MA201 Mathematics III
Solutions to Complex Analysis Tutorial 01

Complex Numbers and Complex Algebra

1. Find the modulus, argument, principal value of the argument, and polar form of the given complex number:

   (i) $\sqrt{3}+i$ (ii) $\dfrac{1-i\sqrt{3}}{2}$ (iii) $\dfrac{1-i}{1+i}$ (iv) $\dfrac{(2+i)^2}{(3-i)^2}$ (v) -100 (vi) $-3i$

   Answer:

   (i) $\sqrt{3}+i$

   Modulus: $|\sqrt{3}+i| = \sqrt{(\sqrt{3})^2+1^2} = \sqrt{4} = 2.$

   argument: $\arg(\sqrt{3}+i) = \tan^{-1}(1/\sqrt{3}) + 2k\pi = \dfrac{\pi}{6} + 2k\pi$ where $k \in \mathbb{Z}$.

   Principal value of argument: $\operatorname{Arg}(\sqrt{3}+i) = \dfrac{\pi}{6}$.

   Polar form: $(r,\ \theta) = \left(2,\ \dfrac{\pi}{6}\right)$.

   (ii) $\dfrac{1-i\sqrt{3}}{2}$

   Modulus: $\left|\dfrac{1-i\sqrt{3}}{2}\right| = \sqrt{\left(\dfrac{1}{2}\right)^2 + \left(\dfrac{\sqrt{3}}{2}\right)^2} = \sqrt{1} = 1.$

   argument: $\arg\left(\dfrac{1-i\sqrt{3}}{2}\right) = \tan^{-1}\left(-\sqrt{3}\right) + 2k\pi = \dfrac{-\pi}{3} + 2k\pi$ where $k \in \mathbb{Z}$.

   Principal value of argument: $\operatorname{Arg}\left(\dfrac{1-i\sqrt{3}}{2}\right) = \dfrac{-\pi}{3}$.

   Polar form: $(r,\ \theta) = \left(1,\ \dfrac{-\pi}{3}\right)$.

   (iii) $\dfrac{1-i}{1+i}$

   Observe that $\dfrac{1-i}{1+i} = \dfrac{1-i}{1+i} \times \dfrac{1-i}{1-i} = -i$

   Modulus: $\left|\dfrac{1-i}{1+i}\right| = |-i| = \sqrt{(-1)^2} = \sqrt{1} = 1.$

   argument: $\arg\left(\dfrac{1-i}{1+i}\right) = \arg(-i) = \dfrac{-\pi}{2} + 2k\pi$ where $k \in \mathbb{Z}$.

   Principal value of argument: $\operatorname{Arg}\left(\dfrac{1-i}{1+i}\right) = \operatorname{Arg}(-i) = \dfrac{-\pi}{2}$.

   Polar form: $(r,\ \theta) = \left(1,\ \dfrac{-\pi}{2}\right)$.

   (iv) $\dfrac{(2+i)^2}{(3-i)^2}$

   Observe that $\dfrac{(2+i)^2}{(3-i)^2} = \dfrac{3+4i}{8-6i} = \dfrac{3+4i}{8-6i} \times \dfrac{8+6i}{8+6i} = \dfrac{i}{2}$

Modulus: $\left|\dfrac{(2+i)^2}{(3-i)^2}\right| = \left|\dfrac{i}{2}\right| = \sqrt{\left(\dfrac{1}{2}\right)^2} = \sqrt{\dfrac{1}{4}} = \dfrac{1}{2}$.
argument: $\arg\left(\dfrac{(2+i)^2}{(3-i)^2}\right) = \arg\left(\dfrac{i}{2}\right) = \dfrac{\pi}{2} + 2k\pi$ where $k \in \mathbb{Z}$.
Principal value of argument: $\operatorname{Arg}\left(\dfrac{(2+i)^2}{(3-i)^2}\right) = \operatorname{Arg}\left(\dfrac{i}{2}\right) = \dfrac{\pi}{2}$.
Polar form: $(r,\ \theta) = \left(\dfrac{1}{2},\ \dfrac{\pi}{2}\right)$.

(v) -100
Modulus: $|-100| = \sqrt{(-100)^2} = \sqrt{10000} = 100$.
argument: $\arg(-100) = \pi + 2k\pi = (2k+1)\pi$ where $k \in \mathbb{Z}$.
Principal value of argument: $\operatorname{Arg}(-100) = \pi$.
Polar form: $(r,\ \theta) = (100,\ \pi)$.

(vi) $-3i$
Modulus: $|-3i| = \sqrt{(-3)^2} = \sqrt{9} = 3$.
argument: $\arg(-3i) = \dfrac{-\pi}{2} + 2k\pi$ where $k \in \mathbb{Z}$.
Principal value of argument: $\operatorname{Arg}(-3i) = \dfrac{-\pi}{2}$.
Polar form: $(r,\ \theta) = \left(3,\ \dfrac{-\pi}{2}\right)$.

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2. Show that if $|z| = 2$, then $\left|\dfrac{1}{z^4 - 4z^2 + 3}\right| \leq \dfrac{1}{3}$.
Answer:
Observe that $|4z^2 - 3| \leq 4|z|^2 + 3 = 16 + 3 = 19$ for $|z| = 2$.
$\left|z^4 - 4z^2 + 3\right| = \left|z^4 - (4z^2 - 3)\right| \geq \left|\,|z|^4 - |4z^2 - 3|\,\right| \geq |2^4 - 19| = 3$ for $|z| = 2$.
Therefore, $\left|\dfrac{1}{z^4 - 4z^2 + 3}\right| \leq \dfrac{1}{3}$.

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3. If either $|z_1| = 1$ or $|z_2| = 1$, but not both, then prove that $\left|\dfrac{z_1 - z_2}{1 - \overline{z_1}\, z_2}\right| = 1$. What exception must be made for the validity of the above equality when $|z_1| = |z_2| = 1$?
Answer:
Case I: $|z_1| = 1$ and $|z_2| \neq 1$

$$\left|\frac{z_1 - z_2}{1 - \overline{z_1}\, z_2}\right|^2 = \frac{|z_1|^2 + |z_2|^2 - 2\Re(z_1\overline{z_2})}{1 + |\overline{z_1} z_2|^2 - 2\Re(z_1\overline{z_2})} = \frac{1 + |z_2|^2 - 2\Re(z_1\overline{z_2})}{1 + |z_2|^2 - 2\Re(z_1\overline{z_2})} = 1\ .$$

Observe that the denominator $1 + |\overline{z_1} z_2|^2 - 2\Re(z_1\overline{z_2}) \neq 0$ if $|z_1| = 1$ and $|z_2| \neq 1$.

Case II: $|z_2| = 1$ and $|z_1| \neq 1$
It can be worked out similarly as in the previous case.

Case III: $|z_1| = 1$ and $|z_2| = 1$

Then, $\left|\dfrac{z_1 - z_2}{1 - \overline{z_1}\, z_2}\right|^2 = \dfrac{2 - 2\Re(z_1\overline{z_2})}{2 - 2\Re(z_1\overline{z_2})} = 1$ if the denominator $2-2\Re(z_1\overline{z_2}) \neq 0$. That is, $\Re(z_1\overline{z_2}) \neq 1$ if and only if $z_1 \neq z_2$. So, the exception is to be made for the validity of the above equality in this case is $z_1 \neq z_2$.

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4. If z_1, z_2, z_3 and z_4 are complex numbers of unit modulus, prove that

$$|z_1 - z_2|^2|z_3 - z_4|^2 + |z_1 + z_4|^2|z_3 - z_2|^2 = |z_1(z_2 - z_3) + z_3(z_2 - z_1) + z_4(z_1 - z_3)|^2$$

Answer:
Since $|z_i| = 1$ for $i = 1$, 2, 3 and 4, we have $|z_i|^2 = 1 \Longrightarrow \overline{z_i} = 1/z_i$.
Let $u = (z_1 - z_2)(z_3 - z_4)$, $v = (z_1 + z_4)(z_3 - z_2)$ and $w = z_1(z_2 - z_3) + z_3(z_2 - z_1) + z_4(z_1 - z_3)$.
Then, we need to show that $|u|^2 + |v|^2 = |w|^2$.
First observe that,

$$u+v = (z_1-z_2)(z_3-z_4)+(z_1+z_4)(z_3-z_2) = -\left(z_1(z_2 - z_3) + z_3(z_2 - z_1) + z_4(z_1 - z_3)\right) = -w\ .$$

Now,

$$\begin{aligned}|w|^2 = w\overline{w} &= \left(-(u+v)\right)\overline{\left(-(u+v)\right)}\\ &= (u+v)\left(\overline{u}+\overline{v}\right)\\ &= u\overline{u} + v\overline{v} + u\overline{v} + \overline{u}v\\ &= |u|^2 + |v|^2 + u\overline{v} + \overline{u}v\end{aligned}$$

Now, we will show below that $u\overline{v} + \overline{u}v = 0$.

$$\begin{aligned}u\overline{v} + \overline{u}v &= (z_1 - z_2)(z_3 - z_4)(\overline{z_1} + \overline{z_4})(\overline{z_3} - \overline{z_2}) + (\overline{z_1} - \overline{z_2})(\overline{z_3} - \overline{z_4})(z_1 + z_4)(z_3 - z_2)\\ &= (z_1 - z_2)(z_3 - z_4)(\frac{1}{z_1} + \frac{1}{z_4})(\frac{1}{z_3} - \frac{1}{z_2}) + (\frac{1}{z_1} - \frac{1}{z_2})(\frac{1}{z_3} - \frac{1}{z_4})(z_1 + z_4)(z_3 - z_2)\\ &= \frac{1}{z_1z_2z_3z_4}\left[(z_1 - z_2)(z_3 - z_4)(z_1 + z_4)(z_2 - z_3) + (z_1 + z_4)(z_3 - z_2)(z_2 - z_1)(z_4 - z_3)\right]\\ &= \frac{1}{z_1z_2z_3z_4} \times 0\\ &= 0\end{aligned}$$

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5. Prove that equation of the circle whose diameter is formed by joining z_1 and z_2 is
$$2z\overline{z} - z(\overline{z_1} + \overline{z_2}) - \overline{z}(z_1 + z_2) + z_1\overline{z_2} + \overline{z_1}z_2 = 0.$$

Answer:
Let z be any point lying on the circle. Then, the points z, z_1 and z_2 form a right angle triangle with hypotenuse as the diameter which is the line segment joining z_1 and z_2.
By Pythagorean theorem,

$$|z_1 - z_2|^2 = |z - z_1|^2 + |z - z_2|^2$$

$$(z_1 - z_2)(\overline{z_1} - \overline{z_2}) = (z - z_1)(\overline{z} - \overline{z_1}) + (z - z_2)(\overline{z} - \overline{z_2})$$

$$z_1\overline{z_1} - z_1\overline{z_2} - z_2\overline{z_1} + z_2\overline{z_2} = z\overline{z} - z\overline{z_1} - z_1\overline{z} + z_1\overline{z_1} + z\overline{z} - z\overline{z_2} - z_2\overline{z} + z_2\overline{z_2}$$

$$-z_1\overline{z_2} - z_2\overline{z_1} = 2z\overline{z} - z\overline{z_1} - z_1\overline{z} - z\overline{z_2} - z_2\overline{z}$$

$$2z\overline{z} - z(\overline{z_1} + \overline{z_2}) - \overline{z}(z_1 + z_2) + z_1\overline{z_2} + \overline{z_1}z_2 = 0$$

Thus, the equation of the circle whose diameter is formed by joining z_1 and z_2 is

$$2z\overline{z} - z(\overline{z_1} + \overline{z_2}) - \overline{z}(z_1 + z_2) + z_1\overline{z_2} + \overline{z_1}z_2 = 0\ .$$

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6. Interpret geometrically the following relations:
   (i) $\{z \in \mathbb{C} : |\Re(z)| + |\Im(z)| = 1\}$.
   (ii) $|z - a| - |z + a| = 2c$ where a and c are real constants with $c > 0$
   Answer:
   (i) $\{z \in \mathbb{C} : |\Re(z)| + |\Im(z)| = 1\}$
   In the first quadrant, $x + y = 1$. It is a straight line segment joining $(0,\ 1)$ and $(1,\ 0)$.
   In the second quadrant, $-x + y = 1$. It is a straight line segment joining $(-1,\ 0)$ and $(0,\ 1)$.
   In the third quadrant, $-x - y = 1$. It is a straight line segment joining $(-1,\ 0)$ and $(0,\ -1)$.
   In the fourth quadrant, $x - y = 1$. It is a straight line segment joining $(0,\ -1)$ and $(1,\ 0)$.

   (ii) $|z - a| - |z + a| = 2c$ where a and c are real constants with $c > 0$
   First we observe that $|2a| = |-2a| = |(z-a) - (z+a)| \geq ||z-a| - |z+a|| = |2c|$. This gives that $|a| \geq |c| = c > 0$.
   Let $z = x + iy$.

$$|z - a| - |z + a| = 2c \implies |z - a| = 2c + |z + a|$$

   That is,

$$\sqrt{(x-a)^2 + y^2} = 2c + \sqrt{(x+a)^2 + y^2}$$

$$(x-a)^2 + y^2 = 4c^2 + (x+a)^2 + y^2 + 4c\sqrt{(x+a)^2 + y^2}$$

$$x^2 + a^2 - 2ax + y^2 = 4c^2 + x^2 + a^2 + 2ax + y^2 + 4c\sqrt{(x+a)^2 + y^2}$$

$$-4ax = 4c^2 + 4c\sqrt{(x+a)^2 + y^2}$$

$$\frac{-ax - c^2}{c} = \sqrt{x^2 + a^2 + 2ax + y^2}$$

$$\frac{a^2x^2}{c^2} + c^2 + \frac{2axc}{c} = x^2 + a^2 + 2ax + y^2$$

$$x^2\left(\frac{a^2}{c^2} - 1\right) = (a^2 - c^2) + y^2$$

$$\frac{x^2}{c^2}(c^2 - a^2) + y^2 = c^2 - a^2$$

$$\frac{x^2}{c^2} + \frac{y^2}{(c^2 - a^2)} = 1$$

   The above equation represents a hyperbola.
   If $a = c$ then the equation $x^2\left(\frac{a^2}{c^2} - 1\right) = (a^2 - c^2) + y^2$ gives that $y = 0$. Further $|x-c| - |x+c| = 2c$ gives that $x \leq -c$. Thus, the given set describes the set $\{z = x + iy \in \mathbb{C}\ :\ x \leq -c \text{ and } y = 0\}$.
   If $a = -c$ then the given set describes the set $\{z = x + iy \in \mathbb{C}\ :\ x \geq c \text{ and } y = 0\}$.

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7. Find all the roots or all the values of the following:
(i) Cube roots of i (ii) Fourth roots of $(-2\sqrt{3}-2i)$ (iii) Fourth roots of (-1) (iv) Sixth roots of 8 (v) The values of $(i)^{\frac{2}{3}}$
Answer:
(i) Cube roots of i

$$i = 1 \left[\cos\left(\frac{\pi}{2} + 2k\pi\right) + i \sin\left(\frac{\pi}{2} + 2k\pi\right)\right] \qquad \text{where } k \in \mathbb{Z} .$$
$$(i)^{\frac{1}{3}} = 1^{\frac{1}{3}} \left[\cos\left(\frac{\frac{\pi}{2} + 2k\pi}{3}\right) + i \sin\left(\frac{\frac{\pi}{2} + 2k\pi}{3}\right)\right] \qquad \text{where } k = 0,\ 1,\ 2 .$$

$$k = 0 \quad \text{gives that} \quad \cos\left(\frac{\pi}{6}\right) + i \sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + \frac{i}{2}$$
$$k = 1 \quad \text{gives that} \quad \cos\left(\frac{5\pi}{6}\right) + i \sin\left(\frac{5\pi}{6}\right) = \frac{-\sqrt{3}}{2} + \frac{i}{2}$$
$$k = 2 \quad \text{gives that} \quad \cos\left(\frac{3\pi}{2}\right) + i \sin\left(\frac{3\pi}{2}\right) = -i$$

Thus, the cube roots of i are $\frac{\sqrt{3}}{2} + \frac{i}{2}$, $\frac{-\sqrt{3}}{2} + \frac{i}{2}$ and $-i$.

(ii) Fourth roots of $(-2\sqrt{3} - 2i)$

$$-2\sqrt{3} - 2i = 4 \left[\cos\left(\frac{7\pi}{6} + 2k\pi\right) + i \sin\left(\frac{7\pi}{6} + 2k\pi\right)\right] \qquad \text{where } k \in \mathbb{Z} .$$
$$\left(-2\sqrt{3} - 2i\right)^{\frac{1}{4}} = 4^{\frac{1}{4}} \left[\cos\left(\frac{\frac{7\pi}{6} + 2k\pi}{4}\right) + i \sin\left(\frac{\frac{7\pi}{6} + 2k\pi}{4}\right)\right] \qquad \text{where } k = 0,\ 1,\ 2,\ 3 .$$

$$k = 0 \quad \text{gives that} \quad \sqrt{2}\left[\cos\left(\frac{7\pi}{24}\right) + i \sin\left(\frac{7\pi}{24}\right)\right]$$
$$k = 1 \quad \text{gives that} \quad \sqrt{2}\left[\cos\left(\frac{19\pi}{24}\right) + i \sin\left(\frac{19\pi}{24}\right)\right]$$
$$k = 2 \quad \text{gives that} \quad \sqrt{2}\left[\cos\left(\frac{31\pi}{24}\right) + i \sin\left(\frac{31\pi}{24}\right)\right]$$
$$k = 3 \quad \text{gives that} \quad \sqrt{2}\left[\cos\left(\frac{43\pi}{24}\right) + i \sin\left(\frac{43\pi}{24}\right)\right]$$

(iii) Fourth roots of (-1)

$$-1 = 1 \ [\cos(\pi + 2k\pi) + i \sin(\pi + 2k\pi)] \qquad \text{where } k \in \mathbb{Z} .$$
$$(-1)^{\frac{1}{4}} = 1^{\frac{1}{4}} \left[\cos\left(\frac{(2k+1)\pi}{4}\right) + i \sin\left(\frac{(2k+1)\pi}{4}\right)\right] \qquad \text{where } k = 0,\ 1,\ 2,\ 3 .$$

$$k=0 \quad \text{gives that} \quad \cos\left(\frac{\pi}{4}\right) + \ i\ \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \ \frac{i}{\sqrt{2}}$$

$$k=1 \quad \text{gives that} \quad \cos\left(\frac{3\pi}{4}\right) + \ i\ \sin\left(\frac{3\pi}{4}\right) = \frac{-1}{\sqrt{2}} + \ \frac{i}{\sqrt{2}}$$

$$k=2 \quad \text{gives that} \quad \cos\left(\frac{5\pi}{4}\right) + \ i\ \sin\left(\frac{5\pi}{4}\right) = \frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$$

$$k=3 \quad \text{gives that} \quad \cos\left(\frac{7\pi}{4}\right) + \ i\ \sin\left(\frac{7\pi}{4}\right) = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$$

Thus, the fourth roots of (-1) are $\frac{1}{\sqrt{2}} + \ \frac{i}{\sqrt{2}}, \frac{-1}{\sqrt{2}} + \ \frac{i}{\sqrt{2}}, \frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$.

(iv) Sixth roots of 8

$$8 = 8\ \left[\cos(0+2k\pi) + \ i\ \sin(0+2k\pi)\right] \qquad \text{where } k \in \mathbb{Z}\ .$$

$$(8)^{\frac{1}{6}} = 8^{\frac{1}{6}}\ \left[\cos\left(\frac{2k\pi}{6}\right) + \ i\ \sin\left(\frac{2k\pi}{6}\right)\right] \qquad \text{where } k = 0,\ 1,\ \cdots\ 5\ .$$

$$k=0 \quad \text{gives that} \quad 8^{\frac{1}{6}}\ \left[\cos(0) + \ i\ \sin(0)\right] = 8^{\frac{1}{6}}$$

$$k=1 \quad \text{gives that} \quad 8^{\frac{1}{6}}\ \left[\cos\left(\frac{\pi}{3}\right) + \ i\ \sin\left(\frac{\pi}{3}\right)\right] = 8^{\frac{1}{6}}\ \left[\frac{1}{2} + \ i\ \frac{\sqrt{3}}{2}\right]$$

$$k=2 \quad \text{gives that} \quad 8^{\frac{1}{6}}\ \left[\cos\left(\frac{2\pi}{3}\right) + \ i\ \sin\left(\frac{2\pi}{3}\right)\right] = 8^{\frac{1}{6}}\ \left[\frac{-1}{2} + \ i\ \frac{\sqrt{3}}{2}\right]$$

$$k=3 \quad \text{gives that} \quad 8^{\frac{1}{6}}\ \left[\cos(\pi) + \ i\ \sin(\pi)\right] = -\,(8)^{\frac{1}{6}}$$

$$k=4 \quad \text{gives that} \quad 8^{\frac{1}{6}}\ \left[\cos\left(\frac{4\pi}{3}\right) + \ i\ \sin\left(\frac{4\pi}{3}\right)\right] = 8^{\frac{1}{6}}\ \left[\frac{-1}{2} - \ i\ \frac{\sqrt{3}}{2}\right]$$

$$k=5 \quad \text{gives that} \quad 8^{\frac{1}{6}}\ \left[\cos\left(\frac{5\pi}{3}\right) + \ i\ \sin\left(\frac{5\pi}{3}\right)\right] = 8^{\frac{1}{6}}\ \left[\frac{1}{2} - \ i\ \frac{\sqrt{3}}{2}\right]$$

Thus, the sixth roots of 8 are
$8^{\frac{1}{6}}, 8^{\frac{1}{6}}\ \left[\frac{1}{2} + \ i\ \frac{\sqrt{3}}{2}\right], 8^{\frac{1}{6}}\ \left[\frac{-1}{2} + \ i\ \frac{\sqrt{3}}{2}\right], -\,(8)^{\frac{1}{6}}, 8^{\frac{1}{6}}\ \left[\frac{-1}{2} - \ i\ \frac{\sqrt{3}}{2}\right]$ and $8^{\frac{1}{6}}\ \left[\frac{1}{2} - \ i\ \frac{\sqrt{3}}{2}\right]$.

(v) The values of $(i)^{\frac{2}{3}}$
$(i)^{\frac{2}{3}} = \left(i^2\right)^{\frac{1}{3}} = (-1)^{\frac{1}{3}}$.

$$-1 = 1\ \left[\cos(\pi+2k\pi) + \ i\ \sin(\pi+2k\pi)\right] \qquad \text{where } k \in \mathbb{Z}\ .$$

$$(-1)^{\frac{1}{3}} = 1^{\frac{1}{3}}\ \left[\cos\left(\frac{(2k+1)\pi}{3}\right) + \ i\ \sin\left(\frac{(2k+1)\pi}{3}\right)\right] \qquad \text{where } k = 0,\ 1,\ 2\ .$$

$$k=0 \quad \text{gives that} \quad \cos\left(\frac{\pi}{3}\right) + \ i\ \sin\left(\frac{\pi}{3}\right) = \frac{1}{2} + \ i\ \frac{\sqrt{3}}{2}$$

$$k = 1 \quad \text{gives that} \quad \cos(\pi) + \ i \ \sin(\pi) = -1$$
$$k = 2 \quad \text{gives that} \quad \cos\left(\frac{5\pi}{3}\right) + \ i \ \sin\left(\frac{5\pi}{3}\right) = \frac{1}{2} - \ i \ \frac{\sqrt{3}}{2}$$

The values of $(i)^{\frac{2}{3}}$ are $\frac{1}{2} + \ i \ \frac{\sqrt{3}}{2}$, $\quad -1$ and $\frac{1}{2} - \ i \ \frac{\sqrt{3}}{2}$.

Limit and Continuity

8. Show that $\lim_{z\to 0} \frac{|z|^2}{z} = 0$.
   Answer:
   We know that as $z \to 0$, $\overline{z} \to 0$. To see this, for a given $\epsilon > 0$, choose $\delta = \epsilon$. Then, $0 < |z - 0| = |z| < \delta = \epsilon$ implies that $|\overline{z} - 0| = |\overline{z}| = |z| < \epsilon$.
   Now,
$$\lim_{z\to 0} \frac{|z|^2}{z} = \lim_{z\to 0} \frac{z\ \overline{z}}{z} = \lim_{z\to 0} \overline{z} = 0\ .$$

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9. Let $f(z) = z^2/|z|^2$.
   (a) Find the value of limit of $f(z)$ as $z = (x + iy) \to 0$ along the line $y = x$.
   (b) Find the value of limit of $f(z)$ as $z = (x + iy) \to 0$ along the line $y = 2x$.
   (c) Find the value of limit of $f(z)$ as $z = (x + iy) \to 0$ along the path $y = x^2$.
   (d) What can you conclude about the limit of $f(z)$ as $z \to 0$.
   Answer:
   (a)
$$\lim_{\substack{(x,\,y)\to(0,\,0)\\ y=x}} f(z) = \lim_{\substack{(x,\,y)\to(0,\,0)\\ y=x}} \frac{x^2 - x^2}{x^2 + x^2} + \ i \ \frac{2x^2}{x^2 + x^2} = \lim_{\substack{(x,\,y)\to(0,\,0)\\ y=x}} 0 + \ i \ = \ i\ .$$
   (b)
$$\lim_{\substack{(x,\,y)\to(0,\,0)\\ y=2x}} f(z) = \lim_{\substack{(x,\,y)\to(0,\,0)\\ y=2x}} \frac{x^2 - 4x^2}{x^2 + 4x^2} + \ i \ \frac{4x^2}{x^2 + 4x^2} = \frac{-3}{5} + \ i \ \frac{4}{5}.$$
   (c)
$$\lim_{\substack{(x,\,y)\to(0,\,0)\\ y=x^2}} \frac{x^2 - x^4}{x^2 + x^4} + \ i \ \frac{2x^3}{x^2 + x^4} = \lim_{\substack{(x,\,y)\to(0,\,0)\\ y=x^2}} \frac{x^2(1 - x^2)}{x^2(1 + x^2)} + \ i \ \frac{2x^3}{x^2(1 + x^2)} = 1\ .$$
   (d) Since $f(z)$ approaches different values as z approaches 0 along different paths, we conclude that the limit of $f(z)$ as $z \to 0$ does not exist.

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10. The following functions are defined for $z \neq 0$. Which of these functions can be defined at $z = 0$ so that it becomes continuous at $z = 0$.
    (a) $\frac{\Re(z)}{|z|}$ (b) $\frac{z}{|z|}$ (c) $\frac{z\Re(z)}{|z|}$ (d) $\frac{\Re(z^2)}{|z|^2}$ (e) $\frac{z^2}{|z|}$

Answer:

(a) $\dfrac{\Re(z)}{|z|} = \dfrac{x}{\sqrt{x^2+y^2}}$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x}} \frac{x}{\sqrt{x^2+x^2}} = \frac{1}{\sqrt{2}} \qquad \text{and} \qquad \lim_{\substack{(x,y)\to(0,0)\\ y=2x}} \frac{x}{\sqrt{x^2+4x^2}} = \frac{1}{\sqrt{5}}.$$

Since $f(z)$ approaches different values as z approaches 0 along different paths, we conclude that $\lim_{z\to 0} f(z)$ does not exist and hence $f(z)$ can not be continuous at $z=0$.

(b) $\dfrac{z}{|z|} = \dfrac{x+iy}{\sqrt{x^2+y^2}}$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x}} \frac{x+i\,x}{\sqrt{x^2+x^2}} = \frac{1+i}{\sqrt{2}} \qquad \text{and} \qquad \lim_{\substack{(x,y)\to(0,0)\\ y=2x}} \frac{x+i\,2x}{\sqrt{x^2+4x^2}} = \frac{1+2\,i}{\sqrt{5}}.$$

Since $f(z)$ approaches different values as z approaches 0 along different paths, we conclude that $\lim_{z\to 0} f(z)$ does not exist and hence $f(z)$ can not be continuous at $z=0$.

(c) $\dfrac{z\Re(z)}{|z|} = \dfrac{(x+i\,y)x}{\sqrt{x^2+y^2}} = \dfrac{x^2+i\,xy}{\sqrt{x^2+y^2}}$

Suppose that we define $f(0)=0$. Then,

$$|f(z)-f(0)| = \left|\frac{z\Re(z)}{|z|} - 0\right| = \frac{|z|\,|\Re(z)|}{|z|} = |\Re(z)|\ .$$

We know that $|\Re(z)| \le |z|$.

Given $\epsilon > 0$, choose $\delta = \epsilon$. Then,

$$|z-0| = |z| < \delta = \epsilon \qquad \text{implies that} \qquad |f(z)-f(0)| = |\Re(z)| \le |z| < \epsilon\ .$$

Therefore, by defining $f(0)=0$, we can make the function $f(z) = \dfrac{z\Re(z)}{|z|}$ is continuous at $z=0$.

(d) $\dfrac{\Re(z^2)}{|z|^2} = \dfrac{x^2-y^2}{x^2+y^2}$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x}} \frac{x^2-x^2}{x^2+x^2} = 0 \qquad \text{and} \qquad \lim_{\substack{(x,y)\to(0,0)\\ y=2x}} \frac{x^2-4x^2}{x^2+4x^2} = \frac{-3}{5}.$$

Since $f(z)$ approaches different values as z approaches 0 along different paths, we conclude that $\lim_{z\to 0} f(z)$ does not exist and hence $f(z)$ can not be continuous at $z=0$.

(e) $\dfrac{z^2}{|z|} = \dfrac{(x^2-y^2)+\ i\,2xy}{\sqrt{x^2+y^2}}$

Suppose that we define $f(0)=0$. Then,

$$|f(z)-f(0)| = \left|\frac{z^2}{|z|} - 0\right| = \frac{|z^2|}{|z|} = \frac{|z|^2}{|z|} = |z|\ .$$

Given $\epsilon > 0$, choose $\delta = \epsilon$. Then,

$$|z - 0| = |z| < \delta = \epsilon \qquad \text{implies that} \qquad |f(z) - f(0)| = |z| < \epsilon \ .$$

Therefore, by defining $f(0) = 0$, we can make the function $f(z) = \dfrac{z^2}{|z|}$ is continuous at $z = 0$.
