Root Locus Methodology

- *Root Locus as s – Domain Design Tool*
- *Root Locus Procedure*
- *Root Locus Features*
- *Root Locus Special Cases*

Root Locus Solution

Root Locus Solution

Let '$K.G_c(s)\ G(s) = -1$' be **rewritten as follows.**

$$KG_c(s)G(s) = \frac{K\prod_{i=1}^{m}(s - z_i)}{\prod_{j=1}^{n}(s - p_j)} = -1$$

$$\prod_{j=1}^{n}(s - p_j) + K\prod_{i=1}^{m}(s - z_i) = 0$$

It is **clear** from the above **relation** that values of **'s'** satisfying the condition, **depend** on the values of **'K'.**

Root Locus Solution

E.g., if **K = 0**, then we get,

$$\prod_{j=1}^{n}(s-p_j)=0 \rightarrow s_j = p_j$$

I.e. plant **poles** are the **starting values** for the closed loop poles.

Similarly, if **K = ∞**, we get,

$$\prod_{i=1}^{m}(s-z_i)=0 \rightarrow s_i = z_i$$

I.e. plant **zeros** are the **end points** of the root locus.

We also know that **path,** in the vicinity of **'p_j' & 'z_i',** can be determined by **value of slope** at these points.

Root Locus Solution

Therefore, there are **many** such simple **rules** which can be used to capture **gross** features of root locus**, graphically.**

This task is made **simpler** by firstly, splitting '**K.G_c G(s)**', as **magnitude** |K.G_c G(s)| & **angle** ∠K.G_c G(s), and next, imposing the **closed loop** condition as follows.

$$\left|KG_c(s)G(s)\right| = 1$$
$$\angle KG_c(s)G(s) = \pm 180^\circ(2k+1), k = 0,1,2\cdots$$

The method involves **checking** for these **conditions** to determine **contiguous** line segments that denote **root locus.**

Straight Line Root Locus Concept

In this manner, we can **arrive** at an overall closed loop **pole map**, with reasonable computational **effort**, which essentially consists of **straight lines.**

We find that as our **primary** focus is the **dominant** closed loop **poles**, we can further **minimize** the effort by **focusing** on the map close to the **imaginary axis.**

Summary

s – domain design procedure **aims** to arrive at **controller** to achieve the **dominant** closed loop poles.

Root locus is a **pictorial view** of closed loop **system poles**, based on the **plant** characteristics.

Root Locus Generation Procedure

Applicable Line Segments

F**ollowing** broad steps are **employed** to generate the **dominan**t root locus **branch.**

Root locus **starting** points; Root locus on **real axis**, Break-away/**break**-in points; **Asymptotes**; Root locus on **imaginary axis**; Angle of arrival/**departure**; Root locus **ending** points.

Root Locus Procedure

Root Locus Starting: $s = -p_j$

Root Locus Ending: $s = -z_i$

Root locus on real axis

(n + m) to the right of root locus, **an odd number.**

Root Locus Procedure

Asymptotes (Case of s $\to \infty$)

$$\lim_{s\to\infty}\frac{K(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)} \approx \frac{K}{s^{n-m}} \to \angle = \frac{\pm(2k+1)180^o}{n-m}$$

$$\lim_{s\to\infty}\frac{K(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)} \approx \frac{K}{s-\dfrac{\sum_{j=1}^{n}p_j-\sum_{i=1}^{m}z_i}{n-m}}; \quad \sigma = \frac{\sum_{j=1}^{n}p_j-\sum_{i=1}^{m}z_i}{n-m}$$

Root Locus Procedure

Break-away & Break-in Points (Double Real Roots)

$$KG(s) = \frac{K(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)^2(s-p_3)\cdots(s-p_n)} = \frac{KA(s)}{(s-p_1)^2 B(s)} = -1$$

$$K(s) = -\frac{(s-p_1)^2 B(s)}{A(s)} \rightarrow \frac{dK}{ds} = (s-p_1)F(s) = 0 \rightarrow s = p_1$$

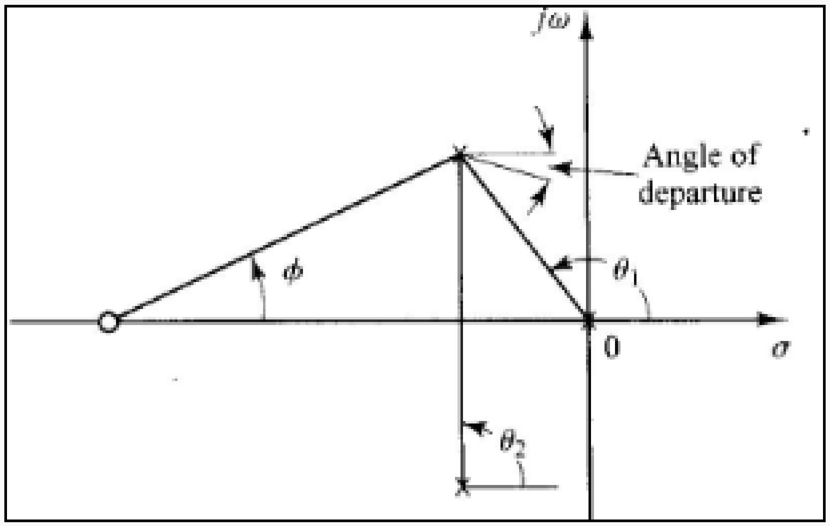
Root Locus Intersection with Imaginary Axis

Use Routh's method to determine K & poles.

Root Locus Procedure

<u>Departure / Arrival Angles</u>

From a **complex** pair of **poles** and at pair of **zeros.**



For any point 's' in vicinity

$$\varphi_d = 180^o - \sum_{j=1}^{n} \theta_j + \sum_{i=1}^{m} \varphi_i$$

$$\varphi_a = 180^o + \sum_{j=1}^{n} \theta_j - \sum_{i=1}^{m} \varphi_i$$

<u>Value of Gain K_p at a point 's'</u>

Use magnitude condition, $|K.G(s)| = 1$.

Root Locus Application

As **root locus** is **plot** of all possible **closed loop** poles as 'loop' **gain** is varied, it provides information about poles **closest** to the imaginary axis (**dominant**).

Thus, **root locus** is useful for **addressing** closed loop **requirements** in terms of the **dominant** poles.

Root Locus Example

Sketch the **asymptotic** root locus for the **following** plant.

$$G(s) = \frac{K}{s(s+1)(s+2)}; \quad H(s) = 1$$

Open Loop Poles & Zeros:

$n = 3, p_1 = 0, p_2 = -1, p_3 = -2; \quad m = 0; \quad (n - m) = 3$

Root locus on Real Axis:

$(+\infty, 0) \rightarrow$ No Root Locus; $(0, -1) \rightarrow$ Root Locus

$(-1, -2) \rightarrow$ No Root Locus; $(-2, -\infty) \rightarrow$ Root Locus

Asymptotes: Angles $\pm 60^o$ & -180^o; Intersection $\sigma = -1$

Root Locus Example

Break-away Point: $K(s) = -\left(s^3 + 3s^2 + 2s\right) \rightarrow$

$$\frac{dK}{ds} = -\left(3s^2 + 6s + 2\right) = 0 \rightarrow s = -0.4226$$

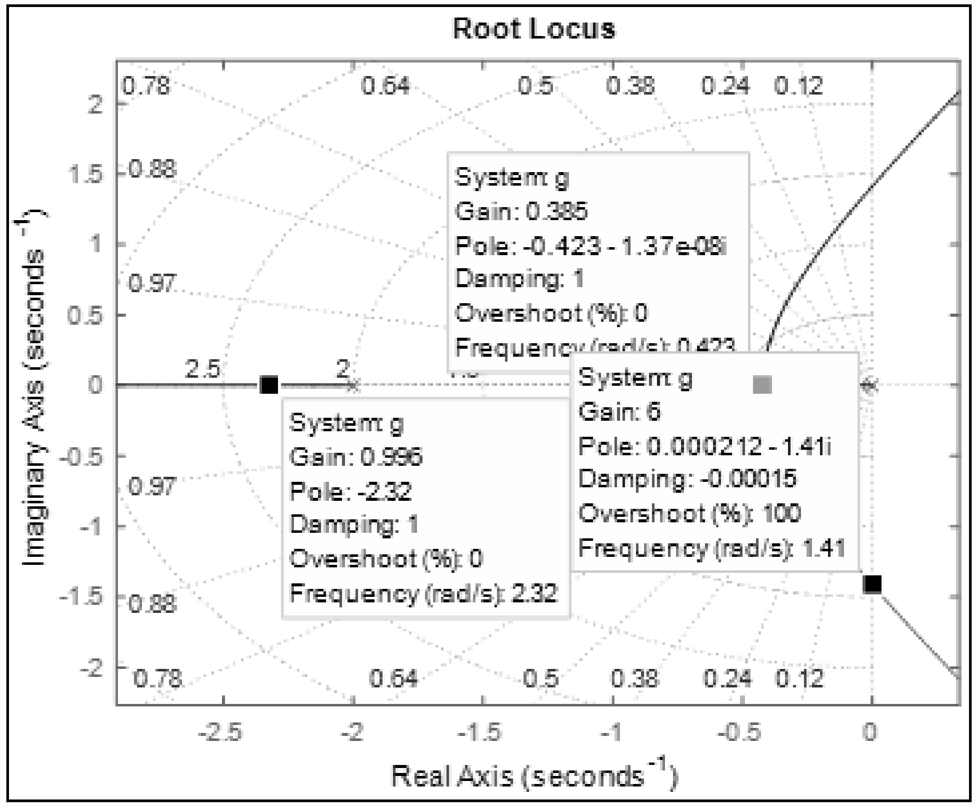
Imaginary Axis Crossing:

$$s^3 + 3s^2 + 2s + K = 0$$

$$6 - K = 0 \rightarrow\rightarrow \omega = \pm\sqrt{2}\,j$$

Gain at any point on Root Locus:

$$K = \frac{1}{|G(s)|}$$



Straight Line Root Locus Benefits

In many cases of **gross** assessment of closed loop **behaviour,** a straight line (or **asymptotic**) root locus is considered **adequate.**

A more **accurate** plot can be **generated** by taking **additional** points between these segments, through **trial** and error **method.**

Summary

Asymptotic root locus is a simple aid for **visualizing** the dominant closed loop **behaviour**.

Root Locus Features

Root Locus Features

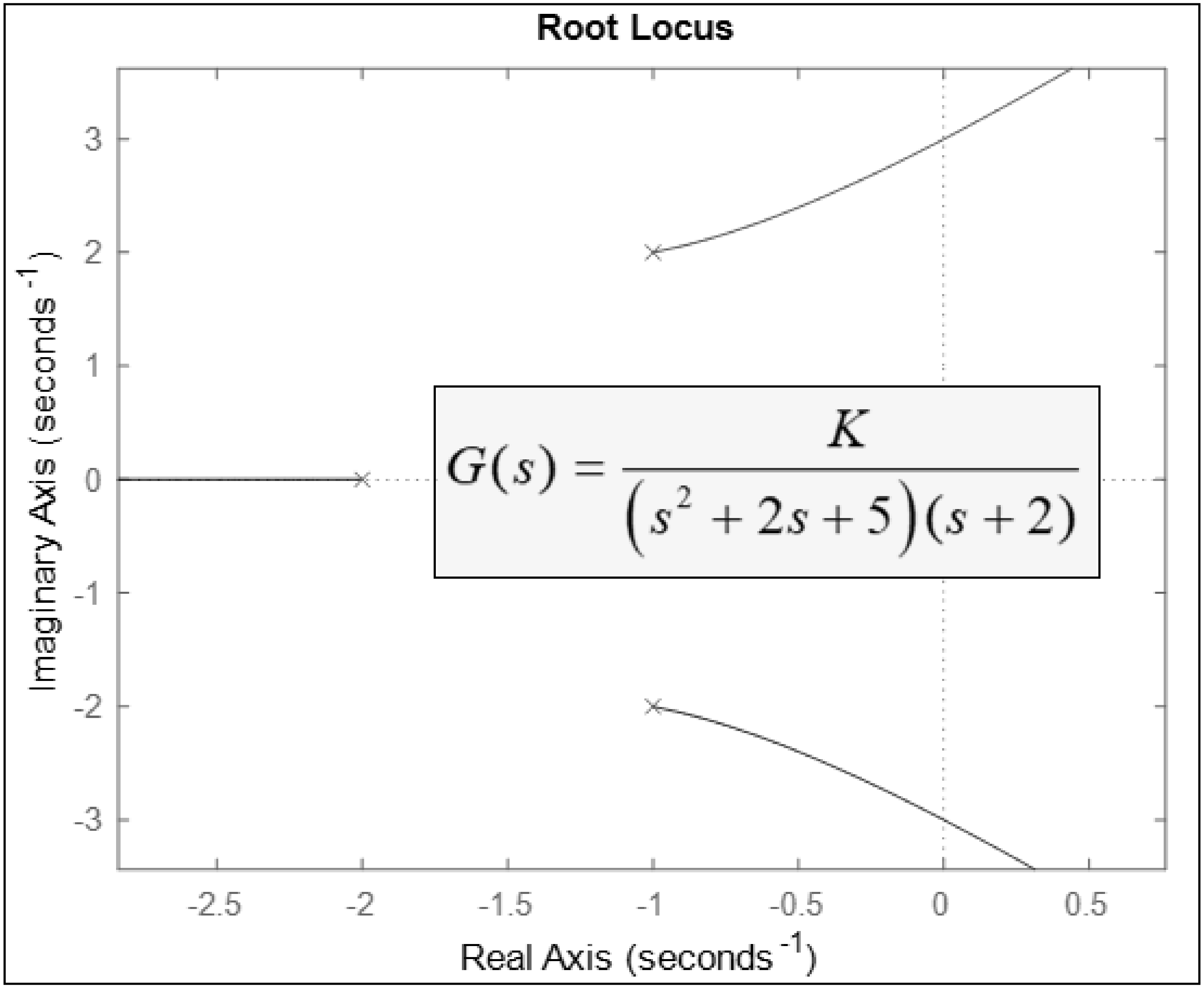
Root Locus contains as many **lines** as the number of **plant poles.**

Root locus is **symmetric** about the **real axis**, due to poles being complex **conjugate**, and hence, only **one half** needs to be **generated.**

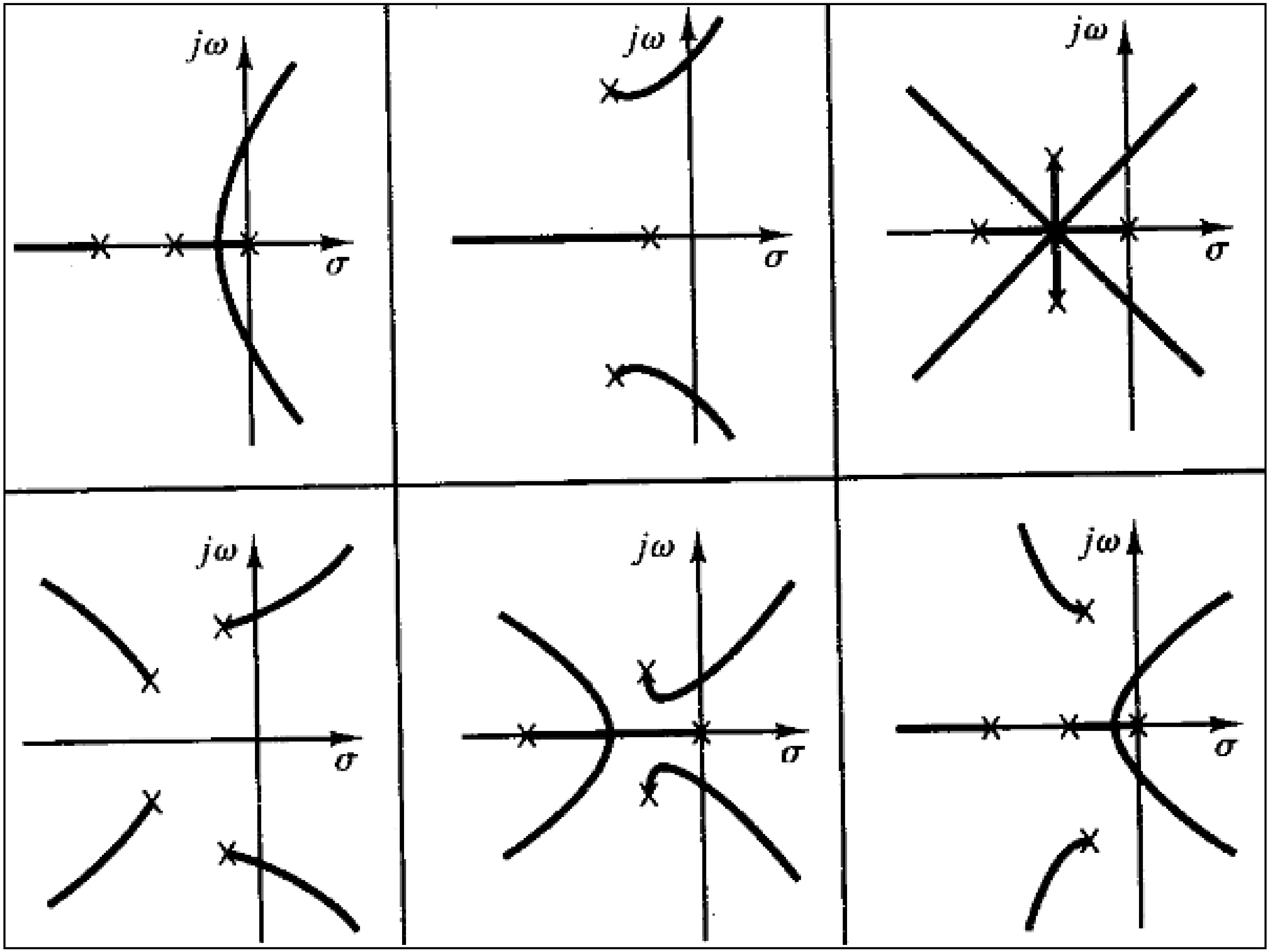
Root locus **lines** never cross each other, except at **break-away**/break-in points.

An **asymptote** indicates a **zero** lying at **infinity.**

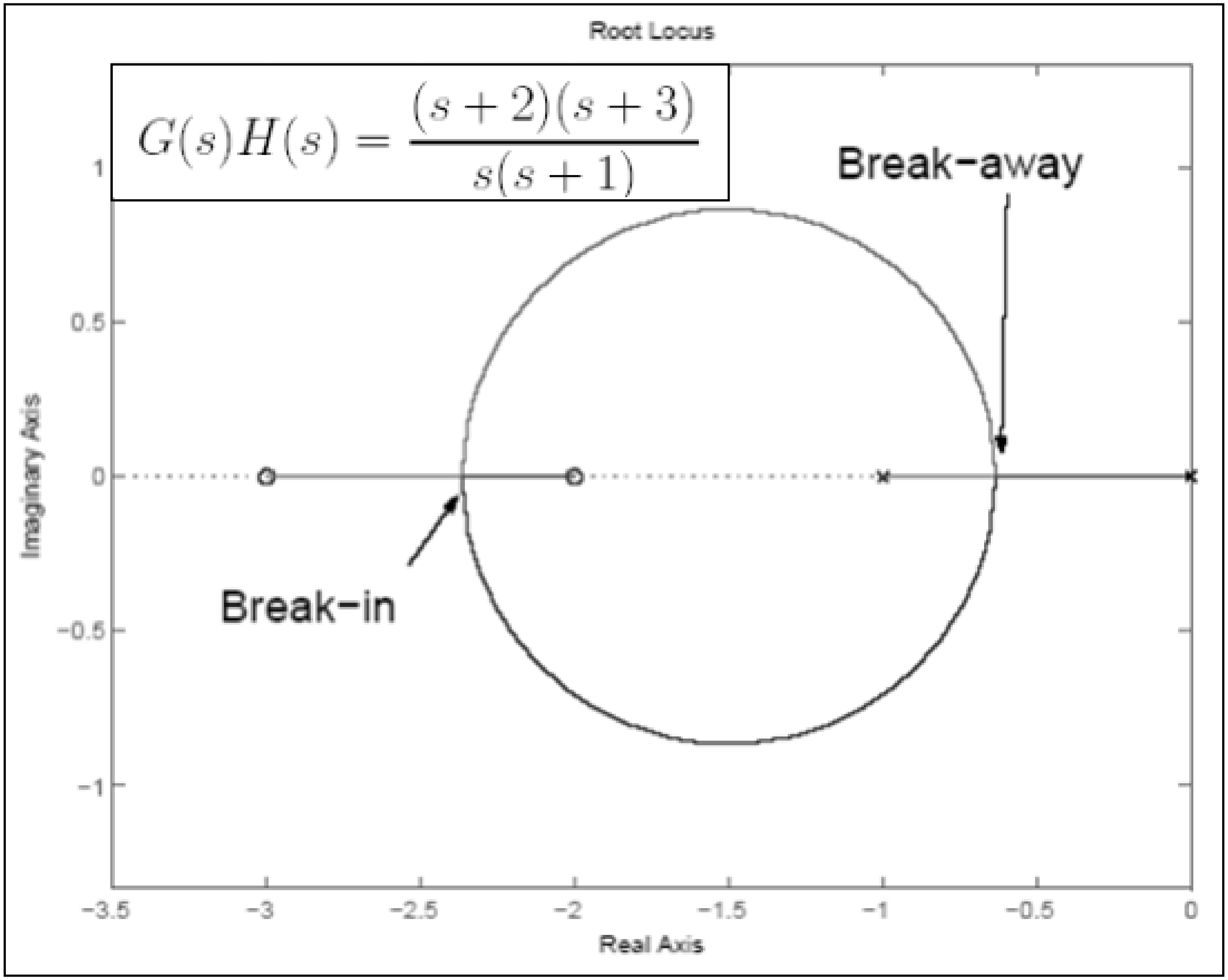
All Pole Plant Root Locus



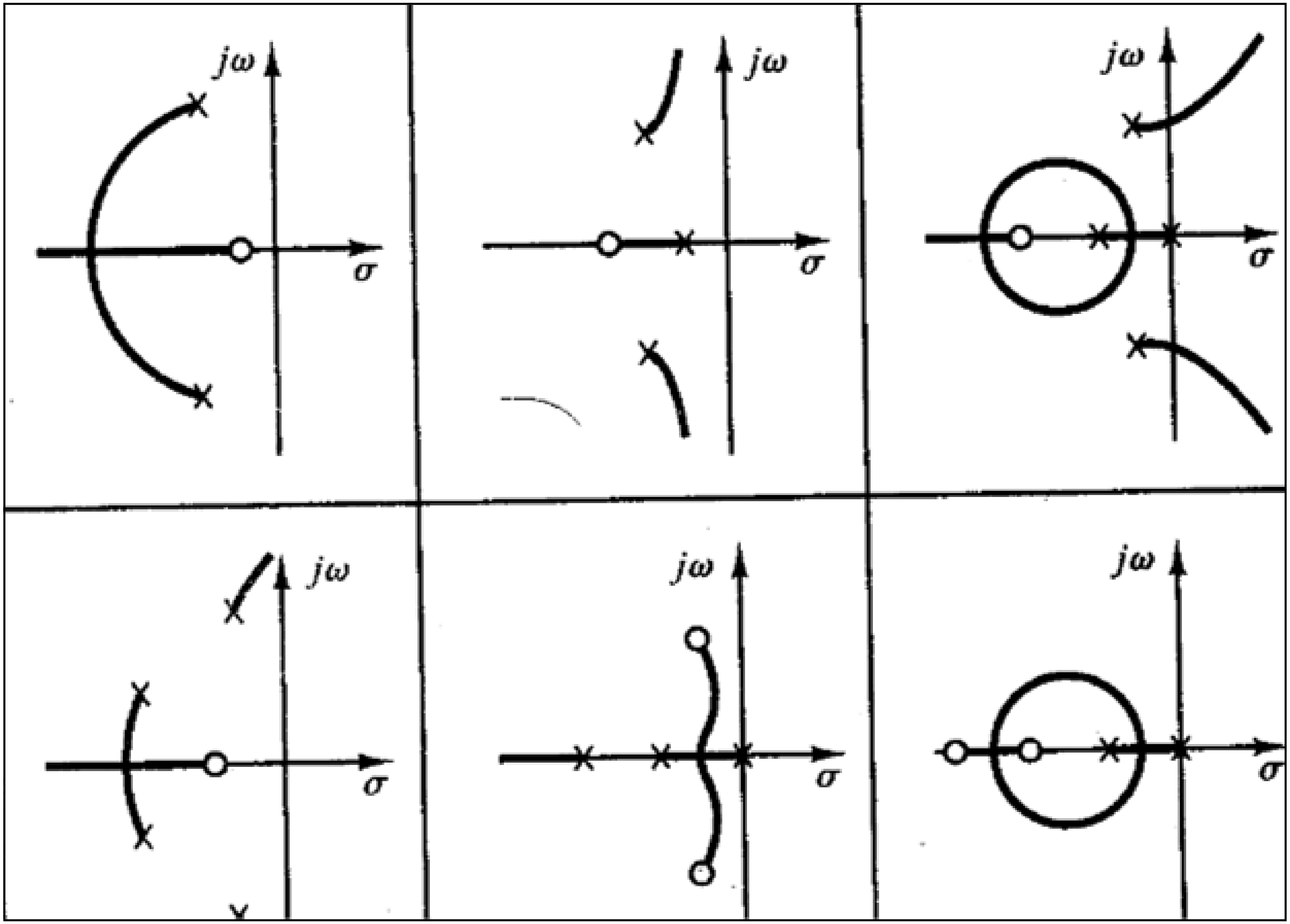
General Rules for All Pole Plants



Pole – Zero Plant Root Locus



General Rules for Pole – Zero Plants



Root Locus Special Cases

Non-unity Feedback Systems

There are many **practical situations** when a **gain** appears in **feedback path**. This is common when there are **sensors / amplifiers** for output signal. **Closed loop** transfer function in **such cases** is as given below.

$$\boxed{\frac{C(s)}{R(s)} = \left(\frac{G(s)}{1+KG(s)} \right)}$$

It can be seen that the **characteristic polynomial** is same as **that obtained** with gain $\mathbf{K_p}$ in **cascade** so that **both** structures have the **same** root locus.

Systems with Positive Feedback

In case of **positive** feedback, following **rules change.**

$\angle G(s)\ H(s) = 0^o \pm k\ 360^o$

(n + m) to the right is even

Asymptotes' angle = $\pm$ k 360^o / (n – m)

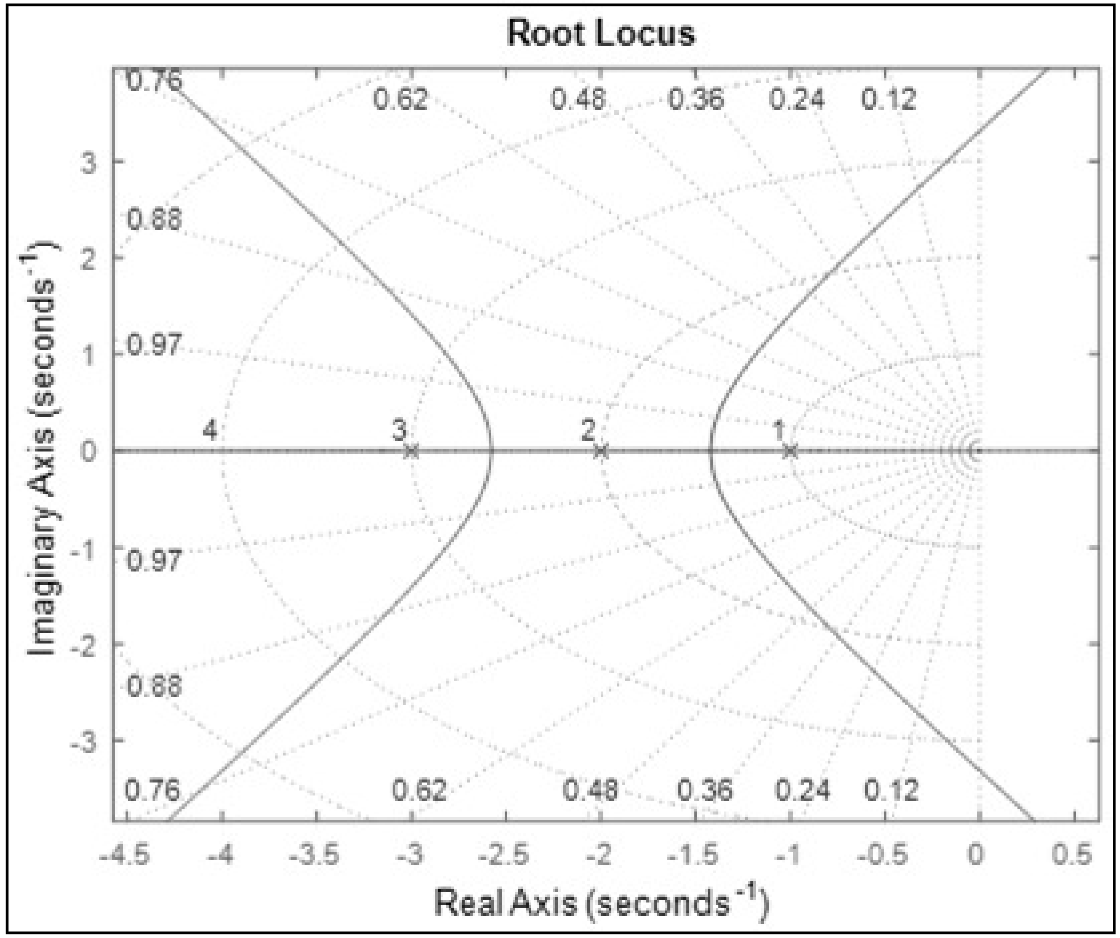
ϕ_a (or ϕ_d) = $0^o \pm (\phi_i - \theta_j)$

Positive feedback root locus is also termed '**negative**' root locus, as this is obtained by putting **K < 0**.

Positive Feedback Example

Generate both **positive & negative feedback** root loci for the **system** given below.

$$G(s) = \frac{K}{(s+1)(s+2)(s+3)}$$



Non – minimum Phase Systems

There are a large **class of plants** whose poles (or zeros) are in the **right half** of s – plane.

While, the **rules** for all such **systems** are same as those for **negative feedback**, angle contributions change leading to **changes** in root locus **shape.**

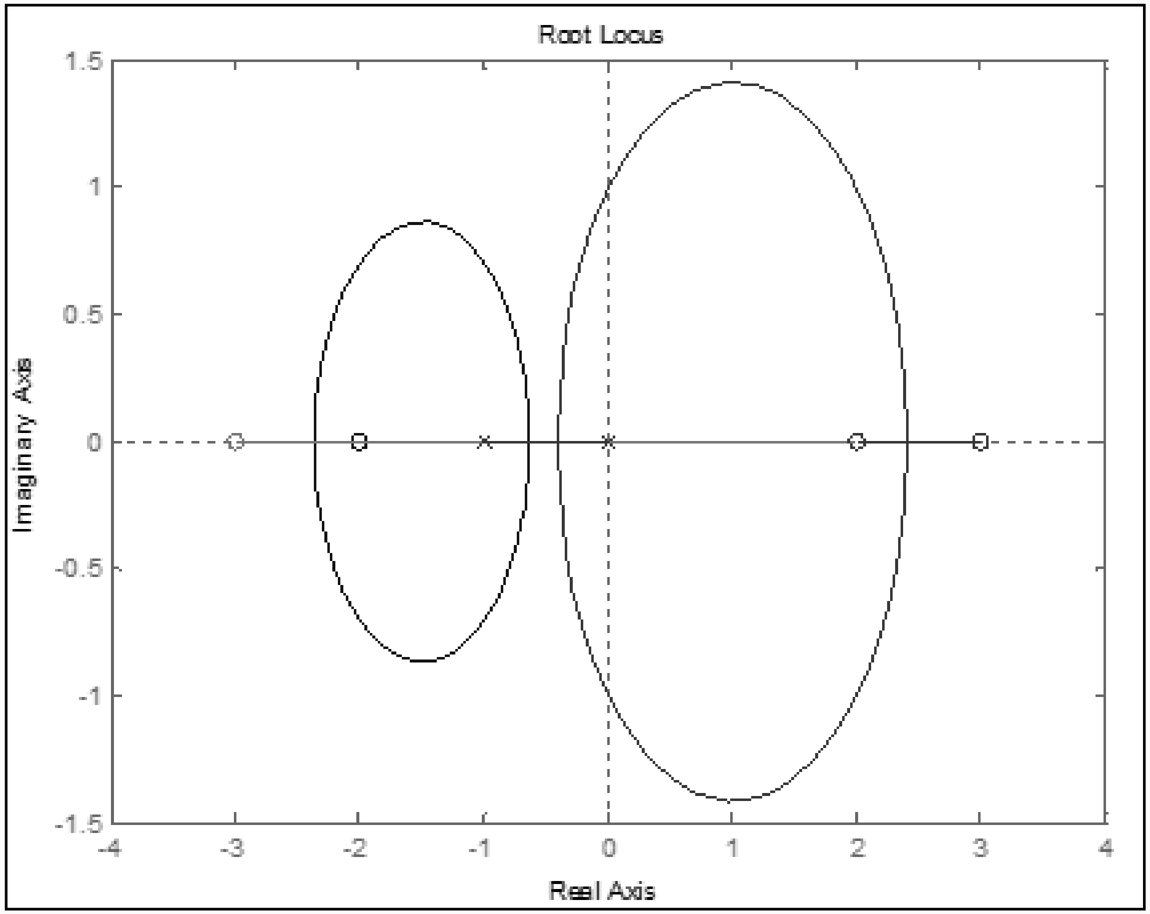
Non – minimum Phase Root Locus

Generate **root loci** and bring out important features.

$$G_1(s) = \frac{K(s+2)(s+3)}{s(s+1)}$$

$$G_2(s) = \frac{K(s-2)(s+3)}{s(s+1)}$$

$$G_3(s) = \frac{K(s-2)(s-3)}{s(s+1)}$$



Summary

Gain in **feedback** path results in the **same** root locus as that obtained for **gain** in cascade.

Negative root locus is applicable for **positive** feedback systems.

Zeros in right half impact angle **relations** adversely, thereby, **impacting** shape of root locus.