Complex Analysis: Lecture-04

MA201 Mathematics III

MGPP, AC, ST, SP

IIT Guwahati

DIFFERENTIABLE FUNCTIONS

Differentiability

Definition

Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ where D is an open set. Let $z_0 \in D$. If

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists, then f is said to be differentiable at the point z_0, and the number

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

is called the derivative of f at z_0.

If we write $\Delta z = z - z_0$, then the above definition can be expressed in the form

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z)}{\Delta z}$$

We can also use the Leibnitz notation for the derivative, $\frac{df}{dz}(z_0)$, or $\frac{df}{dz}|_{z=z_0}$.

Note: In the complex variable case there are infinitely many directions in which a variable can approach a point z_0. But, in the real case, there are only two directions, namely, left and right to approach. So the statement that a function of a complex variable has a derivative is **stronger** than the same statement about a function of a real variable. For example, the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$ is differentiable on $\mathbb{R} \setminus \{0\}$. But, if we consider the same function as a function of complex variable, that is, $f : \mathbb{C} \to \mathbb{R}$ given by $f(z) = |z|$, then it is nowhere differentiable in $\mathbb{C}$ (which will be proved later). Therefore, it has lost the differentiability on the set $\mathbb{R} \setminus \{0\}$ in the complex case.

Examples

Example 1: By using the definition of derivative, let us compute $f'(z)$ at an arbitrary point $z_0 \in \mathbb{C}$ for the function $f(z) = z$.

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{z - z_0}{z - z_0} = \lim_{z \to z_0} 1 = 1 \ .$$

Therefore, the derivative of $f(z) = z$ is $f'(z) = 1$ for any $z \in \mathbb{C}$.

Example 2: By using the definition of derivative, let us compute $f'(z)$ at an arbitrary point $z_0 \in \mathbb{C}$ for the function $f(z) = z^2$.

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{z^2 - z_0^2}{z - z_0} = \lim_{z \to z_0} (z + z_0) = 2z_0 \ .$$

Therefore, the derivative of $f(z) = z^2$ is $f'(z) = 2z$ for any $z \in \mathbb{C}$.

Examples

Example 3:
Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = \bar{z}$ for $z \in \mathbb{C}$.
Examine the differentiability of $f(z)$ at each point z of $\mathbb{C}$.
Answer: $f(z) = \bar{z}$ is not differentiable at any point of $\mathbb{C}$ (No where differentiable in $\mathbb{C}$).
Details are worked out on the board.

Example 4:
Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = (x, -y)$ for $(x, y) \in \mathbb{R}^2$.
Examine the Frechet differentiability of f on $\mathbb{R}^2$ (Differentiability in $\mathbb{R}^2$ from multivariable calculus)
Compare with the above example (Example 3).
Details are worked out on the board.

Examples

Example 5:

On $\mathbb{C}$, examine the differentiability of $f(z) = |z|^2$ for $z \in \mathbb{C}$.

Answer: The $f(z) = |z|^2$ is not differentiable in $\mathbb{C} \setminus \{0\}$. Further it is differentiable at $z = 0$.
That is, $|z|^2$ is differentiable only at the point 0 in $\mathbb{C}$.

Comparison: The function $f(x) = |x|^2$ for $x \in \mathbb{R}$ is (real) differentiable at each point of $\mathbb{R}$.
Details are worked out on the board.

Example 6:

On $\mathbb{C}$, examine the differentiability of $f(z) = |z|$ for $z \in \mathbb{C}$.

Answer:The $f(z) = |z|$ is not differentiable at any point of $\mathbb{C}$.
That is, $|z|$ is nowhere differentiable in $\mathbb{C}$.

Comparison: The function $f(x) = |x|$ for $x \in \mathbb{R}$ is (real) differentiable at each point of $\mathbb{R} \setminus \{0\}$.
Details are worked out on the board.

Results and Properties

1. If $f(z)$ is differentiable at z_0, then f is continuous at z_0.
2. If $f(z) \equiv c$ is a constant function, then $f'(z) = 0$.

Theorem

Let $f(z)$ and $g(z)$ be two differentiable functions. Then,

1. *Sum:* $\dfrac{d}{dz}[f(z) + g(z)] = f'(z) + g'(z)$
2. *Product:* $\dfrac{d}{dz}[f(z)g(z)] = f'(z)g(z) + f(z)g'(z)$
3. *Quotient:* $\dfrac{d}{dz}\left[\dfrac{f(z)}{g(z)}\right] = \dfrac{f'(z)g(z) - f(z)g'(z)}{(g(z))^2}$ *provided* $g(z) \neq 0$
4. *Composition:* $\dfrac{d}{dz}[f(g(z))] = f'(g(z))\, g'(z)$

Total Derivative of f and Partial Derivatives of Component Functions

Let $f(z) = u(x, y) + iv(x, y)$ be a complex function defined on an open set $G \subseteq \mathbb{C}$. Then, the function $u(x, y)$ and $v(x, y)$ are functions from the set $G \subseteq \mathbb{R}^2$ to $\mathbb{R}$. Suppose that $f(z)$ is differentiable at a point $z_0 = x_0 + iy_0 \in G$. Is there any relation between $f'(z_0)$ and the partial derivatives of $u(x, y)$ and $v(x, y)$ at the point (x_0, y_0)?

The answer to the above question was discovered independently by the French mathematician A. L. Cauchy (1789-1857) and the German mathematician G. F. B. Riemann (1826-1866).

Cauchy

Riemann

Cauchy-Riemann Equations

Theorem

Necessary Condition for Differentiability:
Let $f(z) = f(x+iy) = u(x,\ y) + iv(x,\ y)$ be differentiable at the point $z_0 = x_0 + iy_0$. Then, the first order partial derivatives of $u(x,\ y)$ and $v(x,\ y)$ exist at the point $z_0 = (x_0,\ y_0)$ and satisfy the following equations

$$u_x(x_0,\ y_0) = v_y(x_0,\ y_0) \quad \text{and} \quad u_y(x_0,\ y_0) = -v_x(x_0,\ y_0)\ .$$

The above equations are called the Cauchy-Riemann Equations or briefly CR equations.

Proof: Now, $\lim\limits_{z\to z_0} \dfrac{f(z)-f(z_0)}{z-z_0}$ exists and equals $f'(z_0)$ (whichever way z approaches z_0). Let us make z to approach z_0 along two specific paths (i) Horizontally and (ii) Vertically.

Continuation of Proof

- Horizontally: $z = x + iy_0$ approaches $z_0 = x_0 + iy_0$ as x approaches x_0. So,

$$f'(z_0) = \lim_{x \to x_0} \frac{u(x, y_0) - u(x_0, y_0)}{x - x_0} + i \lim_{x \to x_0} \frac{v(x, y_0) - v(x_0, y_0)}{x - x_0},$$

  i.e., $f'(z_0) = u_x(x_0, y_0) + i\, v_x(x_0, y_0).$ (i)

- Vertically: $z = x_0 + iy$ approaches $z_0 = x_0 + iy_0$ as y approaches y_0. So,

$$f'(z_0) = \lim_{y \to y_0} \frac{u(x_0, y) - u(x_0, y_0)}{i(y - y_0)} + i \lim_{y \to y_0} \frac{v(x_0, y) - v(x_0, y_0)}{i(y - y_0)},$$

  i.e., $f'(z_0) = -i\, u_y(x_0, y_0) + v_y(x_0, y_0).$ (ii)

From (i) and (ii) we have at $z_0 = (x_0, y_0)$, $u_x = v_y, v_x = -u_y$.

From the proof of the previous theorem, one can observe that $f'(z_0)$ can be written in terms of the partial derivatives of u and v as follows:

$$\begin{aligned} f'(z_0) &= u_x(x_0,\ y_0) + \ i\, v_x(x_0,\ y_0)\ . \\ \textit{Also } f'(z_0) &= v_y(x_0,\ y_0) - \ i\, u_y(x_0,\ y_0)\ . \end{aligned}$$

Example: We know that $f(z) = z^2 = x^2 - y^2 + i2xy$ is differentiable at each point $z \in \mathbb{C}$. Then, $u(x,\ y) = x^2 - y^2$, $v(x,\ y) = 2xy$ so that $f(z) = u + iv$. The first order partial derivatives of u and v at a point z are $u_x = 2x$, $u_y = -2y$, $v_x = 2y$ and $v_y = 2x$. Therefore, it follows that at each point z

$$u_x = 2x = v_y \qquad \text{and} \qquad u_y = -2y = -v_x\ .$$

Thus, $f(z)$ satisfies the Cauchy-Riemann equations at each point $z \in \mathbb{C}$. Observe that $f'(z) = u_x + i\, v_x = 2x + i\, 2y = 2z = v_y - i\, u_y$.

Not satisfying Cauchy-Riemann equations $\Longrightarrow$ Not differentiable

The idea of "**Not satisfying Cauchy-Riemann equations $\Longrightarrow$ Not differentiable**" can be used as one of the methods to show the non-differentiability of complex functions.

Example: Let $f(z) = \bar{z} = x - i\,y$ for $z \in \mathbb{C}$. Set $u(x,\ y) = x$ and $v(x,\ y) = -y$. This gives that $u_x = 1$, $u_y = 0$, $v_x = 0$ and $v_y = -1$. Now, $1 = u_x \neq v_y = -1$ and $0 = u_y = -v_x = 0$ at any point $z \in \mathbb{C}$. Therefore, the function $f(z)$ does **not** satisfy the Cauchy-Riemann equations at any point $z \in \mathbb{C}$ and consequently, it is **not** differentiable at any point $z \in \mathbb{C}$.

Now, a serious question arising in our mind is that Will the satisfaction of the Cauchy-Riemann equations make the function differentiable? The answer is non-affirmative. The next example shows that the mere satisfaction of the Cauchy-Riemann equations is not a sufficient criterion to guarantee the differentiability of a function.

Function satisfying CR equations, but not differentiable

Let $f(z) = \bar{z}^2/z$ for $z \neq 0$ and $f(0) = 0$. First we test the differentiability of $f(z)$ at the point $z = 0$.

$$\lim_{z\to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{(x,\, y)\to(0,\, 0)} \frac{\left(\frac{x^3-3xy^2}{x^2+y^2} + i\,\frac{y^3-3x^2y}{x^2+y^2}\right) - 0}{x + iy - 0}$$

Let z approach 0 along the x-axis. Then, we have

$$\lim_{(x,\, 0)\to(0,\, 0)} \frac{x - 0}{x - 0} = 1 \ .$$

Let z approach 0 along the line $y = x$. This gives

$$\lim_{(x,\, x)\to(0,\, 0)} \frac{-x - ix}{x + ix} = -1 \ .$$

Since the limits are distinct, we conclude that f is not differentiable at the origin.

Continuation of previous slide

$$f(x+iy) = \left(\frac{x^3 - 3xy^2}{x^2+y^2} + i\,\frac{y^3 - 3x^2y}{x^2+y^2}\right) \qquad \text{and} \qquad f(0) = 0\ .$$

Now, we verify the Cauchy-Riemann equations at the point $z = 0$.
To calculate the partial derivatives of u and v at $(0,\ 0)$, we use the definitions (Why!).

$$u_x(0,\ 0) = \lim_{x\to 0} \frac{u(0+x,\ 0) - u(0,\ 0)}{x} = \lim_{x\to 0} \frac{x-0}{x} = 1\ .$$

$$v_y(0,\ 0) = \lim_{y\to 0} \frac{v(0,\ 0+y) - v(0,\ 0)}{y} = \lim_{y\to 0} \frac{y-0}{y} = 1\ .$$

In a similar fashion, one can show that

$$u_y(0,\ 0) = 0, \qquad \text{and} \qquad v_x(0,\ 0) = 0\ .$$

Hence the function satisfies the Cauchy-Riemann equations at the point $z = 0$.