Laplace Transform Method

– <u>*Laplace Transform as Solution Tool*</u>

Laplace Transform Based Approach

Laplace Transform Definition

Laplace transform is an important **tool** for solving **LTI** systems without explicit integration. **Laplace** transform is an integral **transform**, defined for a function **f(t),** as below.

$$F(s) = \int_0^\infty e^{-st} f(t)dt = L(f)$$

The above **integral**, and the Laplace **transform**, exists if the integrand **'e^{-st} f(t)'** goes to **zero** fast enough as $t \to \infty$.

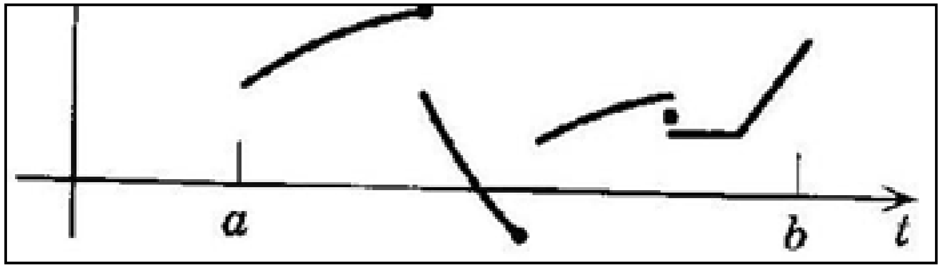
Here, '**s**' is called **Laplace** variable and is a complex quantity defined as '**σ ±jω**', where, 'σ' represents the **real axis** while, 'jω' represents the **imaginary axis** (s – plane).

Properties of Laplace Transform

Laplace transform is a **linear operation** and hence it is **applicable** only in the context of **LTI** systems.

Laplace transform, by virtue of **integral**, replaces every operation of **calculus** by an **algebraic** operation.

However, **f(t)** needs to be at least **piece-wise continuous.**



Typical Laplace Transforms

	$f(t)$	$F(s)$
1	Unit impulse $\delta(t)$	1
2	Unit step $1(t)$	$\frac{1}{s}$
3	t	$\frac{1}{s^2}$
4	$\frac{t^{n-1}}{(n-1)!} \quad (n = 1, 2, 3, \ldots)$	$\frac{1}{s^n}$
5	$t^n \quad (n = 1, 2, 3, \ldots)$	$\frac{n!}{s^{n+1}}$
6	e^{-at}	$\frac{1}{s+a}$
7	te^{-at}	$\frac{1}{(s+a)^2}$
8	$\frac{1}{(n-1)!} t^{n-1} e^{-at} \quad (n = 1, 2, 3, \ldots)$	$\frac{1}{(s+a)^n}$
9	$t^n e^{-at} \quad (n = 1, 2, 3, \ldots)$	$\frac{n!}{(s+a)^{n+1}}$

Transform table is **bi-directional** i.e. we can **get F(s)** for a given f(t) or **vice versa.**

Laplace Transform as Solution Tool

As **Laplace** transform involves **integration**, we can use it to **convert differential** equations into **algebraic** equations, using the following **properties.**

$$\mathcal{L}_{\pm}\left[\frac{d^n}{dt^n} f(t)\right] = s^n F(s) - \sum_{k=1}^{n} s^{n-k} \overset{(k-1)}{f}(0\pm)$$

$$\text{where } \overset{(k-1)}{f}(t) = \frac{d^{k-1}}{dt^{k-1}} f(t)$$

$$\mathcal{L}\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$$

Laplace Transform Example

Consider the following **2nd order** LTI system.

$$\boxed{\ddot{c}(t) + 2\zeta\omega_n \dot{c}(t) + \omega_n^2 c(t) = \omega_n^2 r(t)}$$

We can take **term-by-term** Laplace transform of the above **model** to arrive at the **algebraic** equation, as shown below.

$$\left[s^2 C(s) - \dot{c}(0) - sc(0) \right] + 2\zeta\omega_n \left[sC(s) - c(0) \right] + \omega_n^2 C(S) = \omega_n^2 R(s)$$

$$\left[s^2 + 2\zeta\omega_n s + \omega_n^2 \right] C(s) = \left\{ \dot{c}(0) + \left(s + 2\zeta\omega_n \right) c(0) \right\} + \omega_n^2 R(s)$$

The above **algebraic system** can be suitably **manipulated** and c(t) can be **obtained** through the table of **transforms.**

Summary

Laplace **transform** provides an **elegant way** of converting **differential** equations into **algebraic** form, which are significantly **easier to manipulate.**

Transfer Function Based Solution

TF as LTI System Solution

TF is an important **solution** building block and is defined as **ratio** of the Laplace transforms of **output and input** for a system (under **zero initial** conditions), as detailed below.

$$a_0 \frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_{n-1} \frac{dy}{dt} + a_n y = b_0 \frac{d^m u}{dt^m} + b_1 \frac{d^{m-1} u}{dt^{m-1}} + \cdots + b_{m-1} \frac{du}{dt} + b_m u$$

$$\text{Transfer Function: } G(s) = \frac{L(\text{Output})}{L(\text{Input})} \Big|_{\text{Zero Initial Conditions}} = \frac{Y(s)}{U(s)}$$

$$\left(a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n\right) Y(s) = \left(b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m\right) U(s)$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

Transfer Function Features

Transfer function **G(s)** is the s-domain **unit impulse response**, as shown below.

$$\boxed{Y(s) = G(s)\cdot U(s) \rightarrow \text{ For } U(s) = \delta(s) = 1, \quad Y(s) = G(s)}$$

In general, **G(s)** is represented in **polynomial** & **factored** forms, as shown below.

$$G(s) = K\,\frac{(s^m + b_1 s^{m-1} + \ldots.. + b_{m-1}s + b_m)}{(s^n + a_1 s^{n-1} + \ldots.. + a_{n-1}s + a_n)}$$

$$= K\,\frac{(s - z_1)(s - z_2)\cdots(s - z_m)}{s^k (s - p_{k+1})(s - p_{k+2})\cdots(s - p_n)}$$

p_i's: **poles**; **roots** of the **denominator** polynomial

z_j's: **zeros; roots** of the **numerator** polynomial.

K: **Gain** parameter

k: System **Type**

Transfer Function Example

Consider the **Laplace** transform of a **2nd** order system.

$$\left[s^2 + 2\zeta\omega_n s + \omega_n^2\right]C(s) = \left\{\dot{c}(0) + \left(s + 2\zeta\omega_n\right)c(0)\right\} + \omega_n^2 R(s)$$

We can **write** the corresponding **transfer function**, by applying its **definition** as follows.

$$G(s) = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{\left[s^2 + 2\zeta\omega_n s + \omega_n^2\right]} = \frac{\omega_n^2}{\left[\left(s + \zeta\omega_n\right)^2 + \omega_d^2\right]}$$

$$\omega_d^2 = \omega_n^2\left(1 - \zeta^2\right); \quad G(s) = \frac{\omega_n^2}{\left(s + \zeta\omega_n + j\omega_d\right)\left(s + \zeta\omega_n - j\omega_d\right)}$$

LTI System Responses Using TF

Transfer functions are used to generate time **responses** of LTI systems based on the principle of **superposition**.

This involves (1) **decomposing** Y(s) into its characteristic components and (2) **mapping** these components to their time domain **counterparts.**

Decomposition is based on the **premise** that any complex **LTI** system can be **synthesized** as a linear **combination** of 1^{st} and 2^{nd} order **terms.**

Partial fractions is standard method for **decomposing.**

Partial Fractions Concept

Partial fraction decomposition uses method of **residues** to **decompose** an **'n^{th}'** order fraction into a set of **'n' 1^{st} order** fractions.

Consider **n^{th} order** system, along with its **decomposed form,** as given below.

$$G(s) = K\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)} = \frac{A_1}{s-p_1} + \frac{A_2}{s-p_2} + \cdots + \frac{A_n}{s-p_n}$$

Here, **A_1 to A_n** are called the **residues.**

Partial Fractions Concept

The **residues** represent the **contributions** of each of the **factors** to the total **response** and are obtained as follows.

$$\text{Distinct Poles: } A_i = \left[\left(s - p_i\right) Y(s)\right]\Big|_{s=p_i}; \quad i = 1, n$$

$$\text{Multiple Poles: } G(s) = K\,\frac{(s - z_1)(s - z_2)\cdots(s - z_m)}{(s - p_1)^k (s - p_{k+1})\cdots(s - p_n)}$$

$$Y(s) = \frac{A_1}{s - p_1} + \frac{A_2}{\left(s - p_1\right)^2} + \cdots + \frac{A_k}{\left(s - p_1\right)^k} + \cdots + \frac{A_i}{s - p_i}$$

$$A_j = \frac{1}{(k - j)!}\frac{d^{k-j}}{ds^{k-j}}\left[\left(s - p_1\right)^k Y(s)\right]\Big|_{s=p_1}; \quad j = 1, k; \quad i = k+1, n$$

We can also **compare** the coefficients of applicable **numerator** polynomial in **certain cases.**

Partial Fractions Example – Distinct

Obtain **unit impulse response** of the given TF.

$$G(s) = \frac{(s+3)}{(s+1)(s+2)}$$

$$Y(s) = \frac{A_1}{s+1} + \frac{A_2}{s+2}$$

$$= \frac{(A_1 + A_2)s + (2A_1 + A_2)}{(s+1)(s+2)}$$

$$A_1 + A_2 = 1; \quad 2A_1 + A_2 = 3$$

$$A_1 = 2; \quad A_2 = -1$$

$$p_1 = -1; \quad p_2 = -2; \quad z_1 = -3; \quad K = 1$$

$$Y(s) = \frac{(s+3)}{(s+1)(s+2)} = \frac{A_1}{s+1} + \frac{A_2}{s+2}$$

$$A_1 = \left[(s+1)Y(s)\right] = \left[\frac{(s+3)}{(s+2)}\right]\Big|_{s=-1} = 2$$

$$A_2 = \left[(s+2)Y(s)\right] = \left[\frac{(s+3)}{(s+1)}\right]\Big|_{s=-2} = -1$$

$$Y(s) = \frac{2}{s+1} - \frac{1}{s+2} = \frac{(s+3)}{(s+1)(s+2)}$$

$$y(t) = L^{-1}Y(s) = 2e^{-t} - e^{-2t}$$

Partial Fractions Example – Multiple

Obtain **unit impulse response** of the given system.

$$G(s)=\frac{s^2+2s+3}{(s+1)^3}$$

$$p_1=-1,-1,-1;\quad z_1=-1\pm j\sqrt{2};\quad K=1$$

$$Y(s)=\frac{b_1}{s+1}+\frac{b_2}{(s+1)^2}+\frac{b_3}{(s+1)^3}$$

$$b_3=\left[(s+1)^3\frac{B(s)}{A(s)}\right]_{s=-1}=2$$

$$b_2=\frac{d}{ds}\left[(s+1)^3\frac{B(s)}{A(s)}\right]_{s=-1}=0$$

$$b_1=\frac{1}{2!}\frac{d^2}{ds^2}\left[(s+1)^3\frac{B(s)}{A(s)}\right]_{s=-1}=1$$

$$Y(s)=\frac{1}{s+1}+\frac{2}{(s+1)^3}\rightarrow y(t)=\left(1+t^2\right)e^{-t}$$

$$b_1s^2+\left(2b_1+b_2\right)s$$
$$+\left(b_1+b_2+b_3\right)$$
$$=s^2+2s+3$$
$$b_1=1;\quad b_2=0;\quad b_3=2$$

Summary

Transfer function is the **building block** for obtaining the response of **LTI systems** and its decomposition using **partial fractions** provides a **convenient** methodology.