MA201 MATHEMATICS III

Monsoon Semester of Academic Year 2023 - 2024

TUTORIAL PROBLEMS

COMPLEX ANALYSIS

The main aim of the tutorial classes is to help the students by providing them only hints/ideas/techniques to solve the problems, and not the complete solution. Therefore, students are expected to work out the tutorial problems before coming to the tutorial class. Students are encouraged to get clarification/ explanations in the tutorial class for better understanding of the concepts/techniques related to the tutorial problems.

Attendance in the tutorial classes is mandatory.

Tutorial Date	**Problems that will be Discussed**
31-July-2023	Problem Nos. 1 to 10
07-August-2023	Problem Nos. 11 to 24
14-August-2023	Problem Nos. 25 to 33
21-August-2023	Problem Nos. 34 to 43
28-August-2023	Quiz I
04-September-2023	Problem Nos. 44 to 54
11-September-2023	Problem Nos. 55 to 61

Complex Numbers and Complex Algebra

1. Find the modulus, argument, principal value of the argument, and polar form of the given complex number:
(i) $\sqrt{3}+i$ (ii) $\dfrac{1-i\sqrt{3}}{2}$ (iii) $\dfrac{1-i}{1+i}$ (iv) $\dfrac{(2+i)^2}{(3-i)^2}$ (v) -100 (vi) $-3i$

2. Show that if $|z|=2$, then $\left|\dfrac{1}{z^4-4z^2+3}\right| \leq \dfrac{1}{3}$.

3. If either $|z_1|=1$ or $|z_2|=1$, but not both, then prove that $\left|\dfrac{z_1-z_2}{1-\overline{z_1}\,z_2}\right|=1$. What exception must be made for the validity of the above equality when $|z_1|=|z_2|=1$?

4. If z_1, z_2, z_3 and z_4 are complex numbers of unit modulus, prove that
$$|z_1-z_2|^2|z_3-z_4|^2+|z_1+z_4|^2|z_3-z_2|^2=|z_1(z_2-z_3)+z_3(z_2-z_1)+z_4(z_1-z_3)|^2$$

5. Prove that equation of the circle whose diameter is formed by joining z_1 and z_2 is
$$2z\overline{z}-z(\overline{z_1}+\overline{z_2})-\overline{z}(z_1+z_2)+z_1\overline{z_2}+\overline{z_1}z_2=0.$$

6. Interpret geometrically the following relations:
(i) $\{z\in\mathbb{C}:|\Re(z)|+|\Im(z)|=1\}$.
(ii) $|z-a|-|z+a|=2c$ where a and c are real constants with $c>0$

7. Find all the roots or all the values of the following:
(i) Cube roots of i (ii) Fourth roots of $(-2\sqrt{3}-2i)$ (iii) Fourth roots of (-1) (iv) Sixth roots of 8 (v) The values of $(i)^{\frac{2}{3}}$

Limit and Continuity

8. Show that $\lim\limits_{z\to 0}\dfrac{|z|^2}{z}=0$.

9. Let $f(z)=z^2/|z|^2$.
(a) Find the value of limit of $f(z)$ as $z=(x+iy)\to 0$ along the line $y=x$.
(b) Find the value of limit of $f(z)$ as $z=(x+iy)\to 0$ along the line $y=2x$.
(c) Find the value of limit of $f(z)$ as $z=(x+iy)\to 0$ along the path $y=x^2$.
(d) What can you conclude about the limit of $f(z)$ as $z\to 0$.

10. The following functions are defined for $z\neq 0$. Which of these functions can be defined at $z=0$ so that it becomes continuous at $z=0$.
(a) $\dfrac{\Re(z)}{|z|}$ (b) $\dfrac{z}{|z|}$ (c) $\dfrac{z\Re(z)}{|z|}$ (d) $\dfrac{\Re(z^2)}{|z|^2}$ (e) $\dfrac{z^2}{|z|}$

Differentiability and CR Equations

11. Let $f(z)=\dfrac{x^3(1+i)-y^3(1-i)}{x^2+y^2}$ for $z=x+iy\neq 0$ and $f(0)=0$. Show that $f(z)$ is continuous at origin but $f'(0)$ does not exist.

12. If $f(z)$ is a real valued function in a domain $D \subseteq \mathbb{C}$, then show that either $f'(z) = 0$ or $f'(z)$ does not exist in D.

13. Let $f(z) = (x^3 y(y - ix)) \,/\, (x^6 + y^2)$ for $z = x + iy \neq 0$ and $f(0) = 0$.
(a) Find the value of $\lim\limits_{\Delta z \to 0} \dfrac{f(\Delta z) - f(0)}{\Delta z}$ as $\Delta z \to 0$ along the line $y = mx$.
(b) Find the value of $\lim\limits_{\Delta z \to 0} \dfrac{f(\Delta z) - f(0)}{\Delta z}$ as $\Delta z \to 0$ along the imaginary axis.
(c) Find the value of $\lim\limits_{\Delta z \to 0} \dfrac{f(\Delta z) - f(0)}{\Delta z}$ as $\Delta z \to 0$ along the path $y = x^3$.
(d) What can you conclude about the existence of $f'(z)$ at $z = 0$.
(e) Show that the Cauchy-Riemann (CR) equations hold true at $(0,\ 0)$.

Analytic Functions

14. Show that the function $f(z) = x^3 + 3xy^2 - 3x + i(y^3 + 3x^2 y - 3y)$ is differentiable only at points that lie on the coordinate axes. Is $f(z)$ analytic at any point lies on the coordinate axes?

15. Show that the function $f(z) = xy + iy$ is continuous everywhere, but not analytic in $\mathbb{C}$.

16. Check whether the function $g(z) = (3x^2 + 2x - 3y^2 - 1) + i(6xy + 2y)$ is satisfying the sufficient conditions to be an analytic function at any point in the complex plane. Write this function in terms of z.

17. Let a function $f(z) = u(x,\ y) + i\,v(x,\ y)$ be analytic in a domain D. Prove that $f(z)$ is constant in D if $f'(z) = 0$ throughout in D.

18. Find the analytic function $f(z) = u(x,\ y) + iv(x,\ y)$ given the following:
(First verify that they are harmonic functions)
(a) $u(x,\ y) = y^3 - 3x^2 y$ (b) $v(x,\ y) = \sin x \cosh y$
(c) $u(x,\ y) - v(x,\ y) = (x - y)(x^2 + 4xy + y^2)$

Elementary Analytic Functions and their Mapping Properties

19. Find the values of z which make the function $f(z) = \exp(z)$ (a) purely real and (b) purely imaginary.

20. Find all solutions of $\exp(z - 1) = 1$.

21. Describe the image of the following sets in the z-plane under the mapping $w = \sin(z)$.
(i) $\{z = x + iy \in \mathbb{C} \ :\ x = (\pi/2),\ -\infty < y < \infty\}$
(ii) $\{z = x + iy \in \mathbb{C} \ :\ x = -(\pi/2),\ -\infty < y < \infty\}$
(iii) $\{z = x + iy \in \mathbb{C} \ :\ |x| \leq (\pi/2),\ y = 0\}$
(iv) $\{z = x + iy \in \mathbb{C} \ :\ x = 0,\ -\infty < y < \infty\}$
(v) $\{z = x + iy \in \mathbb{C} \ :\ x = a \text{ with } |a| < (\pi/2),\ -\infty < y < \infty\}$
(vi) $\{z = x + iy \in \mathbb{C} \ :\ |x| < (\pi/2),\ y = b,\ \ b \neq 0\}$

(vii) $\{z = x + iy \in \mathbb{C} \ : \ |x| < (\pi/2), \ y > 0\}$
(Note that mappings by $\cos z$, $\sinh z$ and $\cosh z$ closely related to the $\sin z$ function are easily obtained once mappings by the sine function are known. Because, $\cos(z) = \sin(z + \frac{\pi}{2})$, $\sinh(z) = -i\sin(iz)$ and $\cosh(z) = \cos(iz)$ and they are the same as the sine transformation preceded by translation or rotation.

22. Evaluate the following:
(i) $\log(3 - 2i)$ (ii) $\mathrm{Log}\ i$ (iii) $(i)^{(-i)}$

23. Determine the domain of analyticity for the function $f(z) = \mathrm{Log}\ (3z - i)$ and compute $f'(z)$.

24. Find the principal branch of the function $\log(2z - 1)$.

Line/Contour Integrals

25. Let $z_1 = -1$, $z_2 = 1$ and $z_3 = i$.
Compute $\displaystyle\int_{[z_1,\, z_2,\, z_3]} \overline{z}\, dz$ and $\displaystyle\int_{[z_1,\, z_3]} \overline{z}\, dz$.

26. Evaluate $\displaystyle\int_C |z|\, \overline{z}\, dz$ where C is a positively oriented simple closed contour consists of (i) the line segment from $-2i$ to $2i$ and (ii) the semi circle $|z| = 2$ in the second and third quadrants.

27. If C is the boundary of the triangle with vertices at the points 0, $3i$ and -4 oriented in the counterclockwise direction then show that $\left| \displaystyle\int_C (e^z - \overline{z})\ dz \right| \leq 60.$

Cauchy's Integral Theorems and its Applications

28. Does Cauchy's theorem hold separately for the real and the imaginary parts of an analytic function $f(z)$. If so, prove that it does, if not give a counter example. (Hint: Think of the identity function and the unit circle contour)

29. Evaluate $\displaystyle\int_C \frac{z^2 - 4}{z^2 + 4}\, dz$ if C is a simple closed contour described in the counterclockwise direction and
(i) The point $2i$ lies inside C, and $-2i$ lies outside C
(ii) The point $-2i$ lies inside C, and $2i$ lies outside C
(iii) The points $\pm 2i$ lie outside C
(iii) The points $\pm 2i$ lie inside C

30. Evaluate $\displaystyle\int_C \frac{\cosh z}{(z - i)^{2n+1}}\, dz$ where $C : |z - i| = 1$.

31. Let f be an entire function such that $|f(z)| \leq A + B|z|^n$ for all $z \in \mathbb{C}$ where A and B are positive real constants and n is a fixed natural number. Show that f is a polynomial of degree at most n. (It is a generalization of Exercise Problem 1 of Section 50, Brown and Churchill, 7th edition)

32. Let $f(z) = (z+1)^2$ for $z \in \mathbb{C}$. Let R be the closed triangular region with vertices at the points $z = 0$, $z = 2$ and $z = i$. Find points in R where $|f(z)|$ has its maximum and minimum values.

33. Let f be analytic in the disk $|z| < 1$. Suppose that $|f(z)| \leq 1$ for $|z| < 1$ and $f(0) = 0$. Show that $|f(z)| \leq |z|$ for $|z| < 1$ and $|f'(0)| \leq 1$. This result is known as *Schwarz Lemma*.

Power Series

34. Find the power series expansion of the following functions about the point $z_0 = 0$ and find its radius of convergence
(i) $f(z) = \cos^2 z$ (ii) $f(z) = \sinh^2 z$ (iii) $f(z) = \log(1+z)$ (iv) $f(z) = \sqrt{z+2i}$
(v) $f(z) = \displaystyle\int_0^z \exp(w^2)\, dw$

35. Find the Maclaurin series expansion of the error function $\operatorname{erf} z = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^z \exp(-t^2)\, dt$ using term by term integration.

36. If the radius of convergence for the series $\sum_{n=0}^{\infty} a_n z^n$ is R, then find the radius of convergence for the following:
(i) $\displaystyle\sum_{n=0}^{\infty} n^3 a_n z^n$ (ii) $\displaystyle\sum_{n=0}^{\infty} a_n^4 z^n$ (iii) $\displaystyle\sum_{n=0}^{\infty} a_n z^{2n}$ (iv) $\displaystyle\sum_{n=0}^{\infty} a_n z^{7+n}$ (v) $\displaystyle\sum_{n=1}^{\infty} n^{-n} a_n z^n$

37. Expand each of the following functions about the point $z = 1$ into a power series and find the radius of convergence:
(i) $\dfrac{z}{z^2 - 2z + 5}$ (ii) $\sin(2z - z^2)$ (iii) $\operatorname{Log}\,(1 + z^2)$

38. Find the Laurent series expansion of the following functions about the given singular points $z = z_0$ or in the given region (specify the region in which the expansion is valid wherever it is necessary).

   (a) $z^2 \exp(1/z)$ in the neighborhood of $z = 0$ and $z = \infty$
   (b) $\operatorname{Log}\left(\frac{z-a}{z-b}\right)$ in the neighborhood of $z = \infty$
   (c) $\dfrac{1}{z^2+1}$ in the neighborhood of $z = -1$ and $z = \infty$
   (d) $f(z) = \dfrac{z+3}{z(z^2 - z - 2)}$ for $|z| < 1$ and for $1 < |z| < 2$.

Zeros and Singularities

39. Find the order of the zero $z = 0$ for the function $z^2\,(\exp(z^2) - 1)$.

40. Find the singular points and investigate the behavior at infinity of the following functions:
(i) $\dfrac{\exp(1/(z-1))}{\exp(z) - 1}$ (ii) $\cot z - (2/z)$

41. Find the residues of the function $\dfrac{1}{z^3 - z^5}$ at all isolated singular points and with respect to point at infinity (provided the later is not a limiting point of the singularities)

42. Find the residues of $f(z) = \dfrac{e^{imz}}{z^2 + a^2}$, ($m$, a real) at its singularities.

43. Show that the residue at the point at infinity for the function $f(z) = \left(\dfrac{z^4}{2z^2 - 1}\right) \sin\left(\dfrac{1}{z}\right)$ is equal to $(-1/6)$.

Contour Integrations using Residue Theorem

44. Evaluate $\displaystyle\int_C \frac{z\,dz}{\cos z}$ where $C: \quad \left|z - \dfrac{\pi}{2}\right| = \dfrac{\pi}{2}$

Application of Residues in Summing the Series

45. Suppose that f is analytic on the plane except for poles $w_1, w_2, \cdots, w_N$, none of which are integers, and suppose that $\lim\limits_{z\to\infty} |zf(z)| = 0$.

    Then we have $\displaystyle\sum_{n=-\infty}^{\infty} f(n) = -\sum_{j=1}^{N} \text{Res}\,(f(z)\pi\cot(\pi z);\ w_j)$. Using it find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + a^2}$ where a is chosen such that none of the denominators vanish.

Argument Principle and Rouche's Theorem

46. Let C denote the unit circle $|z| = 1$, described in the positive sense. Determine the change in the argument of $f(z)$ as z describes C once if $f(z) = (z^3 + 2)/z$.

47. Using Rouche's theorem, find the number of roots of the equation $z^9 - 2z^6 + z^2 - 8z - 2 = 0$ lying in $|z| < 1$.

48. How many roots of the equation $z^4 - 5z + 1 = 0$ are situated in the domain $|z| < 1$? In the annulus $1 < |z| < 2$?

Application of Residues in Evaluating Real Integrals

Type - I

49. Prove that $\displaystyle\int_0^{2\pi} \frac{d\theta}{(1 + 2a\,\cos\theta\, + a^2)^2} = \frac{2\pi(1 + a^2)}{(1 - a^2)^3}$ where $-1 < a < 1$.

50. Prove that $\displaystyle\int_0^{\pi} \frac{a\,d\theta}{a^2 + \sin^2\theta} = \int_0^{\pi} \frac{a\,d\theta}{a^2 + \cos^2\theta} = \frac{\pi}{\sqrt{(1 + a^2)}}$; $\quad (a > 0)$

Type - II

51. Prove that $\displaystyle\int_0^\infty \frac{dx}{x^4+a^4} = \frac{\pi}{2a^3\sqrt{2}}, \qquad (a>0)$

Type - III

52. Prove that $\displaystyle\int_0^\infty \frac{x\sin mx\ \ dx}{x^2+a^2} = \frac{\pi}{2}\exp(-ma);\ m>0$

Type - IV: Indented Contour Integration

53. Prove that $P.V. \displaystyle\int_{-\infty}^\infty \frac{x\ \ dx}{(x^3+1)} = \frac{\pi}{\sqrt{3}}$

54. Prove that $\displaystyle\int_0^\infty \frac{\sin\pi x\ \ dx}{x(1-x^2)} = \pi$

Contour Integration of Branch functions of Multiple-valued Functions

55. Using "Indented contour", show that $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)} = \frac{\pi}{\sqrt{2}}$ by integrating an appropriate branch of the multiple valued function.

56. Using "keyhole contour", show that $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)} = \frac{\pi}{\sqrt{2}}$ by integrating an appropriate branch of the multiple valued function.

Inverse Laplace Transformation

57. Show that the inverse Laplace transform of $Y(s) = 1/(s^2+4)$ is $(1/2)\sin 2t$, by using contour integration method.

Bilinear Transformations

58. Find a bilinear transformation which maps 2, i, -2 onto 1, i, -1.

59. Find a bilinear transformation which maps 0, 1, ∞ onto i, -1, $-i$.

60. Find a bilinear transformation which maps ∞, i, 0 onto 0, i, ∞.

Schwarz-Christoffel Transformations (Optional Topic)

61. Find the Schwarz-Christoffel transformation of the upper half plane U onto the equilateral triangle whose vertices are $(-a,0)$, $(a,0)$, $(0,a\sqrt{3})$ where $a>0$.
