Lecture 28: Estimation - Part I

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Recall

- A population is the set of all possible observations available from a particular probability distribution.
- A sample is a particular subset of the population.

e.g., mean, variance of unknown PDF.

- A parameter is a property of a population or a probability distribution.
- A statistic is a property of a sample from the population.

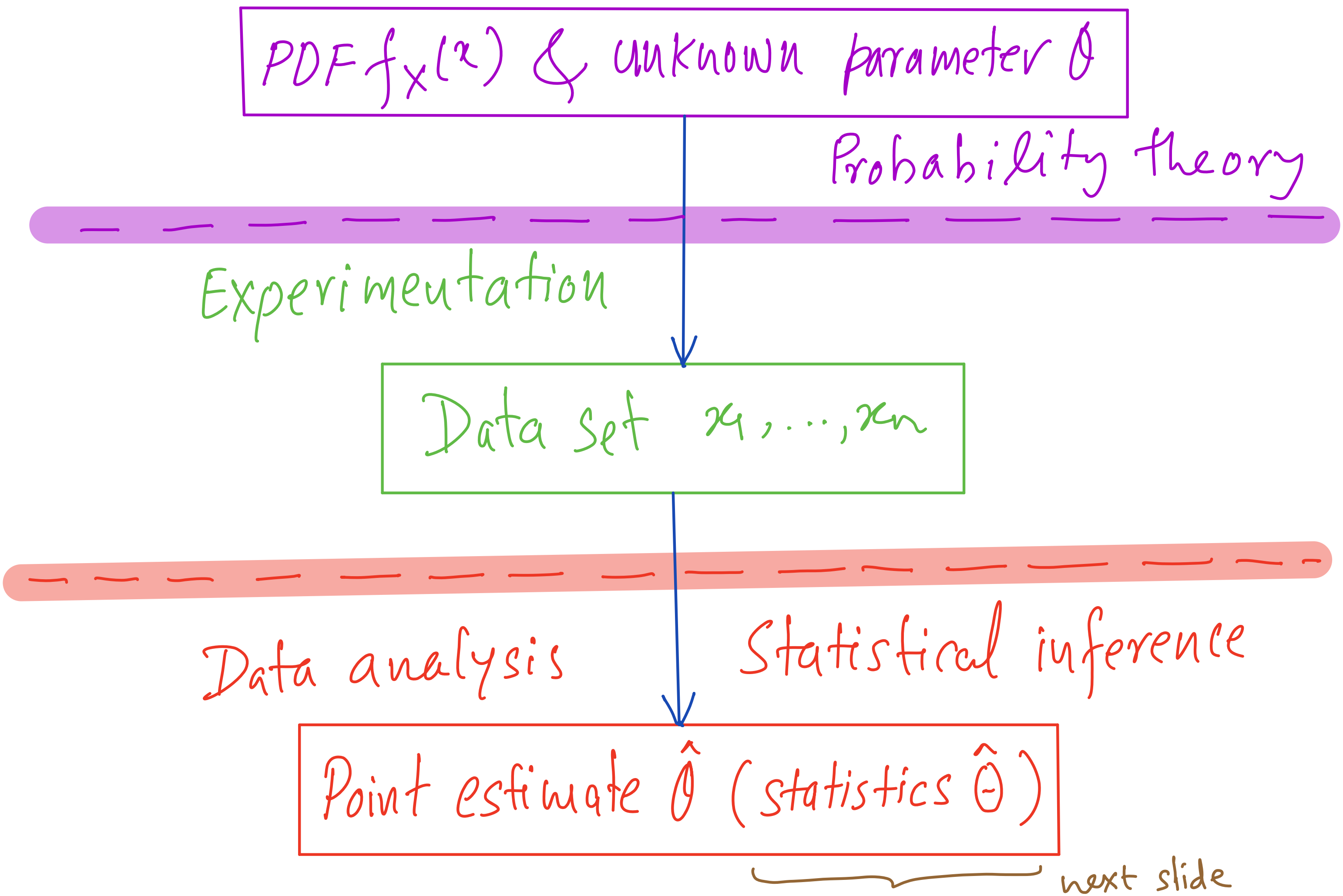
e.g., sample mean, sample variance

Point estimation

- Estimation is a procedure by which the information contained within a sample is used to investigate properties of the population (parameters) from which the sample is drawn.
- A point estimate of an unknown parameter θ is a statistic $\hat{\theta}$ that represents a "best guess" at the value of θ.
- There may be more than one sensible point estimate of a parameter.
- Example: point estimate of the mean μ_X of a probability distribution $f_X(x)$ is the sample mean $\bar{x}$ of data observations obtained from the probability distribution. In this case, $\hat{\mu}_X = \bar{x}$.
- Similarly, two point estimates of the variance σ_X^2 are

$$\underbrace{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}}_{\text{estimate \#1}} \text{ and } \underbrace{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n}}_{\text{estimate \#2}}.$$

Point estimation



Point estimation

- Recall that $\hat{\theta}$ is a point estimate of a parameter θ which is a function $g(x_1, \ldots, x_n)$. Similarly, $\hat{\Theta}$ is a point estimate which is a function $g(X_1, \ldots, X_n)$. Note that $\hat{\theta}$ is a number and $\hat{\Theta}$ is an r.v.
- In general, when there is more than one obvious point estimate for a parameter (e.g., estimates of variance in Slide 3), the following criteria can be used to find desirable point estimate:
- A point estimate $\hat{\Theta}$ for a parameter θ is called unbiased if

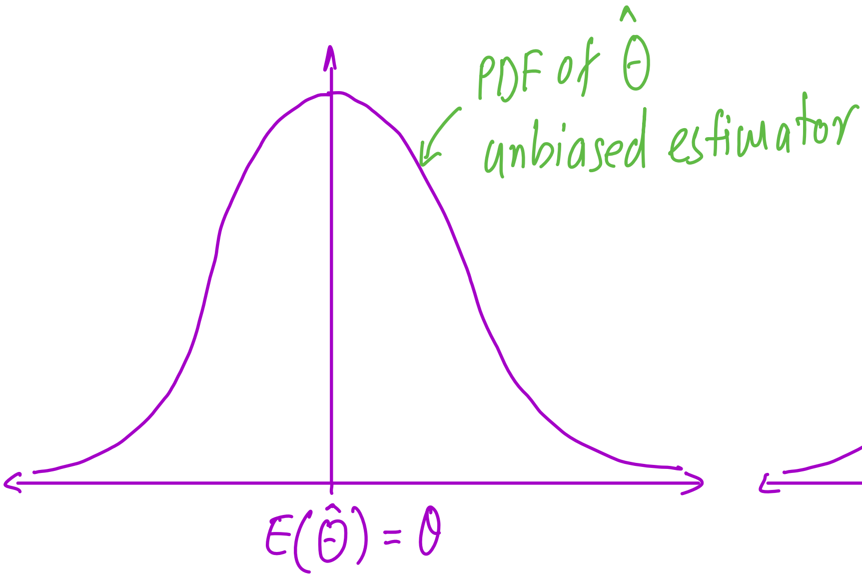
$$E(\hat{\Theta}) = \theta$$

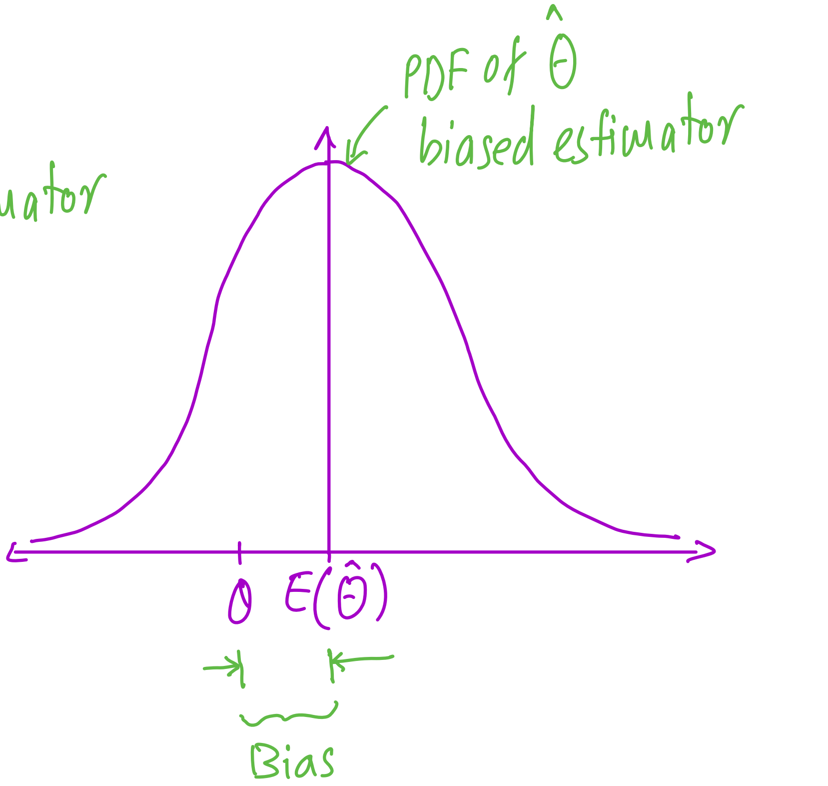
- Unbiasedness is a nice property for a point estimate to possess. If a point estimate is not unbiased, then its bias can be defined as

$$\text{bias} = E(\hat{\Theta}) - \theta.$$

Point estimation

- Unbiased estimator: $E(\hat{\Theta}) = \theta$
- Biased estimator: $E(\hat{\Theta}) \neq \theta, \text{bias} = E(\hat{\Theta}) - \theta$





Point estimation

- Example: If $X_1, \ldots, X_n$ is a sample of observations from a probability distribution with a mean μ_X, then show that the sample mean $\hat{\Theta} = \hat{\mu}_X = \bar{X}$ is an unbiased point estimate of the population mean $\theta = \mu_X$.

Proof: Note that $E(X_i) = \mu_X$ for all $i = 1, \ldots, n$.

Hence,
$$E(\hat{\mu}_X) = E(\bar{X}) = E\left(\frac{X_1 + \ldots + X_n}{n}\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{n\mu_X}{n} = \mu_X.$$

$E(\hat{\mu}_X) = \mu \Rightarrow \hat{\mu}_X$ is unbiased p.e. of μ_X.

Point estimation

- Example: If $X_1, \ldots, X_n$ is a sample of observations from a probability distribution with a variance σ_X^2, then show that the sample variance

$$\hat{\Theta} = \hat{\sigma}_X^2 = S^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1}$$

is an unbiased point estimate of the population variance $\theta = \sigma_X^2$.

Proof: $E(S^2) = \frac{1}{n-1} E\left(\sum_{i=1}^{n} (X_i - \bar{X})^2\right)$

$$= \frac{1}{n-1} E\left(\sum_{i=1}^{n} \left[(X_i - \mu_X) - (\bar{X} - \mu_X)\right]^2\right)$$

$$= \frac{1}{n-1} E\left(\sum_{i=1}^{n} \left[(X_i - \mu_X)^2 - 2(X_i - \mu_X)(\bar{X} - \mu_X) + (\bar{X} - \mu_X)^2\right]\right)$$

$\boxed{\text{You may skip the proof. Won't be asked in the exams}}$

Point estimation

$$= \frac{1}{n-1} E\left(\sum_{i=1}^{n} (X_i - \mu_X)^2 - 2(\bar{X} - \mu_X) \underbrace{\sum_{i=1}^{n} (X_i - \mu_X)}_{= n(\bar{X} - \mu_X)} + n(\bar{X} - \mu_X)^2 \right)$$

$$= \frac{1}{n-1} E\left(\sum_{i=1}^{n} (X_i - \mu_X)^2 - n(\bar{X} - \mu_X)^2 \right)$$

$$= \frac{1}{n-1} \left(\sum_{i=1}^{n} \underbrace{E\left[(X_i - \mu_X)^2\right]}_{Var(X_i) = \sigma_X^2} - n \underbrace{E\left[(\bar{X} - \mu_X)^2\right]}_{Var(\bar{X}) = \sigma_X^2 / n} \right)$$

$$= \frac{1}{n-1} \left(n\sigma_X^2 - n\sigma_X^2/n \right) = \sigma_X^2$$

$\Rightarrow$ s^2 is an unbiased estimator of σ_X^2.

Point estimation

- Homework (Assignment 9 problem): If $X_1, \ldots, X_n$ is a sample of observations from a probability distribution with a variance σ_X^2, then show that the sample variance

Hint: use the fact that

$$\hat{\Theta} = \hat{\sigma}_X^2 = S^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n}$$

$$\sum_{i=1}^{n} \frac{(X_i - \bar{X})^2}{n-1}$$

has the bias of

is an unbiased estimator.

$$-\frac{\sigma_X^2}{n}$$

for point estimate of the population variance $\theta = \sigma_X^2$.

- Hence, we choose the denominator $n-1$ to make sure that the estimator is unbiased.

(Recall: Slide 7, Lecture 27)