MA201 Mathematics III
Solutions to Complex Analysis Tutorial 02

Differentiability and CR Equations

11. Let $f(z) = \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$ for $z = x+iy \neq 0$ and $f(0) = 0$. Show that $f(z)$ is continuous at origin but $f'(0)$ does not exist.
Answer:
$f(z) = \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2} = \dfrac{(x^3-y^3) + \ i\ (x^3+y^3)}{x^2+y^2}$ for $z = x+iy \neq 0$.
Consider
$$|f(z) - f(0)| = \left| \frac{(x^3-y^3) + \ i\ (x^3+y^3)}{x^2+y^2} - 0 \right| \leq \frac{2\ (|x|^3 + |y|^3)}{|x^2+y^2|}$$
Put $x = r\cos\theta$ and $y = r\sin\theta$.
$$|f(z) - f(0)| \leq \frac{2r^3(|\cos\theta|^3 + |\sin\theta|^3)}{r^2} \leq 4r$$
As $z \to 0$, we have $|z| = r \to 0$ and hence $|f(z) - f(0)| \leq r \to 0$. Therefore, f is continuous at $z = 0$.

Consider
$$\lim_{z\to 0} \frac{f(z) - f(0)}{z} = \lim_{z=x+iy\to 0} \frac{\frac{(x^3-y^3) + \ i\ (x^3+y^3)}{x^2+y^2} - 0}{x+iy}$$
Letting z approach 0 along the path $y = 0$ and $x \to 0$, we get
$$\lim_{\substack{y=0\\ x\to 0}} \frac{x^3(1+i)}{x^3} = 1+i\ .$$
Letting z approach 0 along the path $y = x$ and $x \to 0$, we get
$$\lim_{\substack{y=x\\ x\to 0}} \frac{\frac{i\ 2x^3}{2x^2}}{x(1+i)} = \frac{i}{1+i} = \frac{1+i}{2}\ .$$
Since $\dfrac{f(z) - f(0)}{z}$ approaches two different values as $z \to 0$ along two different paths, we conclude that $f'(0) = \lim\limits_{z\to 0} \dfrac{f(z) - f(0)}{z}$ does not exist.

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12. If $f(z)$ is a real valued function in a domain $D \subseteq \mathbb{C}$, then show that either $f'(z) = 0$ or $f'(z)$ does not exist in D.
Answer:
Let $z = x + \ i\, y$ be an arbitrary point in $\mathbb{C}$. We want to examine the differentiability of f at the point z. Set $\Delta z = \ \Delta x + \ i\, \Delta y$. Consider
$$\lim_{\Delta z\to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z=\Delta x+i\Delta y\to 0} \frac{f((x+\Delta x) + \ i\ (y+\Delta y)) - f(x+ \ i\ y)}{\Delta x + \ i\ \Delta y}$$

Letting Δz approach 0 along the path $\Delta y = 0$ and $\Delta x \to 0$, we get

$$\lim_{\substack{\Delta y=0 \\ \Delta x\to 0}} \frac{f((x+\Delta x)+\ i\ y) - f(x+\ i\ y)}{\Delta x} = A \text{ which is a real number .}$$

Reason: Since f is real valued, the numerator quantity is real always. The denominator quantity is real, since $\Delta y = 0$. Therefore, the limiting quantity will be a real number, if it exists.
Letting Δz approach 0 along the path $\Delta x = 0$ and $\Delta y \to 0$, we get

$$\lim_{\substack{\Delta x=0 \\ \Delta y\to 0}} \frac{f(x+\ i\ (y+\Delta y) - f(x+\ i\ y)}{i\ \Delta y} = i\ B \text{ which is a pure imaginary number .}$$

Reason: Since f is real valued, the numerator quantity is real always. The denominator quantity is pure imaginary, since $\Delta x = 0$. Therefore, the limiting quantity will be a pure imaginary number, if it exists.
If $A \neq B$ then $A \neq iB$ and hence $f'(z)$ does not exists.
If f is differentiable at z then $A = iB$ and this is possible provided $A = 0 = B$. Therefore, $f'(z) = 0$.

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13. Let $f(z) = (x^3 y(y - ix)) \,/\, (x^6 + y^2)$ for $z = x + iy \neq 0$ and $f(0) = 0$.
(a) Find the value of $\displaystyle\lim_{\Delta z\to 0} \frac{f(\Delta z) - f(0)}{\Delta z}$ as $\Delta z \to 0$ along the line $y = mx$.
(b) Find the value of $\displaystyle\lim_{\Delta z\to 0} \frac{f(\Delta z) - f(0)}{\Delta z}$ as $\Delta z \to 0$ along the imaginary axis.
(c) Find the value of $\displaystyle\lim_{\Delta z\to 0} \frac{f(\Delta z) - f(0)}{\Delta z}$ as $\Delta z \to 0$ along the path $y = x^3$.
(d) What can you conclude about the existence of $f'(z)$ at $z = 0$.
(e) Show that the Cauchy-Riemann (CR) equations hold true at $(0,\ 0)$.
Answer:
$f(z) = \dfrac{x^3y^2\ -\ i\,x^4y}{(x^6+y^2)} = \dfrac{x^3y^2}{(x^6+y^2)}\ +\ i\ \dfrac{(-x^4y)}{(x^6+y^2)} = u(x,\ y)\ +\ i\ v(x,\ y)$ for $z = x + iy \neq 0$.
Consider

$$\lim_{\Delta z\to 0} \frac{f(\Delta z) - f(0)}{\Delta z} = \lim_{\Delta z=\Delta x+i\Delta y\to 0} \frac{\left(\frac{\Delta x^3\Delta y^2}{(\Delta x^6+\Delta y^2)}\ +\ i\ \frac{(-\Delta x^4\Delta y)}{(\Delta x^6+\Delta y^2)}\right) - 0}{\Delta x + i\Delta y}$$

$$= \lim_{\Delta z=\Delta x+i\Delta y\to 0} \frac{(\Delta x^3\Delta y^2) +\ i\ (-\Delta x^4\Delta y)}{(\Delta x^6 + \Delta y^2)\,(\Delta x + i\Delta y)}$$

(a) Letting $\Delta z \to 0$ along the line $\Delta y = m\Delta x$, we get

$$\lim_{\Delta z\to 0} \frac{f(\Delta z) - f(0)}{\Delta z} = \lim_{\substack{\Delta y=m\Delta x \\ \Delta x\to 0}} \frac{\Delta x^5(m^2 - im)}{\Delta x^3(\Delta x^4 + m^2)(1+im)} = \lim_{\substack{\Delta y=m\Delta x \\ \Delta x\to 0}} \frac{\Delta x^2(m^2 - im)}{(\Delta x^4 + m^2)(1+im)} = 0\ .$$

(b) Letting $\Delta z \to 0$ along the imaginary axis, we get

$$\lim_{\substack{\Delta x = 0 \\ \Delta y \to 0}} \frac{0 \ + \ i\,0}{i\,\Delta y^3} = 0\ .$$

(c) Letting $\Delta z \to 0$ along the path $\Delta y = \Delta x^3$ and $\Delta x \to 0$, we get

$$\lim_{\substack{\Delta y = \Delta x^3 \\ \Delta x \to 0}} \frac{\Delta x^7(\Delta x^2 - \ i)}{2\Delta x^7(1 + i\,\Delta x^2)} = \lim_{\substack{\Delta y = \Delta x^3 \\ \Delta x \to 0}} \frac{(\Delta x^2 - \ i)}{2(1 + i\,\Delta x^2)} = \frac{-i}{2}\ .$$

(d) Since $\dfrac{f(\Delta z) - f(0)}{\Delta z}$ approaches two different values as $\Delta z \to 0$ along two different paths given in (b) and (c), we conclude that $\displaystyle\lim_{\Delta z \to 0} \frac{f(\Delta z) - f(0)}{\Delta z}$ does not exist and hence f is not differentiable at $z = 0$.

(e) Showing f satisfies CR equations at $z = 0$

$$\begin{aligned}
u_x(0,\ 0) = \lim_{x \to 0} \frac{u(x,\ 0) - u(0,\ 0)}{x} &= \lim_{x \to 0} \frac{0 - 0}{x} = 0 \\
u_y(0,\ 0) = \lim_{y \to 0} \frac{u(0,\ y) - u(0,\ 0)}{y} &= \lim_{y \to 0} \frac{0 - 0}{y} = 0 \\
v_x(0,\ 0) = \lim_{x \to 0} \frac{v(x,\ 0) - v(0,\ 0)}{x} &= \lim_{x \to 0} \frac{0 - 0}{x} = 0 \\
v_y(0,\ 0) = \lim_{y \to 0} \frac{v(0,\ y) - v(0,\ 0)}{y} &= \lim_{y \to 0} \frac{0 - 0}{y} = 0
\end{aligned}$$

Thus,

$$\begin{aligned}
u_x(0,\ 0) &= 0 = \ v_y(0,\ 0) \\
u_y(0,\ 0) &= 0 = \ -v_x(0,\ 0)
\end{aligned}$$

Therefore, f satisfies the Cauchy-Riemann equations at $z = 0$, even though f is not differentiable at $z = 0$.

Analytic Functions

14. Show that the function $f(z) = x^3 + 3xy^2 - 3x + i(y^3 + 3x^2y - 3y)$ is differentiable only at points that lie on the coordinate axes. Is $f(z)$ analytic at any point lies on the coordinate axes?

    Answer:
    $f(z) = u(x,\ y)\ +\ i\ v(x,\ y)$ where $u(x,\ y) = x^3 + 3xy^2 - 3x$ and $v(x,\ y) = y^3 + 3x^2y - 3y$ for $z = (x,\ y) \in \mathbb{C}$.

$$\begin{aligned}
u_x(x,\ y) &= 3x^2 + 3y^2 - 3 \\
u_y(x,\ y) &= 6xy \\
v_x(x,\ y) &= 6xy \\
v_y(x,\ y) &= 3x^2 + 3y^2 - 3
\end{aligned}$$

If f is differentiable at a point z then f satisfies the Cauchy-Riemann equations at z. Therefore,

$$\begin{aligned} u_x(x,\ y) &= 3x^2 + 3y^2 - 3 = & v_y(x,\ y) \\ u_y(x,\ y) &= 6xy = -6xy = & -v_x(x,\ y) \end{aligned}$$

$6xy = -6xy$ is possible if and only if either $x = 0$ or $y = 0$. That is, f satisfies the Cauchy-Riemann equations only at points that lie on the coordinate axes $x = 0$ or $y = 0$.
Since the functions $u_x(x,\ y)$, $u_y(x,\ y)$, $v_x(x,\ y)$ and $v_y(x,\ y)$ are continuous at all points in $\mathbb{C}$ (Reason: These are polynomials in two variables and hence continuous), we conclude that f is differentiable only at points on the coordinate axes.

The function f can not be analytic at any point z lies on the coordinate axes, because we can not find a neighborhood $N(z)$ about the point z at which f is differentiable at each point of $N(z)$.

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15. Show that the function $f(z) = xy + iy$ is continuous everywhere, but not analytic in $\mathbb{C}$.
Answer:
Set $u(x,\ y) = xy$ and $v(x,\ y) = y$. Let $z_0 = (x_0,\ y_0)$ be an arbitrary point in $\mathbb{C}$. Observe that

$$\lim_{(x,\ y)\to(x_0,\ y_0)} u(x,\ y) = \lim_{(x,\ y)\to(x_0,\ y_0)} xy = x_0 y_0 = u(x_0,\ y_0)\ .$$

$$\lim_{(x,\ y)\to(x_0,\ y_0)} v(x,\ y) = \lim_{(x,\ y)\to(x_0,\ y_0)} y = y_0 = v(x_0,\ y_0)\ .$$

Therefore $u(x,\ y)$ and $v(x,\ y)$ are continuous at z_0 and hence f is continuous at z_0.

$u_x(x,\ y) = y$, $u_y(x,\ y) = x$, $v_x = 0$ and $v_y = 1$.
$u_x = y = 1 = v_y$ and $u_y = x = 0 = -v_x$ are possible, only if $x = 0$ and $y = 1$.
That is, f satisfies the Cauchy-Riemann equations at $z^* = (0,\ 1)$ only. Therefore, f can not be differentiable in $\mathbb{C} \setminus \{(0,\ 1)\}$ and hence f can not be analytic in $\mathbb{C}$.

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16. Check whether the function $g(z) = (3x^2 + 2x - 3y^2 - 1) + i(6xy + 2y)$ is satisfying the sufficient conditions to be an analytic function at any point in the complex plane. Write this function in terms of z.
Answer:
Set $u(x,\ y) = 3x^2 + 2x - 3y^2 - 1$ and $v(x,\ y) = 6xy + 2y$.
$u_x = 6x + 2$, $u_y = -6y$, $v_x = 6y$ and $v_y = 6x + 2$.
$u_x = 6x + 2 = v_y$ and $u_y = -6y = -v_x$ at all points in $\mathbb{C}$.
u_x, u_y, v_x and v_y are continuous at all points in $\mathbb{C}$.
By sufficient conditions for analyticity, we conclude that f is analytic at all points in $\mathbb{C}$.
Put $y = 0$ and $x = z$, to express f in terms of z. Thus we get

$$g(z) = 3z^2 + 2z - 1 \text{ for } z \in \mathbb{C}\ .$$

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17. Find the analytic function $f(z) = u(x,\ y) + iv(x,\ y)$ given the following:
(First verify that they are harmonic functions)
(a) $u(x,\ y) = y^3 - 3x^2y$ (b) $v(x,\ y) = \sin x \cosh y$
(c) $u(x,\ y) - v(x,\ y) = (x - y)(x^2 + 4xy + y^2)$

Answer:

Step 1: Verifying that u is harmonic.
Given that $u(x,\ y) = y^3 - 3x^2y$.
$u_x = -6xy$, $u_y = 3y^2 - 3x^2$, $u_{xx} = -6y$, $u_{yy} = 6y$. It gives that $u_{xx} + u_{yy} = 0$. Therefore, u is harmonic in $\mathbb{C}$.

Step 2:
Since f is analytic, f satisfies the Cauchy-Riemann equations.

$$u_x(x,\ y) = -6xy = v_y(x,\ y)\ .$$

Holding x fixed, and integrating both sides with respect to y,

$$\int v_y(x,\ y)\ dy\ =\ \int -6xy\ dy$$

$$v(x,\ y) = -3xy^2 + \phi(x)$$

where $\phi(x)$ is arbitrary function of x.

Step 3:
Differentiating $v(x,\ y)$ with respect to x partially, we get

$$v_x(x,\ y) = -3y^2 + \phi'(x)$$

But $u_y(x,\ y) = -v_x(x,\ y)$ gives that

$$v_x(x,\ y) = -u_y(x,\ y) = -3y^2 + 3x^2$$

Combining the last two equations we get

$$\phi'(x) = 3x^2$$

Integrating it with respect to x, we get

$$\int \phi'(x)\ dx = \int 3x^2\ dx$$

$$\phi(x) = x^3 + c$$

where c is an arbitrary real constant. Therefore

$$v(x,\ y) = x^3 - 3xy^2 + c \text{ where } c \text{ is a real constant .}$$

Step 4: Writing $f(z)$.

$$f(z) = u(x,\ y)\ +\ i\ v(x,\ y) = (y^3 - 3x^2y)\ +\ i\ (x^3 - 3xy^2 + c) \text{ where } c \text{ is a real constant .}$$

Putting $y = 0$ and $x = z$ to express f in terms of z, we get

$$f(z) = \ i \ (z^3 + c) \text{ where } c \text{ is a real constant .}$$

(b) Given that $v(x,\ y) = \sin x \cosh y$.
Do it similarly as done in (a)
Final Answer $u(x,\ y) = -\cos x \sinh y + k$ where k is a real constant and $f(z) = i \sin(z) + k$.

(c) Given that $u(x,\ y) - v(x,\ y) = (x - y)(x^2 + 4xy + y^2)$.

$$\begin{aligned} f(z) &= u(x,\ y) + \ i \ v(x,\ y) \\ i \ f(z) &= -v(x,\ y) + \ i \ u(x,\ y) \\ g(z) = (1+i) \ f(z) &= (u(x,\ y) - v(x,\ y)) + \ i \ (u(x,\ y) + v(x,\ y)) \\ &= U(x,\ y) \ + \ i \ V(x,\ y) \text{ (say)} \end{aligned}$$

If f is analytic then $g = (1+i)f$ is analytic.
Now, the real part $U = u - v$ of g is given. We will find the imaginary part $V = u + v$ of g. After that, we can determine u and v from U and V.
Step 1: Verifying that U is harmonic.
Given that $U(x,\ y) = (x - y)(x^2 + 4xy + y^2)$.
$U_x = 3x^2 + 6xy - 3y^2$, $U_y = 3x^2 - 6xy - 3y^2$, $U_{xx} = 6x + 6y$, $U_{yy} = -6x - 6y$. It gives that $U_{xx} + U_{yy} = 0$. Therefore, U is harmonic in $\mathbb{C}$.

Step 2:
Since g is analytic, g satisfies the Cauchy-Riemann equations.

$$U_x(x,\ y) = 3x^2 + 6xy - 3y^2 = V_y(x,\ y) \ .$$

Holding x fixed, and integrating both sides with respect to y,

$$\int V_y(x,\ y) \ dy \ = \ \int (3x^2 + 6xy - 3y^2) \ dy$$

$$V(x,\ y) = 3x^2y + 3xy^2 - y^3 + \phi(x)$$

where $\phi(x)$ is arbitrary function of x.

Step 3:
Differentiating $V(x,\ y)$ with respect to x partially, we get

$$V_x(x,\ y) = 6xy + 3y^2 + \phi'(x)$$

But $U_y(x,\ y) = -V_x(x,\ y)$ gives that

$$V_x(x,\ y) = -U_y(x,\ y) = -3x^2 + 6xy + 3y^2$$

Combining the last two equations we get

$$\phi'(x) = -3x^2$$

Integrating it with respect to x, we get

$$\int \phi'(x)\ dx = \int -3x^2\ dx$$

$$\phi(x) = -x^3 + c$$

where c is an arbitrary real constant. Therefore

$$V(x,\ y) = 3x^2y + 3xy^2 - y^3 - x^3 + c \text{ where } c \text{ is a real constant .}$$

Step 4: Finding $f(z)$.

$$\begin{aligned} U(x,\ y) = u(x,\ y) - v(x,\ y) &= 3x^2y - 3xy^2 - y^3 + x^3 \\ V(x,\ y) = u(x,\ y) + v(x,\ y) &= 3x^2y + 3xy^2 - y^3 - x^3 + c \end{aligned}$$

where c is a real constant.
Putting $y = 0$ and $x = z$ to express g in terms of z, we get

$$g(z) = (1-i)\ z^3 +\ i\ c \text{ where } c \text{ is a real constant .}$$

$$f(z) = \frac{g(z)}{1+i} = \frac{(1-i)\ z^3 +\ i\ c}{(1+i)} = -i\ z^3\ + \frac{(1+i)c}{2}$$

$$f(z) = -i\ z^3 + A \text{ where } A \text{ is a complex constant .}$$
