CS213/293 Data Structure and Algorithms 2025

Lecture 1: Why should you study "data structures?"

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Next course in programming

Are CS101 and SSL not enough to be a programmer?

In CS101, you learned to walk.

In this course, you will learn to dance.

What is data?

Things are not data, but information about them is data.

Example 1.1

Age of people, height of trees, price of stocks, and number of likes.

Data is big!

We are living in the age of big data!

*Image is from the Internet.

Exercise 1.1

1. Estimate the number of messages exchanged for Whatsapp status.
2. How much text data was used to train ChatGPT?

We need to work on data

We process data to solve our problems.

Example 1.2

1. Predict the weather
2. Find a webpage
3. Recognize fingerprint

Disorganized data will need a lot of time to process.

Exercise 1.2

How much time do we need to find an element in an array?

Problem

Definition 1.1

A problem is a pair of an input specification and an output specification.

Example 1.3

The problem of search consists of the following specifications

- Input specification: an array S of elements and an element e
- Output specification: position of e in S if it exists. If it is not found, return -1.

Output specifications refer to the variables in the input specifications

Exercise 1.3

According to the specification, what should happen if e occurs multiple times in S?

Algorithms

Definition 1.2

An algorithm solves a given problem.

- Input $\in$ Input specifications
- Output $\in$ Output specifications

$$\text{Input} \longrightarrow \boxed{\text{Algorithm}} \longrightarrow \text{Output}$$

Note: There can be many algorithms to solve a problem.

Exercise 1.4

1. What truly is an algorithm?
2. How an algorithm is different from a program?

Commentary: An algorithm is a step-by-step process that processes a small amount of data in each step and eventually computes the output. The formal definition of the algorithm will be presented to you in CS310. It took the genius of Alan Turing to give the precise definition of an algorithm.

Example: an algorithm for search

Example 1.4

```
int search( int* S, int n, int e) {
  // n is the length of the array S
  // We are looking for element e in S
  for( int i=0; i < n; i++ ) {
    if( S[i] == e ) {
      return i;
    }
  }
  return -1; // Not found
}
```

Exercise 1.5

What is the running time of the above algorithm if e is not in S?

Commentary: Answer: We count memory accesses, arithmetic operations (including comparisons), assignments, and jumps. The loop in the program will iterate n times. In each iteration, there will be one memory access `S[i]`, three arithmetic operations `i<n`, `S[i] == e` and `i++`, and two jumps. At the initialization, there is an assignment `i=0`. For the loop exit, there will be one more comparison and jump. $Time = nT_{Read} + (3n+2)T_{Arith} + (2n+1)T_{jump} + T_{return}$ Give this program to `https://godbolt.org/` and see the assembly. Check if the above analysis is faithful!

Data needs structure

Storing data as a pile of stuff, will not work. We need structure.

Example 1.5

How do we store data at IIT Bombay Hospital?

Structured data helps us solve problems faster

We can exploit the structure to design efficient algorithms to solve our problems.

The goal of this course!

Example: search on well-structured data I

Example 1.6

Let us consider the problem of search consisting of the following specifications

- Input specification: a non-decreasing array S and an element e
- Output specification: position of e in S if it exists. If it is not found, return -1.

Example: search on well-structured data I

Let us see how we can exploit the structured data!

Let us try to search 68 in the following array.

- Look at the middle point of the array.
- Since the value at the middle point is less than 68, we search only in the upper half.
- We have halved our search space.
- We recursively half the space.

0	1	2	3	4	5	6	7	8	9	10
11	21	35	46	49	60	68	73	81	90	91

A better search

Example 1.7

```
int BinarySearch(int* S, int n, int e){
  // S is a sorted array
  int first = 0, last = n;
  int mid = (first + last) / 2;
  while (first < last) {
    if (S[mid] == e) return mid;
    if (S[mid] > e) {
      last = mid;
    } else {
      first = mid + 1;
    }
    mid = (first + last) / 2;
  }
  return -1;
```

Commentary: Answer: There will be k iterations. In each iteration, the function will follow the same path. In each iteration, there will be

- a memory access `S[mid]`,(why only one?)
- five arithmetic operations `first < last`, `S[mid] == e`, `S[mid] > e`, `first+last`, and `../2`,
- one assignment `last = mid`, (Why?)
- three jumps because of two ifs and a loop ending,

For the loop exit, there will be one additional comparison and a jump at the loop head. In the initialization section, we have two assignments and two arithmetic operations.
$Time = kT_{Read} + (6k+5)T_{Arith} + (3k+1)T_{jump} + T_{return}$

Exercise 1.6

Let $n = 2^{k-1}$. How much time will it take to run the algorithm if `S[0] > e`?

Topic 1.1

Big-O notation

How much resource does an algorithm need?

There can be many algorithms to solve a problem.

Some are good and some are bad.

Good algorithms are efficient in

- time and
- space.

Our method of measuring time is cumbersome and machine-dependent.

We need approximate counting that is machine-independent.

Commentary: Sometimes there is a trade-off between time and space. For example, the inefficient linear search only needed one extra integer, but the binary search used three extra integers. The difference between two integers may be a minor issue, but it illustrates the trade-off.

Input size

An algorithm may have different running times for different inputs.

How should we think about comparing algorithms?

We define the rough size of the input, usually in terms of important parameters of input.

Example 1.8

In the problem of search, we say that the number of elements in the array is the input size.

Please note that the size of individual elements is not considered. (Why?)

Commentary: Ideally, the number of bits in the binary representation of the input is the size, which is too detailed and cumbersome to handle. In the case of search, we assume that elements are drawn from the space of size 2^{32} and can be represented using 32 bits. Therefore, the type of the element was `int`.

Best/Average/Worst case

For a given size of inputs, we may further make the following distinction.

1. Best case: Shortest running time for some input.
2. Worst case: Worst running time for some input.
3. Average case: Average running time on all the inputs of the given size.

Exercise 1.7

How can we modify almost any algorithm to have a good best-case running time?

Example: Best/Average/Worst case

Example 1.9

```
int BinarySearch(int* S, int n, int e){
  // S is a sorted array
  int first = 0, last = n;
  int mid = (first + last) / 2;
  while (first < last) {
    if (S[mid] == e) return mid;
    if (S[mid] > e) {
      last = mid;
    } else {
      first = mid + 1;
    }
    mid = (first + last) / 2;
  }
  return -1;
}
```

In BinarySearch, let $n = 2^{k-1}$.

1. Best case: $e == S[n/2]$
   $T_{Read} + 6T_{Arith} + T_{return}$,
2. Worst case: $e \notin S$
   We have seen the worst case.
3. The average case is roughly equal to the worst case because most often the loop will iterate k times. (Why?)

Commentary: Analyzing the average case is usually involved, which depend on the distribution over inputs. For some important algorithms, we will do detailed average time analyses.

Asymptotic and machine-independent behavior

For short inputs, an algorithm may use a shortcut for better running time.

To avoid such false comparisons, we look at the behavior of the algorithms in limit.

Ignore hardware-specific details

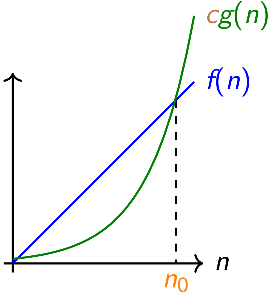
- Rounding numbers: $1000000000000001 \approx 1000000000000000$
- Ignore coefficients: $3kT_{Arith} \approx k$

Big-O notation: an approximate measure

Definition 1.3

Let f and g be functions $\mathbb{N} \to \mathbb{N}$. We say $f(n) \in O(g(n))$ if there are c and n_0 such that

$$\text{for all } n \geq n_0 \qquad f(n) \leq cg(n).$$



- In limit, $cg(n)$ will dominate $f(n)$
- We say $f(n)$ is $O(g(n))$

Exercise 1.8

Which of the following are the true statements?

- $5n + 8 \in O(n)$
- $5n + 8 \in O(n^2)$
- $5n^2 + 8 \in O(n)$
- $n^2 + n \in O(n^2)$
- $50000000000000000000000n^2 \in O(n^2)$
- $50n^2 logn + 60n^2 \in O(n^2 logn)$

Example: Big-O of the worst case of BinarySearch

Example 1.10

In `BinarySearch`, let $n = 2^{k-1}$.

Worst-case running time is $kT_{Read} + (6k+5)T_{Arith} + (3k+1)T_{jump} + T_{return}$.

Therefore, $kT_{Read} + (6k+5)T_{Arith} + (3k+1)T_{jump} + T_{return} \in O(k)$.

Since $k = log\ n + 1$, therefore $k \in O(log\ n)$.

We may also say `BinarySearch` is $O(log\ n)$.

Therefore, the worst-case running time of `BinarySearch` is $O(log\ n)$.

Exercise 1.9

Prove that $f \in O(g)$ and $g \in O(h)$, then $f \in O(h)$.

What does Big O say?

Big O expresses the approximate number of operations executed by the program as a function of input size.

Hierarchy of algorithms

- $O(log\ n)$ algorithm is better than $O(n)$
- We say $O(log\ n) < O(n) < O(n^2) < O(2^n)$

May hide large constants!!

Exercise 1.10

Give formal definition of $<$ over functions.

Complexity of an algorithm

Definition 1.4
The worst-case running time of an algorithm is the complexity of the algorithm.

We may also define average-case complexity as follows.

Definition 1.5
The average-case running time of an algorithm is the average-case complexity of the algorithm.

Example 1.11
The complexity of `BinarySearch` is $O(\log n)$.

Commentary: In a statement of complexity, there are many subtle assumptions as we discussed earlier. We need to be always aware of these assumptions, while applying complexity analysis. The assumptions sometimes limits the applicability of the complexity analysis.

Complexity of a problem

The complexity of a problem is the complexity of the best-known algorithm for the problem.

Exercise 1.11

What is the complexity of the following problem?

- sorting an array $O(n^2)$ ✗
- matrix multiplication $O(n^3)$ ✗

Best algorithm is still not known

Exercise 1.12

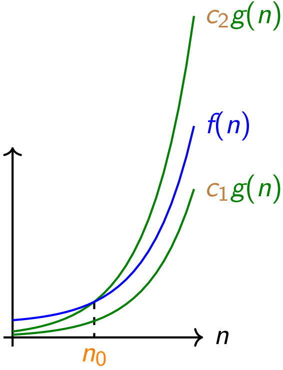
What is the best-known complexity for the above problems?

Commentary: A discussion on the latest developments in matrix multiplication algorithms: https://en.wikipedia.org/wiki/Computational_complexity_of_matrix_multiplication

Names of complexity classes

- Constant: $O(1)$
- Logarithmic: $O(logn)$
- Linear: $O(n)$
- Quadratic: $O(n^2)$
- Polynomial : $O(n^k)$ for some given k
- Exponential : $O(2^n)$

Θ-Notation



Definition 1.6 (Tight bound)

Let f and g be functions $\mathbb{N} \to \mathbb{N}$. We say $f(n) \in \Theta(g(n))$ if there are $c_1 > 0$, $c_2 > 0$, and n_0 such that

$$\text{for all } n \geq n_0 \qquad c_1 g(n) \leq f(n) \leq c_2 g(n)$$

There are more variations of the above definition. Please look at the end.

Exercise 1.13

a. Does the complexity of `BinarySearch` belong to $\Theta(\log n)$?

b. If yes, give c_1, c_2, and n_0 for the application of the above definition on `BinarySearch`.

Topic 1.2

Tutorial Problems

Problem: Compute the exact running time of insertion sort.

Exercise 1.14

The following is the code for insertion sort. Compute the exact worst-case running time of the code in terms of n and the cost of doing various machine operations.

```
for( int j = 1; j < n; j++ ) {
  int key = A[j];
  int i = j-1;
  while( i >= 0 ) {
    if( A[i] > key ) {
      A[i+1] = A[i];
    }else{
      break;
    }
    i--;
  }
  A[i+1] = key;
}
```

Problem: additions and multiplication

Exercise 1.15

What is the time complexity of binary addition and multiplication? How much time does it take to do unary addition?

Problem: hierarchy of complexity

Exercise 1.16
Given $f(n) = a_0 n^0 + ... + a_d n^d$ and $g(n) = b_0 n^0 + ... + b_e n^e$ with $d > e$ and $a_d > 0$ (Why?), show that $f(n) \notin O(g(n))$.

Topic 1.3

Problems

True or False

Exercise 1.17

Mark the following statements True / False. Also provide justification.

1. For any function $f : \mathbb{N} \rightarrow \mathbb{N}$, $O(f) \subseteq \Omega(f)$.
2. For a fixed array of size 2^k for integer k, the binary search always takes the same amount of time in the case of an unsuccessful search.

Order of functions

Exercise 1.18

- If $f(n) \leq F(n)$ and $G(n) \geq g(n)$ (in order sense) then show that $\dfrac{f(n)}{G(n)} \leq \dfrac{F(n)}{g(n)}$.
- Is $f(n)$ the same order as $f(n)|sin(n)|$?

Exercise: an important complexity class!

Exercise 1.19

Prove that $O(\log(n!)) = O(n \log n)$. Hint: Stirling's approximation

Exercise: egg drop problem**

Exercise 1.20

In the dead of night, a master jewel thief is plotting the heist of a lifetime-stealing the most valuable Faberge Egg from a towering 100-story museum. Each floor of the building has an identical egg, but the higher the floor, the more valuable the egg becomes. However, there's a catch. The thief can steal only one egg and she knows that the most valuable egg at the top may not survive a drop from such a great height. To avoid smashing her prized loot, she must identify the highest floor from which an egg can be dropped without breaking. Armed with two replica eggs from the museum's gift shop-perfectly identical but utterly worthless-the thief devises a plan. These two eggs will be her test subjects, sacrificed in the pursuit of the perfect drop. But time is of the essence, and the thief can not afford to be caught by the museum guards. She needs to figure out the minimum number of test drops required to guarantee finding the highest safe floor. Once an egg is broken, it's gone for good-no replacements, no second chances. She cannot use any other method to determine the sturdiness of the eggs.

a. Give an algorithm for the thief to determine, with the least number of drops in the worst case, the highest floor from which an egg can be safely dropped without breaking. (Quiz 2024)

b. Give an algorithm for the best average case, assuming that the probability of the highest safe floor is uniformly distributed.

c. Prove optimality of your algorithm.***

Commentary: https://www.youtube.com/watch?v=NGtt7GJ1uiM

Identities for Big-O (Midsem 2024)

Definition 1.7

Let A and B be subsets of $\mathfrak{p}(\mathbb{N} \rightarrow \mathbb{N})$. $A + B = \{f + g | f \in A \wedge g \in B\}$.

Exercise 1.21

Prove/Disprove the following:

- $O(f)g \subseteq O(fg)$
- $O(f) + O(g) \subseteq O(f + g)$

Exercise 1.22

Can we give examples when the above subset relations are strict?

Topic 1.4

Extra slides: More on complexity

Ω notation

Definition 1.8 (Lower bound)

Let f and g be functions $\mathbb{N} \to \mathbb{N}$. We say $f(n) \in \Omega(g(n))$ if there are c and n_0 such that

$$cg(n) \leq f(n) \qquad \text{for all } n \geq n_0.$$

Small-o,ω notation

Definition 1.9 (Strict Upper bound)

Let f and g be functions $\mathbb{N} \to \mathbb{N}$. We say $f(n) \in o(g(n))$ if for each c, there is n_0 such that

$$f(n) \leq cg(n) \qquad \text{for all } n \geq n_0.$$

Definition 1.10 (Strict Lower bound)

Let f and g be functions $\mathbb{N} \to \mathbb{N}$. We say $f(n) \in \omega(g(n))$ if for each c, there is n_0 such that

$$cg(n) \leq f(n) \qquad \text{for all } n \geq n_0.$$

Exercise 1.23

a. Prove that $f \in o(g)$ implies $f \in O(g)$.

b. Show that $f \in O(g)$ does not imply $f \in o(g)$.

Size of functions

We can define a partial order over functions using the above notations

- $f(n) \in O(g(n))$ implies $f(n) \leq g(n)$
- $f(n) \in o(g(n))$ implies $f(n) < g(n)$
- $f(n) \in \Omega(g(n))$ implies $f(n) \geq g(n)$
- $f(n) \in \omega(g(n))$ implies $f(n) > g(n)$
- $f(n) \in \Theta(g(n))$ implies $f(n) = g(n)$

Exercise 1.24

Show that the partial order is well-defined.

Commentary: Why do we need to prove that the definition is well-defined?

Topic 1.5

Extra slides: Binary search in recursive representation!

Search for ordered keys

If keys are stored in order, then we use the binary search that we have discussed in lecture 1.

Algorithm 1.1: BinarySearch(A, key, low, high)

```
if low > high then
    return -1
mid := (low+high)/2;
if A[mid] == key then
    return mid
if key < A[mid] then
    return BinarySearch(A,key, low, mid-1)
return BinarySearch(A,key, mid+1, high)
```

Exercise 1.25

We earlier saw the iterative version of the Binary search. Can we write any recursive algorithm as iterative algorithm?

End of Lecture 1