1)$, $f(i_2)$, $f(i_3)$, ...

Define a mapping $g : N \to B$ as

$$g(n) = f(i_n)$$

Clearly g is a bijection, and hence B is countable.

This theorem gives us more examples of denumerable sets.

Examples

(i) The set of all prime numbers is denumerable.

(ii) The sets of odd integers and even integers are denumerable.

(iii) For a fixed integer, the set of all integral powers, is denumerable.

4.18.2 Theorem

Let A and B be countable sets. Then $A \cup B$ is countable.

Proof : We shall first assume that A and B are **disjoint** sets. If A and B are finite sets, then $A \cup B$ is countable. If one of them is finite then also $A \cup B$ is countable. We consider the more important case when A and B are both infinite. Since A and B are denumerable sets, there exists bijections $f : N \to A$ and $g : N \to B$. Hence we can express the elements of A as $A = \{a_1, a_2, a_3, ...\}$ where $f(i) = a_i$ and $B = \{b_1, b_2, b_3, ...\}$, where $g(i) = b_i$.

Now define a reverse bijection

$$\begin{aligned} f : A \cup B &\to N \text{ as} \\ f(a_i) &= 2i - 1 \\ \text{and} \quad f(b_i) &= 2i \end{aligned}$$

We have thus put the elements of A in one-one correspondence with the odd numbers and elements of B with the even numbers. As $A \cap B = \phi$, this is really a bijection.

Thus we have proved $A \cup B$ is countable where A and B are disjoint. The theorem is also true if A and B are not necessarily disjoint. This is because we can express $A \cup B = (A - B) \cup (A \cap B) \cup (B - A)$, mutually disjoint union of three sets.

4.18.3 Theorem

If A and B are countable, then $A \times B$ is also countable.

Proof : We prove the theorem for the case when A and B are both infinite, as the remaining cases are easily proved.

Let $A = \{a_1, a_2, ...\}$ and $B = \{b_1, b_2, ...\}$. Construct a bijection $f : N \to A \times B$ inductively as

(i) $f(1) = (a_1, b_1)$

(ii) Suppose $f(p) = (a_m, b_n)$.

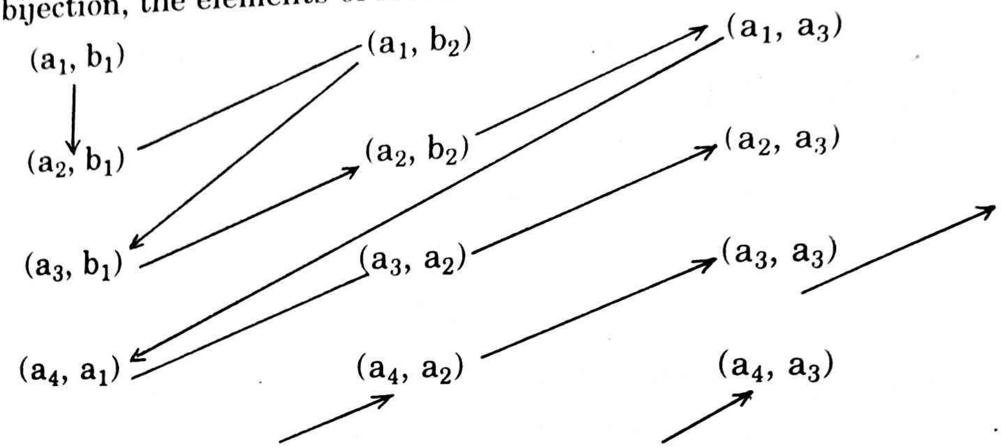
If $m \neq 1$, $f(p+1) = (a_{m-1}, b_{n+1})$

If $m = 1$, $f(p+1) = (a_{n+1}, b_1)$

Hence $f(1) = (a_1, b_1)$, $f(2) = (a_2, b_1)$,

$f(3) = (a_1, b_2)$, $f(4) = (a_3, b_1)$, ...

Since f is a bijection, the elements of A × B are arranged sequentially as



As corollery to the above, theorem if $A_1, A_2, \ldots, A_n$ is a finite collection of countably infinite sets, then $A_1 \times A_2 \times \ldots \times A_n$ is also countably infinite.

Examples

1. The sets $N \times N$, $Z \times Z$, $Z \times N$, $Z \times Z \times Z$ are all countably infinite.
2. Let Q denote the set of rational numbers. Then Q is countable.

Solution : We first make the observation that the rationals can be described as the set of all quotients p/q of integers p and q, where

(i) $q > 0$

(ii) p and q have no positive common factor other than 1.

Define a function $f : Q \to Z \times N$ as $f(p/q) = (p, q)$.

f is an injection. Hence f (Q) is an infinite subset of Z × N and hence countably infinite. Since f(Q) is countably infinite and f is an injection, it follows that Q is also countably infinite.

4.18.4 Theorem

The union of a countable collection of countable sets is countable.

Proof : We prove the theorem only for a countably infinite collection of mutually disjoint countably infinite sets, from which the general result can be easily deduced.

For each set A_i, enumerate its elements as a (i, 1), a (i, 2), a (i, 3), ...

Define a function $f : N \times N \to \bigcup_{i=1}^{\infty} A_i$ as $f(i, j) = a(i, j)$.

As the sets are mutually disjoint, f is a bijection and hence it follows that $\bigcup_{i=1}^{\infty} A_i$ is countably infinite.

The following is an important example from ' formal language '.

Example : Let W = {a, b} and let W^* be the set of all words, formed from a, b of any length k. Then the set W^* is countably infinite.

Solution : We can write W^* as the disjoint union of the following countably infinite collection of countable sets. For each integer $k \geq 0$, let W_k be the set of all words over {a, b} of length k. Then W_R is a finite set and every word in W is an element of W_R for some k. Hence $W = \bigcup_{k=1}^{\infty} W_k$ and therefore countably infinite.

4.18 NON-DENUMERABLE SETS

Thus for all the examples of infinite sets we have seen are countable. One should not however be misled in assuming that every infinite set is countable. We shall now deal with some important sets that are not countable.

4.18.1 Theorem

The set of real numbers **R** is non-denumerable.

Proof : By theorem 4.23.1 it is sufficient to show that there exists a subset of **R** is uncountable. We shall show infact, that the set of real numbers between 0 and 1, i.e. (0, 1) is uncountable. Assume that the set is denumerable (i.e. countably infinite). Then we can arrange the elements as an infinite sequence, each element in the sequence being represented as a unique decimal, without an infinite string of 9's at the end. Thus 0.1239999 will be simply represented as 0.123000. Let the infinite sequence be given by :

$$\begin{aligned}
x_1 &= \cdot a_{11}\, a_{12}\, a_{13} \ldots \\
x_2 &= \cdot a_{21}\, a_{22}\, a_{23} \ldots \\
x_3 &= \cdot a_{31}\, a_{32}\, a_{33} \\
&\vdots \\
x_k &= \cdot a_{k_1}\, a_{k_2}\, a_{k_3} \\
&\vdots \\
&\vdots
\end{aligned}$$

Considering the diagonal elements of the array, construct a number

$$\begin{aligned}
y &= \cdot b_1\, b_2\, b_3 \ldots\ldots \text{ where} \\
b_i &= 0 \text{ if } a_{ii} \neq 0 \\
&= 1 \text{ if } a_{ii} = 0.
\end{aligned}$$

This means that $b_1 \neq a_{11}$, $b_2 \neq a_{22}$, and so on. Hence $b_i \neq a_{ii}$ for any i.

Hence $y \neq x_1$ since it differs from x_1 at the first decimal place,

$y \neq x_2$ since it differs from x_2 at the second place.

and hence in general

$y \neq x_i$ since it differs from x_i at the i^{th} place

Thus we have constructed a number $y \in (0, 1)$ which does not belong to the sequence. Hence the set $(0, 1)$ is non-denumerable, from which it follows that **R** is also non-denumerable.

The cardinality of **R** is denoted by **c.** The choice of **c** is based on the fact that the set $(0, 1)$ is called a **continuum.**

In fact a set A is of cardinality c if there is a bijection from $(0, 1)$ to A.

4.19.2 Corollary

The set of irrational numbers is non-denumerable.

Proof : The set of irrational numbers is $\mathbf{R} - Q = \bar{Q}$.

$$\text{Now } \mathbf{R} = Q \cup \bar{Q}.$$

Q is countable. Hence if $\bar{Q}$ is also countable, then so will be **R**, since union of two countable sets is countable.

Hence the set of irrational numbers is non-denumerable.

The following is another important example of uncountable set.

Examples

The power set of N, P(N) is non-denumerable.

Solution : For any subset A of N, let us represent A by a sequence $(a_1, a_2, a_3, \ldots)$ where $a_k = 1$ if k is an element of A and $a_k = 0$ if k is not an element of A.

Hence {1} is represented by (1, 0, 0, ...), {1, 3} is represented by (1, 0, 1, ...), {3, 5, 6} is represented by (0, 0, 1, 0, 1, 1, ...) and so on.

Now suppose P(N) is denumerable, and let us consider a countably infinite collection of subsets $\{S_1, S_2, \ldots\}$ of N. We shall construct a subset T of N, that is not included in this collection.

Let $k \in T$ iff $k \notin S_k$.

So if $S_3 = (0, 1, 0, \ldots)$, then $3 \notin S_3$ (since if $3 \in S_3$, 1 should appear at the third position of the sequence). Hence $3 \in T$ on the other hand if $S_2 = (1, 1, 0, 0, \ldots)$ then $2 \in S_2$ and hence $2 \notin T$. In this manner, the set T is different from each of the S_i' s and hence not a member of the collection $\{S_1, S_2, \ldots\}$.

Thus any countable collection of subsets of N excludes some subset of N. Hence the set P(n) is non-denumerable.

Note that we have used similar argument to show that the set of real numbers is uncountable. This technique called as 'diagonalisation argument' is used frequently in computability theory.

This technique proves that there are elements which cannot be computed by any computer program, even if a computer of unlimited storage and speed is assumed to exist.

SOLVED EXAMPLES

1. What is the cardinality of the following sets
   (i) $I = \{\ldots -4, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
   (ii) $N \times N$, N is the set of natural numbers.
   (iii) Union of finite number of countable sets.

Solution :

(i) I is countably infinite $\therefore$ $|I| = \aleph_0$.

(ii) $N \times N$ is also countably infinite

$\therefore$ $|N \times N| = \aleph_0$.

(iii) Countably infinite. Hence cardinality is $\aleph_0$.

2. If A and B are denumerable, then prove $A \times B$ is also denumerable.

Solution : See Theorem 4.23.3 for solution.

3. Classify the following into finite, denumerable and non-denumerable.
   (i) Number of trees in India.
   (ii) Set of real numbers between 2 and 3.
   (iii) Number of songs sung by Lata Mangeshkar.
   (iv) Power set of a countably infinite set.

Solution : (i) Since the number of trees is not static but continues to increase, the set is denumerable.

(ii) The set of real number between (2, 3) is not denumerable. We know that (0, 1) is non-denumerable. Define

$$f : (0, 1) \rightarrow (2, 3) \text{ as}$$

$$f(a) = a + 1$$

f is clearly a bijection and since (0, 1) is non-denumerable, (2, 3) must also be non-denumerable.

(iii) The set is finite.

(iv) We know that power set of N is non-denumerable. Hence power set of a countably infinite set is non-denumerable.

4. Define countably infinite and uncountably infinite sets. Define their cardinality numbers. Show that the cardinalities of two open intervals of a real line are equal.

Solution : For definitions, refer to the theory.

Any open interval of the real line is non-denumerable. Hence the cardinalities of two open intervals on the real line are equal, each being the continuum $\subseteq$.

5. Define countably infinite set and prove that the set of positive rational numbers is countably infinite.

Solution : Refer to Example 2 of Examples 4.22.4.

6. Classify the following into countable or uncountable sets.
   (i) $Q \times Q$, Q is the set of rational numbers.
   (ii) The power set of N, where N is the set of natural numbers.
   (iii) All books that can ever be written in English.

Solution : (i) Countably infinite (denumerable).

(ii) P(N) is uncountable.

(iii) Countably infinite : A book is composed of words of varying but finite length. Hence the set of all words in a book is a finite union of finite sets, and hence is finite. Hence all the books that could ever be written in the English language can be thought of as a countably infinite union of countable sets, and is therefore countably infinite.

7. Show that union of a countably infinite number of countably infinite sets is a countably infinite set.

Solution : See theorem 4.23.4 for the solution.

EXERCISE – VI

1. Give an example to show that the cardinality of a set which is the intersection of two countably infinite sets may also be countably infinite.
2. Give an example to show that the cardinality of a set that is the intersection of two countably infinite sets may be finite.
3. Classify the following sets into finite, countably infinite or uncountably infinite.
   (i) Set of all types of trees in India.
   (ii) Set of all prime numbers.
4. State if the following sets are finite, denumerable or non denumerable.
   (i) Class of all possible programs that can be written in any given programming language.
   (ii) Number of fish in the Pacific ocean.
   (iii) All possible books written in the English language.
   (iv) Set of real numbers between 0 and 1.
5. Classify the following sets as finite, denumerable or non-denumerable, giving reason
   (i) The set of all strings (words) in {a, b} of prime length.
   (ii) The set of all strings in {a, b} of length no greater than k.
   (iii) The set of all $m \times n$ matrices with entries from {0, 1, 2, ... k}.
   (iv) The set of all prepositional forms over the prepositional variables p, q, r and s.
   (v) The set of all points in the plane.
   (vi) The set of all points in the plane with positive integer co-ordinates.
6. Determine the cardinalities of the sets
   (a) $A = \left\{ n^7 \mid n \text{ is a positive integer} \right\}$
   (b) $B = \left\{ n^{109} \mid n \text{ is a positive integer} \right\}$
   (c) $A \cup B$
   (d) $A \cap B$.
7. Let N denote the set of all natural numbers. Let S denote the set of all finite subsets of N. What is the cardinality of the set S ? Justify your answer.