Lecture 23:
Continuous Random Variables - Part VI

Satyajit Thakor
IIT Mandi

Continuous random variables

- For two continuous r.v.s, the PDF of the marginal distribution of X is
$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)dy.$$

- Example: For the "mining" problem, find the PDF for zinc content of the ore.

- Recall : X : zinc content Y : iron content

$$f_{X,Y}(x,y) = \frac{39}{400} - \frac{17(x-1)^2}{50} - \frac{(y-25)^2}{10000}$$

for $0.5 \leq x \leq 1$, $20 \leq y \leq 35$ and 0 elsewhere.

- We want to find $f_X(x)$.

Continuous random variables

$$f_X(x) = \int_{y=-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$= \int_{20}^{35} \left(\frac{39}{400} - \frac{17(x-1)^2}{50} - \frac{(y-25)^2}{10000} \right) dy$$

$$= \frac{39y}{400} - \frac{17(x-1)^2 y}{50} - \frac{(y-25)^3}{3 \cdot 10000} \Bigg|_{y=20}^{y=35}$$

$$= \frac{39 \cdot 15}{400} - \frac{17 \cdot 15\,(x-1)^2}{50} - \frac{375}{10000}$$

$$= \frac{14625 - 375}{10000} - \frac{51(x-1)^2}{10}$$

$$= \frac{57}{40} - \frac{51(x-1)^2}{10}, \quad 0.5 \le x \le 1.5$$

and 0 elsewhere.

Continuous random variables

- If two continuous r.v.s X and Y are jointly distributed, then the conditional distribution of random variable X conditional on the event $Y = y$ has a PDF

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)}, \qquad f_Y(y) > 0.$$

- Note that, in the above expression y is fixed and $f_{X|Y=y}$ is a function of the parameter x.
- Example: Suppose that an ore has zinc content of $X = .55$. What is the conditional PDF of the iron content Y given that its zinc content is $X = .55$?

- we want to find $f_{Y|X=0.55}(y) = \dfrac{f_{X,Y}(0.55, y)}{f_X(0.55)}$

Continuous random variables

- $f_X(0.55) = \frac{57}{40} - \frac{51(0.55-1)^2}{10} = 0.39225$

- Hence,

$$f_{Y|X=0.55}(y) = \frac{f_{X,Y}(0.55, y)}{0.39225}$$

$$= \frac{39}{400 \cdot 0.39225} - \frac{17 \cdot (0.55-1)^2}{50 \cdot 0.39225} - \frac{(y-25)^2}{10000 \cdot 0.39225}$$

$$= 0.073 - \frac{(y-25)^2}{3922.5}$$

for $20 \leq y \leq 35$ and 0 elsewhere.

Continuous random variables

- Continuous r.v.s X and Y are independent if
  $$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$
  for all values of x and y.
- If two r.v.s are independent, then
  $$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{f_X(x) f_Y(y)}{f_Y(y)} = f_X(x).$$
- That is, the conditional distributions do not depend upon the value conditioned upon, and they are equal to the marginal distributions.

Continuous random variables

- Example: Suppose that X and Y have the PDF

$$f_{X,Y}(x,y) = 6xy^2$$

for $0 \leq x \leq 1$ and $0 \leq y \leq 1$ and $f_{X,Y}(x,y) = 0$ elsewhere. Are X and Y independent?

- Lets find f_X, f_Y:

$$f_X(x) = \int_{y=0}^{1} 6xy^2\,dy = \frac{6xy^3}{3}\Big|_{y=0}^{1} = 2x$$

$$f_Y(y) = \int_{x=0}^{1} 6xy^2\,dx = \frac{6x^2y^2}{2}\Big|_{x=0}^{1} = 3y^2$$

- check: $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y) = 6xy^2$.

- Hence, X and Y are independent.

Continuous random variables

- Find the covariance of X and Y with the PDF

$$f_{X,Y}(x,y) = 6xy^2$$

for $0 \leq x \leq 1$ and $0 \leq y \leq 1$ and $f_{X,Y}(x,y) = 0$ elsewhere.

- Sol^n 1: from the previous example, $\text{cov}(X,Y) = 0$ since X and Y are independent.

- Sol^n 2: $\text{Cov}(X,Y) = E(XY) - E(X) \cdot E(Y)$

$$E(XY) = \int_{x=0}^{1}\int_{y=0}^{1} x \cdot y \cdot 6xy^2 \, dy \, dx$$

$$= \int_{x=0}^{1} \frac{6x^2y^4}{4} \Big|_{y=0}^{y=1} dx$$

Continuous random variables

$$= \int_{x=0}^{1} \frac{6x^2}{4}\, dx = \frac{6x^3}{12}\Big|_{x=0}^{x=1} = \frac{1}{2}.$$

$$- E(X) = \int_{x=0}^{1} x f_X(x)\, dx = \int_{x=0}^{1} x \cdot 2x\, dx = \frac{2x^3}{3}\Big|_{x=0}^{1} = \frac{2}{3}.$$

$$- E(Y) = \int_{y=0}^{1} y f_Y(y)\, dy = \int_{y=0}^{1} y\, 3y^2\, dy = \frac{3y^4}{4}\Big|_{y=0}^{y=1} = \frac{3}{4}.$$

$$- \mathrm{Cov}(X,Y) = E(XY) - E(X) - E(Y)$$

$$= \frac{1}{2} - \frac{2}{3} \cdot \frac{3}{4} = 0.$$