Lecture 25:
Weak law of large numbers &
Central limit theorem
Part I

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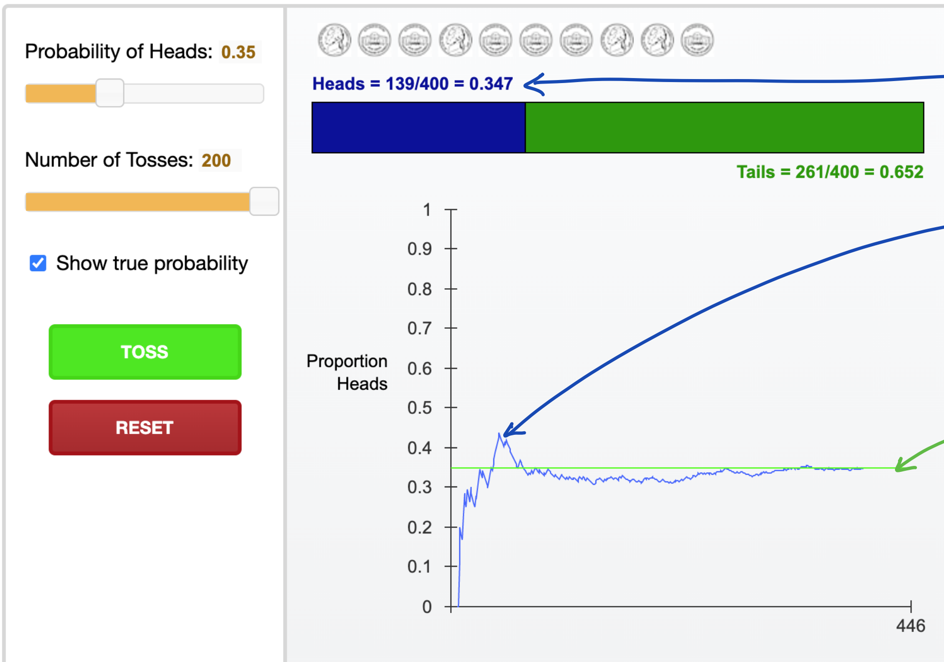
Weak law of large numbers

- The following statement seems correct by intuition: When the same experiment is repeated many times, the average value obtained over the experiments is close to the expectation of the experiment.
- Example: How do you know whether a given coin is fair?
- Another example: consider a Bernoulli random variable with probability of success (i.e., outcome is 1) p. If we conduct n Bernoulli trials then the average of the n outcomes will be very close to p (the expected value of the Bernoulli random variable) for large value of n.
- This phenomenon is called the weak law of large numbers.

Weak law of large numbers

- Example: If we toss a biased coin 400 times with probability of head = .35, it is very likely that approximately 140 times it will be head (i.e., success or outcome 1).



$\frac{\text{no. of Heads}}{\text{no. of trials}}$

average of the outcomes.

Expected value.

X is Bernoulli with $P(X=1) = .35$

$P(X=0) = .65$

$\Rightarrow E(X) = \sum x P(X=x)$

$= 0.35.$

Source: http://digitalfirst.bfwpub.com/stats_applet/stats_applet_10_prob.html

Visit this webpage to play with the applet.

Weak law of large numbers

- Let $X_1, X_2, \ldots,$ be a sequence of independent and identically distributed r.v.s, each having mean $E(X_i) = \mu$. Then, for any $\epsilon > 0$,

a random variable.

$$P\left(\left|\frac{X_1 + \ldots + X_n}{n} - \mu\right| > \epsilon\right) \to 0 \text{ as } n \to \infty.$$

Proof: Note that

$$E\left(\frac{X_1 + \ldots + X_n}{n}\right) = \frac{1}{n}\left[E(X_1) + \ldots + E(X_n)\right] = \frac{1}{n} \cdot n\mu = \mu$$

($\because$ linearity of exp., recall Lecture 14)

$$\mathrm{Var}\left(\frac{X_1 + \ldots + X_n}{n}\right) = \left(\frac{1}{n}\right)^2 \cdot \mathrm{Var}(X_1 + \ldots + X_n)$$

($\because \mathrm{Var}(cX) = c^2 \mathrm{Var}(X)$, recall Lecture 16)

Weak law of large numbers

$$= \frac{1}{n^2}\left[\mathrm{Var}(X_1) + \dots + \mathrm{Var}(X_n)\right]$$ ($\because X_1, \dots, X_n$ are independent recall Lecture 16)

$$= \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}.$$

Now, apply Chebyshev inequality:

$$P\left(\left|\frac{X_1+\dots+X_n}{n} - \mu\right| > \epsilon\right) \leq \frac{\mathrm{Var}\left(\frac{X_1+\dots+X_n}{n}\right)}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2}.$$

Note that as $n \to \infty$, the R.H.S. $\frac{\sigma^2}{n\epsilon^2} \to 0$. Then, as $n \to \infty$, $P\left(\left|\frac{X_1+\dots+X_n}{n} - \mu\right| > \epsilon\right) \to 0$ since $P(\cdot)$ is lower bounded by 0 ($\because P(\cdot)$ is a non-neg. function) and upper bounded by $\frac{\sigma^2}{n\epsilon^2}$.

A result on normal distributions

- Before we study the central limit theorem, let's look at the relation between a normal distribution $\mathcal{N}(\mu, \sigma^2)$ and the standard normal distribution $\mathcal{N}(0, 1)$.
- Recall that if we want to solve problems on the standard normal distribution then we refer to its CDF table.
- What to do if we want to solve a problem on a normal distribution $\mathcal{N}(\mu, \sigma^2)$?
- Relation: If $Z \sim \mathcal{N}(0, 1)$, then $X = \mu + \sigma Z$ have the normal distribution $\mathcal{N}(\mu, \sigma^2)$. (proof: next slide)
- An application of this relation: note that $\frac{X-\mu}{\sigma} \sim \mathcal{N}(0, 1)$. Hence, the CDF of X can be found from the CDF of the standard normally distributed Z.

A result on normal distributions

- Relation: If $Z \sim \mathcal{N}(0,1)$, then $X = \mu + \sigma Z$ have the normal distribution $\mathcal{N}(\mu, \sigma^2)$.

Proof:

$$F_X(x) = P(X \le x)$$
$$= P(\mu + \sigma Z \le x)$$
$$= P\left(Z \le \frac{x-\mu}{\sigma}\right)$$
$$= \Phi_Z\left(\frac{x-\mu}{\sigma}\right)$$

$$\Rightarrow \quad f_X(x) = \frac{d}{dx}\Phi_Z\left(\frac{x-\mu}{\sigma}\right)$$
$$= \frac{1}{\sigma}\phi_Z\left(\frac{x-\mu}{\sigma}\right)$$
$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

chain rule: composite function

$$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x)$$

$$\Rightarrow X \sim \mathcal{N}(\mu, \sigma^2)$$

A result on normal distributions

μ σ^2

- Example (application of the relation): Let $X \sim \underbrace{\mathcal{N}(-1, 4)}_{\text{normal but not standard normal.}}$. What is $P(|X| < 3)$?

- We know that if $X \sim N(\mu, \sigma^2)$ then $X = \mu + \sigma Z$ where $Z \sim N(0,1)$. $\Rightarrow Z = \dfrac{X-\mu}{\sigma} \Rightarrow Z = \dfrac{X+1}{2}$.

Now, $P(|X| < 3) = P(-3 < X < 3)$

$= P\left(\dfrac{-3+1}{2} < \dfrac{X+1}{2} < \dfrac{3+1}{2}\right)$

$= P(-1 < Z < 2)$ $\left(\begin{array}{c} Z \sim N(0,1) \\ \Rightarrow \text{use the CDF table} \end{array}\right)$

$= F_Z(2) - F_Z(-1)$

$= .9772 - 0.1587 = 0.8185$

Example: Let $X = \mu + \sigma Z$ & $Z \sim N(0,1)$.

Use this fact to show that $E(X) = \mu$ & $Var(X) = \sigma^2$.

Soln:

$$\begin{aligned} & E(X) \\ &= E(\mu + \sigma Z) \\ &= \underbrace{E(\mu) + \sigma E(Z)}_{\text{linearity of exp.}} \\ &= \mu + \sigma \cdot 0 \\ &= \mu \end{aligned}$$

recall lecture 14

$$\begin{aligned} & Var(X) \\ &= Var(\mu + \sigma Z) \\ &= Var(\sigma Z) \quad \because Var(X+c) = Var(X) \\ &= \sigma^2 \underbrace{Var(Z)}_{1} \quad \because Var(cX) = c^2 Var(X) \\ &= \sigma^2 \end{aligned}$$

recall lecture 16

recall lecture 16

- In Lecture 21, we discussed that if X & Y are independent then $\mathrm{corr}(X,Y)=0$.
- However, the converse is not true. That is, $\mathrm{corr}(X,Y)=0$ does not imply that X & Y are independent.

<u>Example</u>: Consider r.v.s.

X with $p_X(-1) = p_X(0) = p_X(1) = 1/3$.

$$Y = \begin{cases} 1 & \text{if } X=0 \\ 0 & \text{otherwise} \end{cases}$$

Y : - a function of X
- an indicator function
- $p_Y(1) = 1/3$, $p_Y(0) = 2/3$

Verify that X & Y are uncorrelated but dependent.

<u>Sol^n</u>: We need to find $p_{X,Y}(x,y)$.

$p_{Y|X=-1}$: $p_{Y|X=0}(0) = 1$, $p_{Y|X=0}(1) = 0$

$p_{Y|X=0}$: $p_{Y|X=0}(0) = 0$, $p_{Y|X=0}(1) = 1$

$p_{Y|X=1}$: $p_{Y|X=0}(0) = 1$, $p_{Y|X=0}(1) = 0$

Recall: $p_{X,Y}(x,y) = p_{Y|X=x}(y) \cdot \underbrace{p_X(x)}$

$p_X(0) = 1/3$

$p_X(1) = 1/3$

$p_X(-1) = 1/3$

$\Rightarrow$

Y \ X	-1	0	1
0	1/3	0	1/3
1	0	1/3	0

$$E(X) = 0\cdot\tfrac{1}{3} + 1\cdot\tfrac{1}{3} + (-1)\cdot\tfrac{1}{3} = 0$$

$$E(Y) = \tfrac{1}{3}$$

$$E(XY) = \sum_{(x,y)} x\cdot y \cdot p_{X,Y}(x,y) = 0$$

$$\Rightarrow \quad cov(X,Y) = E(XY) - E(X)E(Y)$$
$$= 0$$

$$\Rightarrow \quad corr(X,Y) = 0.$$

- X & Y are not independent:

Approach 1: Note that $p_{Y|X=x} \neq p_Y$.

$\Rightarrow$ X & Y are not independent

Approach 2: Note that

$$p_{X,Y}(0,0) = 0,$$
$$p_X(0) = \tfrac{1}{3}, \quad p_Y(0) = \tfrac{2}{3}$$

$$\Rightarrow \quad p_{X,Y}(0,0) \neq p_X(0)\cdot p_Y(0)$$

$$\Rightarrow \quad p_{X,Y}(x,y) \neq p_X(x)\cdot p_Y(y) \text{ for all } (x,y)\in\mathbb{R}^2.$$

$\Rightarrow$ X & Y are not independent