EE 325 Midterm-1 Exam, Autumn 2019

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- This is a closed book exam. You are allowed ONE SIDE of an A4-sheet with handwritten formulas.
- You have 60 minutes to finish this exam.
- **Show partial work to receive credit.**
- Calculators/gadgets/cellphones are **not allowed**.
- There are FIVE questions in total. Please ensure that there are FIVE printed pages.
- You may use the empty pages to do your work.

NAME: Model Solution.

ROLL NO.: ______

Question No	1	2	3	4	5		Total
Total	3	4	4	4	3		18
Score							

1. A research team needs uncorrelated but not independent random variables for their experiments. The pointy hair boss of the team instructs to begin with two random variables (X, Y) taking values only in the set $\{0, 1\}$. The puffy-hair man in their team asserts that any uncorrelated random variables (X, Y) on the binary set $\{0, 1\}$ will be independent. Is the puffy hair man correct in his assertion? Explain your answer for credit. *(3 marks)*

Let $\mathbb{P}(X = x, Y = y) = a, \quad (x,y) = (0,0)$

$= b, \quad (x,y) = (0,1)$

$= c, \quad (x,y) = (1,0)$

$= d, \quad (x,y) = (1,1)$

be the joint distribution.

Then, $a+b+c+d = 1$. $\mathbb{P}(X=0) = (a+b) = 1 - \mathbb{P}(X=1)$.

$\mathbb{P}(Y=0) = (a+c) = 1 - \mathbb{P}(Y=1)$.

$\mathbb{E}(XY) = d$; $\quad \mathbb{E}(X) \cdot \mathbb{E}(Y) = (1-a-b)(1-a-c)$

$= (c+d)(b+d)$

If (X,Y) are uncorrelated, then $(c+d)(b+d) = d$.

Now, $\mathbb{P}(X=0, Y=0) = a$; $\mathbb{P}(X=0) \cdot \mathbb{P}(Y=0) = (a+b)(a+c)$.

$\mathbb{P}(X=1) \cdot \mathbb{P}(Y=1) = \mathbb{E}(X) \cdot \mathbb{E}(Y) = \mathbb{E}(XY) = \mathbb{P}(X=1, Y=1)$. — (1)

From ①, we see that $X=1, Y=1$ events are independent. The same will be verified for $X=1, Y=0$, $X=0, Y=1$, and $X=0, Y=0$.

$$\mathbb{P}(X=1, Y=0) = -\mathbb{P}(X=1, Y=1) + \mathbb{P}(X=1)$$

$$= \mathbb{P}(X=1) - \mathbb{P}(X=1).\mathbb{P}(Y=1) \quad \{\text{from } ①$$

~~white, P~~

$$= \mathbb{P}(X=1)\{1 - \mathbb{P}(Y=1)\}$$

$$= \mathbb{P}(X=1).\mathbb{P}(Y=0). \quad ②.$$

Similar to ②,

$$\mathbb{P}(X=0, Y=1) = \mathbb{P}(Y=1) - \mathbb{P}(X=1, Y=1)$$

$$= \mathbb{P}(Y=1) - \mathbb{P}(X=1)\mathbb{P}(Y=1)$$

$$= \mathbb{P}(Y=1).\mathbb{P}(X=0). \quad -③.$$

Finally,

$$\mathbb{P}(X=0, Y=0) = \mathbb{P}(X=0) - \mathbb{P}(X=0, Y=1)$$

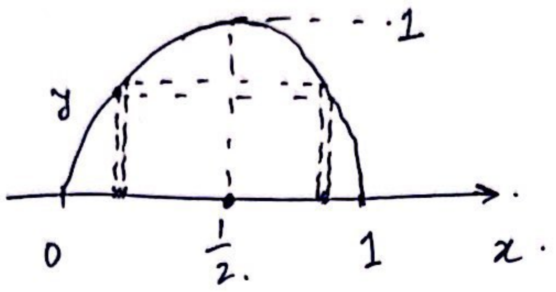
$$= \mathbb{P}(X=0) - \mathbb{P}(X=0)\mathbb{P}(Y=1) \quad \{\text{from } ③$$

$$= \mathbb{P}(X=0).\{1-\mathbb{P}(Y=1)\}$$

$$= \mathbb{P}(X=0).\mathbb{P}(Y=0).$$

Thus, X and Y are independent.
The puffy hair man is correct.

2. Let $U \sim \text{Uniform}[0,1]$ be a random variable. Let $Y = 4U(1-U)$. Derive and sketch the probability density function of Y. Box your final answers. Explain your answer for credit. *(4 marks)*



Let $h(x) = 4x(1-x)$. It is sketched.

For $0 \le U \le 1$, observe that $0 \le Y \le 1$.

For $0 < y < 1$, $\mathbb{P}(Y \in [y, y+dy)) = f_Y(y)\,dy$.

Further, $\mathbb{P}(Y \in [y, y+dy)) = \mathbb{P}(U \in [u_1, u_1 + du_1)) + \mathbb{P}(U \in [u_2, u_2 - du_2))$

$\left\{ \text{where } u_1, u_2 \text{ are the solutions of } 4u(1-u) = y, \quad 0 < y < 1 \right\}$

i.e., $4u^2 - 4u + y = 0$, or $u = \dfrac{4 \pm \sqrt{16 - 16y}}{8} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1-y}$.

$= f_U(u_1) \cdot du_1 + f_U(u_2) \cdot du_2$

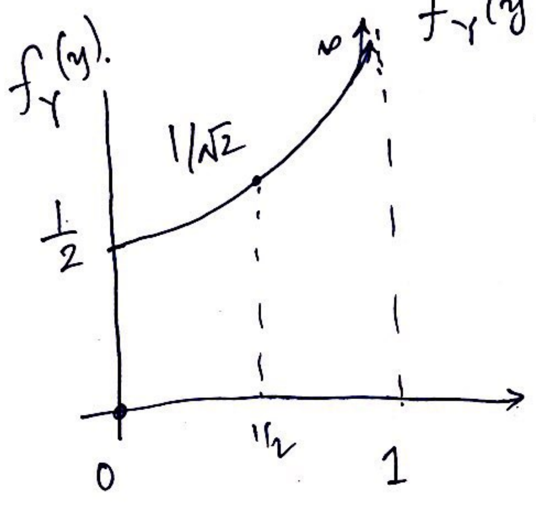
$= du_1 + du_2$.

Note that since $f_U(u) = 1$, $0 \le u \le 1$, so only du_1, du_2 matter. So,

$$f_Y(y) = \left|\frac{1}{dy/du_1}\right| + \left|\frac{1}{dy/du_2}\right|$$, where u_1, u_2 are the solutions of $y = 4u(1-u)$.

$$= \left|\frac{1}{4 - 8u_1}\right| + \left|\frac{1}{4 - 8u_2}\right|.$$

$$= \frac{1}{4\sqrt{1-y}} + \frac{1}{4\sqrt{1-y}} = \frac{1}{2\sqrt{1-y}}$$



3. For a random variable X, it is known that $\mathbb{P}(X \in [1,3]) = 1/3$ and $\mathbb{P}(X \in [2,4]) = 1/2$. The pointy hair boss wants to understand the range of $\mathbb{P}(X \in [1,4])$ from this given data. The puffy hair man suggests that $\mathbb{P}(X \in [1,4]) \leq \frac{5}{6}$ while the bald man suggests that $\mathbb{P}(X \in [1,4]) \geq \frac{2}{3}$. Which of these statements are (always) correct? Explain your answer for credit. *(4 marks)*

Observe that $[1,3] \cup [2,4] = [1,4]$.

So, $\mathbb{P}(X \in [1,4]) = \mathbb{P}(X \in [1,3] \cup [2,4])$.

Further, $\mathbb{P}(X \in [1,4]) = \mathbb{P}(X \in [1,3] \cup (3,4])$

$= \mathbb{P}(X \in [1,3]) + \mathbb{P}(X \in (3,4])$ {since $[1,3]$ and $(3,4]$ are disjoint sets.}

$\leq \mathbb{P}(X \in [1,3]) + \mathbb{P}(X \in [2,4])$.

{since $[2,4] \supset (3,4]$, so $\mathbb{P}(X \in [2,4]) \geq \mathbb{P}(X \in (3,4])$}.

$= \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$.

The puffy hair man is correct (always).

Bald man is wrong, as seen from a counterexample.

Let X be a rv such that $\mathbb{P}(X \in [2,3]) = \frac{1}{3}$, $\mathbb{P}(X \in (3,4]) = \frac{1}{6}$ and $\mathbb{P}(X \in [1,2)) = 0$.

Then, $\mathbb{P}(X \in [1,4]) = \frac{1}{3} + \frac{1}{6} + 0 = \frac{1}{2}$ which is not larger than $\frac{2}{3}$.

4. Consider a wire of resistance R and length of 1 unit. The wire is cut at a uniformly distributed point $U \sim \text{Uniform}[0,1]$ on the wire, and the smaller portion is selected. Let $Y = \min\{U, 1-U\}$ be the length of the smaller portion of wire. Next, the wire of length Y is again cut at a uniformly distributed point $V \sim \text{Uniform}[0,Y]$ on the wire, and the smaller portion is selected. Let $Z = \min\{V, Y-V\}$. Find and sketch the CDF of resistance of the smaller portion Z of wire? Assume that wire length and resistance are linearly related. Explain your answer for credit. *(4 marks)*

Note that $f_U(u) = 1$ for $0 \le u \le 1$

$= 0$ otherwise.

Next, $\mathbb{P}(Y \le y) = 0$ for $y < 0$

$= 1$ for $y \ge \frac{1}{2}$.

$Y \le \frac{1}{2}$ as it is smaller than U or $1-U$ both

$= \underbrace{\mathbb{P}(U \le y \text{ or } U \ge 1-y)}_{= 2y.}$ for $0 \le y \le \frac{1}{2}$

So, $f_Y(y) = 2$; $0 \le y \le \frac{1}{2}$.

Next, $f_{V|Y}(v|y) = \frac{1}{y}$; $0 \le v \le y$

$= 0$ otherwise.

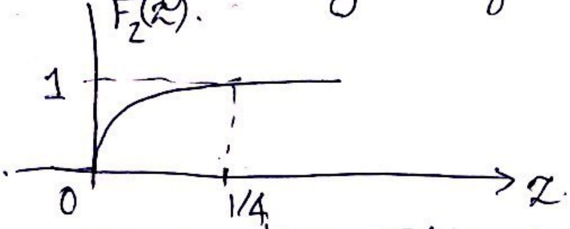
So, $f_{Z|Y}(z|y) = \frac{2}{y}$; $0 \le z \le \frac{y}{2}$

$= 0$ otherwise. (as in prev. case).

Finally, $f_{Z,Y}(z,y) = f_{Z|Y}(z|y) \cdot f_Y(y)$

$= \frac{4}{y}$; $0 \le z \le \frac{y}{2}$, $0 \le y \le \frac{1}{2}$

marginalizing Y give $f_Z(z) = \int_{2z}^{1/2} \frac{4}{y}\, dy = 4\log\frac{1}{2} - 4\log(2z)$



$= 4\log\frac{1}{4} - 4\log z$; $(0 \le z \le \frac{1}{4})$

Integrating, $F_Z(z) = 4z\log\frac{1}{4} - 4(z\log z - z)$; The CDF of resistance is scaled by R.

5. Let (X, Y) be jointly distributed as a Uniform$([-2,2] \times [-2,2])$. That is, their joint density is

$$f(x, y) = \frac{1}{16} \text{ for } -2 \le x, y \le 2$$

and zero otherwise. Insert the correct relation operator $\{>, <, =\}$ in the following blank:

$$F(0,0) + F(-1,-1) - F(-1,0) - F(0,-1) \quad \_\_\_ \quad F(1.1) + F(0,0) - F(0,1) - F(1,0)$$

where $F(x, y)$ is the joint cdf of (X, Y). Explain your answer for credit. *(3 marks)*

Observe that $f_{X,Y}(x,y)$ factorized as

$f_X(x) = \frac{1}{4}$ for $-2 \le x \le 2$

and $f_Y(y) = \frac{1}{4}$ for $-2 \le y \le 2$.

Therefore X and Y are independent.

Further observe that.

$F(a,b) + F(c,d) - F(a,d) - F(c,b)$

$= \mathbb{P}((X,Y) \in (a,c] \times (b,d])$ where $a<c$, $b<d$.

With these observations,

$\text{LHS} = \mathbb{P}((X,Y) \in (-1,0] \times (-1,0])$

$= \mathbb{P}(X \in (-1,0]) \cdot \mathbb{P}(Y \in (-1,0])$ (indep.)

$= \frac{1}{4} \cdot \frac{1}{4}.$

$\text{RHS} = \mathbb{P}((X,Y) \in (0,1] \times (0,1])$

$= \mathbb{P}(X \in (0,1]) \cdot \mathbb{P}(Y \in (0,1])$ (indep.

$= \frac{1}{4} \cdot \frac{1}{4}.$

Operator is $=$.