Lecture 20:
Continuous Random Variables - Part IV

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Continuous random variables

- The normal or Gaussian distribution has a PDF

$$\boxed{f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}}$$

  for $-\infty < x < \infty$, depending upon two parameters, the mean and the variance

$$E(X) = \mu \text{ and } \mathrm{Var}(X) = \sigma^2$$

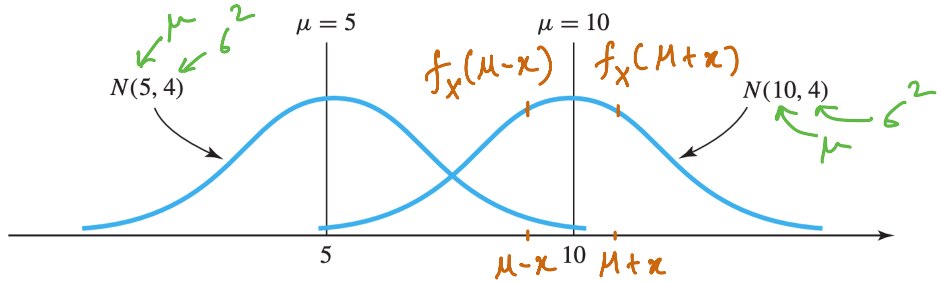
  of the distribution.
- Notation: $\underline{X \sim N(\mu, \sigma^2)}$ means that X has a normal distribution with mean μ and variance σ^2.

Continuous random variables

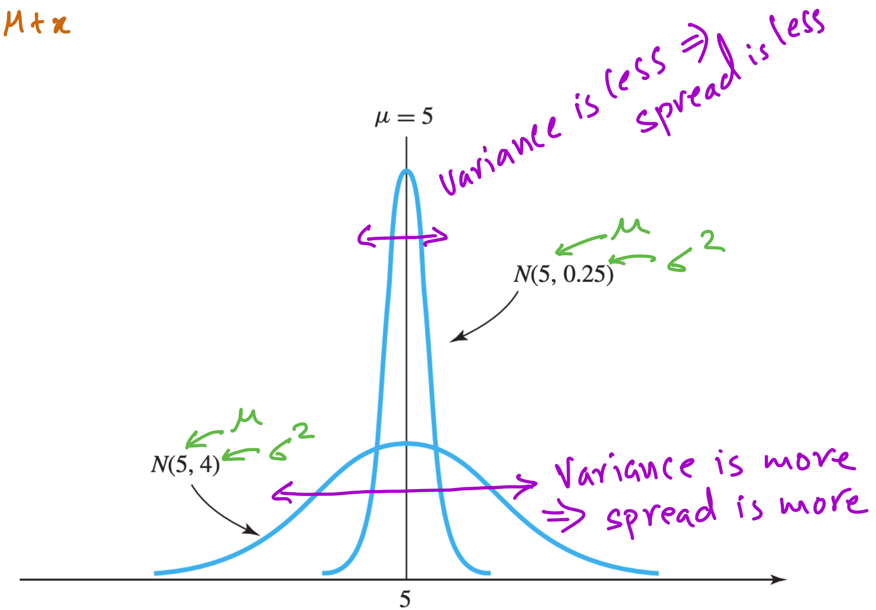
verify: homework

$$f_X(\mu+x) = f_X(\mu-x), \ \forall x$$

- The PDF is a bell-shaped curve that is symmetric about μ:



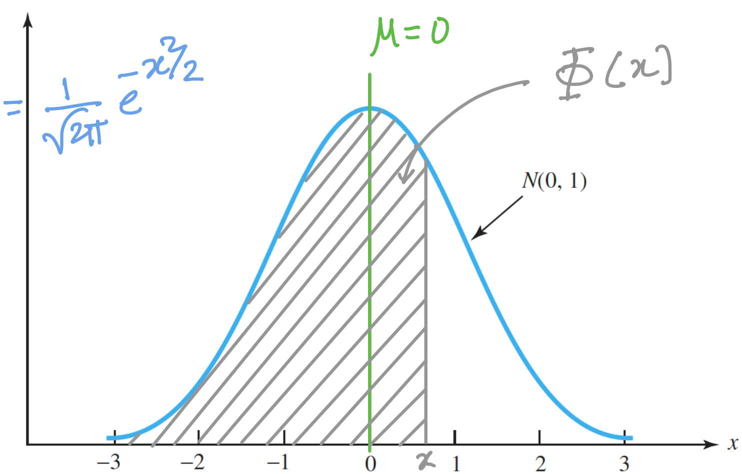
- Effect of change in variance to the PDF:



Continuous random variables

- If $X \sim N(\overset{\mu}{0}, \overset{\sigma^2}{1})$ then X is said to have the standard normal distribution.
- Its PDF is denoted $\phi(x)$: $$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$
- Its CDF is denoted $\Phi(x)$: $$\Phi(x) = \int_{-\infty}^{x} \phi(t)dt$$ closed form ?

$f_X(x) = \phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$



By symmetry,

$\phi(x) = \phi(-x)$.

Continuous random variables

- The exponential distribution is often used to model waiting times (e.g., of a customer in a queue).
- Its PDf is

$$\boxed{f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0; \\ 0, & x < 0, \end{cases}}$$

where $\lambda > 0$ is the parameter for the distribution.
- The CDF is

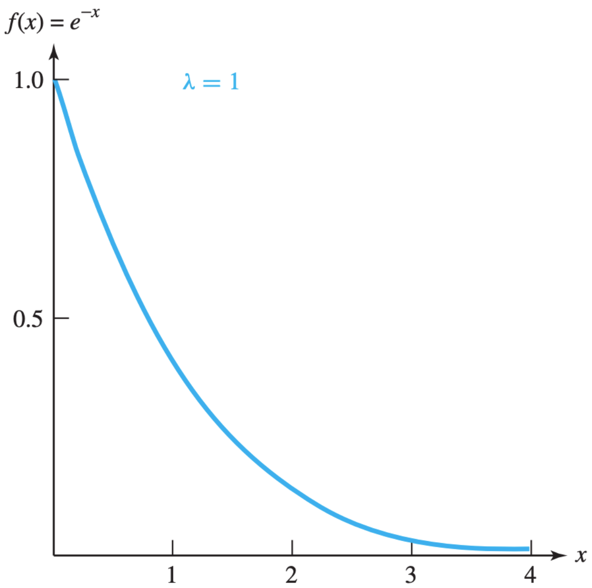
$$F_X(x) = \int_0^x \lambda e^{-\lambda t}\,dt \,, \quad 0 \leq x < \infty$$

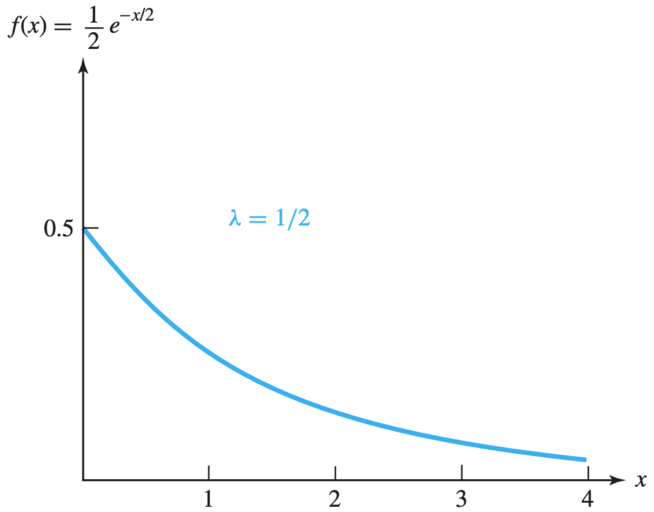
$$= \frac{\lambda e^{-\lambda t}}{-\lambda}\Bigg|_{t=0}^{t=x} = 1 - e^{-\lambda x}$$

$$\Rightarrow F_X(x) = \begin{cases} 0 & x < 0, \\ 1 - e^{-\lambda x} & x \geq 0. \end{cases}$$

Continuous random variables

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0. \end{cases}$$





Continuous random variables

- Mean of an r.v. with exponential distribution:

$$E(X) = \int_0^\infty \underbrace{t}_{u} \cdot \underbrace{\lambda e^{-\lambda t} \, dt}_{dv}$$

$$= t \cdot \left(-e^{-\lambda t}\right)\Big|_0^\infty - \int_0^\infty -e^{-\lambda t} \, dt$$

$$= 0 - \frac{1}{+\lambda} e^{-\lambda t}\Big|_0^\infty$$

$$= 0 - \left[0 - \frac{1}{\lambda}\right]$$

$$= \frac{1}{\lambda}$$

Integration by parts:

$$\int u \, dv = uv - \int v \, du$$

$$u = t \Rightarrow du = dt$$

$$v = -e^{-\lambda t} \Rightarrow dv = \lambda e^{-\lambda t} dt$$

Definite integral:

$$\int_a^b u \, dv = uv\Big|_a^b - \int_a^b v \, du$$

$$\int_a^\infty u \, dv = \lim_{b \to \infty} \left[uv\Big|_a^b - \int_a^b v \, du \right]$$

Continuous random variables

- Variance of an r.v. with exponential distribution:

$$E(X^2) = \int_0^\infty \underbrace{t^2}_{u} \cdot \underbrace{\lambda e^{-\lambda t} \, dt}_{dv}$$

Integration by parts:

$$\int u \, dv = uv - \int v \, du$$

$$u = t^2 \Rightarrow du = 2t \, dt$$

$$v = -e^{-\lambda t} \Rightarrow dv = \lambda e^{-\lambda t} dt$$

$$= t^2 \cdot (-e^{-\lambda t}) \Big|_0^\infty - \int_0^\infty -2t e^{-\lambda t} \, dt$$

$$= 0 + \frac{2}{\lambda} \underbrace{\int_0^\infty \lambda t e^{-\lambda t} \, dt}_{E(X)} = \frac{2}{\lambda} \cdot \frac{1}{\lambda} = \frac{2}{\lambda^2}$$

$$\Rightarrow Var(X) = E(X^2) - E(X)^2$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

Continuous random variables

- Median of an r.v. with exponential distribution:

We need to solve $F_X(x) = 0.5$

$\Rightarrow 1 - e^{-\lambda x} = 0.5$

$\Rightarrow e^{-\lambda x} = 0.5$

$\Rightarrow -\lambda x = \ln 0.5$

$\Rightarrow -\lambda x = -0.693$

$\Rightarrow x = \dfrac{0.693}{\lambda}$

$\Rightarrow x = 0.693 \cdot E(X)$

Note

- Source of figures: reference books