Lecture 14:
Expectation and Variance - Part I

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Expectation

- Let X be a random variable describing the amount won in a game.
- A question is: How much does a person win "on average" in a game?
- For example, Let X be the amount won and is equal to the outcome of a fair dice. Then how much does a person win "on average"? or What is the "expected value" of winning amount?

$$\frac{1+2+3+4+5+6}{6} = \frac{21}{6} = 3.5.$$

- This is the same as the arithmetic mean of n real numbers $x_1, \ldots, x_n$ defined as
$$\frac{x_1 + \ldots + x_n}{n}.$$
- If the dice is biased (not fair) then how much a person wins "on average" in a game?

Expectation

- If X is a discrete r.v. taking on the possible values $x_1, x_2, \ldots$, then the expectation or expected value or mean of X, denoted by $E(X)$, is defined as

$$E(X) = \sum_i x_i P(X = x_i).$$

- Example: If the dice is biased (not fair) such that

$$P(X = 1) = \underline{0}, P(X = 2) = P(X = 3) = P(X = 4) = \underline{1/6},$$

$$P(X = 5) = P(X = 6) = \underline{1/4},$$

then how much a person wins "on average" in a game?

$$E(X) = 0 \cdot 1 + (2 + 3 + 4) \cdot \frac{1}{6} + (5 + 6) \cdot \frac{1}{4}$$

$$= \frac{3}{2} + \frac{11}{4} = \frac{17}{4} = 4.25.$$

Expectation

- Example: Find $E(X)$ if $X \sim \text{Bern}(p)$.

$$E(X) = 1 \cdot p + 0 \cdot (1-p) = p.$$

- Example: What is the expected value of a binomially distributed r.v.?

- Recall: $X \sim \text{Bin}(n, p)$ if

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k} \quad \text{for } k = 0, 1, \ldots, n$$

$$= 0 \quad \text{otherwise.}$$

- We will use the following identity:

$$k \binom{n}{k} = k \frac{n!}{k!\,(n-k)!} = \frac{\cancel{k} \cdot n \cdot (n-1)!}{\cancel{k}\,(k-1)!\,(n-k)!} = n \cdot \frac{(n-1)!}{(k-1)!\,(n-k)!} = n \binom{n-1}{k-1}$$

Expectation

- Now, $E(X) = \sum_{k=0}^{n} k P(X=k)$

$$= \sum_{k=1}^{n} k \binom{n}{k} p^{k} (1-p)^{n-k} \quad \left(\because k=0 \Rightarrow kP(X=k)=0\right)$$

$$= n \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k} (1-p)^{n-k} \quad \left(\because \text{using the identity}\right)$$

$$= np \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k}$$

$$= np \underbrace{\sum_{j=0}^{n-1} \underbrace{\binom{n-1}{j} p^{j} (1-p)^{n-1-j}}_{\text{PMF of Bin}(n-1,\,p) \text{ for } j = 0, 1, \ldots, n-1.}}_{\text{PMF always sums up to 1}}$$

$$= np.$$

Properties of expectation

Law of the unconscious statistician

- Expectation of a function of r.v.: If X is a discrete r.v. and $g : \mathbb{R} \to \mathbb{R}$ is a function, then

$$E(g(X)) = \sum_x g(x)P(X = x) = \sum_x g(x)p_X(x)$$

where the sum is taken over all possible values of X.

- How?: Recall that an r.v. X is a function $X : S \to \mathbb{R}$.

Hence,

$$E(g(X)) = \sum_{s \in S} g(X(s))P(\{s\})$$

$$= \sum_{x \in \mathbb{R}} \sum_{s : X(s) = x} g(X(s))P(\{s\})$$

$$= \sum_{x \in \mathbb{R}} g(x) \sum_{s : X(s) = x} P(\{s\})$$

$$= \sum_x g(x) P(X = x).$$

See the example in the last slide for understanding / demonstration.

Properties of expectation

- Similarly, for a function of two r.v.s, we have

$$E(g(X,Y)) = \sum_x \sum_y g(x,y) P(X = x, Y = y)$$
$$= \sum_x \sum_y g(x,y) p_{X,Y}(x,y).$$

Example:

- Suppose X has the following PMF

$$p_X(0) = .2, p_X(1) = .5, p_X(2) = .3.$$

Find $E[X^2]$.

Let $Y = g(X) = X^2$. Then,

$$E(X^2) = E(Y) = 0^2 \cdot (0.2) + 1^2 (0.5) + 2^2 (0.3)$$
$$= 1.7.$$

Properties of expectation

- **Linearity**: Expectation is linear, i.e., for r.v.s X and Y and a constant c,
  1. $E(X+Y) = E(X) + E(Y)$, Let $g(X,Y) = X+Y$.
  2. $E(cX) = cE(X)$.

$$E(X+Y) = E(g(X,Y)) = \sum_x \sum_y g(x,y)\, p_{X,Y}(x,y)$$

$$= \sum_x \sum_y (x+y)\, p_{X,Y}(x,y)$$

$$= \sum_x \sum_y x\, p_{X,Y}(x,y) + \sum_x \sum_y y\, p_{X,Y}(x,y)$$

$$= \sum_x x \sum_y p_{X,Y}(x,y) + \sum_y y \sum_x p_{X,Y}(x,y)$$

$$= \sum_x x\, p_X(x) + \sum_y y\, p_Y(y)$$

$$= E(X) + E(Y)$$

$$E(cX) = \sum_x cx\, p_X(x)$$

$$= c\sum_x x\, p_X(x)$$

$$= cE(X).$$

Properties of expectation

- Example: A construction firm has sent in bids for 3 jobs worth (in profits) 10, 20, and 40 (lakh) rupees. If its probabilities of winning the jobs are respectively .2, .8, and .3, what is the firm's expected total profit?

- Let X_i be the profit from job i, $i \in \{1, 2, 3\}$.
- Total profit $= X_1 + X_2 + X_3$.
- Expected total profit $= E(X_1 + X_2 + X_3)$

$$= E(X_1) + E(X_2) + E(X_3)$$

$$= 10 \cdot 0.2 + 20 \cdot 0.8 + 40 \cdot 0.3$$

$$= 2 + 16 + 12$$

$= 30$ Lakh rupees.

Example:

Let $S = \{a, b, c, d, e, f\}$

$X : S \to \mathbb{R}$ such that $X(a) = 0, X(b) = 0$
$X(c) = 1, X(d) = 2$
$X(e) = 3, X(f) = 3$

Let $P : S \to [0,1]$ such that $P(s) = \frac{1}{6}$, $s \in S$.

Then, $p_X(0) = \frac{1}{3} = p_X(3)$, $p_X(1) = p_X(2) = \frac{1}{6}$.

Let g be the function of X such that

$$g(0) = g(1) = 10$$
$$g(2) = g(3) = 20$$

Then,
$$E(g(X)) = \sum_{s \in S} g(X(s)) P(s)$$
$$= g(X(a))P(a) + g(X(b))P(b) + g(X(c))P(c) + g(X(d))P(d) + g(X(e))P(e) + g(X(f))P(f)$$
$$= g(0)P(a) + g(0)P(b) + g(1)P(c) + g(2)P(d) + g(3)P(e) + g(3)P(f)$$
$$= \sum_{x=0}^{3} \sum_{s : X(s) = x} g(x) P(s)$$
$$= \sum_{x=0}^{3} g(x) \underbrace{P(X = x)}_{\sum_{s : X(s) = x} P(s)}$$