Lecture 10: Discrete Random Variables - Part IV & Mini Quiz - I

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Discrete random variables

- If X is a random variable, then X^2, e^X, and $\sin(X)$ are also random variables.
- Definition (Function of an r.v.): For an experiment with sample space S, an r.v. X, and a function $g : \mathbb{R} \to \mathbb{R}$, $g(X)$ is the r.v. that maps s to $g(X(s))$ for all $s \in S$.
- Example: $g(x) = |\sqrt{x}|$ (here the notation $|a|$ means the absolute value.)

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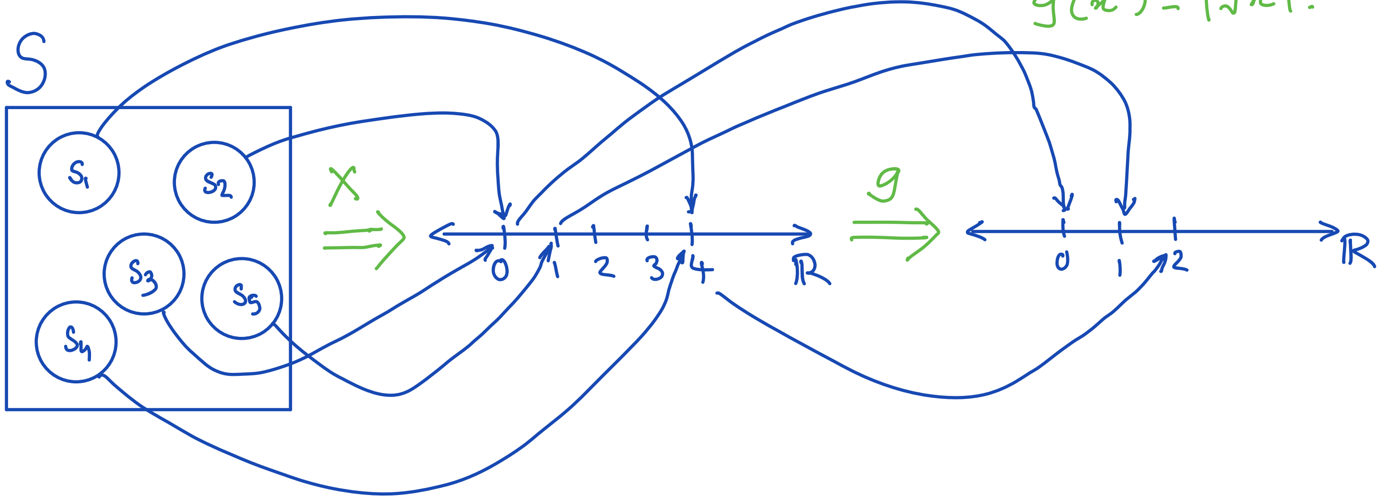
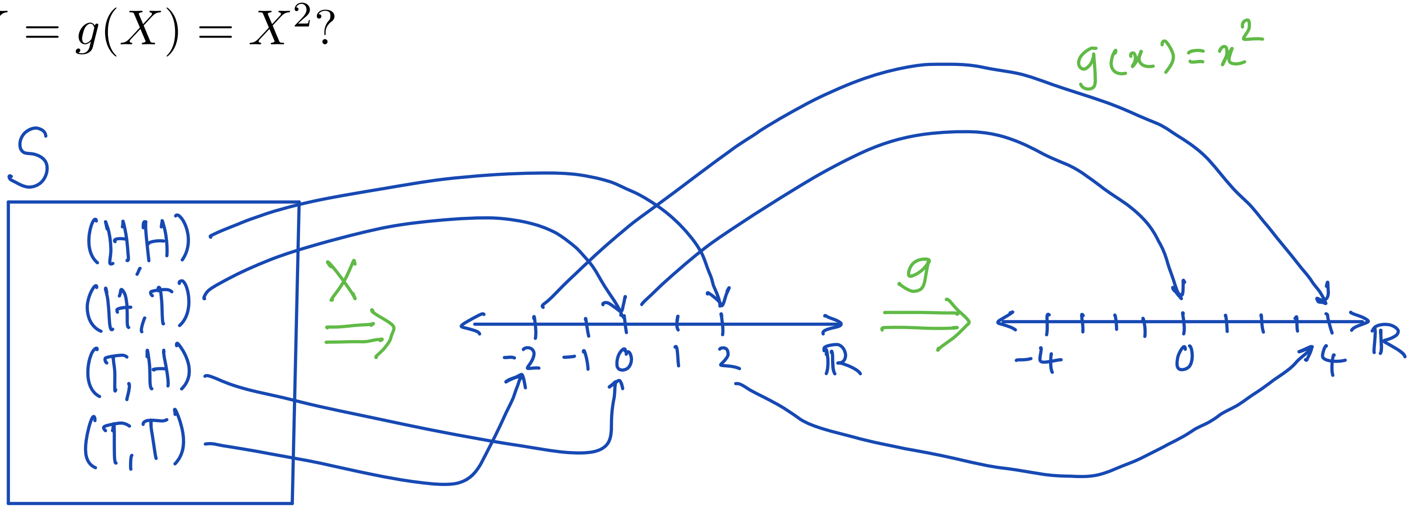


Discrete random variables

- Example: Two fair coins are tossed. Let X denote the following r.v.: whenever a head occurs Rs 1 is gained and whenever a tail occurs the same amount is lost. What is the PMF for $Y = g(X) = X^2$?



$$p_Y(0) = P(g(X)=0) = P(X=0) = P\big((H,T) \text{ or } (T,H)\big) = \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2}.$$

$$p_Y(4) = P(g(X)=4) = P\big(\{X=-2\} \cup \{X=2\}\big) = P\big((T,T) \text{ or } (H,H)\big)$$
$$= \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2}.$$

Discrete random variables

- Let X be an r.v. with PMF $p_X(x)$ and $Y = g(X)$. If g is a one-to-one function what is the PMF of Y?

– Let $p_i = P(X = x_i)$ and $y_i = g(x_i)$. Then, $\underbrace{P(Y = y_i) = P(X = x_i)}_{\because\ g \text{ is a one-to-one } f^n.} = p_i$

- Let X be an r.v. with PMF $p_X(x)$ and $Y = g(X)$. If g is a function (not necessarily one-to-one) what is the PMF of Y?

$$p_Y(y) = P(Y = y) = P(g(X) = y)$$

$$= P\Big(\bigcup_{\underbrace{x : g(x) = y}} \{X = x\}\Big)$$

union of all x s.t. $g(x) = y$.

$$= \sum_{\underbrace{x : g(x) = y}} P(X = x)$$

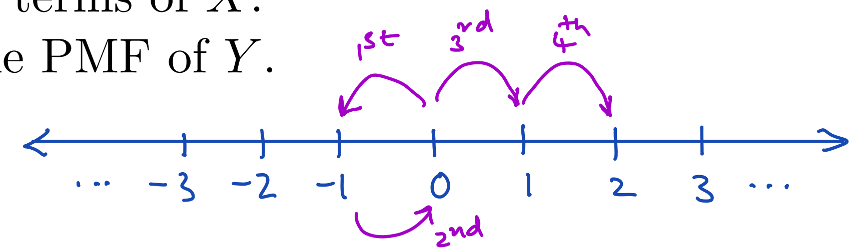
sum of all x s.t. $g(x) = y$.

$$= \sum_{x : g(x) = y} p_X(x)$$

Discrete random variables

Example :

- A particle moves n steps on a number line. The particle starts at 0, and at each step it moves 1 unit to the right or to the left, with equal probabilities. Assume all steps are independent.
  (a) Let X be the number of steps taken to the right. Find the PMF of X.
  (b) Let Y denote the location of the particle after n steps. Write Y in terms of X.
  (c) Find the PMF of Y.



- The support of X is $\{0, 1, 2, \ldots, n\}$.
- What is the PMF of X? : $\text{Bin}(n, \frac{1}{2})$.
  - Consider Bernoulli trial at each step (n Bernoulli trials)
  - Consider moving to right as "success" and left as "failure".

Discrete random variables

- $P(X=i)$ is the probability of i successes in n Bernoulli trials. $\Rightarrow X \sim Bin(n, 1/2)$.

- When $X=i$, what is the position of the particle?
  ↳ particle moved right i times and left $n-i$ times.

  $\Rightarrow$ Final position $= i-(n-i)$
  $= 2i-n$ for $i \in \{0,1,2,\ldots,n\}$.

- Now, let Y be the r.v.: final position of the particle.
  Then, $Y = 2X-n$, support of $Y = \{-n,\ldots,0,\ldots,n\}$.

$$P(Y=j) = P(2X-n=j) = P\left(X = \frac{n+j}{2}\right)$$

$$= \binom{n}{\frac{n+j}{2}} \left(\frac{1}{2}\right)^n \text{ for } -n \leq j \leq n$$

and 0 otherwise.

$(\because X \sim Bin(n, 1/2))$

Discrete random variables

- Let D be the particle's distance from the origin after n steps. Find the PMF of D.

Note that, $D = g(Y) = |Y|$

support of $D = \{0, 1, \dots, n\}$

$$p_D(0) = P(D=0) = P(Y=0) = \binom{n}{n/2}\left(\tfrac{1}{2}\right)^n.$$

$$p_D(k) = P(D=k) = P(\{Y=k\} \text{ or } \{Y=-k\}),\ k \in \{1, \dots, n\}.$$

$$= P(Y=-k) + P(Y=k)$$

$$= \left[\binom{n}{\frac{n-k}{2}} + \binom{n}{\frac{n+k}{2}}\right]\left(\tfrac{1}{2}\right)^n$$

for $k \in \{1, \dots, n\}$.

Mini-Quiz (9 points, 17 mins)

- If $P(E) = 0.1$ and $P(F) = 0.2$, then find the best lower and upper bounds for $P(E \cup F)$, i.e., find biggest l and smallest u such that $l \leq P(E \cup F) \leq u$. [1 point]
- Define partition of a set S. [1.5 points]
- Bag A contains 3 red balls and 7 blue balls. Bag B contains 8 red and 4 blue balls. Bag C contains 5 red and 11 blue balls. A bag is chosen at random, and then a ball is chosen at random from that bag. Calculate the probabilities:
  (a) A red ball is chosen. [1 point] (b) A red ball from bag B is chosen. [1 point] (c) If it is known that a red ball is chosen, what is the probability that it comes from bag A? [1.5 points]
- A fair coin is tossed three times. A player wins Rs. 1 if the first toss is a head, but loses Rs. 1 if the first toss is a tail. Similarly, the player wins Rs. 2 if the second toss is a head, but loses Rs. 2 if the second toss is a tail, and wins or loses Rs. 3 according to the result of the third toss. Let the random variable X be the total winnings after the three tosses. Find its PMF [2 points] and plot it [1 point].

$$P(A\cup B) = P(A)+P(B)-P(A\cap B) \leq P(A)+P(B)$$

① $$\underbrace{0.2 \leq P(E\cup F)}_{\text{If } A\subseteq B,\ P(A)\leq P(B).} \leq 0.3$$

② The subsets $A_1, \dots, A_n$ of S forms a partion of S if $\underbrace{A_i \cap A_j = \phi}_{\text{i.e., } A_1,\dots,A_n \text{ are disjoint.}}$, for all $i\neq j$ and $\bigcup_{i=1}^{n} A_i = S$.

③ Let R be the event that a red ball is chosen.
Let B_i be the event that bag i is chosen.

(a) $$P(R) = \sum_{i\in\{A,B,C\}} P(R|B_i)\cdot P(B_i)$$
$$= \tfrac{1}{3}\cdot \tfrac{3}{10} + \tfrac{1}{3}\cdot \tfrac{8}{12} + \tfrac{1}{3}\cdot \tfrac{5}{16} \approx 0.42.$$

(b) $$P(R\cap B_B) = P(B_B)\cdot P(R|B_B)$$
$$= \tfrac{1}{3}\cdot \tfrac{8}{12} = \tfrac{2}{9}.$$

(c) $$P(B_A|R) = \frac{P(B_A\cap R)}{P(R)} = \frac{P(B_A)\cdot P(R|B_A)}{P(R)} = \frac{\tfrac{1}{3}\cdot \tfrac{3}{10}}{0.42} \approx 0.23$$

④

outcome	amount won (X)
HHH	6
HHT	0
HTH	2
THH	4
HTT	-4
THT	-2
TTH	0
TTT	-6

$$p_X(-6) = p_X(-4) = p_X(-2) = P(X=2) = P(X=4)$$
$$= P(X=6) = \tfrac{1}{8}.$$
$$p_X(0) = P(\text{HHT or TTH}) = \tfrac{1}{4}.$$