Homework 4: characteristic functions, probability inequalities

EE 325: Probability and Random Processes, Autumn 2019

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Instructions: Both problems from Set-A has to be submitted by 5:00pm on 04/10/2019 (Friday). Please follow the given instructions carefully.

If solutions from two or more students resemble or are copied from each other, all the concerned students will get -5 marks. You *must* write your own solution in your own words.

Each question should be submitted on a different sheet. Write your name+roll no. on each page of your submitted solution. Make a photocopy of your homework submission. Submit your Homework + photocopy (homework submission and its copy should not be stapled together) in the box kept in EE-Office for EE325 Homework. The box will become available on Friday morning. *If you have queries, then meet the instructor or the TA during office hours.*

Set-A

1. Let $B_1, B_2, \ldots, B_n$ be IID Bernoulli(p) random variables with $0 < p < 1$. The task is to estimate p.
   (a) Find a rule for obtaining the maximum likelihood estimate of $\widehat{p}_{ML}$ for p. The estimate should only depend on the realization $b_1, b_2, \ldots, b_n$.
   (b) Assume that $\varepsilon > 0$ is a tolerance parameter during estimation. Find the Chernoff bound on $\mathbb{P}(|\widehat{p}_{ML} - p| > \varepsilon)$. How does n grow to ensure that this upper bound is $\leq \delta$ for a given $\varepsilon > 0$?
2. Let $X \sim \mathcal{N}(0, \sigma^2)$ be a zero mean Gaussian random variable.
   (a) Let $X_1, X_2, \ldots$ be IID random variables with the same distribution as X. Let $S_n = X_1 + X_2 + \ldots + X_n$. For $\varepsilon > 0$, establish the Chernoff bound on $\mathbb{P}(|S_n| \geq n\varepsilon)$.
   (b) Let Z be a random variable such that $Z = 1$ with probability $1/2$ and $Z = -1$ with probability $1/2$. Show that Z is a sub-Gaussian random variable and find its parameter.
   (c) Let $Z_1, Z_2, \ldots$ be IID random variables. Using the sub-Gaussian property or otherwise, find the Chernoff bound on
   $$\mathbb{P}\left(\left|\sum_{i=1}^{n} Z_i\right| > n\varepsilon\right). \tag{1}$$

Set-B

1. Let $X_1, \ldots, X_n$ be random variables which are not necessarily independent. Let there be a $\sigma > 0$ such that
   $$\mathbb{E}(e^{tX_i}) \leq \exp(t^2\sigma^2/2) \text{ for all } t > 0.$$
   Then show that
   $$\mathbb{E}\left(\max_{1 \leq i \leq n} X_i\right) \leq \sigma\sqrt{2\log n}.$$
   Note that a zero-mean Gaussian random variable with variance σ^2 will satisfy this inequality.
2. Assume that X is a continuous r.v. with $\phi_X(t)$ as the characteristic function. Are $\mathrm{Re}[\phi_X(t)]$, $\mathrm{Im}[\phi_X(t)]$, and $|\phi_X(t)|^2$ valid characteristic functions? For a complex number z, $\mathrm{Re}[z]$ and $\mathrm{Im}[z]$ represent its real and imaginary parts.

3. Let Y be a zero-mean random variable with variance σ^2. Show the one-sided inequality,

$$\mathbb{P}(Y \geq a) \leq \frac{\sigma^2}{\sigma^2 + a^2}$$

for $a > 0$. (Hint: use the fact that $Y \geq a \Leftrightarrow Y + c \geq a + c$, for any $c \in \mathbb{R}$.)

4. For this problem you may require the Schwarz inequality. Given any two rv X and Y with finite variances, the Schwarz inequality states that

$$[\mathbb{E}(XY)]^2 \leq \left[\mathbb{E}(X^2)\mathbb{E}(Y^2)\right].$$

For a rv Z which is positive, i.e. $Z \geq 0$, show that

$$\mathbb{P}(Z > a) \geq \frac{(\mathbb{E}(Z) - a)^2}{\mathbb{E}(Z^2)},$$

where $a > 0$ is any arbitrary constant. (**Hint:** think of a rv which converts into a probability upon taking expectations.)

5. Construct examples of distributions for X such that,

   (a) The Markov inequality is tight, i.e., there exists a distribution $F_X(x)$ and a point $a \in \mathbb{R}$ such that $\mathbb{P}(X \geq a) = (\mathbb{E}(X)/a)$.

   (b) The Chebyshev inequality is tight, i.e., there exists a distribution $F_X(x)$ and a point $a > 0, a \in \mathbb{R}$ such that $\mathbb{P}(|X - \mathbb{E}(X)| \geq a) = (\sigma_X^2/a^2)$.

6. (Gallager 1.38) If $Y > 0$ and $\mathbb{E}(Y) < \infty$, then show that $\lim_{y\to\infty} y\mathbb{P}(Y \geq y) = 0$.

7. Let $\{X_i\}_{i=1}^{\infty}$ be an IID sequence of random variables, distributed according to the exponential distribution $\text{Exp}(\lambda)$. Show that,

$$\mathbb{P}\left(\sum_{i=1}^{n} X_i \geq n\left(\frac{1}{\lambda} + \epsilon\right)\right) \leq \exp\left(n[\ln(1 + \lambda\epsilon) - \lambda\epsilon]\right).$$

Show that the bound is non-trivial or the RHS of the inequality is not equal to 1 for $\epsilon > 0$. (Hint: use Chernoff bound formulation.)

8. (Kullback Leibler divergence between discrete random variables:) Let X and Y be two discrete random variables with non-zero pmf defined on the set of integers $\{1, 2, \ldots, m\}$.

   (a) Show that the function $g(x) = \ln x - x + 1 \leq 0$. (Hint: Show that $g''(x) \leq 0$ and find its unique maxima to establish the inequality.)

   (b) Using $\ln x \leq x - 1$ from part (a), show that,

$$\sum_{i=1}^{m} p_X(i) \ln \frac{p_X(i)}{p_Y(i)} \geq 0, \tag{2}$$

   with equality occuring only if $p_X(i) = p_Y(i)$ for all $i = 1, 2, \ldots, m$. (Hint: Work with the negative of expression on the LHS of the inequality.)