Homework 6 Solution Outlines

Problem 2

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} \tag{1}$$

1. Defining inner product as $\langle X, Y \rangle = E[XY]$ and using cauchy-schwarz inequality that $|\langle X, Y \rangle|^2 \leq \langle X, X \rangle \langle Y, Y \rangle$, we get $|Cov(X,Y)|^2 \leq Var(X)Var(Y)$. The inequality $|\rho_{X,Y}| \leq 1$ follows.

2. If X and Y are independent, then $Cov(X,Y) = E[XY] - E[X]E[Y] = 0$. Hence $\rho_{X,Y} = 0$.

3. Given $Y = aX + b$. So, $\sigma_Y^2 = a^2\sigma_X^2$ and $Cov(X,Y) = E[(X - E[X])(Y - E[Y])]$. Using $E[Y] = aE[X] + b$, we get $Cov(X,Y) = a\sigma_X^2$.
   $\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = \frac{a\sigma_X^2}{|a|\sigma_X^2} = 1$,if $a > 0$ and is equal to -1 when a is negative as σ_X and σ_Y are positive.

4. $Cov(X,Y) = E(XY) - E(X)E(Y)$. Using $Y = X^2$ where X is uniformly distributed in [-1,1], we get $E(XY) = E(X^3) = 0$. Also, $E(X) = 0$. So we get $Cov(X,Y) = 0$. Hence, $\rho_{X,Y} = 0$.