Lecture 12:
Discrete Random Variables - Part VI

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Independent r.v.s

- The random variables that are independent and have the same distribution are called independent and identically distributed, or i.i.d.
- There are four possibilities for two random variables:
  1. Independent and identically distributed
  2. Independent and not identically distributed
  3. Dependent and identically distributed
  4. Dependent and not identically distributed
- Example 1 (independent and identically distributed): Let X be the result of a dice roll, and let Y be the result of a second, independent dice roll. Then X and Y are i.i.d.

  same dice

  fair dice : support : $\{1,2,3,4,5,6\}$ $P(X=x)=\frac{1}{6}, \forall x \in \{1,\dots,6\}$

  $P(Y=y)=\frac{1}{6}, \forall y \in \{1,\dots,6\}$
- If $X \sim \text{Bin}(n,p)$, then we can write $X = X_1 + \ldots + X_n$ where X_i's are i.i.d. $\text{Bern}(p)$.

Independent r.v.s

fair dice : support : $\{1,2,3,4,5,6\}$ $P(X=x)=\frac{1}{6}, \forall x\in\{1,\dots,6\}$

fair coin : support : $\{0,1\}$ $P(Y=y)=\frac{1}{2}, \forall y\in\{0,1\}$.

- Example 2 (independent and not identically distributed): Let X be the result of a dice roll, and let Y be the number of heads in one coin flip (1 for H and 0 for T). Then X and Y provide no information about each other, and X and Y do not have the same distribution.

- Example 3 (dependent and identically distributed): Let X be the number of H's in n independent fair coin tosses, and let Y be the number of T's in those same n tosses. Then X and Y are both distributed $\text{Bin}(n, 1/2)$, but they are highly dependent: if we know X, then we know Y perfectly.

e.g., $P(X=0)=(1-p)^n$, $P(Y=0)=p^n$

$\Rightarrow P(X=0)\cdot P(Y=0) = p^n(1-p)^n$

However, $P(X=0,Y=0)=0$ $\Rightarrow P(X=0)\cdot P(Y=0) \neq P(X=0,Y=0)$

$\Rightarrow$ X and Y are not independent

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- Example 4 (dependent and not identically distributed): Let X be the number on a fair dice and let Y be the indicator random variable whether the number is odd. Then X and Y are dependent, and X and Y do not have the same distribution.

- X and Y are dependent: In fact, Y is a function of X

$$\underbrace{P(Y=\text{"odd"}\mid X=2)=0 \neq P(Y=\text{"odd"})} = P(\{1\}\text{ or }\{3\}\text{ or }\{5\}) = \frac{3}{6} = \frac{1}{2}$$

$$P(Y=y\mid X=x) = \frac{P(Y=y, X=x)}{P(X=x)} = \frac{P(Y=y)\cdot P(X=x)}{P(X=x)} = P(Y=y) \Rightarrow X \,\&\, Y \text{ are not independent.}$$

- X and Y do not have the same distribution: X is uniformly distributed over the support $\{1, 2, \ldots, 6\}$ where as $Y \sim \text{Bern}(1/2)$.

follows if X & Y are independent.

Note: conditional PMF will be discussed in Lecture 13 in details.

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Example :

- In a roll of two fair dice, Let X be the number on the first dice and Y be the number on the second dice. Consider functions $g(X,Y) = X + Y$ and $h(X,Y) = X - Y$ and denote them as r.v.s G and H.
  (a) Find p_G, p_H (directly).
  (b) Find $p_{G,H}$ and write in the table form. Then find marginal PMFs p_G, p_H from the joint PMF and verify your solution with the solution of (a).
  (c) Are $G = g(X,Y)$ and $H = h(X,Y)$ independent?

(a): - To find P_G and P_H, we need to know the support of G, H and no. of ways each element can occur.

- The support of $X+Y$ is $\{2,3,\dots,12\}$
  The support of $X-Y$ is $\{-5,-4,\dots,4,5\}$

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$G =$ $X+Y$:	value (g):	(x, y):	no. of ways the value can occur:	$P_G(g)$:
	2	(1,1)	1	1/36
	3	(1,2), (2,1)	2	2/36
	4	(1,3), (3,1), (2,2)	3	3/36
	5	(1,4), (4,1), (2,3), (3,2)	4	4/36
	6	(1,5), (5,1), (2,4), (4,2), (3,3)	5	5/36
	7	(1,6), (6,1), (2,5), (5,2), (3,4), (4,3)	6	6/36
	8	(2,6), (6,2), (3,5), (5,3), (4,4)	5	5/36
	9	(3,6), (6,3), (4,5), (5,4)	4	4/36
	10	(4,6), (6,4), (5,5)	3	3/36
	11	(5,6), (6,5)	2	2/36
	12	(6,6)	1	1/36

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$H = X - Y$:

Value (h):	(x,y):	no. of ways the value can occur:	$p_H(h)$:
-5	$(1,6)$	1	$1/36$
-4	$(1,5),(2,6)$	2	$2/36$
-3	$(1,4),(2,5),(3,6)$	3	$3/36$
-2	$(1,3),(2,4),(3,5),(4,6)$	4	$4/36$
-1	$(1,2),(2,3),(3,4),(4,5),(5,6)$	5	$5/3$
0	$(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)$	6	$6/36$
1	$(2,1),(3,2),(4,3),(5,4),(6,5)$	5	$5/36$
2	$(3,1),(4,2),(5,3),$	4	$4/36$
3	$(4,1),(5,2),(6,3)$	3	$3/36$
4	$(5,1),(6,2)$	2	$2/36$
5	$(6,1)$	1	$1/36$

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Homework: Verify all the values in the table. (Use the reasoning similar to one in red text)

(b):

- Note that $g = 2$ if $(x, y) = (1, 1)$.
- But if $(x, y) = (1, 1)$ then $h = 0$.
- Hence $P(G = 2, H = 0)$ $= P((X, Y) = (1, 1))$ $= 1/36$.
- Note that $P(G = 2, H = i) = 0$ for all $i \neq 0$.
- With this reasoning we have the first column ($g = 2$).

$H = h$ \ $G = g$	2	3	4	5	6	7	8	9	10	11	12	p_H
−5	0	0	0	0	0	1/36	0	0	0	0	0	1/36
−4	0	0	0	0	1/36	0	1/36	0	0	0	0	2/36
−3	0	0	0	1/36	0	1/36	0	1/36	0	0	0	3/36
−2	0	0	1/36	0	1/36	0	1/36	0	1/36	0	0	4/36
−1	0	1/36	0	1/36	0	1/36	0	1/36	0	1/36	0	5/36
0	1/36	0	1/36	0	1/36	0	1/36	0	1/36	0	1/36	6/36
1	0	1/36	0	1/36	0	1/36	0	1/36	0	1/36	0	5/36
2	0	0	1/36	0	1/36	0	1/36	0	1/36	0	0	4/36
3	0	0	0	1/36	0	1/36	0	1/36	0	0	0	3/36
4	0	0	0	0	1/36	0	1/36	0	0	0	0	2/36
5	0	0	0	0	0	1/36	0	0	0	0	0	1/36
p_G	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36	

$p_{G,H}$

- Similarly, we find $p_{GH}(g, h)$ for other values of g and h.

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(c):

- By def^n: If $P(X+Y=x+y, X-Y=x-y) \neq P(X+Y=x+y)\cdot P(X-Y=x-y)$ for some (x,y) then $X+Y$ and $X-Y$ are not independent.

- Intiutively, we can see dependence between $X+Y$ and $X-Y$: For example, if $\underbrace{X+Y=12}_{\text{if } (x,y)=(6,6)}$ then $X-Y$ must be 0.

- Now, note that: $P(X+Y=12, X-Y=1)=0$

$$P(\underbrace{X+Y=12}_{\text{if } (x,y)=(6,6)})\cdot P(\underbrace{X-Y=1}_{(x,y)=(2,1) \text{ or } (3,2) \text{ or } (4,3) \text{ or } (5,4) \text{ or } (6,5).}) = \frac{1}{36}\cdot\frac{5}{36}$$

- Hence, X and Y are not independent.

Example: joint and marginal distributions

- Recall: The marginal PMF of X is $p_X(x) = P(X = x) = \sum_y P(X = x, Y = y)$.
- Similarly, the marginal CDF of X is $F_X(x) = P(X \leq x) = \sum_y P(X \leq x, Y = y)$.
- Example (Two Bernoulli r.v.s X and Y): Find $F_X(0)$.

$Y=$ $X=$	1	0
1	0.05	0.2
0	0.03	0.72

$$
\begin{aligned}
F_X(0) &= P(X \leq 0) \\
&= P(X = 0) \\
&= P(X=0, Y=0) + P(X=0, Y=1) \\
&= 0.72 + 0.03 \\
&= 0.75.
\end{aligned}
$$