Unit-2

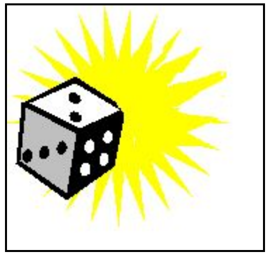
Probability Concepts in Simulation

Basic Concepts of Probability

Probability Experiments

A **probability experiment** is an action through which specific results (counts, measurements or responses) are obtained.

Example:
Rolling a die and observing the number that is rolled is a probability experiment.



The result of a single trial in a probability experiment is the **outcome**.

The set of all possible outcomes for an experiment is the **sample space**.

Example:
The sample space when rolling a die has six outcomes.

$\{1, 2, 3, 4, 5, 6\}$

Events

An **event** consists of one or more outcomes and is a subset of the sample space.

Events are represented by uppercase letters.

Example:
A die is rolled. Event A is rolling an even number.

A **simple event** is an event that consists of a single outcome.

Example:
A die is rolled. Event A is rolling an even number.
This is not a simple event because the outcomes of event A are $\{2, 4, 6\}$.

Classical Probability

Classical (or **theoretical**) **probability** is used when each outcome in a sample space is equally likely to occur. The classical probability for event E is given by

$$P(E) = \frac{\text{Number of outcomes in event}}{\text{Total number of outcomes in sample space}}.$$

Example:

A die is rolled.

Find the probability of Event A: rolling a 5.

There is one outcome in Event A: $\{5\}$

$$P(A) = \frac{1}{6} \approx 0.167$$

"Probability of Event A."

Empirical Probability

Empirical (or **statistical**) **probability** is based on observations obtained from probability experiments. The empirical frequency of an event E is the relative frequency of event E.

$$P(E) = \frac{\text{Frequency of Event } E}{\text{Total frequency}}$$
$$= \frac{f}{n}$$

Example:

A travel agent determines that in every 50 reservations she makes, 12 will be for a cruise.

What is the probability that the next reservation she makes will be for a cruise?

$$P(\text{cruise}) = \frac{12}{50} = 0.24$$

Law of Large Numbers

As an experiment is repeated over and over, the empirical probability of an event approaches the theoretical (actual) probability of the event.

Example:
Sally flips a coin 20 times and gets 3 heads. The empirical probability is $\frac{3}{20}$. This is not representative of the theoretical probability which is $\frac{1}{2}$. As the number of times Sally tosses the coin increases, the law of large numbers indicates that the empirical probability will get closer and closer to the theoretical probability.

Probabilities with Frequency Distributions

Example:
The following frequency distribution represents the ages of 30 students in a statistics class. What is the probability that a student is between 26 and 33 years old?

Ages	**Frequency, *f***
18 – 25	13
26 – 33	8
34 – 41	4
42 – 49	3
50 – 57	2

$\sum f = 30$

$$P\,(\text{age 26 to 33}) = \frac{8}{30}$$

$$\approx 0.267$$

Subjective Probability

Subjective probability results from intuition, educated guesses, and estimates.

Example:
A business analyst predicts that the probability of a certain union going on strike is 0.15.

Range of Probabilities Rule

The probability of an event *E* is between 0 and 1, inclusive. That is

$$0 \le P(A) \le 1.$$

Impossible to occur | **0.5** | **Even chance** | **Certain to occur**

Complementary Events

The **complement of Event *E*** is the set of all outcomes in the sample space that are not included in event E. (Denoted E' and read "E prime.")

$$P(E) + P(E') = 1 \qquad P(E) = 1 - P(E') \qquad P(E') = 1 - P(E)$$

Example:

There are 5 red chips, 4 blue chips, and 6 white chips in a basket. Find the probability of randomly selecting a chip that is not blue.

$$P\text{ (selecting a blue chip)} = \frac{4}{15} \approx 0.267$$

$$P\text{ (}\textit{not}\text{ selecting a blue chip)} = 1 - \frac{4}{15} = \frac{11}{15} \approx 0.733$$

Conditional Probability and the Multiplication Rule

Conditional Probability

A **conditional probability** is the probability of an event occurring, given that another event has already occurred.

$\boldsymbol{P(B \mid A)}$ → "Probability of B, given A"

Example:
There are 5 red chip, 4 blue chips, and 6 white chips in a basket. Two chips are randomly selected. Find the probability that the second chip is red given that the first chip is blue. (Assume that the first chip is not replaced.)

Because the first chip is selected and not replaced, there are only 14 chips remaining.

P (selecting a red chip|first chip is blue) $= \dfrac{5}{14} \approx 0.357$

Conditional Probability

Example:
100 college students were surveyed and asked how many hours a week they spent studying. The results are in the table below. Find the probability that a student spends more than 10 hours studying given that the student is a male.

	Less then 5	5 to 10	More than 10	Total
Male	11	22	16	49
Female	13	24	14	51
Total	24	46	30	100

The sample space consists of the 49 male students. Of these 49, 16 spend more than 10 hours a week studying.

$$P\,(\text{more than 10 hours}|\text{male}) = \frac{16}{49} \approx 0.327$$

Independent Events

Two events are **independent** if the occurrence of one of the events does not affect the probability of the other event. Two events A and B are independent if

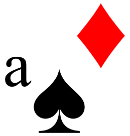
$$P(B|A) = P(B) \text{ or if } P(A|B) = P(A).$$

Events that are not independent are **dependent**.

Example:

Decide if the events are independent or dependent.

Selecting a diamond from a standard deck of cards (A), putting it back in the deck, and then selecting a spade from the deck (B).



$$P(B|A) = \frac{13}{52} = \frac{1}{4} \text{ and } P(B) = \frac{13}{52} = \frac{1}{4}.$$

The occurrence of A does not affect the probability of B, so the events are independent.

Multiplication Rule

The probability that two events, A and B will occur in sequence is

$$P\,(A \text{ and } B) = P\,(A) \cdot P\,(B\,|A).$$

If event A and B are independent, then the rule can be simplified to $P\,(A \text{ and } B) = P\,(A) \cdot P\,(B)$.

Example:

Two cards are selected, without replacement, from a deck. Find the probability of selecting a diamond, and then selecting a spade.

Because the card is not replaced, the events are dependent.

P (diamond and spade) = P (diamond) · P (spade |diamond).

$$= \frac{13}{52} \cdot \frac{13}{51} = \frac{169}{2652} \approx 0.064$$

Multiplication Rule

Example:
A die is rolled and two coins are tossed.
Find the probability of rolling a 5, and flipping two tails.

$P\,(\text{rolling a }5) = \dfrac{1}{6}.$

Whether or not the roll is a 5, $P\,(\text{Tail}\,) = \dfrac{1}{2},$
so the events are independent.

$$P\,(5 \text{ and } T \text{ and } T\,) = P\,(5)\cdot\, P\,(T\,)\cdot\, P\,(T\,)$$

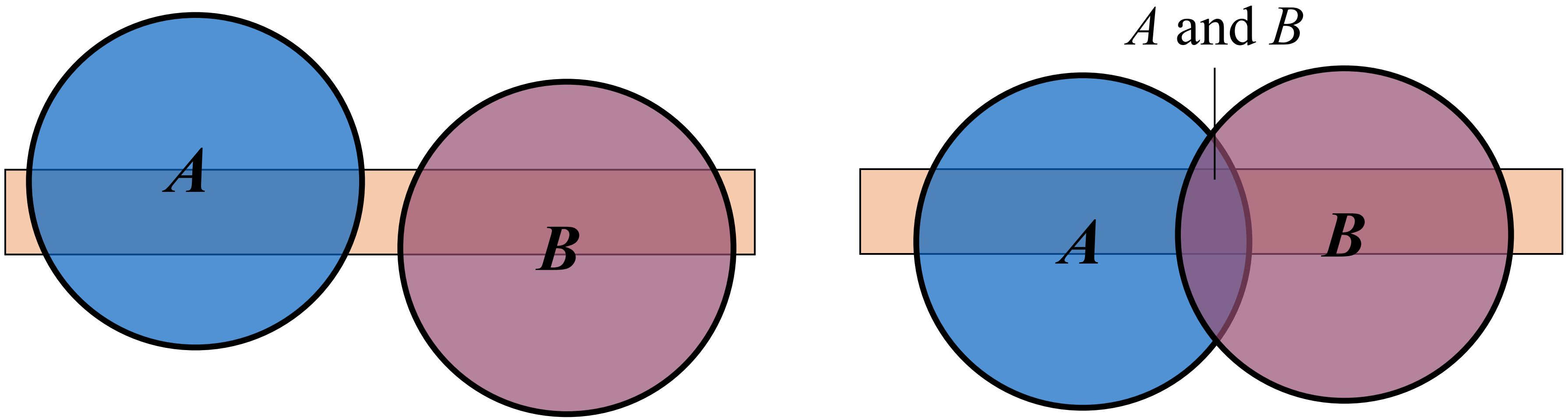
$$= \frac{1}{6}\cdot\frac{1}{2}\cdot\frac{1}{2}$$

$$= \frac{1}{24} \approx 0.042$$

The Addition Rule

Mutually Exclusive Events

Two events, A and B, are **mutually exclusive** if they cannot occur at the same time.



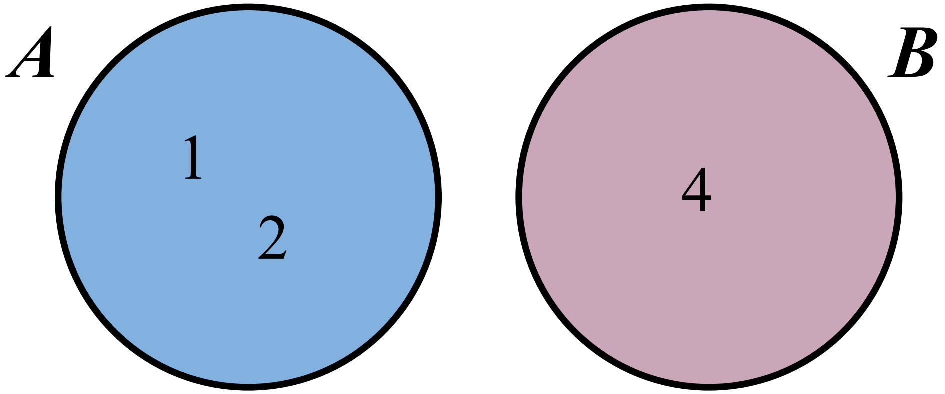
A and B are mutually exclusive.

A and B are <u>not</u> mutually exclusive.

Mutually Exclusive Events

Example:
Decide if the two events are mutually exclusive.

Event *A*: Roll a number less than 3 on a die. Event *B*: Roll a 4 on a die.

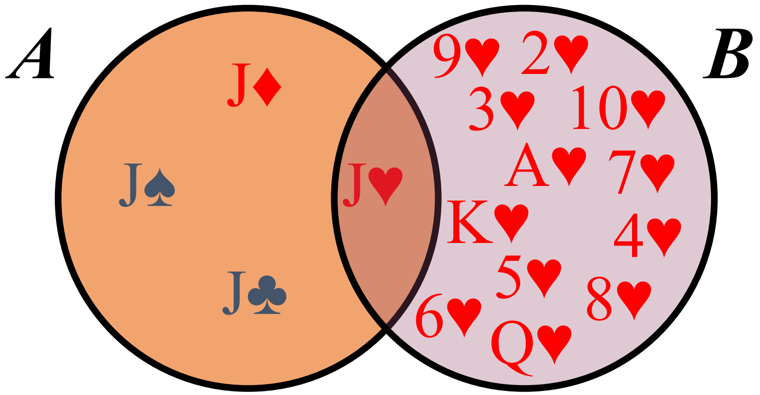


These events cannot happen at the same time, so the events are mutually exclusive.

Mutually Exclusive Events

Example:
Decide if the two events are mutually exclusive.

Event A: Select a Jack from a deck of cards. Event B: Select a heart from a deck of cards.



Because the card can be a Jack and a heart at the same time, the events are not mutually exclusive.

The Addition Rule

The probability that event A or B will occur is given by

$$P\,(A \text{ or } B) = P\,(A) + P\,(B) - P\,(A \text{ and } B\,).$$

If events A and B are mutually exclusive, then the rule can be simplified to $P\,(A \text{ or } B) = P\,(A) + P\,(B)$.

Example:

You roll a die. Find the probability that you roll a number less than 3 or a 4.

The events are mutually exclusive.

P (roll a number less than 3 or roll a 4)

$= P$ (number is less than 3) $+ P$ (4)

$$= \frac{2}{6} + \frac{1}{6} = \frac{3}{6} = 0.5$$

The Addition Rule

Example:

A card is randomly selected from a deck of cards. Find the probability that the card is a Jack or the card is a heart.

The events are not mutually exclusive because the Jack of hearts can occur in both events.

P (select a Jack or select a heart)

$= P\text{ (Jack)} + P\text{ (heart)} - P\text{ (Jack of hearts)}$

$$= \frac{4}{52} + \frac{13}{52} - \frac{1}{52}$$

$$= \frac{16}{52} \approx 0.308$$

The Addition Rule

Example:
100 college students were surveyed and asked how many hours a week they spent studying. The results are in the table below. Find the probability that a student spends between 5 and 10 hours or more than 10 hours studying.

	Less then 5	5 to 10	More than 10	Total
Male	11	22	16	49
Female	13	24	14	51
Total	24	46	30	100

The events are mutually exclusive.

P (5 to10 hours or more than 10 hours) = P (5 to10) + P (10)

$$= \frac{46}{100} + \frac{30}{100} = \frac{76}{100} = 0.76$$

Counting Principles

Fundamental Counting Principle

If one event can occur in m ways and a second event can occur in n ways, the number of ways the two events can occur in sequence is $\boldsymbol{m \cdot n}$. This rule can be extended for any number of events occurring in a sequence.

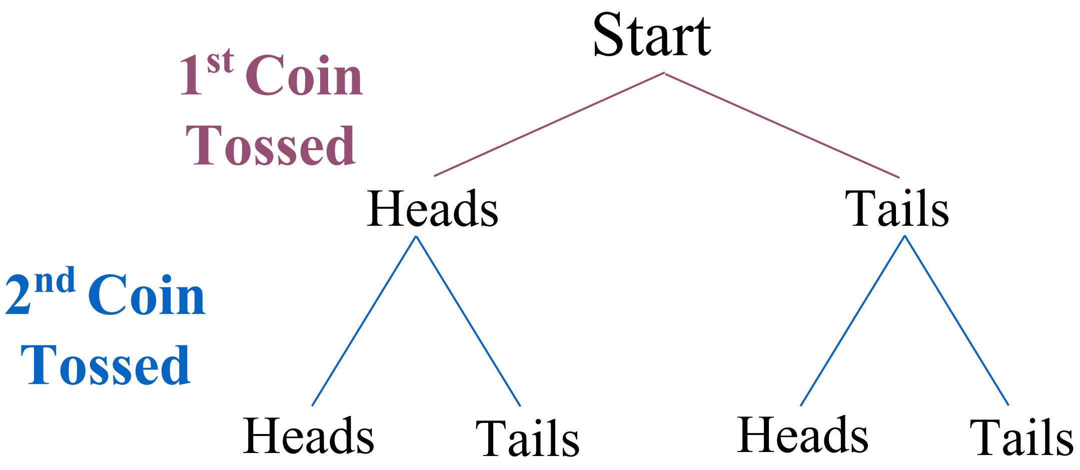
Example:
A meal consists of a main dish, a side dish, and a dessert. How many different meals can be selected if there are 4 main dishes, 2 side dishes and 5 desserts available?

# of main dishes		# of side dishes		# of desserts		
4	×	**2**	×	**5**	=	**40**

There are 40 meals available.

Fundamental Counting Principle

Example:
Two coins are flipped. How many different outcomes are there?
List the sample space.



2 ways to flip the coin

2 ways to flip the coin

There are **2** $\times$ **2** = 4 different outcomes: {HH, HT, TH, TT}.

Fundamental Counting Principle

Example:

The access code to a house's security system consists of 5 digits. Each digit can be 0 through 9. How many different codes are available if

a.) each digit can be repeated?

b.) each digit can only be used once and not repeated?

a.) Because each digit can be repeated, there are 10 choices for each of the 5 digits.

$10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 100{,}000$ codes

b.) Because each digit cannot be repeated, there are 10 choices for the first digit, 9 choices left for the second digit, 8 for the third, 7 for the fourth and 6 for the fifth.

$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = 30{,}240$ codes

Permutations

A **permutation** is an ordered arrangement of objects. The number of different permutations of n distinct objects is $n!$.

"n factorial"

$$\boldsymbol{n! = n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot \ldots \cdot 3 \cdot 2 \cdot 1}$$

Example:
How many different surveys are required to cover all possible question arrangements if there are 7 questions in a survey?

$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040 \text{ surveys}$$

Permutation of *n* Objects Taken *r* at a Time

The number of **permutations of *n* elements taken *r* at a time** is

$$ {}_{n}P_{r} = \frac{n!}{(n-r)!}. $$

in the group

taken from the group

Example:

You are required to read 5 books from a list of 8. In how many different orders can you do so?

$$ {}_{n}P_{r} = {}_{8}P_{5} = \frac{8!}{(8-5)!} = \frac{8!}{3!} = \frac{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1}}{\cancel{3}\cdot \cancel{2}\cdot \cancel{1}} = 6720 \text{ ways} $$

Distinguishable Permutations

The number of **distinguishable permutations** of n objects, where n_1 are one type, n_2 are another type, and so on is

$$\frac{n!}{n_1! \cdot n_2! \cdot n_3! \cdots n_k!}, \text{ where } n_1 + n_2 + n_3 + \cdots + n_k = n.$$

Example:

Jessie wants to plant 10 plants along the sidewalk in her front yard. She has 3 rose bushes, 4 daffodils, and 3 lilies. In how many distinguishable ways can the plants be arranged?

$$\frac{10!}{3!4!3!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4!}{3!4!3!}$$

$$= 4{,}200 \text{ different ways to arrange the plants}$$

Combination of *n* Objects Taken *r* at a Time

A **combination** is a selection of r objects from a group of n things when order does not matter. The number of combinations of r objects selected from a group of n objects is

$$_{n}C_{r} = \frac{n!}{(n-r)!\,r!}.$$

\# in the collection

\# taken from the collection

Example:
You are required to read 5 books from a list of 8. In how many different ways can you do so if the order doesn't matter?

$$_{8}C_{5} = \frac{8!}{3!5!} = \frac{8 \cdot 7 \cdot 6 \cdot 5!}{3!5!}$$

$= 56$ combinations

Application of Counting Principles

Example:
In a state lottery, you must correctly select 6 numbers (in any order) out of 44 to win the grand prize.

a.) How many ways can 6 numbers be chosen from the 44 numbers?

b.) If you purchase one lottery ticket, what is the probability of winning the top prize?

a.) $_{44}C_6 = \frac{44!}{6!38!} = 7{,}059{,}052$ combinations

b.) There is only one winning ticket, therefore,

$$P(\text{win}) = \frac{1}{7059052} \approx 0.00000014$$