Homework 5: convergence of random variables

EE 325: Probability and Random Processes, Autumn 2019
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Instructions: Some of these questions will be asked in a quiz in the class on 04/11/19. *If you have queries, then meet the instructor or the TA during office hours.*

Set-A

1. Let $f(t)$ be a bandlimited Fourier series defined with fundamental period $T = 1$. That is,

$$f(t) = \sum_{k=-b}^{b} a[k] \exp(j2\pi kt), \quad t \in [0,1]$$

where $a[k]$ are the Fourier series coefficients of $f(t)$. From an experiment, $f(U_1), \ldots, f(U_n)$ are obtained where $U_1, \ldots, U_n$ are given i.i.d. realizations of a Uniform$[0,1]$ random variable. Knowing the values of $U_1, \ldots, U_n$ develop an approximation for the Fourier series coefficients $a[k]$. Evaluate the mean-squared error of your approximation for $a[k]$? It would be desirable if the mean-squared error decreases to zero as $n \to \infty$.

2. Let $\{X_1, X_2, X_3, \ldots\}$ be a sequence of *zero-mean* dependent random variables such that,

$$\mathrm{cov}(X_i, X_j) = \frac{1}{n^{|i-j|}}. \tag{1}$$

Notice that as $|i - j|$ increases, the covariance between X_i and X_j decreases. Is it true that $(S_n/n) \xrightarrow{\mathbb{P}} c$, where $S_n = X_1 + X_2 + \ldots + X_n$ and c is some constant? If yes, find the value of c.

3. Let $\{X_1, X_2, X_3, \ldots\}$ be an iid sequence of Unif$[0,1]$ random variables. Let $Y_n = n(1 - X_{(n)})$. Find if $Y_n \xrightarrow{d} Y$. If yes, find the cdf of the limit Y.

4. Assume that $\{Y_n\}_{n\in\mathbb{N}}, \{Z_n\}_{n\in\mathbb{N}}$ are sequences of random variables such that $Y_n \xrightarrow{\mathbb{P}} Y$ and $Z_n \xrightarrow{\mathbb{P}} Z$. Show that $Y_n + Z_n \xrightarrow{\mathbb{P}} Y + Z$. (Hint: You may find the triangle inequality $|x + y| \leq |x| + |y|$ useful.)

5. Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variable. Assume that $X_n \sim$ Poisson$(1/n)$. Show that $X_n \xrightarrow{\mathbb{P}} 0$ and $nX_n \xrightarrow{\mathbb{P}} 0$.

6. Assuming that $Z_n \xrightarrow{\mathbb{P}} Z$, show that $Z_n \xrightarrow{d} Z$. (Hint: You need to show that $\mathbb{P}(Z_n \leq x) \to \mathbb{P}(Z \leq x)$ for all x where $F_Z(x)$ is continuous. If $F_Z(x)$ is continuous at x, then there is an interval $(x - \delta, x + \delta)$ in which $F_Z(x)$ is continuous. Further, $|Z_n - Z| \leq \epsilon$ with high probability. Connect these pieces with suitable inequalities to get the result.)

7. Let $\{X_n\}_{n\in\mathbb{Z}}$ be a sequence of random variables. Assume b to be a real number. Show that $X_n \xrightarrow{\mathcal{L}^2} b$ if and only if,

$$\lim_{n\to\infty} \mathbb{E}(X_n) = b \quad \text{and} \quad \lim_{n\to\infty} \mathrm{var}(X_n) = 0.$$

8. *(Typical sets)* Let $X_1, X_2, \ldots, X_n$ be i.i.d. Bernoulli(p) random variables. Let $p(x), x = 0, 1$ be the pmf of X. Consider the typical set,

$$A_n(\epsilon) := \left\{ x_1^n : \left| -\frac{1}{n} \log_2(p(x_1^n)) - H_2(p) \right| \leq \epsilon \right\}.$$

(a) Show that for any fixed $\epsilon > 0$ and large enough n, $\mathbb{P}((X_1, X_2, \ldots, X_n) \in A_n(\epsilon)) \geq (1 - \epsilon)$.

(b) Let $h_2(p) = -p \log_2 p - (1 - p) \log_2(1 - p)$. Show that for any $(x_1, \ldots, x_n) \in A_n(\epsilon)$,

$$2^{-nh_2(p)-n\epsilon} \leq \mathbb{P}((X_1, \ldots, X_n) = (x_1, \ldots, x_n)) \leq 2^{-nh_2(p)+n\epsilon}.$$

Thus, all typical set sequences have approximately the same probability of $\approx 2^{-nh_2(p)}$.

(c) Show that the number of typical sequences $|A_n(\epsilon)|$ satisfies the following inequality,

$$(1 - \epsilon)2^{nh_2(p)-n\epsilon} \leq |A_n(\epsilon)| \leq 2^{nh_2(p)+n\epsilon}.$$

Thus about $2^{nh_2(p)}$ typical sequences are there and they require $nh_2(p)$ bits for representation. (Hint: use the Union bound.)