Complex Analysis: Lecture-01

MA201 Mathematics III

MGPP, AC, ST, SP

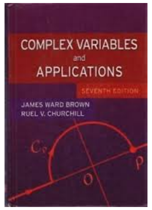
IIT Guwahati

Syllabus of Complex Analysis

- Complex Numbers: Complex numbers and elementary properties.
- Complex Functions: Limits, continuity and differentiation. Cauchy-Riemann equations. Analytic functions, Harmonic functions. Elementary Analytic functions.
- Complex Integration: Contour integrals, Anti-derivatives and path independent of contour integrals.
- Cauchy-Goursat Theorem. Cauchy's integral formula, Morera's Theorem. Liouville's Theorem, Fundamental Theorem of Algebra, Maximum Modulus Principle and its consequences.
- Power Series: Taylor series, Laurent series.
- Zeros and Singularities: Zeros of Analytic Functions, Singularities, Argument Principle, Rouche's Theorem.
- Residues and Applications: Cauchy's Residue Theorem and applications.
- Conformal Mappings: Conformal Mappings, Mobius transformations.

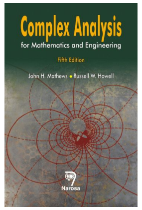
Complex Analysis Books

Text Book:



J. W. Brown and R. V. Churchill, Complex Variables and Applications, 7th or 8th Edition, Mc-Graw Hill, 2004. Note: Any Edition is fine.

Reference Book:



J. H. Mathews and R. W. Howell, Complex Analysis for Mathematics and Engineering, 3rd Edition, Narosa, 1998. Note: Any Edition or Other Publisher is fine.

Topic 01: Learning Outcome

We learn

- Complex Numbers
- Algebraic Operations: Addition, Multiplication, Division
- $\mathbb{C}$ is a field, but not an ordered field
- $x + iy$ form of complex numbers
- Conjugate, Modulus of a complex number
- Basic identities and inequalities
- Nonzero complex numbers: Polar Form, Trigonometric Form, Exponential Form, argument function
- Powers and Roots of Complex Numbers
- Interior Point, Open, Closed, Limit point, boundary point, Exterior Point
- Bounded Set, Connected Set, Compact Set, Convex Set
- Domains, Regions

Recall: $(\mathbb{R}, +, \cdot)$ is a field w. r. t. addition + and multiplication $\cdot$

- Closure Law: For all a and b in $\mathbb{R}$, $a + b \in \mathbb{R}$ and $a \cdot b \in \mathbb{R}$.
- Associative Law: For all a, b and c in $\mathbb{R}$,

$$a + (b + c) = (a + b) + c \qquad \text{and} \qquad a \cdot (b \cdot c) = (a \cdot b) \cdot c\ .$$

- Identity Law: For all a in $\mathbb{R}$,

$$a + 0 = a = 0 + a \qquad \text{and} \qquad a \cdot 1 = a = 1 \cdot a\ .$$

- Law of Additive Inverse: Given $a \in \mathbb{R}$, there exists a unique $x \in \mathbb{R}$ such that $a + x = 0 = x + a$.
- Law of Multiplicative Inverse: Given $a \in \mathbb{R}$ with $a \neq 0$, there exists a unique $x \in \mathbb{R}$ such that $a \cdot x = 1 = x \cdot a$.
- Commutative Law: For all a and b in $\mathbb{R}$,

$$a + b = b + a \qquad \text{and} \qquad \mathrm{a} \cdot b = b \cdot a\ .$$

- Distributive Law: For all a, b and c in $\mathbb{R}$,

$$a \cdot (b + c) = a \cdot b + a \cdot c\ .$$

Why do we need Complex Numbers $\mathbb{C}$?

NOT all polynomial equations have roots in $\mathbb{R}$.
Example: $x^2 + 1 = 0$ has no roots in $\mathbb{R}$.

$(\mathbb{R}, +, \cdot)$ is NOT algebraically closed. There is a need of bigger number system in which all (nonconstant) polynomial equations have roots.

Fact/History: Complex numbers $\mathbb{C}$ were originated when Four Italy mathematicians (Ferro, Tartagila, Cardano, Bombelli) in 16th Century tried to solve cubic equations like $x^3 - 3bx - 2c = 0$, $x^3 - 15x - 4 = 0$ (but not from quadratic equations at that time).
For an interesting article on History of Complex Numbers see:
http://www.math.uri.edu/~merino/spring06/mth562/ShortHistoryComplexNumbers2006.pdf

Advantages (now): Certain real integrals can be computed easily in $\mathbb{C}$. Certain differential equations can be easily solved. A differentiable complex function in an open set (analytic function) has many interesting properties.

Complex Numbers

Definition

A complex number z is defined to be an ordered pair of real numbers x and y as $z = (x,\ y)$. That is, the set of complex numbers is denoted by $\mathbb{C}$ and is given by

$$\mathbb{C} = \{z = (x,\ y)\ :\ x \text{ and } y \text{ are real numbers}\}\ .$$

The ordered pair here means the order in which we write x and y in defining the complex number $z = (x,\ y)$. For example, the number $(1,\ 5)$ is not the same as $(5,\ 1)$.

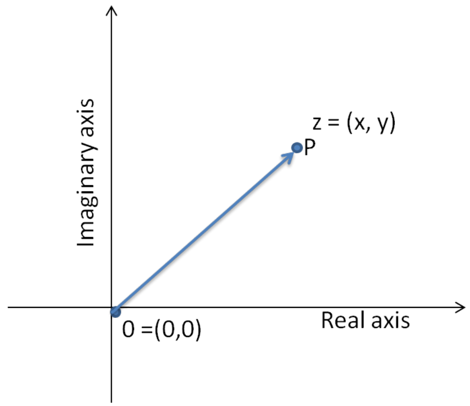
In the complex number $z = (x,\ y)$,

- x is called the real part of z and is denoted by $\Re(z)$ or Re z
- y is called the imaginary part of z and is denoted by $\Im(z)$ or Im z

- The numbers of the form $(0, y)$ are called pure imaginary numbers.
- The numbers of the form $(x, 0)$ are called real numbers.
- The set of real numbers can be identified as a subset $\mathbb{R} = \{z = (x, y) \in \mathbb{C} : x \in \mathbb{R}$ and $y = 0\}$ in $\mathbb{C}$. That is, $\mathbb{R} \subset \mathbb{C}$.
- Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal.

History: The representation of complex numbers in the plane was proposed independently by Casper Wessel (1797), K. F. Gauss (1799) and Jean Robert Argand (1806).

Complex Plane/ z-plane/ Argand Plane



- The complex number $z = (x,\ y)$ can be viewed as a point P having cartesian coordinates $(x,\ y)$ in the plane $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.
- The x-axis and y-axis are called the real axis and the imaginary axis respectively.
- The complex number $z = (x,\ y)$ can also be represented by a vector connecting the origin $0 = (0, 0)$ to the point P.
- This plane is called the complex plane or z-plane. It is also known as the Gauss plane or the Argand Plane. The term Argand diagram is sometimes used.

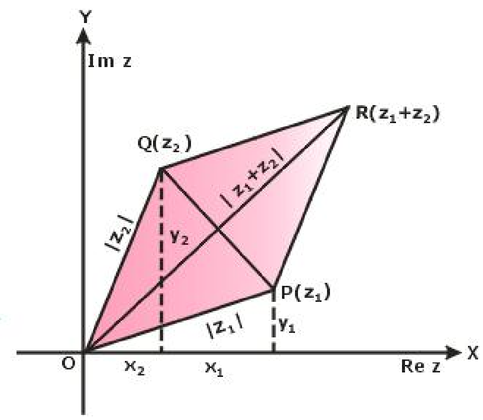
Addition Operation

For any two complex numbers $z_1 = (x_1, \ y_1)$ and $z_2 = (x_2, \ y_2)$, the addition of z_1 and z_2 is defined

$$z_1 + z_2 \ = \ (x_1 + x_2, \quad y_1 + y_2) \ .$$

The sum of any two complex numbers is a complex number.

Geometric Interpretation of Addition of two complex numbers:



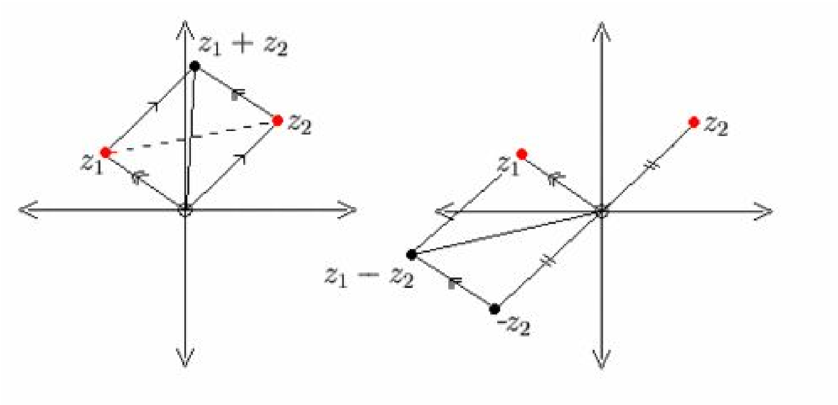
If $\vec{OP}$ and $\vec{OQ}$ are not collinear, then $\vec{OR}$ is the diagonal of the parallelogram with $\vec{OP}$ and $\vec{OQ}$ as adjacent sides.

Subtraction Operation

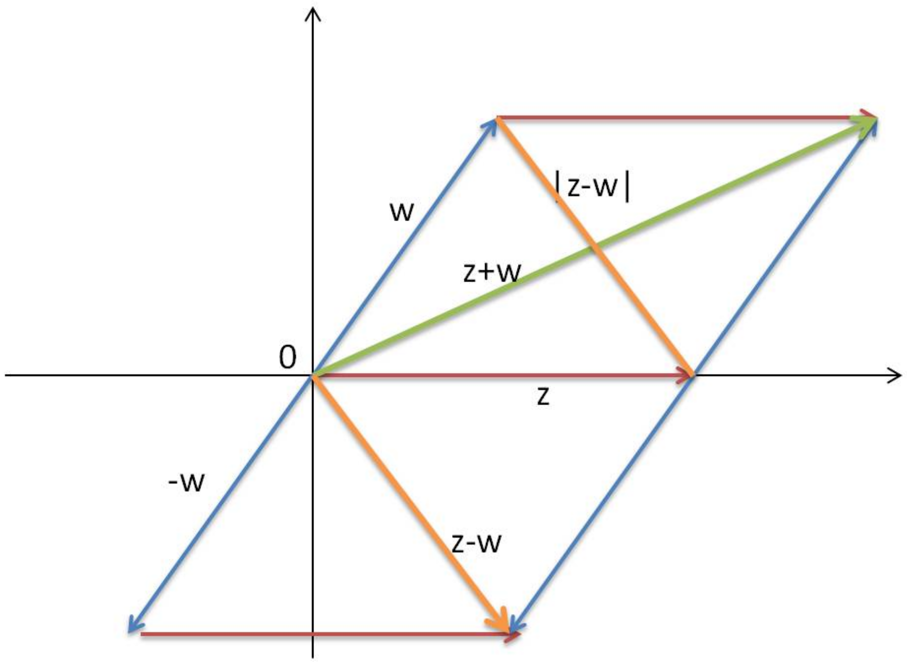
The subtraction $z_1 - z_2$ of the complex numbers $z_1 = (x_1,\ y_1)$ and $z_2 = (x_2,\ y_2)$ is defined as

$$z_1 - z_2 = (x_1 - x_2,\ y_1 - y_2)\ .$$

The subtraction $z_1 - z_2$ can be viewed as the sum of the complex numbers z_1 and $-z_2$.



Geometric Interpretation of Subtraction



This picture will be useful to understand parallelogram law (later on).

$$|z+w|^2 + |z-w|^2 = 2\left(|z|^2 + |w|^2\right)$$

Multiplication and Division

For any two complex numbers $z_1 = (x_1,\ y_1)$ and $z_2 = (x_2,\ y_2)$, the multiplication of z_1 and z_2 is defined by

$$z_1\ z_2 \ = \ (x_1 x_2 - y_1 y_2, \ \ x_1 y_2 + x_2 y_1)\ .$$

The product of any two complex numbers is a complex number.
This multiplication is different from the vector product.

If $z_1 = (x_1,\ y_1)$ and $z_2 = (x_2,\ y_2) \neq 0$ are any two complex numbers, then the complex number z_1 divided by z_2 is defined as

$$\frac{z_1}{z_2} = \left(\frac{1}{x_2^2 + y_2^2}\right)((x_1 x_2 + y_1 y_2),\ (x_2 y_1 - x_1 y_2))\ .$$

The set of complex numbers $\mathbb{C}$ with these operations addition $+$ and multiplication $\cdot$ forms a field. The identity element of $+$ is $(0,0)$ and the identity element of $\cdot$ is $(1,0)$. $\mathbb{R}$ is a subfield of $\mathbb{C}$.

Binomial Formula

Let $0! = 1$ and $n! = 1 \cdot 2 \cdots (n-1) \cdot n$ for $n \in \mathbb{N}$.

Let $nC_k = \frac{n!}{k!\,(n-k)!}$ for $k = 0, \cdots, n$.

Binomial Formula:

For any two complex numbers z_1 and z_2 and for $n \in \mathbb{N}$,

$$(z_1 + z_2)^n = \sum_{k=0}^{n} nC_k \, z_1^{n-k} \, z_2^{k} .$$

The proof is based on mathematical induction and is left as an exercise.

Additional Information: Complex Field is NOT an Ordered Field

We can not define usual order relation like less than, less than or equal to, greater than, greater than or equal to on the set of complex numbers. *That is, the usual ordering of $\mathbb{R}$ can not be taken to $\mathbb{C}$ as such.*

However, we can define in other ways, like dictionary order on $\mathbb{C}$ as follows.

Let $z_1 = x_1 + i\,y_1$ and $z_2 = x_2 + i\,y_2$. The dictionary order is given by:

$$
\begin{array}{llll}
z_1 < z_2 & \textit{if} & x_1 < x_2 & \\
z_1 < z_2 & \textit{if} & x_1 = x_2 & \textit{and if} \quad y_1 < y_2 \\
z_1 = z_2 & \textit{if} & x_1 = x_2 & \textit{and if} \quad y_1 = y_2
\end{array}
$$

The complex field $(\mathbb{C}, +, \cdot)$ can NOT be an ordered field with respect to any (total) order defined on $\mathbb{C}$.
Therefore, the dictionary order is NOT useful in some sense.

Algebraic form (or $x + iy$ notation)

Set

$$i = (0,\ 1)\ .$$

It is called iota.

Electrical engineers use the letter j instead of i.

$$(x,\ y) = (x,\ 0)(1,\ 0) + (0,\ 1)(y,\ 0) = x\ \cdot\ 1\ +\ i\ \cdot\ y = x +\ i\ y$$

$$(x,\ y) = x + i\ y\ ,$$

$$i^2 = (0,\ 1)(0,\ 1) = (-1,\ 0) = -1\ .$$

The form $x + iy$ is called the algebraic form of a complex number.

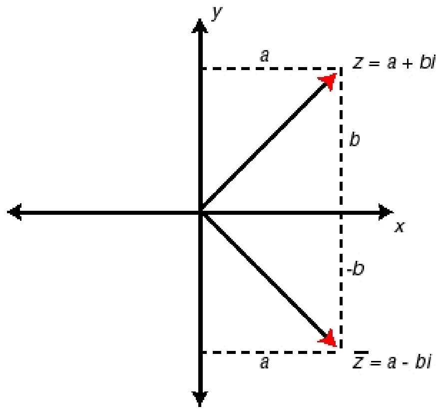
Hereafter, we prefer to use $x + iy$ form instead of ordered pair (x, y) form to write complex numbers.

Conjugate of a Complex Number

The complex conjugate, or simply, the conjugate of a complex number $z = a + ib$ is denoted by $\bar{z}$ and is defined by

$$\bar{z} = a - ib.$$

Geometrically, the point $\bar{z} = a - ib$ is the reflection (mirror image) of the point $z = a + ib$ on the real axis.



Examples: If $z = 3 + 4i$ then $\bar{z} = 3 - 4i$. If $z = -5$ then $\bar{z} = -5$.

Properties of Complex Conjugation

1. $z_1 = z_2$ if and only if $\overline{z_1} = \overline{z_2}$.
2. $\overline{\overline{z}} = z$.
3. $\overline{z} = z$ if and only if z is a real number.
4. $z + \overline{z} = 2\Re(z) = 2x$ if $z = x + iy$.
5. $z - \overline{z} = 2i\,\Im(z) = 2i\,y$ if $z = x + iy$.
6. $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$.
7. $\overline{z_1 z_2} = \overline{z_1}\;\overline{z_2}$.
8. $\overline{(z_1/z_2)} = \overline{z_1}/\overline{z_2}$ provided $z_2 \neq 0$.

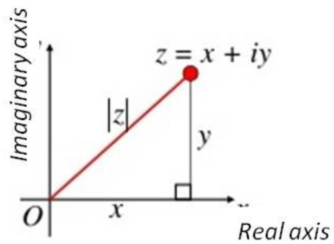
The numbers z and $\overline{z}$ are called the complex conjugate coordinates, or simply the conjugate coordinates corresponding to the point $z = (x,\ y) = x + iy$. Also they have been called the isotropic coordinates and the minimal coordinates of the point.

Modulus of a Complex Number

The modulus or absolute value of a complex number $z = x + iy$ is denoted by $|z|$ and is given by

$$|z| = \sqrt{x^2 + y^2} \,.$$

Here, as usual, the radical stands for the principal (non-negative) square root of $x^2 + y^2$.



Example: The modulus of the complex number $4 + 3i$ is

$$|4 + 3i| = \sqrt{4^2 + 3^2} = \sqrt{25} = 5.$$

Note: $|z| \geq 0$ for all $z \in \mathbb{C}$. $|z| = 0$ if and only if $z = 0$.

Properties - Modulus & Conjugate

1. $|z| \geq 0$ and $|z| = 0$ iff $z = 0$.
2. $|\bar{z}| = |z| = |-z|$.
3. $|z|^2 = z\,\bar{z}$.
4. If $z = x + iy$, $|z| \leq |x| + |y|$.
5. If $z = x + iy$, $|x| \leq |z|$ and $|y| \leq |z|$.
6. $|z_1 z_2| = |z_1|\,|z_2|$.
7. Parallelogram Law: $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right)$
8. Triangle Inequality: $|z_1 + z_2| \leq |z_1| + |z_2|$. (Work out proof in class)
9. $|z_1 - z_2| \leq |z_1| + |z_2|$.
10. $|z_1 - z_2| \geq \big|\,|z_1| - |z_2|\,\big|$. (Work out proof in class)
11. $|z_1 + z_2| \geq \big|\,|z_1| - |z_2|\,\big|$.
12. $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ provided $z_2 \neq 0$.
13. If $n \in \mathbb{N}$, then $|z^n| = |z|^n$. If $-n \in \mathbb{N}$, then $|z^n| = |z|^n$ for $z \neq 0$.

Properties (continuation) - Additional Information

14. Lagrange's Identity: If $\{z_1, z_2, \cdots, z_n\}$ and $\{w_1, w_2, \cdots, w_n\}$ are two sets of n complex numbers ($n \geq 1$), then

$$\left|\sum_{k=1}^{n} z_k \, w_k\right|^2 = \left(\sum_{k=1}^{n} |z_k|^2\right) \left(\sum_{k=1}^{n} |w_k|^2\right) - \sum_{1 \leq j \leq k \leq n} |z_j w_k - z_k w_j|^2 .$$

15. Cauchy-Schwarz Inequality: If $\{z_1, z_2, \cdots, z_n\}$ and $\{w_1, w_2, \cdots, w_n\}$ are two sets of n complex numbers ($n \geq 1$), then

$$\left|\sum_{k=1}^{n} z_k \, w_k\right|^2 \leq \left(\sum_{k=1}^{n} |z_k|^2\right) \left(\sum_{k=1}^{n} |w_k|^2\right)$$

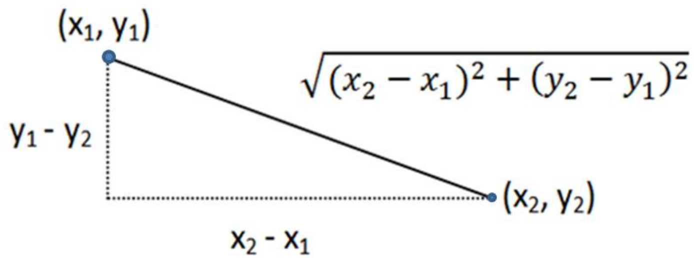
where the equality sign holds iff the z_k are proportional to the $\overline{w_k}$.

Distance between Two Complex Numbers

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be any two complex numbers. Then the (Usual/Euclidean) distance between z_1 and z_2 is defined by

$$d(z_1, z_2) = |z_2 - z_1| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\ .$$

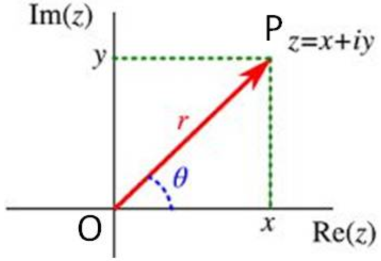
$$= |z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}\ .$$

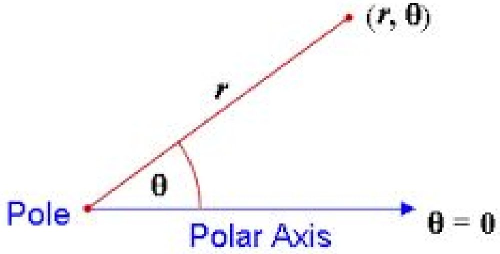


Example: If $z_1 = 1 + i$ and $z_2 = 1 - i$ then $|z_1 - z_2| = \sqrt{(1-1)^2 + (1-(-1))^2} = 2$.

Note: $|z| = d(0, z)$. $(\mathbb{C}, d)$ is a metric space.

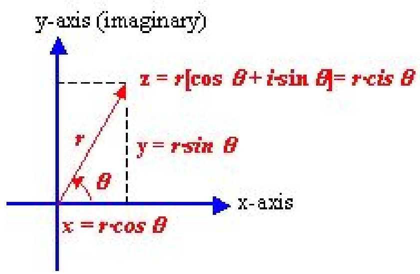
Polar Form of (Non-Zero) Complex Numbers

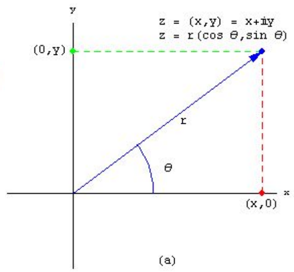


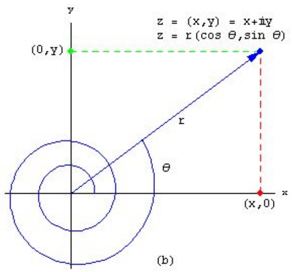


- Each non-zero complex number $z = x + iy = (x,\ y) \neq (0, 0)$ can be represented by the vector from the origin O to the point $P = (x, y)$ in the plane.
- The length r of the vector $\vec{OP}$ is given by $r = \sqrt{x^2 + y^2} = |z| =$ Modulus of z.
- The measure θ in radians of the oriented angle from the positive real axis to the vector $\vec{OP}$ is called the argument or the amplitude of the vector $\vec{OP}$, and we write $\theta = \arg z$.
- For $z \neq 0$, we can write $z = (r,\ \theta)$ where $r = |z|$ and $\theta = \arg(z)$. This representation is called the polar representation of z, and the values of r and θ are called polar coordinates of z.

Trigonometric Form of (Non-Zero) Complex Numbers







- From trigonometry we have, $\cos\theta = \dfrac{x}{r}$ and $\sin\theta = \dfrac{y}{r}$.

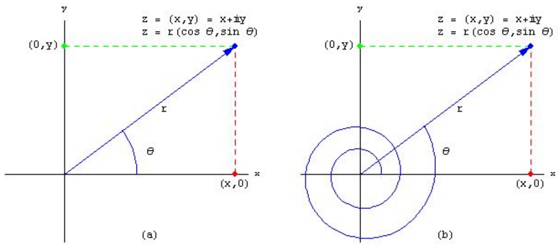
$$z = r\cos\theta + \; i\, r\sin\theta = r(\cos\theta + \; i\; \sin\theta)\ .$$

  This is known as the trigonometric form of the complex number z.

- The number θ is determined only up to multiples of 2π and the set of all such angles is denoted by $\arg z$. However all the values in this set represent the same direction in the complex plane.

Example: Modulus of $(1+i)$ is $\sqrt{2}$ and argument of $(1+i) = \pi/4+$ any multiple of 2π. Polar form of $(1+i)$ is $(\sqrt{2},\ \pi/4)$ or $(\sqrt{2},\ 9\pi/4)$, etc.

About the function $\arg(z)$



- For the complex number $z = 0$, the modulus is 0, but the argument is undefined.
- If a complex number z is written in the polar form or in the trigonometric form then it is understood that it is a non-zero complex number.
- For each nonzero z, $\arg(z)$ takes a set of values. This set is an infinite set. For each nonzero point z, argument function thus assigns a set as value. Therefore, $\arg(z)$ is called a multiple valued function.

Examples: $\arg(5) = \{2n\pi \ : \ n \in \mathbb{Z}\}$; $\arg(-3) = \{(2n+1)\pi \ : \ n \in \mathbb{Z}\}$; $\arg(1+i) = \{(\pi/4) + 2n\pi \ : \ n \in \mathbb{Z}\}$; Compute $\arg(1-i)$.

Principal Value of argument of z: Arg z

Picking one of the values of $\arg(z)$ for computation purpose. For example, if teacher gives some condition, all students should be able to pick up the same (and unique) value for $\arg(1+i)$. How to do it?

e.g., Teacher says: Restrict the value of $\arg(z)$ in the interval $(-\pi,\ \pi]$ and now tell me the value of $\arg(1+i)$.

Answer: $\arg(1+i)$ in the interval $(-\pi, \pi]$ is ______.

For each non-zero z, there is only one value of $\arg z$ say Θ satisfying $-\pi < \Theta \leq \pi$. This value will henceforth be denoted by Arg z and is called the principal value of $\arg z$.

Examples: $\text{Arg}\,(5) = 0$, $\text{Arg}\,(i) = \pi/2$, $\text{Arg}\,(-8) = \pi$, $\text{Arg}\,(-i) = -\pi/2$.

Exercise: Find the largest set in $\mathbb{C}$ on which $\text{Arg}\, z$ is continuous?

Relation between $\arg z$ and Arg z:

$$\arg z = \text{Arg } z + 2\pi k \text{ where } k \text{ is an integer .}$$

Computing Principal Value of argument and argument

Let $z = x + iy \neq 0$.

Compute ϕ = Principal value of $\tan^{-1}(y/x)$ which lies in $(-\pi/2,\ \pi/2)$.

With the value of ϕ and with the information of signs of x and y (which quadrant z lies) we can compute

$$\text{Arg}\,(z) = \begin{cases} \phi & \text{if} \quad x > 0 \\ \phi + \pi & \text{if} \quad x < 0 \text{ and } y \geq 0 \\ \phi - \pi & \text{if} \quad x < 0 \text{ and } y < 0 \\ \pi/2 & \text{if} \quad x = 0 \text{ and } y > 0 \\ -\pi/2 & \text{if} \quad x = 0 \text{ and } y < 0 \end{cases}$$

$$\arg(z) = \text{Arg}\,(z) + 2k\pi \quad \text{where} \quad k \in \mathbb{Z}\,.$$