Homework 3: functions of rv, total expectation, mgf

EE 325: Probability and Random Processes, Autumn 2019
Instructor: Animesh Kumar, EE, IIT Bombay

Instructions: Some of these questions will be asked in a quiz in the class on 12 Sep 2019 (Thursday). *If you have queries, then meet the instructor or the TA during office hours.*

1. Let Y be a log-normal random variable. The log-normal property implies that $\log_e Y$ is distributed as $\mathcal{N}(\mu, \sigma^2)$ random variable. Find the variance of Y. (Hint: the mgf of a Gaussian random variable may be useful.)

2. A packet arrives at a router with probability p in an independent and identically distributed fashion. That is, at most one packet arrives at each instant independently, and probability of a packet arrival is p. Assume that the router serves for N clocks (discrete slots), where N is a Poisson(n) random variable. Find the mean and variance of total number of arrived packets at the router. (Hint: use conditional expectation.)

3. Let $X_1, X_2, \ldots, X_5$ be i.i.d. normalized Gaussian rv. Find the pdf of $Z = X_1^2 + X_2^2 + \ldots + X_5^2$ and $Y = X_1 + X_2 + \ldots + X_5$.

4. Let X_1, X_2 be independent Exponential(1) random variables. Find the joint pdf of $Y = X_1 + X_2$ and $Z = X_1 - X_2$.

5. In this problem you have to construct a random variables (i.e., their distributions) so that their mgf satisfies certain properties.

   (a) Describe a random variable X such that $g_X(r)$ is not finite for $r < 0$ but is finite for $r \geq 0$.

   (b) Describe a random variable Y such that $g_Y(r)$ is not finite for $r > 0$ but is finite for $r \leq 0$.

   (c) Assuming that X, Y in the above parts are independent, what is the ROC of $(X + Y)$ for your examples?

6. Let $X \sim \mathcal{N}(0, \sigma^2)$. Using the mgf of X, show that,

$$\mathbb{E}[X^{2k+1}] = 0, \quad \mathbb{E}[X^{2k}] = \frac{(2k)!\sigma^{2k}}{k!2^k}, \text{ where, } k > 0 \text{ and } k \in \mathbb{Z}.$$

7. Assume that the mgf of a random variable X exists (i.e., is finite) in the interval $(r_-, r_+), r_- < 0 < r_+$. Assume that $r \in (r_-, r_+)$ throughout in this problem.

   (a) For any finite constant $c \in \mathbb{R}$, express the moment generating function of $(X - c)$, i.e. $g_{(X-c)}(r)$, in terms of $g_X(r)$ and show that it exists for all $r \in (r_-, r_+)$. Explain why $g''_{(X-c)}(r) \geq 0$.

   (b) Show that $g''_{(X-c)}(r) = [g''_X(r) - 2cg'_X(r) + c^2 g_X(r)]e^{-rc}$.

   (c) Use (a) and (b) to show that $g''_X(r)g_X(r) - [g'_X(r)]^2 \geq 0$. Let $\gamma_X(r) = \ln g_X(r)$. Show that $\gamma''_X(r) \geq 0$.

8. Let X and Y be IID $\mathcal{N}(0, 1)$ random variables. Let $Z \sim \text{Exp}\left(\frac{1}{2}\right)$ be an exponentially distributed random variable with $\lambda = (1/2)$.

   (a) Find $\mathbb{E}(e^{rX^2})$ and $\mathbb{E}(e^{rZ})$, i.e., the MGF of X^2 and Z. Find out the region of convergence or $[r_-(X^2), r_+(X^2)]$ and $[r_-(Z), r_+(Z)]$. Identify whether $g_{X^2}(r)$ and $g_Z(r)$ converges at the boundary points or not.

   (b) It can be shown that if the MGF of a random variable V is identical to that of W in a small neighborhood $r \in (-\delta, \delta)$ with $\delta > 0$, then V and W have the same distribution. Use part (a) to find the distribution of $X^2 + Y^2$.