Gain and Phase Margins

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Definition of Margin

Gain & Phase Margins Concept

Gain & **phase** margins, also called stability **margins**, refer to the **parameter**s that quantitatively **indicate** how far is the **closed** loop from **instability** (i.e. imaginary axis).

Gain Margin (GM) is **defined** as the amount of **gain** that can be **added** to plant, before unity feedback **closed loop** system becomes **unstable**.

Gain & Phase Margins Concept

Similarly, **phase** margin (PM) is the amount of **phase lag** (or negative angle) that **can be added** to the plant before the **closed** loop system becomes **unstable.**

It is clear that these **quantities** also reflect the **stand-off distance** of dominant **closed** loop poles from ‘**jω**’ axis.

Gain Margin Definition

GM is the **reciprocal** of the magnitude at ω_{PCO} and is given as $\mathbf{1/|G(j\omega_{PCO})|}$.

Alternatively, it is negative of **log-magnitude** at ω_{PCO} and is given as, $\mathbf{-20\log_{10}|G(j\omega_{PCO})|}$ in dB units.

It should be noted that **phase** is already $\pm\mathbf{180^o}$ at ω_{PCO}.

Gain margin is considered **positive** if $\mathbf{|G(j\omega_{PCO})|}$ is $\mathbf{< 1}$ (or < 0 dB).

Phase Margin Definition

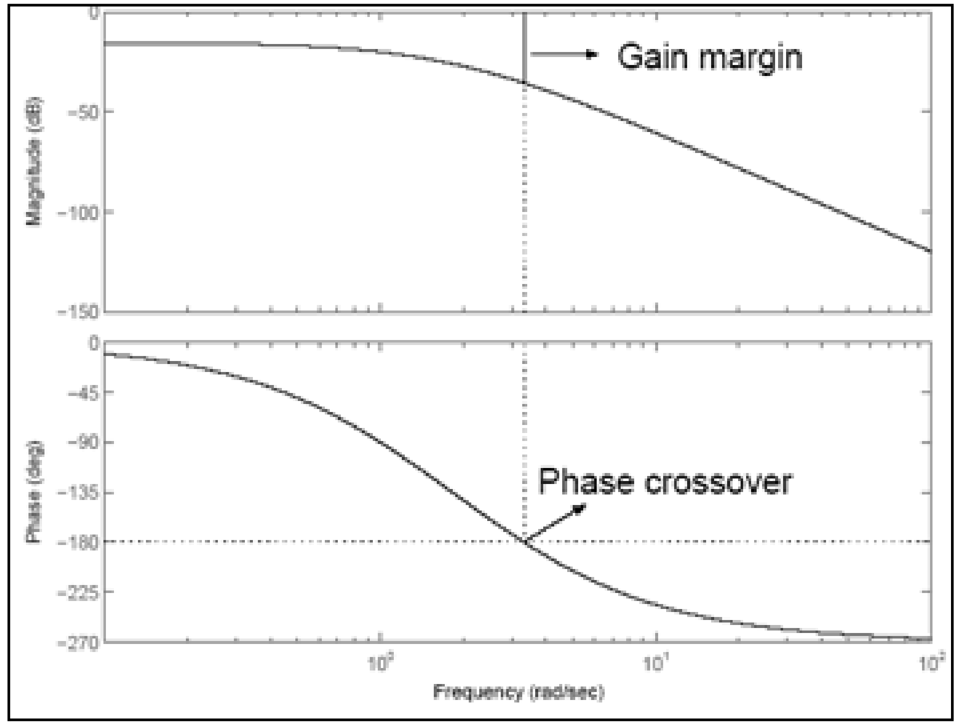
As **negative angle** is measured **clockwise** from positive **real axis**, PM is defined as $\mathbf{180^o + \angle G(j\omega_{GCO})}$.

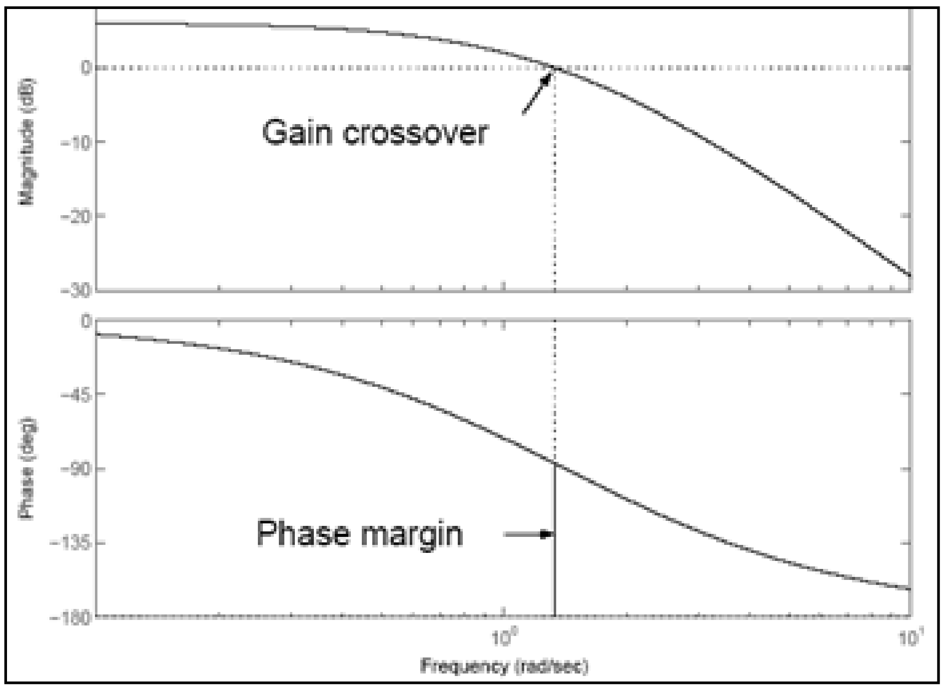
PM is treated as **positive** if negative angle is $\mathbf{< 180^o}$.

In this **context,** it is worth noting that $\mathbf{|G(j\omega)|}$ is already 1.0 at ω_{GCO}.

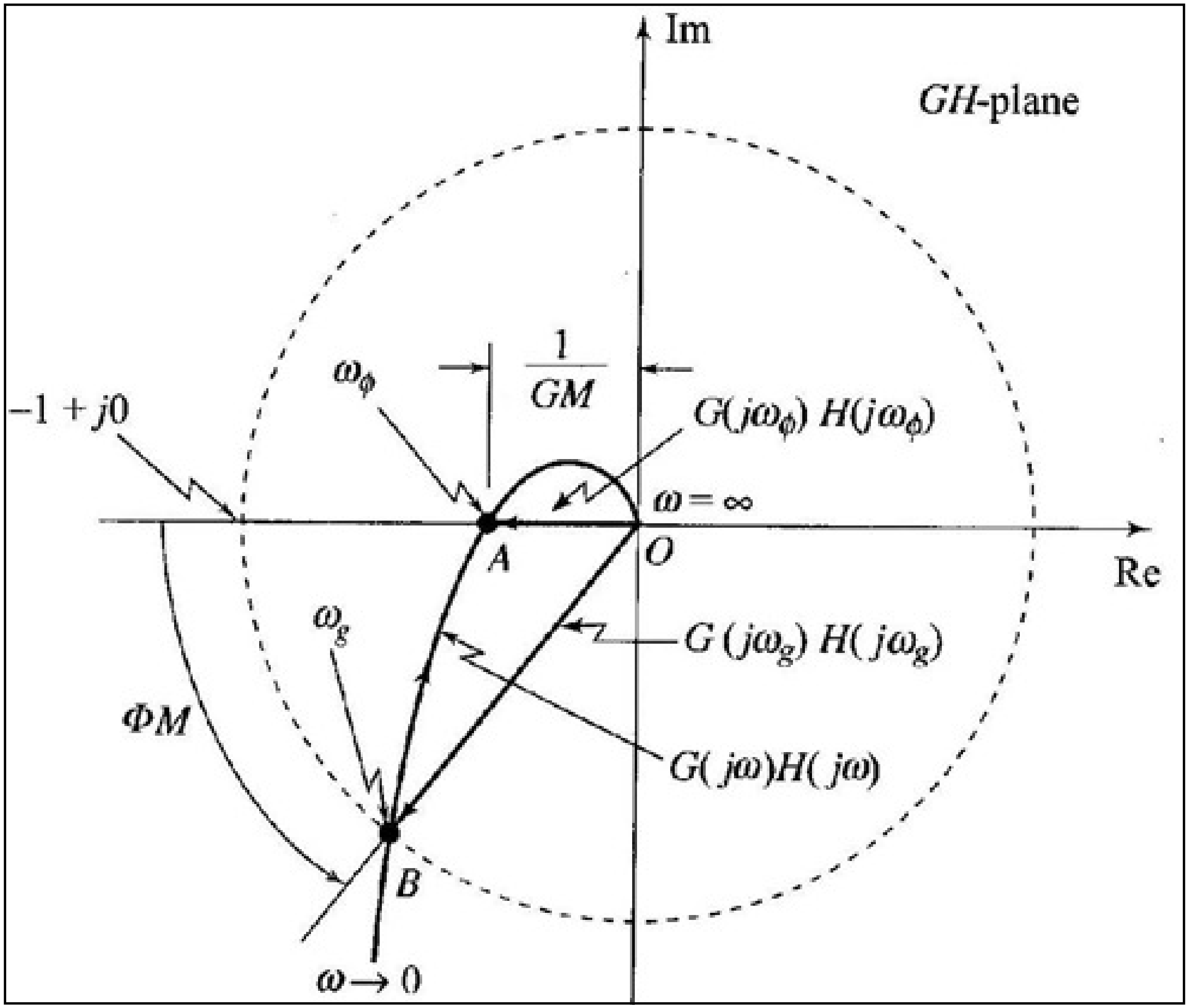
GM & PM from Bode' Plots

GM & **PM** from **bode** plots are as shown in **figures.**





GM & PM from Nyquist Plots



GM/PM Example

Determine GCO, PCO, GM & PM for following plant and **predict** the stability of closed loop **system**. Also, correlate with **closed** loop pole **locations**.

$$G(s) = \frac{10}{s(s+1)(s+5)}$$

$$\left|G(j\omega_{GCO})\right| = 1 \rightarrow \omega^6 + 26\omega^4 + 25\omega^2 - 100 = 0; \quad \omega_{GCO}^2 = 1.506$$

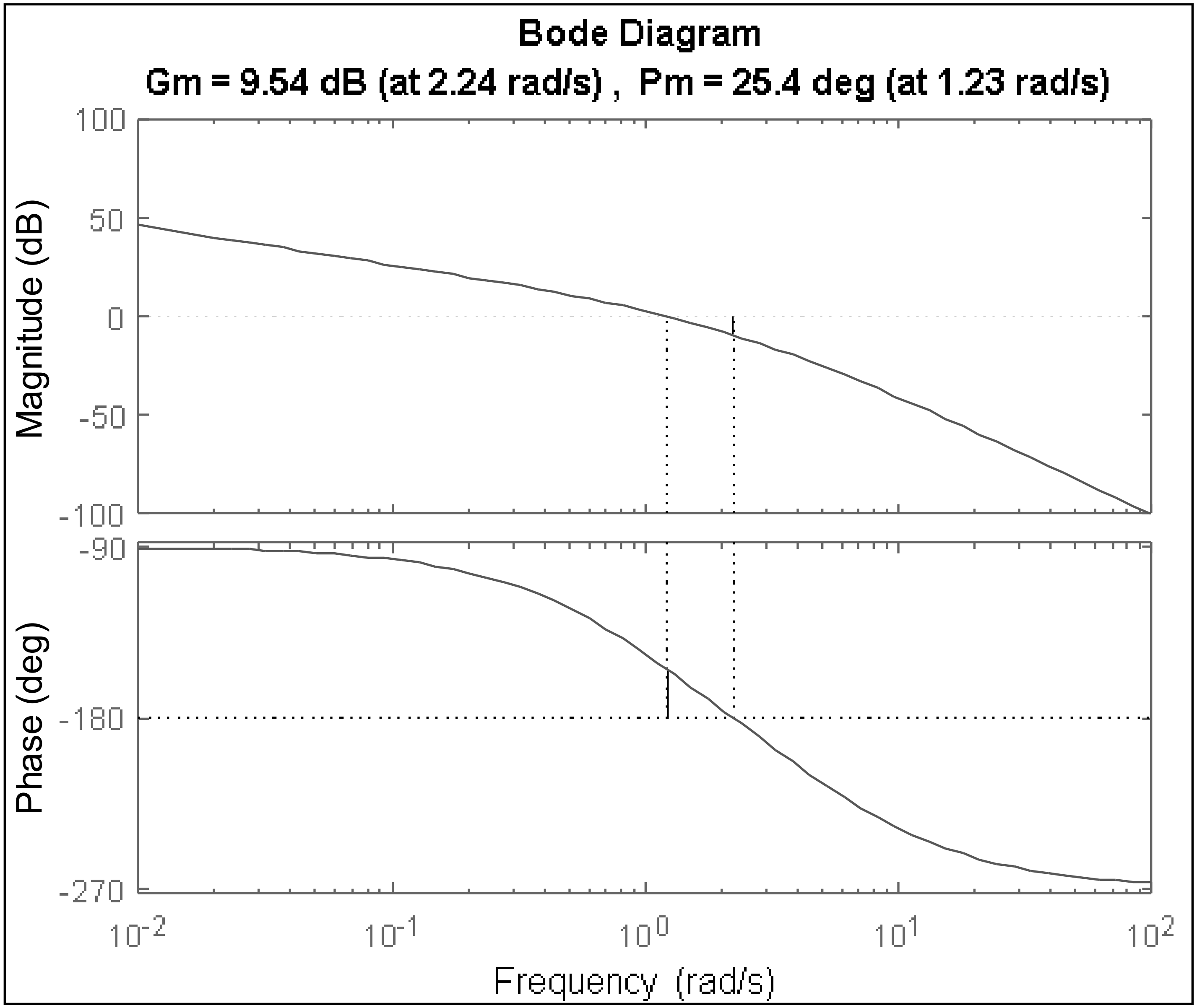
$$\omega_{GCO} = 1.227; \quad \angle G(j\omega_{PCO}) = -180^o \rightarrow \tan\left(\tan^{-1}\omega_{PCO} + \tan^{-1} 0.2\omega_{PCO}\right) = \infty$$

$$\omega_{PCO} = 2.25; \quad GM = \frac{1}{\left|G(j2.25)\right|} = \frac{1}{0.329} = 3.03 \text{ or } 9.63dB$$

$$PM = 180^o + \angle G(j1.227) = 180^o - 90^o - \tan^{-1} 1.227 - \tan^{-1} 0.245 = 25.4^o$$

$$PM, GM > 0 \rightarrow \text{ Stable; } \quad \text{CL Poles: } \quad -5.42, -0.29 \pm j1.33$$

GM & PM Verification



Summary

Gain and Phase margins are useful **quantitative** measures for assessing the relative **stability** of a system in unity **feedback** close loop **configuration.**

Systems with Infinite GM

There are **situations** where the **phase** plot either does **not cross 180^o**, or phase cross over occurs at **$\omega = \infty$.**

In such cases, the **GM** becomes **undefined** and is commonly **interpreted** to be **infinite.**

Implication of such a situation is that **no amount** of increase in **gain** will destabilize the **closed loop** system.

Such a scenario is **beneficial** from tracking point of view, though **transient response** may get adversely affected.

Infinite GM Example

Consider the **plant,** as given below.

$$G(s) = \frac{1}{(s+0.1)(s+1)}$$

The **solutions** for PCO, GCO, GM, PM are **as follows.**

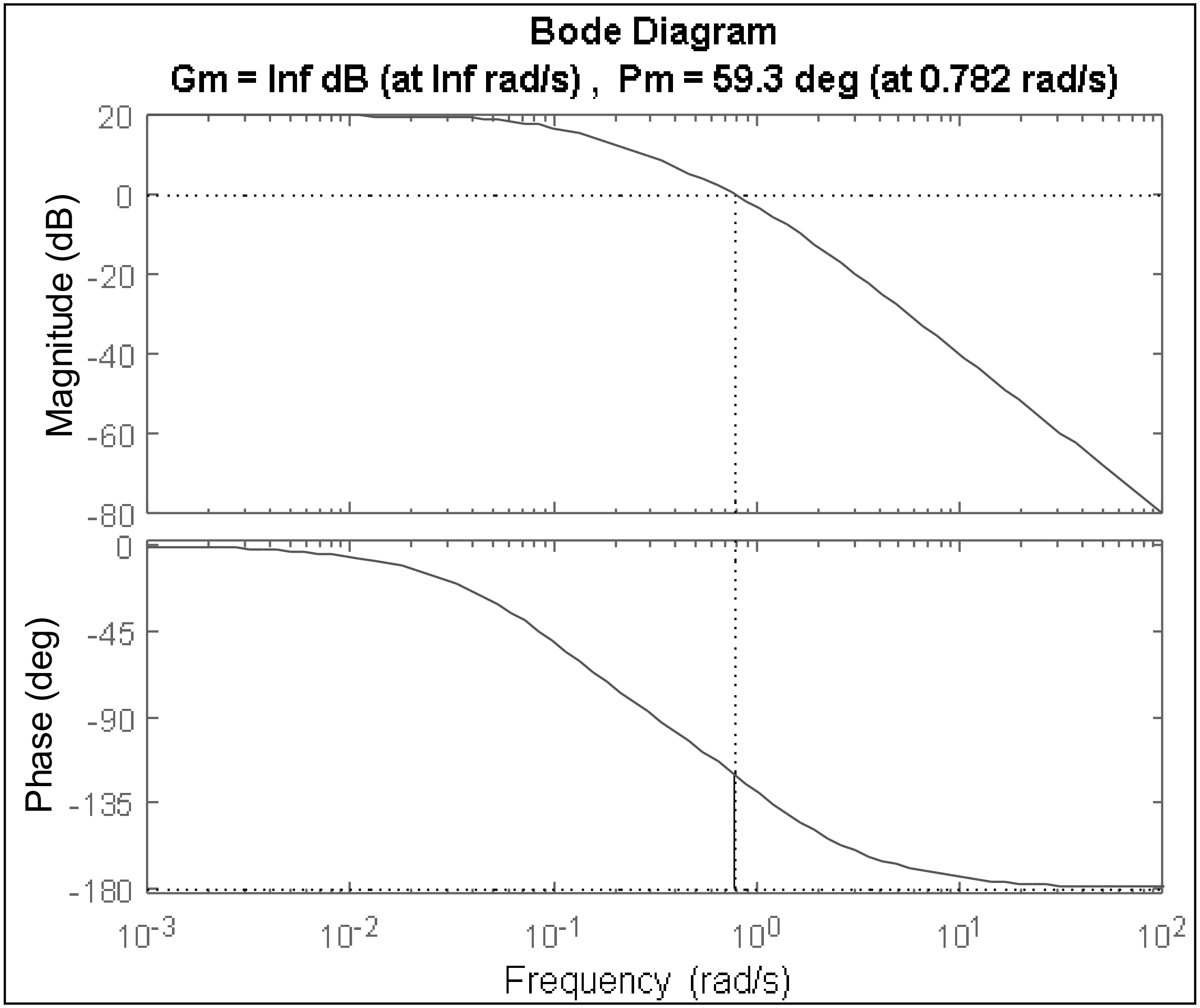
$$\angle G(j\omega_{PCO}) = -180^o \rightarrow \tan^{-1} 10\omega_{PCO} + \tan^{-1}\omega_{PCO} = 180^o$$

$$\frac{11\omega_{PCO}}{1-10\omega_{PCO}^2} = 0 \rightarrow \omega_{PCO} = \pm\infty \rightarrow \omega_{PCO} \text{ is } \infty \rightarrow GM = \infty$$

$$\left|G(j\omega_{GCO})\right| = 1 \rightarrow \omega_{GCO}^4 + 1.01\omega_{GCO}^2 - 0.99 = 0 \rightarrow \omega_{GCO} = 0.781$$

$$PM = 180^o - \tan^{-1} 7.81 - \tan^{-1} 0.781 = 59.28^o$$

Infinite GM Example

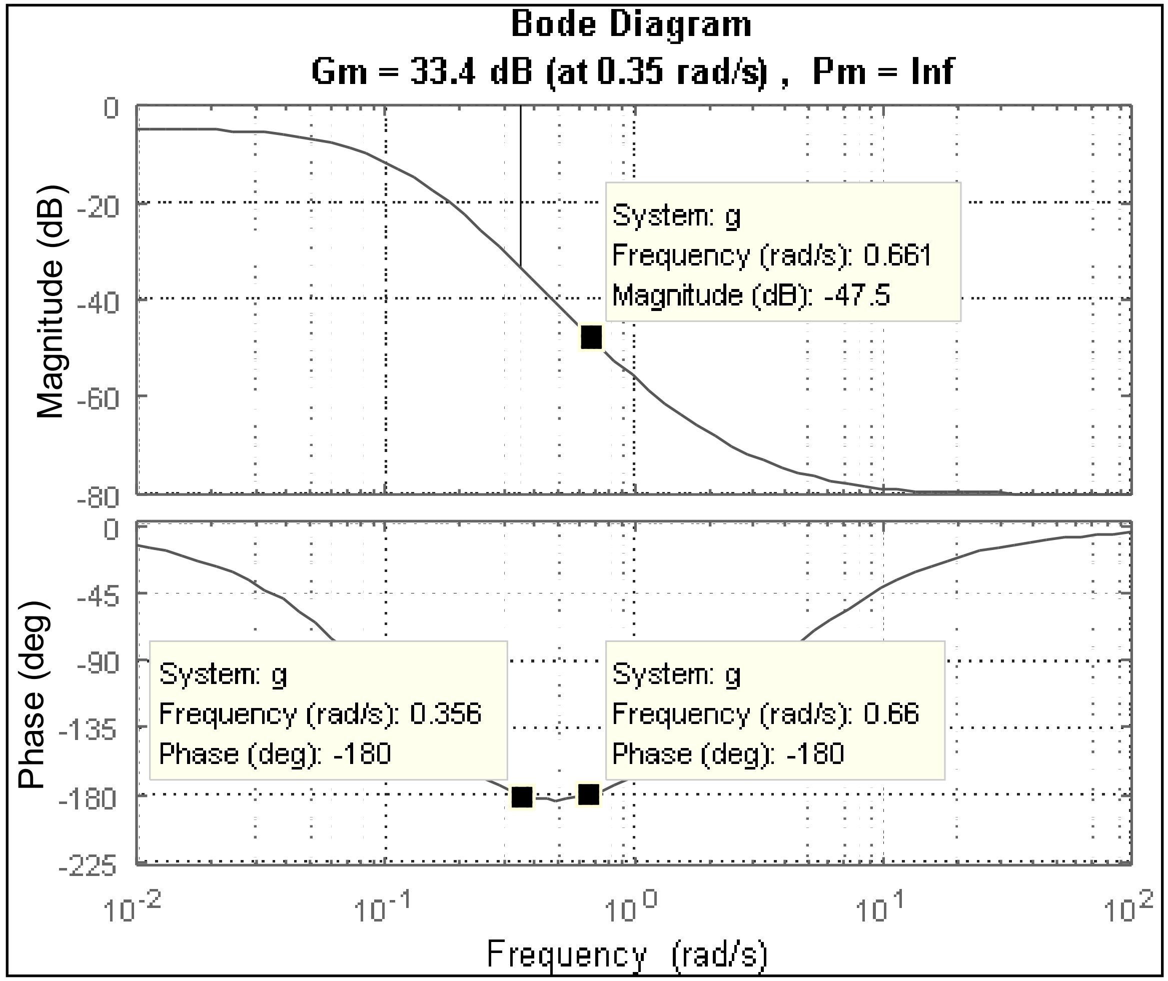


Infinite PM Example

Consider the **plant** as given below, along with its **bode plot**, shown alongside.

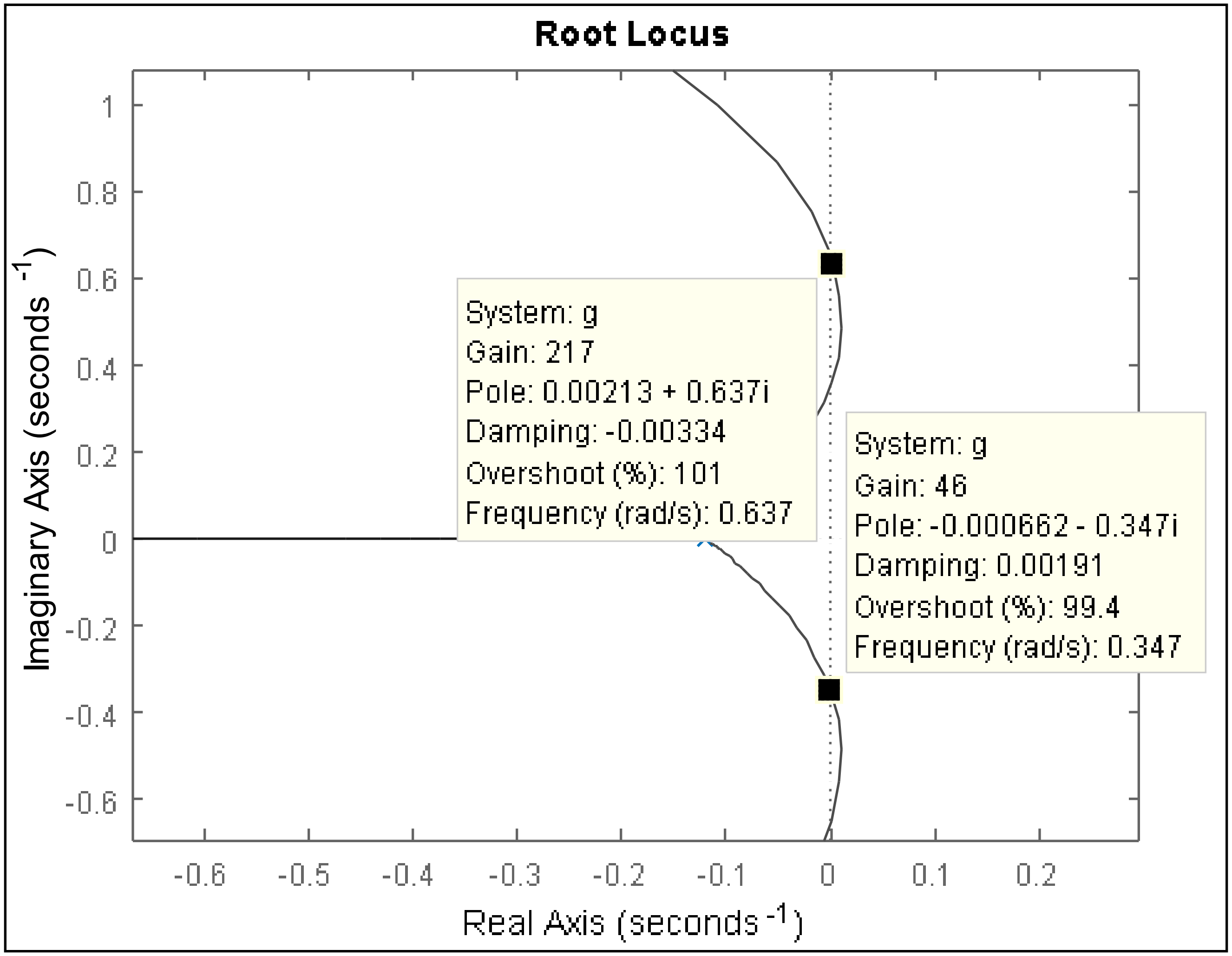
$$G(s) = \frac{0.0001(s+1)(s+2)(s+5)}{(s+0.12)^3}$$

It can be seen that **plant** has no **GCO** and hence, **infinite** phase margin.



Infinite PM Example

Existence of **two PCO,** reflects two **crossings** of 'jω' axis as shown in the **root locus** alongside.



Infinite GM & PM Example

Consider the **plant,** as given below.

$$G(s) = \frac{2(s+1)}{(s+0.5)}$$

The **solutions** for PCO, GCO, GM, PM are **as follows.**

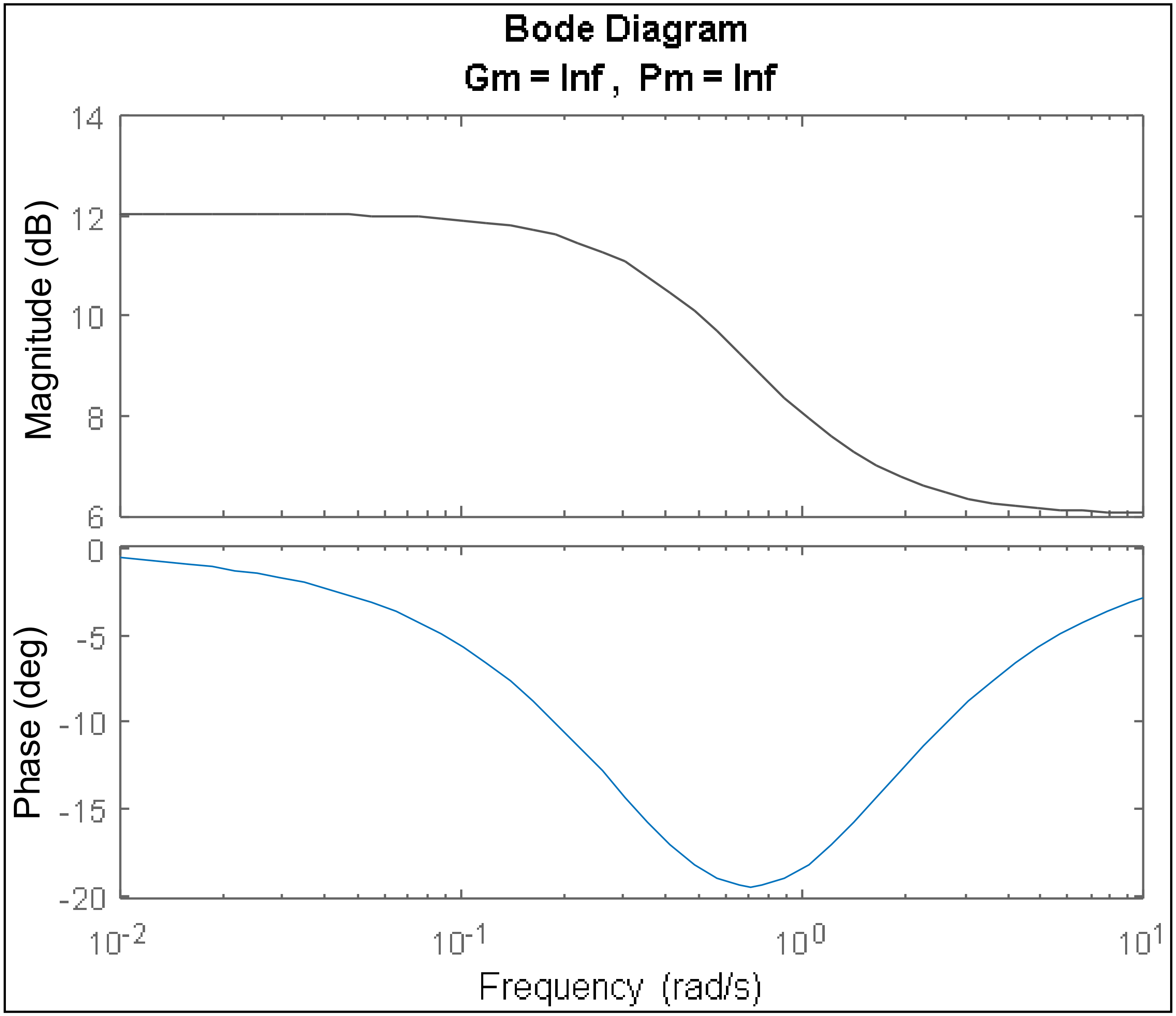
$$\angle G(j\omega_{PCO}) = -180^o = -\tan^{-1} 2\omega_{PCO} + \tan^{-1} \omega_{PCO}$$

$$2\omega_{PCO}^3 - \omega_{PCO} + 1 = 0 \rightarrow \omega_{PCO} \text{ is not a positive real number}$$

$$GM = \infty; \quad \left|G(j\omega_{GCO})\right| = 1 \rightarrow 3\omega_{GCO}^2 + 3.75 = 0$$

$$\omega_{GCO} \text{ is not a positive real number} \rightarrow PM = \infty$$

Infinite GM & PM Example



Non-Minimum Phase GM/PM

We know that **phase** characteristics of **non-minimum** phase systems are significantly **different** and hence it is expected that both **GM & PM** would get **affected.**

As the presence of **RH s-plane** zero adds additional **lag,** it is anticipated that it will **directly** reduce the **PM** and **indirectly** reduce **GM.**

In this context, we **modify** the previous plant to create **non-minimum** phase system, as shown **below.**

$$\boxed{G(s) = \frac{2(s+1)e^{-s}}{(s+0.5)}}$$

Non-minimum Phase GM/PM

We can employ the **1st order** Pade's approximation to get the **rational** transfer function as given below.

$$G(s) = \frac{2(s+1)(1-0.5s)}{(s+0.5)(1+0.5s)}$$

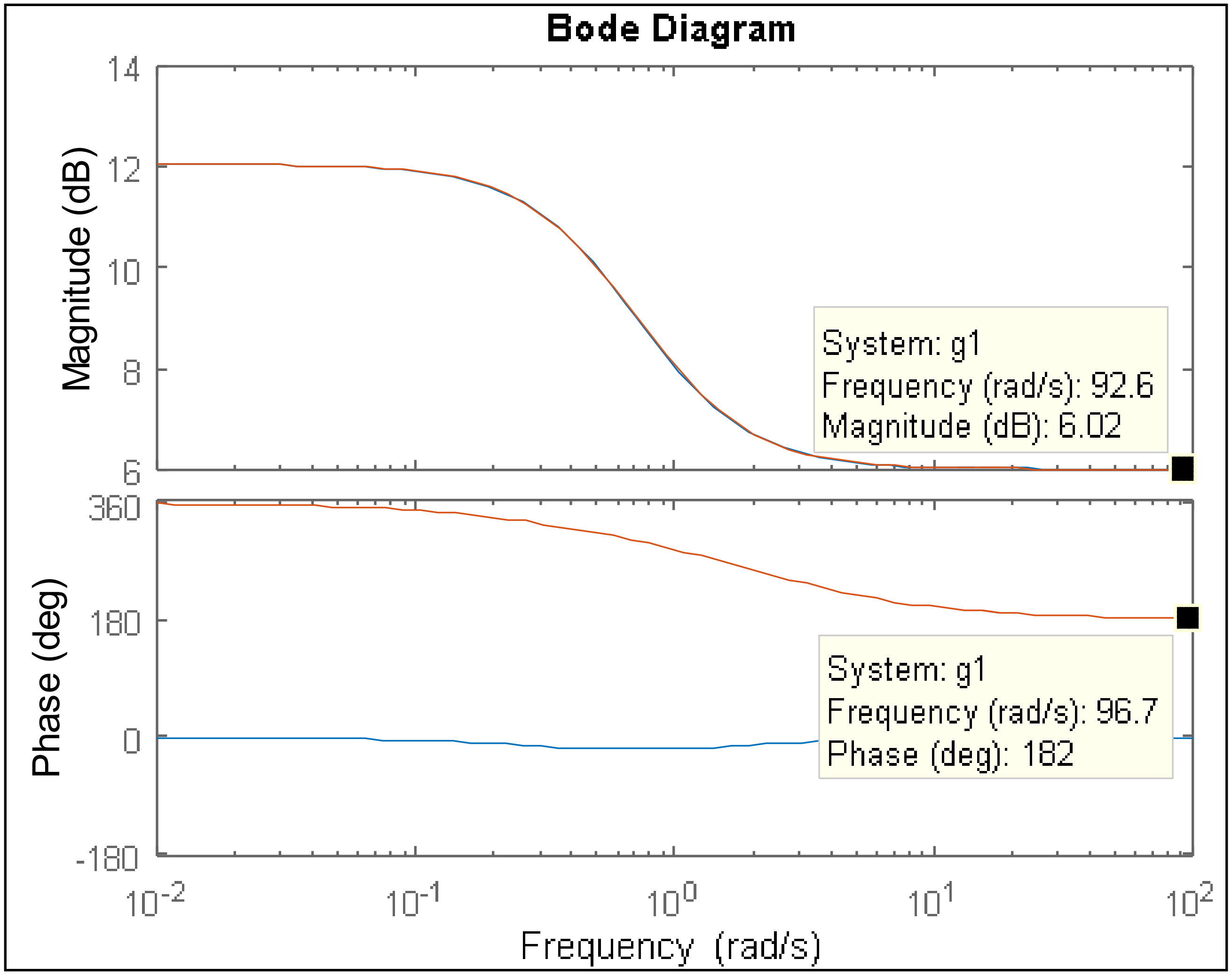
The solution for **GCO, PCO, GM & PM** is as follows.

$$\angle G(j\omega_{PCO}) = -180^o = \tan^{-1}\omega_{PCO} - 180 - \tan^{-1} 2\omega_{PCO}; \quad \frac{-\omega_{PCO}}{1+2\omega_{PCO}^2} = 0^-$$

$$\omega_{PCO} = 0, \infty \rightarrow \omega_{PCO} = 0 \text{ is invalid. } GM = -20\log_{10}\left|G(j\infty)\right| = -6.02dB$$

$$\left|G(j\omega_{GCO})\right| = 1; \quad \omega_{GCO}^2 = -0.083 \rightarrow \text{ No } \omega_{GCO}; \quad PM_{\max} = 0$$

Non-minimum Phase GM/PM



We find that **closed loop poles** of the above **plant** are; **-0.922, 5.42.**

Summary

Infinite gain and phase **margins** provide **greater design freedom.**

Non-minimum phase behaviour has a **significant impact** on the stability **margins.**