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Fundamentals of Digital Control

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1 Introduction

All the modern state-of-the-art control systems implement digital control. With the advances in very large scale integration (VLSI) technology, microprocessors /computers along with peripherals are becoming inexpensive and hence popular for implementation of digital control. Advantages of digital control can be listed as

- Both linear and nonlinear control laws can be implemented
- There is a lot of flexibility in changing the control law. One needs just to change the program to change control law as against expensive electronics replacement in the case of analog control implementation.
- Performance enhancement because of use of implementation of higher ended control

Among the disadvantages include higher cost (which is coming down over the years) and discretization problems. Thus it is imperative for a mechatronic engineer to know the fundamentals of the digital control.

When one talks about digital control, the first and foremost issue to be addressed is sampling. In the basic configuration of digital control implementation depicted in Figure 1, the sampling takes place at the A/D converter and at the D/A converter. If we observe the control input signal given to the system from D/A converter, the signal will usually be constant between sampling intervals. Because of this change, the response of system changes drastically from that of equivalent analog system, the new parameter in the response being the sampling time T. How one should go about analyzing such systems? How to choose the sampling time T for optimum performance? are the questions addressed by the development of the digital control theory.

These notes first explain the fundamental Shannon sampling theorem, reconstruction and then present other fundamental aspects of digital control theory. First the effect of sampling on system dynamics is presented. Next the Z-transforms are introduced and their importance is briefly mentioned. Thus all the necessary fundamentals to start analyzing a digital control system are addressed.

2 Shannon Sampling Theorem

Theorem 2.1 *"A continuous time signal with a Fourier transform that is zero outside the interval $(-\omega_0, \omega_0)$ is given uniquely by it's values in equidistant points if the sampling frequency is higher than $2\omega_0$. The continuous time signal can be constructed from the sampled signal by the interpolation formula*

$$F(t) = \sum_{k=-\infty}^{\infty} F(kh) \frac{\sin \omega_s (t - kh)/2}{\omega_s (t - kh)/2} = \sum_{k=-\infty}^{\infty} F(kh) \sin \omega_s (t - kh)/2 \qquad (1)$$

ω_s in the sampling frequency in rad/sec"

Figure 1: Basic block diagram of a digital control system

Thus the theorem gives effect of sampling of the signal and also provides an ideal way to reconstruct the signal from the sampled data. Next we will give the proof of the theorem which is based on Fourier transform and its inverse.

Proof We know by the definition of Fourier transform that,

$$F(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t)dt \qquad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} f(\omega)d\omega$$

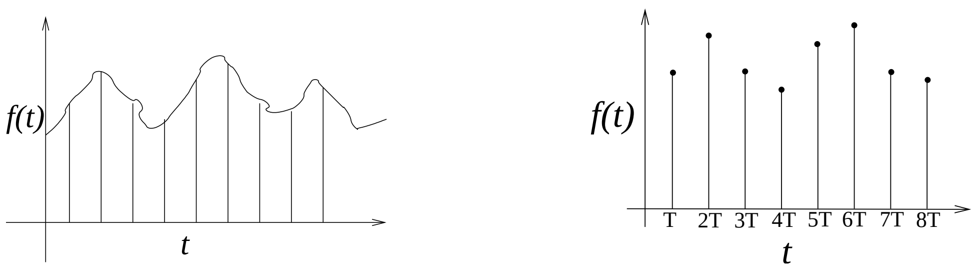

Figure 2: Sampling an analog signal with sampling time T

The sampled signal can be represented in terms of direc δ function as

$$f_s(t) = f(kT)\underbrace{\sum_{k=-\infty}^{\infty}\delta(t-kT)}_{s(t)} \qquad s(t) = \sum_{k=-\infty}^{\infty}\delta(t-kT)$$

$$s(\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} f(\omega - k\omega_s) \qquad \omega_s = \frac{2\pi}{T}$$

Using convolution integral we have

$$
\begin{aligned}
F_s(\omega) &= \frac{1}{2\pi} F(\omega) \star S(\omega) && (2)\\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega - y) S(y) dy && (3)\\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega - y) \times \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} S(\omega - k\omega_s) && (4)\\
&= \frac{1}{T} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} F(\omega - y) S(\omega - k\omega_s) && (5)\\
F_s(\omega) &= \frac{1}{T} \sum_{k=-\infty}^{\infty} F(\omega - k\omega_s) && (6)\\
& && (7)
\end{aligned}
$$

where $\star$ refers to convolution. Thus we see that the frequency domain representation of sampled signal is replica of the frequency content of the original signal mirrored at uniform intervals of ω_s. This fact is demonstrated for two cases in Figure 3

Figure 3: Demonstration of effect of sampling in the frequency domain

Shannon Reconstruction

The reconstructed signal in time domain is given by using inverse Fourier transform,

$$
\begin{aligned}
f_r(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} F_r(\omega)\\
F_r(\omega) &= \frac{T}{T} F(\omega) = F(\omega)
\end{aligned}
$$

Where $F_r(\omega)$ refers to frequency domain representation of reconstructed signal. Using ideal reconstruction filter as depicted in Figure 4 we get $F_r(\omega) = F(\omega)$. Notice that ideal filter has constant magnitude T.

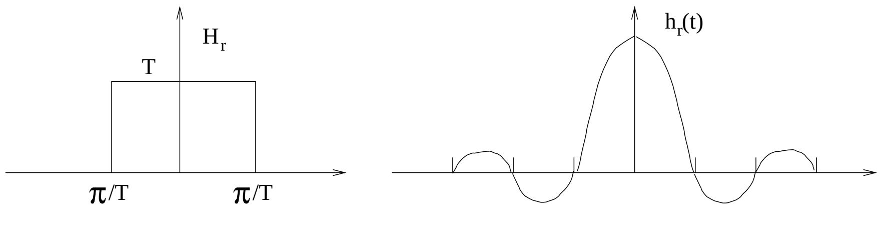

Figure 4: Ideal reconstruction filter a) in frequency domain b) in time domain

$$
\begin{aligned}
f_r(t) &= \frac{1}{2\pi}\int_{-\frac{\omega_s}{2}}^{\frac{\omega_s}{2}} e^{i\omega t}F(\omega)d\omega \\
F_s(\omega) &= \int_{-\infty}^{\infty} e^{-j\omega t}\sum_{k=-\infty}^{\infty} f(kT)\delta(t-kT)dt \\
&= \sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j\omega t}f(kT)\delta(t-kT)dt \\
&= \sum_{k=-\infty}^{\infty} f(kT)\int_{-\infty}^{\infty} e^{-j\omega t}\delta(t-kT)dt \\
F_s(\omega) &= \sum_{k=-\infty}^{\infty} f(kT)e^{-j\omega kt} \qquad \rightarrow \text{origin of DTFT \& FFT} \\
F(\omega) &= TF_s(\omega)\text{from equation (7)}
\end{aligned}
$$

$$
\begin{aligned}
f_r(t) &= \frac{T}{2\pi}\int_{-\frac{\omega_s}{2}}^{\frac{\omega_s}{2}} e^{i\omega t}F_s(\omega)d\omega \\
&= \frac{T}{2\pi}\int_{-\frac{\omega_s}{2}}^{\frac{\omega_s}{2}} e^{i\omega t}\sum_{k=-\infty}^{\infty} f(kT)e^{-j\omega kt}d\omega \\
&= \sum_{k=-\infty}^{\infty}\int_{-\frac{\omega_s}{2}}^{\frac{\omega_s}{2}} \frac{T}{2\pi}e^{i\omega t}f(kT)e^{-j\omega kt}d\omega \\
&= \sum_{k=-\infty}^{\infty}\int_{-\frac{\omega_s}{2}}^{\frac{\omega_s}{2}} \frac{e^{i\omega(t-kt)}}{\frac{2\pi}{T}}f(kT)d\omega \\
&= \sum_{k=-\infty}^{\infty}\frac{f(kT)}{\frac{2\pi}{T}}\int_{-\frac{\omega_s}{2}}^{\frac{\omega_s}{2}} e^{i\omega(t-kt)}d\omega \\
&= \sum_{k=-\infty}^{\infty}\frac{f(kT)}{\frac{2\pi}{T}}\left[\frac{e^{i\omega(t-kt)}}{i(t-kT)}\right]_{-\frac{\omega_s}{2}}^{\frac{\omega_s}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=-\infty}^{\infty} \frac{f(kT)}{\frac{2\pi}{T}} \left[\frac{e^{i\frac{\omega_s}{2}(t-kt)} - e^{-i\frac{\omega_s}{2}(t-kt)}}{2i(t-kT)} \right] \\
&= \sum_{k=-\infty}^{\infty} f(kT) \frac{\sin\frac{\omega_s(t-k)}{2}}{\pi(t-kT)} \\
&= \sum_{k=-\infty}^{\infty} f(kT) \frac{\sin\frac{\pi(t-k)}{T}}{\pi(t-kT)/T}
\end{aligned}
$$

Another perspective of Shanon reconstruction

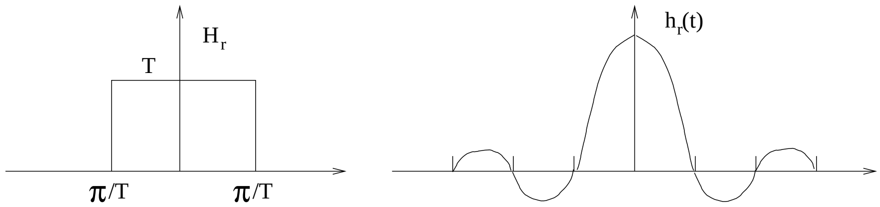

Figure 5: Ideal reconstruction filter in frequency domain and in time domain

It can be shown that

$$h_r = \frac{\sin[\pi t/T]}{\pi t/T}$$

We know that the sampled signal $f_s(t)$ in time domain is given by $\mathrm{f}_s(t) = \sum_{k=-\infty}^{\infty} f(kT)\delta(t-kT)$
The reconstructed signal in time domain will be given by $\mathrm{f}_r(t) = \sum_{k=-\infty}^{\infty} f(kT)h_r(t-kT)$
where $h_r(t)$ is impulse response of the ideal filter designed for Shannon Reconstruction. Using this $f_r(t)$ Figure 6 shows the ideal reconstructed signal

Questions to ponder 1. What other constructions are possible
2. How they will be represented in frequency domain & time domain ?.

3 Zero order hold (ZOH) and other reconstructions

Since ideal Shannon reconstruction requires the future values of all samples in time domain (see Figure 6) for most of the real-life control systems, other types of reconstruction are used. One of the simplest one among them is zero order hold reconstruction. It is very easy to implement in digital circuit using a circuit for latch. Thus the sample value will be kept constant from one sampling instant to the other. Thus zero order hold equation in time domain is

$$
\begin{aligned}
f_r &= \sum f(kT)h_r(t-kT)^{****} \\
f(t) &= f(t_k) \qquad t_k \le t \le t_{k+1}
\end{aligned}
$$

Figure 6: Ideal Shannon reconstruction of original signal from samples

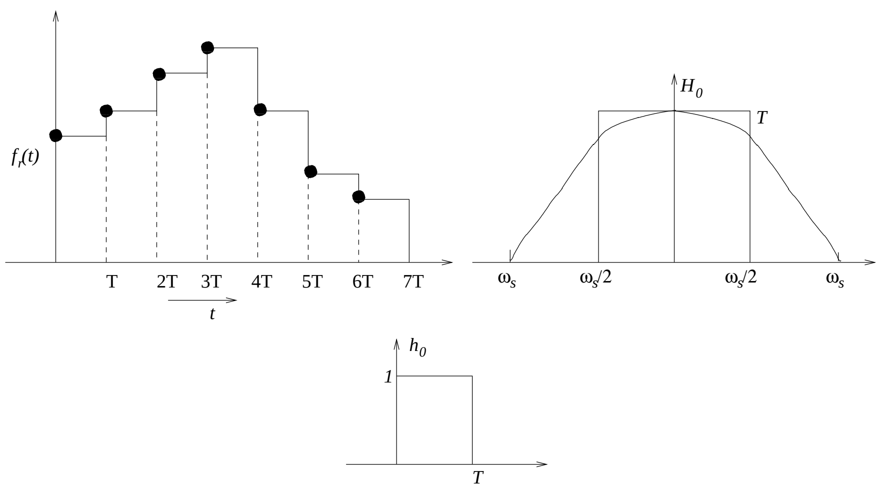

Figure 7: Zero order hold of a sampled signal reconstruction a) time domain b) frequency domain comparison with ideal reconstruction

Q: What happens in the frequency domain?.

To answer this question, we have to see what is frequency domain representation of zero order hold reconstruction function $h(t)$

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} f(n)h_0(t-nT) \qquad h_0 = \text{Impulse response of the ZOH} \\ H_0(\omega) &= \frac{\sin[\omega t/2]}{\omega/2} e^{-j\omega T/2} \end{aligned}$$

$$
\begin{aligned}
|H_0(\omega)| &= \frac{\sin[\omega t/2]}{\omega/2} \\
H_0(\omega) &= \int_{-\infty}^{\infty} e^{-j\omega t} h_0(t) dt \\
&= \int_{T}^{0} e^{-j\omega t} \cdot 1 \quad dt \\
&= \frac{\left[e^{-j\omega t}\right]_0^T}{-j\omega} \\
&= \frac{2\left[e^{-j\omega T/2} - e^{j\omega T/2}\right]}{2j\omega} \times e^{-j\omega T/2} \\
H_0(\omega) &= \frac{2\sin[\omega T/2]}{\omega} e^{-j\omega T/2} \\
|H_0(\omega)| &= \frac{2\sin[\omega T/2]}{\omega} = \frac{\left(2\frac{T}{2}\right)\sin[\omega T/2]}{\omega T/2}
\end{aligned}
$$

Q: How is it possible to ideally reconstruct the signal when ZOH is applied? hint: filter characteristics.
Ans.: modify the frequency domain response of ZOH by multiplying with appropriate function so that the response is constant=T.

$$F(t) = f(t_k) + \frac{t \cdot t_k}{T}\left[f(t_k) - f(t_k - f(t_{k+1})\right] \qquad t_k \leq t \leq t_{k+1}$$

First Order Hold:

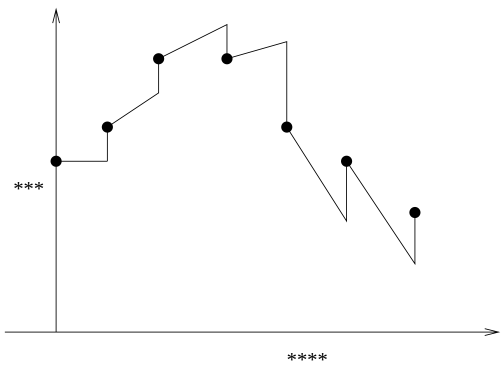

Figure 8: First order hold: method 1: causal reconstruction

First Order Hold: **Another way**

$$f(t) = \sum_{k=-\infty}^{\infty} f(kT) h_1(t - kT)$$

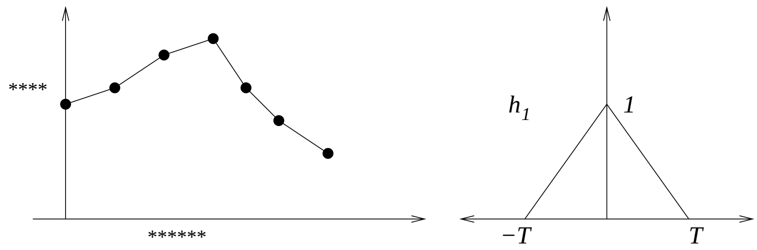

Figure 9: First order hold: method 2: non causal reconstruction

$$
\begin{aligned}
H_1 &= 1 \\
H_1 &= \left(\frac{-1}{T}\right)t + 1 \qquad 0 \le t < T \\
&= \left(\frac{1}{T}\right)t + 1 \qquad -T \le t < 0 \\
H_1(\omega) &= \int_{-\infty}^{\infty} e^{-j\omega t} h_1(t)dt
\end{aligned}
$$

Mention additional references - Books

Sampling rate reduction by integer factor (or downsampling)

Necessity:

$\hookrightarrow$ Analog filters with sharp cut off are costly

$\hookrightarrow$ relative case of implementation of sharp cutoff digital filters

$$
\begin{aligned}
X_s(\omega) &= \frac{1}{T}\sum X(\omega - k\omega_s) \\
X_s(\omega) &= \frac{1}{MT}\sum X(\omega - k\frac{2\pi}{MT})
\end{aligned}
$$

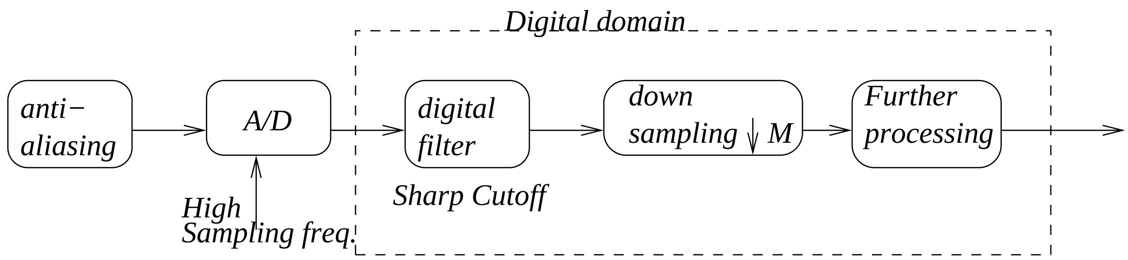

Figure 10: Scheme for reducing the high performance constraint on prefilter (anti-aliasing filter) by using sharp cutoff digital filter and downsampling

Practical Problem Quantisation:

Because of representation of sampled signal as a digital no. - 8 bit data, 3 bit data, 16 bit data, there will be error between the sampled output & it's representation. This error is termed as quantization error. The number representation (hence A/D converter) needs to be more accurate to reduce this error. The more are the number of bits the lesser will be the quantization error.

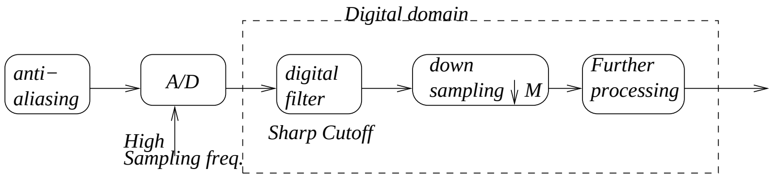

Figure 11: Quantization error

⋆ **Mathematical Analysis**

Quantizer: "It is nonlinear system whose purpose is to transform the input sample into one of a finite set of prescribed values."

$$\hat{x}\,[n] = Q(x[n])$$

Analysis of quantization errors:

$$e[n] = \hat{x}[n] - x[n] \qquad -\frac{\Delta}{2} < e[n] < \frac{\Delta}{2} \qquad \text{with in range of values}$$

- e[n] not known in most of the cases
- problem in modeling.
- Random variable/process type of modeling used in the literature.

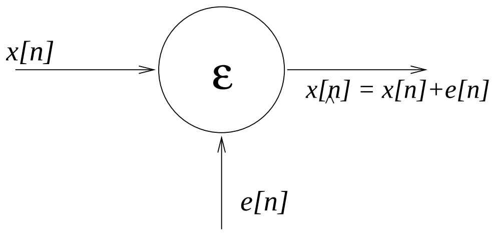

Figure 12: Typical way of introducing quantization error as disturbance to system

Assumptions :-
- e[n] - Stationary random process
- e[n] & x[n] Uncorrelated
- e[n] whole noise

4 Digitization of analog system

Question: What if i am holding u(t) to be constant over sampling instant?
How do i analyze such a system?.

Standard state-space model continuous domain

$$\begin{aligned}\dot{x} &= Ax + Bu(t)\\ \dot{y} &= Cx + Du(t)\end{aligned}$$

Answer

The solution to the standard state space model mentioned above is given by,
x(t) = e$^{A(t-t_k)}x(t_k) + \int_{t_k}^{t} e^{A(t-V)}Bu(V)dV$
Now we know that for digital control system with zero order hold, for $t_k \leq t < t_{k+1}$,
$u(t) = constant = u(t_k)$ hence, we have x

$$\begin{aligned}(t_{k+1}) &= e^{A(t_{k+1}-t_k)}x(t_k) + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-V)}dVBu(t_k)\\ x(t_{k+1}) &= \Phi(t_{k+1}-t_k)x(t_k) + \Gamma(t_{k+1},t_k)u(t_k) \quad where,\\ \Phi &= e^{A(t_{k+1}-t_k)} = e^{AT}\\ \Gamma &= \int_0^{t_{k+1}-t_k} e^{AV}dVB\end{aligned}$$

Hence digital domain system or sampled system,

$$\left.\begin{aligned}x(t_{k+1}) = \Phi x(t_k) + \Gamma u(t_k)\\ y(t_k) = Cx(t_k) + Du(t_k)\end{aligned}\right\}$$ *** valid only for Zero order hold systems (for other holds ?) think

Computation of Φ & Γ:

Cayley-Hamilton Theorem for matrix:
Matrix 'A' satisfies it's own characteristic equation
So

$$\lambda^n + \lambda^{n-1} + \lambda^{n-2} + \cdots\cdots + a_n = 0 \qquad \rightarrow \text{Characteristic Equation}$$

$$\text{Hence,}\quad A^n + a_1A^{n-1} + a_2A^{n-2} + \cdots\cdots + a_nI = 0$$

To evaluate,

$$\begin{aligned}e^{AT} &= C_1I + C_2A + C_3A^2 + \cdots\cdots + C_nA^{n-1}\\ e^{\lambda T} &= C_1 + C_2\lambda + C_3\lambda^2 + \cdots\cdots + C_n\lambda^{n-1} \qquad \text{for all } .\lambda\end{aligned}$$

where λ denote eigenvalues of A. Thus, in all there will be n equations one for each eigen value and there are n unknowns C_i so one can get a solution and hence e^{AT}
Example
1:-

$$\frac{dx}{dt} = \alpha x + \beta a \qquad \text{single DOF}$$

$$\begin{aligned}
\Phi &= e^{\alpha t} \\
\Gamma &= \int_0^T e^{\alpha V} dV \beta \\
&= \beta \int_0^T e^{\alpha V} dV = \beta \left[\frac{e^{\alpha V}}{\alpha}\right]_0^T \\
&= \beta \left[\frac{e^{\alpha V}}{\alpha}\right]_0^T = \frac{\beta}{\alpha}(e^{\alpha T} - 1) \\
x(t_{k+1}) &= e^{\alpha T} x(t_k) + \frac{\beta}{\alpha}(e^{\alpha T} - 1) u(t_k)
\end{aligned}$$

2:- Double Integrator

$$\begin{aligned}
\dot{x} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x \\
\Phi &= e^{AT} = I + AT + \frac{A^2T^2}{2} + \cdots \qquad A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
\Phi &= 1 + AT = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & T \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \\
\Gamma &= \int_0^T \begin{bmatrix} 1 & V \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} dV \\
\Gamma &= \int_0^T \begin{bmatrix} V \\ 1 \end{bmatrix} dV = \begin{bmatrix} T^2/2 \\ T \end{bmatrix}
\end{aligned}$$

2:-

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 3u$$

$$\begin{aligned}
y_1 &= y \qquad \dot{y_1} = \dot{y} = y_2 \\
\dot{y_1} &= y_2 \\
\dot{y_2} &= -3y_2 - 2y_1 + 3u \qquad \text{u=constant} \Longrightarrow \dot{u} = 0 \\
\dot{y} &= Ay + Bu \\
\begin{bmatrix} \dot{y_1} \\ \dot{y_2} \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} u
\end{aligned}$$

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \qquad e^{AT} = c_1 I + c_2 AT$$

Pulse response function

From analysis done so far we know that for zero order hold the digital system response is given by

$$\begin{aligned} x(k+1) &= \Phi(T)x(k) + \Gamma u(k) \\ y(k) &= Cx(k) \end{aligned}$$

Assuming no D term in the output equation for y. Now

$$\begin{aligned} x(k_0+1) &= \Phi(T)x(k_0) + \Gamma u(k_0) \\ x(k_0+2) &= \Phi(T)[\Phi(T)x(k_0) + \Gamma u(k_0)] + \Gamma u(k_0+1) \\ x(k_0+2) &= \Phi^2 x(k_0) + \Phi\Gamma u(k_0) + \Gamma u(k_0+1) \\ &\ldots\ldots\ldots \\ x(k) &= \Phi^{k-k_0} x(k_0) + \sum_{j=k_0}^{k-1} \Phi^{k-j-1}\Gamma u(j) \end{aligned}$$

For $x(k_0) = 0$

$$\begin{aligned} x(k) &= \sum_{j=k_0}^{k-1} \Phi^{k-j-1}\Gamma u(j) \\ hencey(k) &= \sum_{j=k_0}^{k-1} C\Phi^{k-j-1}\Gamma u(j) \end{aligned}$$

Thus the pulse response function is given by

$$h(k) = C\Phi^{k-j-1}\Gamma \tag{8}$$

The pulse response function is similar to the impulse response function in analog domain.

Pulse Transfer function

Question Is there any way of representing digitized system in the form of transfer function as in the case of s domain analog transfer function??

Answer Yes it is indeed possible. Such digital domain transfer function will depend on two important factors:

a) Sampling time

b) kind of reconstruction (if ZOH, or First order hold etc.)

To get such a transfer function for zero order hold that we are working with so far, we introduce a new operator q called pulse transfer operator as q x(k) = x(k+1) From analysis done so far we know that for zero order hold the digital system response is given by

$$\begin{aligned} x(k+1) &= \Phi(T)x(k) + \Gamma u(k) \\ qx(k) &= \Phi(T)x(k) + \Gamma u(k) \end{aligned}$$

qI - $\Phi(T)x(k) = \Gamma u(k)$

$$\begin{aligned} x(k) &= [qI - \Phi(T)]^{-1}\Gamma u(k) \\ y(k) &= C[qI - \Phi(T)]^{-1}\Gamma u(k) \end{aligned}$$

Thus the pulse transfer function between input $u(k)$ and output $y(k)$ is given by H(q) = C [qI - $\Phi(T)]^{-1}\Gamma$ Thus any analog domain continuous system can be converted into its digital equivalent with ZOH by using the pulse transfer function above. Attached with these notes is also a table giving ZOH equivalent of different s domain transfer functions $G(s)$.

Example Consider $G(s) = \frac{1}{s^2}$. Converting this double integrator system in the state space form we get,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$C = [10] \qquad D = 0$$

We have already obtained Φ and Γ for this case as

$$\Phi = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \qquad \Gamma = \begin{bmatrix} T^2/2 \\ T \end{bmatrix}$$

Hence,

$$x(k+1) = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} x(k) \qquad + \begin{bmatrix} T^2/2 \\ T \end{bmatrix} u(k)$$

Thus with pulse transfer operator q we get,

$$\begin{bmatrix} qx_1(k) \\ qx_2(k) \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} T^2/2 \\ T \end{bmatrix} u(k)$$

This can be represented as

$$\begin{bmatrix} q & 0 \\ 0 & q \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} T^2/2 \\ T \end{bmatrix} u(k)$$

Collecting terms of vector $x(k)$, we get

$$\begin{bmatrix} q-1 & -T \\ 0 & q-1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} T^2/2 \\ T \end{bmatrix} u(k)$$

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \frac{1}{(q-1)^2} \begin{bmatrix} q-1 & -T \\ 0 & q-1 \end{bmatrix} \begin{bmatrix} T^2/2 \\ T \end{bmatrix} u(k)$$

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \frac{1}{(q-1)^2} \begin{bmatrix} (q-1)\frac{T^2}{2} + T^2 \\ (q-1)T \end{bmatrix} u[k]$$

Finally the output $y(k)$ can be obtained as

$$y_1(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = x_1(k) = \frac{1}{(q-1)^2}(q+1)\frac{T^2}{2}u(k) = \frac{T^2(q+1)}{2(q-1)^2}u(k)$$

Significance : way for defining new type of transform z- transform, since z-transforms have properties similar to shift operator as we will see in the next section

5 Z-Transforms

Definition

$$F(z) = \sum_{k=0}^{\infty} f(k_h) z^{-k}$$

Z- transform -important points
a) z is a complex variable=$re^{j\omega}$
b) there is a concept of region of convergence (ROC) as explained below.
For any sequence $f(kT)$a set of values of z for which the z transform converges is called the region of convergence(ROC)
Uniform convergence of z transform requires that

$$\sum_{k=-\infty}^{\infty} x[k] \left|z\right|^{-k} < \infty$$

This follows from the mathematical requirement for convergence of sequences
- If $z = z1$ is the ROC then ring defined by $|z| = |z1|$ will be the ROC Geometrically ROC will have boundaries as circles

Poles of zeros of H(z)

$$H(z) = \frac{P(z)}{Q(z)}$$

Poles - values of z for which $x(z) = \infty$
Zeros - values of z for which x(z)=0

Example:

Find the z transform of

$$x[n] = a^n u[n] z^{-n}$$

$$x(z) = \sum_{k=-\infty}^{\infty} a^n u[n] z^{-n}$$ x(z)=$\sum_{k=0}^{\infty}$

For convergence

$$\sum_{n=0}^{\infty} \left|az^{-1}\right|^n < \infty$$

$$\sum_{n=0}^{\infty} |r|^n < \infty$$

So the condition on z plane for ROC is: $\left|az^{-1}\right| < 1$ or $|z| > |a|$
In the ROC the poles and zeros of z transfer function are given by
$X(Z) = \frac{z}{z-a}$ - poles
$|z| = |a|$ zeros z=0
Example:

$$\begin{aligned}
x[n] &= -a^r u[-n-1] \\
X(Z) &= -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} \\
X(Z) &= -\sum_{n=-\infty}^{\infty} a^n z^{-n} \\
X(Z) &= 1 - \sum_{n=-\infty}^{\infty} (a^{-1}z)^n \\
X(Z) &= 1 - \frac{1}{1-a^{-1}z} \\
X(Z) &= \frac{z}{z-a}
\end{aligned}$$

If $(a^{-1}z) < 1$ Thus ROC is defined by $|z| < |a|$
Previous and this example: Poles and zeros are same but ROC is different
Example:

$$x[n] = \begin{cases} a^n & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}
X(z) &= \sum_{n=0}^{N-1} a^n z{-}n \\
X(z) &= \sum_{n=0}^{N-1} (az{-}1)^n
\end{aligned}$$

$$
\begin{aligned}
X(z) &= \frac{1-(az^{-1})^N}{1-az^{-1}} \\
X(z) &= \frac{1}{z^{N-1}}\frac{z^N-a^N}{z-a}
\end{aligned}
$$

ROC

$$\sum_{n-0}^{N-1}(az^{-1})^n < \infty$$

Finite term so $|a| < \infty \qquad z \neq 0$
ROC all values except z=∞
Example:

$$
\begin{aligned}
x[n] &= (\frac{1}{2})^n u[n] + (\frac{-1}{3})^n u[n] \\
X(z) &= \sum_{n=0}^{n=\infty}((\frac{1}{2})^n u[n] + (\frac{-1}{3})^n u[n])z^{-n} \\
X(z) &= \sum_{n=0}^{n=\infty}(\frac{1}{2})^n u[n]z^{-n} + \sum_{n=-\infty}^{n=\infty}(\frac{-1}{3})^n u[n]z^{-n} \\
X(z) &= \sum_{n=0}^{n=\infty}(\frac{1}{2})^n z^{-n} + \sum_{n=-\infty}^{n=\infty}(\frac{-1}{3})^n z^{-n} \\
X(z) &= \frac{1}{1-(\frac{1}{2})z^{-1}} + \frac{1}{1+\frac{1}{3}z^{-1}} \\
X(z) &= \frac{2z(z-\frac{1}{2})}{(z-\frac{1}{2})(z+\frac{1}{3})} \qquad \text{ROC}\,|z| > \frac{1}{2}\,|z| > \frac{1}{3}
\end{aligned}
$$

5.1 Properties of Z-transfom

1. Time shift $Z(q^{-n}f) = z^{-n}F(z)Z(f) = F(z)$
   Proof $f = f(k)$

$$
\begin{aligned}
q^{-n}f &= f(k-n) \\
Z(f(k-n)) &= \sum_{k=-\infty}^{k=\infty} f(k-n)z^{-k} \\
l &= (k-n) \\
Z(f(k-n)) &= \sum_{k=-\infty}^{k=\infty} f(l)z^{-(l+n)}
\end{aligned}
$$

$$
\begin{aligned}
Z(f(k-n)) &= \sum_{k=-\infty}^{k=\infty} f(l)z^{-l}z^{-n} \\
Z(f(k-n)) &= z^{-n}F(z)
\end{aligned}
$$

2. Linearity $Z(\alpha f+\beta g)=\alpha Z(f)+\beta Z(g)$
3. Initial value theorem

   $$f(0)=\lim_{z\to\infty} F(z)$$

   for $f[n]=0 n<0$
4. Final value theorem If $(1-Z^{-1})F(z)$ does not have poles on or outside the unit circle then

   $$\lim_{k\to\infty} f(x)=\lim_{z\to 1}(1-z^{-1})F(z)$$
5. Convolution
   $Z(f\star g)=Z\sum_{n=0} kf(n)g(k-n)=Z(f)Z(g)$

5.2 z transform for digital system

This section demonstrates how one can go about using z-transform properties for defining z-domain transfer function for analog domain control systems. Note also that the z domain functions can directly be used to represent systems without equivalent analog connection. We have our standard ZOH system given by

$$
\begin{aligned}
x(k+1) &= \Phi x(k)+\Gamma(k) \\
qx(k) &= \Phi x(k)+\Gamma(k)
\end{aligned}
$$

Now consider

$$
\begin{aligned}
Z(qx(k)) &= z(X(z)-X_1) \quad \textit{where} \\
X_1 &= \sum_{j=o}^{j=0} x(0)z^{-0} \\
X_1 &= x(0) \\
Z(qx(k)) &= zX(z)
\end{aligned}
$$

Assuming that $x(0)=0$. Thus we see that Z transform of pulse transfer operator results in multiplication of $X(z)$ by z.Thus a connection between pulse transfer function and z transform is established.

Example: Standard second order(spring mass system)
G(s) $=\omega_0^2 \frac{}{s^2+2\zeta\omega_0 s+\omega_0^2}$
From the ZOH pulse transfer functions provided with these notes we have

$$
\begin{aligned}
H(q) &= \frac{b_1 q^{n-1} + b_2 q^{n-2} + \ldots + b_n}{a^n + a_1 q^{n-1} + \ldots + a_n} \\
b_1 &= 1 - \alpha(\beta + \frac{\zeta\omega_0}{\omega}\gamma) \qquad \omega = \omega_0\sqrt{1-\zeta^2} \qquad \zeta < 1 \\
b_2 &= \alpha^2 + \alpha(\frac{\zeta\omega_0}{\omega}\gamma - \beta) \\
\alpha &= e^{-\zeta\omega_0 h} \qquad \beta = \cos{(\omega h)} \qquad \gamma = \sin{(\omega h)} \\
a_1 &= -2\alpha\beta \qquad \omega_0 = 1.83 \qquad a_2 = \alpha^2 \qquad \zeta = 0.5
\end{aligned}
$$

H(q)=H(z) For zero initial condition

Poles

$$
\begin{aligned}
Z^2 + a_1 z + a_2 &= 0 \\
a_1 &= -Ze^{-\zeta\omega_0 T}\cos(\omega_0\sqrt{1-\zeta^2 T}) \\
a_2 = e^{-2\zeta\omega_0 T}
\end{aligned}
$$

5.3 Computation of response of digital control system

Q.How to evaluate step response, ramp response, in general response to any other input? The following sample example demonstrates how one can go about getting response of digital control system to any given input. Let $u[n]$ be given input sequence say step for example then

$$
\begin{aligned}
u[n] &= step \\
U(z) &= \frac{z}{1-z}
\end{aligned}
$$

If this input is applied to a system with $H(z)$ as z-domain transfer function then for

$$
\begin{aligned}
H(z) &= \frac{b_1 z + b_2}{z^n + a_1 z + a_2} \quad weget \\
Y(z) &= h(z)U(z)
\end{aligned}
$$

Now one can take inverse z transform to get the response sequence $y[k]$. The next section explains methods of getting the inverse z transform.

6 Inverse Z-transform

There are two main methods

- By inspection, for example $a^n u[n]$ $\frac{1}{1-az^{-1}}$ $|z| > |a|$
- By partial Fraction
- By definition of z transforms itself (using power series in z)
  Method of partial fractions Let

$$X(z) = \frac{\sum_{k=0}^{k=M} b_k z^{-k}}{\sum_{k=0}^{k=N} a_k z^{-k}}$$

the problem is to obtain inverse z transform by partial fraction.
Case 1: $M < N$, If poles are of first order then we can express $X(z)$ using method of partial fractions as

$$X(z) = \sum_{k=1}^{k=N} \frac{A_k}{1 - d_k z^{-1}}$$

Case 2: $M \geq N$ then we can express $X(z)$ as

$$X(z) = \sum_{r=0}^{r=M-N} B_r z^{-r} + \sum_{k=1}^{k=N} \frac{A_k}{1 - d_k z^{-1}} \qquad B_r - by \quad long \quad division$$

If there are multiple order poles at $z = d_i$, then one can use

$$\begin{aligned} X(z) &= \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1 \neq i}^{k=N} \frac{A_k}{1 - d_k z^{-1}} + \sum_{m=1}^{m=s} \frac{C_m}{(1 - d_i z^{-1})^m} \\ C_m &= \frac{1}{(s-m)!(-d_i)^{s-m}} \left(\frac{d^{s-m}}{d\omega^{s-m}} [(1 - d_i \omega)^s X(\omega^{-1})] \right)_{\omega = d_i^{-1}} \end{aligned}$$

The procedure is illustrated with the following examples. **Example:**

$$\begin{aligned} X(z) &= \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} \qquad = \frac{1 + 2z^{-1} + z^{-2}}{(1 - \frac{1}{2}z^{-1})(1 - z^{-1})} \\ X(z) &= B_0 + \frac{A_1}{(1 - \frac{1}{2}z^{-1})} + \frac{A_2}{(1 - z^{-1})} \\ X(z) &= 2 + \frac{5z^{-1} - 1}{(1 - \frac{1}{2}z^{-1})(1 - z^{-1})} \end{aligned}$$

$$\begin{aligned}
A_1 &= \frac{1+2z^{-1}+z^{-2}}{1-z^{-1}}|_{z^{-1}=2} &&= \frac{1+4+4}{1-2} = -9 \\
A_2 &= \frac{1+2z^{-1}+z^{-2}}{1-\frac{1}{2}z^{-1}}|_{z^{-1}=1} &&= \frac{1+2+1}{\frac{1}{2}} = 8 \\
X(z) &= 2 - \frac{9}{1-\frac{1}{2}z^{-1}} + \frac{8}{1-z^{-1}}
\end{aligned}$$

From Table:

$$2 \longleftrightarrow 2\delta[n]$$
$$\frac{1}{1-\frac{1}{2}z^{-1}} \longleftrightarrow (\frac{1}{2})^n u[n]$$
$$x[n] = 2\delta[n] - 9(\frac{1}{2})^n u[n] + 8u[n]$$
$$\frac{1}{1-z^{-1}} \longleftrightarrow u[n]$$

Power series expansion:
Ex1:-

$$\begin{aligned}
X(z) &= z^2(1-\frac{1}{2}z^{-1})(1+z^{-1})(1-z^{-1}) \\
X(z) &= \sum_{n=-\infty}^{n=\infty} x[n]z^{-n} \\
X(z) &= z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}
\end{aligned}$$

Ex2:-

$$X(z) = log(1+az^{-1}) \qquad |x| < 1 \qquad = \sum_{n=1}^{n=\infty} \frac{(-1)^{n+1}a^n z^{-h}}{n}$$

$$x[n] = \begin{cases} (-1)^{n+1}\frac{a^n}{n} & n \geq 1 \\ 0 & n \leq 0 \end{cases}$$

7 Conclusion

This chapter focused on fundamental understanding of digitization process, and tools for digital domain representation and response evaluation. First part focused on Shannon sampling theorem its proof and reconstruction of sampled signal using different reconstructors as Shannon reconstruction, ZOH reconstruction and so on. Then the concept of digitization of analog domain systems was introduced and pulse transfer function was obtained. Finally, use of z transform for representing a digital control system and evaluating response of digital control system was presented. The tools and fundamentals presented here can be used further to analyze and synthesize control law in digital domain. This advanced topic is beyond the scope of this course. Also these fundamentals can be readily used to develop digital filters.