Lecture 13:
Discrete Random Variables - Part VII

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Conditional distribution

- Recall: Conditional probability is defined for events.
- Now we see its generalization: conditional distributions for r.v.s.
- For discrete r.v.s X and Y, the conditional PMF of Y given $X = x$ is

$$p_{Y|X=x}(y) = P(Y = y|X = x) = \frac{P(X = x, Y = y)}{P(X = x)}, \quad P(X = x) > 0.$$

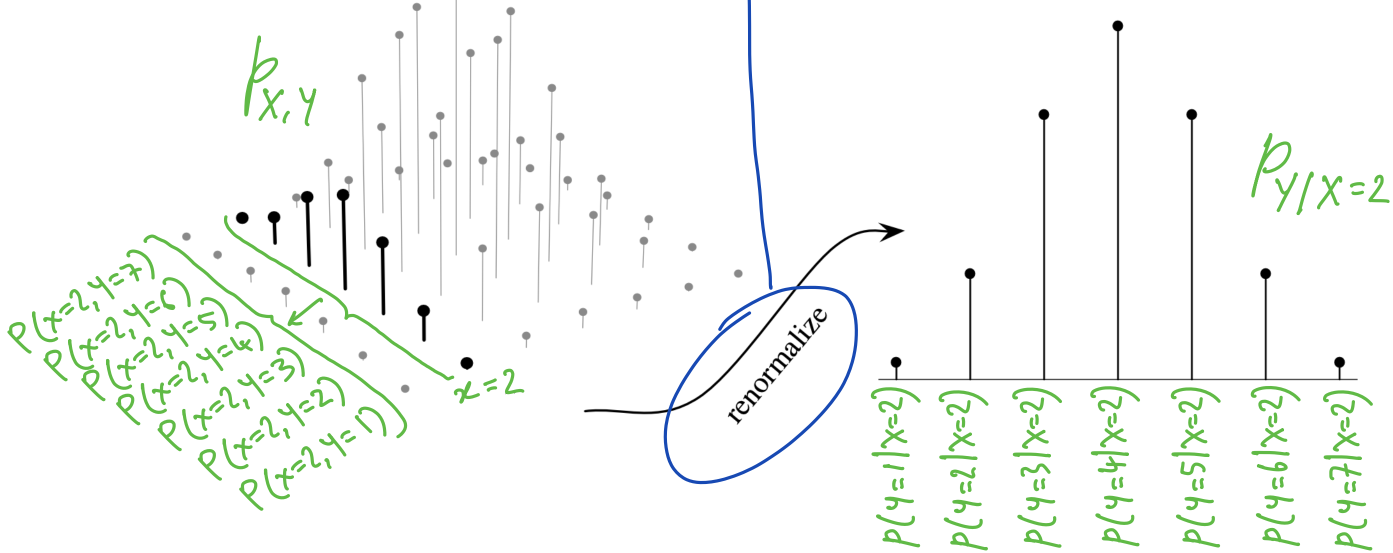
- The conditional PMF is a function of y for fixed x.
- Similar to conditional PDF, the conditional CDF of Y given $X = x$ is defined as

$$F_{Y|X=x}(y) = P(Y \leq y|X = x) = \frac{P(X = x, Y \leq y)}{P(X = x)}, \quad P(X = x) > 0.$$

Conditional distribution

i.e., dividing $p(X=2, Y=y)$ by $p(X=2)$: $p(Y=y \mid X=2) = \dfrac{p(X=2, Y=y)}{p(X=2)}$

where, $y \in \{1,2,3,4,5,6,7\}$.



- By renormalization, note that $\sum_y p(Y=y \mid X=x) = \dfrac{\sum_y p(X=x, Y=y)}{p(X=x)}$
  $= 1.$ (i.e., adds up to 1)
- Hence, conditional PMF too is a PMF.

Conditional distribution

Example:

- Two pens are selected at random from a box that contains 3 blue pens, 2 red pens, and 3 green pens. If X is the number of blue pens selected and Y is the number of red pens selected, find
  (a) the joint PMF $p_{X,Y}$ and the marginal PMFs p_X, p_Y,
  (b) $P((X, Y) \in A)$, where A is the region $\{(x, y) : x + y \leq 1\}$,
  (c) the conditional PMF of X, given that $Y = 1$,
  (c) the conditional CDF $F_{X|Y=1}(1)$.

(a) Note that support of $X = \{0,1,2\}$ = support of Y.

- Hence, (x,y) can take values : $(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2)$.

- But, note that, only two pens are selected in total.

$$\Rightarrow p_{X,Y}(1,2) = p_{X,Y}(2,1) = p_{X,Y}(2,2) = 0.$$

- The cardinality of the sample space is $\binom{3+2+3}{2} = 28$.

Conditional distribution

$$p_{X,Y}(x,y) = \frac{\overbrace{\binom{3}{x}}^{\substack{\text{ways to choose}\\ x \text{ blue pens}}} \overbrace{\binom{2}{y}}^{\substack{\text{ways to choose}\\ y \text{ red pens}}} \overbrace{\binom{3}{2-x-y}}^{\substack{\text{ways to choose}\\ 2-x-y \text{ green pens}}}}{\binom{8}{2}}$$

for $x, y \in \{0,1,2\}$, $x+y \leq 2$.

and 0 otherwise.

$y \backslash x$	0	1	2	p_Y
0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
1	$\frac{6}{28}$	$\frac{6}{28}$	0	$\frac{12}{28}$
2	$\frac{1}{28}$	0	0	$\frac{1}{28}$
p_X	$\frac{10}{28}$	$\frac{15}{28}$	$\frac{3}{28}$	

(b) $P((X,Y) \in A) = P(X+Y \leq 1) = p_{X,Y}(0,0) + p_{X,Y}(0,1) + p_{X,Y}(1,0)$

$$= \frac{3}{28} + \frac{6}{28} + \frac{9}{28} = \frac{18}{28}.$$

Conditional distribution

(c) Note that $p_Y(1) = 12/28 = P(Y=1)$

Hence, $P(X=0 \mid Y=1) = \dfrac{P(X=0, Y=1)}{P(Y=1)} = \dfrac{6/28}{12/28} = \dfrac{1}{2}.$

Similarly, verify that (Homework): $P(X=1 \mid Y=1) = 1/2,$
$P(X=2 \mid Y=1) = 0.$

(d) $F_{X|Y=1}(1) = P(X \leq 1 \mid Y=1) = \dfrac{P(X \leq 1, Y=1)}{P(Y=1)}$

$$= \frac{P(X=0, Y=1) + P(X=1, Y=1)}{P(Y=1)}$$

$$= \frac{6/28 + 6/28}{12/28} = 1.$$

Conditional distribution

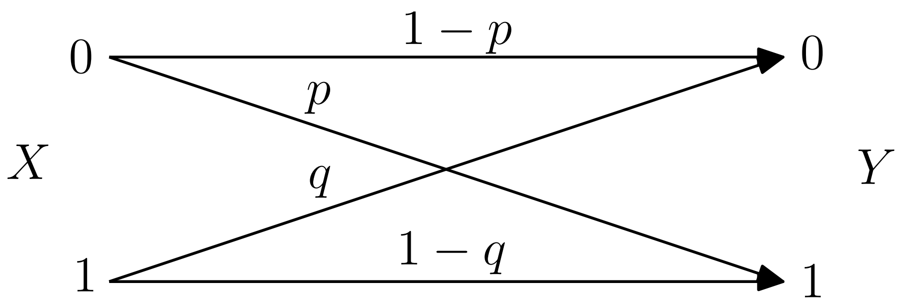
- Wireless communication: In practice, we use channels (e.g., mobile communication via a wireless channel) to communicate information. But in the physical world, the channels are usually not "perfect". That is, due to the noise in the channels, a transmitted message may be received as some other message.
- Example: The input messages to a channel are chosen from the set $\{0, 1\}$ with probability $P(X = 0) = .5$ and $P(X = 1) = .5$. Output of the channel is a stream of messages from the set $\{0, 1\}$ with probability $P(Y = 0)$ and $P(Y = 1)$.
- In the channel, the input message 0 is altered to 1 with probability p and the input message 1 is altered to 0 with probability q.

Conditional distribution

- Hence,

$$P(Y = 0|X = 0) = 1 - p, \tag{1}$$
$$P(Y = 1|X = 0) = p, \tag{2}$$
$$P(Y = 1|X = 1) = 1 - q, \tag{3}$$
$$P(Y = 0|X = 1) = q. \tag{4}$$

- The channel is depicted in the following figure.



- If $p = .25, q = .35$, what is the PMF of Y?

Conditional distribution

Sol^n:

$$p_Y(0) = p_{X,Y}(0,0) + p_{X,Y}(1,0)$$
$$= P(X=0, Y=0) + P(X=1, Y=0)$$
$$= P(Y=0 \mid X=0) \cdot P(X=0) + P(Y=0 \mid X=1) \cdot P(X=1)$$
$$= (1-p) \times 0.5 + q \times 0.5$$
$$= 0.75 \times 0.5 + 0.35 \times 0.5$$
$$= 0.55$$

Similarly,

$$p_Y(1) = P(Y=1 \mid X=0) \cdot P(X=0) + P(Y=1 \mid X=1) \cdot P(X=1)$$
$$= [p + (1-q)] \times 0.5$$
$$= 0.45.$$

Alternatively,

$$p_Y(1) = P(Y=1)$$
$$= 1 - P(Y=0)$$
$$= 1 - 0.55$$
$$= 0.45.$$

$\Big(\because$ Y can take either 0 or 1 value

$\Rightarrow P(Y=1) = 1 - P(Y \neq 1)$
$= 1 - P(Y=0) \Big)$