Closed Loop Response Features

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Closed Loop Response Concept

In a **manner** similar to attributes that **characterize** the absolute and relative **stability** of closed loop system, we need **features** that capture the closed loop **response.**

In a **broad** sense, closed loop **performance** needs to capture overall **objectives** and is usually stated as **requirements** on the time **response.**

In the context of **closed** loop, the time domain **response** features that are **important** are same as **those** seen in case of the **plant**.

Important Response Features

In this regard, it is **worth** noting that **tracking** performance is directly **related** to the error **constants.**

In addition, we know that **except** for small domain around **t = 0,** response is **mainly** due to **dominant poles**.

Therefore, **error** constants and dominant **closed** loop poles are the **desired** response **features.**

Time Response Features

Relative Stability Vs. Time Response

In view of the fact that **stability margins** provide the location of the **dominant** closed loop **poles**, we can ensure a **desirable** response through stability **margins.**

Therefore, in most **control design** tasks, we either specify stability **margins** to achieve transient **response** or vice versa, but **not both.**

Benchmark Time Response

In the context of **time response**, we know that **dominant** behaviour can either be of **1st or of 2nd** order.

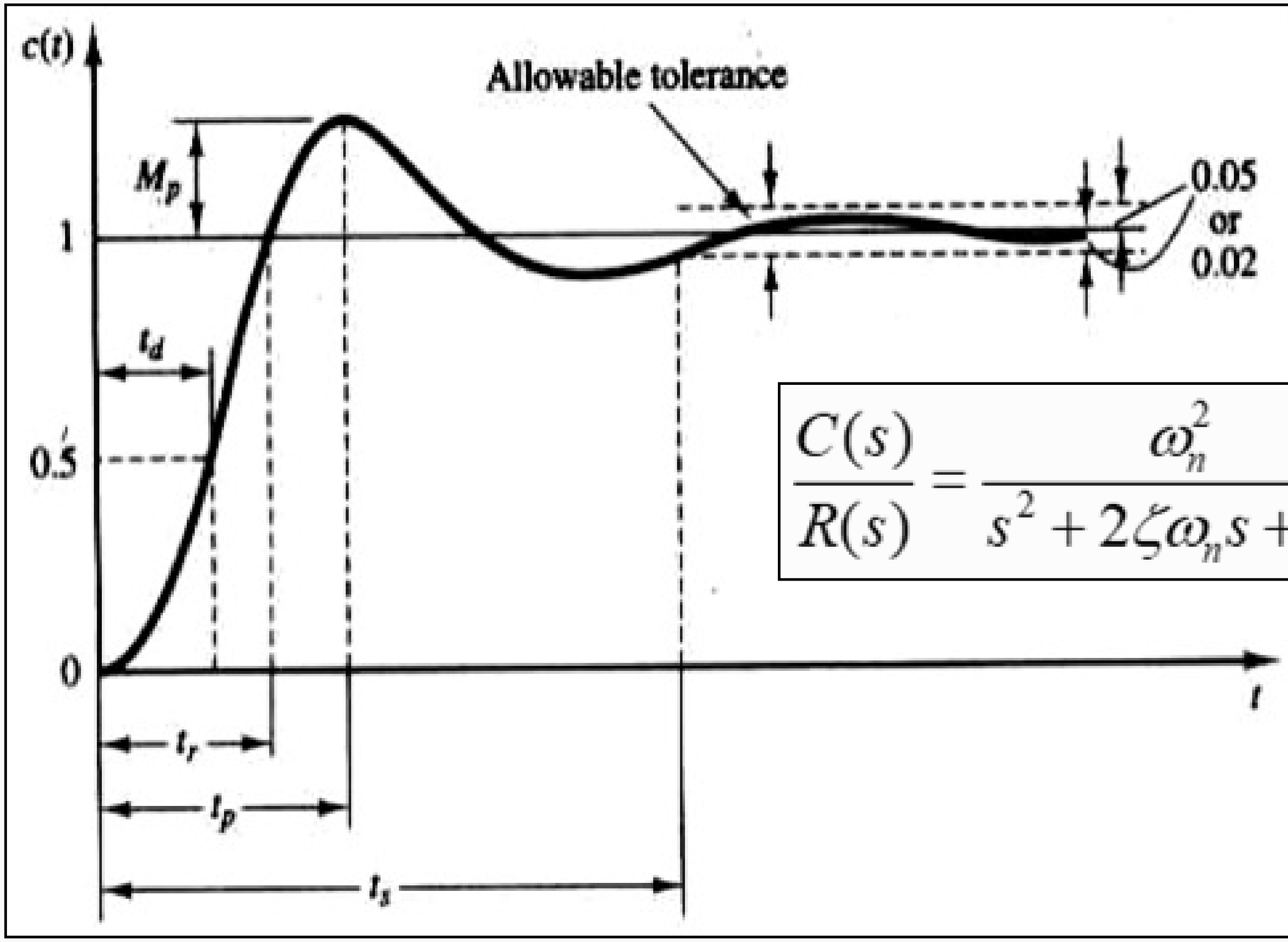
Further, we **note** that for most applications, **step** input as a test signal **extracts** almost all the dynamical **features** e.g. **speed** of response, **overshoot**, settling **time** etc.

Lastly, in most **practical** scenarios, exact **tracking** is required only for **step** input.

Therefore, we **employ** a benchmark **2nd order** dominant response which **exactly** tracks the **step** input.

Benchmark Time Response Features

Consider **applicable** TF & response, as **shown** below.



$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}; \quad R(s) = \frac{1}{s}$$

Time Response – Margin Mapping

In this case, **analytical** expression for **c(t)** & **features** are obtained in terms of **ζ, ω_n, ω_d & σ**, as follows.

$$s_{1,2} = \sigma \pm j\omega_d; \quad \sigma = \xi\omega_n; \quad \eta = \cos^{-1}\zeta; \quad \omega_d = \omega_n\sqrt{1-\xi^2}$$

$$c(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

$$\text{Rise time: } t_r = \frac{1}{\omega_d}\tan^{-1}\left(\frac{\omega_d}{-\sigma}\right); \quad \text{Peak time: } t_p = \frac{\pi}{\omega_d}$$

$$\text{Peak overshoot: } M_p = e^{-\left(\zeta/\sqrt{1-\zeta^2}\right)\pi} = e^{-\left(\sigma/\omega_d\right)\pi}$$

$$\text{Settling time: } t_s \approx \frac{3}{\sigma}(5\%\text{ ripple}) \approx \frac{4}{\sigma}(2\%\text{ ripple})$$

Time Response – Margin Example

Determine the **desired pole** location in respect of the following **transient** specifications.

1. Peak overshoot **= 12%**, Settling time **= 4 sec** (2%)

Settling time: $= 4s = 4/\sigma \rightarrow \sigma = 1.0$

Peak overshoot: $= 0.12 = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \rightarrow \zeta = 0.56$

$$\omega_n = 1.79; \quad \omega_d = 1.48$$

Dominant Pole: $-1.0 \pm j1.48$

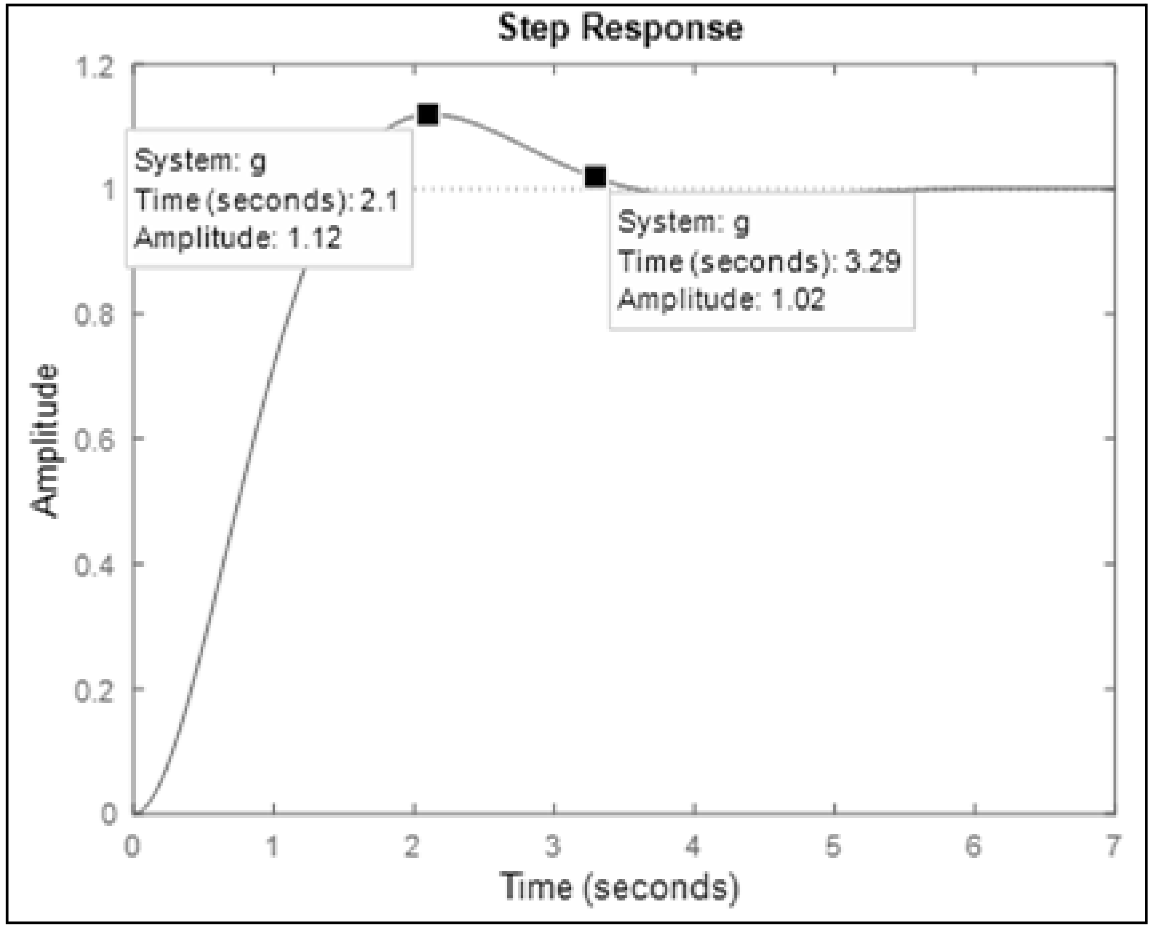
Time Response - Margin Example

The **applicable** closed loop **transfer** function is given below.

$$\frac{C(s)}{R(s)} = \frac{3.19}{s^2 + 2s + 3.19}$$

We see that while **peak** overshoot is **exac**t, the settling **time** is not, as shown alongside.

What is **likely** to happen if **system** is of higher **order?**



Time – Laplace Domain Correlation

We see that **dominant** closed loop pole **location** is closely **related** to the time domain response **features.**

Thus, we realize that **closed** loop relative **stability** also is related to **time domain** response features & in **most** cases, **dominant pole** specifications are **adequate.**

Summary

Closed loop time response is typically **specified** through **peak overshoot**, and settling time, which correlate well with the **dominant** closed loop pole location.

Frequency Response Features

$\omega - t$ Domain Correspondence

We know that **real part** of dominant poles is **strongly** correlated to the stability **margins** in frequency domain.

In view of the above, it is **logical** to expect that **GM, PM**, apart from other **features** e.g. GCO, PCO etc., should also be **related** to the time **response** features.

Closed Loop Response Definition

We can write the **expression** for closed loop frequency **response,** as follows.

$$\frac{C(j\omega)}{R(j\omega)} = \frac{G(j\omega)}{1+G(j\omega)}$$

C/R can be **obtained** from **G(jω)**, as shown next.

Benchmark Closed Loop Response

Similar to **benchmark** time response, it is **sufficient** to consider same **2nd order** transfer function as **closed** loop system, as shown below.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \rightarrow \frac{C(\omega)}{R(\omega)} = \frac{\omega_n^2}{\left(\omega_n^2 - \omega^2\right) + j2\zeta\omega_n\omega}$$

Here, 'ω_n' is natural frequency and 'ζ' is damping ratio.

Margins – Dominant Pole Correlation

We know that **GCO, PCO, GM & PM** are related to the closed loop **frequency** response.

Further, we also know that these **parameters** are related to the s-domain **dominant** pole location.

Therefore, It is **possible** to correlate frequency domain **margins** with the s-domain pole **location**, as shown next.

Benchmark PCO & GM

Consider the **applicable** G(jω) for the **chosen** closed loop response, as shown **below.**

$$G(j\omega) = \frac{C}{R-C} = \frac{\omega_n^2}{j\omega(j\omega + 2\zeta\omega_n)}$$

Thus we see that benchmark G(jω) has no **PCO** and hence, **GM is infinite** and hence not useful.

Benchmark GCO & PM Solution

However, it has **GCO**, and consequently, **PM**, as shown below.

$$\frac{\omega_n^2}{\omega \times \sqrt{\omega^2 + 4\zeta^2\omega_n^2}} = 1 \rightarrow \omega_{GCO} = \omega_n \sqrt{\sqrt{1+4\zeta^4} - 2\zeta^2}$$

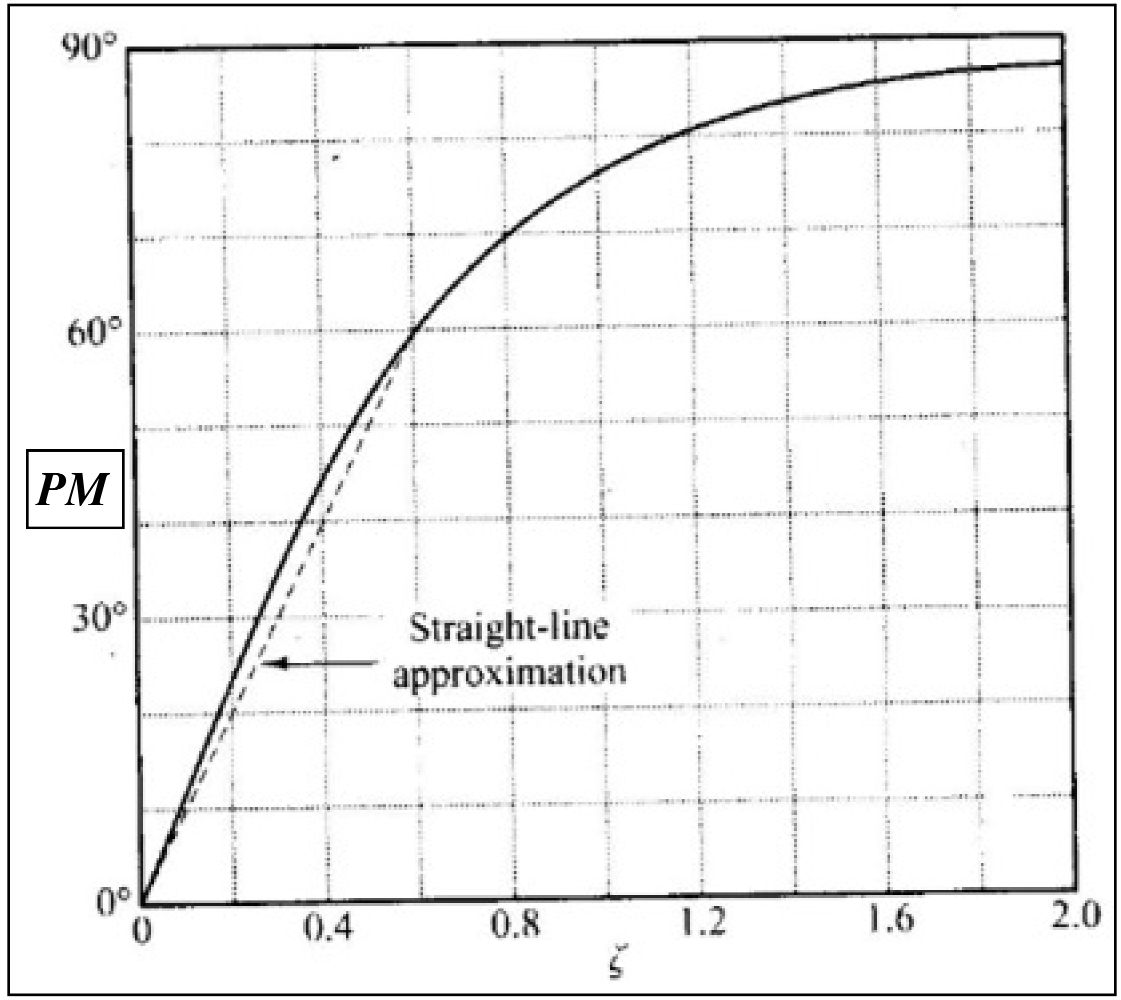
$$PM = 180 + \angle G(j\omega_{GCO}) = \tan^{-1} \frac{2\zeta}{\sqrt{\sqrt{1+4\zeta^4} - 2\zeta^2}}$$

We see that we can **arrive** at the dominant **pole** location from **PM, GCO** and vice versa.

ω-Domain – s-Domain Mapping

Given **alongside** is the plot of **PM vs. ζ,** along with its **linear model** for a range of 'ζ', as shown **below.**

$$\zeta = \frac{PM^o}{100^o}; \quad 0 \le \zeta \le 0.6$$



Margin – Dominant Pole Mapping

We see that **GCO and PM** are important stability related **parameters**, which correlate to both **pole location**, and hence, **time response** features.

Therefore, in general, we pose **design** problems in terms of the desired **PM** and GCO or other frequencies e.g. resonant peak, **bandwidth** etc.

Summary

Dominant 2nd order **model** results in PCO & GM as **infinite.**

However, we can **define GCO** and PM in terms of **s-domain** parameters.