Section 2: Distribution of Random Variables and Some Examples

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1 Random variables

As discussed in the previous section, random variables are used to model uncertainty in engineering systems. In an engineering system, where determining complicated parameters or variables is difficult, random variables are used to model uncertainty. While it is desirable to model uncertainty by random variables, the key purpose for such models is to compute probability of various 'events' associated with random experiments by the use of random variables. For this computability reason, a random variable has to satisfy some properties. We omit the details, but a random variable must satisfy some properties so that probability of any 'event' associated with a random experiment can be computed by an associated random variable. For more details, see the discussion on sample space, sigma fields, probability measure, and random variable as a measurable map in any classical textbook [1].

A random variable is required to satisfy the conditions elucidated in the next definition.

Definition 1.1 (Random variable)**.** *X is a random variable if it satisfies the following conditions:*

1. $\mathbb{P}(X \in [-\infty, \infty]) = 1$*, i.e., the random variable is finite with probability one.*
2. $\mathbb{P}(X \in [a, b))$ *is well defined for all $a < b$ such that $-\infty < a < b < \infty$.*
3. *For a countable number of disjoint intervals $[a_i, b_i), i = 1, 2, \ldots,$*

$$\mathbb{P}\left(X \in \bigcup_{i=1}^{\infty}[a_i, b_i)\right) = \sum_{i=1}^{\infty} \mathbb{P}\left(X \in [a_i, b_i)\right). \tag{1}$$

Once again, it is noted that this definition is built upon the $(\Omega, \mathcal{F}, \mathbb{P})$ triad of probability. With the triad setup, a random variable is a deterministic map $X : \Omega \to \mathbb{R}$. In that setup, it is required that $\{X \leq x\}, x \in \mathbb{R}$ has a pre-image in $\mathcal{F}$. The properties of $\mathcal{F}$ ensure that all these pre-images satisfy closeness under unions and intersections. The third property in the above definition ensures the closeness under unions and intersections of $(-\infty, x], x \in \mathbb{R}$ and its complements. The details are omitted for brevity. In continuation of the definition, the reader can also assume that $\mathbb{P}(x \in (a, b)), \mathbb{P}(x \in [a, b]), \mathbb{P}(x \in (a, b])$ are well defined for all $-\infty < a < b < \infty$. The reader should also verify that for all $a \in \mathbb{R}$:

$$\mathbb{P}(X \in (-\infty, a]) + \mathbb{P}(X \in (a, \infty)) = 1. \tag{2}$$

An example of a random variable construction using the above definition is illustrated next.

Example 1.1 (Uniform Random Variable)**.** *Let U be uniform random variable distributed in $[0, 1]$. We write $U \sim$ Unif$[0, 1]$. The distribution of random variable U is defined as:*

$$\mathbb{P}(U \in [a, b)) = \begin{cases} b - a & \textit{for } 0 \leq a < b \leq 1 \\ 0 & \textit{for } [a, b) \textit{ disjoint with } [0, 1]. \end{cases} \tag{3}$$

The reader is encouraged to properties 1. and 2. for a random variable in Definition 1.1.

2 Distributions functions for random variables

The general method to calculate probabilities of $X \in A$, where $A \subset \mathbb{R}$, involves the cumulative distribution function (or CDF). It is a one stop method to visualize or calculate $\mathbb{P}(X \in A)$. The CDF is defined as follows:

Definition 2.1 (CDF)**.** *The cumulative distribution function (CDF) of random variable X is defined as,*

$$F_X(x) = \mathbb{P}(X \leq x) = \mathbb{P}(X \in (-\infty, x]), \quad x \in \mathbb{R}. \tag{4}$$

This function is well defined by the 2. and 3. in Definition 1.1.

The CDF satisfies the following properties:

1. The CDF increases from 0 at $x = -\infty$ to 1 at $x = \infty$ for any random variable. That is,

$$F_X(-\infty) = \lim_{a \downarrow -\infty} F_X(a) = \lim_{a \downarrow -\infty} \mathbb{P}(X \in (-\infty, a]) = 0 \tag{5}$$

$$\text{and } F_X(\infty) = \lim_{b \uparrow \infty} F_X(b) = \lim_{b \uparrow \infty} \mathbb{P}(X \in (-\infty, b]) = 1. \tag{6}$$

2. The CDF $F_X(x)$ is non-decreasing, i.e.,

$$\text{if } x_1 \leq x_2 \text{ then } F_X(x_1) \leq F_X(x_2). \tag{7}$$

The proof follows since probability is always positive. Further $(-\infty, x_2]$ can be written as a disjoint union as $(-\infty, x_2] = (-\infty, x_1] \cup (x_1, x_2]$. Applying probability on both sides yields the above result.

3. The CDF is right-continuous in x, i.e.,

$$\lim_{\varepsilon \downarrow 0} F_X(x + \varepsilon) = F_X(x). \tag{8}$$

However, $F_X(x)$ may not be left continuous. The reader is encouraged to verify this property with the example of $\text{Ber}(p)$ and $\text{Uniform}[0, 1]$ CDF sketches.

While the CDF is defined for any random variable X, the notion of probability density function (PDF) and probability mass function (PMF) is routinely used in the literature for continuous and discrete random variables, respectively. Random variables that can take only a countable number of values are called discrete. Random variables that can take on uncountable number of values in a union of intervals are called continuous. Some random variables are of mixed type. They will have continuous and discrete component(s). The formal definitions are written next.

Definition 2.2 (Discrete random variable)**.** *If X takes a finite or countably infinite number of points with probability one, then X is discrete. That is, there exist $\{x_i, i \in \mathbb{Z}\} \subset \mathbb{R}$ such that,*

$$\sum_{i \in \mathbb{Z}} \mathbb{P}(X = x_i) = 1 \textit{ and } \mathbb{P}(X = x_i) \neq 0. \tag{9}$$

For discrete random variables, the PMF $p_X(x)$ is defined as

$$p_X(x) := \mathbb{P}(X = x), \quad x \in \mathbb{R}. \tag{10}$$

By definition of discrete random variable, there are a finite or countably infinite number of points where the PMF is non-zero and is zero otherwise. Continuous random variables are defined next.

Definition 2.3 (Continuous random variable)**.** *If the CDF $F_X(x)$ of a random variable X has a finite right derivative as a function of $x \in \mathbb{R}$, then X is a continuous random variable. Also,*

$$f_X(x) = \frac{dF_X(x)}{dx} := \lim_{\varepsilon \downarrow 0} \frac{F_X(x+\varepsilon) - F_X(x)}{\varepsilon}. \tag{11}$$

is called as the probability density function (PDF).

Since $F_X(x)$ is non-decreasing, for continuous random variables the PDF is non negative, that is $f_X(x) \geq 0$.

It is possible to obtain CDF from PDF for continuous random variables and CDF from PMF for discrete random variables. The formulas are written next. For continuous random variables, CDF is obtained from PDF by integration:

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\mathrm{d}t. \tag{12}$$

For discrete random variables, CDF is obtained from PMF by cumulative sum:

$$F_X(x) = \sum_{x_i \leq x} \mathbb{P}(X = x_i). \tag{13}$$

The summation in the above example is not always available in closed form, but can be a part of a computer program. A few examples of random variables are presented next.

Example 2.1 (Switch)**.** *Consider a random variable $X \in \{0, 1\}$ defined as:*

$$\mathbb{P}(X = 0) = 1 - p \quad \textit{and} \quad \mathbb{P}(X = 1) = p \tag{14}$$

where $0 \leq p \leq 1$ is a parameter. This distribution is used to model binary variables such as a probabilistic switch, bit-errors in communication channels, single-bit ADC output against dithered inputs. This random variable is briefly termed as Ber(p) random variable where Ber stands for Bernoulli. The reader is encouraged to sketch the CDF of this random variable and verify the right continuity of its CDF.

Example 2.2 (Phase of a carrier in wireless communication)**.** *In a contemporary wireless communication system, there is a transmitter (BTS) geographically separated from a wireless receiver (UE). In the simplest digital wireless communication setup, the BTS has to send a binary digit (bit) from the set $\{0, 1\}$. The BTS sends a pulse $A_c \cos(2\pi f_c t), 0 \leq t \leq T$ for sending a bit 1 and sends a pulse $-A_c \cos(2\pi f_c t), 0 \leq t \leq T$ for sending a bit 0. The frequency f_c is called as carrier frequency and its selection is based on wireless signal propagation properties as well as regulatory requirements. For instance, for mobile phone systems, f_c is nearly 1 GHz. These frequencies are chosen to ensure good signal propagation in the wireless medium as well as to achieve multiplexing of many users/services.*

The receiver UE is separated from the BTS. As a result, the received signal at the receiver is modeled as

$$r(t) = \pm A_c f(r, \omega, \phi) \cos(2\pi f_c t + \Theta), \tag{15}$$

where Θ is the phase difference (delay) between the BTS and UE. If $\frac{1}{f_c} \ll r$, the phase Θ is well modelled by a uniform random variable: i.e., $\Theta \sim$ Unif$[0, 2\pi]$ with a PDF:

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi} & 0 \leq \theta \leq 2\pi; \\ 0 & \textit{otherwise.} \end{cases} \tag{16}$$

Observe that Θ is a continuous random variable.

Example 2.3 (Poisson Distribution)**.** *The Poisson distribution is often used when a swarm of uncoordinated users send requests (in parallel) to a single server. Such servers are often modeled as queues handling the requests of multiple users.*

Let N be number of requests in a given time interval $[0, T]$. Then N is modelled as Poisson random variable: i.e., $N \sim$ Poisson(λ). The random variable is discrete and its PMF is as follows:

$$\begin{aligned} p_N(k) = \mathbb{P}(N = k) &= \frac{\lambda^k}{k!} e^{-\lambda}; \quad k = 0, 1, \ldots \\ &= 0 \text{ otherwise.} \end{aligned} \tag{17}$$

Observe that $0! = 1$ and the CDF can be obtained by cumulatively summing the PMF values.

References

[1] Robert Gallager: "*Stochastic Processes*", Cambridge University Press.