Quiz 2 Solutions

Communication Systems (EE 308), Autumn'19

QUESTION 1

(a) Note that $X(0) = A$ if $\Theta - \frac{T}{2} \geq 0$, *i.e.*, $\Theta \geq \frac{T}{2}$ and 0 else. So:

$$\begin{aligned}\eta_X(t) &= E\{X(0)\} \\ &= A \times P\left(\Theta \geq \frac{T}{2}\right) \\ &= \frac{A}{2}.\end{aligned}$$

Now, $R_X(\tau) = E\{X(0)X(\tau)\}$. For an integer n, $R_X(\tau+nT) = E\{X(0)X(\tau+nT)\} = E\{X(0)X(\tau)\} = R_X(\tau)$ since $X(t)$ is periodic with period T. Thus, $R_X(\cdot)$ is periodic with period T. Also, $R_X(\cdot)$ is an even function. So $R_X(\cdot)$ is completely determined by its values for $\tau \in \left[0, \frac{T}{2}\right]$.

Fix $\tau \in \left[0, \frac{T}{2}\right]$. We have:

$$\begin{aligned}R_X(\tau) &= E\{X(0)X(\tau)\} \\ &= \begin{cases} A^2, & \text{if } X(0) = X(\tau) = 1 \\ 0, & \text{else.} \end{cases}\end{aligned}$$

But $X(0) = X(\tau) = 1$ iff $\tau \leq \Theta - \frac{T}{2}$, *i.e.*, $\Theta \geq \frac{T}{2} + \tau$. So $P\left(X(0) = X(\tau) = 1\right) = P\left(\Theta \geq \frac{T}{2} + \tau\right) = \frac{T/2-\tau}{T}$. Thus, $R_X(\tau) = A^2 P\left(X(0) = X(\tau) = 1\right) = A^2 \times \frac{T/2-\tau}{T}$.

(b) The cascade is effectively an LTI system with impulse response $h(t) = h_1(t) * h_2(t)$.

As shown in class, $\eta_Y(t) = \eta_X \int_{-\infty}^{\infty} h(\tau)d\tau = \frac{A}{2}\int_{-\infty}^{\infty} h(\tau)d\tau$ and $R_Y(\tau) = R_X(\tau) * h(\tau) * h(-\tau)$.

QUESTION 2

This is problem 4.3 from Madhow, which is part of Homework 5.

QUESTION 3

(a) The PSD of the in-phase and quadrature components is given by:

$$S_{n_I}(f) = S_{n_Q}(f) = \begin{cases} S_n(f - f_c) + S_n(f + f_c), & |f| < 7, \\ 0, & \text{otherwise.} \end{cases}$$

$f_c = 7$ kHz is the mid-band frequency, so that:

$$S_{n_I}(f) = S_{n_Q}(f) = \begin{cases} N_0, & |f| < 7, \\ 0, & \text{otherwise.} \end{cases}$$

The cross-spectral density is given by:

$$S_{n_I,n_Q}(f) = \begin{cases} j\left[S_n(f+f_c) - S_n(f-f_c)\right], & |f| < 7, \\ 0, & \text{otherwise.} \end{cases}$$

However, $S_n(f-f_c) = S_n(f+f_c)$ for $|f| < 7$; so $S_{n_I,n_Q}(f) = 0$.

(b) With $f_c = 6$ kHz:

$$S_{n_I}(f) = S_{n_Q}(f) = \begin{cases} N_0/2, & 3 < |f| < 5, \\ N_0, & |f| < 3, \\ 0, & \text{otherwise.} \end{cases}$$

The cross-spectral density is given by:

$$S_{n_I,n_Q}(f) = \begin{cases} -jN_0/2, & -5 < f < -3, \\ jN_0/2, & 3 < f < 5, \\ 0, & \text{otherwise.} \end{cases}$$

QUESTION 4

(a) True.

Proof: Consider the following joint distribution of the time-shifted process $Y(t+c)$:

$$\begin{aligned} F_{Y(t_1+c),\ldots,Y(t_n+c)}(y_1,\ldots,y_n) &= P(X(t_1+c-Z) \le y_1,\ldots,X(t_n+c-Z) \le y_n) \\ &= \int_{-\infty}^{\infty} P(X(t_1+c-z) \le y_1,\ldots,X(t_n+c-z) \le y_n) f_Z(z)dz. \end{aligned}$$

But $P(X(t_1+c-z) \le y_1,\ldots,X(t_n+c-z) \le y_n) = P(X(t_1) \le y_1,\ldots,X(t_n) \le y_n)$ since the process $X(t)$ is SSS. So $F_{Y(t_1+c),\ldots,Y(t_n+c)}(y_1,\ldots,y_n) = F_{X(t_1),\ldots,X(t_n)}(y_1,\ldots,y_n)$, which is independent of c. Hence, the process $Y(t)$ is SSS.

(b) False. A counterexample is as follows. Let $X(t) = \sin(t+\Theta)$, where Θ is uniformly distributed in $(-\pi,\pi)$ and $Z = \Theta$. It was shown in class that the process $X(t)$ is SSS. But $Y(t) = X(t-Z) = \sin(t)$ is not SSS.