MA201 Mathematics III
Solutions to Complex Analysis Tutorial 03

Elementary Analytic Functions and their Mapping Properties

18. Find the values of z which make the function $f(z) = \exp(z)$ (a) purely real and (b) purely imaginary.
Answer:

$$e^z = e^{x+ i\, y} = e^x\ \left(\cos(y)\ +\ i\ \sin(y)\right)\ .$$

We know that $e^x \neq 0$ for all $x \in \mathbb{R}$.
e^z is a real number iff $\sin y = 0$ iff $y = n\pi$ where $n \in \mathbb{Z}$.
e^z is a pure imaginary number iff $\cos y = 0$ iff $y = \frac{(2n+1)\pi}{2}$ where $n \in \mathbb{Z}$.

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19. Find all solutions of $\exp(z - 1) = 1$.
Answer:
We need to find $z = x +\ i\ y$ such that $\exp(z-1) = \exp(x +\ i\ y - 1) = \exp((x-1)\ +\ i\ y) = e^{(x-1)}\,(\cos y +\ i\ \sin y) = 1$.
That is,

$$e^{(x-1)} \cos y = 1 \qquad \text{and} \qquad e^{(x-1)} \sin y = 0\ .$$

Observe that $e^{(x-1)} \neq 0$ for all $x \in \mathbb{R}$. Therefore, we need a y such that $\sin y = 0$ and $\cos y > 0$. This gives that $y = 2k\pi$ where $k \in \mathbb{Z}$.
When $y = 2k\pi$, $\cos y = 1$ and hence we need a x such that $e^{(x-1)} = 1$. It gives that $x = 1$.
Therefore, the points $z_k = (1,\ 2k\pi)$ where $k \in \mathbb{Z}$ are solutions of the equation $\exp(z-1) = 1$.

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20. Describe the image of the following sets in the z-plane under the mapping $w = \sin(z)$.
(i) $\{z = x + iy \in \mathbb{C}\ :\ x = (\pi/2),\ -\infty < y < \infty\}$
(ii) $\{z = x + iy \in \mathbb{C}\ :\ x = -(\pi/2),\ -\infty < y < \infty\}$
(iii) $\{z = x + iy \in \mathbb{C}\ :\ |x| \leq (\pi/2),\ y = 0\}$
(iv) $\{z = x + iy \in \mathbb{C}\ :\ x = 0,\ -\infty < y < \infty\}$
(v) $\{z = x + iy \in \mathbb{C}\ :\ x = a\ \text{ with }\ |a| < (\pi/2),\ -\infty < y < \infty\}$
(vi) $\{z = x + iy \in \mathbb{C}\ :\ |x| < (\pi/2),\ y = b,\ \ b \neq 0\}$
(vii) $\{z = x + iy \in \mathbb{C}\ :\ |x| < (\pi/2),\ y > 0\}$
(Note that mappings by $\cos z$, $\sinh z$ and $\cosh z$ closely related to the $\sin z$ function are easily obtained once mappings by the sine function are known. Because, $\cos(z) = \sin(z + \frac{\pi}{2})$, $\sinh(z) = -i \sin(iz)$ and $\cosh(z) = \cos(iz)$ and they are the same as the sine transformation preceded by translation or rotation.
Answer:

$$w = u +\ i\ v = \sin z = \sin x\ \cosh y\ +\ i\ \cos x\ \sinh y$$

$$\Longrightarrow \qquad u = \sin x\ \cosh y \qquad \text{and} \qquad v = \cos x\ \sinh y\ .$$

(i) The set given in the z-plane is $\{z = x + iy \in \mathbb{C} \ : \ x = (\pi/2), \ -\infty < y < \infty\}$ and we will find its image under $w = \sin z$

$$u = \sin(\pi/2) \ \cosh y \qquad \text{and} \qquad v = \cos(\pi/2) \ \sinh y$$
$$u = \cosh y \qquad \text{and} \qquad v = 0$$

Range of $\cosh y$ is $[1, \ \infty)$ for $y \in \mathbb{R}$.
Therefore the image set is

$$\{w = u + \ i \, v \in \mathbb{C} \ : \ u \geq 1 \text{ and } v = 0\} \ .$$

It is a part of the u-axis from $[1, \ \infty)$.

(ii) The set given in the z-plane is $\{z = x + iy \in \mathbb{C} \ : \ x = -(\pi/2), \ -\infty < y < \infty\}$ and we will find its image under $w = \sin z$
The image set is

$$\{w = u + \ i \, v \in \mathbb{C} \ : \ u \leq -1 \text{ and } v = 0\} \ .$$

It is a part of the u-axis from $(-\infty, \ -1]$.

(iii) The set given in the z-plane is $\{z = x + iy \in \mathbb{C} \ : \ |x| \leq (\pi/2), \ y = 0\}$ and we will find its image under $w = \sin z$

$$u = \sin(x) \ \cosh(0) \qquad \text{and} \qquad v = \cos(x) \ \sinh(0)$$
$$u = \sin(x) \qquad \text{and} \qquad v = 0$$

Range of $\sin x$ is $[-1, \ 1]$ for $x \in \mathbb{R}$.
Therefore the image set is

$$\{w = u + \ i \, v \in \mathbb{C} \ : \ -1 \leq u \leq 1 \text{ and } v = 0\} \ .$$

It is a part of the u-axis from $[-1, \ 1]$.

(iv) The set given in the z-plane is $\{z = x + iy \in \mathbb{C} \ : \ x = 0, \ -\infty < y < \infty\}$ and we will find its image under $w = \sin z$

$$u = \sin(0) \ \cosh(y) \qquad \text{and} \qquad v = \cos(0) \ \sinh(y)$$
$$u = 0 \qquad \text{and} \qquad v = \sinh y$$

Range of $\sinh y$ is $(-\infty, \ \infty)$ for $y \in \mathbb{R}$.
Therefore the image set is,

$$\{w = u + \ i \, v \in \mathbb{C} \ : \ u = 0 \text{ and } -\infty < v < \infty\} \ .$$

It is the v-axis.

(v) The set given in the z-plane is $\{z = x + iy \in \mathbb{C} \ : \ x = a \ \text{ with } \ |a| < (\pi/2), \ -\infty < y < \infty\}$ and we will find its image under $w = \sin z$

$$u = \sin(a) \ \cosh(y) \qquad \text{and} \qquad v = \cos(a) \ \sinh(y)$$

$$\cosh(y) = \frac{u}{\sin(a)} \qquad \text{and} \qquad \sinh(y) = \frac{v}{\cos(a)}$$

We can eliminate y in the above equations by squaring and using the identity $\cosh^2(y) - \sinh^2(y) = 1$, and it yields that

$$\frac{u^2}{\sin^2(a)} - \frac{v^2}{\cos^2(a)} = 1\ . \tag{1}$$

(Recall from the coordinate geometry: Equation of the hyperbola in cartesian coordinates is given by $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Set $c^2 = a^2 + b^2$. Then, the hyperbola has foci at $(-c,\ 0)$ and $(c,\ 0)$.)
If $a \neq 0$ then the above equation (1) represents a hyperbola in the uv-plane which has foci at the points $(\pm 1,\ 0)$ and passes through the point $(\pm\sin(a),\ 0)$.
If $0 < a < (\pi/2)$ then it describes the right branch of hyperbola.
If $(-\pi/2) < a < 0$ then it describes the left branch of hyperbola.
If $a = 0$ then it is the v-axis.

(vi) The set given in the z-plane is $\{z = x + iy \in \mathbb{C}\ :\ |x| < (\pi/2),\ y > 0\}$
and we will find its image under $w = \sin z$
From the previous part, we know that the image of the set $\{z = x+iy \in \mathbb{C}\ :\ x = a \ \text{ with }\ |a| < (\pi/2),\ -\infty < y < \infty\}$ under the mapping $w = \sin z$ is a hyperbola in the uv-plane which has foci at the points $(\pm 1,\ 0)$ and passes through the point $(\sin(a),\ 0)$. If $y > 0$, then we get $v > 0$. Thus, the image points cover only the upper half plane.
Therefore the image set is

$$\{w = u + \ i\, v \in \mathbb{C}\ :\ -\infty < u < \infty \text{ and } v > 0\,\}\ .$$

It is the (open) upper half plane.

(vii) The set given in the z-plane is $\{z = x + iy \in \mathbb{C}\ :\ |x| < (\pi/2),\ y = b,\ \ b \neq 0\}$
and we will find its image under $w = \sin z$

$$u = \sin(x)\ \cosh(b) \qquad \text{and} \qquad v = \cos(x)\ \sinh(b)$$

$$\sin(x) = \frac{u}{\cosh(b)} \qquad \text{and} \qquad \cos x = \frac{v}{\sinh(b)}$$

We can eliminate y in the above equations by squaring and using the identity $\sin^2(x) + \cos^2(x) = 1$, and it yields that

$$\frac{u^2}{\cosh^2(b)} + \frac{v^2}{\sinh^2(a)} = 1\ . \tag{2}$$

(Recall from the coordinate geometry: Equation of the ellipse in cartesian coordinates is given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Set $c^2 = a^2 - b^2$. Then, the ellipse has foci at $(-c,\ 0)$ and $(c,\ 0)$.)
If $b \neq 0$ then the above equation (2) represents an ellipse in the uv-plane which has foci at the points $(\pm 1,\ 0)$ and passes through the points $(\pm\cosh(b),\ 0)$ and $(0,\ \pm\sinh(b))$.
If $b > 0$ then it describes the portion of the ellipse that lies in the upper half plane.
If $b < 0$ then it describes the portion of the ellipse that lies in the lower half plane.

21. Evaluate the following:
(i) $\log(3-2i)$ (ii) Log i (iii) $(i)^{(-i)}$
Answer:
We know that if $z \neq 0$, then $\log(z) = \ln|z| \ + \ i \ \arg(z)$.
(i)

$$\log(3-2i) = \ln|3-2i| \ + \ i \ \arg(3-2i) = \ln|\sqrt{13}| \ + \ i\ (\alpha + 2n\pi)$$

where $\alpha = \tan^{-1}(-2/3)$ and $n \in \mathbb{Z}$.
Therefore,

$$\log(3-2i) = \frac{1}{2} \ \ln(13) \ + \ i\ (\alpha + 2n\pi)$$

where $\alpha = \tan^{-1}(-2/3)$ and $n \in \mathbb{Z}$.

We know that if $z \neq 0$ then $\operatorname{Log}(z) = \ln|z| \ + \ i \ \operatorname{Arg}(z)$ where Arg denotes the principal value of the argument.
(ii)

$$\operatorname{Log}(i) = \ \ln|i| \ + \ i \ \operatorname{Arg}(i) = \ \ln(1) \ + \ i\ \frac{\pi}{2} = \frac{i\,\pi}{2} \ .$$

(iii)

$$\begin{aligned} (i)^{(-i)} &= \exp\left((-i)\log(i)\right) = \exp\left((-i)\left[\ln|i| \ + \ i \ \arg(i)\right]\right) \\ &= \exp\left((-i)\left[\ln(1) \ + \ i \ \left(\frac{\pi}{2} + 2n\pi\right)\right]\right) \\ &= \exp\left((-i)\left[i \ \left(\frac{\pi}{2} + 2n\pi\right)\right]\right) \\ &= \exp\left(\frac{\pi}{2} + 2n\pi\right) \end{aligned}$$

where $n \in \mathbb{Z}$.

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22. Determine the domain of analyticity for the function $f(z) = \operatorname{Log}(3z - i)$ and compute $f'(z)$.
Answer:
We know the domain of analyticity of $\operatorname{Log}(z)$ is $D^* = \{z = re^{i\,\theta} \ : \ r > 0 \text{ and } -\pi < \theta < \pi\} = \mathbb{C} \setminus \{z = x + iy \ : \ x \leq 0 \text{ and } y = 0\}$ and its derivative is $1/z$ on D^*.
Now,

$$3z - i = 3(x+iy) - i = 3x + \ i\ (3y-1) \ .$$

$$\Re(3z-i) \leq 0 \Longleftrightarrow x \leq 0 \qquad \text{and} \qquad \Im(3z-i) = 0 \Longleftrightarrow y = \frac{1}{3} \ .$$

Therefore, the domain of analyticity of $\operatorname{Log}(3z-i)$ is $D^{**} = \mathbb{C} \setminus \{z = x + iy \ : \ x \leq 0 \text{ and } y = \frac{1}{3}\}$. In the domain D^{**}, the derivative of $\operatorname{Log}(3z-i)$ is equal to $3/(3z-i)$.

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23. Find the principal branch of the function $\log(2z-1)$.
Answer:
The principal branch of $\log(z)$ is $\operatorname{Log}(z)$ and is defined by

$$\operatorname{Log}(z) \ = \ \ln|z| \ + \ i \ \operatorname{Arg}(z) \qquad \text{for } z \in D^*$$

where $D^* = \{z = re^{i\,\theta} \ : \ r > 0 \text{ and } -\pi < \theta < \pi\} = \mathbb{C} \setminus \{z = x + iy \ : \ x \le 0 \text{ and } y = 0\}$.
Now,

$$2z - 1 = 2(x + iy) - 1 = (2x - 1) + \ i\,y\ .$$

$$\Re(2z-1) \le 0 \iff x \le \frac{1}{2} \qquad \text{and} \qquad \Im(2z-1) = 0 \iff y = 0\ .$$

The principal branch of $\log(2z-1)$ is $\operatorname{Log}(2z-1)$ and is defined by

$$\operatorname{Log}(2z-1) \ = \ \ln|2z-1| \ + \ i\,\operatorname{Arg}(2z-1) \qquad \text{for } z \in D^{**}$$

where $D^{**} = \mathbb{C} \setminus \{z = x + iy \ : \ x \le \frac{1}{2} \text{ and } y = 0\}$.

Line/Contour Integrals

24. Let $z_1 = -1$, $z_2 = 1$ and $z_3 = i$.
Compute $\displaystyle\int_{[z_1,\,z_2,\,z_3]} \overline{z}\,dz$ and $\displaystyle\int_{[z_1,\,z_3]} \overline{z}\,dz$.
Answer:
Step 1: On the line segment L_1 from $z_1 = -1$ to $z_2 = 1$
Parametric equation of L_1 is $z(t) = -1 + 2t$ for $t \in [0,\ 1]$. This gives that $z'(t) = 2$ for $t \in [0,\ 1]$.

$$\int_{L_1} \overline{z}\,dz \ = \ \int_0^1 (-1+2t)\ (2)\ dt = \left[-2t + 2t^2\right]_{t=0}^1 = 0\ .$$

Step 2: On the line segment L_2 from $z_2 = 1$ to $z_3 = i$
Parametric equation of L_2 is $z(t) = (1-t) + \ i\,t$ for $t \in [0,\ 1]$. This gives that $z'(t) = (i-1)$ for $t \in [0,\ 1]$.

$$\int_{L_2} \overline{z}\,dz \ = \ \int_0^1 ((1-t) - \ i\,t)\ (i-1)\ dt = \int_0^1 ((2t-1) + \ i\ )\ dt = \left[t^2 - t + i\,t\right]_{t=0}^1 = i\ .$$

Step 3: On the line segment L_3 from $z_1 = -1$ to $z_3 = i$
Parametric equation of L_3 is $z(t) = (-1+t) + \ i\,t$ for $t \in [0,\ 1]$. This gives that $z'(t) = (1+i)$ for $t \in [0,\ 1]$.

$$\int_{L_3} \overline{z}\,dz \ = \ \int_0^1 ((-1+t) - \ i\,t)\ (1+i)\ dt = \int_0^1 ((2t-1) + \ i(-1)\ )\ dt = \left[t^2 - t - i\,t\right]_{t=0}^1 = -i\ .$$

Therefore,

$$\int_{[z_1,\,z_2,\,z_3]} \overline{z}\,dz \ = \int_{[z_1,\,z_2]} \overline{z}\,dz \ + \ \int_{[z_2,\,z_3]} \overline{z}\,dz \ = \ 0 + i \ = \ i\ ,$$

and $\displaystyle\int_{[z_1,\,z_3]} \overline{z}\,dz \ = \ -i.$

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25. Evaluate $\int_C |z|\,\overline{z}\,dz$ where C is a positively oriented simple closed contour consists of (i) the line segment from $-2i$ to $2i$ and (ii) the semi circle $|z| = 2$ in the second and third quadrants.
Answer:
Step 1: On the line segment L from $z_1 = -2i$ to $z_2 = 2i$
Parametric equation of L is $z(t) = i\,t$ for $t \in [-2,\ 2]$. This gives that $z'(t) = i$ for $t \in [-2,\ 2]$.

$$\int_L |z|\,\overline{z}\,dz \;=\; \int_{-2}^{2} (|t|\,(-i\,t))\ (i)\,dt = \int_{-2}^{0} (-t^2)\,dt \;+\; \int_0^2 t^2\,dt \;= 0\,.$$

Step 2: On the semi circle $\gamma :\ |z| = 2$ from $z_2 = 2i$ to $z_1 = -2i$
Parametric equation of γ is $z(t) = 2e^{i\,t}$ for $t \in [\pi/2,\ 3\pi/2]$. This gives that $z'(t) = 2i\ e^{i\,t}$ for $t \in [\pi/2,\ 3\pi/2]$.

$$\int_\gamma |z|\,\overline{z}\,dz \;=\; \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(2 \times 2e^{-i\,t}\right)\ (2i\ e^{i\,t})\,dt = 8i\ \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \ dt = 8\pi\ i\,.$$

Step 3:

$$\int_C |z|\,\overline{z}\,dz \;=\; \int_L |z|\,\overline{z}\,dz \;+\; \int_\gamma |z|\,\overline{z}\,dz \;=\; 0 + 8\pi i = 8\pi i\,.$$

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26. If C is the boundary of the triangle with vertices at the points 0, $3i$ and -4 oriented in the counterclockwise direction then show that $\left|\int_C (e^z - \overline{z})\ dz\right| \;\le\; 60$.
Answer:
Observe that the length of the curve C is 12.
Let R be the triangular region consists of the boundary curve C and its interior. Observe that the function $|f(z)| = |e^z - \overline{z}| \le |e^z| + |\overline{z}| = e^{\Re(z)} + |z|$. In the closed region R, we have $|f(z)| \le e^{\Re(z)} + |z| \le 1 + |z| = 5$. Therefore, $|f(z)|$ is bounded by 5 in the closed region R.

$$\begin{aligned}\left|\int_C (e^z - \overline{z})\ dz\right| \;&\le\; \int_C |e^z - \overline{z}|\ dz \\ &\le\; 5\int_C dz \\ &=\; 5 \times 12 = 60\end{aligned}$$

Cauchy's Integral Theorems and its Applications

27. Does Cauchy's theorem hold separately for the real and the imaginary parts of an analytic function $f(z)$. If so, prove that it does, if not give a counter example. (Hint: Think of the identity function and the unit circle contour)
Answer:
"Question is to be interpreted as: If f is analytic on and inside a simple closed contour C then we know that $\int_C f(z)\,dz \;= 0$. Is it true that $\int_C Re(f(z))\,dz = 0$ and $\int_C Im(f(z))\,dz = 0$?"

The answer is **NO**.
Consider $f(z) = z$ for all $z \in \mathbb{C}$ and $C : |z| = 1$ (positively oriented). Then,

$$\int_C \mathrm{Re}(f(z))\, dz \;=\; \int_{\theta=0}^{2\pi} \cos\theta\, ie^{i\theta}\, d\theta \;=\; i \int_{\theta=0}^{2\pi} \left(\cos^2\theta + \; i\; \cos\theta\sin\theta\right)\, d\theta$$

$$= (-1)\int_{\theta=0}^{2\pi} \cos\theta\sin\theta\, d\theta \;+\; i\int_{\theta=0}^{2\pi} \cos^2\theta\, d\theta \;=\; 0 + \pi i = \pi i \neq 0\ .$$

Similarly,

$$\int_C \mathrm{Im}(f(z))\, dz \;=\; \int_{\theta=0}^{2\pi} \sin\theta\, ie^{i\theta}\, d\theta \;=\; 2\pi \neq 0\ .$$

Thus, Cauchy's theorem does not hold separately for the real and the imaginary part of an analytic function $f(z)$.

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28. Evaluate $\displaystyle\int_C \frac{z^2-4}{z^2+4}\, dz$ if C is a simple closed contour described in the counterclockwise direction and
(i) The point $2i$ lies inside C, and $-2i$ lies outside C
(ii) The point $-2i$ lies inside C, and $2i$ lies outside C
(iii) The points $\pm 2i$ lie outside C
(iii) The points $\pm 2i$ lie inside C
Answer:
Observe that $z^2 + 4 = (z + 2i)(z - 2i)$.
(i) the point $2i$ lies inside C and $-2i$ lies outside C

$$\begin{aligned}\int_C \frac{z^2-4}{z^2+4}\, dz &= \int_C \frac{\frac{z^2-4}{z+2i}}{z-2i}\, dz \\ &= 2\pi\, i \left[\frac{z^2-4}{z+2i}\right]_{z=2i} \\ &= -4\pi\end{aligned}$$

(ii) the point $-2i$ lies inside C and $2i$ lies outside C

$$\begin{aligned}\int_C \frac{z^2-4}{z^2+4}\, dz &= \int_C \frac{\frac{z^2-4}{z-2i}}{z-(-2i)}\, dz \\ &= 2\pi\, i \left[\frac{z^2-4}{z-2i}\right]_{z=-2i} \\ &= 4\pi\end{aligned}$$

(iii) the points $\pm 2i$ lie outside C
Since the function $f(z) = \frac{z^2-4}{z^2+4}$ is analytic on and inside C, by the Cauchy-Goursat theorem,

$$\int_C \frac{z^2-4}{z^2+4}\, dz = 0\ .$$

(iv) the points $\pm 2i$ lie inside C

Observe that

$$\frac{z^2-4}{z^2+4} = 1 + \frac{\frac{-2}{i}}{z-2i} + \frac{\frac{2}{i}}{z+2i}\,.$$

$$\int_C \frac{z^2-4}{z^2+4}\,dz \;=\; \int_C dz \;+\; \int_C \left(\frac{\frac{-2}{i}}{z-2i} + \frac{\frac{2}{i}}{z+2i}\right)\,dz$$

Let $C_1 : |z-2i| = r$ and $C_2 : |z+2i| = r$ where $r > 0$ is sufficiently small so that C_1 and C_2 lie interior to C and they are disjoint. Further the curves C_1 and C_2 do not have points common to their interior bounded domains. By the Cauchy's integral theorem for multiply-connected domains, it follows that

$$\begin{aligned}
\int_C \frac{z^2-4}{z^2+4}\,dz &= \int_C dz - \frac{2}{i}\int_{C_1:|z-2i|=r} \frac{dz}{z-2i} + \frac{2}{i}\int_{C_2:|z+2i|=r} \frac{dz}{z+2i} \\
&= 0 - \frac{2}{i}\times 2\pi i + \frac{2}{i}\times 2\pi i \\
&= 0
\end{aligned}$$

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29. Evaluate $\displaystyle\int_C \frac{\cosh z}{(z-i)^{2n+1}}\,dz$ where $C : |z-i| = 1$.

Answer:

Let $f(z) = \cosh z$ for $z \in \mathbb{C}$.

$$\begin{aligned}
\int_C \frac{\cosh z}{(z-i)^{2n+1}}\,dz &= \frac{2\pi i}{(2n)!}\int_C \frac{f(z)\,dz}{(z-i)^{2n+1}} \\
&= \frac{2\pi i}{(2n)!}\left[\frac{d^{2n}}{dz^{2n}}\cosh(z)\right]_{z=i} \\
&= \frac{2\pi i}{(2n)!}\cosh(i) \\
&= \frac{2\pi i}{(2n)!}\cos(1)
\end{aligned}$$

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30. Let f be an entire function such that $|f(z)| \le A + B|z|^n$ for all $z \in \mathbb{C}$ where A and B are positive real constants and n is a fixed natural number. Show that f is a polynomial of degree at most n. (It is a generalization of Exercise Problem 1 of Section 50, Brown and Churchill, 7th edition)

Answer:

We want to show that f is a polynomial of degree at most n. So we will show that the $(n+1)$-th derivative $f^{(n+1)}(z) = 0$ for all $z \in \mathbb{C}$.

Let z_0 be an arbitrary point in $\mathbb{C}$.

Claim: $f^{(n+1)}(z_0) = 0$.

Let C_r denote the positively oriented circle $|z - z_0| = r$.

By the Cauchy Integral formula for the derivatives, we get

$$f^{(n+1)}(z_0) = \frac{(n+1)!}{2\pi i}\int_{C_r} \frac{f(z)\,dz}{(z-z_0)^{n+2}}\,.$$

Then,

$$\begin{aligned}
\left|f^{(n+1)}(z_0)\right| &= \left|\frac{(n+1)!}{2\pi i}\int_{C_r}\frac{f(z)\,dz}{(z-z_0)^{n+2}}\right| \\
&\leq \frac{(n+1)!}{2\pi}\int_{C_r}\frac{|f(z)|}{|z-z_0|^{n+2}}\,dz \\
&\leq \frac{(n+1)!}{2\pi}\int_{C_r}\frac{A+B|z|^n}{r^{n+2}}\,dz
\end{aligned}$$

Observe that $|z| \leq |z-z_0| + |z_0| = r + |z_0|$ on C_r. It gives that

$$\begin{aligned}
\left|f^{(n+1)}(z_0)\right| &\leq \frac{(n+1)!\ \left(A + B\,(r+|z_0|)^n\right)}{(2\pi)\ r^{n+2}}\int_{C_r} dz \\
&= \frac{K_{n+1}}{r^{n+1}} + \frac{K_n}{r^n} + \cdots + \frac{K_2}{r^2} + \frac{K_1}{r} \qquad \text{where } K_i'\text{ s are constants}
\end{aligned}$$

The above inequality is true for every circle $C_r : |z-z_0| = r$. Letting $r \to \infty$, it follows that $|f^{(n+1)}(z_0)| = 0$ and hence $f^{(n+1)}(z_0) = 0$.
This completes the proof of the claim.
