Lecture 8:
Discrete Random Variables - Part II

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Recall: summary of main ideas

- A $\underbrace{\text{sample space}}_{}$ S is the set of all possible outcomes

  of an experiment

  $S = \{ s : s \text{ is an outcome} \}$

- "An event A occurred" if

  the outcome of the experiment is in A.

- Probability is

  a function from the set of events to the set $[0,1]$ satisfying Axioms 1 & 2.

  $P : \{ A \subseteq S : A \text{ is an event} \} \to [0,1]$.

- A random variable X is

  a function from the sample space S to the set of real numbers.

  $X : S \to \mathbb{R}$.

Recall: summary of main ideas

- Probability mass function (PMF) p_X of a discrete r.v. X is
  the function $p_X(x) = P(X = x)$.

  $$p_X : \mathbb{R} \to [0, 1].$$

- Support of a discrete r.v. X is the set of all values x s.t.
  $p_X(x) > 0$.

  $$\text{support of } X = \{ x \in \mathbb{R} : p_X(x) > 0 \}.$$

- Caution: For precise definition of the terms refer to the earlier lecture slides.

Discrete random variables

- **Example**: Roll two fair 6-sided dice. Let $\underbrace{T = X + Y}_{T \text{ is a } f^n \text{ of } X \,\&\, Y.}$ be the total of the two rolls, where X and Y are the individual rolls. What is the PMF of T?

– What values can T take? support of $T = \{2, 3, \dots, 12\}$

– $p_T(2) = P(T=2) = P(\{X=1, Y=1\})$
$= 1/36$

– $p_T(3) = P(T=3) = P(\underbrace{\{X=1, Y=2\}}_{\text{disjoint}} \overset{\text{"}\cup\text{"}}{\text{or}} \underbrace{\{X=2, Y=1\}}_{\text{events}})$

$= P(\{X=1, Y=2\}) + P(\{X=2, Y=1\})$

$= \frac{2}{36} = p_T(11)$

$= P(\{X=5, Y=6\} \text{ or } \{X=6, Y=5\})$

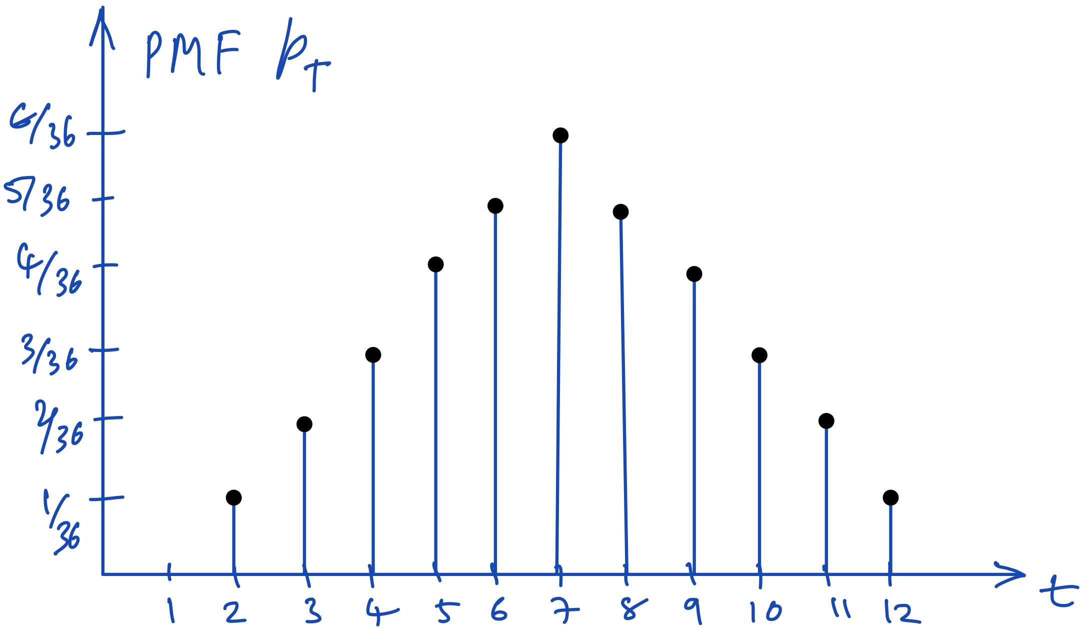
Discrete random variables

- Similarly, $p_T(4) = 3/36 = p_T(10)$ $(1,3)(2,2)(3,1)$

  $p_T(5) = 4/36 = p_T(9)$ $(1,4)(2,3)(3,2),(4,1)$

  $p_T(6) = 5/36 = p_T(8)$ $(1,5),(2,4),(3,3),(4,2),(5,1)$

  $p_T(7) = 6/36$ $(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)$



Discrete random variables

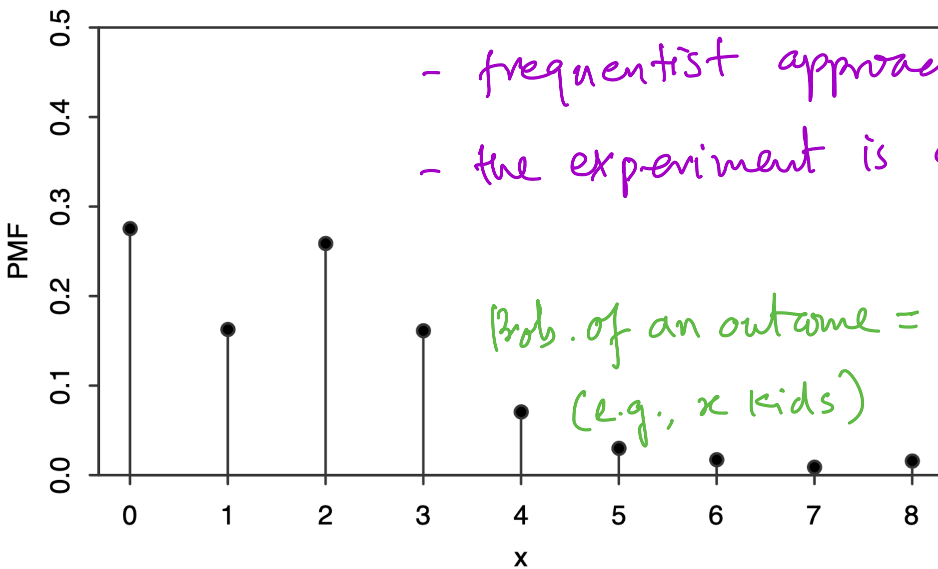
frequentist approach

- **Example of data:** Suppose we choose a family in a town at random. Let X be the number of children in the chosen family. Since X can only take on integer values, it is a discrete r.v. The probability that X takes on the value x is proportional to the number of families in the town with x children, i.e.,

$$p_X(x) = P(X = x) = \underbrace{\frac{\text{no. of families with } x \text{ kids}}{\text{total no. of families}}}.$$

- frequentist approach to probability.

- the experiment is conducted n times



Prob. of an outcome (e.g., x kids) $= \dfrac{\text{no. of time the outcome occurs}}{\text{total no. of experiments}}$

Discrete random variables

- Theorem: Let X be a discrete r.v. with support $\{x_1, x_2, \ldots\}$. The PMF p_X of X must satisfy the following two criteria:
  1. Nonnegative: $p_X(x) > 0$ if $x = x_j$ for some j, and $p_X(x) = 0$ otherwise;
  2. Sums to 1: $\sum_{i=1}^{\infty} p_X(x_i) = 1$

Proof: ①: follows from the def^n of support.

②: $\sum_{i=1}^{\infty} p_X(x_i) = P\left(\bigcup_{i=1}^{\infty} \{X = x_i\}\right)$ ($\because$ Axiom 2)

$= P(\{X = x_1\} \text{ or } \{X = x_2\} \text{ or } \ldots)$

$= 1.$

- Conversely, if distinct values $\{x_1, x_2, \ldots\}$ are specified and we have a function satisfying the two criteria above, then this function is the PMF of some r.v.

– follows from the def^n of discrete r.v.

Discrete random variables

- Example: Let T be the sum of two fair die rolls. We have already calculated the PMF of T. What is the probability that T is in the interval $[1, 4]$?

$$P(T \in [1,4]) = P(\{T=2\} \text{ or } \{T=3\} \text{ or } \{T=4\})$$

$$= 1/36 + 2/36 + 3/36$$

$$= 1/6.$$

Discrete random variables

- An r.v. X is said to have the Bernoulli distribution with parameter p if $P(X = 1) = p$ and $P(X = 0) = 1 - p$, where $0 < p < 1$. We write this as $X \sim \text{Bern}(p)$.
- The symbol $\sim$ is read "is distributed as".
- The number p in $\text{Bern}(p)$ is called the parameter of the distribution.
- Examples: $X \sim \text{Bern}(1/3)$, $Y \sim \text{Bern}(1/8)$
- The indicator random variable of an event A is the r.v. which equals 1 if A occurs and 0 otherwise.
- The indicator r.v. of A is denoted I_A or $I(A)$.
- Note that $I_A \sim \text{Bern}(p)$ with $p = P(A)$.

Discrete random variables

- An experiment that can result in either a "success" or a "failure" (but not both) is called a Bernoulli trial.
- A Bernoulli random variable can be thought of as the indicator of success in a Bernoulli trial: it equals 1 if success occurs and 0 if failure occurs in the trial.

- Suppose that n independent Bernoulli trials are performed, each with the same success probability p. Let X be the number of successes. The distribution of X is called the Binomial distribution with parameters n and p.
- $X \sim \text{Bin}(n, p)$ denotes that X has the Binomial distribution with parameters n and p, where n is a positive integer and $0 < p < 1$.

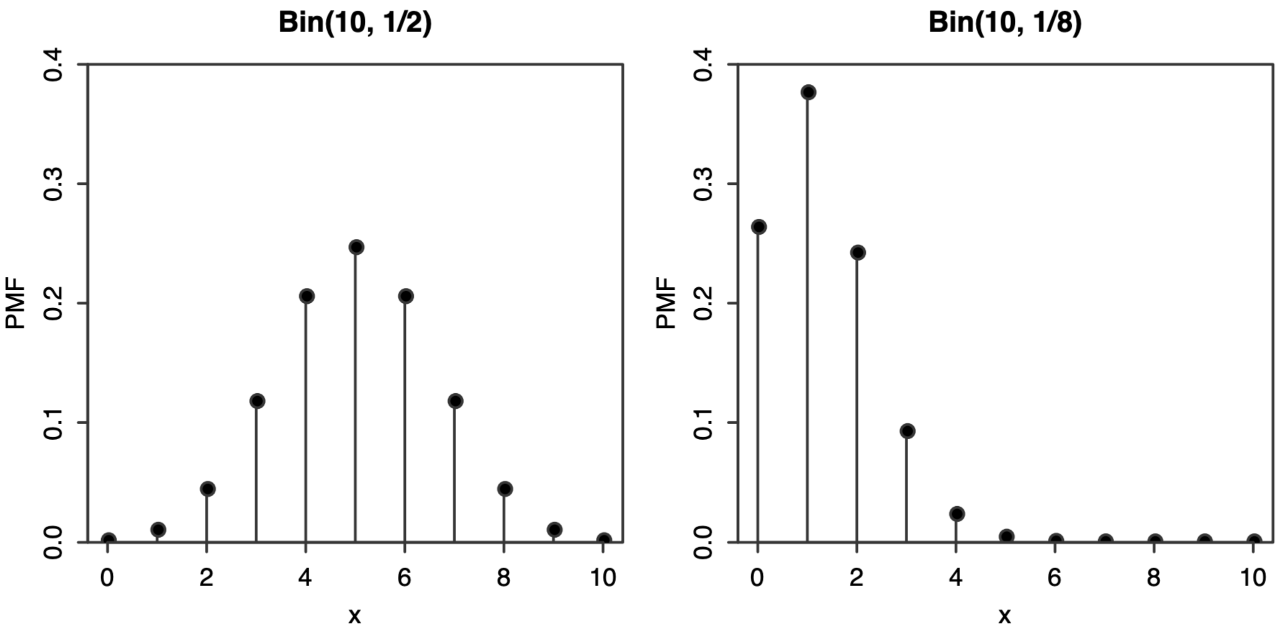
Discrete random variables

- Theorem: If $X \sim \text{Bin}(n, p)$, then the PMF of X is

$$p_X(k) = P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

for $k = 0, 1, \ldots, n$ and $p_X(k) = P(X = k) = 0$ otherwise.

Discrete random variables



- Note that the PMF of the $\text{Bin}(10, 1/2)$ distribution is symmetric about 5, but when the success probability is not $1/2$, the PMF is skewed.

Discrete random variables

- Consider a sequence of independent Bernoulli trials, each with the same success probability $p \in (0, 1)$, with trials performed until a success occurs. Let X be the number of failures before the first successful trial. Then X has the Geometric distribution with parameter p; denoted $X \sim \text{Geom}(p)$.
- Theorem: If $X \sim \text{Geom}(p)$, then the PMF of X is

$$P(X = k) = (1 - p)^k p, \qquad \text{for } k = 0, 1, 2, \ldots$$

- Note that, the support of a geometric r.v. has infinite cardinality.

Discrete random variables

- Theorem: $\mathrm{Geom}(p)$ is a (valid) PMF.