Homework 6: Gaussian random vectors

EE 325: Probability and Random Processes, Autumn 2019
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Instructions: These problems are a part of the syllabus for the final exam. They are not to be submitted. *If you have queries, then meet the instructor or the TA during office hours.*

1. Let X be a sign random variable with the distribution:
$$\mathbb{P}(X = 1) = 0.5; \quad \mathbb{P}(X = -1) = 0.5. \tag{1}$$
Let
$$\vec{X}(i) := \frac{1}{\sqrt{n}} [X(i, 1), \dots, X(i, n)]^T \quad \text{for } i = 1, 2, 3. \tag{2}$$
The elements $\sigma(i, j), 1 \le i \le 3, 1 \le j \le n$ are i.i.d. with the same distribution as X. Let $Y_1, Y_2, Y_3, \dots$ be an i.i.d. sequence of random variables with mean zero and variance 1. Let $\vec{Y} = (Y_1, \dots, Y_n)^T$ be a random vector. Answer the following:
   (a) What will be the distribution of $Z_1 := \vec{Y}^T \vec{X}(1)$ as n becomes large?
   (b) What will be the distribution of the vector $[Z_1, Z_2, Z_3]^T$ as n becomes large? Here $Z_i = \vec{Y}^T \vec{X}(i)$.
2. Assume that $m < n$. Show that if $\vec{Z}$ is an n-dimensional jointly Gaussian random vector and B is a rectangular $m \times n$ matrix, then $B\vec{Z}$ is jointly Gaussian.
3. If two jointly Gaussian random vectors $\vec{X}$ and $\vec{Y}$ are uncorrelated, show that they are also independent. (BONUS) Will this be true if $\vec{X}$ and $\vec{Y}$ are not jointly Gaussian but marginally Gaussian?
4. Let $U^T = (\vec{X}^T, \vec{Y}^T)$ be a jointly Gaussian random vector of size $(n + m)$. Show that if $K_{\vec{U}}$ is non-singular, then both $K_{\vec{X}}$ and $K_{\vec{Y}}$ are non-singular. Further, show that if K_U is non-singular and if $K_U^{-1} = \begin{bmatrix} B & C \\ C^T & D \end{bmatrix}$, then B and D are also non-singular and positive definite.
5. We have seen earlier that if $X \sim \mathcal{N}(0, \sigma_X^2)$ and $Y \sim \mathcal{N}(0, \sigma_Y^2)$ are independent Gaussian random variables, then $X + Y$ is a Gaussian random variable as well. Using induction, show that any linear combination of the components of an IID normalized Gaussian random vector $\vec{W} \sim \mathcal{N}(\vec{0}, I_n)$ is also a Gaussian random variable. (This exercise confirms that $\vec{W}$ is jointly Gaussian.)
6. Let X and Y be zero-mean jointly Gaussian random variables with $\mathbb{E}(X^2) = \sigma_X^2$, $\mathbb{E}(Y^2) = \sigma_Y^2$, and $\mathbb{E}(XY) = \rho\sigma_X\sigma_Y$.
   (a) Find the conditional probability density function $f_{X|Y}(x|y)$.
   (b) Let $V = Y^3$. Find the conditional probability density function $f_{X|V}(x|v)$. (Hint: think carefully before calculations.)
   (c) Let $Z = Y^2$. Find the conditional probability density function $f_{X|Z}(x|z)$. (Hint: first understand why this is more difficult than (b).)
7. Let X and Y be zero-mean and jointly Gaussian random variables with variances σ_X^2, σ_Y^2 and covariance $\rho\sigma_X\sigma_Y$. Find a 2×2 transformation matrix A such that $\vec{V} = A[X, Y]^T$ has independent components V_1 and V_2.
8. Let K be the following matrix,
$$K = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}. \tag{3}$$

(a) Find the eigenpairs of K.

(b) Find Q and Λ such that $K = Q\Lambda Q^T$, and $QQ^T = I_2$.

(c) Find the eigenpairs of K^n, where n is a natural number.

(d) What will be the eigenpairs of K^{-1}?