PID Control Analysis & Design

- *Overall Performance with PID Control*
- *PID Design with Zeigler-Nichols Methods*
- *PID Design in Frequency Domain*

Limitations of P, PI & PD

With the design of **P, PI & PD** controllers, we are in a position to **ensure** a wide range of **tracking and transient responses** for any given plant.

However, the **above assurance** is usually under the condition that either **tracking** or **transient** response features drive the design of **control element.**

In reality, we are likely to **encounter** a combination of steady-state and transient **response** specifications, so that **employing** any one of these would **not be** adequate.

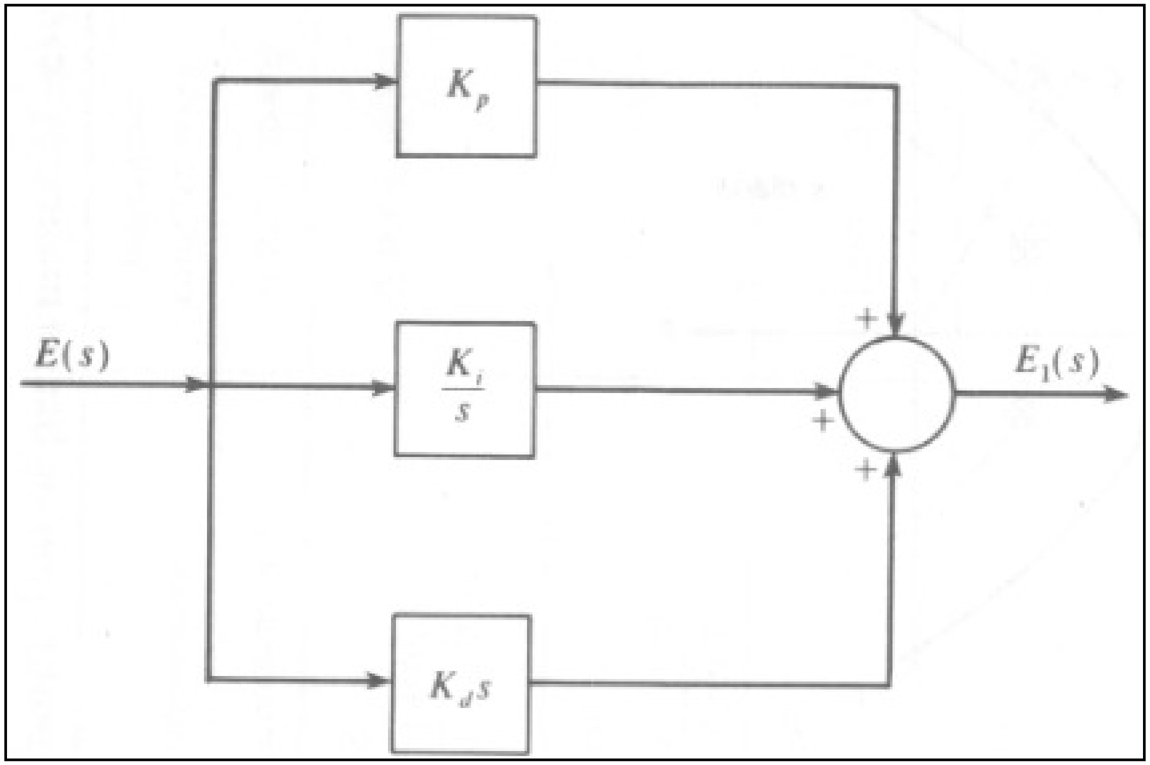
Role of PID Controllers

PID controller aims to achieve **both** tracking and transient response **simultaneously** and, hence, is a **better** option in comparison to either **PI** or **PD**.

This is so because it **includes** all three **actions** which help in achieving a **wide range** of performance.

Further, it **manages** the overall design **effort well**, while **increasing** the overall design **degrees of freedom.**

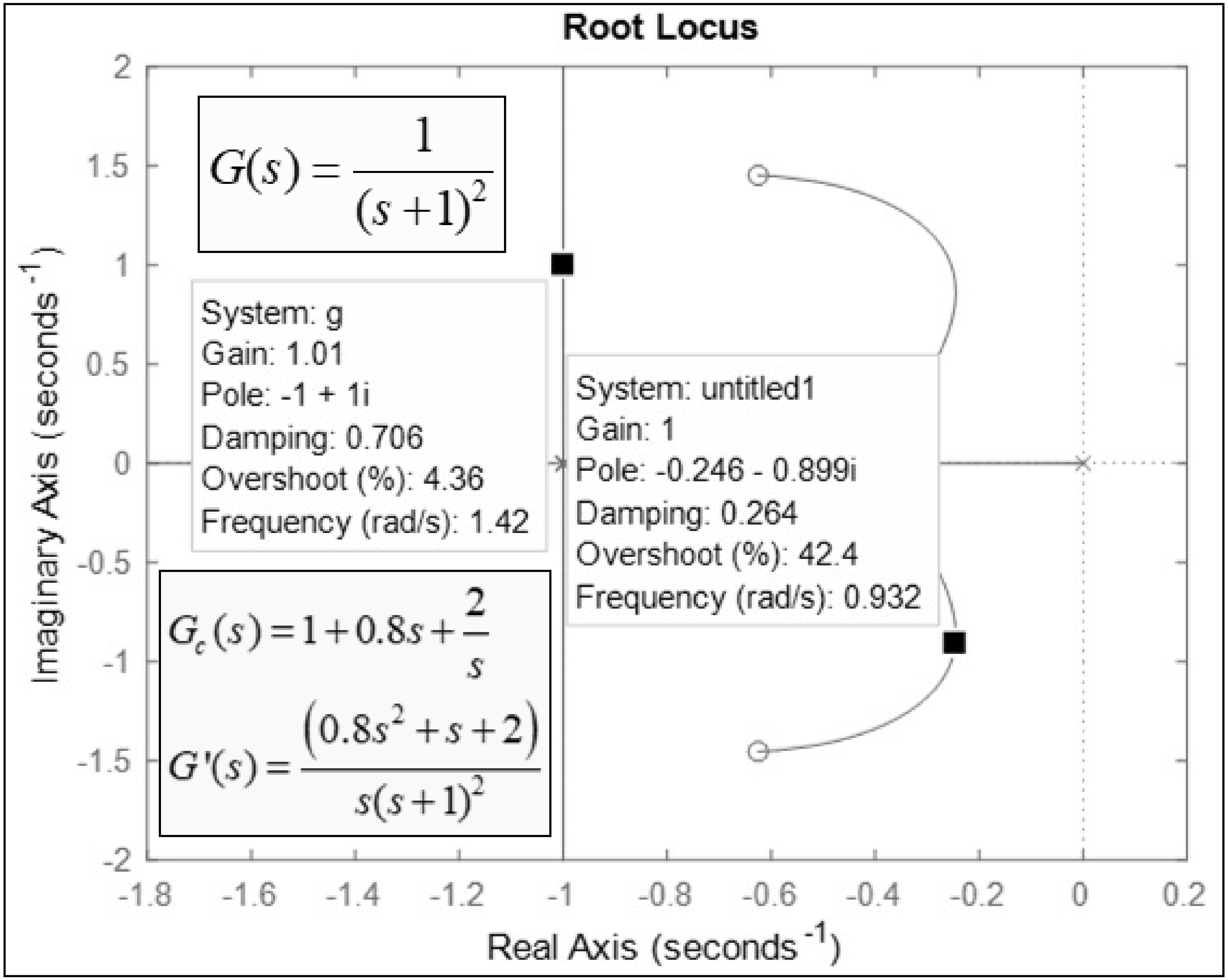
PID Controller Concept



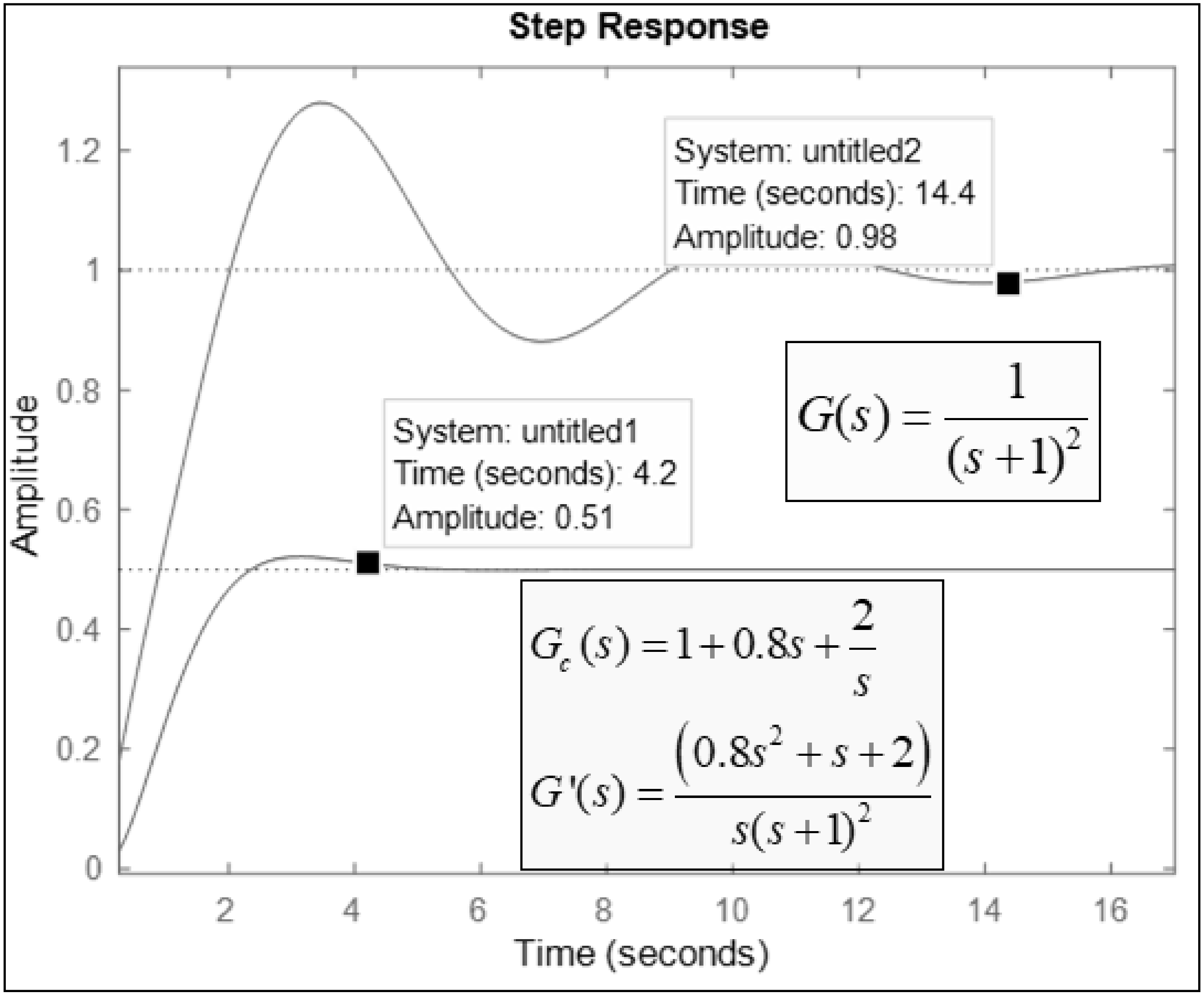
$$G_{PID}(s) = \frac{K_d s^2 + K_p s + K_i}{s}$$

Most **industrial** controllers are PID, **due to** their **versatility** and **least** No. of design **parameters**.

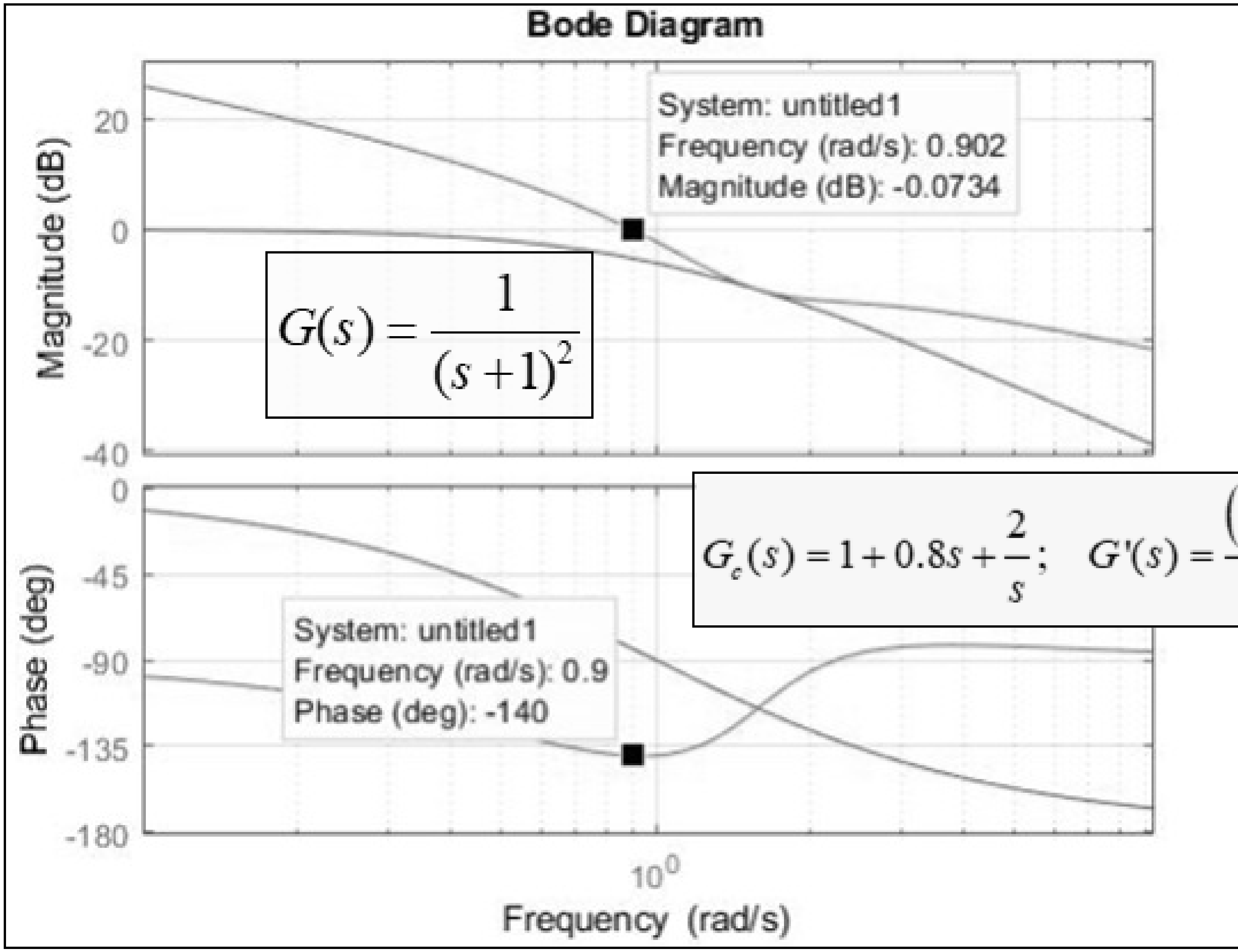
Effect of PID Control on Root Locus



Effect of PID Control on Step Response



Effect of PID Control on Bode Plot



$$G(s) = \frac{1}{(s+1)^2}$$

$$G_c(s) = 1 + 0.8s + \frac{2}{s}; \quad G'(s) = \frac{\left(0.8s^2 + s + 2\right)}{s(s+1)^2}$$

PID Design Strategies

The **simplest** and most **intuitive** way is to use **PI & PD** controllers, in **cascade**, as shown below.

$$G_{PID}(s) = G_{PI}(s) \times G_{PD}(s) = \left(K_{PI} + \frac{K_I}{s} \right) \times \left(K_{PD} + K_D s \right)$$

$$G_{PID}(s) = \left(K_{PI} K_{PD} + K_I K_D \right) + \frac{K_I K_{PD}}{s} + \left(K_{PI} K_D \right) s$$

We find that **PID action** is coupled to **various gains** that are designed **independently** and hence needs **care.**

PID Design Strategies

In case we have **specifications** in the form a **3rd** order characteristic **equation**, we can also determine the **gains** by comparison of **coefficients** of the two **equations.**

However, in **general**, we use a **methodology** proposed by **Zeigler-Nichols** to synthesize the **PID** controller, which is based on plant **step response** features.

PID Control from Zeigler-Nichols

Zeigler-Nichols PID Design

Zeigler – Nichols is a methodology for **designing** PID controllers, based on the **specific** assumptions about the **unit step** response of open / partial closed loop **plants.**

The **controller TF** is rewritten in **terms** of the overall gain $\mathbf{K_p}$ and time constants $\mathbf{T_i}$ **&** $\mathbf{T_d}$ **,** as shown below.

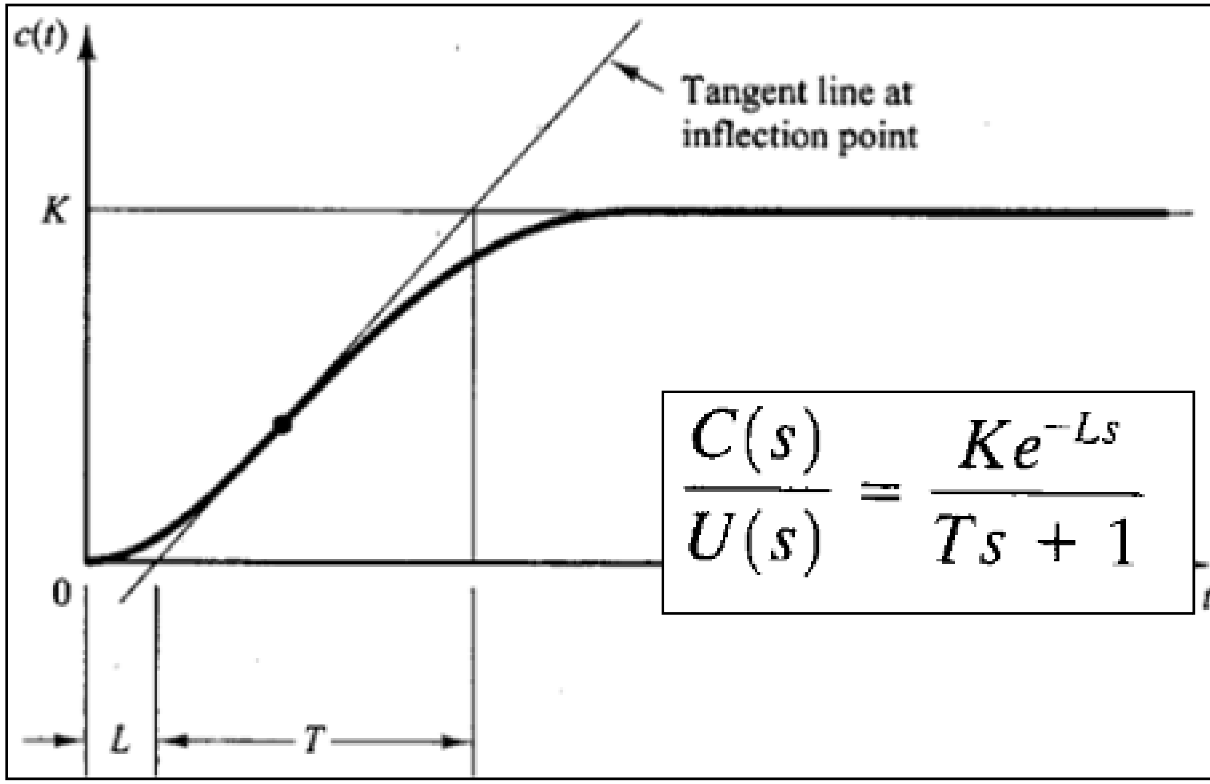
$$\boxed{G_c(s) = K_p\left(1 + \frac{1}{T_i s} + T_d s\right)}$$

There are **two** methods for **arriving** at the **controller.**

First Method Concept

In this method, **step response** of stable plants is **mapped** to a 1st order **TF with delay**, as shown alongside.

Here, **response** is a dominant **1st order type**, with **inflection point** in transient.

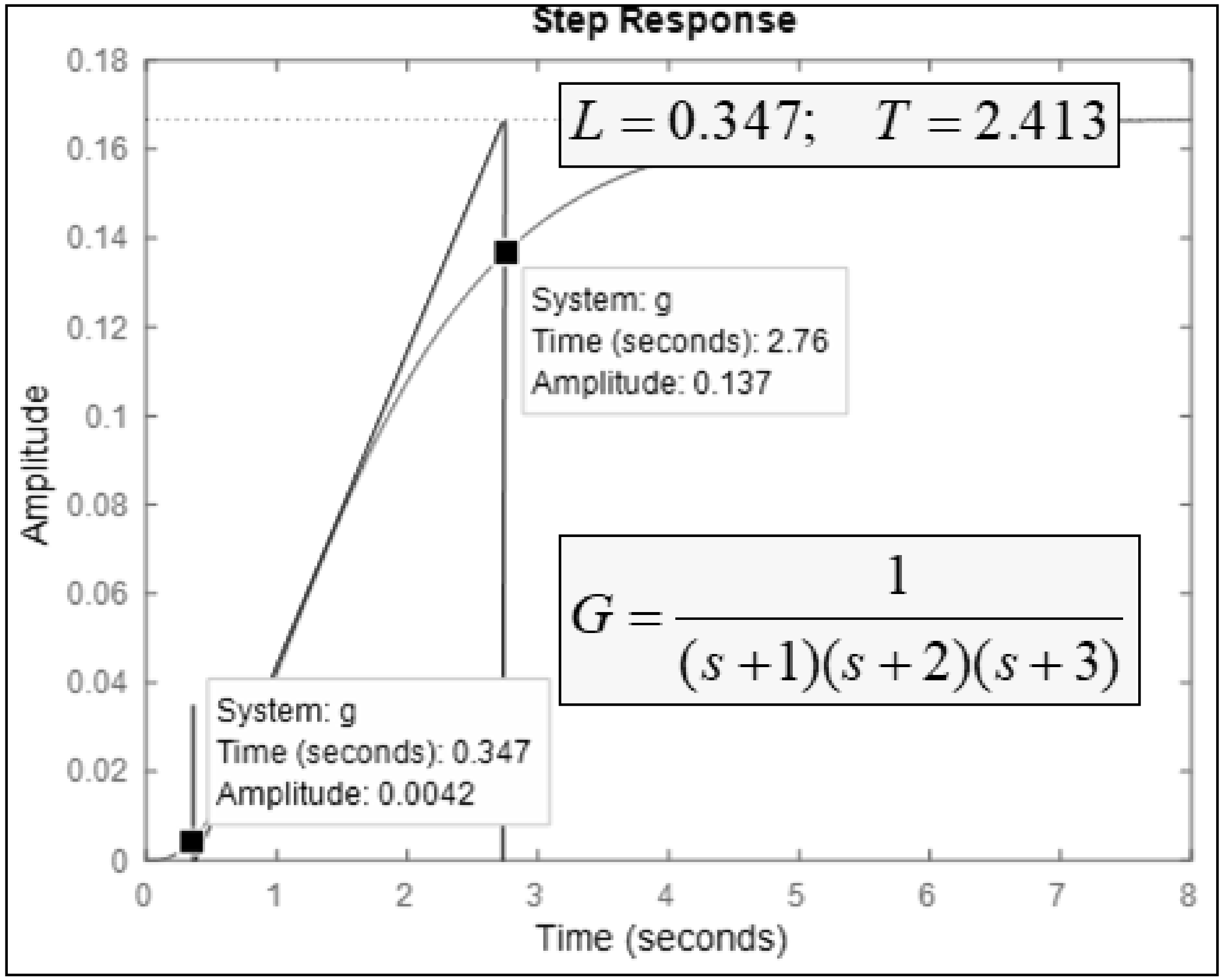


$$\frac{C(s)}{U(s)} = \frac{Ke^{-Ls}}{Ts+1}$$

First Method Design Strategy

The **way** of obtaining '**L**' & '**T**' is **demonstrated** in the figure alongside.

Here, while '**L**' is like '**dead time**', '**T**' is an indicator of **time constant** (i.e. time to reach **~66%** of the final **output**).



First Method Control Formulation

A set of **controllers**, including PID, **is proposed**, based on response **features,** as shown **alongside.**

We can obtain PID **transfer function,** as shown alongside.

We see that **controller** has both the '**zeros**' at '-1/L', while its **gain** is proportional to '**T**'.

Type of Controller	K_p	T_i	T_d
P	$\frac{T}{L}$	∞	0
PI	$0.9\frac{T}{L}$	$\frac{L}{0.3}$	0
PID	$1.2\frac{T}{L}$	$2L$	$0.5L$

$$G_c(s) = K_p\left(1 + \frac{1}{T_i s} + T_d s\right)$$

$$= 1.2\frac{T}{L}\left(1 + \frac{1}{2Ls} + 0.5Ls\right)$$

$$= 0.6T\frac{\left(s + \frac{1}{L}\right)^2}{s}$$

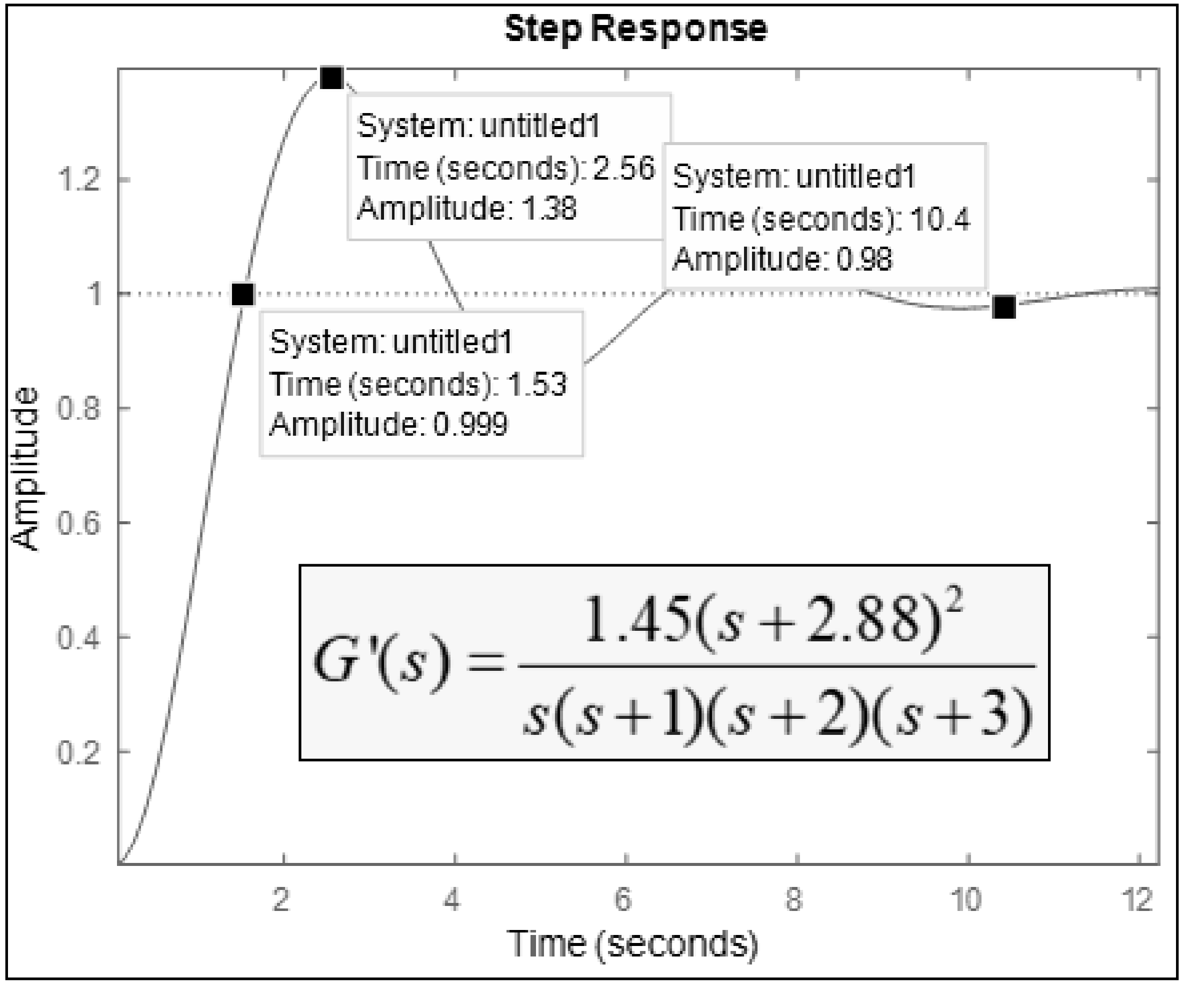
Design Example with First Method

PID controller in the **present case** is given below.

$$G_{PID}(s) = 0.6T\frac{\left(s+\frac{1}{L}\right)^2}{s} = \frac{1.45(s+2.88)^2}{s}$$

Resulting response is **shown alongside.**

We see that **controller** achieves a **response** that needs **adjustments.**

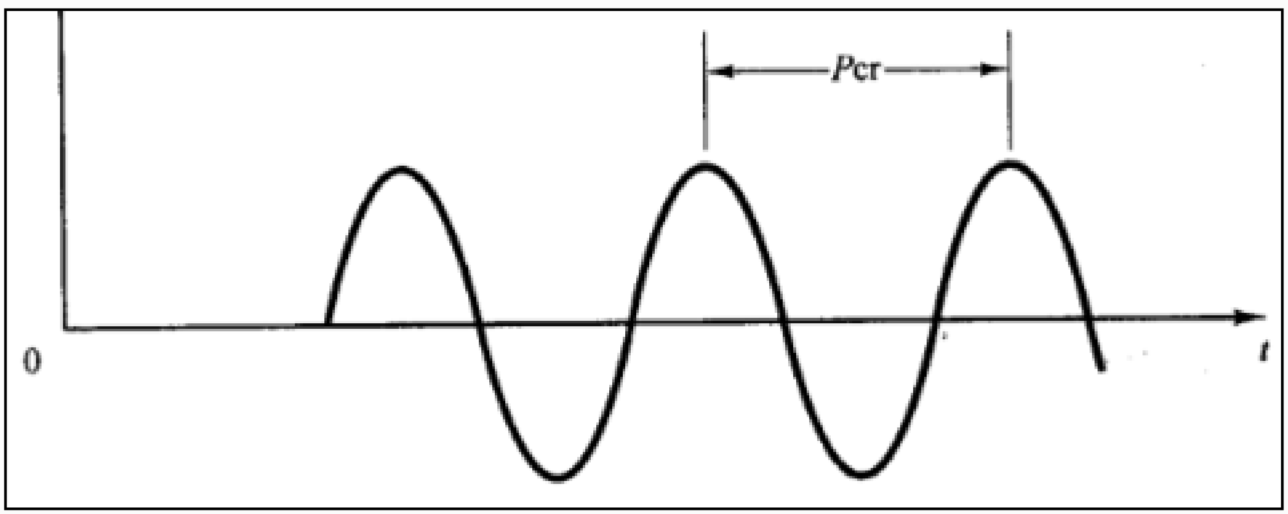


$$G'(s) = \frac{1.45(s+2.88)^2}{s(s+1)(s+2)(s+3)}$$

Second Method Concept

In this method, **initially**, control is only **through K_P**, which is increased from **0** to a **critical value** for which the **dominant** closed loop **poles** lie on the **'jω' axis**.

Time period of the sustained **oscillations**, achieved in this case, **P_{cr},** can be obtained from **impulse** response.



Second Method Design Strategy

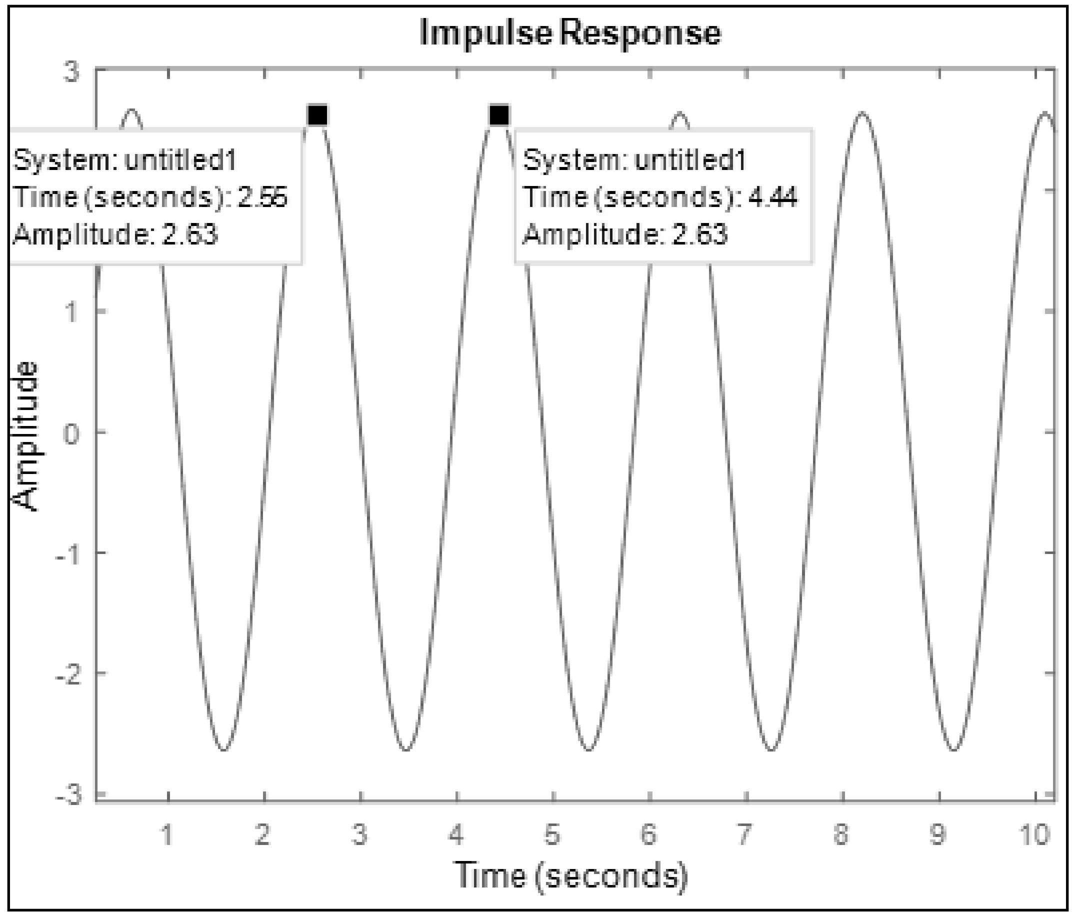
$$G(s) = \frac{1}{(s+1)(s+2)(s+3)}; \quad \frac{C}{R} = \frac{K_{cr}}{s^3 + 6s^2 + 11s + 6 + K_{cr}}$$

$$K_{cr} = 60; \quad P_{cr} = 1.89$$

We employ the **Routh's method** to obtain, '$\mathbf{P_{cr}}$', as shown alongside.

Partial closed loop impulse **response** is as shown **alongside.**

It is to be noted that $\mathbf{K_{cr}}$ is an indicator of **GM**, while $\mathbf{P_{cr}}$ is an indicator of **bandwidth**.



Second Method Formulation

In this case, **proposed controllers** are shown alongside.

Resulting **transfer function** is shown **alongside.**

We find that **similar** to first method, we get a **controller** that has both **zeros** at the **same location,** '-4/P_{cr}'.

Type of Controller	K_p	T_i	T_d
P	$0.5K_{cr}$	∞	0
PI	$0.45K_{cr}$	$\frac{1}{1.2}P_{cr}$	0
PID	$0.6K_{cr}$	$0.5P_{cr}$	$0.125P_{cr}$

$$
\begin{aligned}
G_c(s) &= K_p\left(1 + \frac{1}{T_i s} + T_d s\right) \\
&= 0.6K_{cr}\left(1 + \frac{1}{0.5P_{cr}s} + 0.125P_{cr}s\right) \\
&= 0.075K_{cr}P_{cr}\frac{\left(s + \frac{4}{P_{cr}}\right)^2}{s}
\end{aligned}
$$

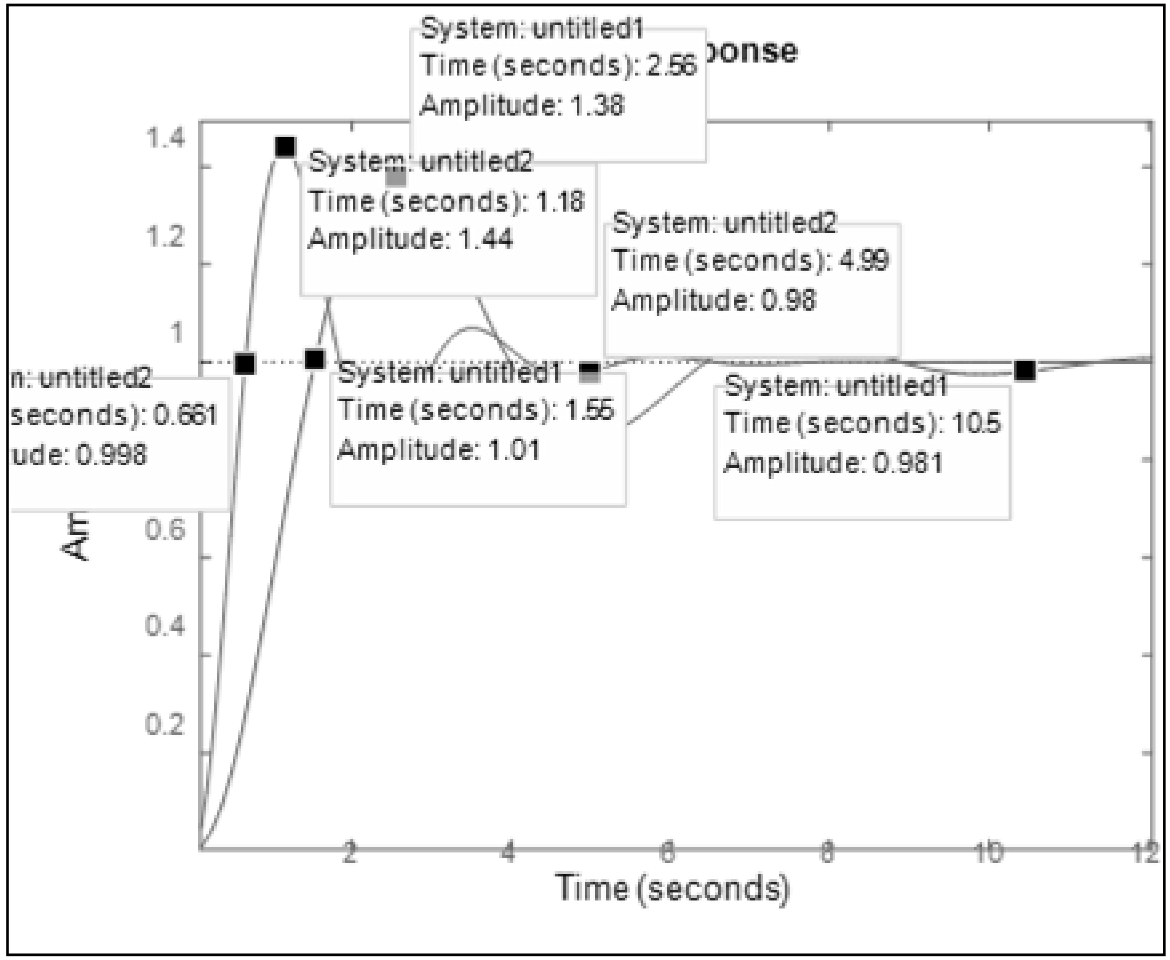
Design Example with Second Method

PID controller TF in the **present case** is given below.

$$G_{c2}(s) = \frac{8.51(s+2.11)^2}{s}$$

We **find** that in this case, **zeros** are '**closer**' to origin, while **gain** is much **larger.**

A **comparison** of **two controllers** in terms of **step response**, is shown alongside.



Summary

PID controllers can be **intuitively** synthesized using **PI** and **PD** design **strategies.**

Nichols – Zeigler tuning results in good **speed** of response but poor **transient** behaviour in closed loop and are used to get **only** the initial **controllers**, which are then **tuned**.

PID Control from Bode Plots

PID Controller Design with Bode

Zeigler-Nichols method of tuning broadly **aims** to arrive at a **stable** closed loop system with acceptable **transient** response.

However, a **more** focused **design** can be done using **frequency** domain methods, which **take** into account the **design** specifications.

In this method, **following** generic form of the **PID** controller is **assumed.**

$$G_{PID}(s) = \frac{K(as+1)(bs+1)}{s}$$

PID Controller Design with Bode

Design with bode starts with first satisfying the **K_V** specification and then the **PM requirement.**

The procedure is **demonstrated** through an **example**. Consider a system as given below.

$$G(s) = \frac{1}{s^2 + 1}$$

Design a PID controller so that K_V **is 4,** PM is at **least 50^o** and GM is more **than 10 dB**.

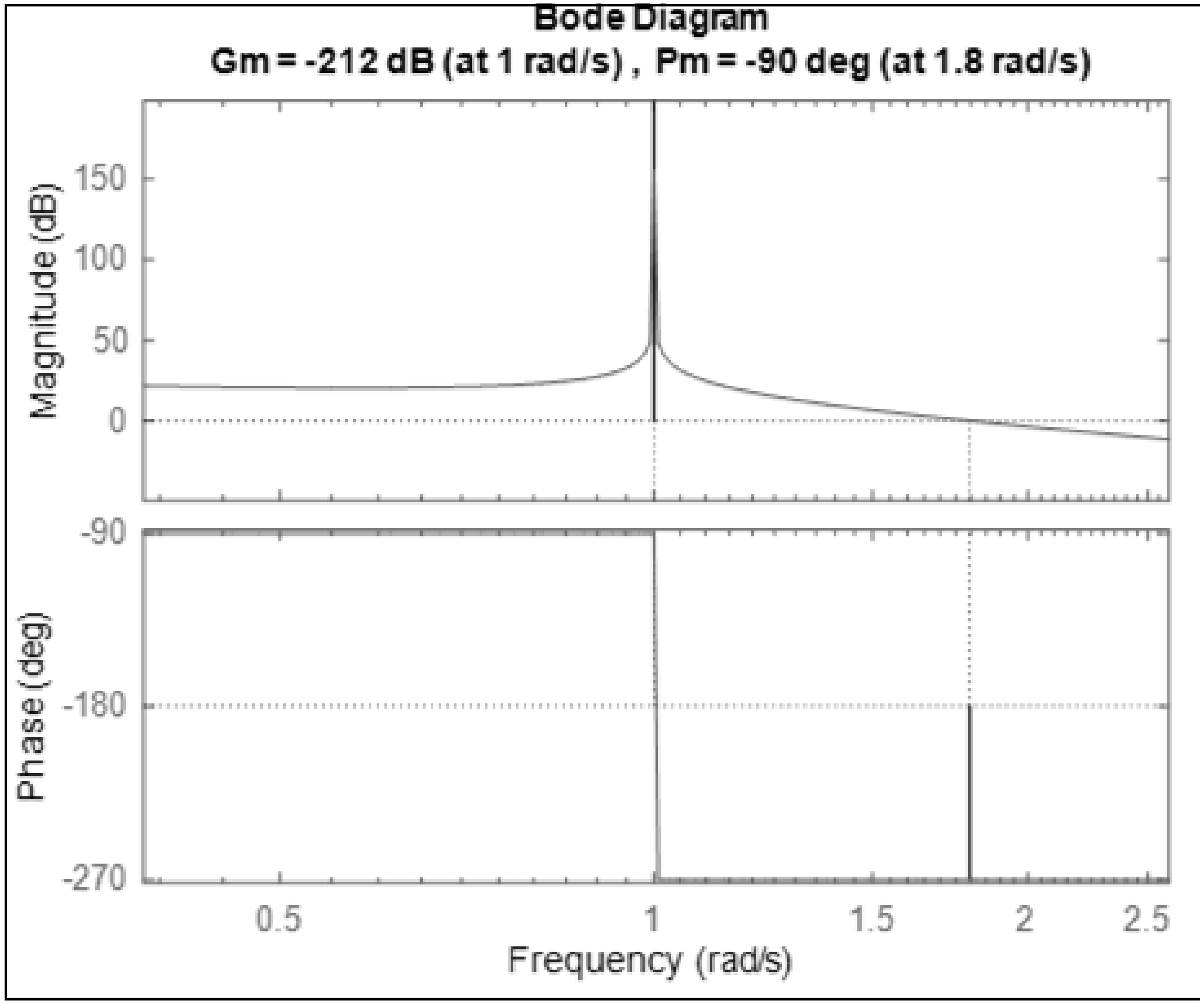
We first find that **K_V requirement** can be met with **K = 4.**

Gain – Integral Adjusted Bode Plot

We also **change type**, so that **modified** plant is as given below, and **bode plot** is alongside.

$$G'(s) = \frac{4}{s\left(s^2+1\right)}$$

We see that we need to add a **large** positive phase at the **GCO.**



PID Controller Design with Bode

This can be done by **first choosing** value of **'a'** to be **large (5)** (i.e. zero at -0.2) so that **large** phase lead will be added around **GCO of 1.8.**

This also creates **a doublet** with **pole at origin** and acts as the **PI** controller, **as shown below.**

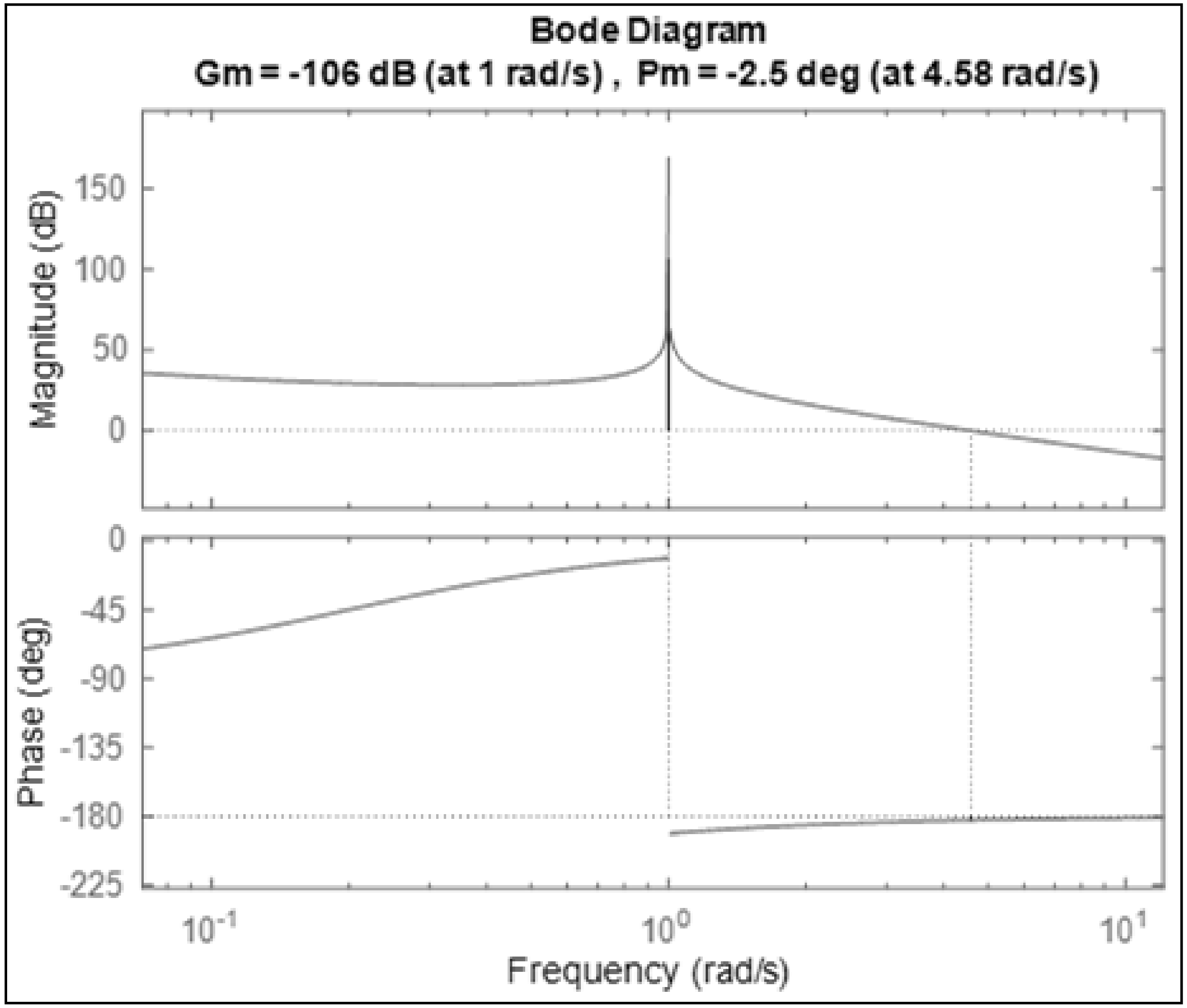
$$G''(s) = \frac{(5s+1)}{s} \times \frac{4}{\left(s^2+1\right)}$$

Bode Plot with Pole & First Zero

Given alongside is the **new bode plot.**

We see that, while **GCO** increases from **1.8 to 4.58**; PCO remains **unchanged.**

However, both **gain & phase** plots change significantly in **low 'ω' range**.



PID Controller Design with Bode

We now **choose 'b'** so that PM **requirement is met.** This also will **change PCO** by adding more **positive phase** in **low** frequency regime.

A simple **trial** suggests that **b = 0.25,** results in 2nd zero to be **placed at -4,** and is adequate for achieving the required **PM.** Resultant **compensated** system is as shown below.

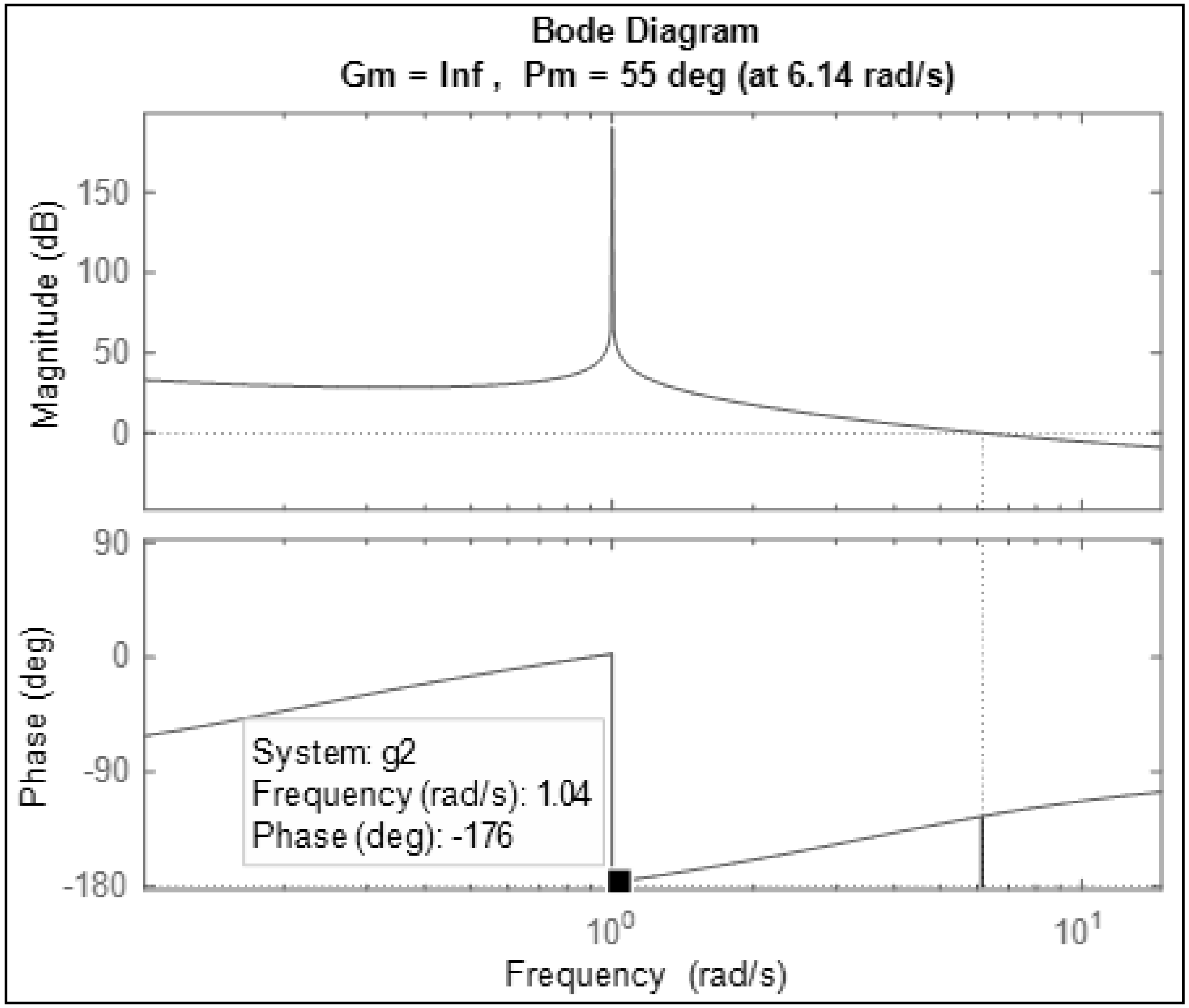
$$\tan^{-1}(4.58b) = 50^o; \quad G'''(s) = \frac{4(5s+1)(0.25s+1)}{s\left(s^2+1\right)}$$

Final Compensated Bode Plot

Final **compensated** bode **plot** is given alongside.

We see that, as **GCO increases** to 6.14, **PM** is more than **required.**

Further, as there is **no PCO**, GM becomes **infinite.**



Summary

PID Controllers in frequency domain are designed for ramp error constant and **phase margin.**

The **1st zero** is placed to achieve the **gain crossover** frequency and the **2nd zero adds** the required phase margin.

Lag-Lead Compensator Concept

Lag – Lead Compensators

Lag – Lead compensators have the **same** action as **PID** controllers and are normally **employed** when plant **type** & relative degree (**n-m**) are to be **preserved**.

However, these require **five** parameters to be **adjusted**, as against **three** in case of **PID**, as shown below.

$$G_c(s) = K_c \left(\frac{s + \frac{1}{T_1}}{s + \frac{\gamma}{T_1}} \right) \left(\frac{s + \frac{1}{T_2}}{s + \frac{1}{\beta T_2}} \right); \quad \gamma = \frac{1}{\alpha}; \quad \beta, \quad \gamma > 1$$

Lag – Lead compensators can be **tuned** better as there are **more** parameters for **adjustment**.

Lag – Lead Design with Root Locus

Case – 1: $\beta \neq \gamma$

$$G_c(s) = K_c \left(\frac{s + \frac{1}{T_1}}{s + \frac{\gamma}{T_1}} \right) \left(\frac{s + \frac{1}{T_2}}{s + \frac{1}{\beta T_2}} \right)$$

Determine dominant closed loop poles.

Determine ϕ_c from G(s) and use it to **determine** T_1 **&** γ (Choice is **non-unique**) and K_c from **|G'| = 1.**

Determine β from K_v.

Choose T_2 such that lag **magnitude is ~1** and its **angle** contribution is between **0^o to -5^o** at dominant poles.

Lag – Lead Design from Root Locus

Case – 2: $\beta = \gamma$

$$G_c(s) = K_c \left(\frac{s + \frac{1}{T_1}}{s + \frac{\beta}{T_1}} \right) \left(\frac{s + \frac{1}{T_2}}{s + \frac{1}{\beta T_2}} \right); \quad \beta > 1$$

Determine closed loop poles from transient specifications.

Determine K_c from error constant specification.

Determine ϕ_c from G(s) and use it, along with **magnitude condition**, to determine T_1 **&** β**.**

Determine T_2 using β such that **lag magnitude** is ~1 and its **angle** contribution is **between** 0^o **to** -5^o**.**

Lag – Lead Design from Bode

Design of **Lag–Lead** compensator in **frequency** domain is a broad **extension** of lag and lead **design strategies.**

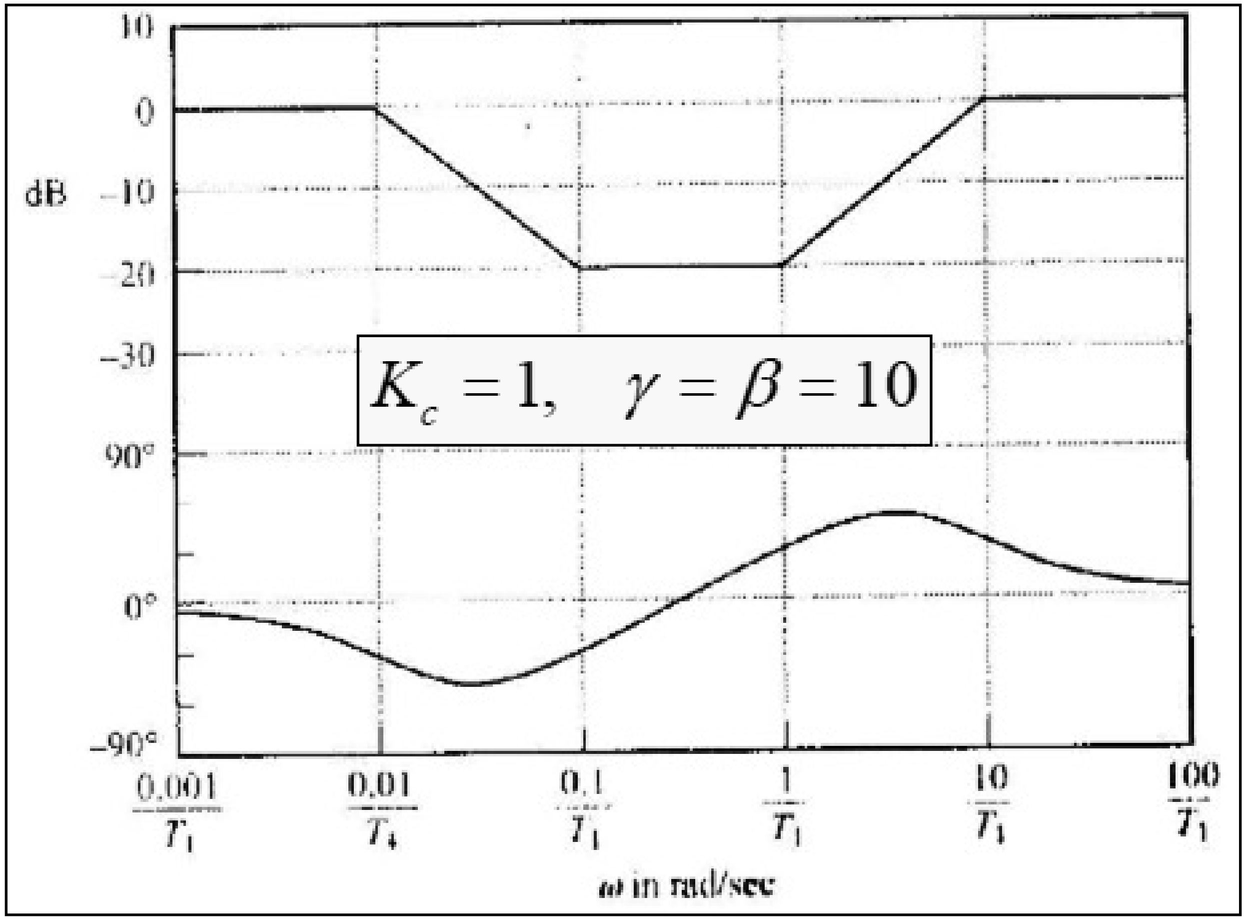
This is possible as **role of lead** part is to increase the **phase margin** and bandwidth, while that of **lag part** is to maintain **low frequency** gain.

In this context, it should be **noted** that it is a good **design philosophy** to keep additional **phase margin** from the compensator **~50^o.**

Lag – Lead Compensator Bode Plot

Given **alongside** is the typical **bode plot** of a **lag-lead** compensator.

We see that it behaves like a **lag compensator** in **low** frequency range & as a **lead compensator** in **high** frequency range.



Lag – Lead Bode Design Steps

Case-2: $\beta = \gamma$

Set gain from K_v requirements.

Locate new **GCO**, around existing **PCO,** and obtain $\mathbf{T_2}$ from lag **corner frequency** (1-decade lower than **GCO**).

Determine β from **phase lead** to be added at **new GCO**, which also **fixes the pole** of lag part.

Determine attenuation required at **GCO** from lead part and **obtain** the lead **corner frequencies.**

Summary

Lag-lead compensator is **counterpart** of PID controller that **preserves** both **type** and relative **degree**.

While design **effort** is significantly **higher**, additional degrees-of-freedom **help** achieve a better **design**.