Model & Modelling Basics

– *Model Types*
– *Mathematical Modelling Process*
– *Mathematical Modelling Examples*

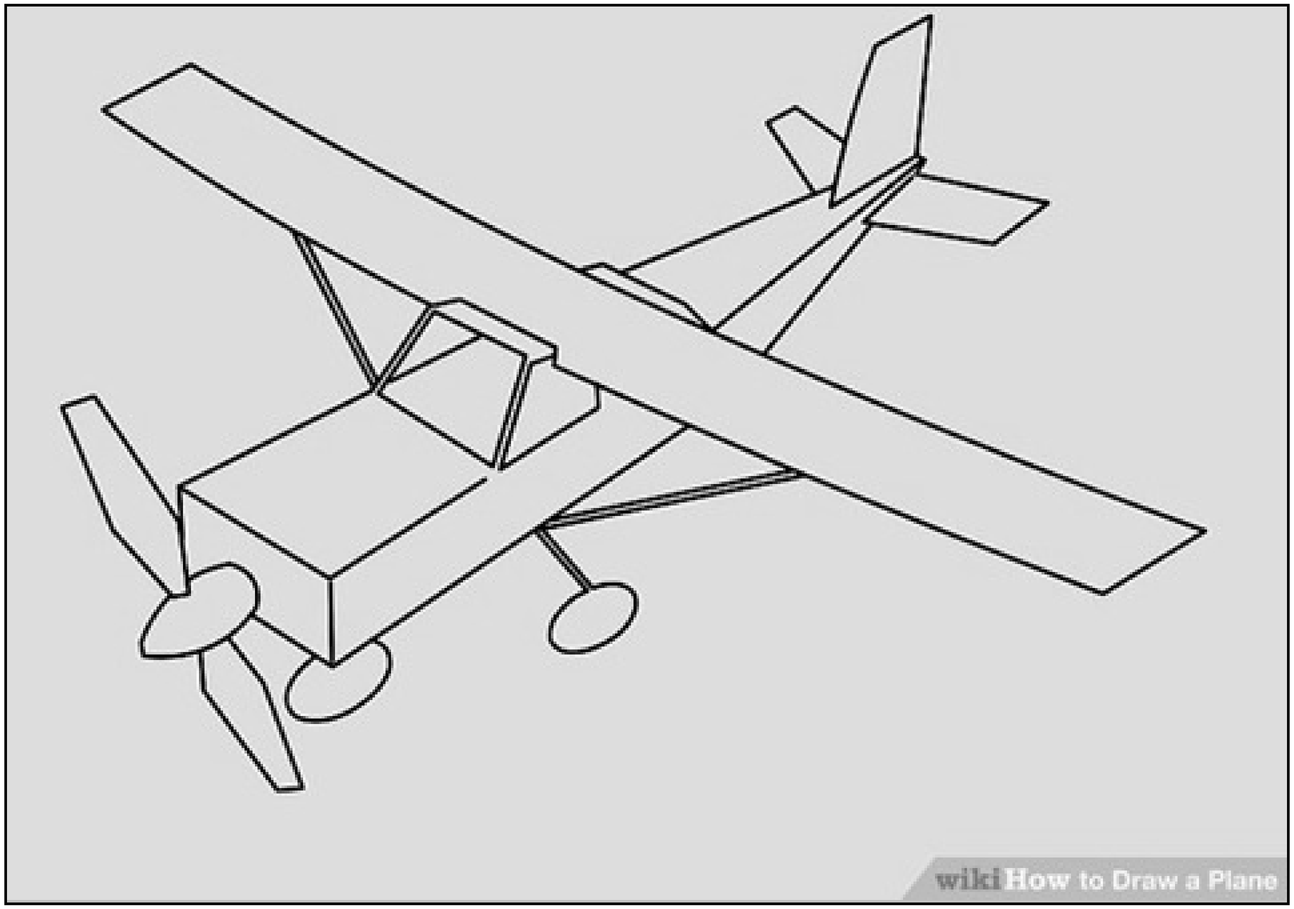
Types of Models

Model Types & Choice

Models of a system, for serving specific objectives, can be of **various types.**

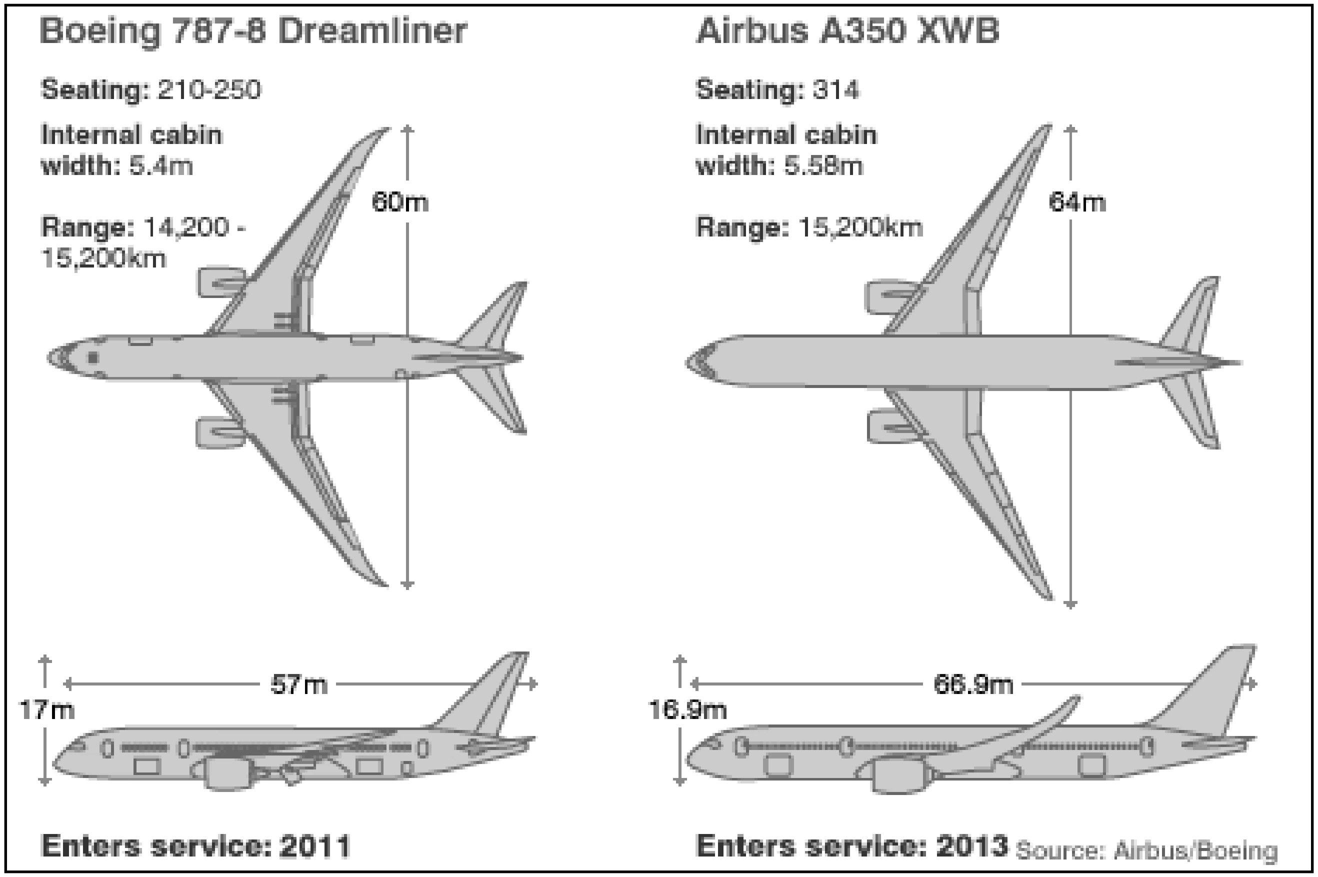
Also, **choice of a model** depends largely on the problem on hand as well as the **resources** for the **modelling task.**

Sketch as Model



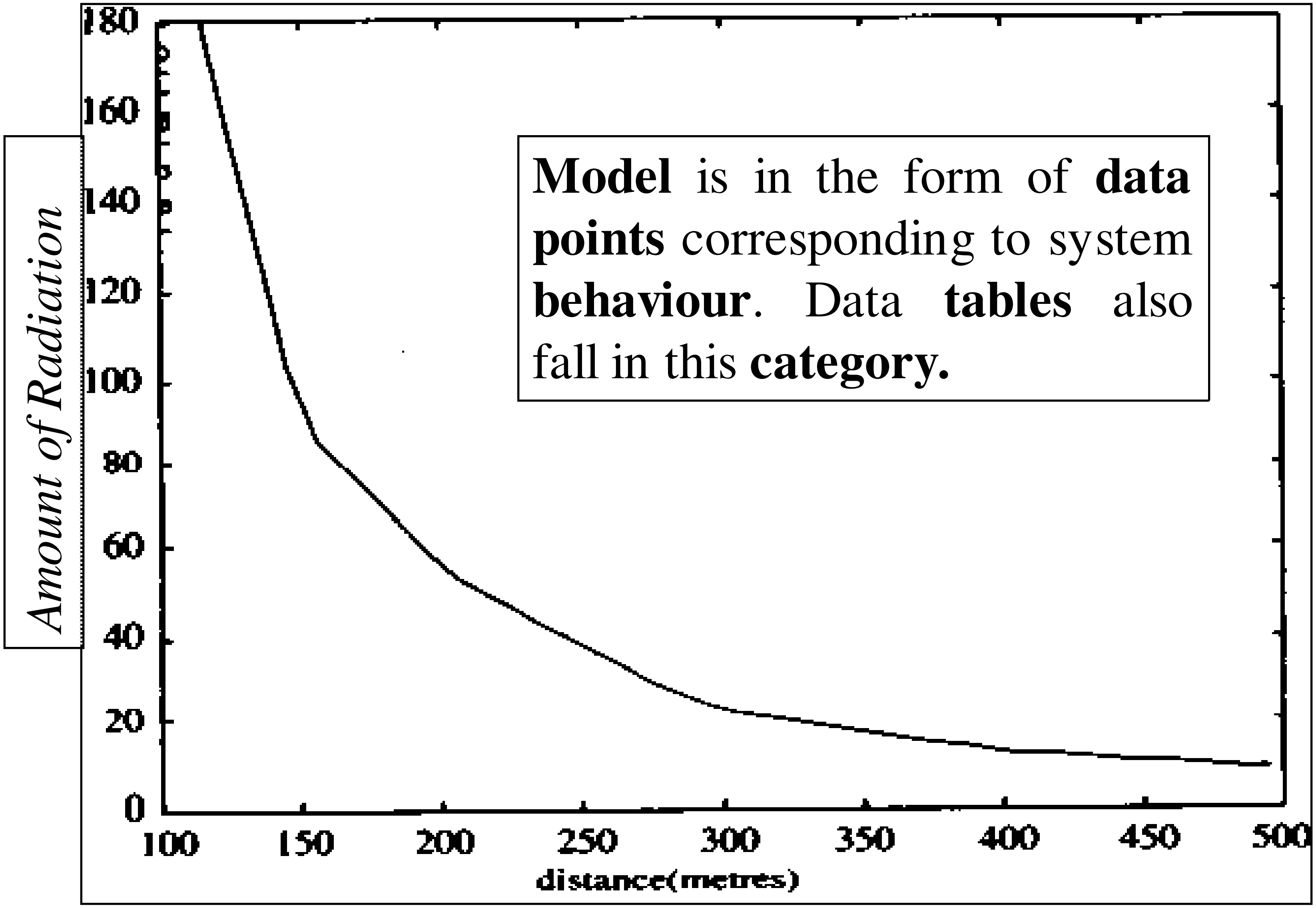
Sketches are the most **common form** of models that most of us deal with. These are **useful** while explaining **the concept.**

Drawing as Model



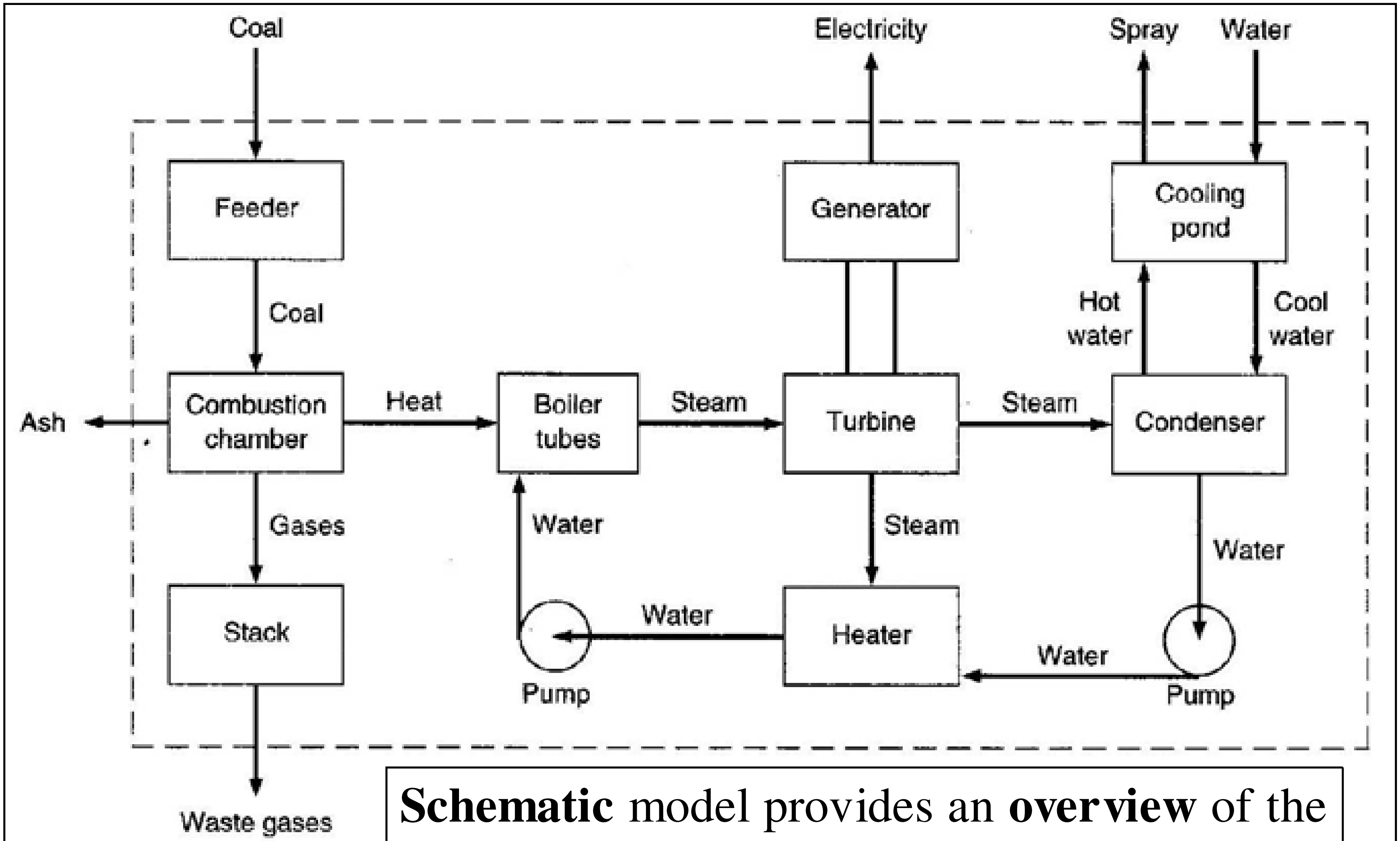
3-view drawings help in overall **visualization** and also drive the **manufacturing process.**

Design Data Model



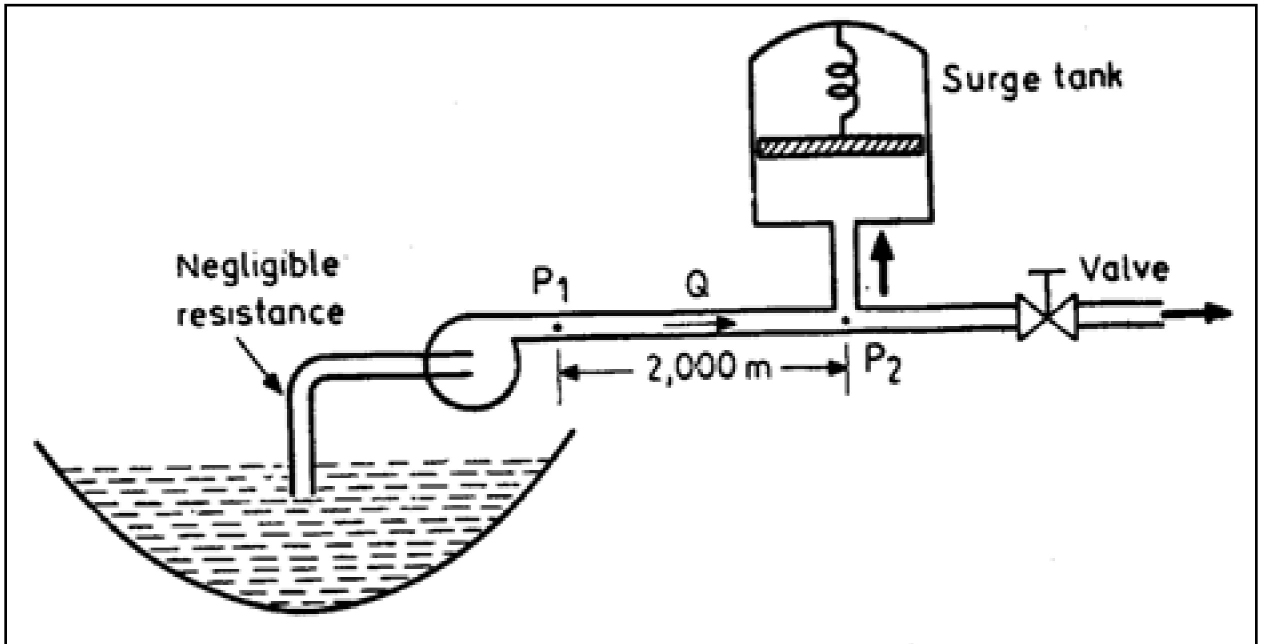
Model is in the form of **data points** corresponding to system **behaviour**. Data **tables** also fall in this **category.**

Schematic Model

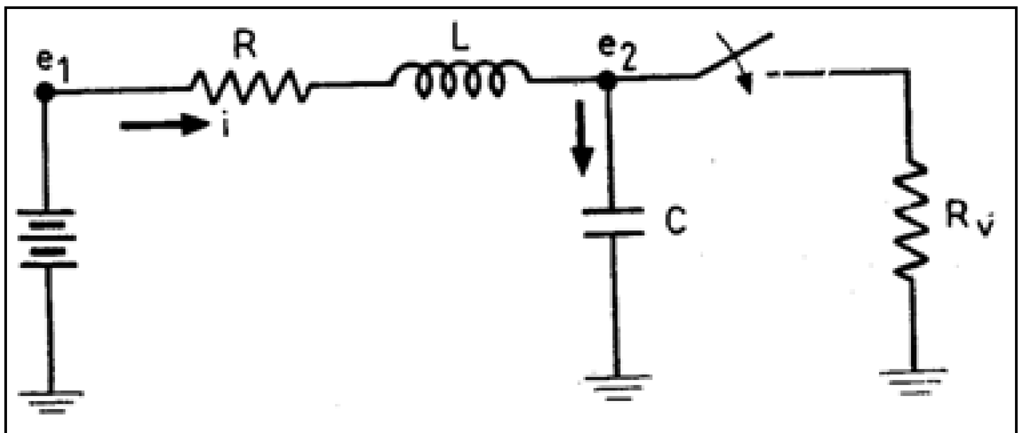


Schematic model provides an **overview** of the system **process /** components, including **data / information flow** within the system.

Analogy Based Model



(a) A simple pumping system



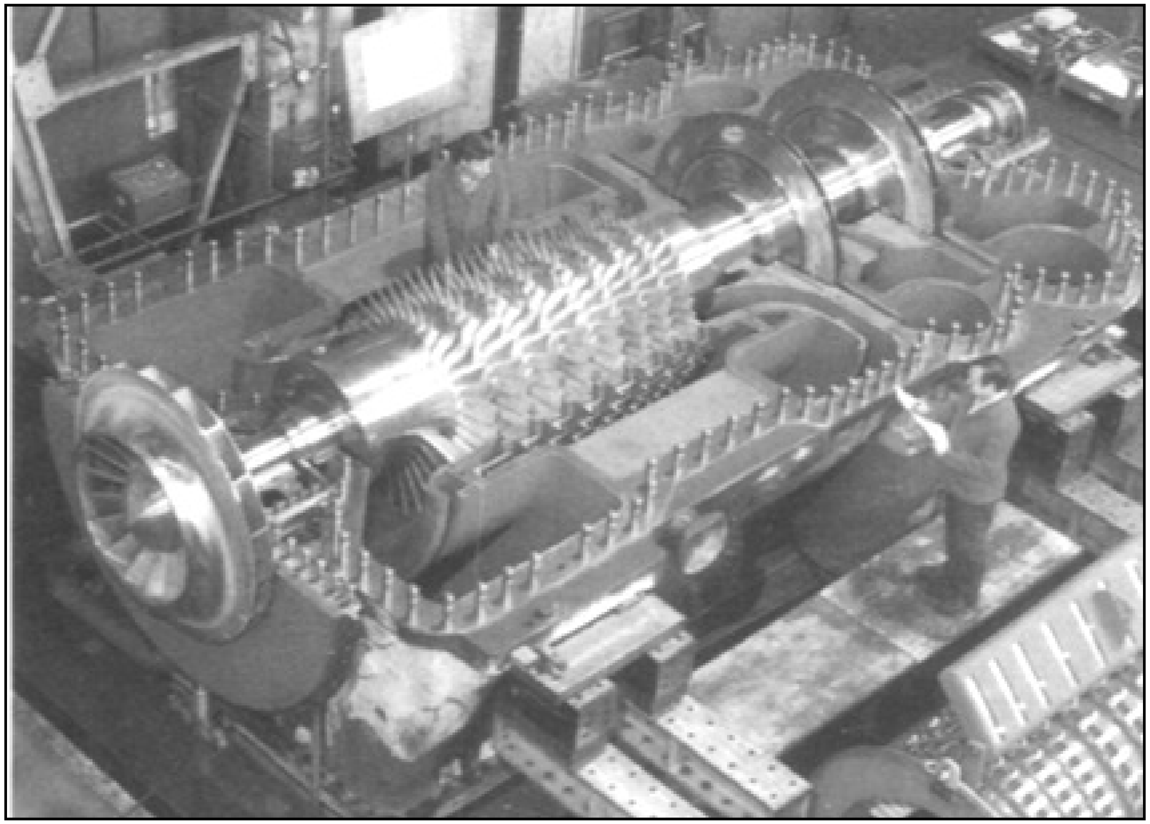
(b) Electrical analog of the pumping system

Analogy based models bring out **equivalence** between different **disciplines** and help in **quick assessment** of performance at **low cost.**

Mockup Model

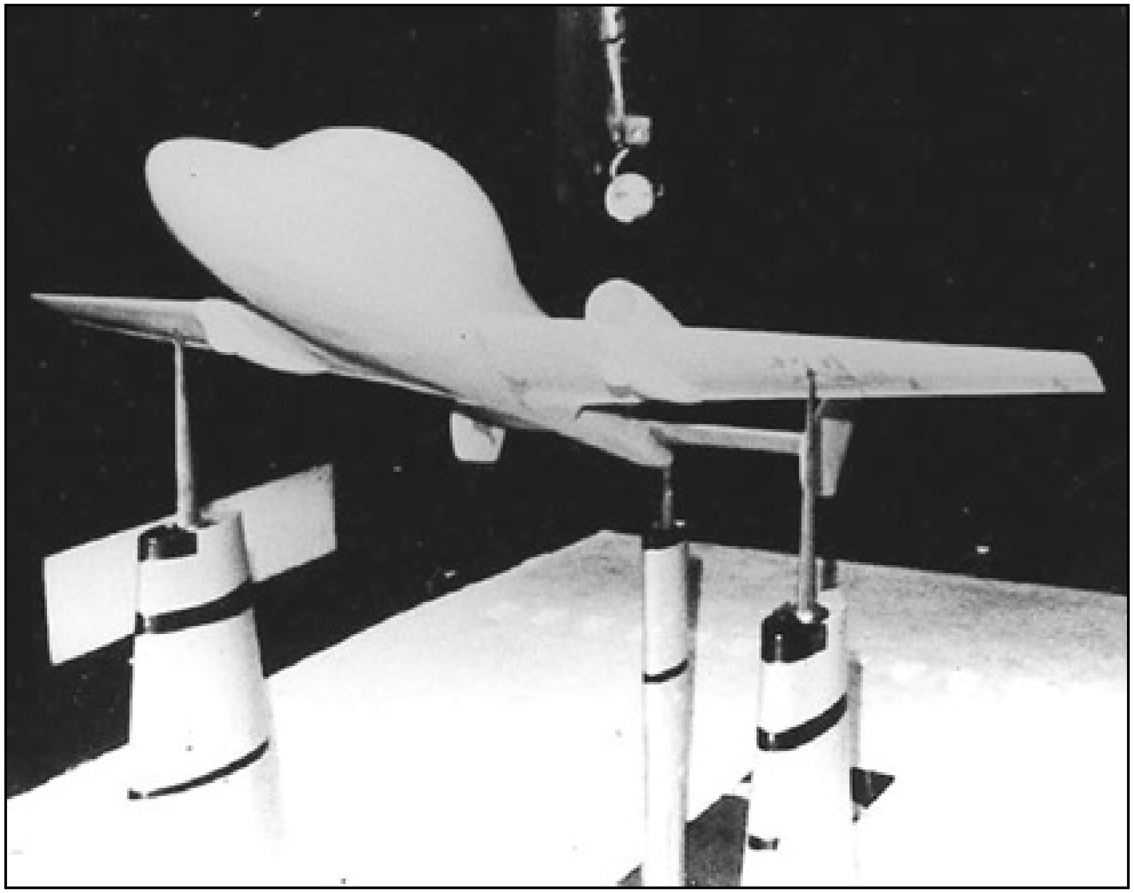
Mock-up models provide a **full scale size feel**, along with **topological** information and **aid in design.**

Cut-section Component Model



Cut-section (or cut-away) models provide details on **internal layout** and help in **re-engineering** of systems.

Scaled Test Model



Scaled models are important **aid** in **verifying** new **designs / concepts** through less expensive **lab level test.**

Mathematical Model

$$X - mg\sin\theta = m(\dot{u}^E + qw^E - rv^E)$$
$$Y + mg\cos\theta\sin\phi = m(\dot{v}^E + ru^E - pw^E)$$
$$Z + mg\cos\theta\cos\phi = m(\dot{w}^E + pv^E - qu^E)$$
$$L = I_x\dot{p} - I_{zx}\dot{r} + qr(I_z - I_y) - I_{zx}pq + qh'_z - rh'_y$$
$$M = I_y\dot{q} + rp(I_x - I_z) + I_{zx}(p^2 - r^2) + rh'_x - ph'_z.$$
$$N = I_z\dot{r} - I_{zx}\dot{p} + pq(I_y - I_x) + I_{zx}qr + ph'_y - qh'_x$$
$$p = \dot{\phi} - \dot{\psi}\sin\theta$$
$$q = \dot{\theta}\cos\phi + \dot{\psi}\cos\theta\sin\phi$$
$$r = \dot{\psi}\cos\theta\cos\phi - \dot{\theta}\sin\phi$$
$$\dot{\phi} = p + (q\sin\phi + r\cos\phi)\tan\theta$$
$$\dot{\theta} = q\cos\phi - r\sin\phi$$
$$\dot{\psi} = (q\sin\phi + r\cos\phi)\sec\theta$$

Mathematical models aim to capture **system** features in the **mathematical framework** so that mathematical **tools** can be employed to characterize the **behaviour.**

Choice of Model Type

In the context of **control**, models are generally **mathematical or experimental** and the choice depends on **knowledge** base and resources.

- **Mathematical Models Used When**
  - A valid & solvable theory exists, along with necessary computational resources.
- **Experimental Models Used When**
  - Mathematical techniques are inadequate.

Comparison of Model Types

Mathematical Models are easier to build and less expensive but are usually less accurate.

Experimental Models are more realistic but also more difficult to synthesize & expensive.

As a first step, models employed for control analysis and design are mathematical in nature.

Mathematical Models

Mathematical Models

Mathematical models are forms which use applicable **mathematical relations** between input and output.

These relations can be **algebraic**, differential, integral, **logical** etc.

In general, such **models** can be created from **first principles**, if clearly known and understood.

We can also **obtain** models from **I/O methods**, in cases where **actual /** analogous **hardware** is available.

First Principles Based Models

In **early stage**s of system development, we have **idea** of only the physics of the **process**, so that we can **employ** the basic **physical laws** that govern the process**.**

As all **processes** involve, mechanics, **elasticity**, fluid dynamics, thermodynamics, **electricity** and magnetism, we can **synthesize** models from laws governing **these effects.**

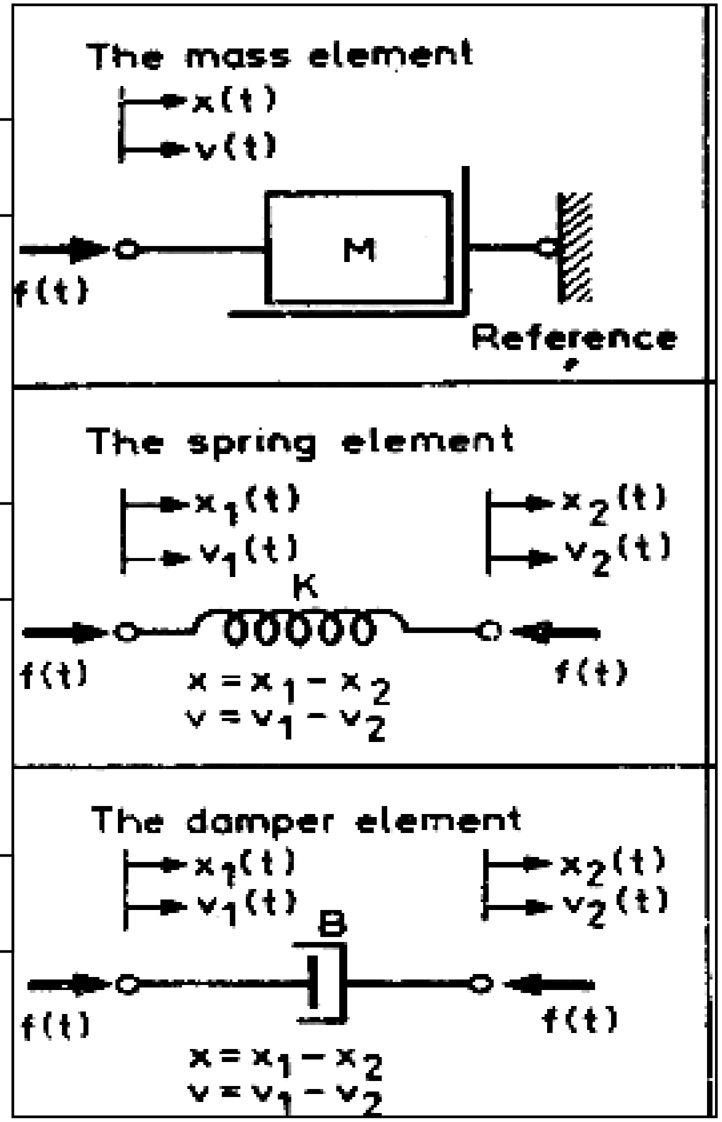
Generally, **idealized versions** of these effects are **employed** to capture the **dominant** features of the process, ignoring the **non-essential** features.

Idealized Mechanical/Elastic Effects

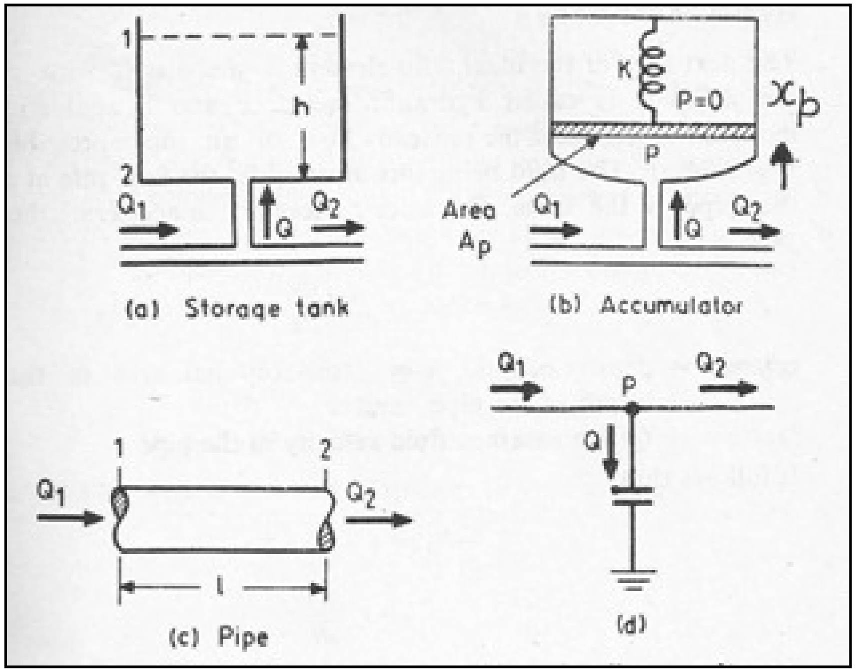
Newton's Law

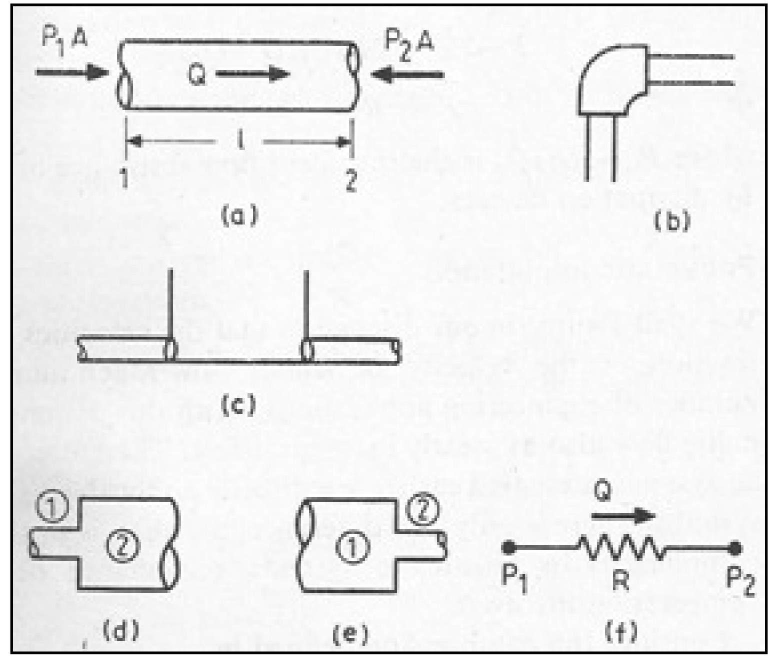
Hooke's Law

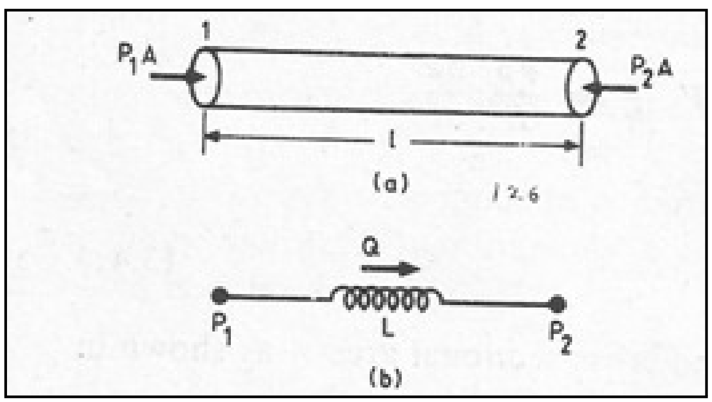
Damping Law



Idealized Fluidic Effects

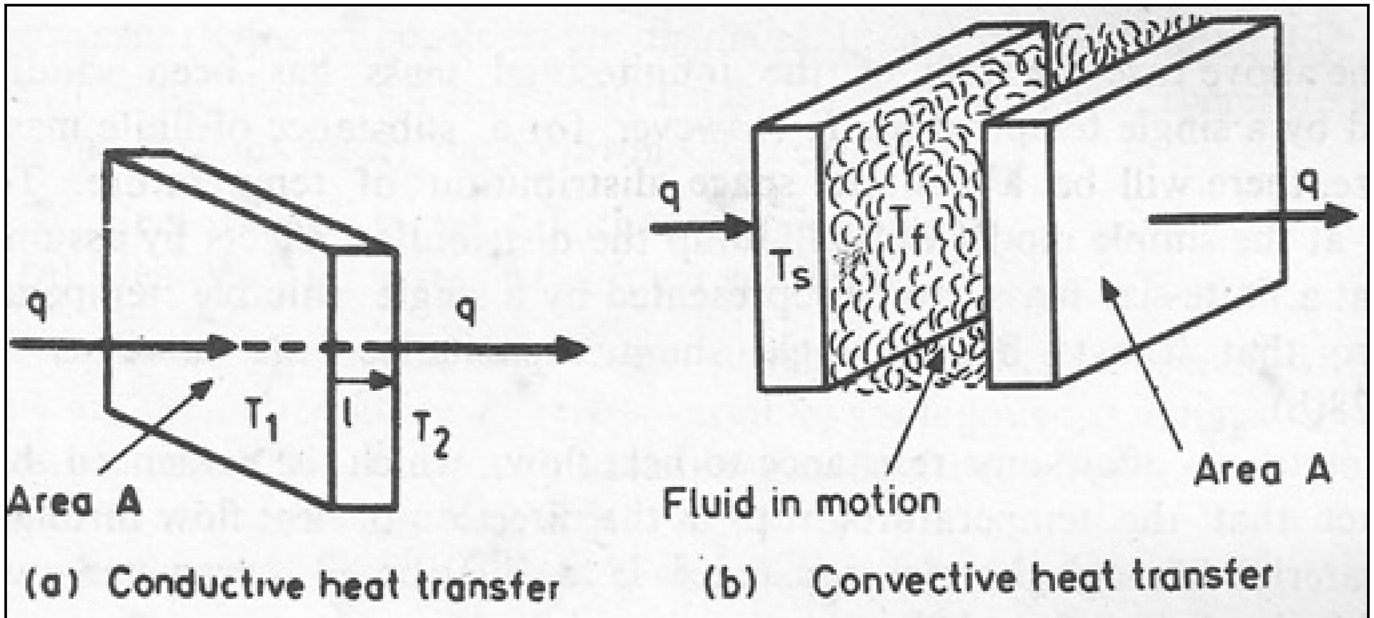






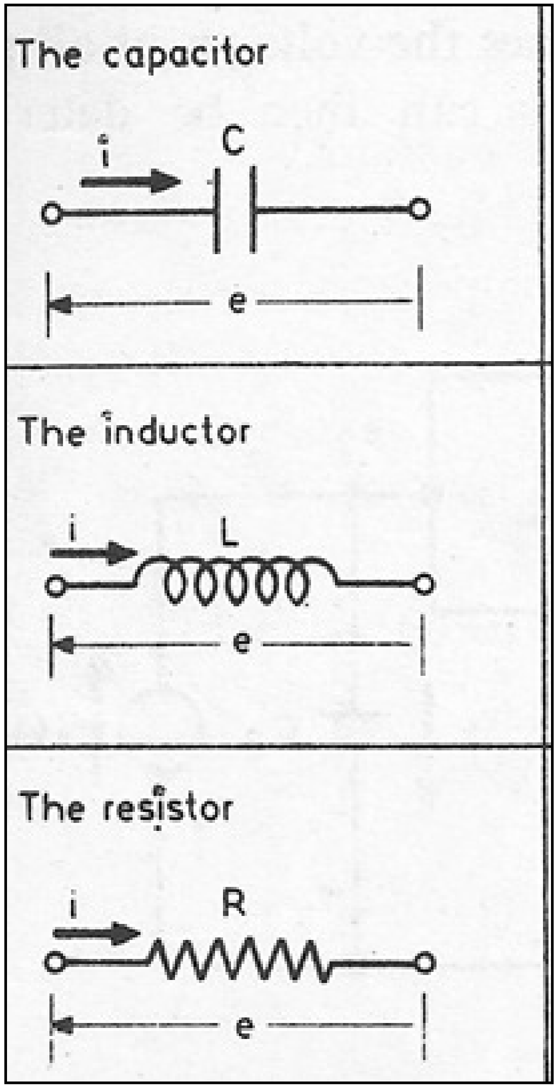
Laws of Fluid Mechanics

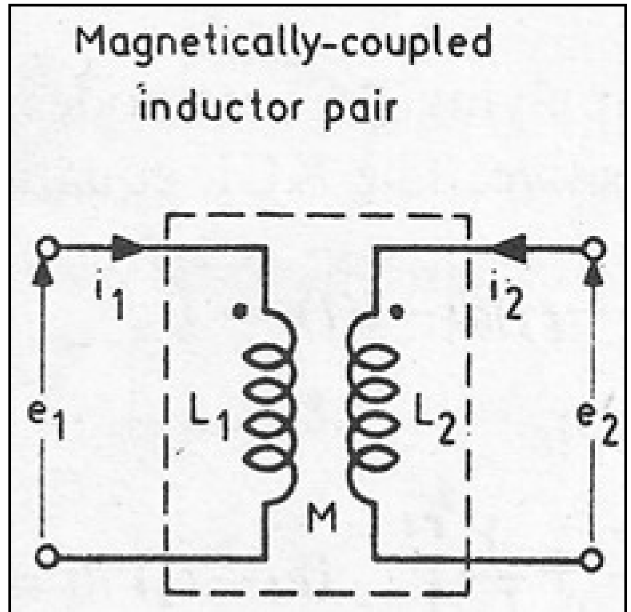
Idealized Thermal Effects



Thermodynamic Laws

Idealized Electrical & Magnetic Effects





$$e_1 = L_1 \frac{di_1}{dt} \pm M \frac{di_2}{dt}$$

$$e_2 = L_2 \frac{di_2}{dt} \pm M \frac{di_1}{dt}$$

Farraday's Law

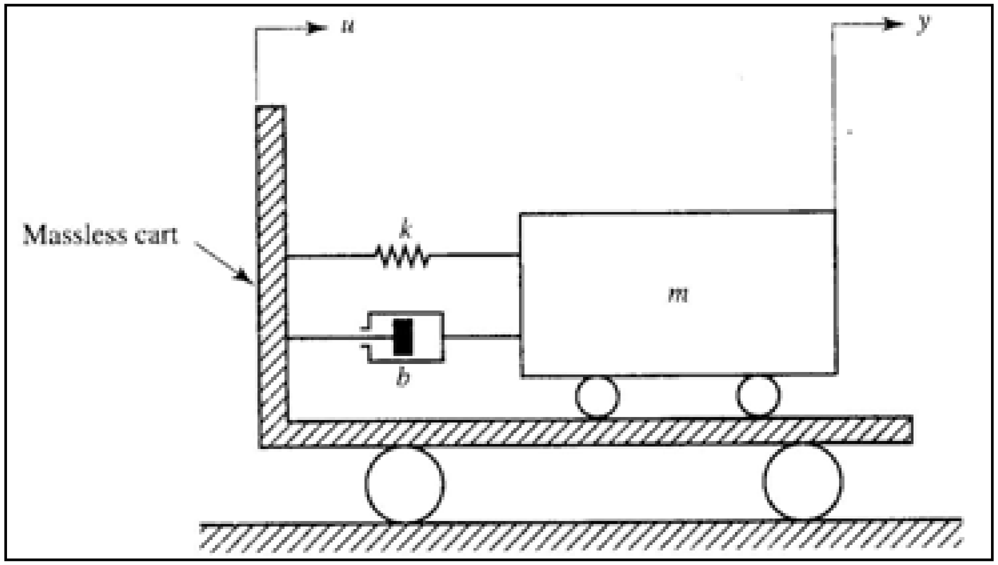
Ohm's Law

Kirchoff's Laws

Mathematical Model Examples

Spring-Mass-Damper System

A **spring-mass-damper** system is shown below.



Equation governing the motion of mass, **'y'**, subjected to input motion '**u**', is as follows.

$$\boxed{k(u-y)+b(\dot{u}-\dot{y})=m\ddot{y}; \quad m\ddot{y}+b\dot{y}+ky=b\dot{u}+ku}$$

Liquid Storage Tank System

A **tank** with cross-sectional area, A(H), is storing water up to a **height, H**. If it is known that outflow (Q) through tap connected to the tank is described by the relation, $Q = K\sqrt{H}$, **the differential equation** for height, H, is as follows.

$$A(-dH) = Qdt \rightarrow \frac{dH}{dt} = -\frac{Q}{A}; \quad Q = K\sqrt{H}; \quad V = AH$$

$$\frac{dH}{dt} + \frac{K\sqrt{H}}{A} = 0; \quad R = \frac{dH}{dQ} = \frac{2H}{Q}; \quad C = \frac{dV}{dH} = A$$

Pressurized Gas System

In a **pressurized gas system,** gas flows through a valve into **a chamber**. If the expansion is **polytropic** process, equation of **gas pressure** variation is obtained as follows.

$$P\left(\frac{V}{m}\right)^n = \frac{P}{\rho^n} = K; \quad PV = R_{gas}T; \quad R = \frac{d(\Delta P)}{\left(dm/dt\right)}$$

$$C = \frac{dm}{dP} = V\frac{d\rho}{dP} = \frac{V\rho}{nP} = \frac{V}{nR_{gas}T}$$

$$CdP = dm \rightarrow RC\frac{dP}{dt} + \Delta P = 0$$

Heating System

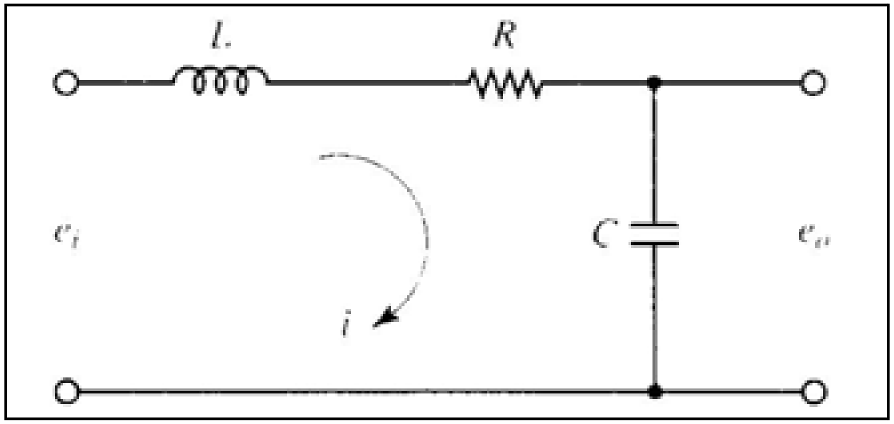
A **block of mass**, M with specific heat, c, is receiving hot **air at temperature, T_1** while its own temperature is at, T. If the convective heat transfer rate (q) is related to the change in temperature, dT, as q = K dT, the **differential equation for T** is as follows.

$$q = K(T_1 - T); \quad qdt = McdT$$

$$K(T_1 - T)dt = McdT \rightarrow \frac{dT}{dt} + \frac{K}{Mc}T = \frac{K}{Mc}T_1$$

Electrical System

An **RLC circuit** is to be used as a **power supply**, as shown below.



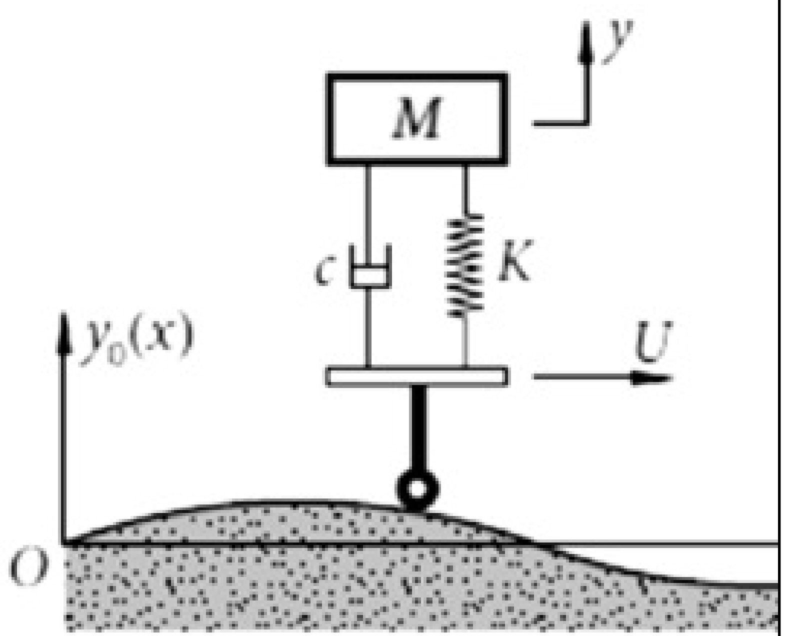
The applicable **differential equation** for e_0 is as follows.

$$e_0 = e_i - L\frac{di}{dt} - Ri; \quad \frac{de_0}{dt} = \frac{i}{C}$$

$$e_i = e_0 + LC\frac{d^2e_0}{dt^2} + RC\frac{de_0}{dt}$$

Bicycle Suspension System

Model a **suspension** system of a **bi-cycle** to characterize the effect of road **bumps.**



Inertia: $M\ddot{y}$; Spring: $k\left(y-y_0\right)$

Damper: $c\left(\dot{y}-\dot{y}_0\right)$; $\sum F=0$

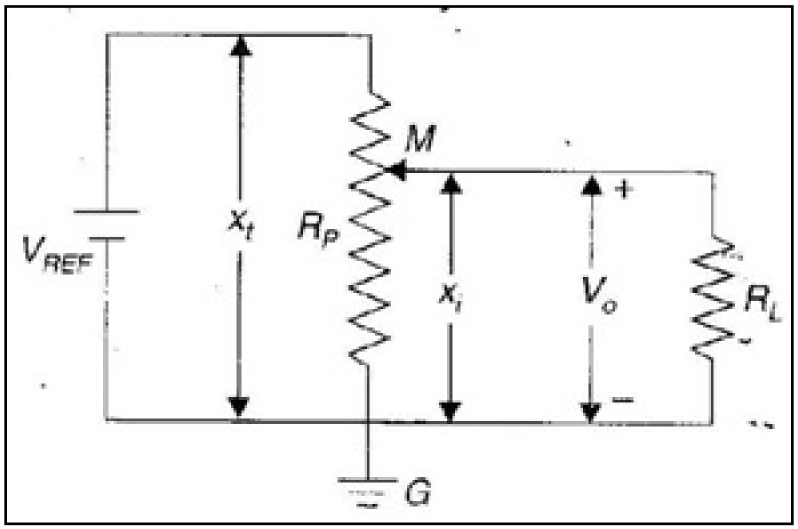
$$M\ddot{y}+c\left(\dot{y}-\dot{y}_0\right)+k\left(y-y_0\right)=0$$

$$M\ddot{y}+c\dot{y}+ky=c\dot{y}_0+ky_0$$

$y_0 \rightarrow$ Road Bump Profile

Potentiometric Sensor System

Model a **potentiometer** that measures **linear displacement** across an electrical **load.**



$$\bar{x}_i = \frac{x_i}{x_t}; \quad \bar{R}_p = \frac{R_p}{R_L}; \quad R_i = \bar{x}_i R_p$$

$$R_{eq} = R_i \parallel R_L = \frac{\bar{R}_p \bar{x}_i}{\left[1 + \bar{x}_i \bar{R}_p\right]} R_L$$

$$V_o = \frac{V_{REF} R_{eq}}{R_{eq} + R_p\left(1 - \bar{x}_i\right)} = K\left(\bar{x}_i\right)\bar{x}_i$$

$$K\left(\bar{x}_i\right) = \left[\frac{V_{REF}}{1 + \left(\bar{R}_p \bar{x}_i\right)\left(1 - \bar{x}_i\right)}\right]$$

$$K\left(\bar{x}_i\right) \rightarrow \text{ Calibration Constant}$$

Summary

Mathematical models of dynamical systems are in the form of **differential equations.**

First Principles based **modelling** needs disciplinary knowledge as well as **assumptions** regarding the process.

Mathematical models of **engineering** systems involve **assembly** of idealized **elements** corresponding to the participating **physical** concepts / processes.