Stability Analysis Techniques

- *Absolute Stability Hypothesis*
- *TF Based Routh-Hurwitz Method*
- *Frequency Domain Based Nyquist Method*
- *Conformal Mapping & Nyquist Criterion*

Feedback Control Applications

We have seen that **feedback control** concepts are able to **achieve** the control **tasks** in the context of **ensuring** stability, input **trackin**g and disturbance **rejection.**

However, prior to **setting up** control systems, we first **need** to setup tools & **methods** which establish the **performance** of plant in terms of these **attributes.**

In this regard, we **first** take up the **tools** for the **stability** analyses.

Stability Characterization

We have seen that **nature** of real part of **poles** decides the **manner** in which the response **evolves** over time, and hence, decide the nature of **stability.**

We also know that **poles** are roots of **equation** formed using the **denominator** of transfer function i.e. **D(s) = 0**.

This is also called **characteristic equation** as it provides roots that determine **characteristic** of system response.

Stability Analysis Strategy

Thus, we need to **extract** the real part of the **poles** in order to find out the **dominant** pole and its **location** with respect to the **imaginary** axis.

In general, this **task** can be done by **solving** the applicable **algebraic** equation.

This is **easy** if the **order** of system is **small**, but becomes quite **tedious** for systems with **higher** order.

Concept of Absolute Stability

As a **first** step, we want to **know** if the system is **stable** or unstable and **later** we can **quantify** its level.

This task can be **easily** achieved by **knowing** whether or not the **system** has any pole with **positive real part**, as any one such **pole** will make the system **unstable.**

This is the concept of **absolute stability**, which is examined in a **simple** manner by extracting **only the sign** of real part of poles (or poles lying in **RH s-plane**).

Routh's Procedure is a **tool** that helps us in this **matter.**

Routh's Procedure

We know that **coefficients** of characteristic polynomial contain **information** about its **roots**.

Routh & Hurwitz criterion aims to examine the **stability**, by **manipulating the coefficients** of the characteristic polynomial, which is specified in the form of **necessary** and **sufficient** conditions.

Necessary condition states that the **polynomial** must be **complete,** with all coefficients **having same sign**.

Sufficient condition examines **Hurwitz determinants** for their **sign** in order to conclude about system **stability.**

Routh's Tabulation

s^n	a_n	a_{n-2}	a_{n-4}	----------	a_0
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	----------	a_1
s^{n-2}	b_1	b_2	b_3	--------	
.					
.					
.					
s^2	e_1	e_2			
s^1	f_1				
s^0	g_1				

1st two rows based on even/odd powers

Zeros used to complete a row

How to get remaining rows?

$b_1 = (a_{n-1}\, a_{n-2} - a_n\, a_{n-3}) / a_{n-1}$

$b_2 = (a_{n-1}\, a_{n-4} - a_n\, a_{n-5}) / a_{n-1}$

$e_1 = (d_1\, c_2 - c_1\, d_2) / d_1$ and so on ….

Routh's Example – Stable

$$s^4 + 8s^3 + 18s^2 + 16s + 5 = 0$$

s^4	1	18	5
s^3	8	16	0
s^2	16	5	
s^1	27/2	0	
s^0	5		

Complete Polynomial with all coefficients having **same sign.**

No sign changes, so no poles in the right half of s – plane.

Poles → -5.0, - 1.0, -1.0, -1.0

Routh's Example – Unstable

$$s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$$

s^4	1	3	5
s^3	2	4	0
s^2	1	5	
s^1	-6	0	
0	5		

Complete polynomial with all coefficients having **same sign.**

Two sign changes, so two poles in the right half of s – plane.

Poles

0.2878 + 1.4161i
0.2878 - 1.4161i
-1.2878 + 0.8579i
-1.2878 - 0.8579i

Special Condition – ‘0’ in 1^{st} Column

Quite often we encounter **‘0’ in 1^{st} column**, which needs to be **resolved**, which is done as follows.

Replace ‘0’ by $\in$ (+ve) and **proceed.** Sign changes indicate **unstable poles.**

$$D(s) = s^3 + 2\, s^2 + s + 2$$

S^3	1	1
S^2	2	2
S^1	0 ($\in$)	0
S^0	2	

Complete polynomial with all coefficients having **same sign.**

No sign changes, so no poles in right half s – plane.

-2.0000
0.0000 - 1.0000i
0.0000 + 1.0000i

Zero element indicates **poles** on the **imaginar**y axis.

Special Condition – A Zero Row

Some times, the **whole row becomes zero**, which is **resolved** by forming $D_a(s)$ using coefficients of **previous** non-zero **row** and **differentiating** it.

$$D(s) = s^5 + 2s^4 + 24s^3 + 48\, s^2 - 25s - 50$$

S^5	1	24	-25
S^4	2	48	-50
S^3	0 (8)	0 (96)	0
S^2	24	-50	0
S^1	112.7	0	
S^0	-50		

Complete polynomial with some coefficients having different **sign.**

One sign change, so one pole in right half s – plane.

0.0000 + 5.0000i
0.0000 - 5.0000i
-2.0000
-1.0000
1.0000

Zero row indicates **roots** symmetric about s-plane **origin.**

Summary

Routh's is a **simple method** to examine the **stability** of both open and closed loop **systems.**

However, it only **provides** the information about the **absolute** stability (qualitative).

Nyquist Based Absolute Stability

Frequency Domain Absolute Stability

As **frequency response** contains the **essential features** of **G(s)H(s),** it can be used to analyze the **absolute stability** of the corresponding unity feedback **closed loop system.**

The methodology, while similar in its **approach** to Routh's criterion, uses a more **rigorous mathematical** approach to arrive at the **stability result.**

Nyquist plots are used as the basic tool for **setting up** the stability **analysis process.**

Nyquist Stability Analysis Concept

It is a **method** for analyzing **absolute stability** of a unity feedback closed loop system from plant **Nyquist plot** & **poles**.

Consider the **closed loop** characteristic equation, **D(s)** = 1 + G(s)H(s) = **F(s) = 0**

We know that for **closed loop stability**, it is necessary that all roots of **F(s) = 0** must lie in **left half** of s-plane.

Nyquist Stability Analysis Concept

Nyquist stability analysis relates **G(jω)H(jω)** & No. of **poles of G(s)H(s)** that lie in the **right half** of s-plane, to predict No. of **roots of F(s) = 0 in right half** of s – plane.

It uses the **conformal mapping** hypothesis which states that any **point in s- plane,** not passing through a singularity, **maps uniquely** into a point in **F(s) plane**.

Conformal Mapping Scenario

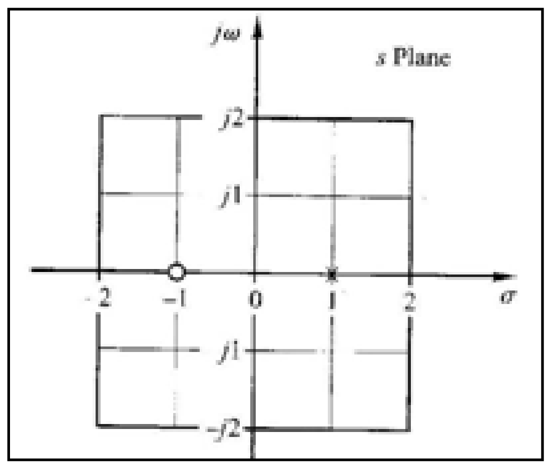
$$G(s)H(s) = \frac{2}{s-1}$$

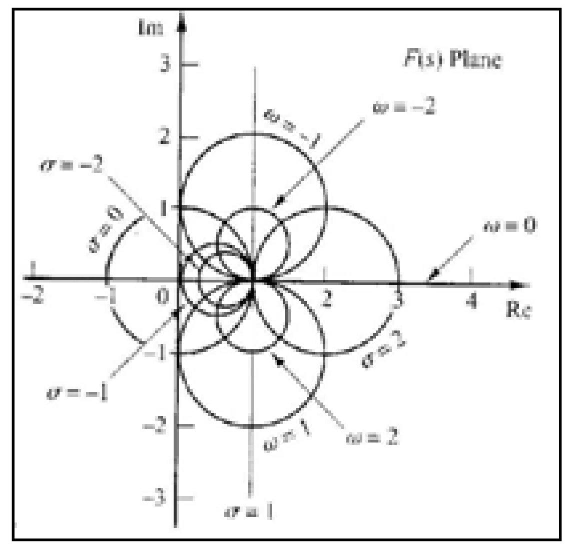
Consider a **plant** as shown **alongside.**

Expression for F(s) can be **obtained** as shown.

$$F(s) = 1 + G(s)H(s) = \frac{s+1}{s-1}$$

We can now **map** the grid **lines in s-plane** into the corresponding **curves in F(s)-plane**, as shown alongside.

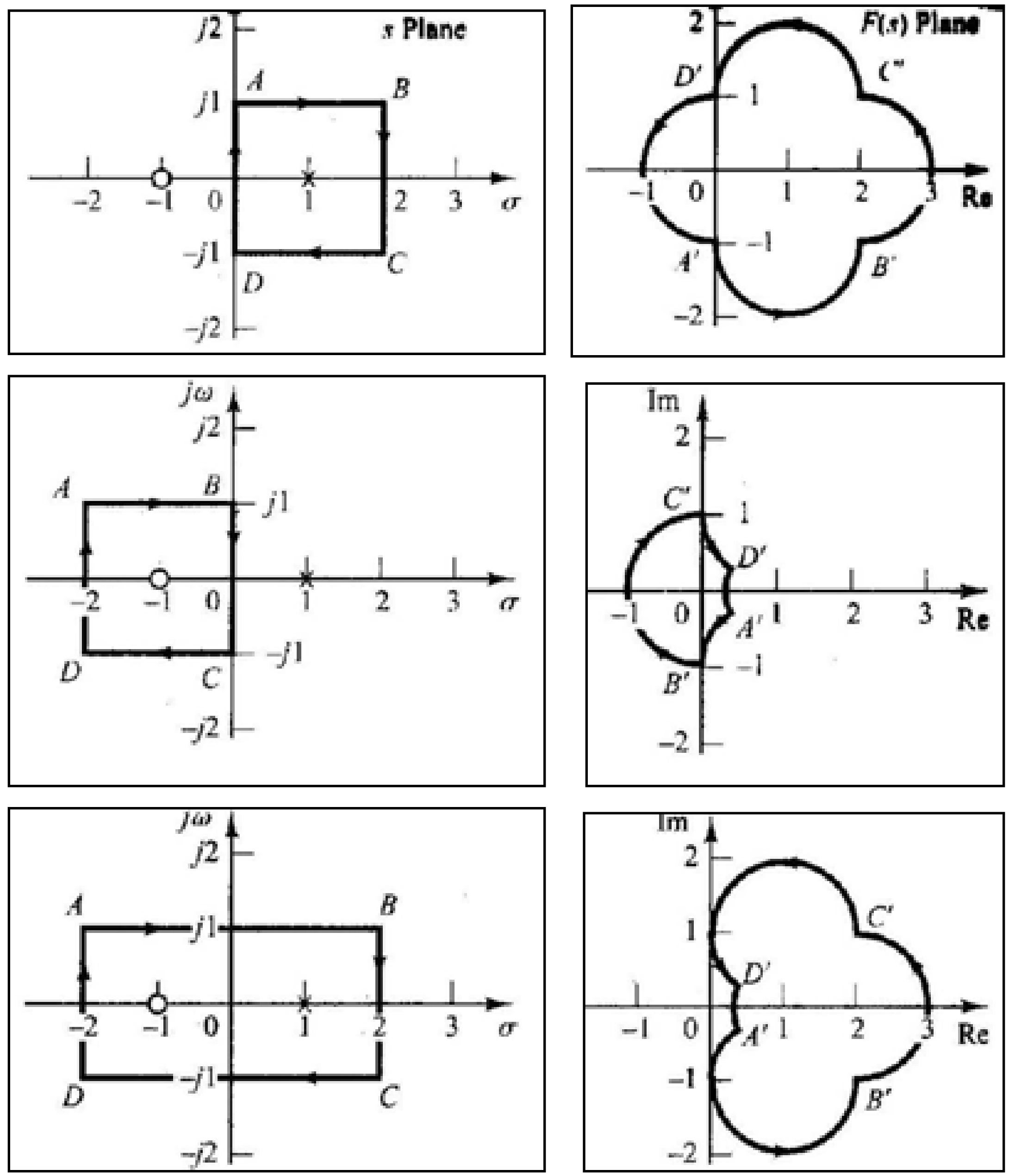




Mapping of Closed Curves

We can **extend** the above **mapping** to closed **curves** in s-plane, which are **directional** in nature, as shown **alongside.**

We see that the **relation for directionality** and encirclement of origin, is dependent on **enclosing of poles/zeros** by the s-plane curve.



Closed Curve Mapping Features

We can see that **direction of encirclement** of origin of F(s) – plane, by locus of F(s), **depends** on whether the contour in s – plane **encloses a 'zero' or a 'pole**'.

For example, if it **encloses a pole**, the direction of **curve is reversed**, while it remains same, if a **zero is enclosed.**

We also see that if closed **contour encloses equal** No. of **poles and zeros**, then F(s) locus **does not encircle the origin** of F(s) – plane.

This has **led to** the formulation of the **Nyquist** mapping **theorem**, which can now **be stated as** follows.

Nyquist Hypothesis

Let F(s) be a **ratio** of two **polynomials** in ‘s’.

Also, let ‘**P**’ be No. of **poles** and ‘**Z**’ be No. of **zeros** of F(s) that **lie inside** some closed **contour in ‘s’ plane**, (Including Multiplicity).

Lastly, let the **contour** be such that it does not **pass** through any **poles of G(s)H(s).**

Then, **contour** in ‘s’ – plane **maps** into another contour **in ‘F(s)’ – plane** such that No. of clockwise encirclements (**N**) of **origin of F(s)** plane is equal to **(Z – P).**

Nyquist Hypothesis

It should be noted that '**P**' is the number of **poles of G(s)H(s)** inside the **s-plane contour.**

Further, '**Z**' is nothing but the number of **closed loop poles** inside some **contour in s-plane.**

Lastly, '**N**' is the number of **clock-wise encirclements** of F(s) plane origin by the **mapped contour** and can be both positive or **negative** (anti-clockwise).

Interpretation of Mapping Theorem

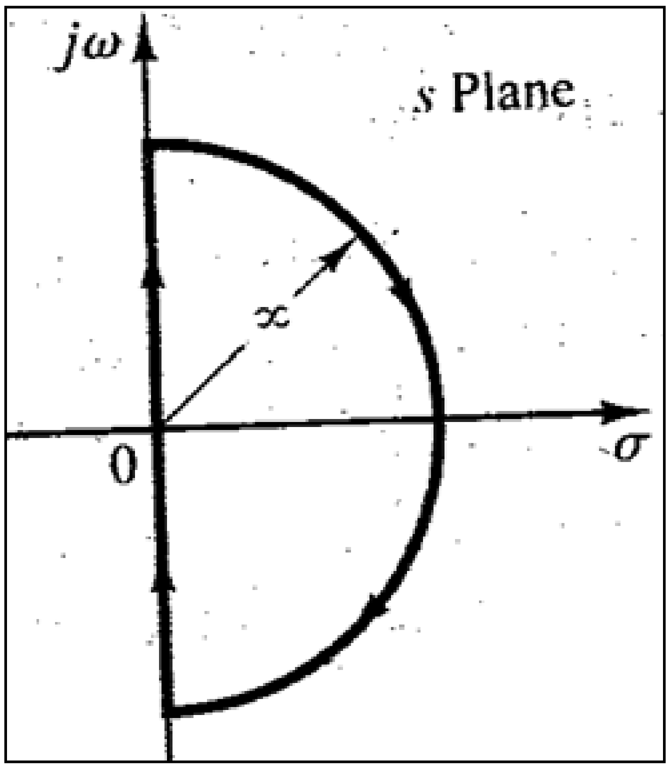
Here, **positive 'N'** indicates that **Z > P** while a **negative 'N'** indicates **Z < P**. However, both 'Z' & 'P' are ≥ **0.**

In control theory, '**P**' is readily **known** from **G(s)H(s)** itself, while '**N**' can be obtained from the **plot of F(s)**, so that we can **determine** the closed loop poles, or '**Z**'.

It is to be noted that **exact shape** of contour in 's' plane is **not important** to encirclement of origin of F(s) – plane and also that for **stability**, we need to **know only 'Z'.**

s-plane (Nyquist) Curve Concept

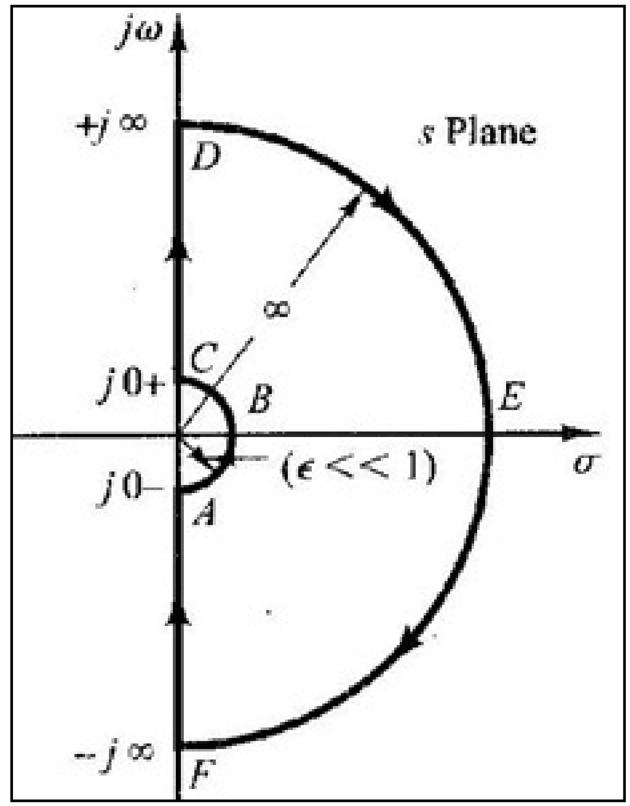
As we want to **determine** the poles in **RH s-plane** to establish absolute **stability**, a curve containing **complete RH s-plane** is the most logical **choice**, as shown below.



$$
\begin{array}{ccc}
A \rightarrow & B & \rightarrow C \\
\uparrow & & \downarrow \\
F \leftarrow & E & \leftarrow D
\end{array}
$$

$$s = \varepsilon e^{j\theta}$$

$$-90 < \theta < +90$$



Imaginary Axis as Nyquist Curve

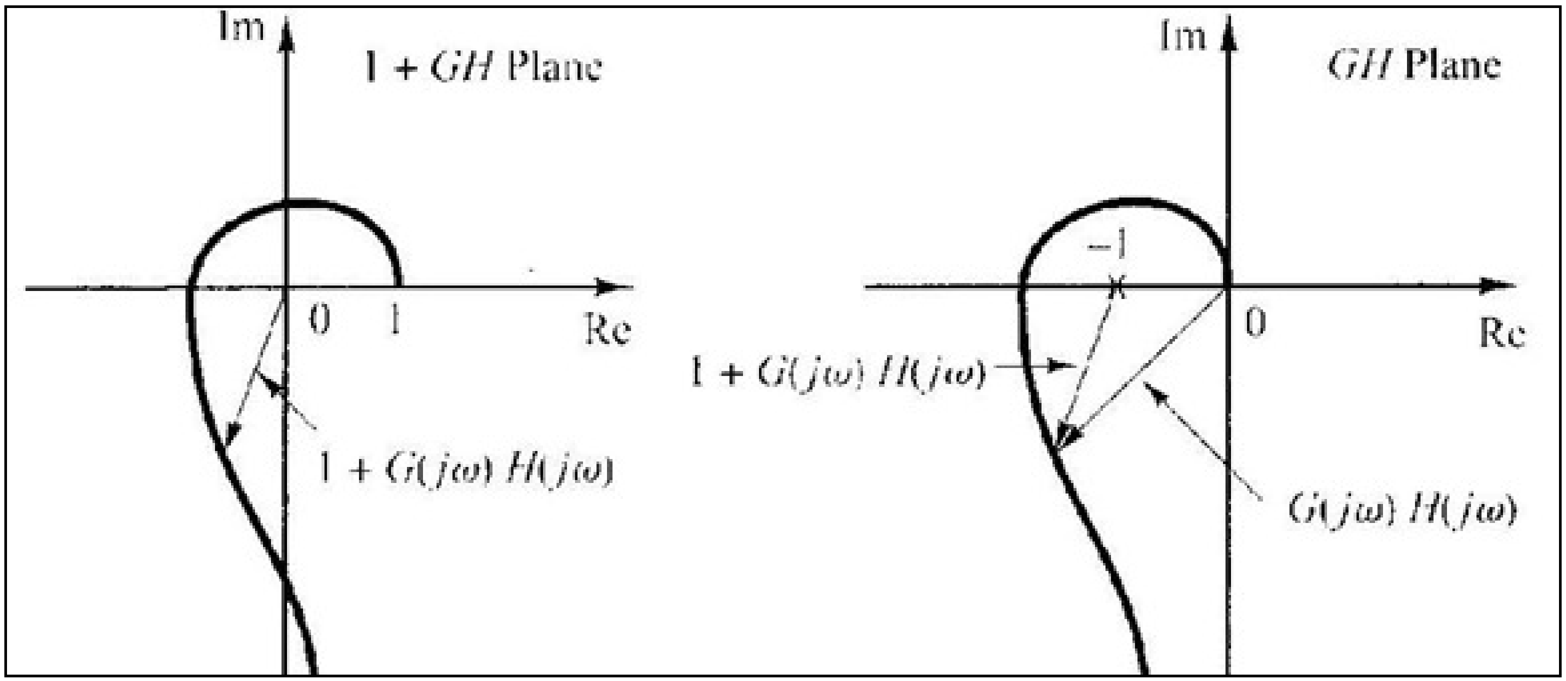
As most systems are **physically realizable**, F(s) is either **'0' or a constant** along semi-circle, so that it is **enough** if we consider **only the 'jω' axis**, as defined below.

$$\begin{array}{ccc} A\rightarrow & B\rightarrow & C \\ \uparrow & & \downarrow \\ \leftarrow & F\leftarrow & D \end{array}$$

Thus, we can **replace** the F(s) plane mapping with **F(jω)** plane mapping such that **F(jω) = 1 + GH(jω).**

Application of Mapping Theorem

Further, as **F = 0** is same as **GH = -1**, we can instead use **encirclement** of point **-1+j0** by GH plot in **GH-plane**, for stability **analysis**, as shown below.



Nyquist Stability Conditions

Closed loop system **stability** is ensured only if **Z = 0**, which puts **restrictions** on both '**N**' and '**P**', as below.

If **<u>P ≠ zero</u>**, then for a **stable** closed loop, **N = -P**, i.e. there must be **as many anti-clockwise** encirclements of -1 + j0 as there are **right half poles** of G(s)H(s).

If **<u>P = 0</u>**, then for a **stable** closed loop, **Z = N**, which means that there must be **no encirclement** of -1 + j0.

Unstable Closed Loop Scenarios

If **N > 0**, then **no matter** what **'P'** is, Z > 0 and in such cases, **closed loop** will always be **unstable**.

If the **Nyquist plot** passes through the **point -1 + j0**, it means that **closed loop** poles lie on the **imaginary axis**.

Case – 1: Stable Closed Loop & $P \neq 0$

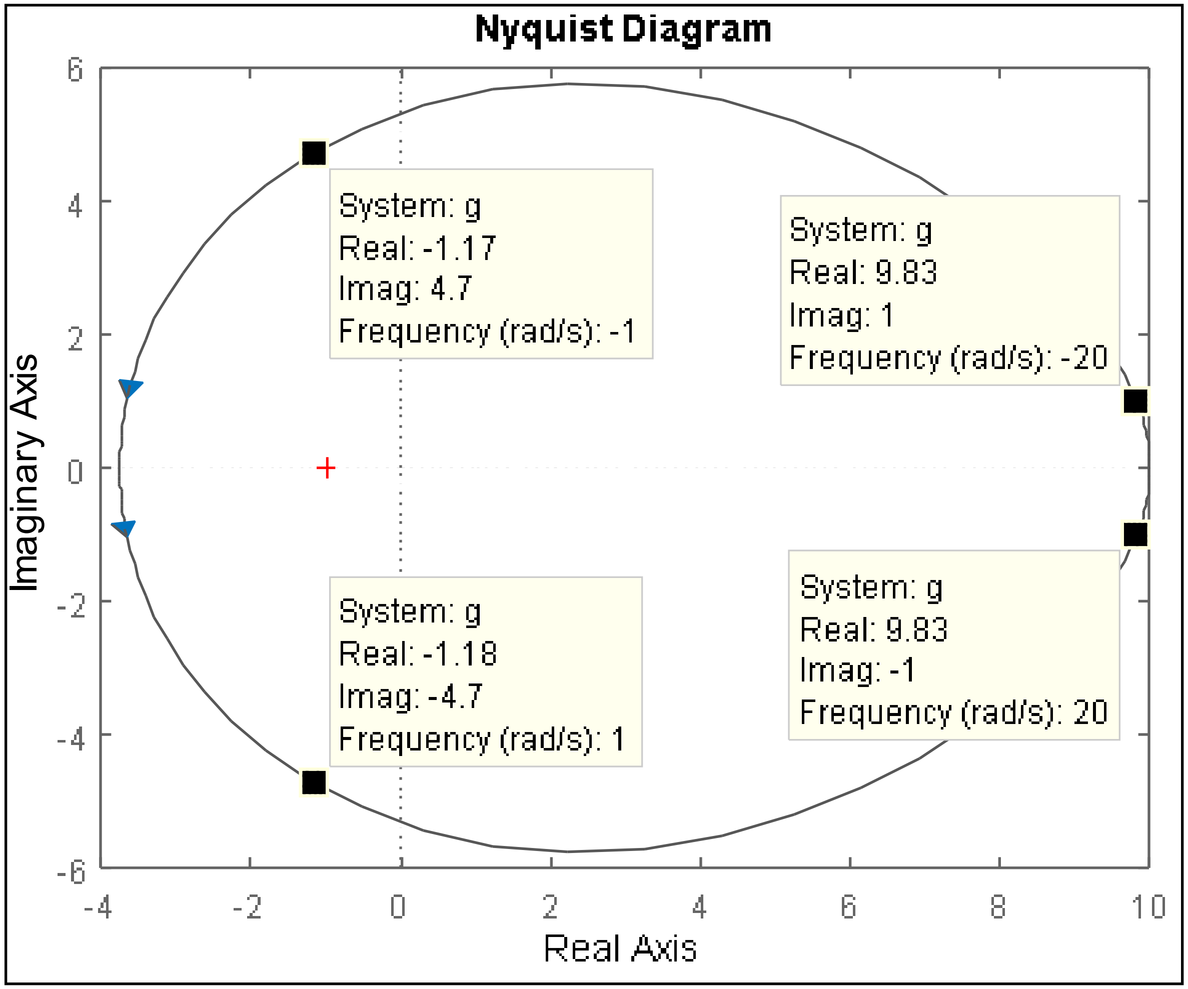
$$G(s) = \frac{10(s+1)(s+3)}{(s-2)(s+4)}$$

Consider the **unstable** plant shown alongside.

Nyquist plot of it is shown alongside.

We see that **N = -1** & P = 1, so that **Z = 0.**

We find that **closed loop poles** are -0.627 & -3.19.



Case – 2: Stable Closed Loop & P = 0

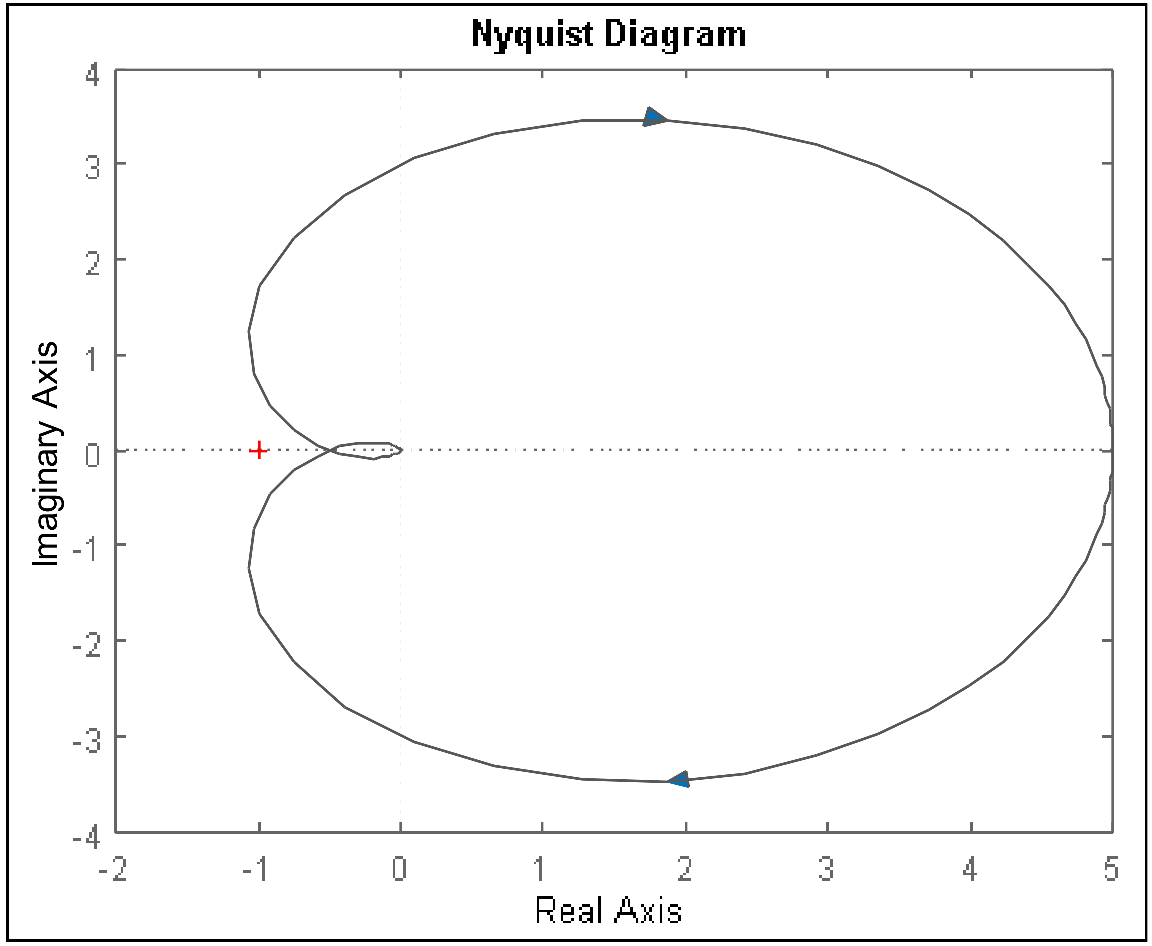
$$G(s) = \frac{3}{(s+1)(s+2)(s+3)}$$

Consider the **stable** plant shown **alongside**.

Nyquist plot of it is shown alongside.

We see that **N = 0** & P = 0, so that **Z = 0.**

We find that **closed loop poles** are -5.21 & -0.393±j2.6.



Case – 3: Unstable Closed Loop

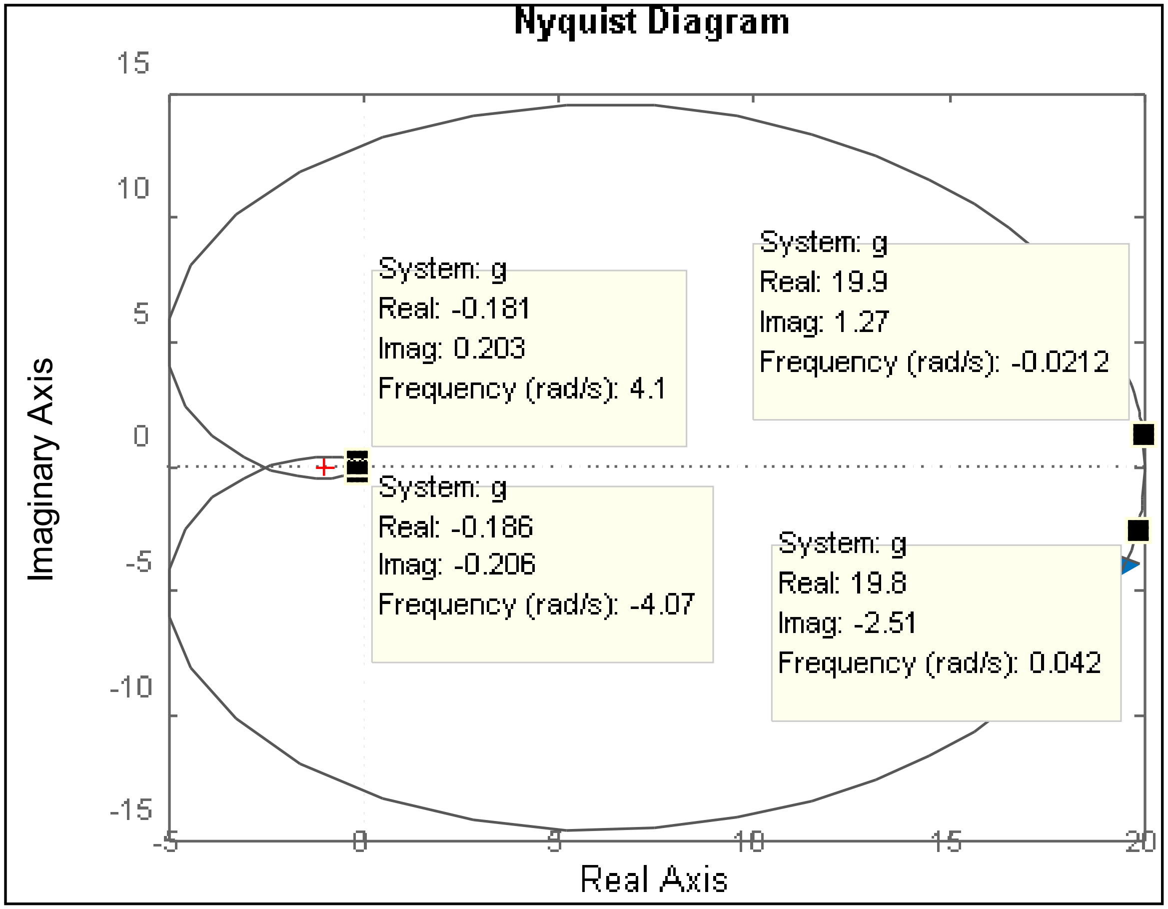
$$G(s) = \frac{20}{(s+1)^3}$$

Consider the **stable** plant shown alongside.

Nyquist plot of it is as shown.

We see that **N = 2** & P = 0, so that **Z = 2.**

We find that **closed loop poles** are -3.71 & 0.357±j2.35.



Case – 4: Poles on Imaginary Axis

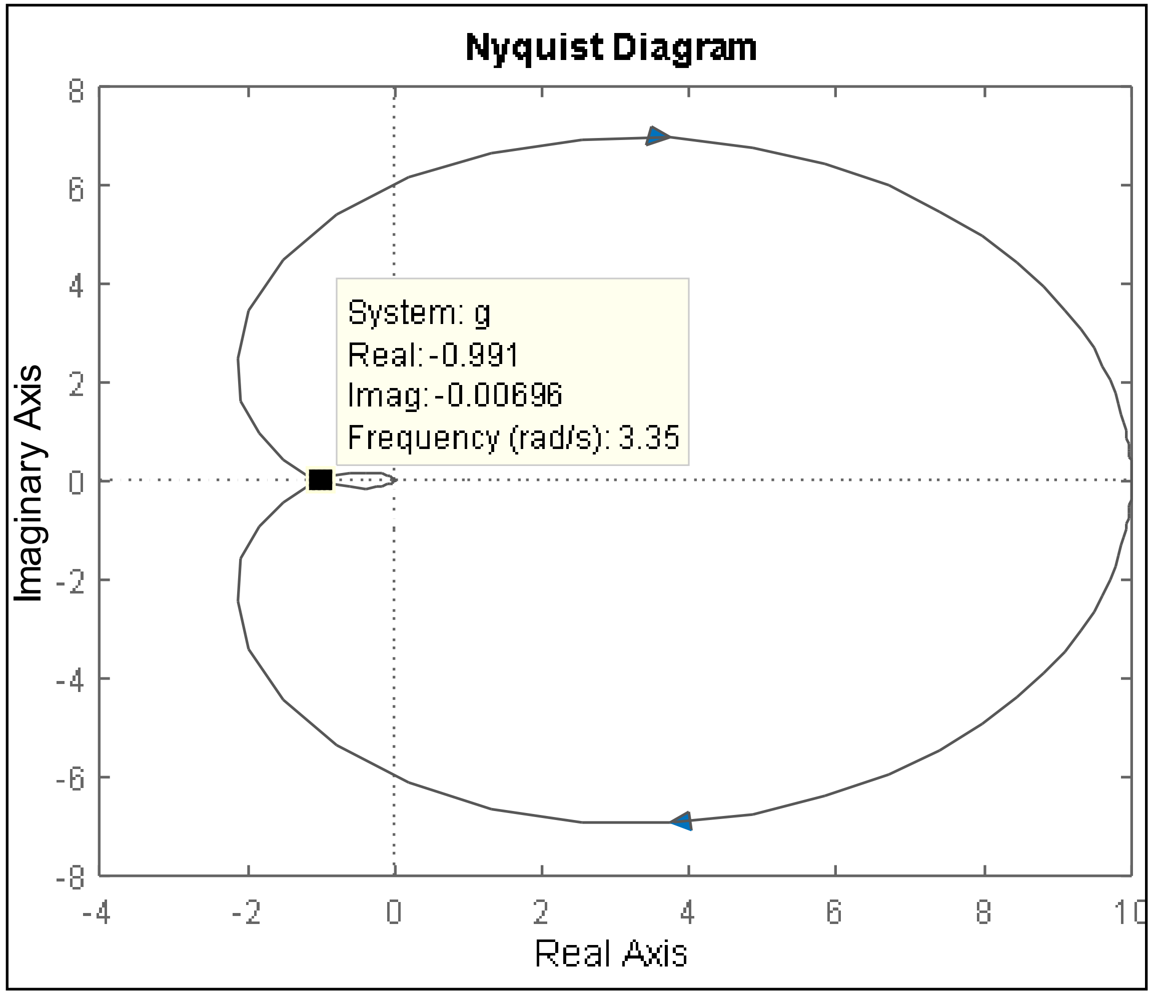
$$G(s) = \frac{60}{(s+1)(s+2)(s+3)}$$

Consider the **stable** plant shown alongside.

Nyquist plot of it is shown alongside.

We see that **N = 0** & P = 0, so that **Z = 0.**

We find that **closed loop poles** are -6.0 & 0.0±j3.32.



Summary

Nyquist stability criterion is an elegant **methodology** for analyzing the absolute **stability** of unity feedback **closed loop** systems.

The method **makes** use of conformal **mapping** theorem for extracting **information** about singular points of a **complex** function.

The method **does not** need to extract **exact** closed loop **poles** and is quite similar to **Routh's** methodology.