Topic 01
Complex Numbers and its Algebra, Topology of Sets

MA201 Mathematics III

MGPP, AC, ST, SP

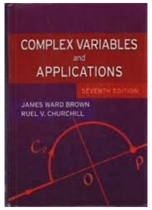
IIT Guwahati

Syllabus of Complex Analysis

- Complex Numbers: Complex numbers and elementary properties.
- Complex Functions: Limits, continuity and differentiation. Cauchy-Riemann equations. Analytic functions, Harmonic functions. Elementary Analytic functions.
- Complex Integration: Contour integrals, Anti-derivatives and path independent of contour integrals.
- Cauchy-Goursat Theorem. Cauchy's integral formula, Morera's Theorem. Liouville's Theorem, Fundamental Theorem of Algebra, Maximum Modulus Principle and its consequences.
- Power Series: Taylor series, Laurent series.
- Zeros and Singularities: Zeros of Analytic Functions, Singularities, Argument Principle, Rouche's Theorem.
- Residues and Applications: Cauchy's Residue Theorem and applications.
- Conformal Mappings: Conformal Mappings, Mobius transformations.

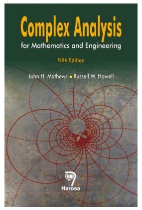
Complex Analysis Books

Text Book:



J. W. Brown and R. V. Churchill, Complex Variables and Applications, 7th or 8th Edition, Mc-Graw Hill, 2004. Note: Any Edition is fine.

Reference Book:



J. H. Mathews and R. W. Howell, Complex Analysis for Mathematics and Engineering, 3rd Edition, Narosa, 1998. Note: Any Edition or Other Publisher is fine.

Topic 01: Learning Outcome

We learn

- Complex Numbers
- Algebraic Operations: Addition, Multiplication, Division
- $\mathbb{C}$ is a field, but not an ordered field
- $x + iy$ form of complex numbers
- Conjugate, Modulus of a complex number
- Basic identities and inequalities
- Nonzero complex numbers: Polar Form, Trigonometric Form, Exponential Form, argument function
- Powers and Roots of Complex Numbers
- Interior Point, Open, Closed, Limit point, boundary point, Exterior Point
- Bounded Set, Connected Set, Compact Set, Convex Set
- Domains, Regions

Recall: $(\mathbb{R}, +, \cdot)$ is a field w. r. t. addition $+$ and multiplication $\cdot$

- Closure Law: For all a and b in $\mathbb{R}$, $a + b \in \mathbb{R}$ and $a \cdot b \in \mathbb{R}$.
- Associative Law: For all a, b and c in $\mathbb{R}$,

$$a + (b + c) = (a + b) + c \qquad \text{and} \qquad a \cdot (b \cdot c) = (a \cdot b) \cdot c\ .$$

- Identity Law: For all a in $\mathbb{R}$,

$$a + 0 = a = 0 + a \qquad \text{and} \qquad a \cdot 1 = a = 1 \cdot a\ .$$

- Law of Additive Inverse: Given $a \in \mathbb{R}$, there exists a unique $x \in \mathbb{R}$ such that $a + x = 0 = x + a$.
- Law of Multiplicative Inverse: Given $a \in \mathbb{R}$ with $a \neq 0$, there exists a unique $x \in \mathbb{R}$ such that $a \cdot x = 1 = x \cdot a$.
- Commutative Law: For all a and b in $\mathbb{R}$,

$$a + b = b + a \qquad \text{and} \qquad \mathrm{a} \cdot b = b \cdot a\ .$$

- Distributive Law: For all a, b and c in $\mathbb{R}$,

$$a \cdot (b + c) = a \cdot b + a \cdot c\ .$$

Why do we need Complex Numbers $\mathbb{C}$?

NOT all polynomial equations have roots in $\mathbb{R}$.
Example: $x^2 + 1 = 0$ has no roots in $\mathbb{R}$.

$(\mathbb{R}, +, \cdot)$ is NOT algebraically closed. There is a need of bigger number system in which all (nonconstant) polynomial equations have roots.

Fact/History: Complex numbers $\mathbb{C}$ were originated when Four Italy mathematicians (Ferro, Tartagila, Cardano, Bombelli) in 16th Century tried to solve cubic equations like $x^3 - 3bx - 2c = 0$, $x^3 - 15x - 4 = 0$ (but not from quadratic equations at that time).
For an interesting article on History of Complex Numbers see:

http://www.math.uri.edu/~merino/spring06/mth562/ShortHistoryComplexNumbers2006.pdf

Advantages (now): Certain real integrals can be computed easily in $\mathbb{C}$. Certain differential equations can be easily solved. A differentiable complex function in an open set (analytic function) has many interesting properties.

Complex Numbers

Definition

A complex number z is defined to be an ordered pair of real numbers x and y as $z = (x,\ y)$. That is, the set of complex numbers is denoted by $\mathbb{C}$ and is given by

$$\mathbb{C} = \{z = (x,\ y)\ :\ x \text{ and } y \text{ are real numbers }\}\ .$$

The ordered pair here means the order in which we write x and y in defining the complex number $z = (x,\ y)$. For example, the number $(1,\ 5)$ is not the same as $(5,\ 1)$.

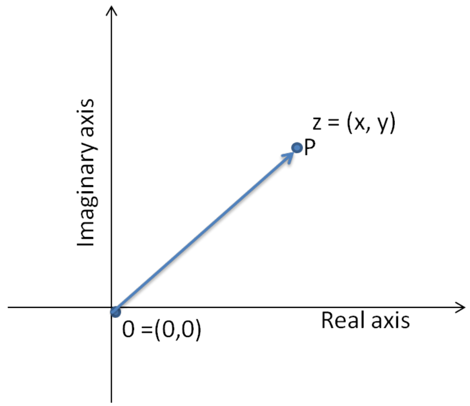
In the complex number $z = (x,\ y)$,

- x is called the real part of z and is denoted by $\Re(z)$ or Re z
- y is called the imaginary part of z and is denoted by $\Im(z)$ or Im z

- The numbers of the form $(0, y)$ are called pure imaginary numbers.
- The numbers of the form $(x, 0)$ are called real numbers.
- The set of real numbers can be identified as a subset $\mathbb{R} = \{z = (x, y) \in \mathbb{C} : x \in \mathbb{R}$ and $y = 0\}$ in $\mathbb{C}$. That is, $\mathbb{R} \subset \mathbb{C}$.
- Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal.

History: The representation of complex numbers in the plane was proposed independently by Casper Wessel (1797), K. F. Gauss (1799) and Jean Robert Argand (1806).

Complex Plane/ z-plane/ Argand Plane



- The complex number $z = (x,\ y)$ can be viewed as a point P having cartesian coordinates $(x,\ y)$ in the plane $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.
- The x-axis and y-axis are called the real axis and the imaginary axis respectively.
- The complex number $z = (x,\ y)$ can also be represented by a vector connecting the origin $0 = (0, 0)$ to the point P.
- This plane is called the complex plane or z-plane. It is also known as the Gauss plane or the Argand Plane. The term Argand diagram is sometimes used.

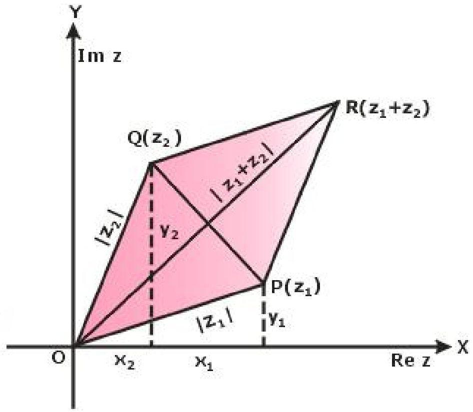
Addition Operation

For any two complex numbers $z_1 = (x_1,\ y_1)$ and $z_2 = (x_2,\ y_2)$, the addition of z_1 and z_2 is defined

$$z_1 + z_2 \;=\; (x_1 + x_2,\;\; y_1 + y_2)\ .$$

The sum of any two complex numbers is a complex number.

Geometric Interpretation of Addition of two complex numbers:



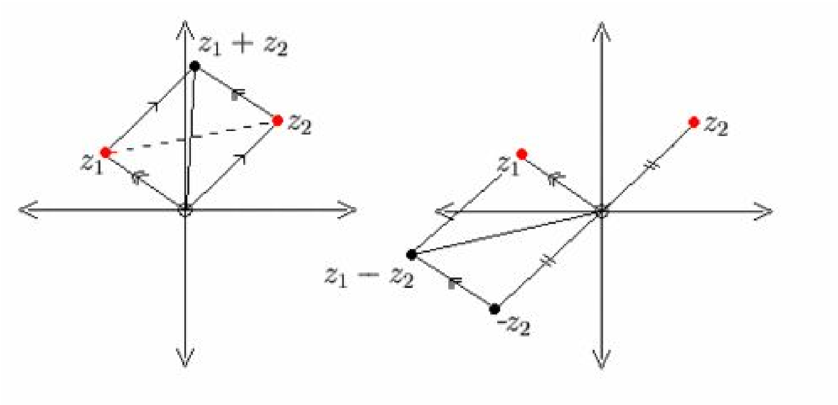
If $\vec{OP}$ and $\vec{OQ}$ are not collinear, then $\vec{OR}$ is the diagonal of the parallelogram with $\vec{OP}$ and $\vec{OQ}$ as adjacent sides.

Subtraction Operation

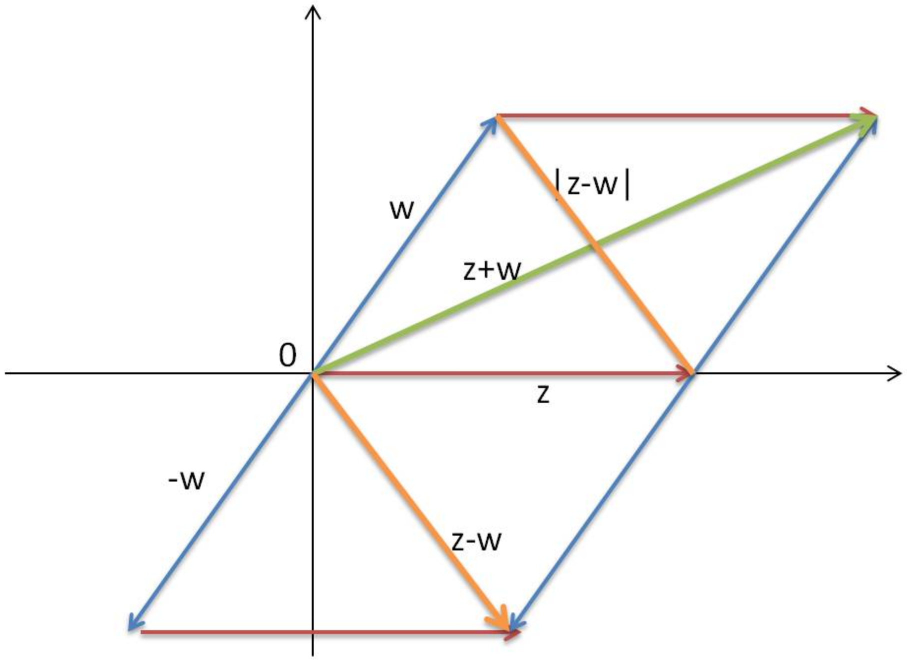
The subtraction $z_1 - z_2$ of the complex numbers $z_1 = (x_1,\ y_1)$ and $z_2 = (x_2,\ y_2)$ is defined as

$$z_1 - z_2 = (x_1 - x_2,\ y_1 - y_2)\ .$$

The subtraction $z_1 - z_2$ can be viewed as the sum of the complex numbers z_1 and $-z_2$.



Geometric Interpretation of Subtraction



This picture will be useful to understand parallelogram law (later on).

$|z + w|^2 + |z - w|^2 = 2\left(|z|^2 + |w|^2\right)$

Multiplication and Division

For any two complex numbers $z_1 = (x_1,\ y_1)$ and $z_2 = (x_2,\ y_2)$, the multiplication of z_1 and z_2 is defined by

$$z_1\ z_2 \;=\; (x_1 x_2 - y_1 y_2,\ \ x_1 y_2 + x_2 y_1)\ .$$

The product of any two complex numbers is a complex number.
This multiplication is different from the vector product.

If $z_1 = (x_1,\ y_1)$ and $z_2 = (x_2,\ y_2) \neq 0$ are any two complex numbers, then the complex number z_1 divided by z_2 is defined as

$$\frac{z_1}{z_2} = \left(\frac{1}{x_2^2 + y_2^2}\right)((x_1 x_2 + y_1 y_2),\ (x_2 y_1 - x_1 y_2))\ .$$

The set of complex numbers $\mathbb{C}$ with these operations addition $+$ and multiplication $\cdot$ forms a field. The identity element of $+$ is $(0,0)$ and the identity element of $\cdot$ is $(1,0)$. $\mathbb{R}$ is a subfield of $\mathbb{C}$.

Binomial Formula

Let $0! = 1$ and $n! = 1 \cdot 2 \cdots (n-1) \cdot n$ for $n \in \mathbb{N}$.

Let $nC_k = \frac{n!}{k!\,(n-k)!}$ for $k = 0, \cdots, n$.

Binomial Formula:

For any two complex numbers z_1 and z_2 and for $n \in \mathbb{N}$,

$$(z_1 + z_2)^n = \sum_{k=0}^{n} nC_k \, z_1^{n-k} \, z_2^k .$$

The proof is based on mathematical induction and is left as an exercise.

Additional Information: Complex Field is NOT an Ordered Field

We can not define usual order relation like less than, less than or equal to, greater than, greater than or equal to on the set of complex numbers. *That is, the usual ordering of $\mathbb{R}$ can not be taken to $\mathbb{C}$ as such.*

However, we can define in other ways, like dictionary order on $\mathbb{C}$ as follows.

Let $z_1 = x_1 + i\,y_1$ and $z_2 = x_2 + i\,y_2$. The dictionary order is given by:

$$\begin{aligned} z_1 < z_2 &\quad \textit{if} \quad x_1 < x_2 \\ z_1 < z_2 &\quad \textit{if} \quad x_1 = x_2 \quad \textit{and if} \quad y_1 < y_2 \\ z_1 = z_2 &\quad \textit{if} \quad x_1 = x_2 \quad \textit{and if} \quad y_1 = y_2 \end{aligned}$$

The complex field $(\mathbb{C}, +, \cdot)$ can NOT be an ordered field with respect to any (total) order defined on $\mathbb{C}$.
Therefore, the dictionary order is NOT useful in some sense.

Algebraic form (or $x + iy$ notation)

Set

$$i = (0,\ 1)\ .$$

It is called iota.

Electrical engineers use the letter j instead of i.

$$(x,\ y) = (x,\ 0)(1,\ 0) + (0,\ 1)(y,\ 0) = x\ \cdot\ 1\ +\ i\ \cdot\ y = x +\ i\ y$$

$$(x,\ y) = x + i\ y\ ,$$

$$i^2 = (0,\ 1)(0,\ 1) = (-1,\ 0) = -1\ .$$

The form $x + iy$ is called the algebraic form of a complex number.

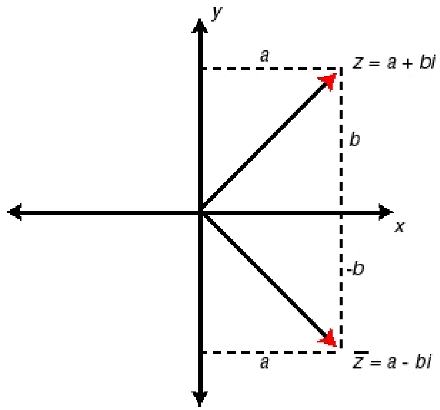
Hereafter, we prefer to use $x + iy$ form instead of ordered pair (x, y) form to write complex numbers.

Conjugate of a Complex Number

The complex conjugate, or simply, the conjugate of a complex number $z = a + ib$ is denoted by $\bar{z}$ and is defined by

$$\bar{z} = a - ib\,.$$

Geometrically, the point $\bar{z} = a - ib$ is the reflection (mirror image) of the point $z = a + ib$ on the real axis.



Examples: If $z = 3 + 4i$ then $\bar{z} = 3 - 4i$. If $z = -5$ then $\bar{z} = -5$.

Properties of Complex Conjugation

1. $z_1 = z_2$ if and only if $\overline{z_1} = \overline{z_2}$.
2. $\overline{\overline{z}} = z$.
3. $\overline{z} = z$ if and only if z is a real number.
4. $z + \overline{z} = 2\Re(z) = 2x$ if $z = x + iy$.
5. $z - \overline{z} = 2i\,\Im(z) = 2i\,y$ if $z = x + iy$.
6. $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$.
7. $\overline{z_1 z_2} = \overline{z_1}\;\overline{z_2}$.
8. $\overline{(z_1/z_2)} = \overline{z_1}/\overline{z_2}$ provided $z_2 \neq 0$.

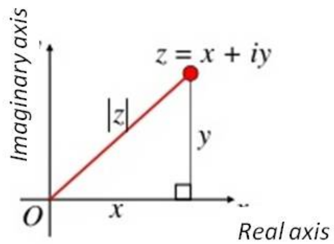
The numbers z and $\overline{z}$ are called the complex conjugate coordinates, or simply the conjugate coordinates corresponding to the point $z = (x,\ y) = x + iy$. Also they have been called the isotropic coordinates and the minimal coordinates of the point.

Modulus of a Complex Number

The modulus or absolute value of a complex number $z = x + iy$ is denoted by $|z|$ and is given by

$$|z| = \sqrt{x^2 + y^2}\,.$$

Here, as usual, the radical stands for the principal (non-negative) square root of $x^2 + y^2$.



Example: The modulus of the complex number $4 + 3i$ is

$$|4 + 3i| = \sqrt{4^2 + 3^2} = \sqrt{25} = 5.$$

Note: $|z| \geq 0$ for all $z \in \mathbb{C}$. $|z| = 0$ if and only if $z = 0$.

Properties - Modulus & Conjugate

1. $|z| \geq 0$ and $|z| = 0$ iff $z = 0$.
2. $|\bar{z}| = |z| = |-z|$.
3. $|z|^2 = z\,\bar{z}$.
4. If $z = x + iy$, $|z| \leq |x| + |y|$.
5. If $z = x + iy$, $|x| \leq |z|$ and $|y| \leq |z|$.
6. $|z_1 z_2| = |z_1|\,|z_2|$.
7. Parallelogram Law: $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right)$
8. Triangle Inequality: $|z_1 + z_2| \leq |z_1| + |z_2|$. (Work out proof in class)
9. $|z_1 - z_2| \leq |z_1| + |z_2|$.
10. $|z_1 - z_2| \geq |\,|z_1| - |z_2|\,|$. (Work out proof in class)
11. $|z_1 + z_2| \geq |\,|z_1| - |z_2|\,|$.
12. $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ provided $z_2 \neq 0$.
13. If $n \in \mathbb{N}$, then $|z^n| = |z|^n$. If $-n \in \mathbb{N}$, then $|z^n| = |z|^n$ for $z \neq 0$.

Properties (continuation) - Additional Information

14. Lagrange's Identity: If $\{z_1,\ z_2,\ \cdots,\ z_n\}$ and $\{w_1,\ w_2,\ \cdots,\ w_n\}$ are two sets of n complex numbers ($n \geq 1$), then

$$\left|\sum_{k=1}^{n} z_k\ w_k\right|^2 = \left(\sum_{k=1}^{n} |z_k|^2\right) \left(\sum_{k=1}^{n} |w_k|^2\right) - \sum_{1 \leq j \leq k \leq n} |z_j w_k - z_k w_j|^2 .$$

15. Cauchy-Schwarz Inequality: If $\{z_1,\ z_2,\ \cdots,\ z_n\}$ and $\{w_1,\ w_2,\ \cdots,\ w_n\}$ are two sets of n complex numbers ($n \geq 1$), then

$$\left|\sum_{k=1}^{n} z_k\ w_k\right|^2 \leq \left(\sum_{k=1}^{n} |z_k|^2\right) \left(\sum_{k=1}^{n} |w_k|^2\right)$$

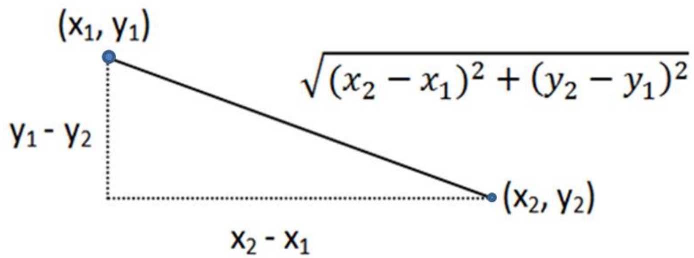
where the equality sign holds iff the z_k are proportional to the $\overline{w_k}$.

Distance between Two Complex Numbers

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be any two complex numbers. Then the (Usual/Euclidean) distance between z_1 and z_2 is defined by

$$d(z_1, z_2) = |z_2 - z_1| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\ .$$

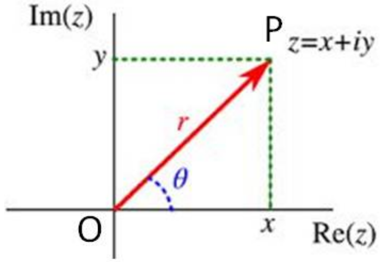
$$= |z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}\ .$$

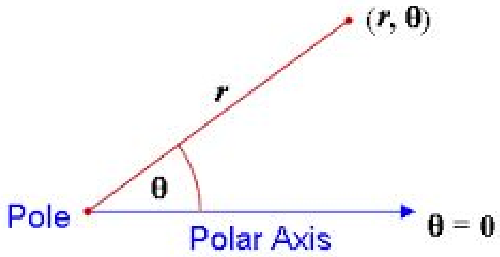


Example: If $z_1 = 1 + i$ and $z_2 = 1 - i$ then $|z_1 - z_2| = \sqrt{(1-1)^2 + (1-(-1))^2} = 2$.

Note: $|z| = d(0, z)$. $(\mathbb{C}, d)$ is a metric space.

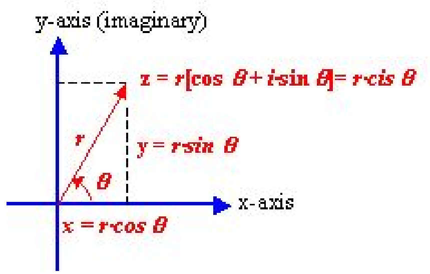
Polar Form of (Non-Zero) Complex Numbers

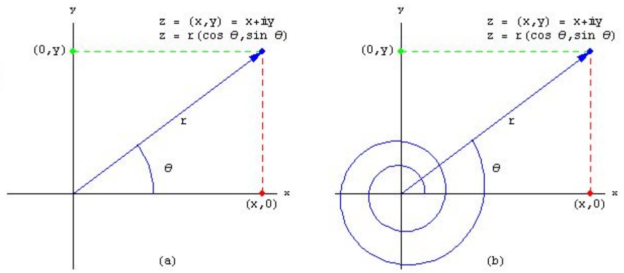




- Each non-zero complex number $z = x + iy = (x,\ y) \neq (0, 0)$ can be represented by the vector from the origin O to the point $P = (x, y)$ in the plane.
- The length r of the vector $\vec{OP}$ is given by $r = \sqrt{x^2 + y^2} = |z| =$ Modulus of z.
- The measure θ in radians of the oriented angle from the positive real axis to the vector $\vec{OP}$ is called the argument or the amplitude of the vector $\vec{OP}$, and we write $\theta = \arg z$.
- For $z \neq 0$, we can write $z = (r,\ \theta)$ where $r = |z|$ and $\theta = \arg(z)$. This representation is called the polar representation of z, and the values of r and θ are called polar coordinates of z.

Trigonometric Form of (Non-Zero) Complex Numbers





- From trigonometry we have, $\cos\theta = \dfrac{x}{r}$ and $\sin\theta = \dfrac{y}{r}$.

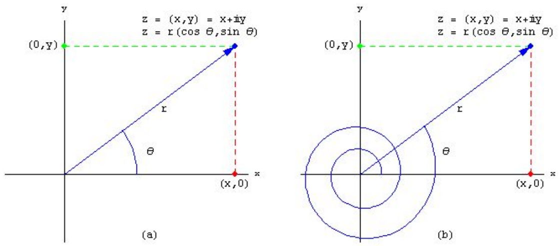
$$z = r\cos\theta + \ i\,r\sin\theta = r(\cos\theta + \ i\ \sin\theta)\ .$$

  This is known as the trigonometric form of the complex number z.

- The number θ is determined only up to multiples of 2π and the set of all such angles is denoted by $\arg z$. However all the values in this set represent the same direction in the complex plane.

Example: Modulus of $(1+i)$ is $\sqrt{2}$ and argument of $(1+i) = \pi/4+$ any multiple of 2π.
Polar form of $(1+i)$ is $(\sqrt{2},\ \pi/4)$ or $(\sqrt{2},\ 9\pi/4)$, etc.

About the function $\arg(z)$



- For the complex number $z = 0$, the modulus is 0, but the argument is undefined.
- If a complex number z is written in the polar form or in the trigonometric form then it is understood that it is a non-zero complex number.
- For each nonzero z, $\arg(z)$ takes a set of values. This set is an infinite set. For each nonzero point z, argument function thus assigns a set as value. Therefore, $\arg(z)$ is called a multiple valued function.

Examples: $\arg(5) = \{2n\pi \ : \ n \in \mathbb{Z}\}$; $\arg(-3) = \{(2n+1)\pi \ : \ n \in \mathbb{Z}\}$; $\arg(1+i) = \{(\pi/4) + 2n\pi \ : \ n \in \mathbb{Z}\}$; Compute $\arg(1-i)$.

Principal Value of argument of z: Arg z

Picking one of the values of $\arg(z)$ for computation purpose. For example, if teacher gives some condition, all students should be able to pick up the same (and unique) value for $\arg(1+i)$. How to do it?

e.g., Teacher says: Restrict the value of $\arg(z)$ in the interval $(-\pi, \ \pi]$ and now tell me the value of $\arg(1+i)$.

Answer: $\arg(1+i)$ in the interval $(-\pi, \pi]$ is _____.

For each non-zero z, there is only one value of $\arg z$ say Θ satisfying $-\pi < \Theta \leq \pi$. This value will henceforth be denoted by Arg z and is called the principal value of $\arg z$.

Examples: Arg $(5) = 0$, Arg $(i) = \pi/2$, Arg $(-8) = \pi$, Arg $(-i) = -\pi/2$.

Exercise: Find the largest set in $\mathbb{C}$ on which Arg z is continuous?

Relation between $\arg z$ and Arg z:

$$\arg z = \text{Arg } z + 2\pi k \text{ where } k \text{ is an integer .}$$

Computing Principal Value of argument and argument

Let $z = x + iy \neq 0$.

Compute ϕ = Principal value of $\tan^{-1}(y/x)$ which lies in $(-\pi/2,\ \pi/2)$.

With the value of ϕ and with the information of signs of x and y (which quadrant z lies) we can compute

$$\text{Arg}\,(z) = \begin{cases} \phi & \text{if} \quad x > 0 \\ \phi + \pi & \text{if} \quad x < 0 \text{ and } y \geq 0 \\ \phi - \pi & \text{if} \quad x < 0 \text{ and } y < 0 \\ \pi/2 & \text{if} \quad x = 0 \text{ and } y > 0 \\ -\pi/2 & \text{if} \quad x = 0 \text{ and } y < 0 \end{cases}$$

$$\arg(z) = \text{Arg}\,(z) + 2k\pi \quad \text{where} \quad k \in \mathbb{Z}\,.$$

Exponential form of Non-Zero Complex Numbers

- Let $z = x + iy \neq 0$ be written in the trigonometric form as $z = r(\cos\theta + i\ \sin\theta)$ where r is the modulus and θ is the argument of z.
- The Euler's formula says that

$$e^{i\theta} = \cos\theta + \ i\ \sin\theta$$

where θ is measured in radians.

If $z \neq 0$ then using Euler's formula, we can write z as

$$z = re^{i\theta}$$

where $r = |z|$ and $\theta = \arg(z)$ which is known as the exponential form of a complex number z.

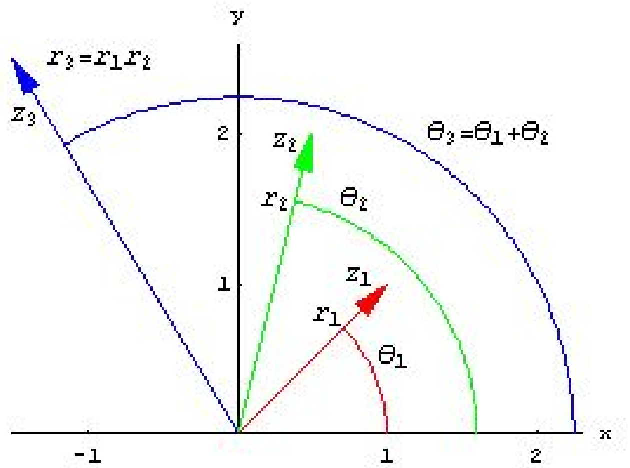
Examples: $1 + i = \sqrt{2}e^{i\pi/4}$, $\quad -i = e^{-i\pi/2}$, $\quad -8 = 8e^{i\pi} = 8e^{i3\pi}$.

Geometrical Interpretation of Multiplication

Let $z_1 \neq 0$ and $z_2 \neq 0$. Then,

$$z_i = r_i(\cos\theta_i + \ i \ \sin\theta_i), \quad i = 1, \ 2 \ .$$

$$\begin{aligned} z_1 z_2 &= r_1 r_2 \left[(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + i\,(\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2)\right] \\ &= r_1 r_2 \ \left[\cos(\theta_1 + \theta_2) \ + \ i \ \sin(\theta_1 + \theta_2)\right] \end{aligned}$$



$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

The above identity is to be interpreted as saying that if values of two of these three (multiple valued) arguments are specified, then there is a value of the third such that the above equation holds.

Example: If $3 = 3e^{2\pi i}$ and $-2 = 2e^{3\pi i}$ then $-6 = 6e^{i\theta_3}$ with $\theta_3 = 5\pi$ (one of the values of $\arg(-6)$ plus a suitable multiple of 2π is to be taken) so that the identity holds.

In the above identity, if we replace $\arg(z)$ by Arg (z), then identity is in general NOT true. If z_1 and z_2 lies in the first quadrant then it will be true.

$$\textit{Arg}\,(z_1 z_2) \neq \textit{Arg}\,(z_1) + \textit{Arg}\,(z_2) \qquad \textit{(in general)}.$$

If $0 \neq z = re^{i\theta}$ then $(1/z) = (1/r)e^{-i\theta}$ and hence $\arg(1/z) = -\arg(z)$.

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2).$$

Powers of Complex Numbers

Let z be a complex number and let n be an integer.

- If $z = 0$, we have $z^n = 0$ if $n \in \mathbb{N}$.
- If $z \neq 0$, then setting $z = re^{i\theta}$ and using $e^{t_1}e^{t_2} = e^{t_1+t_2}$ by mathematical induction one can prove that

$$z^n = r^n e^{in\theta} \qquad \text{for} \qquad n = 0,\ 1,\ 2,\ 3,\ \cdots\ .$$

- If n is negative integer, then set $m = -n$ and apply the above equation to $(1/z)^m$ to get $z^n = r^n e^{in\theta}$.
- If $r = 1$ then we get $\left(e^{i\theta}\right)^n = e^{in\theta}$.
- Rewriting it, we get following de Moivre's formula.

$$(\cos\theta + i\ \sin\theta)^n = \cos(n\theta) + \ i\ \sin(n\theta) \qquad \text{for} \qquad n \in \mathbb{Z}\ .$$

- Example: $(\sqrt{3} + i)^7 = (2e^{i\pi/6})^7 = 2^7 e^{i7\pi/6} = (2^6 e^{i\pi})(2e^{i\pi/6}) = -64(\sqrt{3} + i)$.

n-th Roots of Unity ($1^{1/n}$)

Find the solutions of the equation $z^n = 1$ where n is a positive integer.
Let $z = re^{i\theta}$ be a solution to $z^n = 1$.
Then, $z^n = r^n(e^{i\theta})^n = r^n e^{i\,n\theta} = 1 \cdot e^{i0}$ which implies

$$r^n = 1, \qquad n\theta = 0 + 2k\pi \text{ where } k \text{ is an integer .}$$

We get n distinct solutions to $z^n = 1$ by setting $k = 0, 1, \cdots, n-1$ as

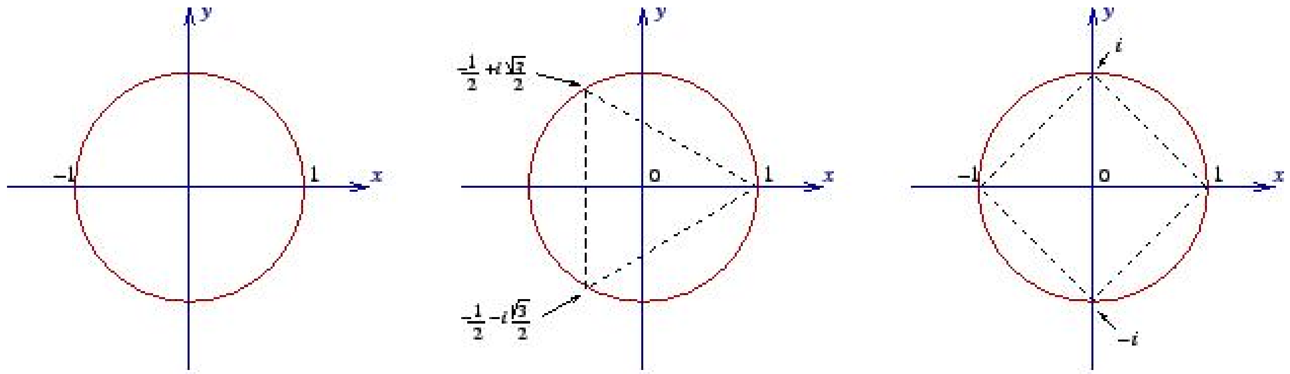
$$z_k = e^{i\frac{2k\pi}{n}} = \cos\left(\frac{2k\pi}{n}\right) + i\ \sin\left(\frac{2k\pi}{n}\right)$$

where $k = 0, 1, \cdots, n-1$ and are called the n-th roots of unity.

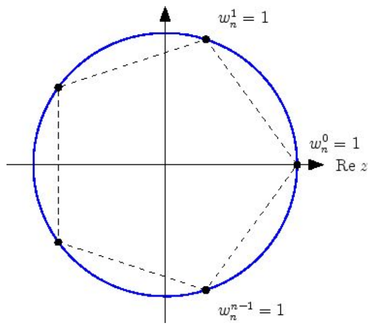
Set $\omega_n = e^{i2\pi/n}$ (primitive n-th root of unity). By De Moivre's formula, the n-th roots of unity can be expressed as $1, \omega_n, \omega_n^2, \omega_n^3, \cdots, \omega_n^{n-1}$.

Properties of n-th Roots of Unity

Geometrically, the n-th roots of unity are equally spaced points that lie on the unit circle $\{z \ : \ |z| = 1\}$ and form the vertices of a regular polygon with n sides.



(c) $n = 2$ (d) $n = 3$ (e) $n = 4$



n-th Roots of Nonzero Complex Number $W^{1/n}$

Find the solutions of the equation $z^n = W$ where n is a positive integer.
Let $z = re^{i\theta}$ be a solution to $z^n = W = \rho e^{i\phi}$.
$z^n = r^n e^{in\theta} = W = \rho e^{i\phi}$ gives that

$$r^n = \rho \qquad \text{and} \qquad n\theta = \phi + 2k\pi \qquad \text{where } k \in \mathbb{Z}\,.$$

By setting $k = 0, 1, \cdots, n-1$,we get n distinct solutions to $z^n = W$ as

$$z_k = \rho^{\frac{1}{n}}\ e^{i\frac{\phi+2k\pi}{n}} = \rho^{\frac{1}{n}}\left[\cos\left(\frac{\phi+2k\pi}{n}\right) + \ i\ \sin\left(\frac{\phi+2k\pi}{n}\right)\right]$$

for $k = 0, 1, \cdots, n-1$.
If c is any n-th root of W then all the n-th roots of W are given by $c, c\omega_n, c\omega_n^2, \cdots, c\omega_n^{n-1}$ where ω_n is a primitive n-th root of unity.
Example: Cube roots of $64i$ are $z_0 = 4e^{i\pi/6} = 2\sqrt{3} + i\,2$, $z_1 = 4e^{i5\pi/6} = -2\sqrt{3} + i\,2$ and $z_3 = 4e^{i3\pi/2} = -4i$.

Computing W^{α} where $W \neq 0$ and $\alpha \in \mathbb{Q}$

Let W be a nonzero complex number.
Let $\alpha = m/n$ where m and n are integers with gcd $(m, n) = 1$.
Then,

$$W^{\alpha} = W^{m/n} = (W^m)^{1/n} \ .$$

Since m is an integer, W^m will be a single complex number.
Then, taking n-th root of W^m, we get n distinct complex numbers z_k satisfying $z_k^n = W^m$.

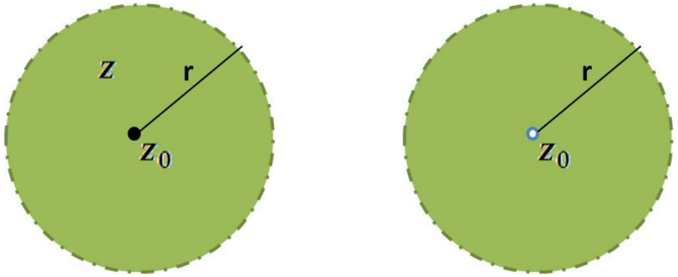
Exercise: Find all values of $(-8i)^{2/3}$.
Exercise: From real function to complex function what is happening? Compare domain of definition and range of real function $x_0^{1/n}$ and complex function $z_0^{1/n}$.

Sets in $\mathbb{C}$ (Planar Sets)

Identify the following sets / Find the Locus of the Points satisfying the equations / Interpret geometrically the following relations:

1. $\{z \in \mathbb{C} : |z-1| - |z+1| = 0\}$.
2. $\{z \in \mathbb{C} : |\Re(z)| + |\Im(z)| = 1\}$.
3. $|z-a| - |z+a| = 2c$ where a and c are real constants with $c > 0$.
4. $z = a + tb$ for $t \in \mathbb{R}$ where a and $b \neq 0$ are complex constants.
5. $\{z \in \mathbb{C} : \text{Im}\left(\frac{z-a}{b}\right) > 0\}$ where a and $b \neq 0$ are complex constants.

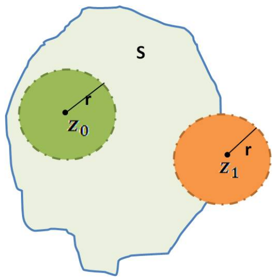
Open Ball/Neighorhood, Puncture Neighborhood



$|z - z_0| < R$ $\qquad$ $0 < |z - z_0| < R$

- Open Disk/Open Ball centered at the point z_0 with radius r is denoted by $B_r(z_0)$ (or $B(z_0)$ or $B(z_0,\ r)$) and is defined by $B_r(z_0) = \{z \in \mathbb{C}\ :\ |z - z_0| < r\}$.
- Let z_0 be a point in $\mathbb{C}$. Any open ball with center at z_0 and radius $r > 0$ is called an open neighborhood of z_0 or simply a neighborhood of z_0 and is usually denoted by $N_r(z_0)$ or $N(z_0)$ or $N(z_0,\ r)$.
- A punctured or deleted neighborhood of a point z_0 is given by $B_r(z_0) \setminus \{z_0\} = \{z \in \mathbb{C}\ :\ 0 < |z - z_0| < r\}$.

Interior Points, Interior of a Set



In the above picture z_0 is an interior point. z_1 is not an interior point.

Definition

Let $S \subseteq \mathbb{C}$ be a set. A point $z_0 \in \mathbb{C}$ is said to be an interior point of the set S if there exists an open neighborhood $N(z_0)$ of z_0 such that $N(z_0) \subset S$.

The set of all interior points of S is called called the interior set of S and is denoted by S° or Int(S).

Examples:

Let $S : |z| < 2$. Then $1 + i$ is an interior point of S, but 2 is not an interior point of S.

Open Set, Closed Set

Definition

A set $S \subseteq \mathbb{C}$ is said to be an open set in $\mathbb{C}$ if every point of S is an interior point of S.

Examples of Open Sets:

$\{z \in \mathbb{C} \ : \ |z - z_0| < r\}$ with $r > 0$ is an open set.

$\{z \in \mathbb{C} \ : \ \Re(z) > 0\}$ is an open set.

Definition

A set $S \subseteq \mathbb{C}$ is said to be a closed set in $\mathbb{C}$ if the complement set $\mathbb{C} \setminus S$ is an open set.

Examples of Closed Sets:

$\{z \in \mathbb{C} \ : \ |z - z_0| = r\}$ with $r > 0$ is a closed set.

$\{z \in \mathbb{C} \ : \ \Re(z) \geq 0\}$ is a closed set.

- The empty set $\emptyset$ and the whole set $\mathbb{C}$ are both open and closed.
- There are sets which are neither open nor closed in $\mathbb{C}$. For example, $S = \{z = x + iy \in \mathbb{C} \ : \ x \in (-1, 1) \text{ and } y = 0\}$ is neither open nor closed in $\mathbb{C}$ (Why?).
- Examples of Open Sets:
  $\{z \ : \ |z - (1 + i)| < 5\}$,
  $\{z \ : \ \mathrm{Im}(z) \neq 0\}$,
  $\{z \ : \ \mathrm{Im}(z) > 0\}$,
  $\{z \ : \ 2 < |z - (1 + i)| < 5\}$.
- Examples of Closed Sets: $\{z \ : \ |z - (1 + i)| \leq 5\}$,
  $\{z \ : \ |z - (1 + i)| = 5\}$,
  $\{z \ : \ \mathrm{Im}(z) \geq 0\}$,
  $\{z \ : \ 2 \leq |z - (1 + i)| \leq 5\}$.

Draw the pictures of the above sets and explore whether it is open or closed or not?

Limit Point, Closure

Definition

Let $S \subseteq \mathbb{C}$ be a set. A point $z_0 \in \mathbb{C}$ is said to be a limit point or accumulation point of the set S if every deleted neighborhood $N(z_0)$ of z_0 contains at least one point of S.

Example: Let $S = \{z \in \mathbb{C} : |z| < 1\}$. Then each point z with $|z| \leq 1$ is a limit point of S.

A set S is closed iff S contains all its limit points.
If S is a finite set then S has no limit points.

The set of all limit points of S is called the derived set of S and is denoted by S' or Der(S).

Definition

A set S together with all its limit points is called the closure of S and is denoted by $\overline{S}$ or Cl (S).

Properties

- The closure of a set is always a closed set.
- The closure of a set S is the smallest closed set containing the set S.
- S is closed if and only if $S = \overline{S}$.
- The interior of a set is always an open set.
- The interior of a set S is the largest open set contained in the set S.
- S is open if and only if $S = S^\circ$.
- Empty set $\emptyset$ and the whole set $\mathbb{C}$ are both open and closed sets.

Properties

- Let $\{A_\alpha : \alpha \in \Lambda\}$ be an arbitrary collection of open sets in $\mathbb{C}$. Then, their union $\bigcup_{\alpha\in\Lambda} A_\alpha$ is an open set. That is, Arbitrary union of open sets is open.
- Let $\{A_\alpha : \alpha \in \Lambda\}$ be an arbitrary collection of closed sets in $\mathbb{C}$. Then, their intersection $\bigcap_{\alpha\in\Lambda} A_\alpha$ is a closed set. That is, Arbitrary intersection of closed sets is closed.
- Let $\{A_i : 1 \leq i \leq m\}$ be a finite collection of open sets in $\mathbb{C}$. Then, their intersection $\bigcap_{i=1}^{m} A_i$ is an open set. That is, Finite intersection of open sets is open.
- Let $\{A_i : 1 \leq i \leq m\}$ be a finite collection of closed sets in $\mathbb{C}$. Then, their union $\bigcup_{i=1}^{m} A_i$ is a closed set. That is, Finite union of closed sets is closed.

Boundary Point, Exterior Point

Let S be a subset of $\mathbb{C}$. The complement of the set S in $\mathbb{C}$ is defined as $S^c = \{z \in \mathbb{C} \ : \ z \notin S\} = \mathbb{C} \setminus S$.

Definition

A point z_0 is said to be a boundary point of S if every neighborhood $N(z_0)$ of z_0 contains at least one point in S and at least one point not in S. That is, every neighborhood of z_0 intersects S and S^c.

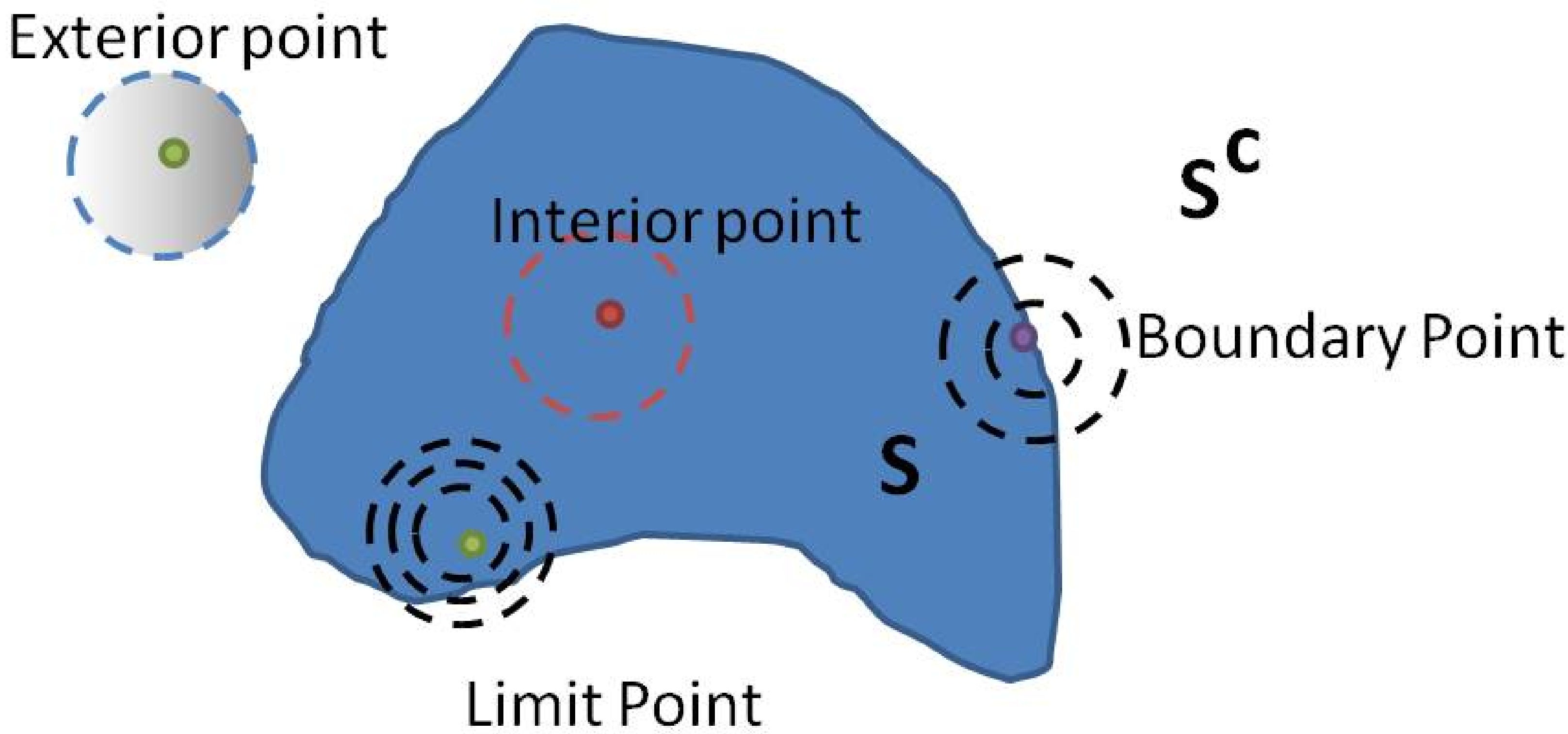
Example: Each point on $|z| = 1$ is a boundary point of the set $|z| < 1$.
The set of all boundary points of S is called the boundary set of S and is denoted by ∂S or Bd(S).

Definition

A point z_0 is said to be an exterior point of S if there is an open neighborhood $N(z_0)$ of z_0 such that $N(z_0) \cap S = \emptyset$.
That is, $N(z_0) \subseteq S^c$ and z_0 is an interior point of S^c.

The set of all exterior points of S is called the exterior set of S and is denoted by

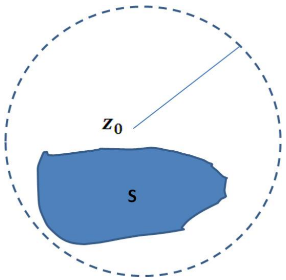
Example: Each point in $|z| > 1$ is an exterior point of the set $|z| < 1$.


Exterior point
Interior point
S
c
Boundary Point
S
Limit Point

Bounded Set, Compact Set

Definition

A set $S \subseteq \mathbb{C}$ is said to be bounded if there exists an open ball $B(z_0,\ r_0)$ for some $z_0 \in \mathbb{C}$ with $r_0 > 0$ such that $S \subset B(z_0,\ r_0)$.
That is, the set S can be put inside an open ball with some center and a finite radius.



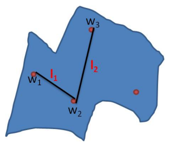
An empty set Ø is bounded.
A set S which is not bounded is called unbounded.

Definition

A set $S \subseteq \mathbb{C}$ is said to be compact if it is closed and bounded.

Connected Set, Domain, Region

Let $w_1, w_2, \cdots, w_{n+1}$ be $n+1$ points in the plane. For each $k = 1, 2, \cdots, n$, let l_k denote the line segment joining w_k to w_{k+1}. Then, the successive line segments $l_1, l_2, \cdots, l_n$ form a continuous chain known as a polygonal path that joins w_1 to w_{n+1}.



Polygonal Path

Definition

An open set $S \subseteq \mathbb{C}$ is said to be connected if every pair of points z_1, z_2 in S can be joined by a polygonal path that lies entirely in S.

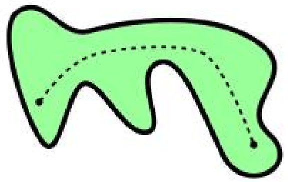
Note: The concept of connecting any two points by a path is actually known as Path Connected and Path Connected $\Longrightarrow$ Connected.

If a set S is connected then its closure $\overline{S}$ is also connected.

Connected Sets and Domain

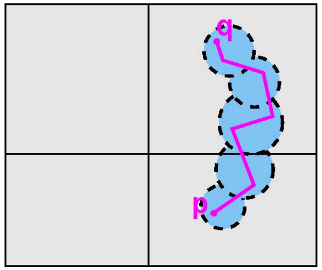
Definition

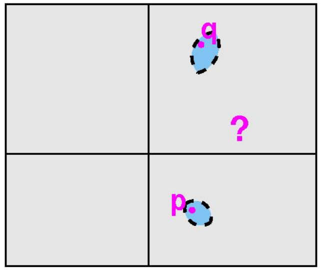
An open, connected set $S \subseteq \mathbb{C}$ is called a domain.
A domain, together with some, none, or all of its boundary points, is called a region.



Connected Set

A set that is not connected is called a disconnected set.



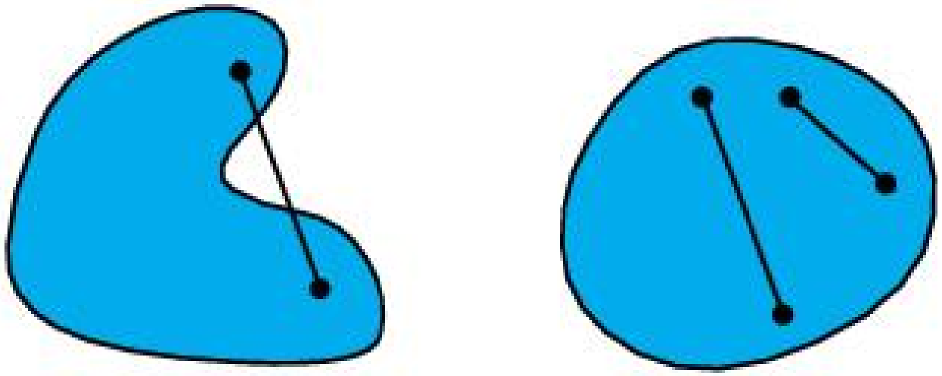


Connected Set Disconnected Set

Convex Set

Definition

A set S is said to be convex if every straight line segment L joining any two points of S lies entirely inside the set S.



Not Convex Convex