Lecture 35:
Hypothesis Testing - Part IV

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Recall

- Example: Consider the null hypothesis that the average weight of students in a college is 68 kgs against the alternative hypothesis that it is unequal to 68:

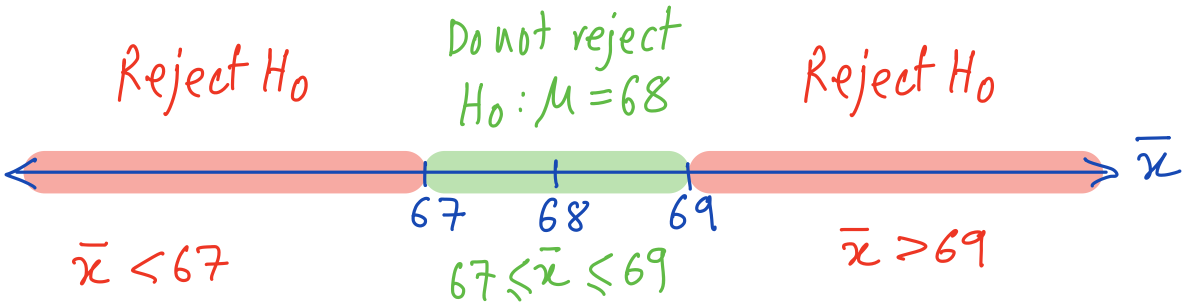
$$\left.\begin{aligned} H_0 &: \mu = 68, \\ H_1 &: \mu \neq 68. \end{aligned}\right\} \text{ two-tailed test}$$

- Assume that the weight is a normally distributed with $\sigma = 3.6$ and the sample size is 36.
  (a) Define a suitable critical region
  (b) Find the probability of Type I error
  (b) Find the probability of Type II error

(a) We define the critical region as: $\bar{x} < 67$, $\bar{x} > 69$.

- That is, do not reject if $67 \leq \bar{x} \leq 69$ and reject otherwise.

Recall



(b) Note that, $n = 36$ (sample size)

- Let X be " weight of a student "
- Let $\bar{X}$ be the sample mean. Then,

$$\mu_{\bar{X}} = \mu_X, \qquad \sigma^2_{\bar{X}} = \sigma^2_X / n = 3.6^2/36 = 0.36.$$

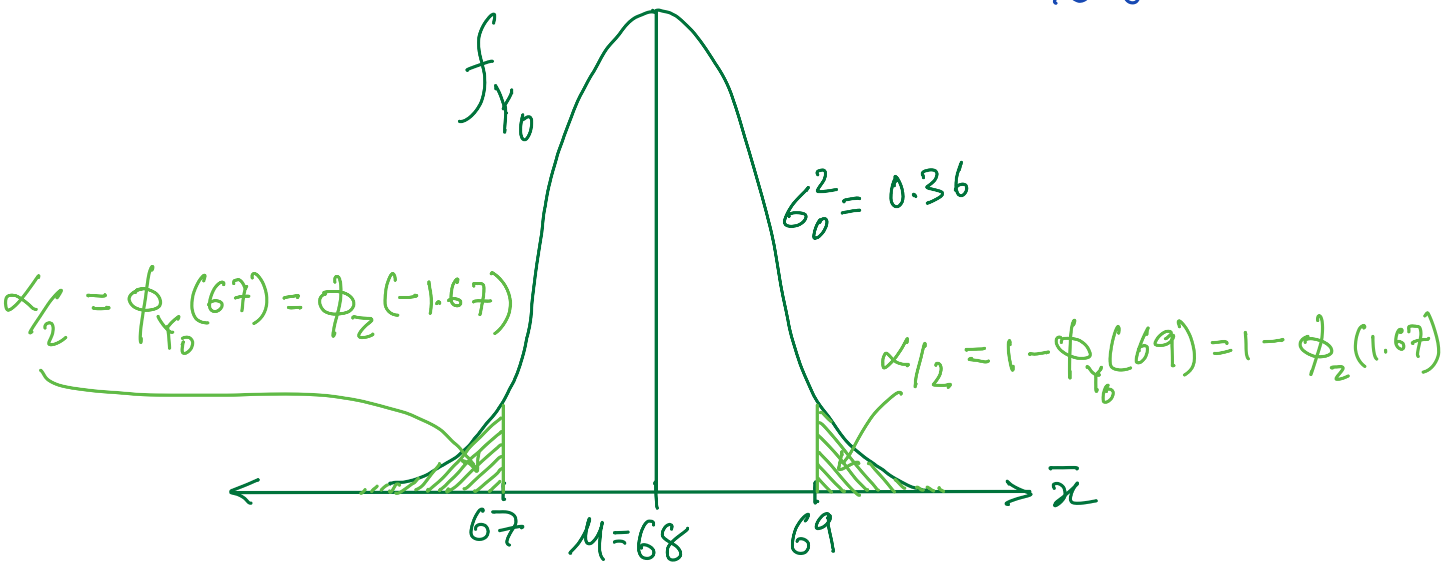
- Then, $\alpha = P(\bar{X} < 67 \text{ when } \mu = 68)$

$$+ P(\bar{X} > 69 \text{ when } \mu = 68)$$

Recall

— Let: $Y_0 \sim N(68, 0.36)$ $\quad Y_0 = 68 + 0.6Z$

$$\Rightarrow \alpha = P(Y_0 < 67) + P(Y_0 > 69)$$
$$= P(Z < -1.67) + P(Z > 1.67) = 2\,\Phi_Z(-1.67)$$
$$\approx 0.0950.$$



Hypothesis testing

(c) To compute β, let the alternative hypothesis be $\mu = 70$.

- Then, $\beta = P(67 \leq \bar{X} \leq 69 \text{ when } \mu = 70)$

- Let: $Y_1 \sim N(70, 0.36)$, $Y_1 = 70 + 0.6Z$.

$$\Rightarrow \beta = P(67 \leq Y_1 \leq 69)$$

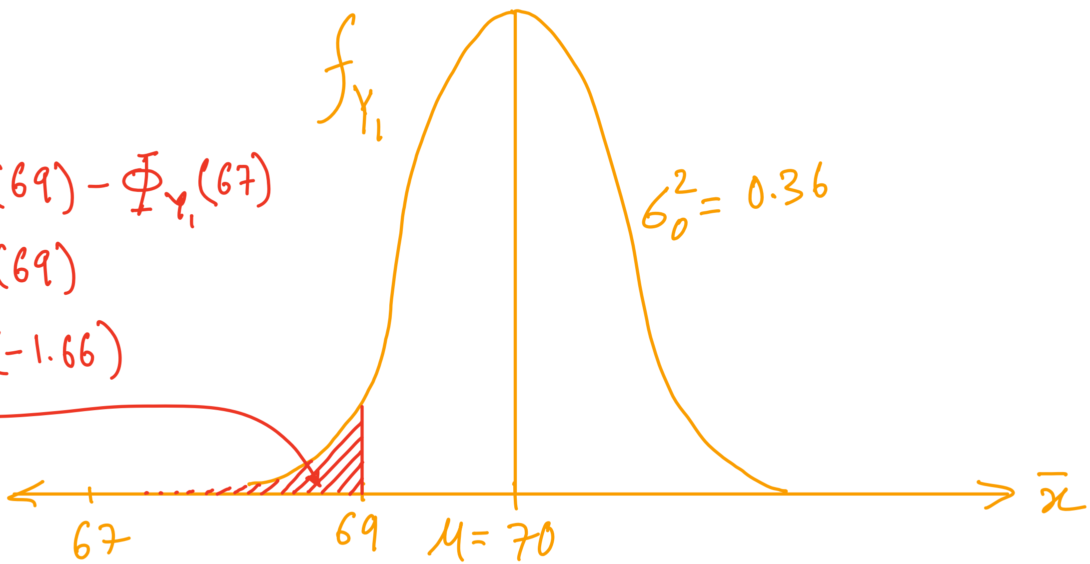
$$= P\left(\frac{-3}{.6} \leq Z \leq \frac{-1}{.6}\right)$$

$$= \Phi_Z(-1.66) - \Phi_Z(-5)$$

$$\approx 0.0485 - 0$$

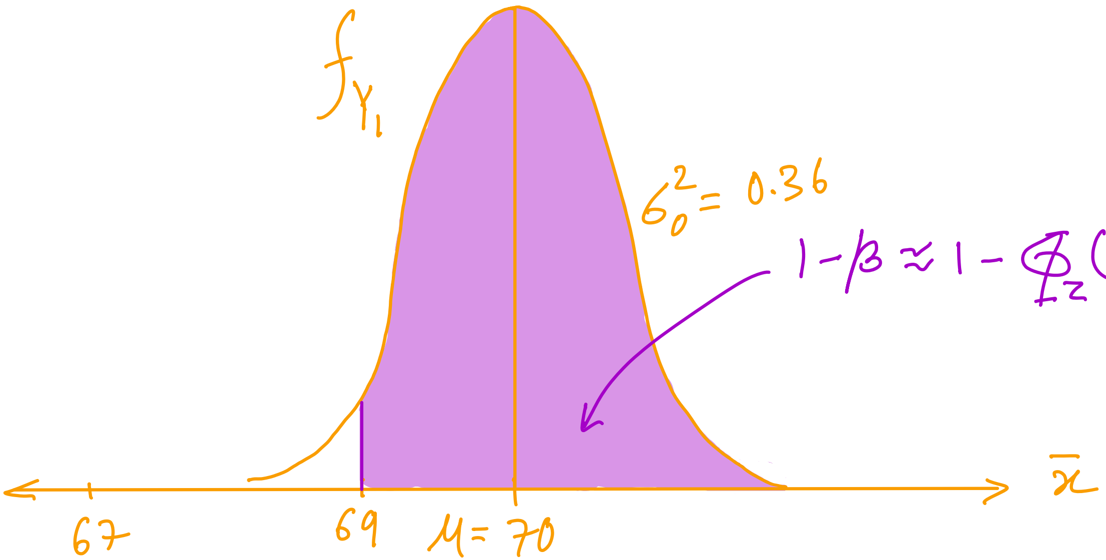
Hypothesis testing

$$\beta = \Phi_{Y_1}(69) - \Phi_{Y_1}(67)$$
$$\approx \Phi_{Y_1}(69)$$
$$= \Phi_Z(-1.66)$$



Hypothesis testing

- The power of a test is the probability of rejecting H_0 given that a specific alternative is true, i.e., $1-\beta$.



Hypothesis testing

- In general, one wants to control the probability of committing type I or type II error or both.
- It is customary to choose $\alpha = .05$, or in some tests, $\alpha = .01$.
- p-value is an important parameter to decide whether to reject the null hypothesis.

- p-values are the probability of obtaining an effect at least as extreme as the one in your sample data, assuming the truth of the null hypothesis.

- Question: If the p-value for a given sample is very low, shall we accept or reject H_0?

- Answer: reject! (why?: let's understand by example)

Hypothesis testing

- Example: Suppose the observed value of $\bar{X}$ is 69.3.

- Then,

$$
\begin{aligned}
p\text{-value} &= P(\bar{X} \geq 69.3 \text{ when } \mu = 68) \\
&\quad + P(\bar{X} \leq 66.7 \text{ when } \mu = 68) \\
&= P(Y_0 \geq 69.3) + P(Y_0 \leq 66.7) \\
&= 2P(Y_0 \leq 66.7) \quad (\because f_{Y_0} \text{ is symmetric around } 68) \\
&= 2P\left(Z \leq -\frac{1.3}{.6}\right) \\
&= 2\Phi_Z(-2.16) \\
&= 2 \cdot 0.0154 = 0.0308.
\end{aligned}
$$

Hypothesis testing

- That is, there is 3% chance that we get the sample mean 69.8 or worst when $\mu = 68$.

- Hence, it is more likely that $\mu \neq 68$ and so we should reject H_0.

- In practice, $\underbrace{H_0 \text{ is rejected if p-value} < 0.05.}_{\text{A typical practice but not always.}}$