Homework 2: expectation, conditional distributions, functions of rv

EE 325: Probability and Random Processes, Autumn 2019

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Instructions: The homework is a part of the syllabus for Midterm 1. *If you have queries, then meet the instructor or the TA during office hours.*

1. Assume that X is a continuous random variable with,
$$f_X(x) = \frac{c}{1+|x|^6}, x \in \mathbb{R}.$$
The constant c is selected such that $\int_{\mathbb{R}} f_X(x)\mathrm{d}x = 1$. Find the values of $\mathbb{E}(X)$ and $\mathbb{E}(X^5)$.

2. Assume that $\mathbb{E}(X^2) < \infty$. Show that $\alpha = \mathbb{E}(X)$ is the unique value of α that minimizes $\mathbb{E}((X-\alpha)^2)$.

3. If the random variables (X, Y) are independent, then show that (X^2+X, Y^2+2Y) are also independent.

4. Let $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$, with $\lambda, \mu > 0$. Assume that X and Y are independent, and n is a non-negative integer.
   (a) Find the pmf of $Z = X + Y$.
   (b) Find the conditional distribution of Y conditioned on $Z = n$, i.e., the pmf $p_{Y|Z}(y|n)$.

5. Let $\{X_i\}_{i=1}^n$ be a sequence of i.i.d. continuous random variables with probability density function $f(x)$.
   (a) Find $\mathbb{P}(X_1 \leq X_2)$.
   (b) Find $\mathbb{P}(X_1 \leq X_2, X_1 \leq X_3)$.
   (c) Let N be a new integer-valued random variable defined as follows. N is the index of the first random variable that is less than X_1, that is,
   $$\mathbb{P}(N = n) = \mathbb{P}(X_1 \leq X_2, X_1 \leq X_3, \ldots, X_1 \leq X_{n-1}, X_1 > X_n). \tag{1}$$
   Find $\mathbb{P}(N > n)$ as a function of n.
   (d) Show that $\mathbb{E}(N) = \infty$

6. Let X_1 and X_2 be IID Gaussian random variables with $X_i \sim \mathcal{N}(0, \sigma^2), i = 1, 2$. Let $Y = X_1 + X_2$. Then answer the following questions.
   (a) Find the distribution of Y by using 'functions of random variable' approach. You can use the convolution of pdf formula if it is required.
   (b) Find the conditional distribution of X_1 given $Y = y$. Interpret the result obtained. What will you expect the conditional distribution of X_2 given $Y = y$ to be?

7. Let X, Y be a continuous random variables having a cumulative distribution function $F(x, y)$. Let their marginal (cumulative) distributions be $G(x)$ and $H(y)$.
   (a) Show that $G(X)$ is uniformly distributed in $(0, 1)$.
   (b) Suppose that you have access to a random variable U uniformly distributed in $(0, 1)$ (for example, in MATLAB or C, you will have access to a uniform random variable). How would you use it to simulate a continuous random variable X having a distribution function G? Justify rigorously.
   (c) Suppose now you have two IID random variables U_1 and U_2 distributed uniformly in $(0, 1)$. How would you use them to simulate random variable pair (X, Y) having a joint distribution $F(x, y)$?