Convolution Integral Method

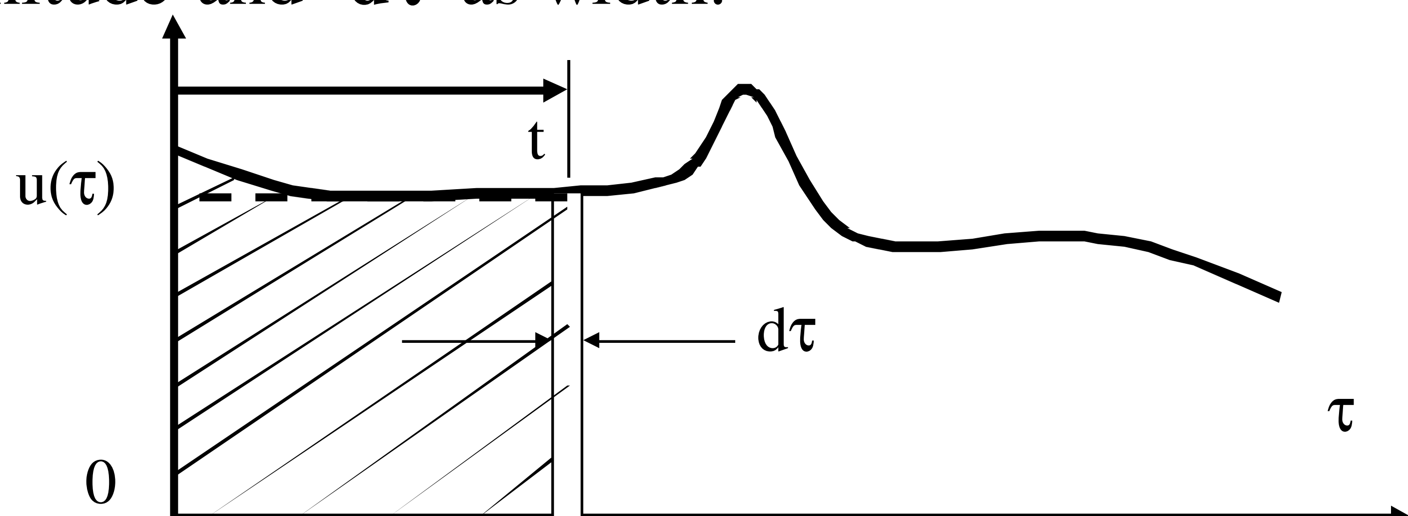
– *Concept of Convolution*

Convolution Integral Approach

Convolution Concept

Forced response of LTI systems can be shown to be a **result** of the **convolution process**, as demonstrated below.

Let **u(t)** be an arbitrary input, which is **represented** as a collection of **pulses** at different time instants 't' with '**u(τ)**' as magnitude and '**dτ**' as width.



Input – Response Connection

We know that an **impulse** with a time **delay**, acting on a **LTI** system, generates a **response** with same time **delay.**

Therefore, all the **impulses** generate **responses**, separated from **each other** by time interval **‘dτ’,** so that the **integral** corresponds to the response to the **complete input.**

However, in order for the **methodology** to work, we need **closed form** expression for the unit **impulse response.**

Impulse Input Description

Practical form of impulse is the **Dirac delta** function, as shown alongside.

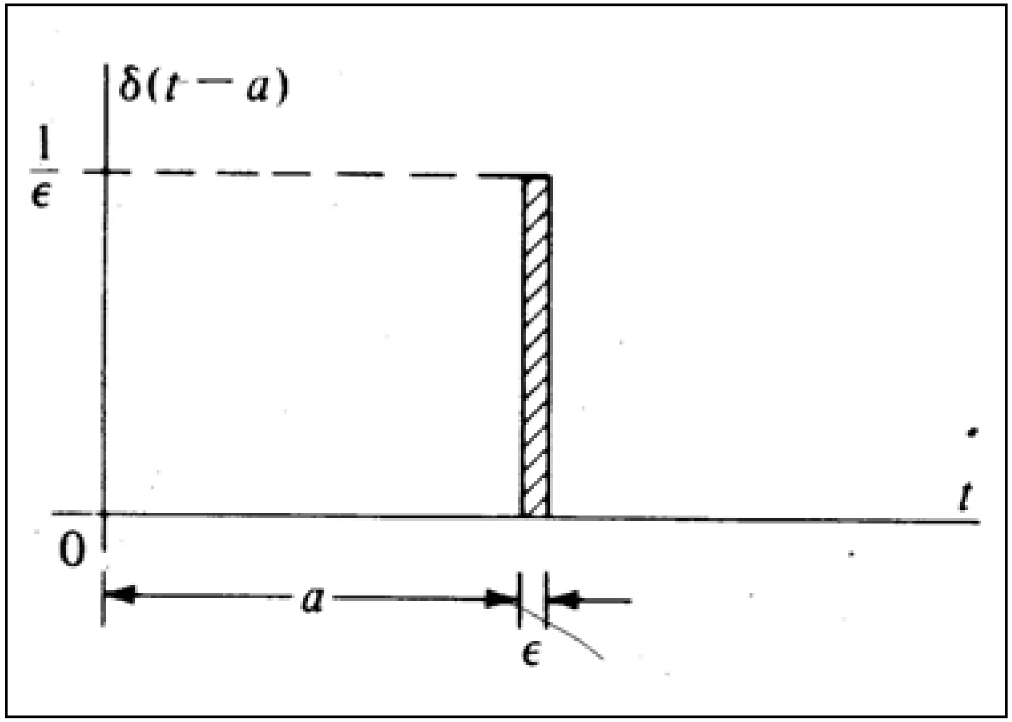
Mathematically, it can be **represented** through the following relations.

$$\delta(t-a) = 0 \quad \text{for } t \neq a$$

$$\delta(t-a) = \frac{1}{\in} \quad \text{for } t = a$$

$$\text{such that } \int_{-\infty}^{+\infty} \delta(t-a)dt = 1$$

We assume that **g(t-a)** is the **response** of system to **δ(t-a).**



1^{st} Order Impulse Response

Impulse response (i.e. g(t)) of **1st order** systems **can be obtained** in closed form, as follows.

$$\text{1st Order: } c\dot{g} + kg = \delta(t); \quad \lim_{\in \to 0} \int_0^{\in} \delta(t)\,dt = 1$$

$$\lim_{\in \to 0} \int_0^{\in} \left(c\dot{g} + kg \right) dt = \lim_{\in \to 0} c\,g(t)\,|_0^{\in} + \lim_{\in \to 0} kg(0) \in = 1$$

$$\lim_{\in \to 0} c\left[g(\in) - g(0) \right] + 0 \to g(0+) = \frac{1}{c}$$

$$g(t) = g(0+) e^{-\left(t/\tau\right)} = \frac{1}{c} e^{-\left(t/\tau\right)}; \quad \tau = \frac{c}{k}$$

$g(0+) \to$ Amount of displacement in time interval $\in$

2^{nd} Order Impulse Response

Impulse response (i.e. g(t)) of **2^{nd} order** systems **can be obtained** in closed form, as follows.

$$\text{2nd Order: } m\ddot{g} + c\dot{g} + kg = \delta(t); \quad \lim_{\in \to 0} \int_0^{\in} \delta(t)\,dt = 1$$

$$\lim_{\in \to 0} \int_0^{\in} \left(m\ddot{g} + c\dot{g} + kg \right) dt = \lim_{\in \to 0} m\dot{g} \,\Big|_0^{\in} + \lim_{\in \to 0} cg \,\Big|_0^{\in} + \lim_{\in \to 0} kg(0) \in = 1$$

$$m\dot{g}(0+) + 0 + 0 = 1 \to \dot{g}(0+) = \frac{1}{m}; \quad g(t) = \frac{1}{m\omega_d} e^{-\zeta\omega t} \sin \omega_d t$$

$$c = 2m\zeta\omega_n; \quad k = m\omega_n^2; \quad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$\dot{g}(0+) \to \text{ Amount of velocity in time interval } \in$$

Convolution Formulation

As we know that ‘**δ(t)**’, shifted by **‘dτ’,** generates ‘**g(t)**’ also shifted by **‘dτ’** (time invariance), we can **set up** a process for **assembling** all such responses through **convolution.**

Thus, ‘**convolution**’ process essentially becomes an assembly of **time shifted** responses, as shown below.

$$y(t) = \int_0^t g(t-\tau)\, u(\tau)\, d\tau = \int_0^t g(t)\, u(t-\tau)\, d\tau; \quad t \geq 0$$

Convolution Example – 1st Order

1st order system subjected to unit **step input** is as below.

$$T\dot{c}(t) + c(t) = u(t) = 1(t); \quad g(t) = \frac{1}{T} e^{-t/T}$$

Generate its unit **step response.**

$$c_{\text{step}}(t) = \int_0^t g(t-\tau)\, 1(\tau)\, d\tau = \frac{1}{T}\int_0^t e^{-t/T} \cdot e^{\tau/T}\, d\tau = e^{-t/T}\int_0^t e^{\tau/T}\, d\tau$$

$$c_{\text{step}}(t) = e^{-t/T}\,\frac{1}{T}\left[T e^{\tau/T}\right]_0^t = e^{-t/T}\left[e^{t/T} - 1\right] = \left(1 - e^{-t/T}\right)$$

Convolution Example – 2nd Order

2nd order system subjected to unit **step input** is as below.

$$\ddot{c}(t) + 2\zeta\omega_n \dot{c}(t) + \omega_n^2 c(t) = \frac{1}{m}u(t) = \frac{1}{m}1(t)$$

Generate its unit **Step response**.

$$u(t) = 1 \quad \text{for t} \geq 0; \quad \sigma = \frac{c}{2m}; \quad \zeta = \frac{c}{2m\omega_n}; \quad \omega_n = \sqrt{\frac{k}{m}}; \quad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$c(t) = \frac{1}{m\omega_d}\int_0^t e^{-\sigma(t-\tau)} \sin\omega_d (t-\tau) d\tau = \frac{e^{-\sigma t}}{m\omega_d}\int_0^t e^{\sigma\tau} \sin\omega_d (t-\tau) d\tau$$

$$c(t) = -\frac{e^{-\sigma t}}{m\omega_d \times \left(\sigma^2 + \omega_d^2\right)}\left[e^{\sigma\tau}\left(\omega_d \sin\omega_d (t-\tau) + \sigma\cos\omega_d (t-\tau)\right)\right]_0^t$$

$$c(t) = \frac{1}{k}\left[1 - e^{-\sigma t}\left(\cos\omega_d t + \frac{\sigma}{\omega_d}\sin\omega_d t\right)\right]$$

Convolution Limitations

Convolution approach is generally **feasible** only for **1st or 2nd** order systems and also only for **simple inputs.**

We see that as **order** increases and/or **input** function becomes **complex**, integration process becomes **tedious.**

Further, as we **need** to generate **responses** repeatedly during the **design** phase, it is necessary to have a **strategy** that can do the task **quickly** for all systems & **inputs.**

Laplace transform & **transfer function** based techniques are **part** of such a **solution** strategy.

Laplace Transform Example

Consider the following **2nd order** LTI system.

$$\boxed{\ddot{c}(t) + 2\zeta\omega_n \dot{c}(t) + \omega_n^2 c(t) = \omega_n^2 r(t)}$$

We can take **term-by-term** Laplace transform of the above **model** to arrive at the **algebraic** equation, as shown below.

$$\left[s^2 C(s) - \dot{c}(0) - sc(0)\right] + 2\zeta\omega_n \left[sC(s) - c(0)\right] + \omega_n^2 C(S) = \omega_n^2 R(s)$$

$$\left[s^2 + 2\zeta\omega_n s + \omega_n^2\right] C(s) = \left\{\dot{c}(0) + \left(s + 2\zeta\omega_n\right) c(0)\right\} + \omega_n^2 R(s)$$

The above **algebraic system** can be suitably **manipulated** and c(t) can be **obtained** through the table of **transforms.**

Summary

Though, **convolution integral** can be used for generating time response, the **integration process** can become tedious for **higher order systems** with arbitrary inputs.