Lecture 4:
Counting - Part III
&
Conditional Probability - Part I

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Counting

- De Montmort's matching problem: Consider a well-shuffled deck of n cards, labelled 1 through n. You flip over the cards one by one, saying the numbers 1 through n as you do so. You win the game if, at some point, the number you say aloud is the same as the number on the card being flipped over (for example, if the 7th card in the deck has the label 7). What is the probability of winning?
- What is your guess: How the probability will grow as $n \to \infty$?
- Hint: To solve the problem employ the IEF.

Sol^n: - Let A_i be the event that i^{th} card has the no. i written on it.

- We want to find $\underbrace{P(A_1 \cup A_2 \cup \cdots \cup A_n)}$

Prob. of winning = Prob. that there is at least 1 card s.t. no. said = no. written

Counting

$$P(A_i) = \frac{(n-1)!}{n!} = \frac{1}{n}$$

$$P(A_i \cap A_j) = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}$$

$$P(A_i \cap A_j \cap A_\ell) = \frac{(n-3)!}{n!} = \frac{1}{n(n-1)(n-2)}$$

In general,

$$P\Big(\bigcap_{\substack{i \in I : |I| = k, \\ I \subseteq \{1,\dots,n\}}} A_i\Big) = \frac{1}{n(n-1)\cdot(n-2)\cdot \ldots (n-k+1)}$$

IEF: $$P\Big(\bigcup_{i=1}^{n} A_i\Big) = \sum_{\underbrace{i=1}_{n \text{ possibilities}}}^{n} P(A_i) - \sum_{\underbrace{\{i,j\}}_{\binom{n}{2} \text{ possibilities}} \subseteq \{1,\dots,n\}} P(A_i \cap A_j)$$

$$+ \sum_{\underbrace{\{i,j,\ell\}}_{\binom{n}{3} \text{ possibilities}} \subseteq \{1,\dots,n\}} P(A_i \cap A_j \cap A_\ell) - \ldots + (-1)^{n+1} P\Big(\bigcap_{i=1}^{n} A_i\Big)$$

Counting

$$= \frac{n}{n} - \frac{\binom{n}{2}}{n(n-1)} + \frac{\binom{n}{3}}{n(n-1)(n-2)} - \cdots + (-1)^{n+1}\frac{1}{n!}$$

$$= 1 - \frac{n(n-1)/2!}{n(n-1)} + \frac{n(n-1)(n-2)/3!}{n(n-1)(n-2)} - \cdots + \frac{(-1)^{n+1}}{n!}$$

$$= 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \cdots + \frac{(-1)^{n+1}}{n!}$$

But the Taylor series expansion for $1/e$ is:

$$\frac{1}{e} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots$$

$$= 1 - \underbrace{\left(1 - \frac{1}{2!} + \frac{1}{3!} - \cdots\right)}_{}\quad P\left(\bigcup_{i=1}^{\infty} A_i\right)$$

$\Rightarrow$ as $n \to \infty$, $P(\text{winning}) = 1 - \frac{1}{e} \approx 1 - \frac{1}{2.71} \approx 0.63.$

Conditional probability

- Roughly speaking, conditional probability is the concept that addresses this fundamental question: how should we update our beliefs in light of the evidence we observe?
- Conditional probability is essential for reasoning in many fields, e.g., scientific, legal etc.

an event of interest

- Example 1: What is the probability of rain? What is the probability of rain given that the sky is clear?

this event is evidence/observation.

- Example 2: What is the probability that John has stolen a car? What is the probability that John has stolen a car given that John has been convicted stealing 5 cars in the past and that car's tire marks are found at John's property?

two evidence/observations.

Conditional probability

- If A and B are events with $P(B) > 0$, then the conditional probability of A given B, denoted by $P(A|B)$, is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

- $P(A|B)$: the probability of the event A provided that the event B (an evidence) has occurred.
- $P(A)$ is called the prior probability of A and $P(A|B)$ is called the posterior probability of A.
- "prior" means before updating based on the evidence, and "posterior" means after updating based on the evidence.
- For any event A, $P(A|A) = P(A \cap A)/P(A) = 1$. That is, upon observing that A has occurred, our updated probability for A is 1.

makes sense!

Conditional probability

- Example: A standard deck of cards is shuffled well. Two cards are drawn randomly, one at a time without replacement. Let A be the event that the first card is a heart, and B be the event that the second card is red. Find $P(A|B)$ and $P(B|A)$.

Sol^n:

$$P(A \cap B) = \frac{13 \cdot 25}{52 \cdot 51} = \frac{25}{204}.$$

$$P(A) = \frac{13}{52} = \frac{1}{4}.$$

$$P(B) = \frac{26}{52} = 1/2.$$

A: card drawn is a heart.

B: card drawn is red.

Conditional probability

$$\Rightarrow \quad P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{25/204}{1/2} = \frac{25}{102},$$

$$P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{25/204}{1/4} = \frac{25}{51}.$$

Observe:

① $P(A|B) \neq P(B|A)$, i.e., conditioning is not symmetrical.

② Easy to see why $P(B|A) = \frac{25}{51}$ (direct argument)

③ $P(A|B)$: not straightforward to see.

"red has occured" does not help much

Conditional probability

A statement/problem that leads to a contradiction.

- Two children paradox: Martin Gardner posed the following puzzle in the 1950s:
  (1) Mr. Jones has two children. The older child is a girl. What is the probability that both children are girls?
  (2) Mr. Smith has two children. At least one of them is a boy. What is the probability that both children are boys?
- What is your guess: Should the answer to (1) and (2) be the same? Why?

$$(1)\ P(GG \mid \text{elder } G) = \frac{P(GG \ \& \ \text{elder } G)}{P(\text{elder } G)} = \frac{1/4}{1/2} = \frac{1}{2}$$

$$(2)\ P(BB \mid \text{at least } 1B) = \frac{P(BB, \text{a.l. } 1B)}{P(\text{a.l. } 1B)} = \frac{1/4}{3/4} = \frac{1}{3}$$

This contradits intuition: Having one child as B shold not have any influence on the second child as B.

Conditional probability

- Contradiction occurs since it is not clear how the event "at least 1B" is (or should be) generated.

- Alternative way to generate "a. l. 1B":

Older child	Younger child	P(this family) $\times$	P(a.l.1B given this family) $=$	P(a.l.1B & this family)
G	G	1/4	0	0
G	B	1/4	1/2 or 1	1/8 or 1/4
B	G	1/4	1/2 or 1	1/8 or 1/4
B	B	1/4	1	1/4

$$P(a.l.1B) = \frac{1}{8}+\frac{1}{8}+\frac{1}{4} = \frac{1}{2}$$

or $$P(a.l.1B) = \frac{1}{4}+\frac{1}{4}+\frac{1}{4} = \frac{3}{4}$$

This implies from the law of total probability

Moral of the story (two children paradox)

- If a problem is not well-defined or an event is not well-defined, it may lead to a contradicting / inconsistent conclusions.

– For more information on two children paradox and similar paradoxes, read the following paper:

Lynch, Peter. "The Two-Child Paradox: Dichotomy and Ambiguity." Irish Mathematical Society Bulletin (2011):