Observer

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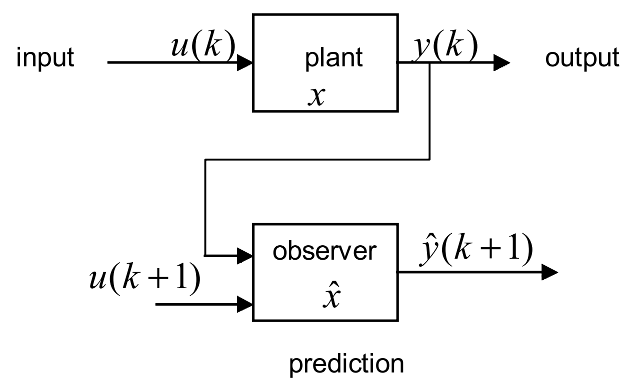
Object of observer

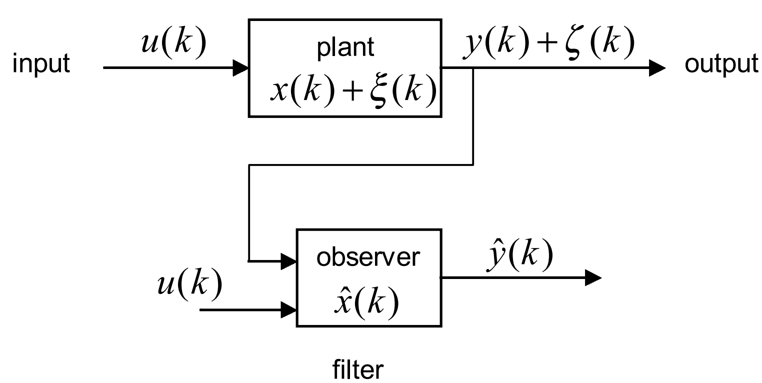
(1) prediction

$$\widehat{y}(k+1) = f_1\left[u(k+1), y(k), \Theta_1\right]$$



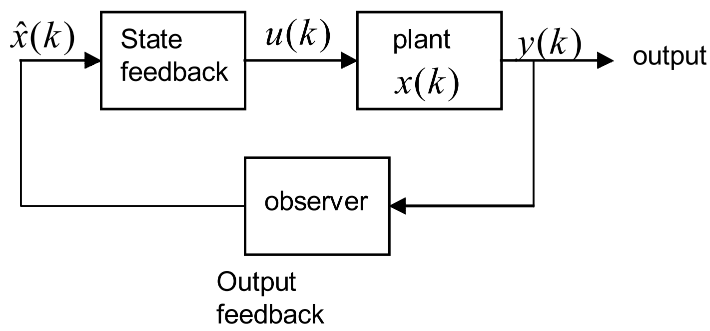
(2) filter (analysis of system)

$$\begin{aligned}\widehat{y}(k) &= f_2\left[u(k), y(k), \Theta_2\right] \\ y(k) &= \overline{y}(k) + \varsigma(k)\end{aligned}$$



(3) output feedback

$$u(k) = K\widehat{x}(k) = f_3\left[u(k), y(k), \Theta_3\right]$$



Conditions for building observer

(1) The plant is observable

(2) Input/output are known

(3) The plant is known (generally)

Linear observer

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

The solution for x is

$$x(t) = \phi(t, t_0)\, x(t_0) + \int_{t_0}^{t} \phi(t, \tau)\, Bu(\tau)\, d\tau, \qquad \phi(t, t_0) = e^{A(t-t_0)}$$
$$y(t) = C\phi(t, t_0)\, x(t_0) + C\int_{t_0}^{t} \phi(t, \tau)\, Bu(\tau)\, d\tau$$

where ϕ is state transition matrix of $\dot{x} = Ax$.

If $x(t_0)$ is known, $A \to \phi(t, t_0)$, B, C, $u(t)$ are known, $x(t)$ can be calculated.

If $x(t_0)$ is unknown,

$$C\phi(t, t_0)\, x(t_0) = y(t) - C \int_{t_0}^{t} \phi(t, \tau)\, Bu(\tau)\, d\tau \qquad (1)$$

$$\begin{aligned} x(t_0) &= (C\phi)^{-1} \left[y - C \textstyle\int_{t_0}^{t} \phi Bud\tau \right] \\ &= f(A, B, C, y, u) \end{aligned}$$

Theorem 1 *If and only if* $[C\phi(t, t_0)]^{-1}$ *exist (has* n *linear independent column on* (t, t_0)*), the linear system is observable.*

Proof. Multiply $\phi^T C^T$ and integrate from t_0 to t_1 in (1)

$$\left(\int_{t_0}^{t_1} \phi^T C^T C\phi dt \right) x(t_0) = \left[\int_{t_0}^{t_1} (C\phi)^T C\phi dt \right] x(t_0) = V(t_0, t_1)\, x$$

If all columns of $C\phi$ are linear independent on $[t_0, t_1]$, $V(t_0, t_1)$ is non-singular, so

$$x(t_0) = V^{-1}(t_0, t_1) \int_{t_0}^{t_1} (C\phi)^T \left[y - C \int_{t_0}^{t} \phi Bud\tau \right] dt$$

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For **time-invariance case**

$V = \left[C, CA, \cdots CA^{n-1} \right]^T$, if rank($V$)$=n$, the linear system is observable.

Example 1 *Linear system*

$$\begin{cases} \dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{cases}$$

$V = [C, CA] = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, *rank(V)=* 1, *it is not observable. Because*

$$y = x_1$$
$$\dot{x}_1 = x_1, \quad \dot{x}_2 = x_2$$

we cannot find a relation between y *and* x_2. *But if* $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $V = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, *rank(V)=* 2, *it is observable, because*

$$y = x_1$$
$$\dot{x}_1 = x_1 + x_2, \quad \dot{x}_2 = x_2$$

There exist a relation between y *and* x_2.

Luenberger observer

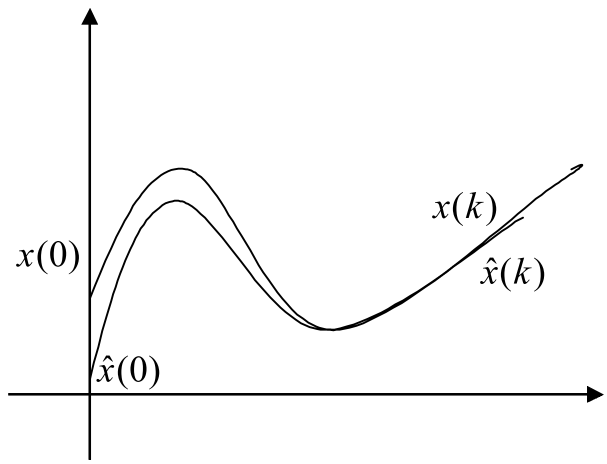
observer=(copy of plant)+(modification term).

For linear plant

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \qquad x(t_0) = a$$

Luenberger observer

$$\begin{cases} \dot{\widehat{x}} = A\widehat{x} + Bu + L\left(y - \widehat{y}\right) \\ \widehat{y} = C\widehat{x} \end{cases} \qquad \widehat{x}\left(t_0\right) = b$$



Luenberger observer is used only for the initial condition error

Asymptotic observer

observer error is $\widetilde{x} = x - \widehat{x}$,

$$\dot{\widetilde{x}} = (A - LC)\,\widetilde{x}$$

If we select L such that $(A - LC)$ is asymptotic stable (the eigenvalues of $(A - LC)$ are negative), $\lim_{t\to\infty} \widetilde{x} = 0$.

Theorem 2 *If (A, C) is observable (rank($\left[C, CA, \cdots CA^{n-1}\right]^T$)$=n$), then all eigenvalues of $(A - LC)$ can be arbitrarily assigned by select L.*

Example 2

$$\begin{cases} \dot{x} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u & x(0) = [0, 0] \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{cases}$$

it is observable.

$$\begin{cases} \dot{\widehat{x}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \widehat{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} (y - \widehat{y}) & \widehat{x}(0) = [1, 1] \\ \widehat{y} = \begin{bmatrix} 1 & 0 \end{bmatrix} \widehat{x} \end{cases}$$

$A - LC = \begin{bmatrix} 1 - l_1 & 1 \\ -l_2 & 1 \end{bmatrix}$, *eigenvalues satisfy* $[sI - (A - LC)] = 0 = s^2 + (l_1 - 2)s + 1 - l_1 + l_2$, *we hope* $s_1 = -1$, $s_1 = -2$, $L = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

Reduced order observer

If the linear system can be transformed into

$$\begin{cases} \dot{x} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \\ y = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{cases}$$

i.e.

$$\begin{cases} \dot{x}_1 = A_{11}x_2 + A_{12}x_2 + B_1 u \\ \dot{x}_2 = A_{22}x_2 + A_{21}x_1 + B_2 u \\ y = x_1 \end{cases}$$

Define new output

$$w = \dot{y} - A_{11}x_2 - B_1 u$$

The plant becomes

$$\begin{cases} \dot{x}_2 = A_{22}x_2 + (A_{21}x_1 + B_2 u) \\ w = A_{12}x_2 \end{cases}$$

The reduced order observer

$$\begin{cases} \dot{\widehat{x}}_2 = A_{22}\widehat{x}_2 + \overline{U} + L\left(w - \widehat{w}\right) \\ \widehat{w} = A_{12}\widehat{x}_2 \end{cases}$$

where $\overline{U} = A_{21}x_1 + B_2 u$. The observability of (A, C) is the same as the observability of (A_{22}, A_{12}) .

Canonical form of observer

Theorem 3 *If n dimensional linear time-invariant system is observable, then is can be transformed into*

$$\begin{cases} \dot{\overline{x}} = \begin{bmatrix} 0 & \cdots & 0 & -\alpha_n \\ 1 & & \vdots & \vdots \\ & \ddots & 0 & \\ 0 & & 1 & -\alpha_1 \end{bmatrix} \overline{x} + \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} u \\ y = \begin{bmatrix} 0 & 1 \end{bmatrix} \overline{x} \end{cases}$$

where α_i and β_i satisfy

$$G(s) = C\left(sI - A\right)^{-1} B = \frac{\beta_n s^{n-1} + \cdots + \beta_1}{s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n}$$

Output feedback

Linear plant

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

we hope the state feedback control is $u = Kx$. If the plant is known, but x is not measurable, Luenberger observer is used

$$\begin{cases} \dot{\widehat{x}} = A\widehat{x} + Bu + L\left(y - \widehat{y}\right) \\ \widehat{y} = C\widehat{x} \end{cases} \qquad \widehat{x}\left(t_0\right) = b$$

The control is $u = K\widehat{x}$.

We have two problems: 1) observer

$$\dot{\widetilde{x}} = \left(A - LC\right)\widetilde{x}$$

2) feedback control

$$\dot{x} = Ax + BK\widehat{x} = Ax + BK\left(x - \widetilde{x}\right) = Ax + BKx - BK\widetilde{x}$$

Combine them together

$$\begin{bmatrix} \dot{x} \\ \dot{\widetilde{x}} \end{bmatrix} = \begin{bmatrix} A + BK & -BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ \widetilde{x} \end{bmatrix}$$

The stability of the system is depended on $(A + BK)$ and $(A - LC)$.

1) we can select K such that $(A + BK)$ is stable if (A, B) is controllable

1) we can select L such that $(A - LC)$ is stable if (A, C) is observable

It is separation property (principle) for linear system