Closed Loop Response – 2

– *Concept of Resonant Peak*

– *Closed Loop Bandwidth*

ω- Domain Physical Parameters

We know that **time domain** is the real **physical** domain in which the **dynamics** of any system **evolves.**

However, **s-domain**, though **synthetic** in nature, does involve 'ζ' **and** 'ω_d' which are physically **measurable.**

ω - domain, a special case of **s-domain**, also contains **closed loop** response **features** e.g. GCO, PCO, GM, PM.

ω- Domain Physical Parameters

Here, it is **worth** noting that while **GCO & PM** are related to 'ζ' and 'ω_n', and, hence, to **time response** features, by themselves, these **do not** have any physical **interpretation.**

In **view** of the above, we need **features** of closed loop in **ω - domain**, which can have **physical** connotations.

In this regard, we consider **resonant** peak, **bandwidth** as two **possible** features that not only have **physical** meaning but also can be **extracted** from closed loop **response.**

ω- Domain Physical Parameters

Resonant peak is the maximum **amplitude** that a system shows in its **frequency** response and is **indicative** of behaviour in **relation** to time domain **inputs.**

Similarly, **bandwidth** is frequency up to which the **system** will show a **significant** response (i.e. > 70%).

Therefore, we can **extract** these features from **closed loop** response and **assess** the possible **time** domain **behaviour.**

Resonant Frequency and Peak

Resonant Frequency & Amplitude

Closed loop systems, comprising **2nd order** factors, exhibit the **resonance** when subjected to **sinusoidal** inputs.

This has **given** rise to the concepts of **resonant** frequency & amplitude, which are important **figures of merit** of the closed loop frequency **response.**

The resonant **frequency** is the frequency at which the **closed** loop system gives **maximum** output.

We know that **peak** magnitude directly impacts the peak **overshoot** if resonant **frequency** is present in the **input**.

Amplitude Formulation

We can **obtain** closed loop **frequency response** as follows.

$$\frac{C(j\omega)}{R(j\omega)} = \frac{\omega_n^2}{-\omega^2 + j2\zeta\omega_n\omega + \omega_n^2} = Me^{j\alpha}$$

$$M = \frac{1}{\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}; \quad \alpha = -\tan^{-1}\left(\frac{2\zeta\frac{\omega}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}\right)$$

Resonance Solution

Maximum value of '**M**' is obtained as follows.

$$\frac{d}{d\omega}\left(\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(2\zeta\frac{\omega}{\omega_n}\right)^2\right)=0 \rightarrow 1-\frac{\omega_r^2}{\omega_n^2}=2\zeta^2$$

$$\omega_r=\omega_n\sqrt{1-2\zeta^2};\quad M_r=\frac{1}{2\zeta\sqrt{1-\zeta^2}};\quad \alpha_r=-\tan^{-1}\left(\frac{\sqrt{1-2\zeta^2}}{\zeta}\right)$$

Resonance Solution Attributes

We note that **resonant peak** is directly related to 'ζ' while resonant **frequency** is dependent on both 'ζ' **& 'ω_n'.**

Hence, we can obtain the **dominant** pole location, so also the time **response** features, from **resonance** condition.

We further see that for ζ **= 1/√2**, $\omega_r = 0$, indicating that there is **no resonance** beyond this damping **value.**

Lastly, we see that at this **damping**, phase angle is **zero.**

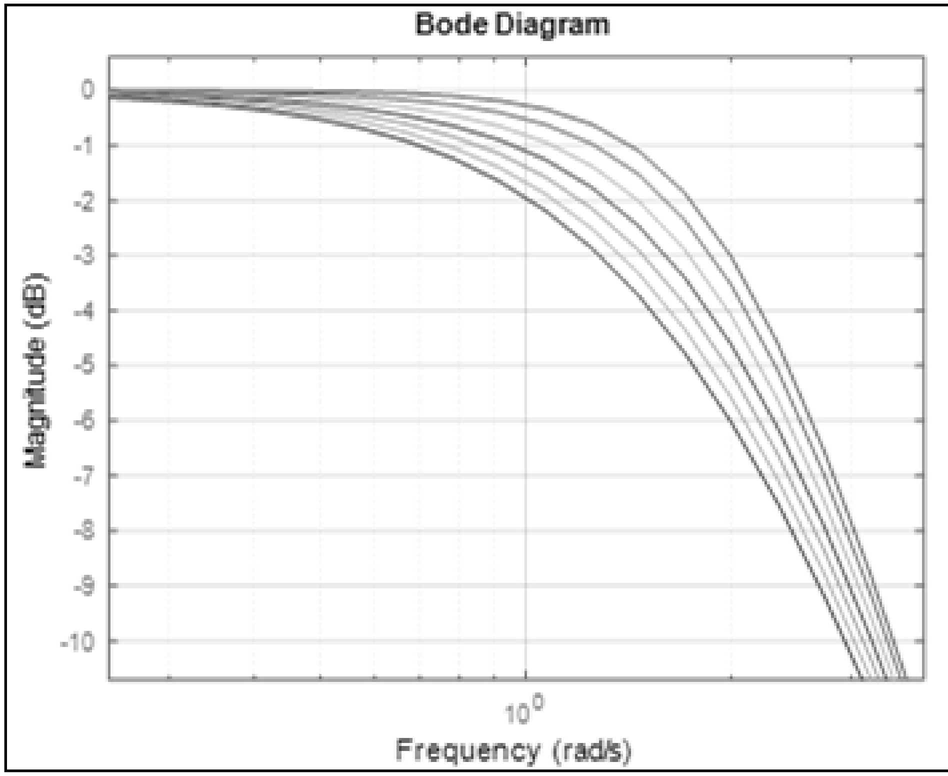
Resonance Solution Attributes

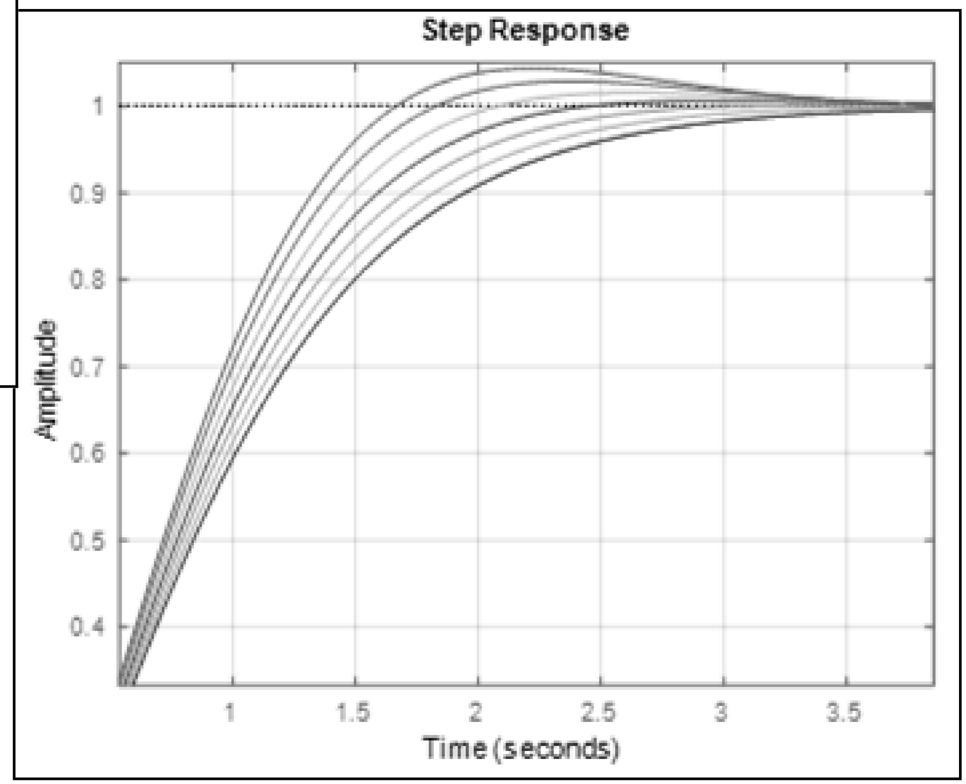
We find that we can get the **same** solution by putting **M_r = 1**, indicating **no** resonance.

We also find that as 'ζ' increases beyond **0.707**, the system shows a **1st** order type frequency **response**, even **though** the poles in **s-domain** are **complex conjugate.**

Above behaviour of '**M_r**' is found to be **correlated** with the variation of '**M_p**' with 'ζ' and is **commonly** used as **indicator** of the time **response** directly.

$M_r - M_p$ Correlation





Summary

Closed loop response is typically **specified** through resonant **peak**, which correlates **well** with the **peak overshoot** of time response.

Cut-off Frequency & Bandwidth

Bandwidth As Response Attribute

In addition to **GCO**, ω_n and ω_d, **bandwidth** of closed loop system is also an **important** response feature that is strongly **correlated** to time response.

It is worth noting that **bandwidth** indicates that closed loop **time response** will be **substantial** at all those **frequencies** which are **within** the bandwidth and present in the **input.**

Cut-off Frequency & Bandwidth

Most physical **systems** are described using a **proper** transfer function so that $|G(j\omega)| \to 0$ as $\boldsymbol{\omega \to \infty}$**.**

Therefore, it is clear that **high frequency** input will generate **negligible output.**

This behaviour is **commonly** quantified through the concepts of **cut-off** frequency and **bandwidth.**

Cut-off Frequency & Bandwidth

Bandwidth is actually an **indicator** of the frequencies of input **signals** that the system is **capable** of tracking.

In this context, it is **worth noting** that all inputs in **time domain** can be represented through an **infinite series** of harmonic components, also called the '**Fourier Series**'.

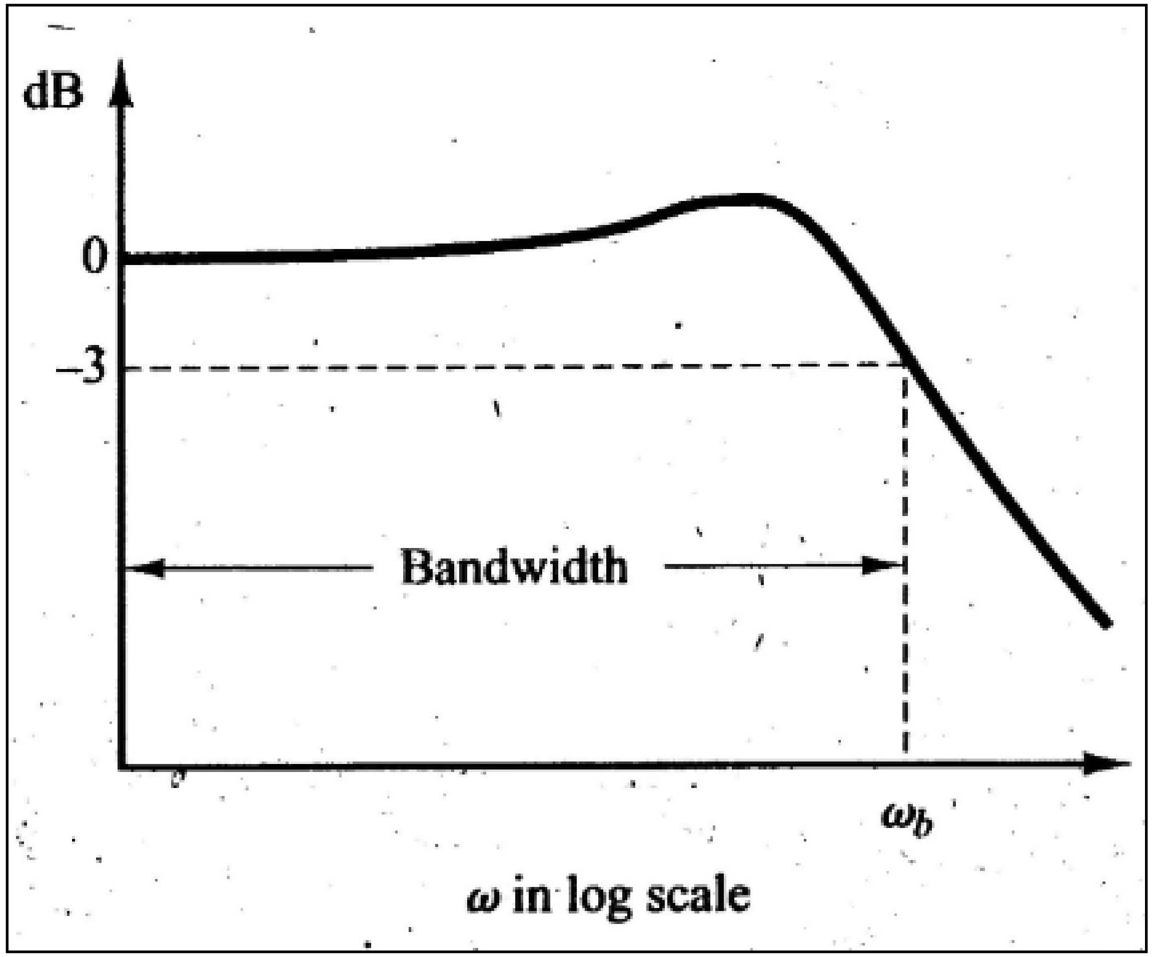
Thus, it is clear that a **specific input** will not be tracked **correctly,** unless the system **bandwidth** contains all the frequencies that **comprise** that input.

Cut-off Frequency & Bandwidth

Cut-off frequency is defined as **frequency** above which, bode **magnitude** is **below** ‘0’ dB by 3 dB or more**.**

Cut-off frequency indicates that beyond this value, $\mathbf{P_o \leq 0.5P_i}$ (Half-power point).

Bandwidth is range of **frequencies** up to the **cut-off** frequency.



Bandwidth as Response Feature

We can use closed loop **bandwidth** as a response feature, which is **related** to 'ω_n', 'ζ', and is derived **as follows.**

$$20\log_{10}\left|\frac{C(j\omega_b)}{R(j\omega_b)}\right| = -3dB \rightarrow \bar{\omega}_b = \frac{\omega_b}{\omega_n}; \quad \frac{1}{\sqrt{\left(1-\bar{\omega}_b^2\right)^2 + \left(2\zeta\bar{\omega}_b\right)^2}} = \frac{1}{\sqrt{2}}$$

$$\left(1-\bar{\omega}_b^2\right)^2 + \left(2\zeta\bar{\omega}_b\right)^2 = 2 \rightarrow \omega_b = \omega_n\sqrt{\sqrt{\left(1-2\zeta^2\right)^2+1}+\left(1-2\zeta^2\right)}$$

We **note** that for $\zeta = 1/\sqrt{2}$, $\boldsymbol{\omega_b = \omega_n}$.

ω- Domain Specification Example

Determine the **frequency domain** equivalent of the following **time domain** specifications.

1. Peak overshoot = **12%**, Settling time = **4 sec** (2%)

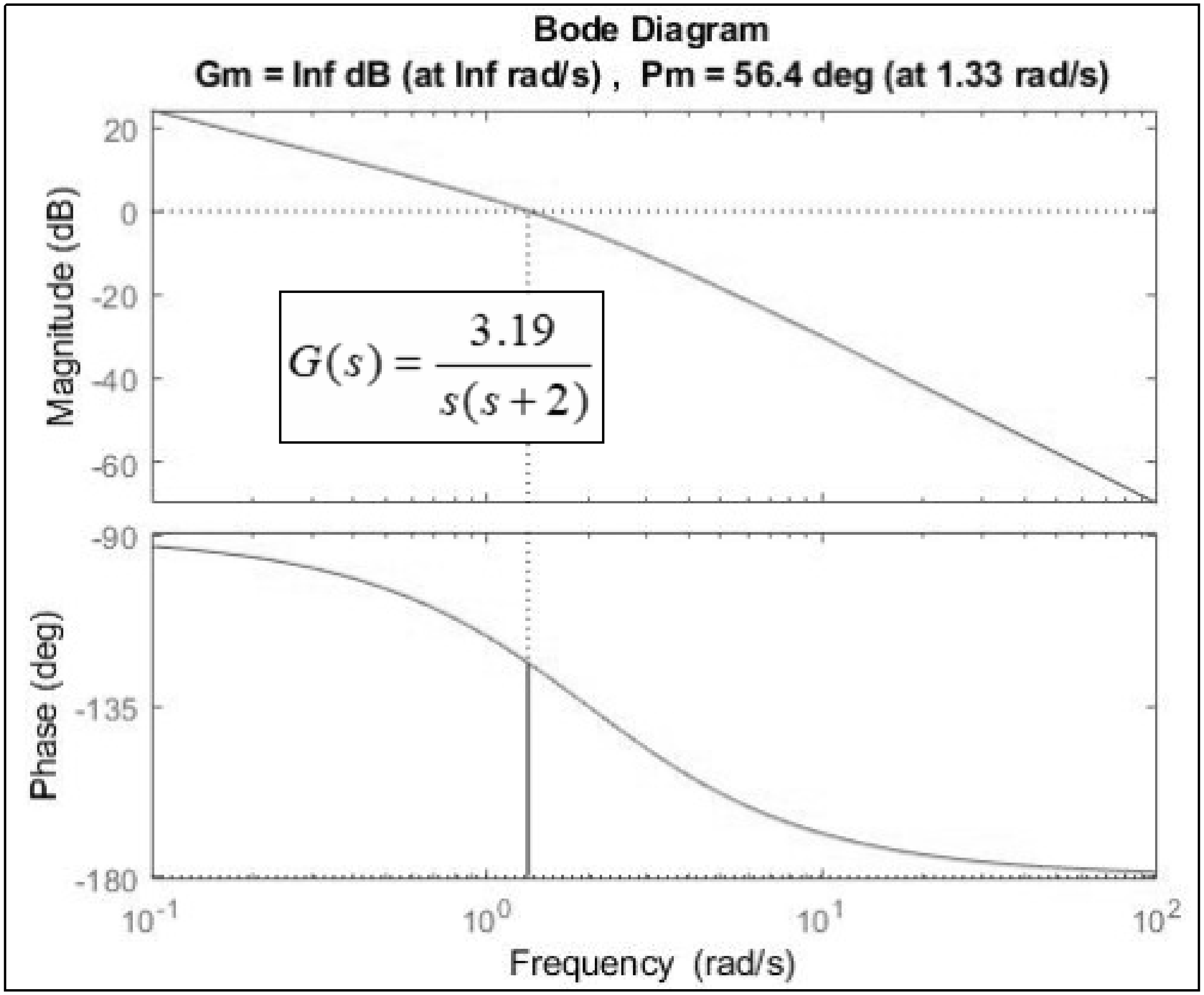
s-Domain Parameters: $\zeta = 0.56$, $\omega_n = 1.786$, $\omega_d = 1.48$

GCO & PM: $\omega_{GCO} = 1.328$, PM = 56.4^o

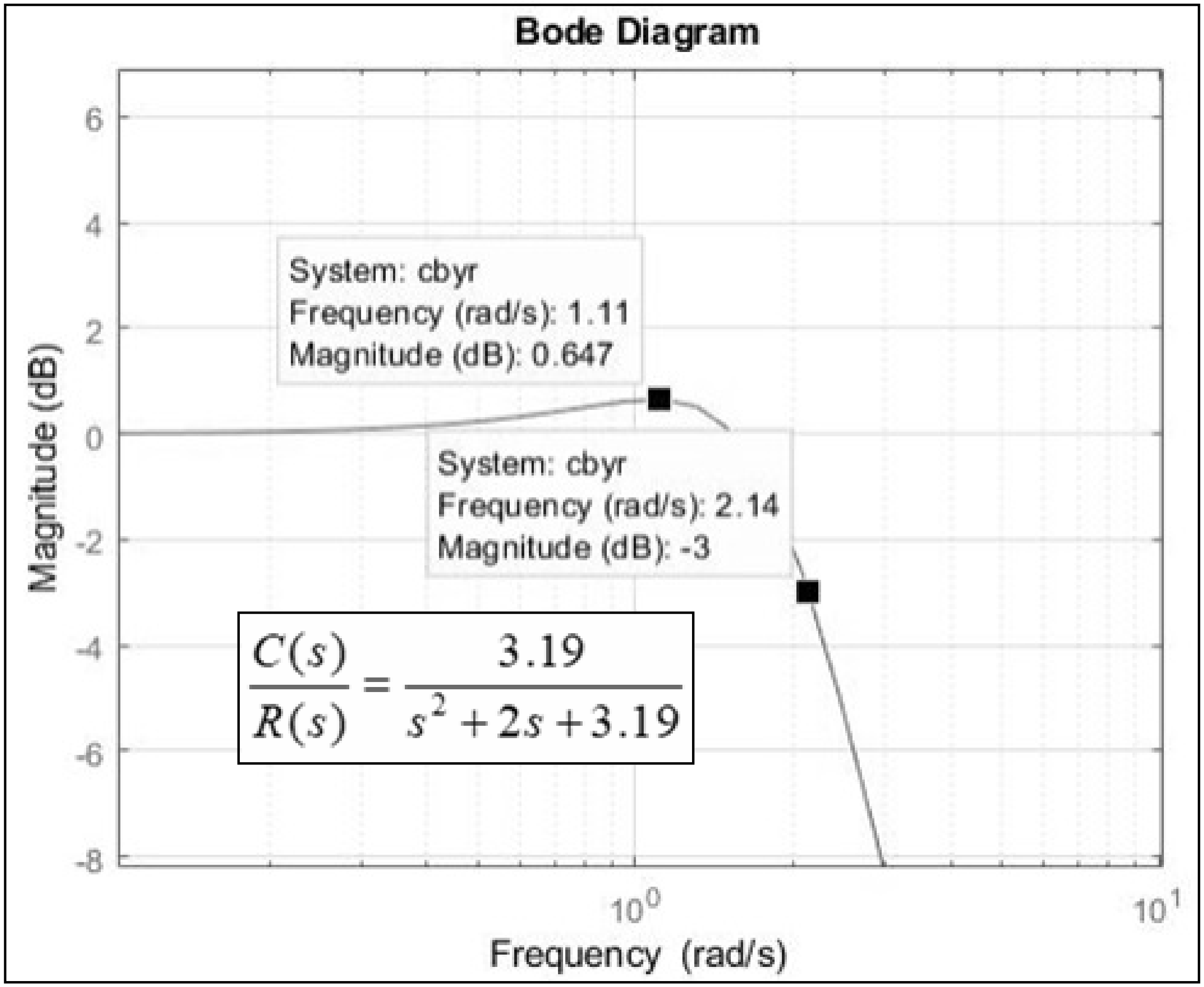
Resonant Frequency & Peak: $\omega_r = 1.09$, $M_r = 1.078$

Bandwidth: $\omega_b = 2.14$

Verification of GCO, PM



Verification of M_r , ω_r and ω_b



Summary

Bandwidth is an important response **attribute** which is closely correlated to **speed** of response in time domain and **undamped** natural frequency in **Laplace** domain.