Homework 5 solutions: convergence of random variables

EE 325: Probability and Random Processes, Autumn 2019

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1. Let $f(t)$ be a bandlimited Fourier series defined with fundamental period $T = 1$. That is,

$$f(t) = \sum_{k=-b}^{b} a[k] \exp(j2\pi kt), \quad t \in [0, 1]$$

where $a[k]$ are the Fourier series coefficients of $f(t)$. From an experiment, $f(U_1), \ldots, f(U_n)$ are obtained where $U_1, \ldots, U_n$ are given i.i.d. realizations of a Uniform$[0, 1]$ random variable. Knowing the values of $U_1, \ldots, U_n$ develop an approximation for the Fourier series coefficients $a[k]$. Evaluate the mean-squared error of your approximation for $a[k]$? It would be desirable if the mean-squared error decreases to zero as $n \to \infty$.

Solution: In this problem, we will use a technique named as Monte-Carlo integration. Let $g_k(U) = f(U) \exp(-j2\pi kU)$ be a function of a Uniform$[0, 1]$ random variable U. The expected value of $g_k(U)$ is given by

$$\mathbb{E}[g_k(U)] = \int_0^1 f(u) \exp(-j2\pi ku) du. \tag{1}$$

The Fourier series coefficients of $f(t)$ are given by

$$a[k] = \int_0^1 f(t) exp(-j2\pi kt) dt = \mathbb{E}[g_k(U)]. \tag{2}$$

That is, an estimate of $\mathbb{E}(g_k(U))$ will lead us to an estimate of the Fourier series coefficient $a[k]$. By the weak law of large numbers,

$$\widehat{A}[k] := \frac{1}{n} \sum_{i=1}^{n} g_k(U_i) \xrightarrow{\mathbb{P}} a[k] \tag{3}$$

as $n \to \infty$. Since $g_k(U_i), i = 1, \ldots, n$ are i.i.d., therefore,

$$\mathbb{E}\left[\left|\widehat{A}[k] - a[k]\right|^2\right] = \frac{\text{var}(|g_k(U))|}{n}.$$

The careful reader would notice that $g_k(U_i)$ is a complex variable, so the mean-squared error involves $|g_k(U)|$ and not $g_k(U)$. As long as $\text{var}(g_k(U))$ is bounded, the mean-squared error between $\widehat{A}[k]$ and $a[k]$ decreases to zero with n.

Finally, $\text{var}(g_k(U))$ is bounded since

$$\text{var}(|g_k(U)|) \leq \mathbb{E}[|g_k(U)|^2] = \mathbb{E}(|f(U)|^2) = \int_0^1 |f(u)|^2 \mathrm{d}u$$

the energy of $f(t)$ in one period is bounded.

2. Let $\{X_1, X_2, X_3, \ldots\}$ be a sequence of *zero-mean* dependent random variables such that,

$$\text{cov}(X_i, X_j) = \frac{1}{n^{|i-j|}}. \tag{4}$$

Notice that as $|i - j|$ increases, the covariance between X_i and X_j decreases. Is it true that $(S_n/n) \xrightarrow{\mathbb{P}} c$, where $S_n = X_1 + X_2 + \ldots + X_n$ and c is some constant? If yes, find the value of c.

Solution: First it will be shown that $\mathrm{var}(S_n/n) \to 0$. Then, by Chebychev inequality, the convergence in probability will be established. Note that

$$\mathrm{var}\left(\frac{S_n}{n}\right) = \frac{1}{n^2}\mathrm{var}(S_n).$$

Unless $\mathrm{var}(S_n)$ grows quadratically with n, we can expect $\mathrm{var}(S_n/n)$ to decrease to zero. And $\mathrm{var}(S_n)$ will grow quadratically if $X_1, X_2, \ldots, X_n$ are heavily correlated with each other. The given covariance condition hints otherwise. That is why there is hope for $\mathrm{var}(S_n/n)$ to converge to zero. Formally, $\mathrm{var}(S_n)$ will be computed next.

$$\begin{aligned}
\mathrm{var}(S_n) &= \mathrm{var}\Big(\sum_{i=1}^{n} X_i\Big) \\
&= \mathbb{E}\Bigg[\Big(\sum_{i=1}^{n}(X_i - \mu_i)\Big)^2\Bigg] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\Big[(X_i-\mu_i)(X_j-\mu_j)\Big] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\mathrm{cov}(X_i, X_j) \\
&= n + \frac{2(n-1)}{n} + \frac{2(n-2)}{n^2} + \ldots + \frac{2}{n^{n-1}} \qquad (5)
\end{aligned}$$

Using (5),

$$\begin{aligned}
\mathrm{var}\left(\frac{S_n}{n}\right) &= \frac{1}{n} + \frac{2}{n^2} - \frac{2}{n^3} + \frac{2(n-2)}{n^4} + \ldots + \frac{2}{n^{n+1}}. \\
\text{Hence, } \lim_{n\to\infty}\mathrm{var}\left(\frac{S_n}{n}\right) &= 0. \qquad (6)
\end{aligned}$$

Since $\{X_n\}_{n\in\mathbb{N}}$ are zero-mean, therefore $\mathbb{E}(S_n) = 0$. By Chebyshev's inequality,

$$0 \le \mathbb{P}\left(\left|\frac{S_n}{n} - 0\right| > \epsilon\right) \le \tfrac{1}{\epsilon^2}\mathrm{var}\left(\tfrac{S_n}{n}\right).$$

Using (6), as $n \to \infty$, $\mathbb{P}\left(\left|\frac{S_n}{n} - 0\right| > \epsilon\right) \to 0$. Hence, $S_n \xrightarrow{\mathbb{P}} 0$.

3. Let $\{X_1, X_2, X_3, \ldots\}$ be an iid sequence of Unif$[0,1]$ random variables. Let $Y_n = n(1 - X_{(n)})$. Find if $Y_n \xrightarrow{d} Y$. If yes, find the cdf of the limit Y.

   Solution: Recall that $X_{(n)} = X_{n:n} = \max\{X_1, X_2, \ldots, X_n\}$. Its cdf is given by,

$$F_{X_{(n)}}(x) = \mathbb{P}(X_{(n)} \le x) = [F_{X_1}(x)]^n, \text{ since all } \{X_n\}_{n\in\mathbb{N}} \text{ are i.i.d. random variables .} \qquad (7)$$

Since X_n are i.i.d. Uniform random variables, therefore,

$$\begin{aligned}
F_{X_{(n)}}(x) &= 0 \quad \text{for } x \le 0 \\
&= x^n \quad \text{for } 0 < x \le 1 \\
&= 1 \quad \text{for } x > 1 \qquad (8)
\end{aligned}$$

It is given that $Y_n = n(1 - X_{(n)})$, then $F_{Y_n}(y)$ can be worked out as:

$$F_{Y_n}(y) = \mathbb{P}(Y_n \le y) = \mathbb{P}(1 - X_{(n)} \le y/n) = 1 - F_{X_{(n)}}(1 - y/n).$$

Using (8),

$$\begin{aligned}
F_{Y_n}(y) &= 0 \qquad \text{for } y < 0 \\
&= 1 - (1 - y/n)^n \quad \text{for } 0 \le y \le n \\
&= 1 \qquad \text{for } y > n
\end{aligned}$$

As $n \to \infty$, the interval $y > n$ reduces to ∞. Thus, $F_{Y_n}(y)$ can be simplified as

$$F_Y(y) = \lim_{n\to\infty} F_{Y_n}(y) = 0 \qquad \text{for } y < 0$$
$$= 1 - e^{-y} \qquad \text{for } y \geq 0.$$

Hence, $Y_n \xrightarrow{d} Y$, where Y is an Expoential(1) random variable.

4. Assume that $\{Y_n\}_{n\in\mathbb{N}}, \{Z_n\}_{n\in\mathbb{N}}$ are sequences of random variables such that $Y_n \xrightarrow{\mathbb{P}} Y$ and $Z_n \xrightarrow{\mathbb{P}} Z$. Show that $Y_n + Z_n \xrightarrow{\mathbb{P}} Y + Z$. (Hint: You may find the triangle inequality $|x + y| \leq |x| + |y|$ useful.)

Solution: Y_n converges in probability to Y, i.e., $Y_n \xrightarrow{\mathbb{P}} Y$. This is equivalent to,

$$\lim_{n\to\infty} \mathbb{P}(|Y_n - Y| > \epsilon/2) = 0, \text{ for any } \epsilon > 0.$$

Define a set D_Y as follows:

$$D_Y(n) = \{|Y_n - Y| \leq \epsilon/2\} = \{\omega : |Y_n(\omega) - Y(\omega)| \leq \epsilon/2\}.$$

Similarly, let

$$D_Z(n) = \{|Z_n - Z| \leq \epsilon/2\}.$$

Using the triangle inequality it is known that,

$$|(Y_n + Z_n) - (Y + Z)| \leq |Y_n - Y| + |Z_n - Z| \tag{9}$$

Now consider an event D_{YZ} as follows:

$$D_{YZ}(n) = \{|Y_n + Z_n - (Y + Z)| \leq \epsilon\}$$

Using the triangle inequality, observe that $D_Y(n) \cap D_Z(n) \subseteq D_{YZ}(n)$ for all $n \in \mathbb{N}$.

Since $Y_n \xrightarrow{\mathbb{P}} Y$, it is noted that $\mathbb{P}(D_Y(n)) \to 1$ as $n \to \infty$. Similarly, since $Z_n \xrightarrow{\mathbb{P}} Z$, it is noted that $\mathbb{P}(D_Z(n)) \to 1$ as $n \to \infty$. It is further noted that $\mathbb{P}(D_Z(n) \cap D_Y(n)) \to 1$ as $n \to \infty$. (Use $\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B)$ to derive the last part.)

Therefore $1 \geq \mathbb{P}(D_{YZ}(n)) \geq \mathbb{P}(D_Y(n) \cap D_Z(n))$. As RHS tends to 1, $\mathbb{P}(D_{YZ}(n))$ will converge to 1. Thus, $(Y_n + Z_n) \xrightarrow{\mathbb{P}} (Y + Z)$.

5. Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variable. Assume that $X_n \sim \text{Poisson}(1/n)$. Show that $X_n \xrightarrow{\mathbb{P}} 0$ and $nX_n \xrightarrow{\mathbb{P}} 0$.

Solution: Since $X_n \sim \text{Poisson}(1/n)$ therefore $\mathbb{E}(X_n) = 1/n$. Using the Markov inequality,

$$\mathbb{P}(|X_n| \geq \epsilon) \leq \frac{\mathbb{E}(X_n)}{\epsilon} = \frac{1}{n\epsilon} \to 0 \text{ as } n \to \infty.$$

Thus,

$$\lim_{n\to\infty} \mathbb{P}(|X_n - 0| \geq \epsilon) = 0 \text{ or } X_n \xrightarrow{\mathbb{P}} 0.$$

For showing $nX_n \xrightarrow{\mathbb{P}} 0$, the Markov inequality is not useful since $\mathbb{E}(nX_n) = 1$. Directly probability computation will be used. For any $\epsilon > 0$,

$$\mathbb{P}(nX_n > \epsilon) = \mathbb{P}(X_n > \epsilon/n) = 1 - \mathbb{P}(nX_n = 0) = 1 - \mathbb{P}(X_n = 0) = 1 - e^{-1/n}.$$

Applying limits,

$$\lim_{n\to\infty} \mathbb{P}(|nX_n - 0| \geq \epsilon) = 0 \text{ for all } \epsilon > 0$$

which proves the desired result.

6. Assuming that $Z_n \xrightarrow{\mathbb{P}} Z$, show that $Z_n \xrightarrow{d} Z$. (Hint: You need to show that $\mathbb{P}(Z_n \leq x) \to \mathbb{P}(Z \leq x)$ for all x where $F_Z(x)$ is continuous. If $F_Z(x)$ is continuous at x, then there is an interval $(x-\delta, x+\delta)$ in which $F_Z(x)$ is continuous. Further, $|Z_n - Z| \leq \epsilon$ with high probability. Connect these pieces with suitable inequalities to get the result.)

Solution: To show that $F_{Z_n}(x) \to F_Z(x)$, we need to show that $\lim_{n\to\infty} \mathbb{P}(Z_n \leq x) \to \mathbb{P}(Z \leq x)$ when $F_Z(x)$ is continuous. Using the law of total probability, first note that

$$\mathbb{P}(Z_n \leq x) - \mathbb{P}(Z \leq x) = \{\mathbb{P}(Z_n \leq x, Z > x+\delta) + \mathbb{P}(Z_n \leq x, Z \leq x+\delta)\} - \mathbb{P}(Z \leq x).$$

Similarly, $F_Z(x)$ can be rewritten as,

$$\mathbb{P}(Z \leq x) = \mathbb{P}(Z_n \leq x+\delta, Z \leq x) + \mathbb{P}(Z_n > x+\delta, Z \leq x).$$

Using the above results we can write

$$\begin{aligned}\mathbb{P}(Z_n \leq x) - \mathbb{P}(Z \leq x) = \mathbb{P}(Z_n \leq x, Z > x+\delta) + \mathbb{P}(Z_n \leq x, Z \leq x+\delta) - \mathbb{P}(Z_n \leq x+\delta, Z \leq x)\\ -\mathbb{P}(Z_n > x+\delta, Z \leq x).\end{aligned}$$

The first and the fourth term imply that $|Z_n - Z| > \delta$. Since $Z_n \xrightarrow{\mathbb{P}} Z$, therefore $\mathbb{P}(|Z_n - Z| > \delta) \leq \epsilon$ for large enough n. Thus,

$$\begin{aligned}0 \leq \mathbb{P}(Z_n \leq x) - \mathbb{P}(Z \leq x) &\leq \epsilon + \mathbb{P}(Z_n \leq x, Z \leq x+\delta) - \mathbb{P}(Z_n \leq x+\delta, Z \leq x)\\ &\leq \epsilon + \mathbb{P}(Z_n \leq x+\delta, Z \leq x+\delta) - \mathbb{P}(Z_n \leq x+\delta, Z \leq x),\end{aligned}$$

In the last step we use the fact that $\{Z_n \leq x\} \subseteq \{Z_n \leq (x+\delta)\}$. Finally,

$$\begin{aligned}0 \leq \mathbb{P}(Z_n \leq x) - \mathbb{P}(Z \leq x) &\leq \epsilon + \mathbb{P}(Z_n \leq x+\delta, x \leq Z \leq x+\delta)\\ &\leq \epsilon + \mathbb{P}(x < Z \leq x+\delta) \text{ since } \mathbb{P}(A \cap B) \leq \mathbb{P}(A).\end{aligned}$$

Since $F_Z(x)$ is continuous at x, $\mathbb{P}(x < Z \leq x+\delta) = F_Z(x+\delta) - F_Z(x) \to 0$ as $\delta \to 0$. As $n \to \infty$, δ (and ϵ) can be made arbitrarily small. Thus, we have $0 \leq \mathbb{P}(Z_n \leq x) - \mathbb{P}(Z \leq x) \leq \epsilon + \epsilon_1$, where both $\epsilon, \epsilon_1 \to 0$ as $n \to \infty$. This proves the result.

Observe that the continuity of F_Z at x is critical for arguing that $\epsilon_1 \to 0$.

7. Let $\{X_n\}_{n\in\mathbb{Z}}$ be a sequence of random variables. Assume b to be a real number. Show that $X_n \xrightarrow{\mathcal{L}^2} b$ if and only if,

$$\lim_{n\to\infty} \mathbb{E}(X_n) = b \quad \text{and} \quad \lim_{n\to\infty} \text{var}(X_n) = 0.$$

Solution: First note that for any r.v. X, $\mathbb{E}((X-b)^2) = \text{var}(X) + (b - \mathbb{E}(X))^2$. Thus,

$$\mathbb{E}(|X_n - b|^2) = \text{var}(X_n) + (b - \mathbb{E}(X_n))^2.$$

Observe that both the terms on the RHS of (7) are positive. Thus, $\lim_{n\to\infty} \mathbb{E}(|X_n - b|^2) = 0$ if and only if $\lim_{n\to\infty} \mathbb{E}(X_n) = b$ **and** $\lim_{n\to\infty} \text{var}(X_n) = 0$.

8. *(Typical sets)* Let $X_1, X_2, \ldots, X_n$ be i.i.d. Bernoulli(p) random variables. Let $p(x), x = 0, 1$ be the pmf of X. Consider the typical set,

$$A_n(\epsilon) := \left\{x_1^n : \left|-\frac{1}{n}\log_2(p(x_1^n)) - H_2(p)\right| \leq \epsilon\right\}.$$

   (a) Show that for any fixed $\epsilon > 0$ and large enough n, $\mathbb{P}((X_1, X_2, \ldots, X_n) \in A_n(\epsilon)) \geq (1-\epsilon)$.

   (b) Let $h_2(p) = -p\log_2 p - (1-p)\log_2(1-p)$. Show that for any $(x_1, \ldots, x_n) \in A_n(\epsilon)$,

$$2^{-nh_2(p)-n\epsilon} \leq \mathbb{P}((X_1, \ldots, X_n) = (x_1, \ldots, x_n)) \leq 2^{-nh_2(p)+n\epsilon}.$$

   Thus, all typical set sequences have approximately the same probability of $\approx 2^{-nh_2(p)}$.

(c) Show that the number of typical sequences $|A_n(\epsilon)|$ satisfies the following inequality,

$$(1-\epsilon)2^{nh_2(p)-n\epsilon} \leq |A_n(\epsilon)| \leq 2^{nh_2(p)+n\epsilon}.$$

Thus about $2^{nh_2(p)}$ typical sequences are there and they require $nh_2(p)$ bits for representation. (Hint: use the Union bound.)

Solution:

(a) By WLLN and using $p(X_1^n) = \Pi_{i=1}^n p(X_i)$,

$$-\frac{1}{n}\log_2(p(X_1^n)) \xrightarrow{\mathbb{P}} H_2(p)$$

Thus,

$$\lim_{n\to\infty} \mathbb{P}\left\{x_1^n : \left|-\frac{1}{n}\log_2(p(x_1^n)) - H_2(p)\right| \leq \epsilon\right\} = 1$$

i.e., $\lim_{n\to\infty} \mathbb{P}(A_n(\epsilon)) \to 1$. Hence for any $\epsilon > 0$, there exists an n_0, such that $\mathbb{P}(A_n(\epsilon)) \geq (1-\epsilon)$ for all $n \geq n_0$.

(b) On the set $A_n(\epsilon)$, we have,

$$\left\{\left|-\frac{1}{n}\log_2(p(x_1^n)) - H_2(p)\right|\right\} \leq \epsilon$$

which can be rewritten as shown below:

$$\begin{aligned}
-\epsilon \leq -\frac{1}{n}\log_2(p(x_1^n)) - H_2(p) &\leq \epsilon \\
-n\epsilon \leq -\log_2(p(x_1^n)) - nH_2(p) &\leq n\epsilon \\
-n\epsilon - nH_2(p) \leq \log_2(p(x_1^n)) &\leq n\epsilon - nH_2(p) \\
2^{(-n\epsilon - nH_2(p))} \leq p(x_1^n) &\leq 2^{(n\epsilon - nH_2(p))}
\end{aligned}$$

This proves the desired result.

(c) We know that $\sum_{x_1^n \in A_n(\epsilon)} p(x_1^n) = \mathbb{P}(A_n(\epsilon)) \leq 1$. From part(b), $p(x_1^n) \geq 2^{(-n\epsilon - nH_2(p))}, x_1^n \in A_n(\epsilon)$. Using these results, the following can be derived:

$$\begin{aligned}
\sum_{x_1^n \in A_n(\epsilon)} 2^{(-n\epsilon - nH_2(p))} \leq \sum_{x_1^n \in A_n(\epsilon)} p(x_1^n) &\leq 1, \text{ or} \\
2^{(-n\epsilon - nH_2(p))} \sum_{x_1^n \in A_n(\epsilon)} 1 &\leq 1, \text{ or} \\
2^{(-n\epsilon - nH_2(p))} |A_n(\epsilon)| &\leq 1.
\end{aligned}$$

Thus $|A_n(\epsilon)| \leq 2^{(n\epsilon + nH_2(p))}$.
For deriving the lower bound on $|A_n(\epsilon)|$, consider $\mathbb{P}(A_n(\epsilon)) \geq 1-\epsilon$ and $p(x_1^n) \leq 2^{(n\epsilon - nH_2(p))}$.

$$\begin{aligned}
\sum_{x_1^n \in A_n(\epsilon)} p(x_1^n) &\geq 1-\epsilon \text{ which leads to} \\
\sum_{x_1^n \in A_n(\epsilon)} 2^{(-n\epsilon + nH_2(p))} &\geq 1-\epsilon.
\end{aligned}$$

By rearranging the terms, we get $|A_n(\epsilon)| \geq (1-\epsilon)2^{(n\epsilon - nH_2(p))}$.