Homework 3: functions of rv, total expectation, mgf

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1. Let Y be a log-normal random variable. The log-normal property implies that $\log_e Y$ is distributed as $\mathcal{N}(\mu, \sigma^2)$ random variable. Find the variance of Y. (Hint: the mgf of a Gaussian random variable may be useful.)

   Solution: Let $V \sim \mathcal{N}(\mu, \sigma^2)$. From the lectures, the mgf of the Gaussian random variable V is given by

   $$g_V(r) = \exp\left(\mu r + \frac{\sigma^2 r^2}{2}\right); \quad r \in \mathbb{R}.$$

   The random variable Y is lognormal; that is, $V = \log_e Y$ where $V \sim \mathcal{N}(\mu, \sigma^2)$. We will use the mgf of V to find the first and second moments of Y conveniently. It begins by observing that,

   $$g_V(r) = \mathbb{E}(e^{rV}) = \mathbb{E}(e^{r \log_e Y}) = \mathbb{E}(Y^r), \quad r \in \mathbb{R}.$$

   Then, $\mathbb{E}(Y)$ and $\mathbb{E}(Y^2)$ can be obtained by substituting $r = 1$ and $r = 2$, respectively, in $g_V(r)$. So,

   $$\begin{aligned} \mathbb{E}(Y) &= e^{\mu + \frac{\sigma^2}{2}}, \\ \mathbb{E}(Y^2) &= e^{2(\mu+\sigma^2)}, \\ \text{and, } \operatorname{var}(Y) &= \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 = e^{2(\mu+\sigma^2)} - e^{2\mu+\sigma^2}. \end{aligned}$$

2. A packet arrives at a router with probability p in an independent and identically distributed fashion. That is, at most one packet arrives at each instant independently, and probability of a packet arrival is p. Assume that the router serves for N clocks (discrete slots), where N is a Poisson(n) random variable. Find the mean and variance of total number of arrived packets at the router. (Hint: use conditional expectation.)

   Solution: Let X_i be a random variable that is 1 if a packet arrives at the i-th instant, and otherwise is 0. That is,

   $$\begin{aligned} X_i &= 1 \text{ with probability } p \\ &= 0 \text{ with probability } (1-p), \end{aligned}$$

   where X_i are IID random variables. Since $X \sim$Ber(p), so $\mathbb{E}(X_i) = p$ and $\operatorname{var}(X_i) = p(1-p)$. Let S_N be the random variable indicating the total number of packets that have arrived after N clocks. Therefore,

   $$S_N = \sum_{i=1}^{N} X_i$$

   where N is a Poisson(n) random variable. Now the total expectation rule will be invoked to solve this problem.[1] From Poisson distribution properties, we know that $\mathbb{E}(N) = n$ and $\operatorname{var}(N) = n$. By the total expectation rule

   $$\mathbb{E}(\mathbb{E}(S_N|N)) = \mathbb{E}(S_N).$$

   The expected value of S_N given $N = k$ can be written as,

   $$\mathbb{E}(S_N|N = k) = \mathbb{E}(\sum_{i=1}^{k} X_i|N = k) = \mathbb{E}(\sum_{i=1}^{k} X_i)$$

[1] An alternate and messy approach is to find the distribution of S_N and then calculate the desired expectations.

as X_i and N are independent of each other. Since X_i are i.i.d. random variables and expectation of a sum of random variables is the sum of their expectations,

$$\mathbb{E}(S_N|N=k) = k\mathbb{E}(X_1) = k\mathbb{E}(X).$$

Therefore,$\mathbb{E}(S_N|N) = N\mathbb{E}(X)$ and by the total expectation rule $\mathbb{E}(S_N) = \mathbb{E}(N)\mathbb{E}(X) = np$.
Similarly, for $\mathbb{E}[S_N^2]$

$$\begin{aligned}
\mathbb{E}(S_N^2|N=k) &= \mathbb{E}\left(\left(\sum_{i=1}^{k} X_i\right)^2\right) \\
&= \text{var}\left(\sum_{i=1}^{k} X_i\right) + \left(\mathbb{E}\left(\sum_{i=1}^{k} X_i\right)\right)^2 \\
&= k\text{var}(X_i) + k^2(\mathbb{E}(X_i))^2
\end{aligned}$$

Therefore, $\mathbb{E}[S_N^2|N] = N\text{var}(X) + N^2(\mathbb{E}(X))^2$. By total expectation rule,

$$\begin{aligned}
\mathbb{E}(S_N^2) &= \mathbb{E}(\mathbb{E}(S_N^2|N)) \\
&= \mathbb{E}(N)\text{var}(X) + \mathbb{E}(N^2)(\mathbb{E}(X))^2 = np + n^2p^2
\end{aligned}$$

Since $\text{var}(S_N) = \mathbb{E}(S_N^2) - (\mathbb{E}(S_N))^2$ therefore $\text{var}(S_N) = np + n^2p^2 - (np)^2 = np$. The answer is complete.

3. Let $X_1, X_2, \ldots, X_5$ be i.i.d. normalized Gaussian rv. Find the pdf of $Z = X_1^2 + X_2^2 + \ldots + X_5^2$ and $Y = X_1 + X_2 + \ldots + X_5$.

Solution: From the lectures, the MGF of $X \sim \mathcal{N}(\mu, \sigma^2)$ is given by

$$g_X(t) = \exp\left(t\mu + \frac{1}{2}\sigma^2 t^2\right) \tag{1}$$

Therefore, for i.i.d. normalized Gaussian random variables $X_1, X_2, \ldots, X_5$,

$$g_{X_i}(t) = \exp\left(\frac{t^2}{2}\right), \quad i = 1, 2, \ldots, 5.$$

Since X_1, X_2, …, X_5 are independent, so $Y = X_1 + X_2 + \ldots + X_5$ has an mgf given by

$$g_Y(t) = \prod_{i=1}^{5} g_{X_i}(t) = \exp\left(\frac{5t^2}{2}\right). \tag{2}$$

By comparison with (1), $Y \sim \mathcal{N}(0, 5)$ and

$$f_Y(y) = \frac{1}{\sqrt{10\pi}} \exp\left(-\frac{y^2}{10}\right).$$

Let $Z_i = X_i^2$ for $1 \le i \le 5$. The cdf of Z_i for each i is

$$F_{Z_i}(z) = P(Z_i \le z) = P(X^2 \le z) = P(|X| \le z).$$

This results in

$$\begin{aligned}
F_{Z_i}(z) &= F_X(\sqrt{z}) - F_X(-\sqrt{z}) && \text{for } z \ge 0 \\
&= 0 && \text{for } z < 0.
\end{aligned}$$

The pdf of Z_i is obtained by differentiation of the cdf for $z > 0$, that is,

$$f_{Z_i}(z) = \frac{f_X(\sqrt{z})}{2\sqrt{z}} + \frac{f_X(-\sqrt{z})}{2\sqrt{z}} = \frac{1}{\sqrt{2\pi z}} e^{-z/2} \text{ for } z \ge 0.$$

For $z < 0$, the pdf is zero. As discussed in class $Z_i \sim \chi_1^2$. Its mgf can be obtained as follows:

$$\begin{aligned} g_{Z_1}(t) &= \mathrm{E}[e^{tZ_1}] \\ &= \int_0^\infty e^{tz} f_{Z_1}(z)\mathrm{d}z \\ &= \int_0^\infty e^{tz} \frac{1}{\sqrt{2\pi z}} e^{-z/2}\mathrm{d}z. \end{aligned}$$

Let $\left(\frac{1}{2} - t\right) z = u$, where $t < 1/2$ for the convergence of the integral. Then,

$$\begin{aligned} g_{Z_1}(t) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-u} \left(\frac{u}{\frac{1}{2} - t}\right)^{-\frac{1}{2}} \frac{du}{\frac{1}{2} - t} \\ &= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{\frac{1}{2} - t}\right)^{1/2} \int_0^\infty u^{-1/2} e^{-u} \mathrm{d}u. \end{aligned}$$

Let $c = 1/(\sqrt{2\pi}) \int_0^\infty u^{-1/2} e^{-u}\mathrm{d}u$. Then,

$$g_{Z_1}(t) = c\left(\frac{1}{\frac{1}{2} - t}\right)^{1/2}.$$

Since $g_{Z_1}(0) = 1$, therefore $c = 1/\sqrt{2}$ and

$$g_{Z_1}(t) = \left(\frac{\frac{1}{2}}{\frac{1}{2} - t}\right)^{1/2} \quad \text{for } t < 1/2.$$

Since Z_1, Z_2, …, Z_5 are independent, so

$$g_Z(t) = [g_{Z_1}(t)]^5 = \left(\frac{\frac{1}{2}}{\frac{1}{2} - t}\right)^{5/2} \quad \text{for } t < 1/2.$$

It turns out the above distribution is related to class of distributions, called gamma distributions, having support on the positive real line. The pdf of a gamma distribution G with parameters (θ, k) is given by

$$f_G(x) = \frac{1}{\theta^k \Gamma(k)} x^{k-1} e^{-\frac{x}{\theta}}, \quad x \geq 0.$$

Its mgf is known to be $g_G(t) = (1 - \theta t)^{-k}$, where $t \leq 1/\theta$. Therefore, the pdf of Z will be the pdf of Gamma distribution with parameters $\theta = 2$ and $k = 5/2$. Accordingly,

$$f_Z(x) = \frac{1}{2^{5/2}\Gamma(5/2)} x^{3/2} e^{-\frac{x}{2}}, \quad x \geq 0.$$

4. Let X_1, X_2 be independent Exponential(1) random variables. Find the joint pdf of $Y = X_1 + X_2$ and $Z = X_1 - X_2$.

Solution: This solution will utilize properties of linear maps and linear algebra. Consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Then,

$$\begin{bmatrix} Y \\ Z \end{bmatrix} = A \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

that is, the map from X_1, X_2 to Y, Z is linear. Next, the idea is that if $\vec{X} := \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \in \mathrm{d}\vec{x}$, an infinitesimal volume at the point $\vec{x}$, then correspondingly $\vec{V} := \begin{bmatrix} Y \\ Z \end{bmatrix} \in \mathrm{d}\vec{v}$, an infinitesimal volume at the point $\vec{v} = A\vec{x}$. Since A is invertible, so there is one to one relationship between $\vec{X}$ and $\vec{V}$.

As a result,

$$\begin{aligned}
&\mathbb{P}(\vec{V} \in d\vec{v}) = \mathbb{P}(\vec{X} \in d\vec{x}) \\
\text{or } & f_{\vec{V}}(\vec{v})|\mathrm{d}\vec{v}| = f_{\vec{X}}(\vec{x})|\mathrm{d}\vec{x}| \\
\text{or } & f_{Y,Z}(y,z) = \frac{1}{|\det(A)|} f_{X_1,X_2}(x_1,x_2).
\end{aligned}$$

This pdf will have a range $0 \leq y \leq \infty, -\infty < z < \infty$, and $y > z$. Note that $\det(A) = 2$ in the above expression. Do remember to substitute $x_1 = (y+z)/2$ and $x_2 = (y-z)/2$ in your answers.

5. In this problem you have to construct a random variables (i.e., their distributions) so that their mgf satisfies certain properties.

   (a) Describe a random variable X such that $g_X(r)$ is not finite for $r < 0$ but is finite for $r \geq 0$.

   (b) Describe a random variable Y such that $g_Y(r)$ is not finite for $r > 0$ but is finite for $r \leq 0$.

   (c) Assuming that X, Y in the above parts are independent, what is the ROC of $(X+Y)$ for your examples?

At first, two symmetric r.v. U and V will be constructed. The r.v. U will have an mgf $g_U(r)$ that exists for all $r \in \mathbb{R}$ and the r.v. V will have an mgf $g_V(r)$ that exists only for $r = 0$. Then the two r.v. will be suitably used to generate pdfs which satisfy the conditions asked in (a) and (b).

Consider $U \sim \mathcal{N}(0,1)$. Then $g_U(r) = \exp(r^2/2)$ for all $r \in \mathbb{R}$.

For V, consider the Cauchy distribution, i.e.,

$$f_V(y) = \frac{1}{\pi(1+v^2)}, \tag{3}$$

for which mgf is given by $g_V(0) = 1$ and $g_V(r) = \infty$ if $r \neq 0$. This can be shown by the following chain of inequalities for $r > 0$:

$$\begin{aligned}
g_V(r) = \mathbb{E}(\exp(rV)) &= \int_{-\infty}^{\infty} \frac{\exp(rv)}{\pi(1+v^2)} \mathrm{d}v \\
&\geq \int_{1}^{\infty} \frac{\exp(rv)}{\pi(1+v^2)} \mathrm{d}v \\
&\geq \int_{1}^{\infty} \frac{rv}{2v^2} \mathrm{d}v \text{ since } \exp(rv) \geq rv \text{ and } 1+v^2 < 2v^2 \text{ for } v > 1 \\
&= \frac{r}{2} \ln v \Big|_1^{\infty} = \infty
\end{aligned}$$

Similarly, we can prove that $g_V(r)$ also does not exist for $r < 0$. That is mgf for cauchy distribution is finite only for $r = 0$.

Also observe that one side of Cauchy distribution, i.e., the function $\frac{1}{\pi(1+v^2)} \exp(rv)$ is integrable for $r \leq 0$. This is because $\exp(rv) < 1$ for $v < 0$ and $r > 0$.

(a) Now we will construct a random variable X such that

$$\begin{aligned}
f_X(x) &= \frac{1}{\pi(1+x^2)} \qquad \text{for } x < 0 \\
&= \frac{1}{\sqrt{2\pi}} \exp\left(-x^2/2\right) \qquad \text{for } x \geq 0
\end{aligned}$$

In above distribution $g_X(r)$ is finite for $r \geq 0$ but infinite for $r < 0$.

(b) In a similar manner as in part (a), construct a r.v. Y with the following pdf:

$$\begin{aligned}
f_Y(y) &= \frac{1}{\pi(1+y^2)} \qquad \text{for } y \geq 0 \\
&= \frac{1}{\sqrt{2\pi}} \exp\left(-y^2/2\right) \qquad \text{for } y < 0
\end{aligned}$$

For the above pdf, $g_Y(r)$ is finite for $r \leq 0$.

(c) Now, consider $Z = X + Y$, where X and Y have the pdf as defined in (a) and (b), and they are independent. Then,

$$g_Z(r) = g_X(r)g_Y(r), \tag{4}$$

with the ROC being the intersection of ROC of X and Y. That is, $g_Z(r)$ will be finite in the ROC common to $g_X(r)$ and $g_Y(r)$ which is $r = 0$.

6. Let $X \sim \mathcal{N}(0, \sigma^2)$. Using the mgf of X, show that,

$$\mathbb{E}[X^{2k+1}] = 0, \quad \mathbb{E}[X^{2k}] = \frac{(2k)!\sigma^{2k}}{k!2^k}, \text{ where, } k > 0 \text{ and } k \in \mathbb{Z}.$$

Solution: Recall that the mgf of Gaussian random variable is given by $g_X(r) = e^{r^2\sigma^2/2}$. Expanding it in the power series form, we get

$$\exp(r^2\sigma^2/2) = 1 + \frac{r^2\sigma^2}{2 \cdot 1!} + \frac{r^4\sigma^4}{2^2 \cdot 2!} + \frac{r^6\sigma^6}{2^3 \cdot 3!} + \dots \tag{5}$$

On the other hand, by using the definition of $g_X(r)$ we obtain that

$$\begin{aligned} g_X(r) = \mathbb{E}\left(e^{rX}\right) &= \mathbb{E}\left(1 + \frac{rX}{1!} + \frac{r^2X^2}{2!} + \frac{r^3X^3}{3!} + \dots\right) \\ &= 1 + \frac{r\mathbb{E}(X)}{1!} + \frac{r^2\mathbb{E}(X^2)}{2!} + \frac{r^3\mathbb{E}(X^3)}{3!} + \dots \end{aligned} \tag{6}$$

By equating the odd powers of r in (5) and (6) we get $\mathbb{E}(X^{2k+1}) = 0$. By equating the even powers of r in (5) and (6) we get,

$$\frac{\sigma^{2k}}{k!2^k} = \frac{\mathbb{E}(X^{2k})}{(2k)!} \quad \text{or} \quad \mathbb{E}(X^{2k}) = \frac{(2k)!\sigma^{2k}}{k!2^k}.$$

7. Assume that the mgf of a random variable X exists (i.e., is finite) in the interval $(r_-, r_+), r_- < 0 < r_+$. Assume that $r \in (r_-, r_+)$ throughout in this problem.

(a) For any finite constant $c \in \mathbb{R}$, express the moment generating function of $(X - c)$, i.e. $g_{(X-c)}(r)$, in terms of $g_X(r)$ and show that it exists for all $r \in (r_-, r_+)$. Explain why $g''_{(X-c)}(r) \geq 0$.

(b) Show that $g''_{(X-c)}(r) = [g''_X(r) - 2cg'_X(r) + c^2 g_X(r)]e^{-rc}$.

(c) Use (a) and (b) to show that $g''_X(r)g_X(r) - [g'_X(r)]^2 \geq 0$. Let $\gamma_X(r) = \ln g_X(r)$. Show that $\gamma''_X(r) \geq 0$.

Solution:

(a) First, we will find the mgf of $Y = (X - c)$.

$$g_Y(r) = g_{(X-c)}(r) = \mathbb{E}(e^{r(X-c)}) = \mathbb{E}(e^{rX}e^{-rc}) = e^{-rc}\mathbb{E}(e^{rX}) = e^{-rc}g_X(r). \tag{7}$$

The function $g_X(r)$ is finite (or it exists) for $r_- < r < r_+$. The function e^{-rc} is also finite for $r_- < r < r_+$. Therefore, $g_{(X-c)}(r)$ is also finite in the range $r \in (r_-, r_+)$. Next we will evaluate its first and second derivatives.

$$g'_Y(r) = \frac{\mathrm{d}}{\mathrm{d}r}\mathbb{E}(e^{rY}) = \mathbb{E}(Y.e^{rY}) \quad \text{and} \quad g''_Y(r) = \mathbb{E}(Y^2.e^{rY}).$$

The terms Y^2 and e^{rY} are always non-negative. Thus, the expectation of their product is also non-negative. Hence, $g''_Y(r) = g''_{(X-c)}(r) \geq 0$.

(b) The mgf of Y and the positiveness of $g''_Y(r)$ will be used to show the required result. Using the rule for derivative of product of two functions in (7), it can be shown that

$$g''_Y(r) = g''_{(X-c)}(r) = \left(g''_X(r) - 2cg'_X(r) + c^2g_X(r)\right)e^{-rc}. \tag{8}$$

(c) Since $g''_Y(r) \geq 0$ and $e^{-rc} \geq 0$, thus it follows that

$$g''_X(r) - 2cg'_X(r) + c^2 g_X(r) \geq 0. \tag{9}$$

This is a quadratic expression in c. Since the RHS is independent of the value of c, so we can minimize the LHS as a function of c while preserving the inequality. The quadratic expression is minimized at $c = (g'_X(r)/g_X(r))$. Upon substitution, we get

$$\frac{g''_X(r)g_X(r) - (g'_X(r))^2}{g_X(r)} \geq 0 \quad \text{or} \quad g''_X(r)g_X(r) - (g'_X(r))^2 \geq 0 \tag{10}$$

Consider the semi-invariant function $\gamma_X(r) = \ln g_X(r)$. By taking two derivatives of $\gamma_X(r)$, we get

$$\gamma''_X(r) = \frac{g''_X(r)g_X(r) - [g'_X(r)]^2}{[g_X(r)]^2}. \tag{11}$$

By comparing (13) and (14), we get the desired result that $\gamma''_X(r) \geq 0$.

8. Let X and Y be IID $\mathcal{N}(0,1)$ random variables. Let $Z \sim \text{Exp}\left(\frac{1}{2}\right)$ be an exponentially distributed random variable with $\lambda = (1/2)$.

   (a) Find $\mathbb{E}(e^{rX^2})$ and $\mathbb{E}(e^{rZ})$, i.e., the MGF of X^2 and Z. Find out the region of convergence or $[r_-(X^2), r_+(X^2)]$ and $[r_-(Z), r_+(Z)]$. Identify whether $g_{X^2}(r)$ and $g_Z(r)$ converges at the boundary points or not.

   (b) It can be shown that if the MGF of a random variable V is identical to that of W in a small neighborhood $r \in (-\delta, \delta)$ with $\delta > 0$, then V and W have the same distribution. Use part (a) to find the distribution of $X^2 + Y^2$.

Solution:

Recall the pdf's of exponential ($Z \sim \text{Exp}(0.5)$) and Gaussian ($X \sim \mathcal{N}(0,1)$) random variables.

$$\begin{aligned} f_Z(z) &= 0.5\ e^{-0.5x} \quad ; \forall x \geq 0 \\ f_X(x) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad ; x \in \Re \end{aligned}$$

(a) From the above density functions, we can calculate the MGF's as follows.

$$\begin{aligned} g_Z(r) &= \int_0^\infty 0.5\ e^{-0.5x} e^{rx} dx \\ &= \frac{1}{2} \int_0^\infty e^{-\left(\frac{1-2r}{2}\right)x} dx = \frac{1}{1-2r} \quad \forall\ r < 0.5 \ \text{ i.e. for } r \in (-\infty, 0.5). \end{aligned} \tag{12}$$

$$\begin{aligned} g_{X^2}(r) &= \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} e^{rx^2} dx \\ &= \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2\left(\frac{1}{1-2r}\right)}} dx \\ &= \frac{1}{\sqrt{1-2r}} \quad \forall\ 1 - 2r > 0 \text{ i.e. } r \in (-\infty, 0.5). \end{aligned} \tag{13}$$

Note that the region of convergence for $g_Z(r)$ and $g_{X^2}(r)$ is $(-\infty, 0.5)$. The boundary points don't exist in the region of convergence.

(b) *The result stated in the question is useful in many contexts.* If X, Y are independent, then X^2, Y^2, are also independent. Recall the following property of the MGF, the MGF of the sum of two i.i.d random variables is the square of the MGF of one of them. From (13), we have

$$g_{X^2+Y^2}(r) = \frac{1}{1-2r} \quad \text{for } r \in (-\infty, 0.5).$$

Note that the above expression is the same as $g_Z(r)$, from (12). Hence $X^2 + Y^2$ is Exponential with mean 0.5.