Linearization of Models

- *Response from Models*
- *Linearization Concept & Methodology*
- *Linearization of Dynamic Models*

Behaviour from Mathematical Models

Mathematical models, created from governing laws, **contain** all the features of the **system behaviour** and need a solution **methodology** to characterize the **response.**

In this context, it is seen that **equations** are normally **nonlinear** as well as **time varying.**

Further, for **realistic systems**, we could have a large **number of outputs** of interest, also **coupled** to each other, which make the task fairly **complex.**

Basic Solution Process

As most dynamical **models** are described through **ODEs**, a solution of these **equations** is necessary for obtaining the **information** about the output of **interest.**

In this **context**, it is to be noted that **solution** is obtained subject to **inputs** that the system is **expected** to receive.

In view of the fact that **most realistic** models would be **nonlinear,** the process of solution can be **tedious.**

Direct Integration Method

Let us consider the model of **conical tank**, as seen earlier.

$$\boxed{\frac{dH}{dt}+\frac{0.045\sqrt{H}}{\pi H^{2}}=0}$$

The **exact** analytical **solution** for initial $\mathbf{H_0}$ is as follows.

$$\frac{dH}{dt}=-\frac{0.045\sqrt{H}}{\pi H^{2}}\rightarrow \pi\int H^{3/2}dH=-0.045\int dt$$

$$\pi\times\frac{2}{5}H^{5/2}=-0.045t+C\rightarrow H^{5/2}=H_0^{5/2}-\frac{0.225}{2\pi}\left(t-t_0\right)$$

We see that '**H**' reduces as time '**t**' progresses.

Direct Integration Features

It is **possible** to directly **integrate** the **ODE** in cases where the **equation** is of the first order and **simple.**

However, most dynamical **models** involve many **physical** processes and hence are **likely** to be of much **higher order** as well as may **involve many** different inputs.

In all such cases, it would be **practically impossible** to employ the **direct integration** methodology, and hence, we need to **process** the models to **simplify** the procedure.

Model Processing Tools

The solution **methodology** can be simplified, **without** significantly affecting the **fidelity**, by employing a few **tools**, which also **enable** a **structured approach** to generation of system **response.**

Among the many tools, **linearization** and **block diagram** representation/ manipulation are the most **commonly employed** tools in **control analyses.**

Nonlinear Vs. Linear Systems

Idealization of the physical **effect**, though captures only the dominant discipline, still **generally results** in the mathematical **description** which is **nonlinear** in nature.

While, it is **possible to understand** the dynamics and design **control**, even if the description is **nonlinear**, the process is **tedious** and results **cannot be extrapolated**.

Thus, in **most cases**, a certain level of **accuracy** is sacrificed by creating **approximate** descriptions, in which **input-output** relation is captured through **linear** forms.

Linear System Features

Linear systems greatly simplify the **solution procedures** as well as **allow the extrapolation** of results through application of the principle of **superposition.**

Linearization is a method to arrive at **linear input-output** relations through a structured process that **ignores** the higher order **terms.**

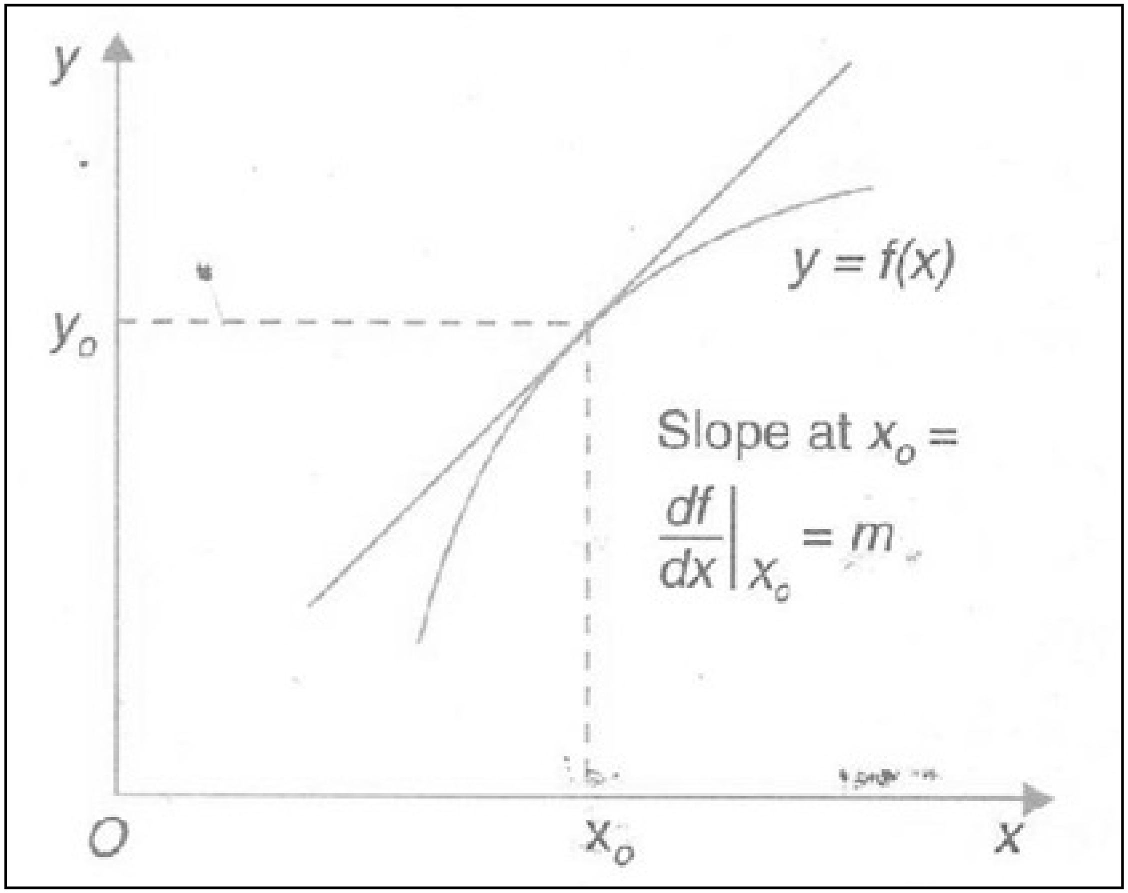
However, in **such cases**, the applicability of results gets **limited** to a **small domain** over which the **linearization** is carried out.

In all such cases, it is **important to assess** accuracy loss.

Linearization

Basic Linearization Process

Consider a general **input-output** relation as shown below.



Operating point, x_0, defines the **condition** that the linearized system is **expected** to achieve in the **steady-state** (or at equilibrium).

Basic Linearization Process

A **linear relation** is obtained by assuming that **variables deviate** only by a **small amount** from the desired **operating** condition.

Under these conditions, it is **possible to express** nonlinear relation, through a **Taylor's series expansion**, as follows.

$$y = f(x); \quad x_0 \rightarrow \text{ Operating Point}; \quad y_0 = f(x_0)$$

$$= y_0 + \frac{df}{dx}\Big|_{x=x_0} (x - x_0) + \frac{1}{2!}\frac{d^2 f}{dx^2}\Big|_{x=x_0} (x - x_0)^2 + \ldots.$$

For **small** $(\mathbf{x} - \mathbf{x_0})$**,** we can ignore **quadratic** & higher **terms**, so that '**y**' becomes '**linear**' with respect to '**x**'.

Concept of Small Disturbance

In such a case, we can **rewrite** the functional **relation** as follows.

$$y = y_0 + K\left(x - x_0\right); \quad K = \frac{df}{dx}\Big|_{x=x_0}$$

$$y - y_0 = K\left(x - x_0\right); \quad \delta y = K\,\delta x$$

$$\delta x = x - x_0; \quad \delta y = y - y_0$$

Thus, we find that **linearized equation** is in terms of new **variables,** 'δy' and 'δx' that define the **small departure** of output and input from the **operating point,** (x_0, y_0).

Single Variable Example

Linearize the following equation, around given operating **point**, and assess its **accuracy** for x = 1.8.

$$y = 0.2x^3; \quad x_0 = 2$$

$$y - y_0 = a\left(x - x_0\right); \quad a = \frac{dy}{dx}\Big|_{x=2} = 2.4;$$

$$y_0 = 0.2x_0^3 = 1.6; \quad y - 1.6 = 2.4(x-2)$$

$$y = 2.4x - 3.2; \quad y(1.8) = 1.12, \quad \text{(Exact: 1.17)}$$

Multivariable Function Linearization

In case the function has **many** independent **variables**, the same **procedure** is applied, except that **partial derivatives** are employed in place of **total derivatives**, as shown below.

$$
\begin{aligned}
y = f(x_1, x_2) \quad &= f(x_{10}, x_{20}) \\
&+ \left[\frac{\partial f}{\partial x_1} \bigg|_{x_1 = x_{10}, x_2 = x_{20}} (x_1 - x_{10}) + \frac{\partial f}{\partial x_2} \bigg|_{x_1 = x_{10}, x_2 = x_{20}} (x_2 - x_{20}) \right] \\
&+ \frac{1}{2!} \left[\frac{\partial^2 f}{\partial x_1^2} \bigg|_{x_1 = x_{10}, x_2 = x_{20}} (x_1 - x_{10})^2 + \frac{\partial^2 f}{\partial x_2^2} \bigg|_{x_1 = x_{10}, x_2 = x_{20}} (x_2 - x_{20})^2 \right] \\
&+ \cdots\cdots
\end{aligned}
$$

$$
\delta y = K_1 \delta x_1 + K_2 \delta x_2; \quad K_1 = \frac{\partial f}{\partial x_1} \bigg|_{x_1 = x_{10}, x_2 = x_{20}}; \quad K_2 = \frac{\partial f}{\partial x_2} \bigg|_{x_1 = x_{10}, x_2 = x_{20}}
$$

Multivariable Linearization Example

Linearize the following equation, around given operating **point**, and assess its **accuracy** for x = 5, y= 10.

$$z = xy; \quad x_0 = 6, y_0 = 11$$

$$z - z_0 = K_1\left(x - x_0\right) + K_2\left(y - y_0\right)$$

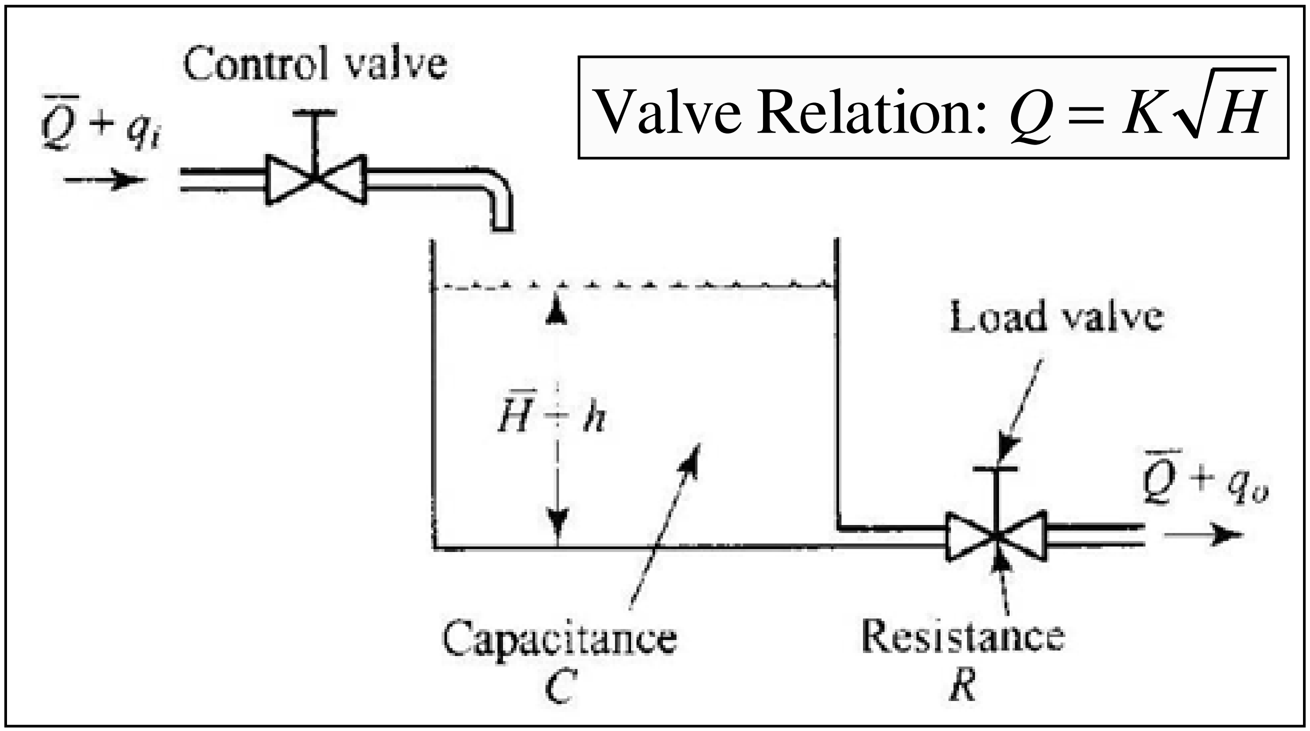
$$K_1 = \frac{\partial z}{\partial x}\Big|_{x_0=6,\, y_0=11} = 11; \quad K_2 = \frac{\partial z}{\partial y}\Big|_{x_0=6,\, y_0=11} = 6$$

$$z_0 = x_0 y_0 = 66; \quad z - 66 = 11\left(x - 6\right) + 6\left(y - 11\right)$$

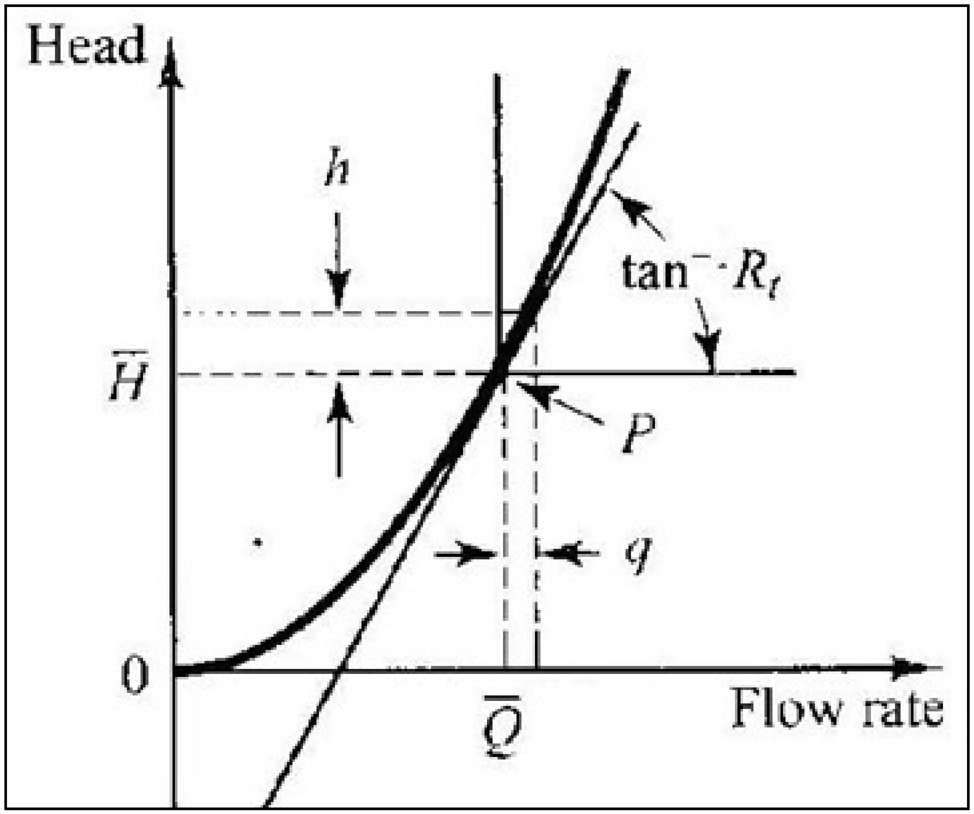
$$z = 11x + 6y - 66; \quad z(5,10) = 49, \quad \text{(Exact: 50)}$$

Linearization of Valve

Obtain the **linear** input-output **relation** of the load **valve** having **non-linear** H-Q relation.



Linearization of Valve



$$Q = \bar{Q} + \frac{dQ}{dH}\Big|_{(\bar{Q},\bar{H})} (H - \bar{H})$$

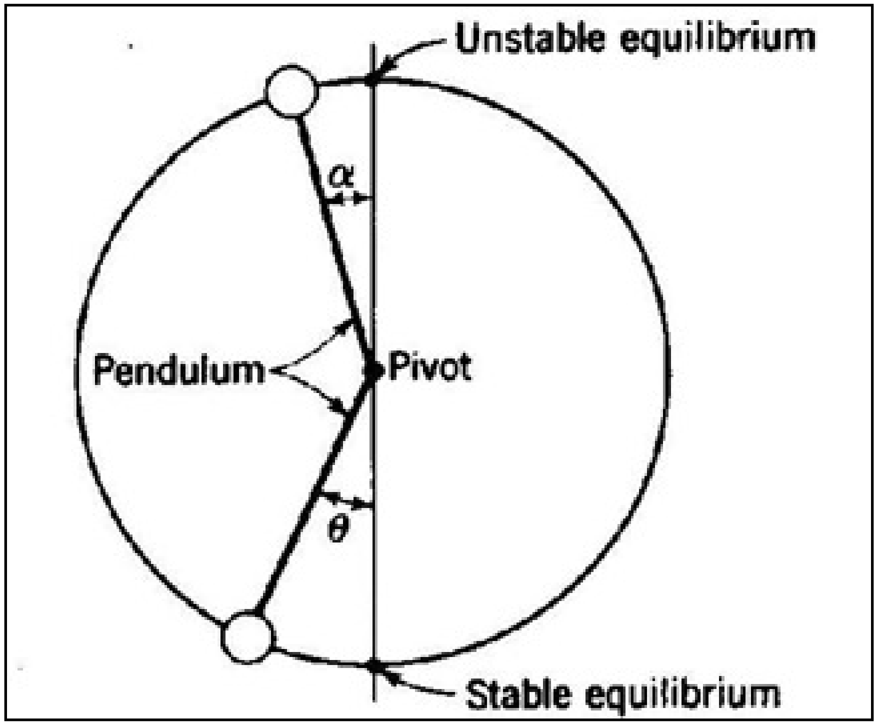
$$H - \bar{H} = h; \quad Q - \bar{Q} = q$$

$$q = \frac{K}{2\sqrt{\bar{H}}} h = \frac{\bar{Q}}{2\bar{H}} h$$

$$\text{OR } h = \frac{2\bar{H}}{\bar{Q}} q = Rq$$

Linearization of Pendulum Motion

Consider the **simple pendulum** as shown below.



Derive the general **non-linear** equation of motion of the pendulum and give its **linear equivalent.**

General **nonlinear** dynamic equation for pendulum is,

$$J\ddot{\theta} + B\dot{\theta} + mgl\sin\theta = 0; \quad m \rightarrow \text{ Mass}; \quad l \rightarrow \text{ Length}$$

$$J \rightarrow \text{ MoI} = ml^2; \quad B \rightarrow \text{ Damping Constant}$$

Linearization of Pendulum Motion

Dynamic equation can be **linearized** as follows.

$$y = J\ddot{\theta} + B\dot{\theta} + mgl\sin\theta = y_0 + \frac{dy}{d\theta}\Big|_{\theta=\theta_0} (\theta - \theta_0)$$

$$y - y_0 = \delta y = \frac{d}{d\theta}\left[J\ddot{\theta} + B\dot{\theta} + mgl\sin\theta\right]\Big|_{\theta=\theta_0} \delta\theta$$

$$\delta y = \left[J\frac{d^2}{dt^2} + B\frac{d}{dt} + mgl\cos\theta\right]\Big|_{\theta=\theta_0} \delta\theta$$

$$= J\delta\ddot{\theta} + B\delta\dot{\theta} + mgl\cos\theta_0\delta\theta = 0$$

An **alternate method** of linearizing dynamical system is to **substitute 'θ_0 + $\delta\theta$'** in place of 'θ' and carry out **the expansion** of various terms.

Alternate Linearization Method

Alternate linearization **technique** is as follows.

$$J\left(\ddot{\theta}_0 + \delta\ddot{\theta}\right) + B\left(\dot{\theta}_0 + \delta\dot{\theta}\right) + mgl\sin\left(\theta_0 + \delta\theta\right) = 0$$

$$J\ddot{\theta}_0 + J\delta\ddot{\theta} + B\dot{\theta}_0 + B\delta\dot{\theta} + mgl\left(\sin\theta_0 + \cos\theta_0\delta\theta\right) = 0$$

$$\left(J\ddot{\theta}_0 + B\dot{\theta}_0 + mgl\sin\theta_0\right) + \left(J\delta\ddot{\theta} + B\delta\dot{\theta} + mgl\cos\theta_0\delta\theta\right) = 0$$

$$\left(J\ddot{\theta}_0 + B\dot{\theta}_0 + mgl\sin\theta_0\right) = 0 \rightarrow \text{ Steady-state Dynamics}$$

$$\left(J\delta\ddot{\theta} + B\delta\dot{\theta} + mgl\cos\theta_0\delta\theta\right) = 0 \rightarrow \text{ Linearized Dynamics}$$

We see that **steady-state equation** is non-linear, while **linearized** equation describes the **disturbed motion.**

Summary

Linearization results in the systems that obey the **principle of superposition**, leading to great simplification of the **solution procedures.**

Linearization process assumes small departures from an operating point, so that **higher order terms are** ignored.