Complex Analysis: Lecture-05

MA201 Mathematics III

MGPP, AC, ST, SP

IIT Guwahati

Sufficient Conditions for Differentiability

Theorem

Sufficient conditions for differentiability: Let $f(z) = u(x, y) + i\, v(x, y)$ be defined in some neighborhood of the point $z_0 = x_0 + i\, y_0$. Suppose that

- the first order partial derivatives u_x, u_y, v_x and v_y exist in a neighborhood of $z_0 = (x_0, y_0)$,
- u_x, u_y, v_x and v_y are continuous at the point (x_0, y_0),
- the Cauchy Riemann equations $u_x = v_y$, $u_y = -v_x$ hold at the point z_0.

Then, the function f is differentiable at z_0 and the derivative
$f'(z_0) = u_x(x_0, y_0) + i\, v_x(x_0, y_0) = v_y(x_0, y_0) - i\, u_y(x_0, y_0)$.

Example

Let $f(x+iy) = e^{-x}\cos y - \ i\, e^{-x}\sin y$ for $z = x+iy \in \mathbb{C}$.
Then,

- the functions

$$u_x = -e^{-x}\cos y\ ,$$
$$u_y = -e^{-x}\sin y\ ,$$
$$v_x = e^{-x}\sin y\ ,$$
$$v_y = -e^{-x}\cos y$$

  are continuous in $\mathbb{C}$, and
- For any $z = x+iy \in \mathbb{C}$, f satisfies the CR equations:

$$u_x = -e^{-x}\cos y = v_y \qquad \text{and} \qquad u_y = -e^{-x}\sin y = -v_x\ .$$

Therefore, by the previous theorem (sufficient conditions for differentiability), we conclude that $f(z)$ is differentiable in $\mathbb{C}$ and $f'(z) = u_x + \ i\, v_x = -e^{-x}\cos y + \ i\, e^{-x}\sin y$ at each point of $\mathbb{C}$.

CR equations in Polar Form

Let $f(z) = f(re^{i\theta}) = u(r,\ \theta) + i\ v(r,\ \theta)$ be differentiable in D.
The polar form of the Cauchy-Riemann equations of f is given by

$$u_r(r,\ \theta) = \frac{1}{r} v_\theta(r,\ \theta) \qquad \text{and} \qquad v_r(r,\ \theta) = \frac{-1}{r} u_\theta(r,\ \theta)\ .$$

Since $x = r\cos\theta$ and $y = r\sin\theta$, we have

$$\begin{aligned} u_r &= \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = u_x\cos\theta + u_y\sin\theta \\ u_\theta &= \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = -u_x r\sin\theta + u_y r\cos\theta. \end{aligned}$$

i.e., $u_r = u_x\cos\theta + u_y\sin\theta,\ u_\theta = -u_x r\sin\theta + u_y r\cos\theta$ (1)

Similarly, $v_r = v_x\cos\theta + v_y\sin\theta,\ v_\theta = -v_x r\sin\theta + v_y r\cos\theta.$ (2)

From CR equations: $u_x = v_y, u_y = -v_x$, (2) becomes

$$v_r = -u_y\cos\theta + u_x\sin\theta,\ v_\theta = u_y r\sin\theta + u_x r\cos\theta. \tag{3}$$

From (1) and (3), $ru_r = v_\theta,\quad u_\theta = -rv_r$ **(CR Equations in polar form)**.

CR Equations in Complex Form

The Cauchy-Riemann equations in complex form is given by

$$\frac{\partial f}{\partial \bar{z}} = 0 \, .$$

Proof:

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}}(u(x,y) \; + \; i \; v(x,y)) = \frac{\partial u}{\partial \bar{z}} \; + \; i \; \frac{\partial v}{\partial \bar{z}}$$

$$= \left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \bar{z}}\right) \; + \; i \; \left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \bar{z}}\right)$$

$$= \left(\frac{\partial u}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial u}{\partial y}\left(\frac{i}{2}\right)\right) \; + \; i \; \left(\frac{\partial v}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial v}{\partial y}\left(\frac{i}{2}\right)\right)$$

$$= \frac{1}{2}\left[\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) \; + \; i \; \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] \; = \; 0$$

Note: A differentiable function $f(z)$ can not contain any terms involving $\bar{z}$ explicitly!

ANALYTIC FUNCTIONS

Analytic Functions (Holomorphic Functions)

Definition

Let $f(z)$ be a function defined on an open set $S \subseteq \mathbb{C}$. Then the function $f(z)$ is said to be analytic in the open set S if $f(z)$ is differentiable at each point of S.

Examples: The functions $f(z) = z$ and $g(z) = z^2$ are analytic in $\mathbb{C}$. The functions $f(z) = \overline{z}$ and $g(z) = |z|^2$ are no where analytic in $\mathbb{C}$.

Definition

Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ and let $z_0 \in D$. Then, the function f is said to be analytic at the point z_0 if there exists an open neighborhood $N(z_0) \subset D$ of z_0 such that f is differentiable at each point of $N(z_0)$.
Further f is said to be analytic in D if f is analytic at each point of D.

Note: The other terminologies for analytic are holomorphic or regular.
Think: Suppose f is analytic at a point z_0. Does it imply that f is analytic in an open set containing z_0?

”Analytic” is a property ”defined over open sets”

- We emphasize that analyticity is a property defined over open sets, while differentiability could conceivable hold at one point only.
- That is, we can have a function which is differentiable at exactly one point in $\mathbb{C}$. But we cannot construct a function which is analytic at exactly one point in $\mathbb{C}$.
- If we say $f(z)$ is analytic in a set S which is not open in $\mathbb{C}$, then it actually means that $f(z)$ is analytic in an open set D which contains S.

Results on Analyticity

Theorem

If $f(z)$ is analytic in an open set D then $f(z)$ is differentiable in D.

Note: Converse of above theorem is not true. Example: $|z|^2$ at $z = 0$.

Theorem

Necessary condition for analyticity:
Let $f(z)$ be analytic in an open set D of $\mathbb{C}$. Then $f(z)$ satisfies the Cauchy-Riemann equations at each point of D.

Theorem

Sufficient conditions for analyticity: Let $f(z) = u(x,\ y) + i\ v(x,\ y)$ be defined in an open set D. If the first order partial derivatives of u and v exist, continuous and satisfy the Cauchy-Riemann equations at all points of D, then f is analytic in D.

In case of analytic function f in an open set D, the previous result becomes necessary and sufficient conditions.

A function $f(z) = u(x,\ y) + i\ v(x,\ y)$ is analytic in an open set $D \subseteq \mathbb{C}$

if and only if

the first order partial derivatives of u and v exist, continuous and satisfy the Cauchy-Riemann equations at all points of D.

Results on Analyticity (continuation)

Theorem

Suppose that $f(z)$ and $g(z)$ are analytic in an open set D of $\mathbb{C}$. Then the functions $f+g$, $f-g$, fg are analytic in D. If $g(z) \neq 0$ for all $z \in D$ then the function f/g is analytic in D.

Theorem

If f is analytic in an open set D and g is analytic in an open set containing $f(D)$, then the composite function $h(z) = g(f(z))$ is analytic in D.

Theorem

Let $f(z)$ be analytic in an open set D of $\mathbb{C}$. Then the derivatives of all orders of $f(z)$ exist in D and they are analytic in D. That is, $f^{(n)}(z)$ for all $n \in \mathbb{N}$ exist and analytic in D.

Proof: Will be proved later.

Theorem

If $f(z)$ is analytic in an open and connected set D in $\mathbb{C}$ and if $f'(z) = 0$ everywhere in D, then $f(z)$ is constant in D.

Proof: Worked out on the board.

Results on Analyticity (continuation)

Theorem

Let $f(z) = u(x,\ y) + \ i\ v(x,\ y)$ be an analytic function in a domain D of $\mathbb{C}$. If any one of the following conditions hold in the domain D, then the function $f(z)$ is constant in D:

1. $f'(z) = 0$ for all $z \in D$.
2. $u(x,\ y)$ is constant in D.
3. $v(x,\ y)$ is constant in D.
4. $f(z)$ is real valued for all $z \in D$.
5. $f(z)$ is pure imaginary valued for all $z \in D$.
6. $|f(z)|$ is constant in D.
7. Arg $(f(z))$ is constant in D.
8. $\overline{f(z)}$ is also analytic in D.
9. $|f(z)|$ is also analytic in D.