PROBLEMS

2.1

(a) Find the Fourier transform of the half-cosine pulse shown in Figure P2.1*a*.

(b) Apply the time-shifting property to the result obtained in part (*a*) to evaluate the spectrum of the half-sine pulse shown in Figure P2.1*b*.

(c) What is the spectrum of a half-sine pulse having a duration equal to aT?

(d) What is the spectrum of the negative half-sine pulse shown in Figure P2.1*c*?

(e) Find the spectrum of the single sine pulse shown in Figure P2.1*d*.

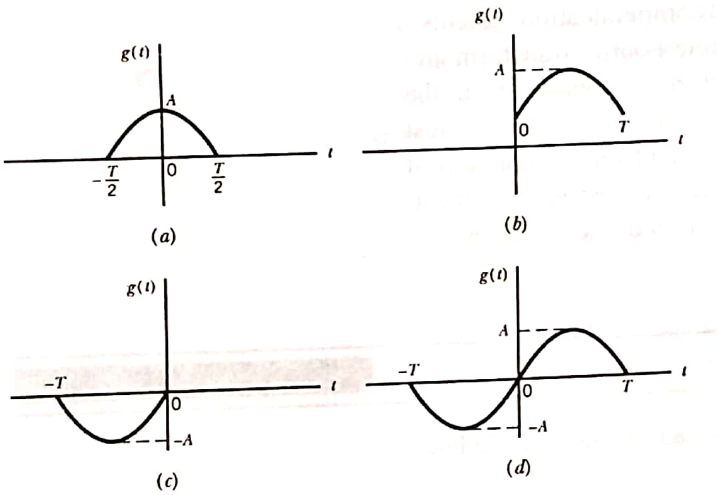

Figure P2.1

2.2 Evaluate the Fourier transform of the damped sinusoidal wave

$$g(t) = \exp(-t)\sin(2\pi f_c t)u(t)$$

where $u(t)$ is the unit step function.

2.3 Any function $g(t)$ can be split unambiguously into an *even part* and an *odd part*, as shown by

$$g(t) = g_e(t) + g_o(t)$$

The even part is defined by

$$g_e(t) = \frac{1}{2}[g(t) + g(-t)]$$

and the odd part is defined by

$$g_o(t) = \frac{1}{2}[g(t) - g(-t)]$$

(a) Evaluate the even and odd parts of a rectangular pulse defined by

$$g(t) = A \operatorname{rect}\left(\frac{t}{T} - \frac{1}{2}\right)$$

(b) What are the Fourier transforms of these two parts of the pulse?

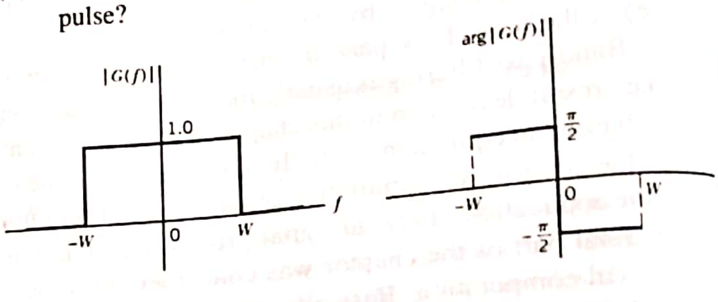

Figure P2.4

2.4 Determine the inverse Fourier transform of the frequency function $G(f)$ defined by the amplitude and phase spectra shown in Figure P2.4.

2.5 The following expression may be viewed as an approximate representation of a pulse with finite rise time:

$$g(t) = \frac{1}{\tau}\int_{t-T}^{t+T} \exp\left(-\frac{\pi u^2}{\tau^2}\right) du$$

where it is assumed that $T >> \tau$. Determine the Fourier transform of $g(t)$. What happens to this transform when we allow τ to become zero? *Hint*: Express $g(t)$ as the superposition of two signals, one corresponding to integration from $t - T$ to 0, and the other from 0 to $t + T$.

2.6 The Fourier transform of a signal $g(t)$ is denoted by $G(f)$. Prove the following properties of the Fourier transform:

(a) If a real signal $g(t)$ is an even function of time t, the Fourier transform $G(f)$ is purely real. If a real signal $g(t)$ is an odd function of time t, the Fourier transform $G(f)$ is purely imaginary.

(b)

$$t^n g(t) \rightleftharpoons \left(\frac{j}{2\pi}\right)^n G^{(n)}(f)$$

where $G^{(n)}(f)$ is the nth derivative of $G(f)$ with respect to

(c) $\displaystyle\int_{-\infty}^{\infty} t^n g(t)\,dt = \left(\frac{j}{2\pi}\right)^n G^{(n)}(0)$

(d) $\displaystyle g_1(t)g_2^*(t) \rightleftharpoons \int_{-\infty}^{\infty} G_1(\lambda)G_2^*(\lambda - f)\,d\lambda$

(e) $\displaystyle\int_{-\infty}^{\infty} g_1(t)g_2^*(t)\,dt = \int_{-\infty}^{\infty} G_1(f)G_2^*(f)\,df$

2.7 The Fourier transform $G(f)$ of a signal $g(t)$ is bounded by the following three inequalities:

$$|G(f)| \le \int_{-\infty}^{\infty} |g(t)|dt$$

$$|j2\pi f G(f)| \le \int_{-\infty}^{\infty} \left|\frac{dg(t)}{dt}\right| dt$$

and

$$|(j2\pi f)^2 G(f)| \le \int_{-\infty}^{\infty} \left|\frac{d^2 g(t)}{dt^2}\right| dt$$

where it is assumed that the first and second derivatives of $g(t)$ exist.

Construct these three bounds for the triangular pulse shown in Figure P2.7 and compare your results with the actual amplitude spectrum of the pulse.

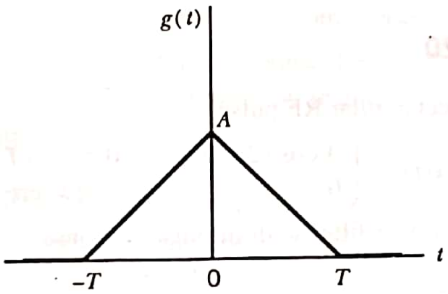

Figure P2.7

2.8 Prove the following properties of the convolution process:

(a) The commutative property:

$$g_1(t) \star g_2(t) = g_2(t) \star g_1(t)$$

(b) The associative property:

$$g_1(t) \star [g_2(t) \star g_3(t)] = [g_1(t) \star g_2(t)] \star g_3(t)$$

(c) The distributive property:

$$g_1(t) \star [g_2(t) + g_3(t)] = g_1(t) \star g_2(t) + g_1(t) \star g_3(t)$$

2.9 Consider the convolution of two signals $g_1(t)$ and $g_2(t)$. Show that

(a) $\displaystyle\frac{d}{dt}[g_1(t) \star g_2(t)] = \left[\frac{d}{dt}g_1(t)\right] \star g_2(t)$

(b) $\displaystyle\int_{-\infty}^{t}[g_1(\tau) \star g_2(\tau)]d\tau = \left[\int_{-\infty}^{t} g_1(\tau)d\tau\right] \star g_2(t)$

2.10 A signal $x(t)$ of finite energy is applied to a square-law device whose output $y(t)$ is defined by

$$y(t) = x^2(t)$$

The spectrum of $x(t)$ is limited to the frequency interval $-W \le f \le W$. Hence, show that the spectrum of $y(t)$ is limited to $-2W \le f \le 2W$. *Hint*: Express $y(t)$ as $x(t)$ multiplied by itself.

2.11 Evaluate the Fourier transform of the delta function by considering it as the limiting form of (1) a rectangular pulse of unit area, and (2) a sinc pulse of unit area.

2.12 The Fourier transform $G(f)$ of a signal $g(t)$ is defined by

$$G(f) = \begin{cases} 1, & f > 0 \\ \dfrac{1}{2}, & f = 0 \\ 0, & f < 0 \end{cases}$$

Determine the signal $g(t)$.

2.13 Show that the two different pulses defined in parts (a) and (b) of Figure P2.1 have the same energy spectral density:

$$\mathcal{E}_g(f) = \frac{4A^2T^2\cos^2(\pi T f)}{\pi^2(4T^2f^2 - 1)^2}$$

2.14

(a) The *root mean-square (rms) bandwidth* of a low-pass signal $g(t)$ of finite energy is defined by

$$W_{\text{rms}} = \left[\frac{\int_{-\infty}^{\infty} f^2|G(f)|^2\,df}{\int_{-\infty}^{\infty} |G(f)|^2\,df}\right]^{1/2}$$

where $|G(f)|^2$ is the energy spectral density of the signal. Correspondingly, the *root mean-square (rms) duration* of the signal is defined by

$$T_{\text{rms}} = \left[\frac{\int_{-\infty}^{\infty} t^2|g(t)|^2\,dt}{\int_{-\infty}^{\infty} |g(t)|^2\,dt}\right]^{1/2}$$

Using these definitions, show that

$$T_{\text{rms}}W_{\text{rms}} \ge \frac{1}{4\pi}$$

Assume that $|g(t)| \to 0$ faster than $1/\sqrt{|t|}$ as $|t| \to \infty$.

(b) Consider a Gaussian pulse defined by

$$g(t) = \exp(-\pi t^2)$$

Show that, for this signal, the equality

$$T_{\text{rms}}W_{\text{rms}} = \frac{1}{4\pi}$$

can be reached.

2.15 Let $x(t)$ and $y(t)$ be the input and output signals of a linear time-invariant filter. Using Rayleigh's energy theorem, show that if the filter is stable and the input signal $x(t)$ has finite energy, then the output signal $y(t)$ also has finite energy. That is, given that

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt < \infty$$

then show that

$$\int_{-\infty}^{\infty} |y(t)|^2 \, dt < \infty$$

2.21 The rectangular RF pulse

$$x(t) = \begin{cases} A\cos(2\pi f_c t), & 0 \le t \le T \\ 0, & \text{elsewhere} \end{cases}$$

is applied to a linear filter with impulse response

$$h(t) = x(T - t)$$

Assume that the frequency f_c equals a large integer multiple o

$1/T$. Determine the response of the filter and sketch it.

Figure P3.4

3.5 Consider the AM signal

$$s(t) = A_c[1 + \mu \cos(2\pi f_m t)]\cos(2\pi f_c t)$$

produced by a sinusoidal modulating signal of frequency f_m. Assume that the modulation factor is $\mu = 2$, and the carrier frequency f_c is much greater than f_m. The AM signal $s(t)$ is applied to an ideal envelope detector, producing the output $v(t)$.

(a) Determine the Fourier series representation of $v(t)$.
(b) What is the ratio of second-harmonic amplitude to fundamental amplitude in $v(t)$?

3.6 Consider a *square-law detector*, using a nonlinear device whose transfer characteristic is defined by

$$v_2(t) = a_1 v_1(t) + a_2 v_1^2(t)$$

where a_1 and a_2 are constants, $v_1(t)$ is the input, and $v_2(t)$ is the output. The input consists of the AM wave

$$v_1(t) = A_c[1 + k_a m(t)]\cos(2\pi f_c t)$$

(a) Evaluate the output $v_2(t)$.
(b) Find the conditions for which the message signal $m(t)$ may be recovered from $v_2(t)$.

3.7 The AM signal

$$s(t) = A_c[1 + k_a m(t)]\cos(2\pi f_c t)$$

is applied to the system shown in Figure P3.7. Assuming that $|k_a m(t)| < 1$ for all t and the message signal $m(t)$ is limited to the interval $-W \le f \le W$, and that the carrier frequency $f_c > 2W$, show that $m(t)$ can be obtained from the square-rooter output $v_3(t)$.

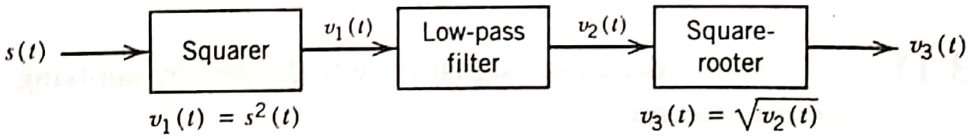

Figure P3.7

3.8 Consider a message signal $m(t)$ with the spectrum shown in Figure P3.8. The message bandwidth $W = 1$ kHz. This signal is applied to a product modulator, together with a carrier wave $A_c \cos(2\pi f_c t)$, producing the DSB-SC modulated signal $s(t)$. The modulated signal is next applied to a coherent detector. Assuming perfect synchronism between the carrier waves in the modulator and detector, determine the spectrum of the detector output when: (a) the carrier frequency $f_c = 1.25$ kHz and (b) the carrier frequency $f_c = 0.75$ kHz. What is the lowest carrier frequency for which each component of the modulated signal $s(t)$ is uniquely determined by $m(t)$?

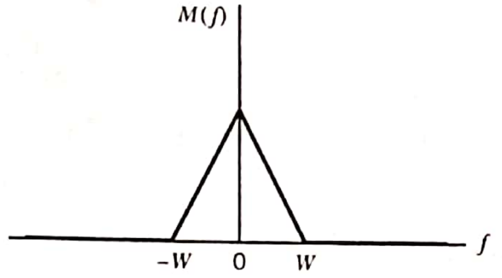

Figure P3.8

3.9 Figure P3.9 shows the circuit diagram of a *balanced modulator*. The input applied to the top AM modulator is $m(t)$, whereas that applied to the lower AM modulator is $-m(t)$; these two modulators have the same amplitude sensitivity. Show that the output $s(t)$ of the balanced modulator consists of a DSB-SC modulated signal.

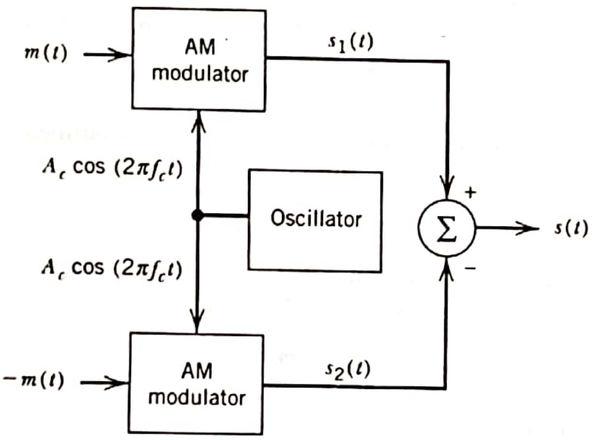

Figure P3.9

3.10 Figure 3.10 shows the circuit details of the ring modulator. Assume that the diodes are identical and the transformers are perfectly balanced. Let R denote the terminating resistance at the input end and output end of the modulator (assuming ideal 1 : 1 transformers). Determine the output voltage of the modulator for each of the two conditions described in Figures 3.10*b* and 3.10*c*. Hence, show that these two output voltages are equal in magnitude and opposite in polarity.

3.11 A DSB-SC modulated signal is demodulated by applying it to a coherent detector.

(a) Evaluate the effect of a frequency error Δf in the local carrier frequency of the detector, measured with respect to the carrier frequency of the incoming DSB-SC signal.
(b) For the case of a sinusoidal modulating wave, show that because of this frequency error, the demodulated signal exhibits *beats* at the error frequency. Illustrate your answer with a sketch of this demodulated signal.

3.12 Consider the DSB-SC signal

$$s(t) = A_c \cos(2\pi f_c t) m(t)$$

where $A_c \cos(2\pi f_c t)$ is the carrier wave and $m(t)$ is the message signal. This modulated signal is applied to a square-law device characterized by

$$y(t) = s^2(t)$$

The output $y(t)$ is next applied to a narrow-band filter with a passband amplitude response of one, mid-band frequency $2f_c$, and bandwidth Δf. Assume that Δf is small enough to treat the spectrum of $y(t)$ as essentially constant inside the passband of the filter.

(a) Determine the spectrum of the square-law device output $y(t)$.
(b) Show that the filter output $v(t)$ is approximately sinusoidal, given by

$$v(t) \simeq \frac{A_c^2}{2} E \, \Delta f \cos(4\pi f_c t)$$

where E is the energy of the message signal $m(t)$.

3.13 Consider the quadrature–carrier multiplex system of Figure 3.16. The multiplexed signal $s(t)$ produced at the transmitter output in Figure 3.16*a* is applied to a communication channel of transfer function $H(f)$. The output of this channel is in turn applied to the receiver input in Figure 3.16*b*. Prove that the condition

$$H(f_c + f) = H^*(f_c - f), \qquad 0 \le f \le W$$

is necessary for recovery of the message signals $m_1(t)$ and $m_2(t)$ at the receiver outputs; f_c is the carrier frequency, and W is the message bandwidth. *Hint:* Evaluate the spectra of the two receiver outputs.

3.14 Suppose that in the receiver of the quadrature-carrier multiplex system of Figure 3.16 the local carrier available for demodulation has a phase error ϕ with respect to the carrier source used in the transmitter. Assuming a distortionless communication channel between transmitter and receiver, show that this phase error will cause *cross-talk* to arise between the two demodulated signals at the receiver outputs. By cross-talk we mean that a portion of one message signal appears at the receiver output belonging to the other message signal, and vice versa.

3.15 A particular version of *AM stereo* uses quadrature multiplexing. Specifically, the carrier $A_c \cos(2\pi f_c t)$ is used to modulate the sum signal

$$m_1(t) = V_0 + m_l(t) + m_r(t)$$

where V_0 is a dc offset included for the purpose of transmitting the carrier component, m_ℓ is the left-hand audio signal, and $m_r(t)$ is the right-hand audio signal. The quadrature carrier $A_c \sin(2\pi f_c t)$ is used to modulate the difference signal

$$m_2(t) = m_l(t) - m_r(t)$$

(a) Show that an envelope detector may be used to recover the sum $m_r(t) + m_l(t)$ from the quadrature-multiplexed signal. How would you minimize the signal distortion produced by the envelope detector?
(b) Show that a coherent detector can recover the difference $m_l(t) - m_r(t)$.
(c) How are the desired $m_l(t)$ and $m_r(t)$ finally obtained?

3.16 The single tone modulating signal $m(t) = A_m \cos(2\pi f_m t)$ is used to generate the VSB signal

$$s(t) = \frac{1}{2}aA_mA_c\cos[2\pi(f_c + f_m)t] + \frac{1}{2}A_mA_c(1-a)\cos[2\pi(f_c - f_m)t]$$

where a is a constant, less than unity, representing the attenuation of the upper side frequency.

(a) If we represent this VSB signal as a quadrature carrier multiplex

$$s(t) = A_c m_1(t)\cos(2\pi f_c t) + A_c m_2(t)\sin(2\pi f_c t)$$

What is $m_2(t)$?

(b) The VSB signal, plus the carrier $A_c\cos(2\pi f_c t)$, is passed through an envelope detector. Determine the distortion produced by the quadrature component, $m_2(t)$.

(c) What is the value of constant a for which this distortion reaches its worst possible condition?

3.17 Using the message signal

$$m(t) = \frac{1}{1+t^2}$$

determine and sketch the modulated waves for the following methods of modulation:

(a) Amplitude modulation with 50 percent modulation.

(b) Double sideband-suppressed carrier modulation.

3.18 The local oscillator used for the demodulation of an SSB signal $s(t)$ has a frequency error Δf measured with respect to the carrier frequency f_c used to generate $s(t)$. Otherwise, there is perfect synchronism between this oscillator in the receiver and the oscillator supplying the carrier wave in the transmitter. Evaluate the demodulated signal for the following two situations:

(a) The SSB signal $s(t)$ consists of the upper sideband only.

(b) The SSB signal $s(t)$ consists of the lower sideband only.

3.19 Figure P3.19 shows the block diagram of *Weaver's method* for generating SSB modulated waves. The message (modulating) signal $m(t)$ is limited to the frequency band $f_a \leq |f| \leq f_b$. The auxiliary carrier applied to the first pair of product modulators has a frequency f_0, which lies at the center of this band, as shown by

$$f_0 = \frac{f_a + f_b}{2}$$

The low-pass filters in the upper and lower branches are identical, each with a cutoff frequency equal to $(f_b - f_a)/2$. The carrier applied to the second pair of product modulators has a frequency f_c that is greater than $(f_b - f_a)/2$. Sketch the spectra at the various points in the modulator of Figure P3.19, and hence show that:

(a) For the lower sideband, the contributions of the upper and lower branches are of opposite polarity, and by adding them at the modulator output, the lower sideband is suppressed.

(b) For the upper sideband, the contributions of the upper and lower branches are of the same polarity, and by adding them, the upper sideband is transmitted.

(c) How would you modify the modulator of Figure P3.19, so that only the lower sideband is transmitted?

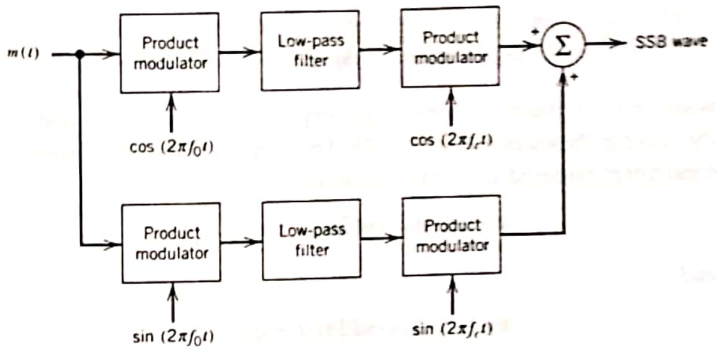

Figure P3.19

3.20

(a) Consider a message signal $m(t)$ containing frequency components at 100, 200, and 400 Hz. This signal is applied to an SSB modulator together with a carrier at 100 kHz, with only the upper sideband retained. In the coherent detector used to recover $m(t)$, the local oscillator supplies a sine wave of frequency 100.02 kHz. Determine the frequency components of the detector output.

(b) Repeat your analysis, assuming that only the lower sidebanu is transmitted.

3.21 The spectrum of a voice signal $m(t)$ is zero outside the interval $f_a \leq |f| \leq f_b$. In order to ensure communication privacy, this signal is applied to a *scrambler* that consists of the following cascade of components: a product modulator, a high-pass filter, a second product modulator, and a low-pass filter. The carrier wave applied to the first product modulator has a frequency equal to f_c, whereas that applied to the second product modulator has a frequency equal to $f_b + f_c$; both of them have unit amplitude. The high-pass and low-pass filters have the same cutoff frequency at f_c. Assume that $f_c > f_b$.

(a) Derive an expression for the scrambler output $s(t)$, and sketch its spectrum.

(b) Show that the original voice signal $m(t)$ may be recovered from $s(t)$ by using an *unscrambler* that is identical to the unit described above.

3.22 A method that is used for carrier recovery in SSB modulation systems involves transmitting two pilot frequencies that are appropriately positioned with respect to the transmitted sideband. This is illustrated in Figure P3.22*a* for the case when only the lower sideband is transmitted. In this case, the pilot frequencies f_1 and f_2 are defined by

$$f_1 = f_c - W - \Delta f$$

and

$$f_2 = f_c + \Delta f$$

3.24 Consider a multiplex system in which four input signals $m_1(t)$, $m_2(t)$, $m_3(t)$, and $m_4(t)$, are respectively multiplied by the carrier waves

$$[\cos(2\pi f_a t) + \cos(2\pi f_b t)]$$
$$[\cos(2\pi f_a t + \alpha_1) + \cos(2\pi f_b t + \beta_1)]$$
$$[\cos(2\pi f_a t + \alpha_2) + \cos(2\pi f_b t + \beta_2)]$$
$$[\cos(2\pi f_a t + \beta_3) + \cos(2\pi f_b t + \beta_3)]$$

and the resulting DSB-SC signals are summed and then transmitted over a common channel. In the receiver, demodulation is achieved by multiplying the sum of the DSB-SC signals by the four carrier waves separately and then using filtering to remove the unwanted components.

(a) Determine the conditions that the phase angles α_1, α_2, α_3 and β_1, β_2, β_3 must satisfy in order that the output of the kth demodulator is $m_k(t)$, where $k = 1, 2, 3, 4$.

(b) Determine the minimum separation of carrier frequencies f_a and f_b in relation to the bandwidth of the input signals so as to ensure a satisfactory operation of the system.