AE775/308 Quiz No. 2 Date: Monday 14th Oct, 2019 Time: 1405-1520
Weight: 10% Examination is closed notes. Only calculator is allowed.
Model Solutions

1. Determine the phase crossover frequency and gain margin for the following plant. (3)

$$G(s) = \frac{20}{s(s+1)(s+5)}$$

$$\angle G(j\omega) = -90 - \tan^{-1}\omega - \tan^{-1}0.2\omega = -180 \to \tan\left(\tan^{-1}\omega + \tan^{-1}0.2\omega\right) = \infty$$

$$\frac{1.2\omega}{1-0.2\omega^2} = \infty \to 1-0.2\omega^2 = 0 \to \omega_{PCO} = \sqrt{5} = 2.236$$

$$\left|G(j\omega_{PCO})\right| = \left|\frac{20}{\omega\times\sqrt{1+\omega^2}\times\sqrt{25+\omega^2}}\right|_{\omega=2.236} = 0.667 \to GM = 1.5 \quad \text{or} \quad 3.52dB$$

Further, if it is known that phase margin is 8.9°, and real closed loop pole is at -5.74, determine the location of closed loop dominant poles.

$$PM^o = 8.9 = 100\zeta \to \zeta = 0.089; \quad D(s) = s^3 + 6s^2 + 5s + 20$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = \frac{D(s)}{(s+5.74)} = s^2 + 0.26s + 3.51 \to \omega_n = 1.87$$

$$\sigma = \zeta\omega_n = 0.13; \quad s_{1,2} = -0.13 \pm j1.86 \quad \text{or} \quad -0.17 \pm j1.86$$

2. Derive the expressions for peak time and peak overshoot in respect of a standard 2nd order closed loop transfer function, whose unit step response expression is as given below. (2)

$$c(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

$$\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} \to \tan\phi = \frac{\sqrt{1-\zeta^2}}{\zeta}; \quad \sin\phi = \sqrt{1-\zeta^2}; \quad \cos\phi = \zeta$$

$$\frac{dc(t)}{dt} = -\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\left[-\zeta\omega_n \sin\left(\omega_d t + \phi\right) + \omega_d \cos\left(\omega_d t + \phi\right)\right] = 0$$

$$\tan\left(\omega_d t + \phi\right) = \frac{\omega_d}{\zeta\omega_n} = \frac{\sqrt{1-\zeta^2}}{\zeta} = \tan\phi \to \frac{\tan(\omega_d t) + \tan\phi}{1 - \tan(\omega_d t)\tan\phi} = \tan\phi \to \tan(\omega_d t)\left[1 + \tan^2\phi\right] = 0$$

$$\omega_d t = \pi \to t_p = \frac{\pi}{\omega_d}; \quad M_p = -\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n \times \pi/\omega_d}\left[\sin\left(\pi + \phi\right)\right] = \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n \times \pi/\omega_d} \sin\phi = e^{-\pi \zeta/\sqrt{1-\zeta^2}}$$

3. Derive the expression for bandwidth of a standard 2nd order closed loop transfer function, as given below, in terms of ζ and ω_n. (1)

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\left|G(j\omega)\right| = \left|\frac{\omega_n^2}{-\omega^2 + 2j\zeta\omega_n\omega + \omega_n^2}\right| = \frac{\omega_n^2}{\sqrt{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\zeta\omega_n\omega\right)^2}} = \frac{1}{\sqrt{2}}$$

$$\left(\omega_n^2 - \omega^2\right)^2 + \left(2\zeta\omega_n\omega\right)^2 = 2\omega_n^4 \rightarrow \omega^4 - 2\left(1-2\zeta^2\right)\omega_n^2\omega^2 - \omega_n^4 = 0$$

$$\omega^2 = \frac{2\left(1-2\zeta^2\right)\omega_n^2 + \sqrt{\left[2\left(1-2\zeta^2\right)\omega_n^2\right]^2 + 4\omega_n^4}}{2}$$

$$\omega^2 = \left(1-2\zeta^2\right)\omega_n^2 + \omega_n^2\sqrt{1+\left(1-2\zeta^2\right)^2}; \quad \omega = \omega_n\sqrt{\left(1-2\zeta^2\right) + \sqrt{1+\left(1-2\zeta^2\right)^2}}$$

4. Consider the following plant.

$$G(s) = \frac{K(s+1)}{s(s+2)(s+3)}$$

Determine the value of parameter, 'K' (> 0), for which dominant pole(s) of unity feedback closed loop system is (are) at -0.5 and determine the nature of the dominant pole(s). (2)

$$1 + G(s) = 1 + \frac{K(s+1)}{s(s+2)(s+3)} = \frac{s^3 + 5s^2 + (6+K)s + K}{s^3 + 5s^2 + 6s} = 0$$

$$s = z - 0.5; \quad D(z) = (z-0.5)^3 + 5(z-0.5)^2 + (6+K)(z-0.5) + K = 0$$

$$D(z) = z^3 + 3.5z^2 + (1.75+K)s + (0.5K - 1.875)$$

$$\begin{array}{|ll} 1 & 1.75+K \\ 3.5 & 0.5K-1.875 \\ \dfrac{3K+8}{3.5} & 0 \\ 0.5K-1.875 & \end{array} \rightarrow 0.5K - 1.875 = 0 \rightarrow K = 3.75$$

5. Determine the benchmark 2nd order closed loop transfer function which has a rise time of 1 sec and 2% ripple settling time of 2.5 sec. (Hint: You may use the following relations as applicable). (1)

$$t_r = \frac{1}{\omega_d}\tan^{-1}\left(\frac{\omega_d}{-\sigma}\right); \quad t_s(2\%) = \frac{4}{\sigma}$$

$$t_s(2\%) = \frac{4}{\sigma} = 2.5 \rightarrow \sigma = 1.6; \quad t_r = \frac{1}{\omega_d}\tan^{-1}\left(\frac{\omega_d}{-1.6}\right) = 1$$

$$\tan\omega_d = \frac{\omega_d}{-1.6} \rightarrow \omega_d = 1.6\tan(\pi - \omega_d) \rightarrow \omega_d = 2.2$$

$$\zeta\omega_n = 1.6; \quad \omega_n\sqrt{1-\zeta^2} = 2.2 \rightarrow \frac{1-\zeta^2}{\zeta^2} = 1.89 \rightarrow \zeta = 0.59$$

$$\omega_n = 2.72; \quad \frac{C(s)}{R(s)} = \frac{7.4}{s^2 + 3.2s + 7.4}$$

6. Derive the equation of constant 'M' trajectories for the unity negative feedback closed loop frequency response of a plant having G(jω) as its frequency response. (Hint: M is the magnitude of closed loop frequency response. You may use the following relation as applicable.). (1)

$$G(j\omega) = X(\omega) + jY(\omega)$$

$$G(j\omega)=X(\omega)+jY(\omega);\quad \frac{C(j\omega)}{R(j\omega)}=\frac{G(j\omega)}{1+G(j\omega)}=\frac{X+jY}{(1+X)+jY}$$

$$M=\left|\frac{C(j\omega)}{R(j\omega)}\right|=\left|\frac{X+jY}{(1+X)+jY}\right|=\frac{|X+jY|}{|(1+X)+jY|}\rightarrow M^2=\frac{X^2+Y^2}{(1+X)^2+Y^2}$$

$$M\neq 1\rightarrow X^2\left(1-M^2\right)-2M^2X-M^2+\left(1-M^2\right)Y^2=0$$

$$X^2+\frac{2M^2}{M^2-1}X+\frac{M^2}{M^2-1}+Y^2=0\rightarrow\left(X+\frac{M^2}{M^2-1}\right)^2+Y^2=\frac{M^2}{\left(M^2-1\right)^2}$$

$$\text{Circle with } R=\frac{M}{\left(M^2-1\right)};\quad \text{Centre: }\left(-\frac{M^2}{M^2-1},0\right);\quad M=1\rightarrow X=-\frac{1}{2}$$

7. Consider the plant given below.

$$G(s)=\frac{2(s-1)}{(s+1)}$$

Determine the cascade controller, G_c to achieve a closed loop damping of $1/\sqrt{2}$ and undamped natural frequency of $\sqrt{2}$ for the benchmark 2nd order response to a unit step input and determine its order and stability. (1)

$$\frac{C(s)}{R(s)}=\frac{2}{s^2+2s+2};\quad G_{eq}=\frac{2}{s(s+2)};\quad G_c=\frac{G_{eq}}{G}=\frac{(s+1)}{s(s-1)(s+2)}$$

3rd order, type '1' unstable controller.

PAPER ENDS