Lecture 18:
Continuous Random Variables - Part II

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Continuous random variables

- Continuous random variables can take any value within an interval (or continuous region).
- Example: Let X be the time to failure of a newly charged battery.
- Failure can be defined to be the moment at which the battery can no longer supply enough energy to operate a certain appliance.
- The r.v. X is continuous since it can take any positive value, i.e., any value in the interval $[0, \infty)$.

– Recall: $f_X(x) = \frac{d}{dx} F_X(x)$ (compute PDF from CDF)

$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$ (compute CDF from PDF)

Continuous random variables

- By definition of CDF,

$$\boxed{P(X \le x) = F_X(x) = \int_{-\infty}^{x} f_X(t)dt.}$$

- Similarly, the probability that X takes values in the interval $[a,b]$ or $(a,b]$ or $[a,b)$ or (a,b) is

$$\boxed{\begin{aligned} P(a \le X \le b) &= P(a < X \le b) = P(a \le X < b) = P(a < X < b) \\ &= F_X(b) - F_X(a) = \underbrace{\int_a^b f_X(x)dx}_{\text{area under the PDF curve from } a \text{ to } b.}. \longrightarrow F_X(x)\Big|_a^b \end{aligned}}$$

- Also,

$$\underbrace{P(X = x) = \int_x^x f_X(t)dt = 0}_{\text{for a continuous r.v. } X,\ P(X=x)=0 \text{ for all } x \in \mathbb{R}.}$$

area of a line segment is zero.

Continuous random variables

- Example: PDF and CDF

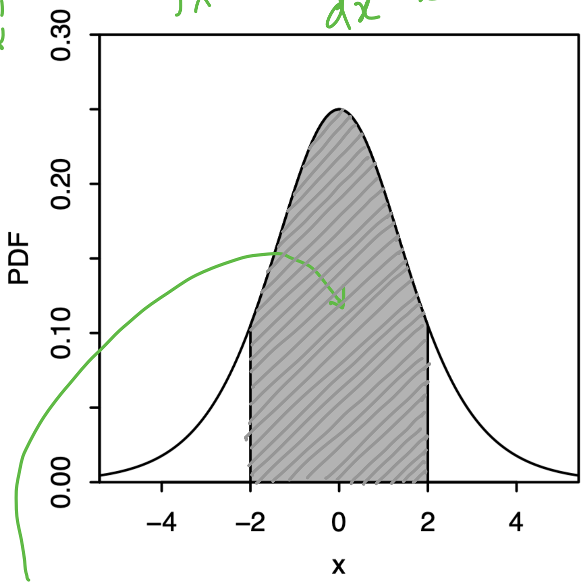
differentiate

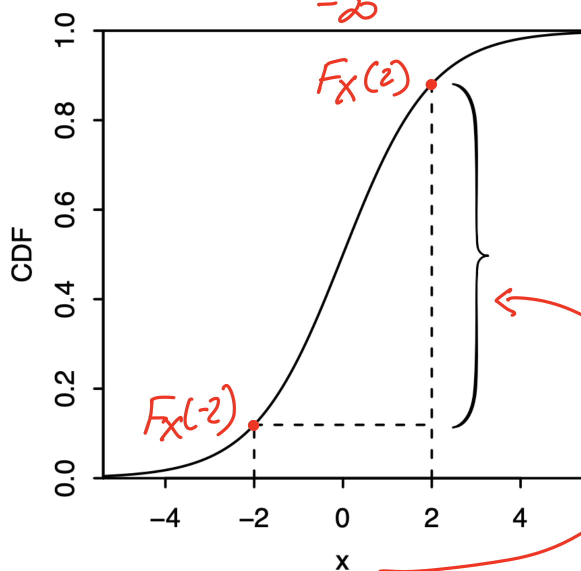
PDF $f_X(x)$ $\rightleftarrows$ CDF $F_X(x)$

integrate

$f_X(x) = \frac{d}{dx} F_X(x)$ $\quad$ $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$

Prob. of $\{-2 \leq X \leq 2\}$ is the area under the PDF curve from $x=-2$ to $x=2$.





Prob. of $\{-2 \leq X \leq 2\}$ is the difference $F_X(2) - F_X(-2)$

$$P(-2 \leq X \leq 2) = \int_{-2}^{2} f_X(x)\,dx$$

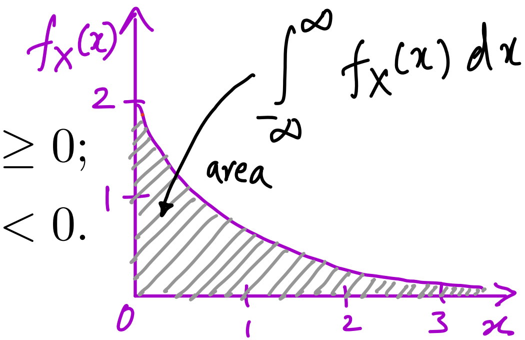
$$P(-2 \leq X \leq 2) = F_X(2) - F_X(-2)$$

Continuous random variables

That is, the area under the PDF curve from $-\infty$ to ∞ is 1.

- Recall: a valid PMF must be nonnegative and sum to 1.
- Similarly, a valid PDF must be nonnegative and integrate to 1.
- Example: Suppose that the battery failure time, measured in hours, has the PDF

$$f_X(x) = \begin{cases} f(x) = \frac{2}{(x+1)^3}, & x \geq 0; \\ 0, & x < 0. \end{cases}$$



- (a) Is this a valid PDF?

(1) Check "non-negativity": Yes, $f_X(x)$ is non-negative for all $x \in \mathbb{R}$

(2) Check "integrate to 1":

$$\int_{-\infty}^{\infty} f_X(x)\, dx = \int_0^{\infty} \frac{2}{(x+1)^3}\, dx = -\frac{1}{(x+1)^2}\Bigg|_{x=0}^{x=\infty}$$

$$= 0 - (-1) = 1$$

From (1) & (2) the PDF is valid.

Let $u = x+1$

$\Rightarrow du = dx$

$\Rightarrow \int_0^{\infty} \frac{2}{(x+1)^3}\, dx$

$= \int_1^{\infty} 2u^{-3}\, du$

$= 2 \cdot \frac{u^{-2}}{-2}\Big|_1^{\infty} = \frac{-1}{(x+1)^2}\Big|_0^{\infty}$

Continuous random variables

Prob. that battery lasts no more than 5 hours.

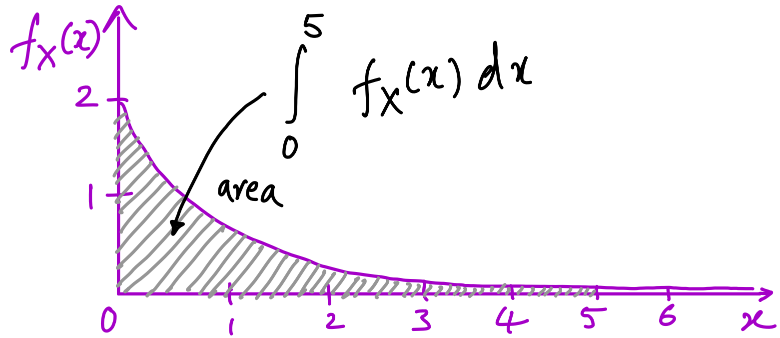
- $\underbrace{\text{(b) Find } P(X \in [0,5])}$ using the PDF.

$$P(X\in[0,5]) = P(0 \le X \le 5)$$

$$= \int_0^5 \frac{2}{(x+1)^3}\,dx$$

$$= \left.\frac{-1}{(x+1)^2}\right|_0^5$$

$$= -\frac{1}{36} - \frac{-1}{1} = \frac{35}{36}.$$

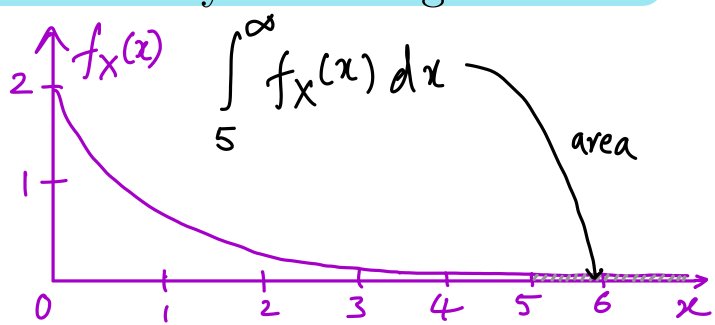


- (c) What is the probability that the battery lasts longer than 5 hours?

$$P(X>5) = 1 - P(X \le 5)$$

$$= 1 - P(0 \le X \le 5)$$

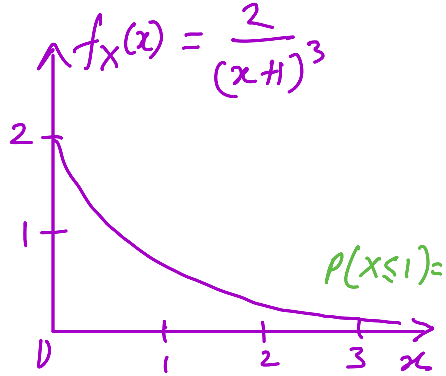
$$= 1 - 35/36 = 1/36$$

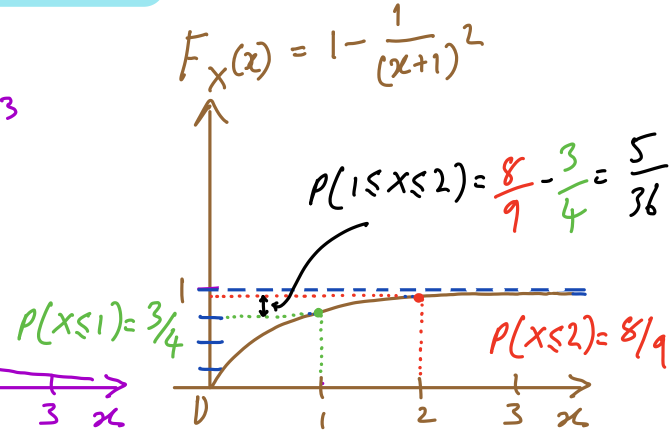


Continuous random variables

- (d) Find the CDF of the battery failure time.

$$F_X(x) = P(X \leq x)$$
$$= \int_0^x \frac{2}{(t+1)^3}\,dt$$
$$= \left.\frac{-1}{(t+1)^2}\right|_{t=0}^{t=x}$$
$$= \frac{-1}{(x+1)^2} - (-1) = 1 - \frac{1}{(x+1)^2}$$





- (e) Find $P(X \in [1, 2])$ using the CDF.

$$P(X \in [1,2]) = P(1 \leq X \leq 2)$$
$$= F_X(2) - F_X(1)$$
$$= 8/9 - 3/4 = 5/36$$

Continuous random variables

- The expected value or expectation or mean of a continuous r.v. with a probability density function $f_X(x)$ is

$$E(X) = \int_{-\infty}^{\infty} x f_X(x) dx.$$

- Example continued: (f) Find the expected battery failure time.

$$E(X) = \int_0^{\infty} \left[x \frac{2}{(x+1)^3} \right] dx$$

$$= \int_0^{\infty} \left[\frac{2}{(x+1)^2} - \frac{2}{(x+1)^3} \right] dx$$

$$= -\frac{2}{(x+1)} + \frac{1}{(x+1)^2} \Bigg|_{x=0}^{x=\infty}$$

$$= 0 - (-1) = 1.$$