Successive differentiation is the process of differentiating a given function successively & result of such differentiation are called successive derivatives.

The higher oder differential coefficients are of utmost importance in scientific & engineering applications.

* If y is function of x i.e $y=f(x)$ then Successive derivatives denoted by $f'(x), f''(x)$ ----- $f^{(n)}(x)$ or $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \dots, \frac{d^ny}{dx^n}$ or

$y', y'', y''', \dots, y^{(n)}$ or $y_1, y_2, y_3 \dots, y_n$

~~Note~~

* n^{th} derivative of some std functions:-

1) If $y=e^{ax}$

$$y_1 = ae^{ax}$$
$$y_2 = a^2e^{ax}$$
$$y_3 = a^3e^{ax}$$
$$\vdots$$
$$y_n = a^ne^{ax}$$

i.e $\boxed{\text{If } y=e^{ax} \text{ then } y_n=a^ne^{ax}}$ — (1)

2) If $y=a^x$

$$y_1 = \log a(a^x)$$
$$y_2 = a^x(\log a)^2$$

$\vdots$

$y_n = a^x (\log a)^n$

i.e $\boxed{\text{If } y = a^x \text{ then } y_n = a^x (\log a)^n}$ — (2)

3) If $y = (ax+b)^m$, where m is any real number.

$y = (ax+b)^m$

$y_1 = m(ax+b)^{m-1} \cdot a$

$y_2 = \cancel{ma}^2 \; m(m-1)a^2(ax+b)^{m-2}$

$y_3 = m(m-1)(m-2)a^3(ax+b)^{m-3}$

$\vdots$

$y_n = m(m-1)(m-2) \cdots (m-(n-1))a^n(ax+b)^{m-n}$

$\therefore$ i.e If $y = (ax+b)^m$, then $\cancel{y_n} =$

$y_n = m(m-1)(m-2) \cdots (m-n+1)\, a^n (ax+b)^{m-n}$ — (3)

4) If $y = (ax+b)^{-m}$, where m is any +ve integer.

$\because y = (ax+b)^{-m}$

$y_1 = -ma(ax+b)^{-m-1} = \cancel{(-1)\, ma}$

$y_2 = -m(-m-1)a^2(ax+b)^{-m-2}$

$= (-1)^2 m(m+1)a^2(ax+b)^{-m-2}$

$y_3 = (-1)^3 m(m+1)(m+2)a^3(ax+b)^{-m-3}$

$\vdots$

$y_n = (-1)^3 m(m+1)(m+2) \cdots (m+(n-1))a^n(ax+b)^{-m-n}$.

i.e If $y = \dfrac{1}{(ax+b)^m}$ where m is +ve integer

then $y_n = \dfrac{(-1)^n m(m+1)(m+2) \cdots (m+(n-1))a^n}{(ax+b)^{m+n}}$ — (4)

Special case :-

(I) If $y = (ax+b)^m$, where m is +ve integer & $m > n$.

then $y_n = m(m-1)(m-2) \cdots (m-n+1)\, a^n (ax+b)^{m-n}$.

This can be written as,

$$y_n = \frac{m(m-1)(m-2)\cdots(m-n+1)(m-n)(m-n-1)\cdots 3.2.1\; a^n(ax+b)^{m-n}}{(m-n)(m-n-1)(m-n-2)\cdots 3.2.1}$$

$$\therefore \boxed{y_n = \frac{m!\, a^n (ax+b)^{m-n}}{(m-n)!}} \quad \text{—— } (*)$$

i.e If $y = (ax+b)^m$ & m is +ve integer, $m > n$

then $y_n = \dfrac{m!\, a^n (ax+b)^{m-n}}{(m-n)!}$

(II) If $m = n$.

then $y_n = \dfrac{n!\, a^n (ax+b)^{n-n}}{(n-n)!}$

i.e $\boxed{y_n = n!\, a^n}$

(III) If $m < n$ $\dfrac{d^n}{dx^n}(ax+b)^m = 0$

If $y = x^n$ then $y_n = n!$ but $y_{n-1} \neq (n-1)!$

To find y_{n-1}, in (*) put $m = n$, $n = n-1$, $a = 1$, $b = 0$.

$$y_{n-1} = \frac{n!\, 1^{n-1}\, x^{n-(n-1)}}{(n-n+1)!}$$

i.e $y_{n-1} = \dfrac{n!\,x}{1}$ & if $\boxed{y = x^n \text{ then } y_{n-1} = n!\,x}$

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~~5) If $y = \log(ax+b)$~~

~~$y_1 = \frac{a}{ax}$~~

case iv) If $y = (ax+b)^{-1}$ i.e $y = \frac{1}{ax+b}$

In formula (4) put $m = 1$

then $y_n = \frac{(-1)^n n! a^n}{(ax+b)^{n+1}}$

(5) i.e $\boxed{\text{If } y = \frac{1}{ax+b} \text{ then } y_n = \frac{(-1)^n n! a^n}{(ax+b)^{n+1}}}$ — (* *)

5) $y = \log(ax+b)$

$y_1 = \frac{a}{ax+b}$

Take $(n-1)^{th}$ derivative of y_1

$\therefore$ ~~$y_n = \frac{a(-1)^n n! a^n}{(ax+b)^{n+1}}$~~ . by (* *) $y_n = \frac{a(-1)^{n-1}(n-1)! a^{n-1}}{(ax+b)^{n-1+1}}$

$\therefore$ $\boxed{\text{If } y = \log(ax+b) \text{ then } y_n = \frac{(-1)^{n-1}(n-1)! a^n}{(ax+b)^n}}$

6) $y = \sin(bx+c)$

$y_1 = b\cos(bx+c) = b\sin(bx+c+\pi/2)$

$y_2 = b^2\cos(bx+c+\pi/2) = b^2\sin(bx+c+\frac{2\pi}{2})$

$\vdots$

$\boxed{y_n = b^n \sin(bx+c+\frac{n\pi}{2})}$

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i.e $\boxed{\text{If } y=\sin(bx+c) \text{ then } y_n = b^n \sin\left(bx+c+\frac{n\pi}{2}\right)}$

$$\boxed{\begin{array}{l}\text{similarly if } y=\cos(bx+c) \\ \quad\quad\quad \text{then } y_n = b^n \cos\left(bx+c+\frac{n\pi}{2}\right)\end{array}}$$ → prove.

7) If $y=e^{ax}\sin(bx+c)$

then $y_n = r^n e^{ax}\sin(bx+c+n\theta)$

where $r=\sqrt{a^2+b^2}$, $\theta=\tan^{-1}\frac{b}{a}$.

8) If $y=e^{ax}\cos(bx+c)$

then $y_n = r^n e^{ax}\cos(bx+c+n\theta)$

where $r=\sqrt{a^2+b^2}$, $\theta=\tan^{-1}\frac{b}{a}$.

y	y_n
1) e^{ax}	$a^n e^{ax}$
2) a^x	$a^x (\log a)^n$
3) $(ax+b)^m$, m is any real number	$m(m-1)(m-2)\cdots(m-n+1)a^n(ax+b)^{m-n}$
4) $\frac{1}{(ax+b)^m}$, m is +ve integer	$\frac{(-1)^n m(m+1)(m+2)\cdots(m+n-1)a^n}{(ax+b)^{m+n}}$
5) $(ax+b)^m$, m is +ve integer $m>n$.	$\frac{m!\,a^n (ax+b)^{m-n}}{(m-n)!}$
6) $(ax+b)^m$, $m=n$.	$n!\,a^n$
7) $(ax+b)^m$, $m<n$	0

y	y_n
$\frac{1}{(ax+b)}$	$\frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}$
$\log(ax+b)$	$\frac{(-1)^{n-1}(n-1)!\, a^n}{(ax+b)^n}$
$\sin(bx+c)$	$b^n \sin(bx+c+\frac{n\pi}{2})$
$\cos(bx+c)$	$b^n \cos(bx+c+n\pi/2)$
$e^{ax}\sin(bx+c)$	$r^n e^{ax}\sin(bx+c+n\theta)$ where $r=\sqrt{a^2+b^2}$, $\theta=\tan^{-1} b/a$

ex. find nth derivative of

1) $y=e^{5x}$, 2) $\sin 2x\cdot\cos 3x$ 3) $\cos^4 x$ 4) a^{2x}.

→ ① $y_n = 5^n e^{5x}$ ⟶ formula

② $y=\sin 2x\cdot\cos 3x$

$= \frac{1}{2}[2\cos 3x\sin 2x]$

$= \frac{1}{2}[\sin 5x - \sin x]$

$\therefore\ y_n = \frac{1}{2}\left[5^n \sin(5x+\frac{n\pi}{2}) - \sin(x+\frac{n\pi}{2})\right]$

③ $\cos^4 x = \left(\frac{1+\cos 2x}{2}\right)^2$

$= \frac{1}{4}[1+2\cos 2x+\cos^2 2x]$

$= \frac{1}{4}\left[1+2\cos 2x+\frac{1+\cos 4x}{2}\right]$

$= \frac{1}{8}[3+4\cos 2x+\cos 4x]$

$\therefore\ y_n = \frac{1}{8}\left[4\cos(2x+\frac{n\pi}{2})+\cos(4x+\frac{n\pi}{2})\right]$

example! find 100^{th} derivative of $y = \sin 2x$

$$y_{100} = 2^{100} \sin\left(2x + \frac{100\pi}{2}\right)$$

$$= 2^{100} \sin(2x + 50\pi)$$

$$\boxed{y_{100} = 2^{100} \sin 2x}$$

example① If $y = (3x+2)^{99}$, then $y_{100} = ?$, $y_{99} = ?$.

example If $y = \log(3x+4)$ then $y_n = ?$.

$$y_n = \frac{(-1)^{n-1}(n-1)!\, 3^n}{(3x+4)^n}.$$

example① find nth derivative $\frac{1}{(x+1)(x+2)}$

$$\because\ y = \frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

$$y = \frac{1}{x+1} - \frac{1}{x+2}$$

$\therefore$ By formula $y = \frac{1}{ax+b}$, $y_n = \frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}$

$$\boxed{y_n = \frac{(-1)^n n!}{(x+1)^{n+1}} - \frac{(-1)^n n!}{(x+2)^{n+1}}}$$

② $y = \frac{1}{1-5x+6x^2}$ then find y_n.

$$\because\ y = \frac{1}{1-5x+6x^2} = \frac{1}{(2x-1)(3x-1)}$$

$$\therefore\ y = \frac{A}{2x-1} + \frac{B}{3x-1}$$

$$= \frac{2}{2x-1} - \frac{3}{3x-1} \text{ ——— formula}$$

$$\therefore\ y_n = \frac{2(-1)^n n!\, 2^n}{(2x-1)^{n+1}} - \frac{3(-1)^n n!\, 3^n}{(3x-1)^{n+1}}$$

Note: If degree of numerator is greater than denominator the take actual division then use partial fraction method.

eg If $y = \dfrac{x^3}{(x-1)(x-2)}$

Here degree of numerator is greater than denominator take actual division

$$\therefore y = \frac{x^3}{(x-1)(x-2)} = \frac{x^3}{x^2-3x+2}$$

$$\begin{array}{r|l} & x+3 \\ x^2-3x+2 & x^3 \\ & -(x^3-3x^2+2x) \\ \hline & 3x^2-2x \\ & -(3x^2-9x+6) \\ \hline & 7x-6 \end{array}$$

$$\therefore x^3 = (x^2-3x+2)(x+3) + (7x-6)$$

$$\therefore y = \frac{x^3}{(x-1)(x-2)} = x+3 + \frac{7x-6}{(x-1)(x-2)} \quad \text{--- (1)}$$

Now $\dfrac{7x-6}{(x-1)(x-2)} = \dfrac{A}{x-1} + \dfrac{B}{x-2}$

$$= \frac{-1}{x-1} + \frac{8}{x-2}$$

$$\therefore y = x - 3 - \frac{1}{x-1} + \frac{8}{x-2}$$

$$\therefore \boxed{y_n = -\left[\frac{(-1)^n n!}{(x-1)^{n+1}}\right] + \frac{8(-1)^n n!}{(x-2)^{n+1}}}$$

HW: find nth derivative of $\dfrac{x^3}{x^2-1}$

eg. $y = \dfrac{1}{(x-1)^2(x-2)}$ find $y_n = ?$

$$y = \frac{A}{x-2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$= \frac{1}{x-2} - \frac{1}{(x-1)} - \frac{1}{(x-1)^2}$$

by formula

$$y_n = \frac{(-1)^n n!}{(x-2)^{n+1}} - \frac{(-1)^n n!}{(x-1)^{n+1}} - \frac{(-1)^n (n+1)!}{(x-1)^{n+2}}$$

eg. find y_n of $y = \dfrac{2x+3}{5x+7}$

$$= \frac{2}{5}\left[\frac{5x+\frac{15}{2}}{5x+7}\right]$$

$$= \frac{2}{5}\left[\frac{5x+7-7+\frac{15}{2}}{5x+7}\right]$$

$$= \frac{2}{5}\left[1 + \frac{1}{2(5x+7)}\right]$$

$$y = \frac{2}{5} + \frac{1}{5(5x+7)}$$

$$\therefore \boxed{y_n = \frac{1}{5}\frac{(-1)^n n!\, 5^n}{(5x+7)^{n+1}}}$$

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* Find n^{th} derivative of $y = \frac{1}{x^2+a^2}$

$\rightarrow$ $$y = \frac{1}{x^2+a^2}$$

$$= \frac{1}{(x+ia)(x-ia)}$$

$$= \frac{A}{x+ia} + \frac{B}{x-ia}$$

$$= \frac{1}{2ia}\left[\frac{1}{x-ia} - \frac{1}{x+ia}\right]$$

$$y_n = \frac{1}{2ia}\left[\frac{(-1)^n n!}{(x-ia)^{n+1}} - \frac{(-1)^n n!}{(x+ia)^{n+1}}\right]$$

$$= \frac{(-1)^n n!}{2ia}\left[\frac{1}{(x-ia)^{n+1}} - \frac{1}{(x+ia)^{n+1}}\right]$$

put $x+ia = r[\cos\theta + i\sin\theta]$, $x-ia = r[\cos\theta - i\sin\theta]$

$\therefore$ $x = r\cos\theta$, $a = r\sin\theta$ & $\theta = \tan^{-1}\frac{a}{x}$.

$$\therefore y_n = \frac{(-1)^n n!}{2iar^{n+1}}\left[\frac{1}{(\cos\theta - i\sin\theta)^{n+1}} - \frac{1}{(\cos\theta + i\sin\theta)^{n+1}}\right]$$

$$= \frac{(-1)^n n!}{2iar^{n+1}}\left[\frac{1}{\cos(n+1)\theta - i\sin(n+1)\theta} - \frac{1}{\cos(n+1)\theta + i\sin(n+1)\theta}\right]$$

$$= \frac{(-1)^n n!}{\cancel{2i}ar^{n+1}}\left[\cancel{2i}\sin[(n+1)\theta]\right]$$

$$= \frac{(-1)^n n! \sin(n+1)\theta}{a\left(\frac{a^{n+1}}{\sin^{n+1}\theta}\right)} \quad \left[\because a = r\sin\theta \therefore r = \frac{a}{\sin\theta}\right]$$

$$\boxed{y_n = \frac{(-1)^n n! \sin(n+1)\theta \sin^{n+1}\theta}{a^{n+2}} \quad \text{where } \theta = \tan^{-1}\frac{a}{x}}$$

i.e If $y = \frac{1}{x^2+a^2}$ then $y_n = \frac{(-1)^n n! \sin(n+1)\theta \sin^{n+1}\theta}{a^{n+2}}$

formula :- ~~If~~ $y = \tan^{-1}\frac{x}{a}$ ~~then~~

$$y_1 = \frac{a}{x^2+a^2}$$

$$y_1 = a\left(\frac{1}{x^2+a^2}\right)$$

Differentiating $(n-1)$ times w.r.t x.

$$\therefore y_n = a\left[\frac{(-1)^{n-1}(n-1)!\sin n\theta \sin^n\theta}{a^{n+1}}\right] \text{ where } \theta = \tan^{-1}\frac{a}{x}$$

$$\therefore \boxed{\text{If } y = \tan^{-1}\frac{x}{a} \text{ then } y_n = \frac{(-1)^{n-1}(n-1)!\sin n\theta \sin^n\theta}{a^n} \text{ where } \theta = \tan^{-1}\frac{a}{x}}$$

— ***

eg. ① If $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ then find y_n.

$\rightarrow$ $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ put $x = \tan\phi$

$$= \tan^{-1}\left(\frac{\sqrt{1+\tan^2\phi}-1}{\tan\phi}\right)$$

$$= \tan^{-1}\left(\frac{\sec\phi - 1}{\tan\phi}\right) = \tan^{-1}\left(\frac{1-\cos\phi}{\sin\phi}\right)$$

$$= \tan^{-1}\left(\frac{2\sin^2\phi/2}{2\sin\phi/2\cos\phi/2}\right)$$

$$= \tan^{-1}(\tan\phi/2)$$

$$= \phi/2$$

$$y = \frac{\tan^{-1}x}{2} \quad \therefore \text{by ***}$$

$$\therefore y_n = \frac{1}{2}(-1)^{n-1}(n-1)!\sin n\theta \sin^n\theta \quad \text{where } \theta = \tan^{-1}\frac{1}{x}.$$

2) find n^{th} derivative of $y = \cos^{-1}\left(\frac{x - x^{-1}}{x + x^{-1}}\right)$

$\because y = \cos^{-1}\left(\frac{x - x^{-1}}{x + x^{-1}}\right)$

$= \cos^{-1}\left(\frac{x^2 - 1}{x^2 + 1}\right)$

put $x = \tan\phi$

$\therefore y = \cos^{-1}\left(\frac{\tan^2\phi - 1}{\tan^2\phi + 1}\right)$

$= \cos^{-1}(-\cos 2\phi)$

$= \cos^{-1}(\cos(\pi + 2\phi))$

$= \pi + 2\phi$

$\therefore y = \pi + 2\tan^{-1}x$.

$\therefore y_n = 2(-1)^{n-1}(n-1)!\sin^n\theta \sin n\theta$, where $\theta = \tan^{-1} 1/x$

HW:— (*) find n^{th} derivative of

(1) $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, (2) $\tan^{-1}\left(\frac{1+x}{1-x}\right)$

(3) $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

* Leibnitz's theorem :-

This is useful to find n^{th} derivative of product of two functions.

statement: If $y = uv$ where u & v are functions of x then n^{th} derivative

$$\boxed{y_n = {}^nC_0 u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \cdots + {}^nC_n u v_n}$$

where suffixes denotethe order of the derivatives. & ${}^nC_0, {}^nC_1, {}^nC_2 \cdots {}^nC_n$ are Binomial coefficients.

~~Ex~~.

Note:- The choice of u & v is important. That function should be chosen as v whose successive derivative vanish earlier & that function should be taken as u whose n^{th} derivative is known.

eg. find n^{th} derivative of

① $y = x^2 e^{ax}$ ② $x^3 \cos x$ ③ $e^x (2x+3)^2$

① $y = x^2 e^{ax}$

$u = e^{ax}$, $v = x^2$

$u_n = a^n e^{ax}$, $v_1 = 2x$

$u_{n-1} = a^{n-1} e^{ax}$, $v_2 = 2$

$u_{n-2} = a^{n-2} e^{ax}$, $v_3 = 0$.

$\therefore y_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \cdots u v_n$

$$\therefore y_n = a^n e^{ax}(x^2) + n(a^{n-1}e^{ax})2x + \frac{n(n-1)}{2!}(a^{n-2}e^{ax})(2)$$

$$y_n = a^n e^{ax}\left[x^2 + \frac{2xn}{a} + \frac{n(n-1)}{a^2}\right]$$

(2) $y = x^3 \cos x$.

$u = \cos x, \quad v = x^3$

$u_n = \cos\left(x + \frac{n\pi}{2}\right), \quad v_1 = 3x^2$

$u_{n-1} = \cos\left(x + \frac{(n-1)\pi}{2}\right), \quad v_2 = 6x$

$u_{n-2} = \cos\left(x + \frac{(n-2)\pi}{2}\right), \quad v_3 = 6.$

$$\therefore y_n = u_n v + n u_{n-1} v_1 + \frac{n(n-1)}{2!} u_{n-2} v_2 + \frac{n(n-1)(n-2)}{3!} u_{n-3} v_3$$

$$y_n = \cos\left(x + \frac{n\pi}{2}\right)x^3 + n\cos\left(x + \frac{(n-1)\pi}{2}\right)3x^2$$
$$+ \frac{n(n-1)}{2!}\cos\left(x + \frac{(n-2)\pi}{2}\right)(6x) + \frac{n(n-1)(n-2)}{3!}\cos\left(x + \frac{(n-3)\pi}{2}\right)(6).$$

$$\therefore y_n = x^3\cos\left(x + \frac{n\pi}{2}\right) + 3nx^2\cos\left(x + \frac{(n-1)\pi}{2}\right) + 3n(n-1)x\cos(\quad)$$
$$+ n(n-1)(n-2)\cos\left(x + \frac{(n-3)\pi}{2}\right).$$

eg. (3) If $y = \frac{\sin^{-1}x}{\sqrt{1-x^2}}$ then show that

$y_2(1-x^2) - 3xy_1 - y = 0$ & hence prove that

$(1-x^2)y_{n+2} - (2n+3)xy_{n+1} - (n+1)^2 y_n = 0.$

$\because \; y = \dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$

$\therefore \; \sqrt{1-x^2}\; y = \sin^{-1}x$.

$\therefore \; (1-x^2)\, y^2 = (\sin^{-1}x)^2$ —— ①

Differentiating both side w.r.t x .

$\therefore \; (1-x^2)\, 2yy_1 - 2xy^2 = 2\sin^{-1}x \left(\dfrac{1}{\sqrt{1-x^2}}\right)$

$\therefore \; (1-x^2)\, y_1 - \cancel{2}xy = \cancel{1}\, 1.$

$\therefore \; (1-x^2)\, y_1 - xy - 1 = 0$. —— ②

Differentiating ② w.r.t x .

$\therefore \; (1-x^2)\, y_2 - 2xy_1 - xy_1 - y \; \cancel{y_1} = 0$

$\therefore \; \boxed{(1-x^2)\, y_2 - 3xy_1 - y = 0}$ —— ③ .

Differentiate ③ n times by leibnitz thm .

$(1-x^2)\, y_{n+2} - n\, 2x\, y_{n+1} + \dfrac{n(n-1)}{2!}(-2)\, y_n$

$- 3\left[x y_{n+1} + n y_n\right] - y_n = 0$.

$\therefore \; (1-x^2)\, y_{n+2} - (2n+3)\, x y_{n+1} - (n^2 - n + 3n + 1)\, y_n = 0$

$\therefore \; (1-x^2)\, y_{n+2} - (2n+3)\, x y_{n+1} - (n^2 + 2n + 1)\, y_n = 0$.

$\therefore \; \boxed{(1-x^2)\, y_{n+2} - (2n+3)\, x y_{n+1} - (n+1)^2 y_n = 0}$ is

required result .

② If $y = \cancel{e}\; e^{a\sin^{-1}x}$ then prove that

$(1-x^2)\, y_{n+2} - (2n+1)\, x y_{n+1} - (n^2 + a^2)\, y_n = 0$.

$$y = e^{a\sin^{-1}x}$$

$$y_1 = \frac{ae^{\sin^{-1}x}}{\sqrt{1-x^2}} = \frac{ay}{\sqrt{1-x^2}}$$

$$\therefore y_1\sqrt{1-x^2} = ay$$

squaring both side :

$$\therefore y_1^2(1-x^2) = a^2y^2$$

Differentiating once again

$$\therefore 2y_1y_2(1-x^2) - 2xy_1^2 = 2a^2yy_1$$

$$\therefore y_2(1-x^2) - xy_1 = a^2y$$

Differentiating n times by using Leibnitz thm .

$$\therefore \left[(1-x^2)y_{n+2} - 2nxy_{n+1} + \frac{n(n-1)}{2}y_n(-2)\right]$$

$$-\left[y_{n+1}x + ny_n\right] = a^2y_n = 0 .$$

$$\therefore \boxed{(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0}$$

is required result .

H.W :- ① If $y = \cos(m\log x)$, show that

$$x^2y_{n+2} + (2n+1)xy_{n+1} + (m^2+n^2)y_n = 0 .$$

② If $x = \sin t$, $y = \sin at$ then show that

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-a^2)y_n = 0 .$$

③ Find 4th derivative of $x^2\sin 3x$.

④ If $x = \tan(\log y)$, Prove that

$$(1+x^2)y_{n+1} + (2nx-1)y_n + n(n-1)y_{n-1} = 0 .$$

Taylor's and Maclaurin's theorems:-

1) Maclaurin's thm:-

Let $f(x)$ be a function of x which can be expanded in asending powers of x (i.e power series in x) If $f(x)$ is term by term differentiable then

$$\boxed{f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots + \frac{x^n}{n!} f^{(n)}(0) + \cdots}$$

is called Maclaurin's series.

* Expansion of some standard functions by using Maclaurin's theorem.

1) $f(x) = \sin x$, $f(0) = 0$

$f'(x) = \cos x \quad f'(0) = 1$

$f''(x) = -\sin x, \quad f''(0) = 0$

$f'''(x) = -\cos x, \quad f'''(0) = -1$

$f^{IV}(x) = \sin x, \quad f^{IV}(0) = 0 \; \cdots$

substituting this value in Maclaurins series then we get series for $\sin x$.

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{IV}(0) + \frac{x^5}{5!} f^{(V)}(0) + \cdots$$

$$\therefore \sin x = 0 + x + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(1) + \cdots$$

$\therefore$ Expansion of $\sin x$ is

$$\boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} \cdots}$$

Simillarly we can find expansion of $\cos x$

2) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$

3) $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots$

4) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$

5) $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$

6) $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots$

7) $\tanh x = x - \frac{x^3}{3} + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \cdots$

8) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$

9) $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots$

Differentiate (8) w.r.t x then

(10) $\frac{1}{1-x} = 1 - x + x^2 - x^3 + x^4 - \cdots$

Differentiate (9) w.r.t x then

(11) $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$

(12) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$

is called Binomial expansion.

This all are standard formulae

example :-

1) Expand $\sqrt{1+\sin x}$ upto x^6.

$y = \sqrt{1+\sin x}$

$= \sqrt{\sin^2 x/2 + \cos^2 x/2 + 2\sin\frac{x}{2}\cos x/2}$

$= \sqrt{\left(\cos x/2 + \sin x/2\right)^2}$

$= \cos x/2 + \sin x/2$

$= \left(1 - \frac{1}{2!}\left(\frac{x}{2}\right)^2 + \frac{1}{4!}\left(\frac{x}{2}\right)^4 - \cdots\right) + \left(\left(\frac{x}{2}\right) - \frac{1}{3!}\left(\frac{x}{2}\right)^3 + \frac{1}{5!}\left(\frac{x}{2}\right)^5 + \cdots\right)$

$= 1 + \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{48} + \frac{x^4}{384} + \frac{x^5}{3840} + \frac{x^6}{46080} + \cdots$

2) Expand $\log(1+x+x^2+x^3)$ upto a term in x^8

$f(x) = \log(1+x+x^2+x^3)$

$= \log\left[(1+x) + x^2(1+x)\right]$

$= \log\left[(1+x)(1+x^2)\right]$

$= \log(1+x) + \log(1+x^2)$

we have std formula for $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

$\therefore \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} + \cdots\right) + \left(x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \frac{x^{10}}{5} - \cdots\right)$

$= x - \frac{x^2}{2} + x^2 + \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^4}{2} + \frac{x^5}{5} + \frac{x^6}{3} - \frac{x^6}{6} + \frac{x^7}{7} - \frac{x^8}{4} - \frac{x^8}{8}$

$= x + \frac{x^2}{2} + \frac{x^3}{3} - \frac{3x^4}{4} + \frac{1}{5}x^5 + \frac{x^6}{6} + \frac{x^7}{7} - \frac{3}{8}x^8 + \cdots$

3) Expand $(1+x)^x$ in ascending powers of x, expansion being correct upto fifth power.

$\rightarrow \quad y = (1+x)^x$

$\therefore \log y = x \log(1+x)$

$$= x\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right)$$

$$= x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \frac{x^6}{5} - \cdots$$

$$= z \text{ (say)}$$

$\therefore \cancel{y = e}$

$\therefore \quad y = e^z$

$$= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$= 1 + \left(x^2 + \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \frac{x^6}{5} - \cdots\right) + \frac{1}{2}\left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \cdots\right)^2$$

$$+ \frac{1}{6}\left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \cdots\right)$$

neglect x^6 & higher powers of x, we get

$$\therefore y = 1 + x^2 - \frac{1}{2}x^3 + \frac{x^4}{3} - \frac{x^5}{4} + \frac{1}{2}(x^4 - x^5) + \cdots$$

$$y = 1 + x^2 - \frac{1}{2}x^3 + \frac{5}{6}x^4 - \frac{3}{4}x^5 + \cdots$$

example :- prove that

4) $\boxed{\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots\cdots}$

Let $y = \tan^{-1}x$

$$\therefore \frac{dy}{dx} = \frac{1}{1+x^2}$$

$\because$ We know $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots$

$\therefore \frac{dy}{dx} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots$

Integrating both side ~~by~~ w.r.t x

$$y = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^8}{8} - \cdots$$

i.e $\boxed{\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^8}{8} - \cdots}$ This is standard formula.

similarly we can find expansion of

$$\sin^{-1}x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} + \frac{1}{2} + \frac{3}{4}\frac{5}{6}\frac{x^7}{7} + \cdots$$

$$\cos^{-1}x = \frac{\pi}{2} - \left[x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} + \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{x^7}{7} + \cdots\right]$$

example ①:

Prove that $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \frac{1}{2}\left(x - \frac{x^3}{3} + \frac{x^5}{5} \cdots\right)$

$$y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$$

put $x = \tan\theta$

$$\therefore y = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{2\sin^2\theta/2}{2\sin\theta/2\cos\theta/2}\right) = \tan^{-1}(\tan\theta/2) = \frac{\theta}{2}$$

$= \frac{1}{2}\tan^{-1}x$

$= \frac{1}{2}\left[x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots\right]$ By standard expansion of $\tan^{-1}x$

(2) Expand $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ in ascending powers of x.

$y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

put $x = \tan\theta$

$\therefore y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$

$y = \sin^{-1}(\sin 2\theta)$

$= 2\theta$

$= 2\tan^{-1}x$

$$\boxed{y = 2\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots\right)}$$

(3) HW: Expand $\cos^{-1}\left(\frac{x - x^{-1}}{x + x^{-1}}\right)$ in ascending powers of x.

(4) Expand $\cos^{-1}(4x^3 - 3x)$ in ascending powers of x.
(Hint put $x = \cos\theta$)

(5) Expand $\sin^{-1}(3x - 4x^3)$ in ascending powers of x.
(Hint put $x = \sin\theta$.

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⑥ Expand $\sec x$ by Maclaurin's theorem as far as the terms upto x^4.

$\rightarrow$ $f(x) = \sec x$, $f(0) = 1$

$f'(x) = \sec x \cdot \tan x$, $f'(0) = 0$

$f''(x) = \sec^3 x + \sec x \tan^2 x$, $f''(0) = 1$

$f'''(x) = 3\sec^2 x \tan x + \sec x \tan^3 x + 2\tan x \sec^3 x$.

$\therefore f'''(0) = 0$

$f^{IV}(x) = 3[\sec^5 x + \tan^2 x \cdot 3\sec^3 x] + \sec^3 x \cdot 3\tan^2 x$

$+ \tan^4 x \sec x + 2[\tan^2 x \cdot 3\sec^3 x + \sec^5 x]$

$\therefore f^{IV}(0) = 5$ ...

put the values of $f(0)$, $f'(0)$, $f''(0)$, $f^{IV}(0)$ ....

in Maclaurin's theorem.

$\therefore f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{IV}(0) + \frac{x^5}{5!} f^{V}(0) + \cdots$

$= 1 + x(0) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(5) + \cdots$

$\therefore \sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{4!} + \cdots$

HW: Expand by Maclaurin's theorem.

① ~~log(1+~~ $e^x \sec x$ ② $e^x \sin x$ ③ $\log \sec x$.

Taylor's Theorem :-

Statement :- Let f(x+h) be a function, which can be expanded in powers of h

i.e $$\boxed{f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots + \frac{h^n}{n!}f^{(n)}(x) + \cdots}$$

Deductions: 1) Replace x by h & h by x then we get

$$f(x+h) = f(h) + xf'(h) + \frac{x^2}{2!}f''(h) + \frac{x^3}{3!}f'''(h) + \cdots + \frac{x^n}{n!}f^{(n)}(h) + \cdots$$

is expansion in powers of x.

2) we can write $x = x - a + a$ then

$f(x) = f(x-a+a)$,

put $y = x - a$ & $h = a$ then

$$\cancel{f(x)} = f(y+h) = f(h) + yf'(h) + \frac{y^2}{2!}f''(h) + \frac{y^3}{3!}f'''(h) + \cdots + \frac{y^n}{n!}f^{(n)}(h) + \cdots$$

i.e $$\boxed{f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots \frac{(x-a)^n}{n!}f^{(n)}(a) + \cdots}$$

is expansion in powers of $x - a$

3) In Deduction ① put $h = 0$ then we get Maclaurin's series

i.e $$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$$

eg. 1 Using Taylor's theorem, express $\cancel{(x-2)^4}$ $(x-2)^4 - 3(x-2)^3 + 4(x-2)^2 + 5$ in powers of x.

Let $f(x+h) = (x-2)^4 - 3(x-2)^3 + 4(x-2)^2 + 5$, where $h = -2$.

by Taylor's thm,

$$f(x+h) = f(h) + xf'(h) + \frac{x^2}{2!} f''(h) + \frac{x^3}{3!} f'''(h) + \frac{x^4}{4!} f^{(iv)}(h) + \cdots$$

Now $f(h) = h^4 - 3h^3 + 4h^2 + 5$, $f(-2) = 61$

$f'(h) = 4h^3 - 9h^2 + 8h$, $f'(-2) = -84$

$f''(h) = 12h^2 - 18h + 8$, $f''(-2) = 92$

$f'''(h) = 24h - 18$, $f'''(-2) = -66$

$f^{(iv)}(h) = 24$ $f^{(iv)}(-2) = 24$.

$$\therefore f(x-2) = 61 - 84x + 92\frac{x^2}{2!} - 66\frac{x^3}{3!} + 24\frac{x^4}{4!}$$

$$\therefore f(x-2) = 61 - 84x + 46x^2 - 11x^3 + x^4$$

eg. 2 Expand $2x^3 + 7x^2 + x - 6$ in powers of $(x-2)$

$f(x) = 2x^3 + 7x^2 + x - 6$

$\rightarrow f(x) = f(2 + (x-2))$

By Taylor's thm

$$f(x) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!} f''(2) + \frac{(x-2)^3}{3!} f'''(3) + \cdots \quad \text{--- (1)}$$

$\because f(x) = 2x^3 + 7x^2 + x - 6$, $f(2) = 40$

$f'(x) = 6x^2 + 14x + 1$, $f'(2) = 53$

$f''(x) = 12x + 14$, $f''(2) = 38$

$f'''(x) = 12$ $\quad f'''(2) = 12$

$f^{iv}(x) = 0$ $\quad f^{iv}(2) = 0$.

put all values in (1)

$\therefore f(x) = 40 + (x-2)(53) + \frac{(x-2)^2}{2!}(38) + \frac{(x-2)^3}{3!}(12) + 0$

$f(x) = 40 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3$

Q-3) Using Taylors thm find expansion of $\tan(x + \frac{\pi}{4})$ in ascending powers of x upto terms in x^4 & find approximately the value of $\tan(43^\circ)$

$\rightarrow$ $f(x+h) = \tan(x + \frac{\pi}{4})$, then $h = \frac{\pi}{4}$.

$f(h) = \tan(h)$, $f(\pi/4) = 1$

$f'(h) = \sec^2(h) = 1 + \tan^2 h$, $f'(\pi/4) = 2$

$f''(h) = 2\tan h \sec^2 h = 2\tan h(1 + \tan^2 h)$; $f''(\frac{\pi}{4}) = 4$.

$= 2[\tan h + \tan^3 h]$

~~$f'''(h) = 2[$~~

$f'''(h) = 2[\sec^2 h + 3\tan^2 h \sec^2 h]$

$= 2[1 + \tan^2 h + 3\tan^2 h(1 + \tan^2 h)]$

$= 2[1 + 4\tan^2 h + 3\tan^4 h]$

$\therefore f'''(\frac{\pi}{4}) = 16$

$f^{iv}(h) = 2\left[8\tan^2 h \sec^2 h + 12 \tan^3 h \sec^2 h\right]$

$\therefore f^{iv}\left(\frac{\pi}{4}\right) = 80.$

Substitute all values in Taylor's thm.

$$f(x+h) = f(h) + x f'(h) + \frac{x^2}{2!} f''(h) + \frac{x^3}{3!} f'''(h) + \frac{x^4}{4!} f^{iv}(h) + \cdots$$

$$\therefore f(x+h) = f\left(\frac{\pi}{4}\right) + x f'(\pi/4) + \frac{x^2}{2!} f''(\pi/4) + \frac{x^3}{3!} f'''(\pi/4) + \frac{x^4}{4!} f^{iv}(\pi/4) + \cdots$$

$$\therefore \tan(x + \pi/4) = 1 + 2x + 2x^2 + \frac{8}{3}x^3 + \frac{10}{3}x^4 + \cdots \quad \text{—— (1)}$$

Now $\tan(43^\circ) = \tan(x+h) = \tan(-2^\circ + 45^\circ)$

$= \tan\left(\frac{-2\pi}{180} + \pi/4\right)$

$\therefore x = -2^\circ = -\frac{2\pi}{180} = -0.0349, \quad h = \pi/4.$

putting $x = -0.0349$ in (1)

$$\therefore \tan(43^\circ) = 1 - 2(0.0349) + 2(0.0349)^2 - \frac{8}{3}(0.0349)^3 + \frac{10}{3}(0.0349)^4 + \cdots$$

$= 0.9326$ approximately

Q.4) Expand e^x in powers of $x-2$

$\because f(x) = e^x$, Now $x = x-2+2$

$\therefore$ By Taylor's thm

$$\therefore f(x) = f((x-2)+2) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!} f''(2) + \frac{(x-2)^3}{3!} f'''(2) + \cdots \quad \text{—— (1)}$$

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$\because f(x) = e^x \quad \therefore f(2) = e^2$

$f'(x) = e^x, \quad f'(2) = e^2$

$f''(x) = e^x, \quad f''(2) = e^2$

$f'''(x) = e^x, \quad f'''(2) = e^2$

$\therefore f(x) = e^2 + (x-2)e^2 + \frac{(x-2)^2}{2!}e^2 + \frac{(x-2)^3}{3!}e^2 + \cdots$

$= \cancel{e^2} E$

$= e^2\left(1 + (x-2) + \frac{(x-2)^2}{2!} + \frac{(x-2)^3}{3!} + \cdots\right).$

Q.5) Use Taylor's thm to find $\sqrt{25.15}$

Here $f(x+h) = \sqrt{x+h}$, where $x = 25$, $h = 0.15$.

$\Rightarrow f(x) = \sqrt{x}, \quad f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2}$

$f''(x) = -\frac{1}{4}x^{-3/2}, \quad f'''(x) = \frac{3}{8}x^{-5/2}, \quad f^{iv}(x) = -\frac{15}{16}x^{-7/2}$

$\therefore$ By Taylors thm,

$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{iv}(x) + \cdots$

$\sqrt{x+h} = \sqrt{x} + \frac{h}{2\sqrt{x}} - \frac{h^2}{8(\sqrt{x})^3} + \frac{h^3}{16(\sqrt{x})^5} - \frac{5h^4}{128(\sqrt{x})^7} + \cdots$

$\therefore$ put $x = 25$, $h = 0.15$.

$\sqrt{25.15} = 5 + \frac{0.15}{2\times5} - \frac{(0.15)^2}{8\times125} + \frac{(0.15)^3}{16\times3125} - \frac{5(0.15)^4}{128\times78125} + \cdots$

$= 5 + 0.015 - 0.000225 + 0.000000675 - \cdots$

$= 5.01478$ approximately

Q.6) show that $\frac{1}{2}[f(x) - f(2a-x)] = (x-a)f'(a)$

$+ \frac{(x-a)^3}{3!} f'''(a) + \frac{(x-a)^5}{5!} f^{(v)}(a) + \cdots$

$\rightarrow$ Let $f(x) = f(a + (x-a))$

By Taylors thm,

$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots$

$\frac{(x-a)^4}{4!} f^{(iv)}(a) + \frac{(x-a)^5}{5!} f^{v}(a)$ — ①

Let $f(2a-x) = f(a + (a-x))$

$\therefore$ By Taylors thm,

$f(2a-x) = f(a) + (a-x)f'(a) + \frac{(a-x)^2}{2!} f''(a) + \frac{(a-x)^3}{3!} f'''(a) + \cdots$

$+ \frac{(a-x)^4}{4!} f^{iv}(a) + \frac{(a-x)^5}{5!} f^{v}(a)$ — ②

$+ \cdots$

substract ② from ①

~~add ① & ②~~

$\therefore$ ~~$f(x) + f(2a-x) = 2f(a)$~~

$f(x) - f(2a-x) = 2\left[(x-a)f'(a) + \frac{(x-a)^3}{3!} f'''(a) + \frac{(x-a)^5}{5!} f^{v}(a)\right]$

$+ \cdots$

$\therefore \frac{1}{2}[f(x) - f(2a-x)] = (x-a)f'(a) + \frac{(x-a)^3}{3!} f'''(a) + \frac{(x-a)^5}{5!} f^{v}(a)$

$+ \cdots$

H.W:- ① Expand $3x^3-2x^2+x-4$ in powers of $(x+2)$

② Expand $x^4-3x^3+2x^2-x+1$ in powers of $x-3$

③ Using Taylors thm express $7+(x+2)+3(x+2)^3+(x+2)^4$ in powers of x.

④ Using Taylors thm obtain $\sqrt{9.12}$

eg ⑤ prove that $\log \sin(x+h) = \log \sin x + h\cot x - \frac{h^2}{2}\operatorname{cosec}^2 x + \frac{h^3}{3}\operatorname{cosec}^2 x \cot x + \cdots$