Lecture 34:
Hypothesis Testing - Part III

Satyajit Thakor
IIT Mandi

Hypothesis testing

- The probability of committing a type II error, denoted by β, is impossible to compute unless we have a specific alternative hypothesis.
- Type II error: Nonrejection of the H_0 when it is false.
- For our example, the type II error occurs if $p > 1/4$ when $X \leq 8$.
- Let the particular alternative hypothesis be $p = .5 > 1/4$. Then

$$\beta = P\{\text{type II error}\}$$

$$= P\left(X \leq 8 \text{ when } p = 1/2\right)$$

$$= \sum_{k=0}^{8} \binom{20}{k} (1/2)^{20} = 0.25172$$

Hypothesis testing

- That is, it is quite likely (type II error prob. is 0.2517) that we shall reject the new vaccine when, in fact, it is superior (50% effective) to what is now in use (compared to 25% effective).
- Now, let the particular alternative hypothesis be $p = .7 > 1/4$. Then,

$$\beta = P\{\text{type II error}\}$$
$$= P(X \leq 8 \text{ when } p = .7)$$
$$= \sum_{k=0}^{8} \binom{20}{k} (.7)^k (.3)^{20-k} = 0.00514.$$

- That is, it is extremely unlikely that the new vaccine would be rejected when it was 70% effective after a period of 2 years.

Hypothesis testing

- How to decrease Type I or Type II error?
- Let's assume that we want to reduce the possibility of Type II error. This can be done by increasing the critical region:

- Let the critical value be 7.
- Now, we test $p = 1/4$ against $p = 1/2$.
- Then,

$$\alpha = \sum_{k=8}^{20} \binom{20}{k} \left(1/4\right)^{k} \left(3/4\right)^{20-k} = 0.1018$$

$$\beta = \sum_{k=0}^{7} \binom{20}{k} \left(1/2\right)^{20} = 0.1316$$

- Thus, by increasing the critical region β is reduced from 0.2517 to 0.1316 but α is increased from 0.0409 to 0.1018.

Hypothesis testing

- How to decrease the probability of both type I and II errors?
- The probability of committing both types of error can be reduced by increasing the sample size.

– Consider a sample of size 100.

– If more than 36 are protected from the virus for 2 years then we reject $H_0 : p = 1/4$.

– Critical region is $37, \dots, 100$.

Recall: CLT (Lec. 26)

– To compute the errors we use normal approximation.

$$X \sim \text{Bin}(\overbrace{100}^{n}, p) \approx \mathcal{N}(\overbrace{\overbrace{100}^{n}p}^{\mu_X}, \overbrace{\overbrace{100}^{n}p(1-p)}^{\sigma_X^2}) \sim Y$$

$$- \underbrace{p = 1/4}_{H_0} \Rightarrow Y_0 \sim \mathcal{N}(\underbrace{25}_{\mu_{Y_0}}, \underbrace{18.75}_{\sigma_{Y_0}^2}) \qquad Y_0 = \mu_{Y_0} + \sigma_{Y_0} \underbrace{Z}_{\mathcal{N}(0,1)}$$

Hypothesis testing

- Then, $\alpha = P\{\text{Type I error}\}$

$$= P(X > 36 \text{ when } p = 1/4)$$

$$\approx P(Y_0 \geq 36.5)$$

$$= P\left(Z \geq \frac{36.5 - 25}{\sqrt{18.75}}\right)$$

$$= P(Z \geq 2.66)$$

$$= 1 - \Phi_Z(2.66) \approx 1 - 0.9961 = 0.0039.$$

- Now, let the alternative hypothesis be $p = 1/2$.

- $\underbrace{p = 1/2}_{H_1} \Rightarrow Y_1 \sim \mathcal{N}(\underbrace{50}_{\mu_{Y_1}}, \underbrace{25}_{\sigma^2_{Y_1}})$, $Y_1 = \mu_{Y_1} + \sigma_{Y_1} \underbrace{Z}_{\mathcal{N}(0,1)}$.

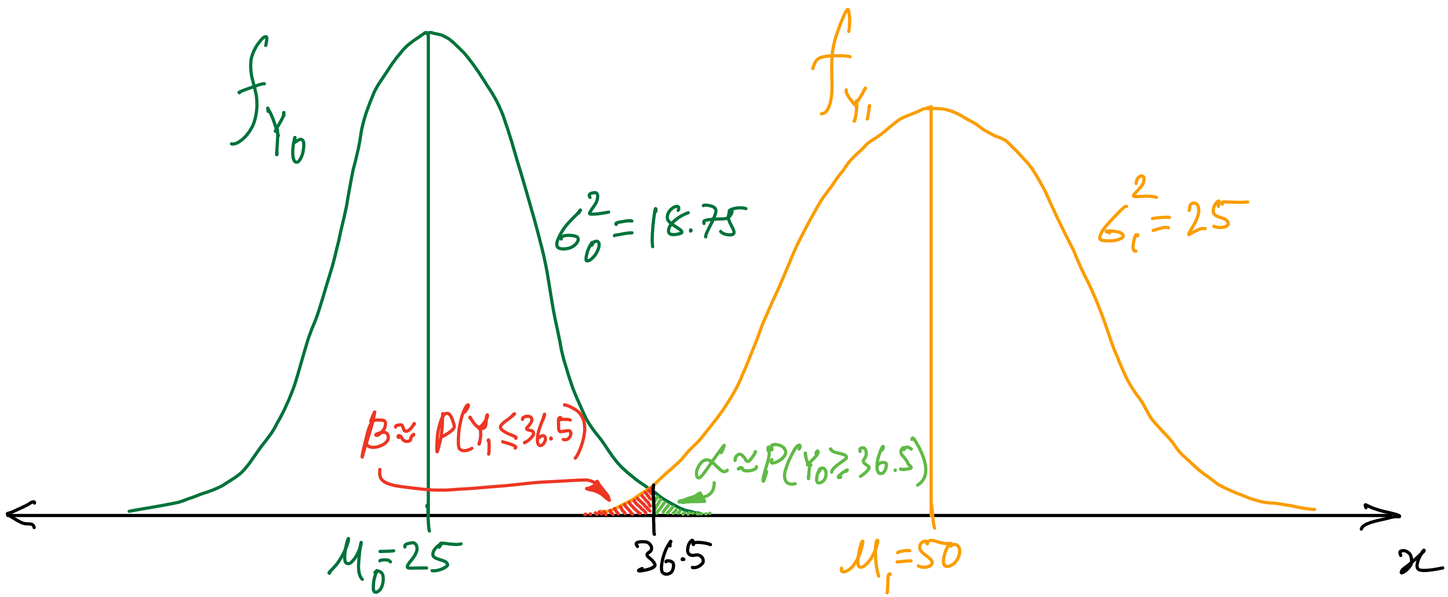
- Then, $\beta = P\{\text{Type II error}\}$

Hypothesis testing

$$= P(X \leq 36 \text{ when } p = 1/2)$$

$$\approx P(Y_1 \leq 36.5)$$

$$= P\left(Z \leq \frac{36.5 - 50}{\sqrt{25}}\right) = P(Z \leq -2.7) = 0.0035$$



Hypothesis testing

- Example: Consider the null hypothesis that the average weight of students in a college is 68 kgs against the alternative hypothesis that it is unequal to 68:

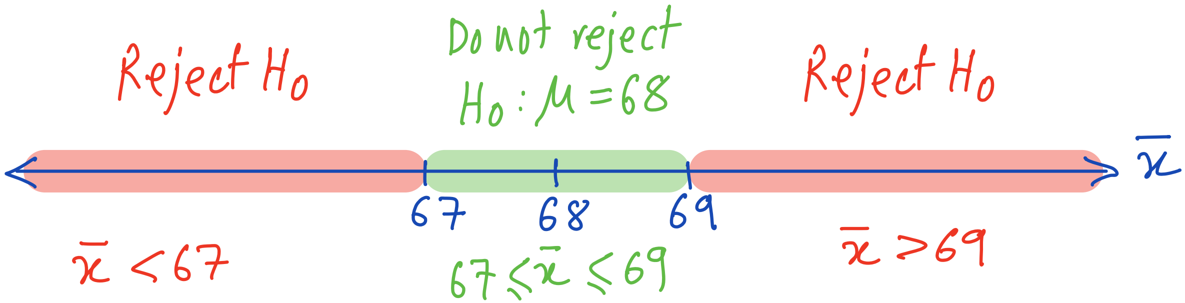
$$\left.\begin{aligned} H_0 &: \mu = 68, \\ H_1 &: \mu \neq 68. \end{aligned}\right\} \text{two-tailed test}$$

- Assume that the weight is a normally distributed with $\sigma = 3.6$ and the sample size is 36.
  (a) Define a suitable critical region
  (b) Find the probability of Type I error
  (b) Find the probability of Type II error

(a) We define the critical region as: $\bar{x} < 67$, $\bar{x} > 69$.

- That is, do not reject if $67 \leq \bar{x} \leq 69$ and reject otherwise.

Hypothesis testing



(b) Note that, $n = 36$ (sample size)

- Let X be "weight of a student"
- Let $\bar{X}$ be the sample mean. Then,

$$\mu_{\bar{X}} = \mu_X, \qquad \sigma^2_{\bar{X}} = \sigma^2_X / n = 3.6^2/36 = 0.36.$$

- Then,

$$\alpha = P(\bar{X} < 67 \text{ when } \mu = 68)$$
$$+ P(\bar{X} > 69 \text{ when } \mu = 68)$$

Hypothesis testing

- Let: $Y_0 \sim N(68, 0.36)$ $\quad Y_0 = 68 + 0.6Z$

$$\Rightarrow \alpha = P(Y_0 < 67) + P(Y_0 > 69)$$
$$= P(Z < -1.67) + P(Z > 1.67) = 2\,\Phi_Z(-1.67)$$
$$\approx 0.0950.$$