Observability

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Abstract

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Observability

Observability is the tool we use to investigate the internal workings of a system. It lets us use what we know about the input $u(t)$ and the output $y(t)$ to **observe** the state of the system $x(t)$.

To understand this concept let's start off with the basic state-space equations describing a system: $x' = Ax + Bu$ $y = Cx + Du$ If we plug the general solution of the state variable, $x(t)$, into the equation for $y(t)$, we'd find the following familiar time-domain equation:

$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t) \tag{1}$$

Without loss of generality, we can assume zero input; this will significantly clarify the following discussion. This assumption can be easily justified. Based on our initial assumption above, the last two terms on the right-hand side of time-domain equation (Equation 1) are known (because we know $u(t)$). We could simply replace these two terms with some function of t. We'll group them together into the variable $y_0(t)$. By moving $y_0(t)$ to the left-hand side, we see that we can again group $y(t) - y_0(t)$ into another replacement function of t, $\overline{y}\ (t)$. This result has the same effect as assuming zero input. $\overline{y}\ (t) = y(t) - y_0(t) = Ce^{At}x(0)$ Given the discussion in the above paragraph, we can now start our examination of observability based on the following formula:

$$y(t) = Ce^{At}x(0) \tag{2}$$

The idea behind observability is to find the state of the system based upon its output. We will accomplish this by first finding the initial conditions of the state based upon the system's output. The state equation solution can then use this information to determine the state variable $x(t)$.

base formula (Equation 2) seems to tell us that as long as we known enough about $y(t)$ we should be able to find $x(0)$. The first question to answer is how much is enough? Since the initial condition of the state $x(0)$ is actually a vector of n elements, we have n unknowns and therefore need n equations to solve the set. Remember that we have complete knowledge of the output $y(t)$. So, to generate these n equations, we can simply take $n-1$ derivatives of base formula (Equation 2). Taking these derivatives is relatively straightforward. On the right-hand side, the derivative operator will only act on the matrix exponential term. Each derivative of it will produce a multiplicative term of A. Then, as we're dealing with these derivatives of $y(t)$ at $t=0$, all of the exponential terms will go to unity ($e^{A0}=1$).

$$\begin{aligned} y(0) &= Cx(0) \\ \frac{d}{dt}y(0) &= CAx(0) \\ \frac{d^2}{dt^2}y(0) &= CA^2x(0) \\ &\vdots \\ \frac{d^{n-1}}{dt^{n-1}}y(0) &= CA^{n-1}x(0) \end{aligned}$$

This can be re-expressed in matrix notation.

$$\begin{pmatrix} y(0) \\ \frac{d}{dt}y(0) \\ \frac{d^2}{dt^2}y(0) \\ \vdots \\ \frac{d^{n-1}}{dt^{n-1}}y(0) \end{pmatrix} = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix} x(0)$$

The first term on the right-hand side is known as the observability matrix, $\sigma(C,A)$:

$$\sigma(C,A) = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix} \quad (3)$$

We call the system **completely observable** if the rank of the observability matrix equals n. This guarantees that we'll have enough independent equations to solve for the n components of the state $x(t)$.

Whereas for controllability we talked about the system's controllable space, for observability we will talk about a system's *un*observable space, X^{unobs}. The unobservable space is found by taking the kernel of the observability matrix. This makes sense because when you multiply a vector in the kernel of the observability matrix by the observability matrix, the result will be 0. The problem is that when we get a zero result for $y(t)$, we cannot say with certainty whether the zero result was caused by $x(t)$ itself being zero or by $x(t)$ being a vector in the nullspace. As we cannot give a definite answer in this case, all of these vectors are said to be unobservable.

One cool thing to note is that the observability and controllability matrices are intimately related:

$$(\sigma(C,A))^T = C\left(A^T, C^T\right) \quad (4)$$