Complex Analysis: Lecture-02

MA201 Mathematics III

MGPP, AC, ST, SP

IIT Guwahati

Exponential form of Non-Zero Complex Numbers

- Let $z = x + iy \neq 0$ be written in the trigonometric form as $z = r(\cos\theta + i\ \sin\theta)$ where r is the modulus and θ is the argument of z.
- The Euler's formula says that

$$e^{i\theta} = \cos\theta + \ i\ \sin\theta$$

where θ is measured in radians.

If $z \neq 0$ then using Euler's formula, we can write z as

$$z = re^{i\theta}$$

where $r = |z|$ and $\theta = \arg(z)$ which is known as the exponential form of a complex number z.

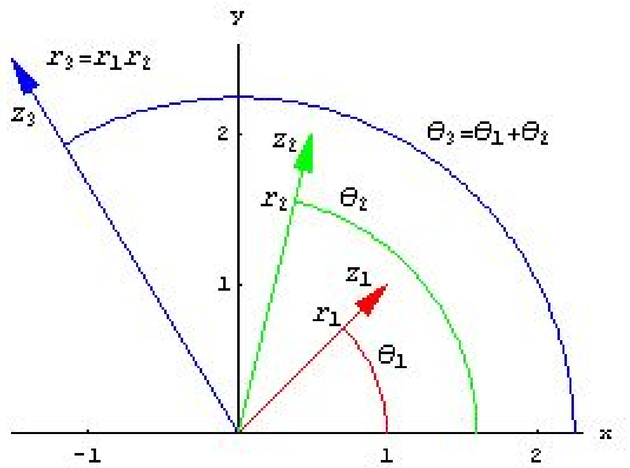
Examples: $1 + i = \sqrt{2}e^{i\pi/4}$, $\quad -i = e^{-i\pi/2}$, $\quad -8 = 8e^{i\pi} = 8e^{i3\pi}$.

Geometrical Interpretation of Multiplication

Let $z_1 \neq 0$ and $z_2 \neq 0$. Then,

$$z_i = r_i(\cos\theta_i + \ i \ \sin\theta_i), \quad i = 1, \ 2 \ .$$

$$\begin{aligned} z_1 z_2 &= r_1 r_2 \left[(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + i\,(\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2)\right] \\ &= r_1 r_2 \ \left[\cos(\theta_1 + \theta_2) \ + \ i \ \sin(\theta_1 + \theta_2)\right] \end{aligned}$$



$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

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The above identity is to be interpreted as saying that if values of two of these three (multiple valued) arguments are specified, then there is a value of the third such that the above equation holds.

Example: If $3 = 3e^{2\pi i}$ and $-2 = 2e^{3\pi i}$ then $-6 = 6e^{i\theta_3}$ with $\theta_3 = 5\pi$ (one of the values of $\arg(-6)$ plus a suitable multiple of 2π is to be taken) so that the identity holds.

In the above identity, if we replace $\arg(z)$ by Arg (z), then identity is in general NOT true. If z_1 and z_2 lies in the first quadrant then it will be true.

$$\textit{Arg}\,(z_1 z_2) \neq \textit{Arg}\,(z_1) + \textit{Arg}\,(z_2) \qquad \textit{(in general)}\ .$$

If $0 \neq z = re^{i\theta}$ then $(1/z) = (1/r)e^{-i\theta}$ and hence $\arg(1/z) = -\arg(z)$.

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)\ .$$

Powers of Complex Numbers

Let z be a complex number and let n be an integer.

- If $z = 0$, we have $z^n = 0$ if $n \in \mathbb{N}$.
- If $z \neq 0$, then setting $z = re^{i\theta}$ and using $e^{t_1}e^{t_2} = e^{t_1+t_2}$ by mathematical induction one can prove that

$$z^n = r^n e^{in\theta} \qquad \text{for} \qquad n = 0,\ 1,\ 2,\ 3,\ \cdots\ .$$

- If n is negative integer, then set $m = -n$ and apply the above equation to $(1/z)^m$ to get $z^n = r^n e^{in\theta}$.
- If $r = 1$ then we get $\left(e^{i\theta}\right)^n = e^{in\theta}$.
- Rewriting it, we get following de Moivre's formula.

$$(\cos\theta + i\ \sin\theta)^n = \cos(n\theta) + \ i\ \sin(n\theta) \qquad \text{for} \qquad n \in \mathbb{Z}\ .$$

- Example: $(\sqrt{3} + i)^7 = (2e^{i\pi/6})^7 = 2^7 e^{i7\pi/6} = (2^6 e^{i\pi})(2e^{i\pi/6}) = -64(\sqrt{3} + i)$.

n-th Roots of Unity ($1^{1/n}$)

Find the solutions of the equation $z^n = 1$ where n is a positive integer.
Let $z = re^{i\theta}$ be a solution to $z^n = 1$.
Then, $z^n = r^n(e^{i\theta})^n = r^n e^{i\,n\theta} = 1 \cdot e^{i0}$ which implies

$$r^n = 1, \qquad n\theta = 0 + 2k\pi \text{ where } k \text{ is an integer .}$$

We get n distinct solutions to $z^n = 1$ by setting $k = 0, 1, \cdots, n-1$ as

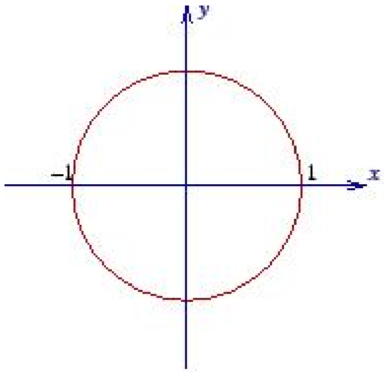
$$z_k = e^{i\frac{2k\pi}{n}} = \cos\left(\frac{2k\pi}{n}\right) + i\ \sin\left(\frac{2k\pi}{n}\right)$$

where $k = 0, 1, \cdots, n-1$ and are called the n-th roots of unity.

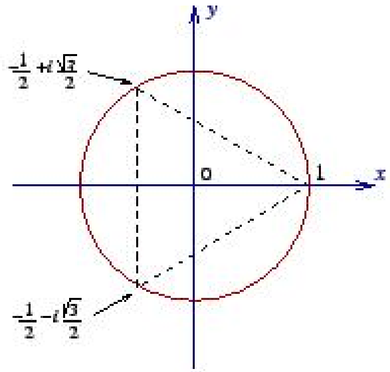
Set $\omega_n = e^{i2\pi/n}$ (primitive n-th root of unity). By De Moivre's formula, the n-th roots of unity can be expressed as $1, \omega_n, \omega_n^2, \omega_n^3, \cdots, \omega_n^{n-1}$.

Properties of n-th Roots of Unity

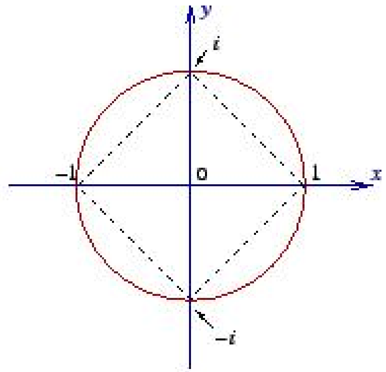
Geometrically, the n-th roots of unity are equally spaced points that lie on the unit circle $\{z \ : \ |z| = 1\}$ and form the vertices of a regular polygon with n sides.



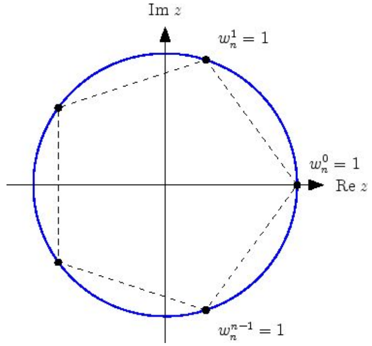
(c) $n = 2$



(d) $n = 3$



(e) $n = 4$



n-th Roots of Nonzero Complex Number $W^{1/n}$

Find the solutions of the equation $z^n = W$ where n is a positive integer.
Let $z = re^{i\theta}$ be a solution to $z^n = W = \rho e^{i\phi}$.
$z^n = r^n e^{in\theta} = W = \rho e^{i\phi}$ gives that

$$r^n = \rho \qquad \text{and} \qquad n\theta = \phi + 2k\pi \qquad \text{where } k \in \mathbb{Z}\,.$$

By setting $k = 0, 1, \cdots, n-1$,we get n distinct solutions to $z^n = W$ as

$$z_k = \rho^{\frac{1}{n}}\; e^{i\frac{\phi+2k\pi}{n}} = \rho^{\frac{1}{n}}\left[\cos\left(\frac{\phi+2k\pi}{n}\right) + \; i\;\sin\left(\frac{\phi+2k\pi}{n}\right)\right]$$

for $k = 0, 1, \cdots, n-1$.
If c is any n-th root of W then all the n-th roots of W are given by $c, c\omega_n, c\omega_n^2, \cdots, c\omega_n^{n-1}$ where ω_n is a primitive n-th root of unity.
Example: Cube roots of $64i$ are $z_0 = 4e^{i\pi/6} = 2\sqrt{3} + i\,2$, $z_1 = 4e^{i5\pi/6} = -2\sqrt{3} + i\,2$ and $z_3 = 4e^{i3\pi/2} = -4i$.

Computing W^α where $W \neq 0$ and $\alpha \in \mathbb{Q}$

Let W be a nonzero complex number.
Let $\alpha = m/n$ where m and n are integers with gcd $(m, n) = 1$.
Then,

$$W^\alpha = W^{m/n} = (W^m)^{1/n} \ .$$

Since m is an integer, W^m will be a single complex number.
Then, taking n-th root of W^m, we get n distinct complex numbers z_k satisfying $z_k^n = W^m$.

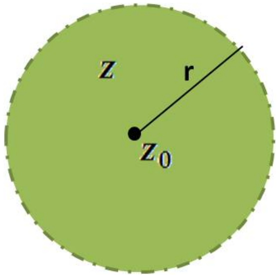
Exercise: Find all values of $(-8i)^{2/3}$.
Exercise: From real function to complex function what is happening? Compare domain of definition and range of real function $x_0^{1/n}$ and complex function $z_0^{1/n}$.

Sets in $\mathbb{C}$ (Planar Sets)

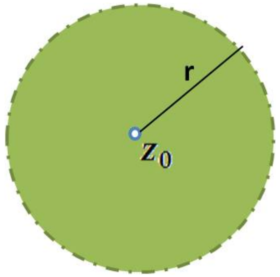
Identify the following sets / Find the Locus of the Points satisfying the equations / Interpret geometrically the following relations:

1. $\{z \in \mathbb{C} : |z-1| - |z+1| = 0\}$.
2. $\{z \in \mathbb{C} : |\Re(z)| + |\Im(z)| = 1\}$.
3. $|z-a| - |z+a| = 2c$ where a and c are real constants with $c > 0$.
4. $z = a + tb$ for $t \in \mathbb{R}$ where a and $b \neq 0$ are complex constants.
5. $\{z \in \mathbb{C} : \text{Im}\left(\frac{z-a}{b}\right) > 0\}$ where a and $b \neq 0$ are complex constants.

Open Ball/Neighorhood, Puncture Neighborhood



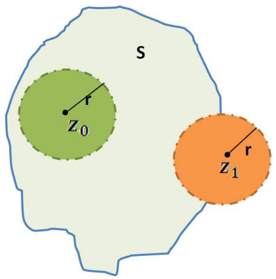
$|z - z_0| < R$



$0 < |z - z_0| < R$

- Open Disk/Open Ball centered at the point z_0 with radius r is denoted by $B_r(z_0)$ (or $B(z_0)$ or $B(z_0,\ r)$) and is defined by $B_r(z_0) = \{z \in \mathbb{C}\ :\ |z - z_0| < r\}$.
- Let z_0 be a point in $\mathbb{C}$. Any open ball with center at z_0 and radius $r > 0$ is called an open neighborhood of z_0 or simply a neighborhood of z_0 and is usually denoted by $N_r(z_0)$ or $N(z_0)$ or $N(z_0,\ r)$.
- A punctured or deleted neighborhood of a point z_0 is given by $B_r(z_0) \setminus \{z_0\} = \{z \in \mathbb{C}\ :\ 0 < |z - z_0| < r\}$.

Interior Points, Interior of a Set



In the above picture z_0 is an interior point. z_1 is not an interior point.

Definition

Let $S \subseteq \mathbb{C}$ be a set. A point $z_0 \in \mathbb{C}$ is said to be an interior point of the set S if there exists an open neighborhood $N(z_0)$ of z_0 such that $N(z_0) \subset S$.

The set of all interior points of S is called called the interior set of S and is denoted by S° or Int(S).

Examples:

Let $S : |z| < 2$. Then $1 + i$ is an interior point of S, but 2 is not an interior point of S.

Open Set, Closed Set

Definition

A set $S \subseteq \mathbb{C}$ is said to be an open set in $\mathbb{C}$ if every point of S is an interior point of S.

Examples of Open Sets:

$\{z \in \mathbb{C} \ : \ |z - z_0| < r\}$ with $r > 0$ is an open set.

$\{z \in \mathbb{C} \ : \ \Re(z) > 0\}$ is an open set.

Definition

A set $S \subseteq \mathbb{C}$ is said to be a closed set in $\mathbb{C}$ if the complement set $\mathbb{C} \setminus S$ is an open set.

Examples of Closed Sets:

$\{z \in \mathbb{C} \ : \ |z - z_0| = r\}$ with $r > 0$ is a closed set.

$\{z \in \mathbb{C} \ : \ \Re(z) \geq 0\}$ is a closed set.

- The empty set $\emptyset$ and the whole set $\mathbb{C}$ are both open and closed.
- There are sets which are neither open nor closed in $\mathbb{C}$. For example, $S = \{z = x + iy \in \mathbb{C} \ : \ x \in (-1, 1) \text{ and } y = 0\}$ is neither open nor closed in $\mathbb{C}$ (Why?).
- Examples of Open Sets:
  $\{z \ : \ |z - (1 + i)| < 5\}$,
  $\{z \ : \ \text{Im}(z) \neq 0\}$,
  $\{z \ : \ \text{Im}(z) > 0\}$,
  $\{z \ : \ 2 < |z - (1 + i)| < 5\}$.
- Examples of Closed Sets: $\{z \ : \ |z - (1 + i)| \leq 5\}$,
  $\{z \ : \ |z - (1 + i)| = 5\}$,
  $\{z \ : \ \text{Im}(z) \geq 0\}$,
  $\{z \ : \ 2 \leq |z - (1 + i)| \leq 5\}$.

Draw the pictures of the above sets and explore whether it is open or closed or not?

Limit Point, Closure

Definition

Let $S \subseteq \mathbb{C}$ be a set. A point $z_0 \in \mathbb{C}$ is said to be a limit point or accumulation point of the set S if every deleted neighborhood $N(z_0)$ of z_0 contains at least one point of S.

Example: Let $S = \{z \in \mathbb{C} \ : \ |z| < 1\}$. Then each point z with $|z| \leq 1$ is a limit point of S.

A set S is closed iff S contains all its limit points.
If S is a finite set then S has no limit points.

The set of all limit points of S is called the derived set of S and is denoted by S' or Der(S).

Definition

A set S together with all its limit points is called the closure of S and is denoted by $\overline{S}$ or Cl(S).

Properties

- The closure of a set is always a closed set.
- The closure of a set S is the smallest closed set containing the set S.
- S is closed if and only if $S = \overline{S}$.
- The interior of a set is always an open set.
- The interior of a set S is the largest open set contained in the set S.
- S is open if and only if $S = S^{\circ}$.
- Empty set $\emptyset$ and the whole set $\mathbb{C}$ are both open and closed sets.

Properties

- Let $\{A_\alpha \ : \ \alpha \in \Lambda\}$ be an arbitrary collection of open sets in $\mathbb{C}$. Then, their union $\bigcup_{\alpha \in \Lambda} A_\alpha$ is an open set. That is, Arbitrary union of open sets is open.
- Let $\{A_\alpha \ : \ \alpha \in \Lambda\}$ be an arbitrary collection of closed sets in $\mathbb{C}$. Then, their intersection $\bigcap_{\alpha \in \Lambda} A_\alpha$ is a closed set. That is, Arbitrary intersection of closed sets is closed.
- Let $\{A_i \ : \ 1 \leq i \leq m\}$ be a finite collection of open sets in $\mathbb{C}$. Then, their intersection $\bigcap_{i=1}^{m} A_i$ is an open set. That is, Finite intersection of open sets is open.
- Let $\{A_i \ : \ 1 \leq i \leq m\}$ be a finite collection of closed sets in $\mathbb{C}$. Then, their union $\bigcup_{i=1}^{m} A_i$ is a closed set. That is, Finite union of closed sets is closed.

Boundary Point, Exterior Point

Let S be a subset of $\mathbb{C}$. The complement of the set S in $\mathbb{C}$ is defined as $S^c = \{z \in \mathbb{C} \ : \ z \notin S\} = \mathbb{C} \setminus S$.

Definition

A point z_0 is said to be a boundary point of S if every neighborhood $N(z_0)$ of z_0 contains at least one point in S and at least one point not in S. That is, every neighborhood of z_0 intersects S and S^c.

Example: Each point on $|z| = 1$ is a boundary point of the set $|z| < 1$.

The set of all boundary points of S is called the boundary set of S and is denoted by ∂S or $\mathrm{Bd}(S)$.

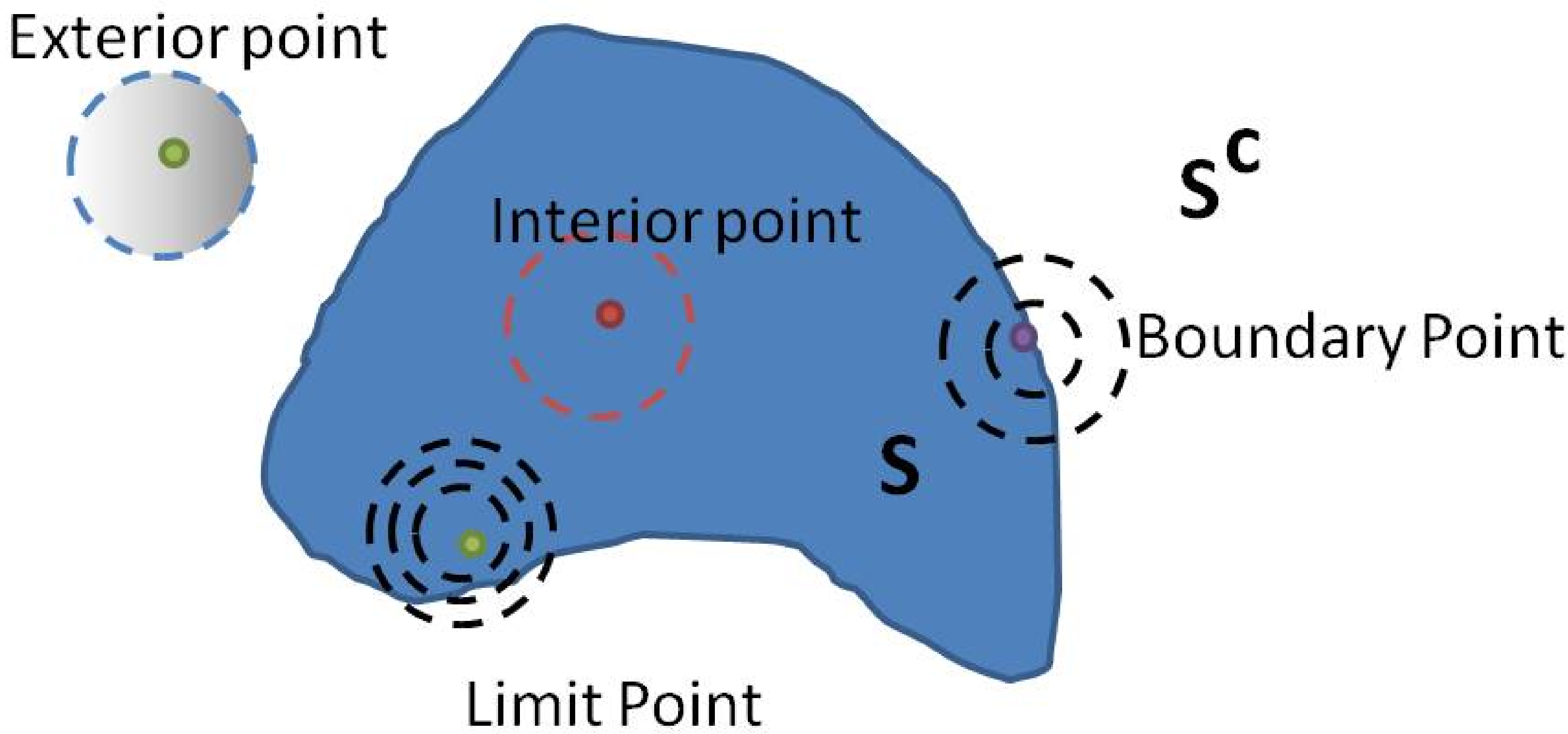
Definition

A point z_0 is said to be an exterior point of S if there is an open neighborhood $N(z_0)$ of z_0 such that $N(z_0) \cap S = \emptyset$.

That is, $N(z_0) \subseteq S^c$ and z_0 is an interior point of S^c.

The set of all exterior points of S is called the exterior set of S and is denoted by

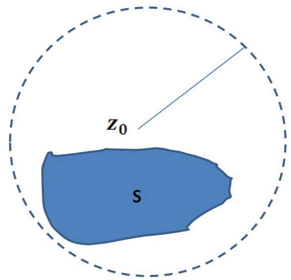
Example: Each point in $|z| > 1$ is an exterior point of the set $|z| < 1$.


Exterior point
Interior point
S^c
Boundary Point
S
Limit Point

Bounded Set, Compact Set

Definition

A set $S \subseteq \mathbb{C}$ is said to be bounded if there exists an open ball $B(z_0,\ r_0)$ for some $z_0 \in \mathbb{C}$ with $r_0 > 0$ such that $S \subset B(z_0,\ r_0)$.
That is, the set S can be put inside an open ball with some center and a finite radius.



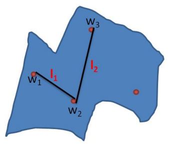
An empty set Ø is bounded.
A set S which is not bounded is called unbounded.

Definition

A set $S \subseteq \mathbb{C}$ is said to be compact if it is closed and bounded.

Connected Set, Domain, Region

Let $w_1, w_2, \cdots, w_{n+1}$ be $n+1$ points in the plane. For each $k = 1, 2, \cdots, n$, let l_k denote the line segment joining w_k to w_{k+1}. Then, the successive line segments $l_1, l_2, \cdots, l_n$ form a continuous chain known as a polygonal path that joins w_1 to w_{n+1}.



Polygonal Path

Definition

An open set $S \subseteq \mathbb{C}$ is said to be connected if every pair of points z_1, z_2 in S can be joined by a polygonal path that lies entirely in S.

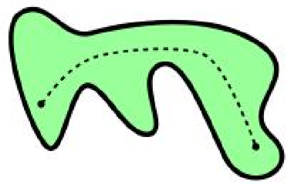
Note: The concept of connecting any two points by a path is actually known as Path Connected and Path Connected $\Longrightarrow$ Connected.

If a set S is connected then its closure $\overline{S}$ is also connected.

Connected Sets and Domain

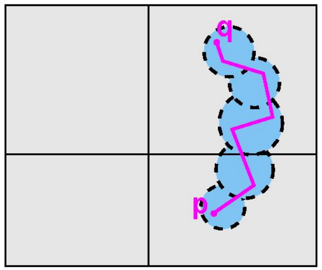
Definition

An open, connected set $S \subseteq \mathbb{C}$ is called a domain.
A domain, together with some, none, or all of its boundary points, is called a region.

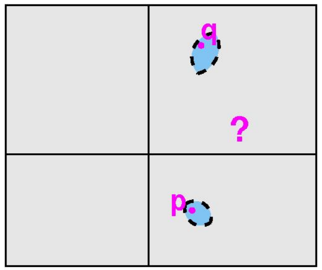


Connected Set

A set that is not connected is called a disconnected set.



Connected Set

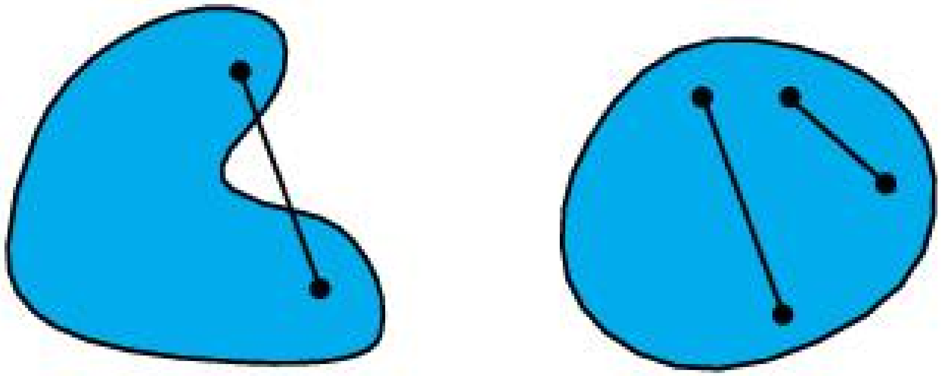


Disconnected Set

Convex Set

Definition

A set S is said to be convex if every straight line segment L joining any two points of S lies entirely inside the set S.



Not Convex Convex