Section 4: Expectation and its Existence

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1 Introduction

Intuitively, the expected value of a random variable is the eventual average value of its IID samples. It is the expected value since it is the weighted sum of its values along with their probabilities. For example, this weighted sum is 7/2 in rolling a six-faced fair dice with labels $\{1, 2, 3, 4, 5, 6\}$. Traditional names such as mean or average are also used for expectation. With some exceptions, only the term expectation or expected value will be used in this course.

Definition 1.1 (Expectation of a random variable)**.** *If X is a discrete random variable then its expectation $\mathbb{E}(X)$ is defined as follows:*

$$\mathbb{E}(X) = \sum_{x:p_X(x)>0} x p_X(x) = \sum_{i \in \mathbb{Z}} x_i p_X(x_i). \tag{1}$$

where, $x_i, i \in \mathbb{Z}$ are the values taken by X with probability one.

If X is a continuous random variable then its expectation $\mathbb{E}(X)$ is defined as follows:

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f_X(x) dx. \tag{2}$$

If X has a finite number of possible sample values, the the sum in (1) will be finite. When X takes a countably infinite number of values, then the sum in (1) may converge or may not converge depending on the properties of $p_X(x)$. This idea is formalized in the next section.

2 Existence of $\mathbb{E}(X)$

Let X be any random variable. Then define two functions of X as below:

$$X^+ = \begin{cases} X & \text{if } X \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad X^- = \begin{cases} -X & \text{if } X < 0 \\ 0 & \textit{otherwise.} \end{cases} \tag{3}$$

Both X^+ and X^- are positive random variables. Further, $X = X^+ - X^-$. By linearity of expectation, we know that,

$$\mathbb{E}(X) = \mathbb{E}(X^+) - \mathbb{E}(X^-). \tag{4}$$

Note that we have used the linearity of expectation which states that

$$\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y) \tag{5}$$

where both sides in the above inequality exist or are well defined. The existence or well-defined nature of $\mathbb{E}(X)$ is explained using X^+ and X^-. The expectation $\mathbb{E}(X)$ exists (as a finite real number) when the sum in (1) or the integral in (2) converges absolutely. From (4) there are following four cases:

- If $\mathbb{E}(X^+) < \infty$ and $\mathbb{E}(X^-) < \infty$, then $\mathbb{E}(X)$ exists (as a finite real number).
- If $\mathbb{E}(X^+) = \infty$ and $\mathbb{E}(X^-) < \infty$, then $\mathbb{E}(X) = +\infty$.
- If $\mathbb{E}(X^+) < \infty$ and $\mathbb{E}(X^-) = \infty$, then $\mathbb{E}(X) = -\infty$.

- If $\mathbb{E}(X^+) = \infty$ and $\mathbb{E}(X^-) = \infty$, then $\mathbb{E}(X)$ is undefined.

The use of the above test is explained using Cauchy random variables in the next example.

Example 2.1 (Expectation and Cauchy random variables)**.** *Let X be a continuous random variable with a PDF*

$$f_X(x) = \frac{1}{\pi(1+x^2)}; x \in \mathbb{R}. \tag{6}$$

Then its CDF is

$$F_X(x) = \frac{1}{\pi}\arctan(x) + \frac{1}{2}; x \in \mathbb{R}. \tag{7}$$

The distributions for X^+ and X^- are defined as

$$F_{X^+}(x) = F_{X^-}(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & x = 0 \\ \frac{1}{\pi}\arctan(x) + \frac{1}{2} & x > 0. \end{cases} \tag{8}$$

It can be shown that, $\mathbb{E}(X^+) = \infty$ and $\mathbb{E}(X^-) = \infty$; therefore, $\mathbb{E}(X)$ does not exists.

This example will be revisited in the context of Cauchy random variables and weak law of large numbers. In the next section, the expectation of a random variable will be obtained from its CDF.

3 Expectation and CDF

An alternate approach to compute expectation is from CDF of the random variable. This approach applies to all the random variables whether they are continuous, discrete, or mixed. For non-negative random variable, $X > 0$, complementary CDF F_X^c of a random variable is defined as

$$F_X^c(x) = 1 - F_X(x) = \mathbb{P}(X > x). \tag{9}$$

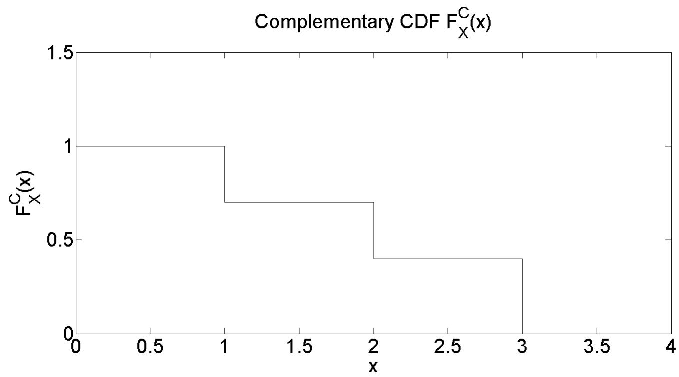

Figure 1: The area under the graph of complement of the CDF is the same as $\mathbb{E}(X)$ for a discrete random variable taking values in the set $\{x_1, x_2, x_3\}$ with respective probabilities p_1, p_2, p_3 such that $p_1 + p_2 + p_3 = 1$.

If $X \geq 0$ then $\mathbb{E}(X)$ is the integral of complemetary CDF for positive values of x. That is,

$$\mathbb{E}(X) = \int_0^\infty F_X^c(x)dx = \int_0^\infty \mathbb{P}(X > x)dx = \int_0^\infty \mathbb{P}(X \geq x)dx. \tag{10}$$

For the discrete case, an illustrative graphical proof is available from the textbooks [1]. This is shown in Figure 1, Let X be a discrete random variable that takes values $\{x_1, x_2, x_3\}$ with probability 1 and $0 < x_1 < x_2 < x_3 < \infty$. Let

$$p_X(x_1) = p_1, \quad p_X(x_2) = p_2, \quad \text{ and } p_X(x_3) = p_3. \tag{11}$$

Then, the expectation of X is

$$\mathbb{E}(X) = p_1x_1 + p_2x_2 + p_3x_3. \tag{12}$$

This is also the area under the curve $F_X^c(x)$, i.e.,

$$\int_0^\infty F_X^c(x)dx = p_1x_1 + p_2x_2 + p_3x_3. \tag{13}$$

For two sided random variable, (i.e. X need not be non-negative):

$$\mathbb{E}(X) = \int_0^\infty F_X^c(x)dx - \int_{-\infty}^0 F_X^c(x)dx \tag{14}$$

The above formula can be used whenever expectation exists.

References

[1] Robert G. Gallager, *Stochastic Processes*, Cambridge University Press.