Assignment 1

Submission Deadline: 24 February, 2022

1. A committee of 3 members is to be formed consisting of one representative each from labor, management, and the public. If there are 3 possible representatives from labor, 2 from management, and 4 from the public, determine how many different committees can be formed. **[3]**

2. In how many ways can 5 differently colored marbles be arranged in a row? **[2]**

3. How many 4-digit numbers can be formed with the 10 digits $0, 1, 2, 3, ..., 9$ if (a) repetitions are allowed, (b) repetitions are not allowed, (c) the last digit must be zero and repetitions are not allowed? **[3]**

4. In the game of poker 5 cards are drawn from a pack of 52 well-shuffled cards. How many elements does the sample space contain? **[1]**

5. A box contains 7 identical marbles, except for color, of which 4 are red and 3 are green. Two marbles are selected at random (a) one by one with replacement; (b) one by one without replacement; (c) two marbles together.

   Compute the numbers of sample points in all these cases. **[3]**

6. Four different mathematics books, six different physics books, and two different chemistry books are to be arranged on a shelf. How many different arrangements are possible if (a) the books in each particular subject must all stand together, (b) only the mathematics books must stand together? **[5]**

7. Prove the identity: **[2]**

$$\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j} \binom{n}{k-j}.$$

Assignment 2

Submission Deadline: 5 March, 2022

1. Assume that n people are seated in a random manner in a row of n seats. What is the probability that two particular people X and Y will be seated next to each other? **[2]**

2. A deck of 52 cards contains four aces. If the cards are shuffled and distributed in a random manner into four sets containing 13 cards each, what is the probability that all four aces will be in the same set? **[3]**

3. A box contains 24 balls, of which 2 are red and remaining white. If a person selects 10 balls at random, without replacement, what is the probability that both red balls will be selected? **[2]**

4. Find the probability that n people ($n \leq 365$) selected at random will have n different birthdays. **[3]**

5. Mr Narayanan is a civil engineer with Kerala government. He is asked to design an over bridge (sky way). The chance that his design is going to be faulty is 60% and the chance that his design will be correct is 40 %. The chance of the over bridge collapsing if the design is faulty is 90 %; otherwise, due to other causes, the chance of the over bridge collapsing is 20%. What is the chance that an over bridge built by Mr Narayanan will collapse? **[3]**

6. Which of the following events has the highest probability?

   A: At least one 6 appears when 6 fair dice are rolled.

   B: At least two 6's appear when 12 fair dice are rolled.

   C: At least three 6's appear when 18 fair dice are **[3]**

7. Suppose that a box contains one blue card and four red cards, which are labelled X, Y, Z, and W. Suppose also that two of these five cards are selected at random, without replacement.

   (i) If it is known that card X has been selected, what is the probability that both cards are red?

   (ii) If it is known that at least one red card has been selected, what is the probability that both cards are red? **[3]**

8. A bag contains one marble which is either green or blue, with equal probabilities. A green marble is put in the bag (so there are 2 marbles now), and then a random marble is taken out. The marble taken out is green. What is the probability that the remaining marble is also green? **[2]**

9. Suppose cards numbered one through ten are placed in a hat, mixed up, and then one of the cards is drawn. If we are told that the number on the drawn card is at least five, then what is the conditional probability that it is ten? **[2]**

10. A card player is dealt a 13-card hand from a well-shuffled, standard deck of cards. What is the probability that the hand has no cards of at least one suit? **[3]**

Assignment 3

IC252 - IIT Mandi

Submission Deadline: 15 March, 2022

1. An office has four copying machines, and the random variable X measures how many of them are in use at a particular moment in time. Suppose that $P(X = 0) = 0.08$, $P(X = 1) = 0.11$, $P(X = 2) = 0.27$, and $P(X = 3) = 0.33$.
   (a) What is $P(X = 4)$? **[1]**
   (b) Draw a line graph of the probability mass function. **[1]**
   (c) Construct and plot the cumulative distribution function. **[1]**
2. If $X \sim \text{Pois}(3.2)$, calculate:
   (a) $P(X = 1)$ **[1]**
   (b) $P(X \leq 3)$ **[1]**
   (c) $P(X \geq 6)$ **[1]**
3. Two cards are drawn at random from a pack of cards with replacement. Let the random variable X be the number of cards drawn from the heart suit.
   (a) Construct the probability mass function. **[2]**
   (b) Construct the cumulative distribution function. **[1]**
   (c) What is the most likely value of the random variable X? **[1]**
4. Two fair dice, one red and one blue, are rolled. A score is calculated to be twice the value of the blue die if the red die has an even value, and to be the value of the red die minus the value of the blue die if the red die has an odd value. Construct and plot the probability mass function and the cumulative distribution function of the score. **[3]**
5. A communication system consists of n components, each of which will, independently, function with probability p. The total system will be able to operate effectively if at least one-half of its components function. For what values of p is a 5-component system more likely to operate effectively than a 3-component system? **[3]**
6. Suppose that a random variable X can take the value 1, 2, or any other positive integer.
   (a) Is it possible that $P(X = i) = c/i^2$ for some value of the constant c? **[1.5]**
   (b) Is it possible that $P(X = i) = c/i$ for some value of c? **[1.5]**

Assignment 4

IC252 - IIT Mandi

Submission Deadline: 27 March, 2022

1. Let X denote the number of times a certain numerical control machine will malfunction: 1, 2, or 3 times on any given day. Let Y denote the number of times a technician is called on an emergency call. Their joint probability distribution is given as

Table 1: $p_{X,Y}(x, y)$

		x		
		1	2	3
y	1	0.05	0.05	0.10
	3	0.05	0.10	0.35
	5	0.00	0.20	0.10

(a) Evaluate the marginal distribution of X. **[1]**

(b) Evaluate the marginal distribution of Y. **[1]**

(c) Find $P(Y = 3|X = 2)$. **[1]**

2. If the joint probability distribution of X and Y is given by

$$p_{X,Y}(x, y) = \frac{x + y}{30},$$

for $x = 0, 1, 2, 3; y = 0, 1, 2$, find

(a) $P(X \leq 2, Y = 1)$; **[2]**

(b) $P(X > 2, Y \leq 1)$; **[2]**

(c) $P(X > Y)$; **[2]**

(d) $P(X + Y = 4)$. **[2]**

3. Each student in a certain BTech program was classified according to her year in university (1st, 2nd, 3rd, 4th year) and according to the number of times that she had visited a certain museum (never, once, or more than once). The proportions of students in the various classifications are given in the following table:

	Never	Once	More than once
1st year	0.08	0.10	0.04
2nd year	0.04	0.10	0.04
3rd year	0.04	0.20	0.09
4th year	0.02	0.15	0.10

(a) If a student selected at random from the BTech program is “3rd year”, what is the probability that she has never visited the museum? **[2]**

(b) If a student selected at random from the BTech program has visited the museum more than once, what is the probability that she is “4th year”? **[2]**

4. A fair coin is tossed four times, and the random variable X is the number of heads in the first three tosses and the random variable Y is the number of heads in the last three tosses.

   (a) What is the joint probability mass function of X and Y? **[2]**

   (b) What are the marginal probability mass functions of X and Y? **[2]**

   (c) Are the random variables X and Y independent? **[1.5]**

5. Two cards are drawn without replacement from a pack of cards, and the random variable X measures the number of hearts drawn and the random variable Y measures the number of clubs drawn.

   (a) What is the joint probability mass function of X and Y ? **[2]**

   (b) What are the marginal probability mass functions of X and Y ? **[2]**

   (c) Are the random variables X and Y independent? **[1.5]**

Assignment 5

IC252 - IIT Mandi

Submission Deadline: 5 April, 2022

1. (Continuation of Problem 4, Assignment 4) A fair coin is tossed four times, and the random variable X is the number of heads in the first three tosses and the random variable Y is the number of heads in the last three tosses.

   What are the expectations and variances of the random variables X and Y ? **[2]**

2. (Continuation of Problem 5, Assignment 4) Two cards are drawn without replacement from a pack of cards, and the random variable X measures the number of hearts drawn and the random variable Y measures the number of clubs drawn.

   What are the expectations and variances of the random variables X and Y ? **[2]**

3. Show that the variance of a binomially distributed r.v. $X \sim \mathrm{Bin}(n, p)$ is **[3]**

$$\mathrm{Var}(X) = np(1-p).$$

4. Suppose that the probabilities are 0.4, 0.3, 0.2, and 0.1, respectively, that IIT Mandi has 0, 1, 2, or 3 power failures in any given month. Find the mean and variance of the random variable X representing the number of power failures at IIT Mandi. **[2]**

5. Let X and Y be independent r.v.s with $\mathrm{Var}(X) = \mathrm{Var}(Y) = 3$. Find $\mathrm{Var}(2X - 3Y + 1)$. **[2]**

6. For a Poisson distributed r.v., find:

   (a) Mean **[2.5]**

   (b) Variance **[2.5]**

Assignment 6

IC252 - IIT Mandi

Submission Deadline: 15 April, 2022

1. Suppose that X is a continuous r.v. whose PDF is given by

$$f_X(x) = \begin{cases} c(4x - 2x^2), & 0 \leq x \leq 2; \\ 0, & \text{elsewhere.} \end{cases}$$

   (a) What is the value of the constant c (for f_X to be a valid PDF)? **[2]**

   (b) Find $P(X > 1)$. **[1]**

2. Milk containers have label printed "2 liters". But, the PDF of the amount of milk deposited in a milk container by a dairy factory is

$$f_X(x) = \begin{cases} 40.976 - 16x - 30e^{-x}, & 1.95 \leq x \leq 2.20; \\ 0, & \text{elsewhere.} \end{cases}$$

   (a) Is f_X a valid PDF? **[2]**

   (b) What is the probability that a container produced by the dairy factory is underweight? **[2]**

3. Consider a random variable measuring the following quantities. In each case state with reasons whether you think it more appropriate to define the random variable as discrete or as continuous.

   (a) A person's height **[1]**

   (b) A student's course grade **[1]**

   (c) The thickness of a metal plate **[1]**

4. A random variable X takes values between 4 and 6 with a probability density function

$$f_X(x) = \begin{cases} \frac{1}{x \log_e(1.5)}, & 4 \leq x \leq 6; \\ 0 & \text{elsewhere.} \end{cases}$$

   (a) Make a plot of the PDF (you may use some programming tools to plot functions). **[1]**

   (b) Check that the total area under the probability density function is equal to 1. **[2]**

   (c) What is $P(4.5 \leq X \leq 5.5)$? **[2]**

   (d) Find the CDF and plot it (you may use some programming tools to plot functions). **[2]**

   (e) What is the expected value of this random variable? **[1.5]**

(f) What is the median of this random variable? **[1.5]**

(g) What is the variance of this random variable? **[2]**

(h) What is the standard deviation of this random variable? **[1]**

5. (a) For $X \sim N(\mu, \sigma^2)$, verify that, its PDF is symmetric around the mean, i.e., **[1]**

$$f_X(\mu - x) = f_X(\mu + x).$$

(b) For $X \sim N(0, 1)$, verify that **[1.5]**

$$\Phi_X(-x) = 1 - \Phi_X(x).$$

6. **Optional (advanced):** For $X \sim N(\mu, \sigma^2)$, i.e., X with PDF

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

verify that

(a) the mean is μ and **[2.5]**

(b) the variance is σ^2. **[2.5]**

Assignment 7

IC252 - IIT Mandi

Submission Deadline: 29 April, 2022

1. Suppose that you are waiting for a friend to call you and that the time you wait in minutes has an exponential distribution with parameter $\lambda = 0.1$.

   (a) What is the expectation of your waiting time? **[1]**

   (b) What is the probability that you will wait longer than 10 minutes? **[1]**

   (c) What is the probability that you will wait less than 5 minutes? **[1]**

2. A new battery supposedly with a charge of 1.5 volts actually has a voltage with a uniform distribution between 1.43 and 1.60 volts.

   (a) What is the expectation of the voltage? **[1]**

   (b) What is the standard deviation of the voltage? **[1]**

   (c) What is the CDF of the voltage? **[1]**

   (d) What is the probability that a battery has a voltage less than 1.48 volts? **[1]**

   (e) If a box contains 50 batteries, what are the expectation and variance of the number of batteries in the box with a voltage less than 1.5 volts? **[2]**

3. Suppose that $Z \sim N(0, 1)$, i.e., Z has the standard normal distribution. Find:

   (a) $P(Z \leq -0.77)$ **[1]**

   (b) $P(Z \geq 0.32)$ **[1]**

   (c) $P(-0.82 \leq Z \leq 1.80)$ **[1]**

   (e) $P(|Z| \geq 0.91)$ **[1]**

   (f) The value of x for which $P(Z \leq x) = 0.23$ **[1]**

   (g) The value of x for which $P(Z \geq x) = 0.51$ **[1]**

   (h) The value of x for which $P(|Z| \geq x) = 0.42$ **[1]**

   For this problem, use the table of CDF for finding numerical solutions:

   `https://www.mathsisfun.com/data/standard-normal-distribution-table.html`

   OR

   `https://www.math.arizona.edu/~rsims/ma464/standardnormaltable.pdf`

4. For the air conditioner maintenance problem discussed in Lecture 21,

   (a) Suppose that a location has only one air conditioner that needs servicing. What is the conditional PMF of the service time required? **[2]**

(b) Suppose that a location requires a service time of two hours. What is the conditional PMF of the number of air conditioner units serviced? **[2]**

(c) find the correlation between X and Y. **[2.5]**

5. Consider the mining problem discussed in Lectures 22-23.

(a) Show that $P(0.8 \leq X \leq 1, 25 \leq Y \leq 30) = 0.092$. **[2.5]**

(b) Show that the iron content has an expected value of 27.36 and a standard deviation of 4.27. **[2.5]**

6. Suppose that two continuous r.v.s X and Y have the joint PDF

$$f_{X,Y}(x, y) = c(e^{x+y} + e^{2x-y})$$

for $1 \leq x \leq 2$ and $0 \leq y \leq 3$, and $f_{X,Y}(x, y) = 0$ elsewhere.

(a) What is the value of c? **[2.5]**

(b) What is $P(1.5 \leq X \leq 2, 1 \leq Y \leq 2)$? **[2.5]**

(c) Construct the marginal PDFs $f_X(x)$ and $f_Y(y)$. **[2.5]**

(d) Are the r.v.s X and Y independent? **[2.5]**

(e) If $Y = 0$, what is the conditional PDF of X? **[2.5]**

Assignment 8

IC252 - IIT Mandi

Submission Deadline: 6 May, 2022

1. The time taken to serve a customer at a fast-food restaurant has a mean of 75.0 seconds and a standard deviation of 7.3 seconds. Use Chebyshev inequality to calculate the time interval that has 89% probability of containing a particular service time. **[2]**

2. A machine produces iron bars whose lengths have a mean of 110.8 cm and a standard deviation of 0.5 cm. Use Chebyshev's inequality to obtain a lower bound on the probability that an iron bar chosen at random has a length between 109.55 cm and 112.05 cm. **[2]**

3. A random variable X has a mean $\mu_X = 8$, a variance $\sigma_X^2 = 9$, and an unknown probability distribution. Find
   (a) $P(-4 \leq X \leq 20)$, **[1]**
   (b) $P(|X - 8| \geq 6)$. **[1]**

4. Homework in Slide 6, Lecture 24: Show that **[1.5]**
$$\int_0^\infty ye^{-\frac{y^2}{2}}dy = 1.$$

5. Cattle are vaccinated to help prevent the spread of diseases among them. Suppose that a particular vaccine has a probability of 0.0005 of causing a serious adverse reaction when administered to an animal. Suppose that the vaccine is to be administered to 500,000 head of cattle. Find the approximate value of the probability that at most 300 animal suffer serious adverse reaction? **[2.5]**

6. Calculate the following probabilities both exactly and by using a normal approximation:
   (a) $P(X \geq 8)$ where $X \sim B(10, 0.7)$ **[2.5]**
   (b) $P(2 \leq X \leq 7)$ where $X \sim B(15, 0.3)$ **[2.5]**

7. A multiple-choice test consists of a series of questions, each with four possible answers. If there are 60 questions, estimate the probability that a student who guesses blindly at each question will get at least 30 questions right. **[2.5]**

Assignment 9

IC252 - IIT Mandi

Submission Deadline: 14 May, 2022

1. The data set for milk container weights (in liters) for 50 randomly chosen containers is as follows:

$$1.958, 1.951, 2.107, 2.092, 1.955, 2.162, 2.168, 2.134, 1.971,$$
$$2.072, 2.049, 2.017, 2.117, 1.977, 2.034, 2.062, 2.110, 1.974,$$
$$1.992, 2.018, 2.135, 2.107, 2.084, 2.169, 2.085, 2.018, 1.977,$$
$$2.116, 1.988, 2.066, 2.126, 2.167, 1.969, 2.198, 2.078, 2.119,$$
$$2.088, 2.172, 2.133, 2.112, 2.066, 2.128, 2.142, 2.042, 2.050,$$
$$2.102, 2.000, 2.188, 1.960, 2.128.$$

   Find the sample mean and sample variance of weight of the container. **[2]**

2. Related to Slide 5, Lecture 26: Show that, if $X_1, \ldots, X_n$ is a sequence of independent identically distributed random variables with a mean μ and a variance σ^2, then $X_1 + \ldots + X_n$ has the mean $n\mu$ and a variance $n\sigma^2$. **[2]**

3. Suppose that $E(X_1) = \mu$, $\mathrm{Var}(X_1) = 10$, $E(X_2) = \mu$, and $\mathrm{Var}(X_2) = 15$, and consider the point estimates (of μ)

$$(1)\ \hat{\Theta} = \frac{X_1}{2} + \frac{X_2}{2}$$
$$(2)\ \hat{\Theta} = \frac{X_1}{6} + \frac{X_2}{3} + 9$$

   (a) Calculate the bias of each point estimate. Is any one of them unbiased? **[2.5]**

   (b) Calculate the variance of each point estimate. Which one has the smallest variance? **[2.5]**

4. Suppose that $X \sim B(n, p)$ (or $\mathrm{Bin}(n, p)$). Then, show that

$$\hat{\Theta} = \frac{X}{n}$$

   is an unbiased point estimate of the success probability p. **[2]**

5. If $X_1, \ldots, X_n$ is a sample of observations from a probability distribution with a variance σ_X^2, then show that the sample variance $\hat{\Theta} = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n}$ has the bias of

$$-\frac{\sigma_X^2}{n}$$

   for point estimate of the population variance $\theta = \sigma_X^2$. **[2]**

Assignment 10

IC252 - IIT Mandi

Submission Deadline: 23 May, 2022

1. Continuation of the example in Slides 8-9 of Lecture 31: The average zinc concentration recovered from a sample of measurements taken in 36 different locations in a river is found to be 2.6 grams per milliliter. Find the 99% confidence intervals for the mean zinc concentration in the river. Assume that the population standard deviation is 0.3 gram per milliliter. **[2]**

2. An electrical firm manufactures light bulbs that have a length of life that is approximately normally distributed with a standard deviation of 40 hours. If a sample of 30 bulbs has an average life of 780 hours, find a 96% confidence interval for the population mean of all bulbs produced by this firm. **[2]**

3. Suppose that, in a biomedical study, 10 rats are injected with cancer cells and then given a cancer drug that is designed to increase their survival rate. The survival times, in months, are 14, 17, 27, 18, 12, 8, 22, 13, 19, and 12. Assume that the exponential distribution applies. Find the maximum likelihood estimate of the mean survival time. **[4]**

   Hints: (1) The numbers 14, 17, 27, 18, 12, 8, 22, 13, 19, and 12 represent a sample $x_1, \ldots, x_{10}$. (2) the survival time is exponentially distributed $\text{Exp}(\lambda)$. (3) First find the general expression for the maximum likelihood estimate of λ and then plug-in the values of the sample to obtain the numerical estimate.

4. The standard deviation of test scores on a certain achievement test is 11.3. If a random sample of 81 students had a sample mean score of 74.6, find a 90% confidence interval estimate for the average score of all students. **[2]**

Assignment 11

IC252 - IIT Mandi

1. A sociologist is concerned about the effectiveness of a training course designed to get more drivers to use seat belts in automobiles.

   (a) What hypothesis is she testing if she commits a type I error by erroneously concluding that the training course is ineffective? **[1]**

   (b) What hypothesis is she testing if she commits a type II error by erroneously concluding that the training course is effective? **[1]**

2. A fabric manufacturer believes that the proportion of orders for raw material arriving late is $p = 0.6$. If a random sample of 10 orders shows that 3 or fewer arrived late, the hypothesis that $p = 0.6$ should be rejected in favor of the alternative $p < 0.6$. Use the binomial distribution.

   (a) Find the probability of committing a type I error if the true proportion is $p = 0.6$. **[2]**

   (b) Find the probability of committing a type II error for the alternatives $p = 0.3, p = 0.4$, and $p = 0.5$. **[3]**

3. An electrical firm manufactures light bulbs that have a lifetime that is approximately normally distributed with a mean of 800 hours and a standard deviation of 40 hours. Test the hypothesis that $\mu = 800$ hours against the alternative, $\mu \neq 800$ hours, if a random sample of 30 bulbs has an average life of 788 hours. Use a p-value for hypothesis testing. **[2]**

4. A dry cleaning establishment claims that a new spot remover will remove more than 70% of the spots to which it is applied. To check this claim, the spot remover will be used on 12 spots chosen at random. If fewer than 11 of the spots are removed, we shall not reject the null hypothesis that $p = 0.7$; otherwise, we conclude that $p > 0.7$.

   (a) Evaluate α, assuming that $p = 0.7$. **[2]**

   (b) Evaluate β for the alternative $p = 0.9$. **[2]**

5. Let X be exponentially distributed with parameter λ. Suppose that we wish to test the hypotheses

$$H_0 : \lambda \geq 1,$$
$$H_1 : \lambda < 1.$$

   Consider the test procedure that rejects H_0 if $X \geq 1$. Determine the power ofthe test. **[2]**

6. Suppose that the proportion p of defective items in a large population of items is unknown, and that it is desired to test the following hypothesis:

$$H_0 : p = 0.2,$$
$$H_1 : p \neq 0.2.$$

Consider a random sample of 20 items. Let Y be the number of defective items in the sample, and consider a test procedure such that the critical region contains all the outcomes for which either $Y \geq 7$ or $Y \leq 1$.

Determine the value of the power of the test at the points $p = 0$, 0.4, 0.8 and 1. **[4]**

7. A biologist claims that mice with an average life span of 32 months will live to be about 40 months old when 40% of the calories in their diet are replaced by vitamins and protein. Is there any reason to believe that $\mu < 40$ if 64 mice that are placed on this diet have an average life of 38 months with a standard deviation of 5.8 months? Use a p-value for making the conclusion. **[2]**