Lecture 15:
Expectation and Variance - Part II

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An application of linearity property

- Example: What is the expected value of a binomially distributed r.v.?
- We solved this problem directly in Lecture 14. Here we solve it using the linearity property of expectation.

- Let $X \sim \mathrm{Bin}(n, p)$. Then, $X = X_1 + X_2 + \dots + X_n$

where X_i's are iid $\mathrm{Bern}(p)$.

$$\Rightarrow \quad E(X) = E(X_1 + X_2 + \dots + X_n)$$
$$= E(X_1) + E(X_2) + \dots + E(X_n)$$
$$= \underbrace{p + p + \dots + p}_{n}$$
$$= np.$$

Properties of expectation

- If X and Y are independent r.v.s, then

$$E(XY) = E(X)E(Y).$$

Proof:

$$E(XY) = \sum_x \sum_y xy\, p_{XY}(x,y)$$

$$= \sum_x \sum_y xy\, p_X(x)\, p_Y(y)$$

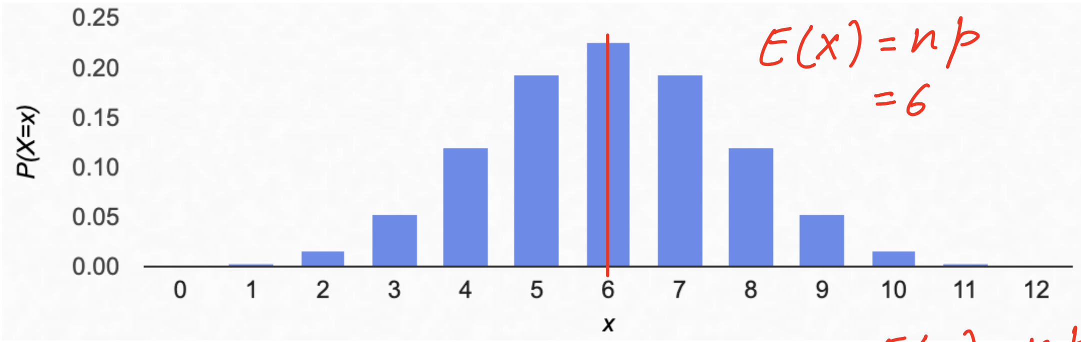
$$= \sum_x x\, p_X(x) \sum_y y\, p_Y(y)$$

$$= E(X)E(Y)$$

Expectation

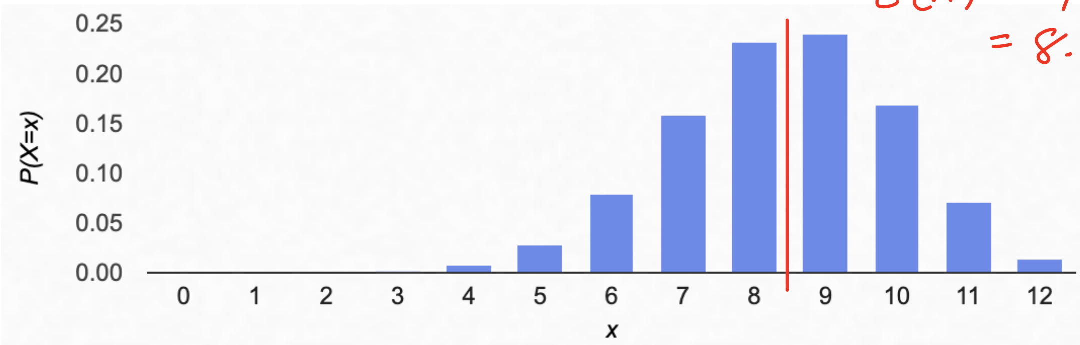
- Expectation of r.v.s with binomial distribution.

$Bin(12, 0.5)$



$E(x) = np$
$= 6$

$Bin(12, 0.7)$



$E(x) = np$
$= 8.4$

- The distribution is "centred around" its expected value.

Expectation

Recall: $X \sim \text{Geom}(p)$ if $P(X=k) = (1-p)^k \cdot p$ for $k = 0, 1, 2, \ldots$

- What is the expected value of a random variable with geometric distribution?

$$E(X) = \sum_{k=0}^{\infty} k \cdot p \cdot (1-p)^k$$

$$= p \sum_{k=1}^{\infty} k (1-p)^k$$

$$= p(1-p) \underbrace{\sum_{k=1}^{\infty} k (1-p)^{k-1}}$$

$$= p(1-p) \frac{1}{p^2}$$

$$= \frac{1-p}{p}$$

$\because$ Recall: for $0 < x < 1$,

$$\sum_{k=1}^{\infty} x^k = \frac{x}{1-x}$$

Take derivative:

$$\frac{d}{dx} \sum_{k=1}^{\infty} x^k = \underbrace{\frac{d}{dx} \frac{x}{1-x}}$$

Homework

$$\Rightarrow \sum_{k=1}^{\infty} k x^{k-1} = \frac{1}{(1-x)^2}$$

Variance

- Recall: Expected value of a random variable is a single value.
- It tells us the center of mass of the distribution of an r.v.
- That is, the distribution is "centred around" its mean value.
- Variance of a random variable is also a single value.
- It tells us how "spread out" the distribution is.
- The variance of an r.v. X is

$$\mathrm{Var}(X) = E[(X - E(X))^2].$$

- The square root of the variance is called the standard deviation:

$$\mathrm{SD}(X) = \sqrt{Var(X)}.$$

Variance

- $E(X)$: Mean of an r.v. X is often denoted by μ_X.
- $\mathrm{Var}(X)$: Variance of an r.v. X is often denoted by σ_X^2.
- Theorem: For any r.v. X,

$$\mathrm{Var}(X) = E(X^2) - E(X)^2.$$

Proof: $\mathrm{Var}(X) = E(X - E(X))^2$

$= E(X - \mu_X)^2$

$= E(X^2 - 2\mu_X X + \mu_X^2)$

$= E(X^2) - 2\mu_X E(X) + \mu_X^2$ (∵ Linearity of expectation)

$= E(X^2) - 2\mu_X^2 + \mu_X^2$

$= E(X^2) - \mu_X^2$

$= E(X^2) - E(X)^2$

(Expectation of a constant is the constant. Let $g(X) = c$. Then $E(c) = \sum_x c\, p_X(x) = c$.)

Variance

- Example: Find $\mathrm{Var}(X)$ if $X \sim \mathrm{Bern}(p)$.

- In Lecture 14, we showed that $E(X) = p$.

- Now, $\mathrm{Var}(X) = E(X^2) - E(X)^2$

$$= \sum_x x^2 p_X(x) - p^2$$

$$= 1^2 \cdot p + 0^2 \cdot (1-p) - p^2$$

$$= p - p^2$$

$$= p(1-p).$$

- Assignment problem: show that the variance of a binomially distributed r.v. $X \sim \mathrm{Bin}(n, p)$ is

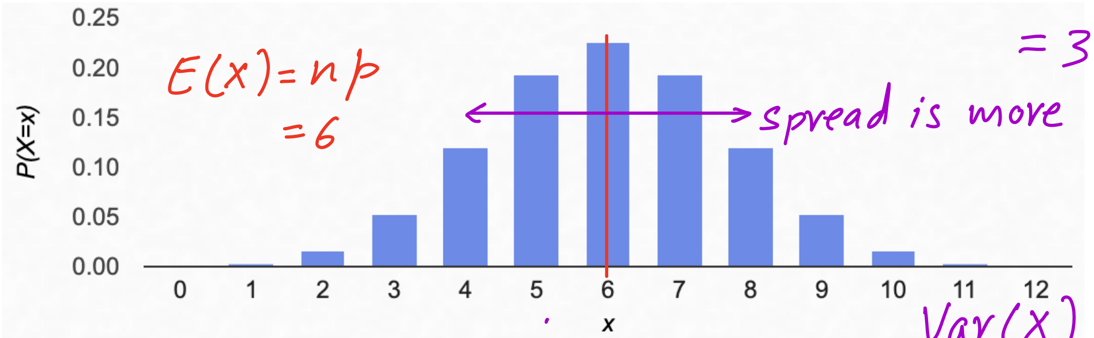
$$\mathrm{Var}(X) = np(1-p)$$

Variance

- Variance of r.v.s with binomial distribution. $Var(X) = np(1-p)$

  $= 3$

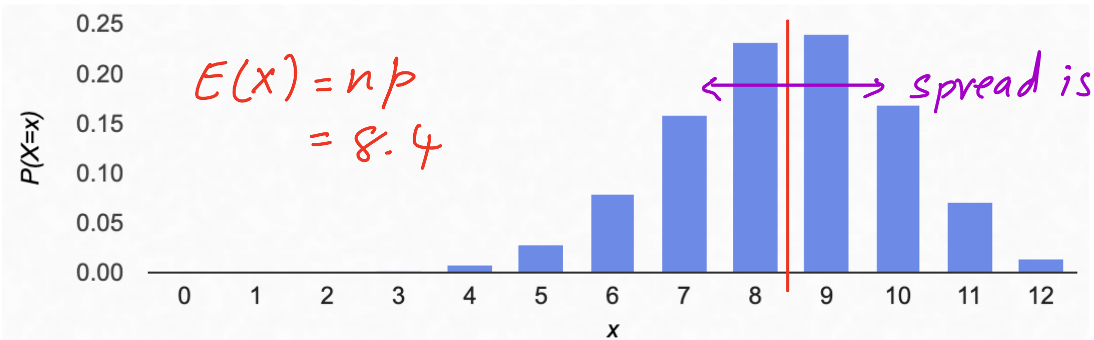
$Bin(12, 0.5)$



$Var(X) = np(1-p)$

$= 2.52$

$Bin(12, 0.7)$



- The distribution is "<u>centred around</u>" its expected value.
- Variance tells us how "<u>spread out</u>" the distribution is.