Lecture 6: Conditional Probability - Part III & Independence

Satyajit Thakor
IIT Mandi

Conditional probability

- Example: A patient named Fred is tested for a disease called conditionitis, a medical condition that afflicts 1% of the population. The test result is positive, i.e., the test claims that Fred has the disease. Let D be the event that Fred has the disease and T be the event that he tests positive. Suppose that the test is "95% accurate", i.e., $P(T|D) = 0.95$ and $P(T^c|D^c) = 0.95$. The quantity $P(T|D)$ is known as the sensitivity or true positive rate of the test, and $P(T^c|D^c)$ is known as the specificity or true negative rate.
Find the conditional probability that Fred has conditionitis, given the evidence provided by the test result.

– We want to find $P(D|T)$.

– Use Baye's rule and LOTP.

Conditional probability

$$P(D|T) = \frac{P(T|D)\cdot P(D)}{P(T)}$$

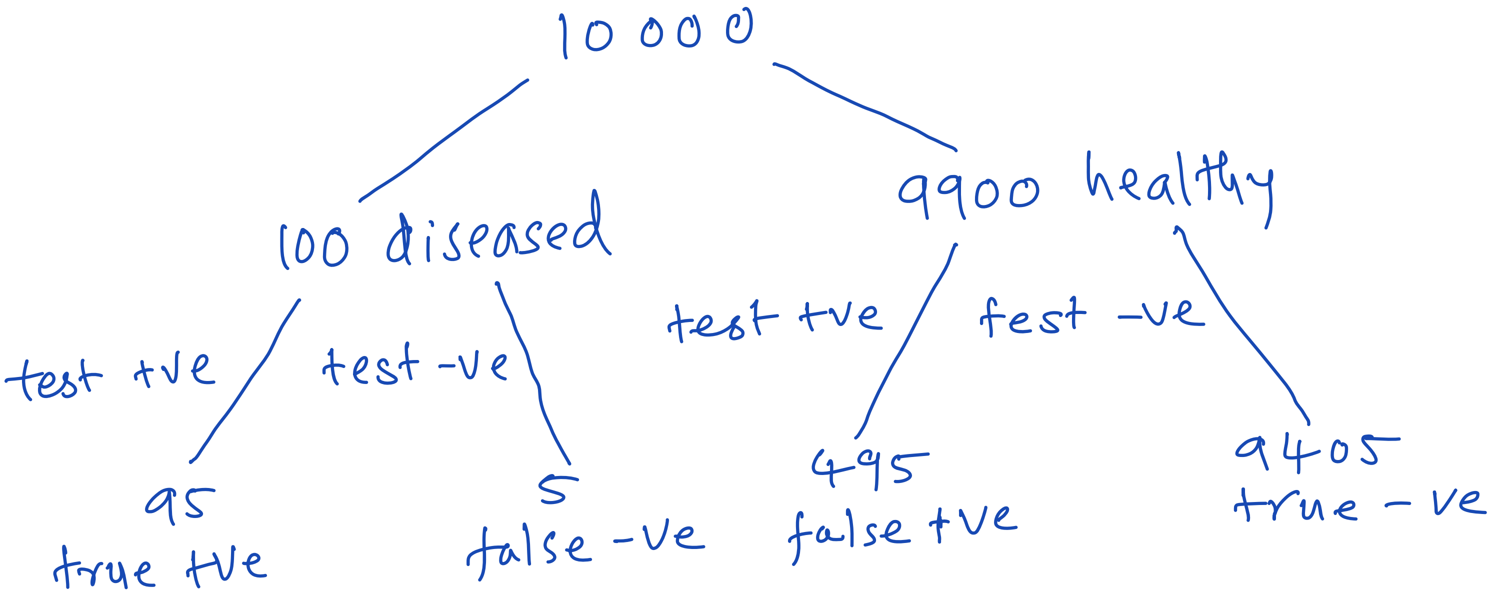
$$= \frac{P(T|D)\cdot P(D)}{P(T|D)\cdot P(D) + P(T|D^c)\cdot P(D^c)}$$

$$= \frac{0.95\cdot 0.01}{(0.95\cdot 0.01) + (0.05\cdot 0.99)}$$

$$\approx 0.16.$$

- Why such low probability? (when the test is 95% acurate)
- Since the disease is <u>rare</u>. (prob. $= .01$)

Conditional probability



- Note that 95 true +ve are far less than 495 false -ve.
- Hence, when tested +ve, the probability of true +ve is small:

$$\frac{\text{no. of true +ve}}{\text{no. of all +ve}} = \frac{95}{590} \approx .16.$$

Independence

- Events A and B are independent if $P(A \cap B) = P(A)P(B)$.
- If $P(A) > 0$ and $P(B) > 0$, then this is equivalent to

$$P(A|B) = P(A),$$
$$P(B|A) = P(B).$$

- I.e., A and B are independent if learning that B occurred does not change the probabilities for A occurring (and vice versa).
- Independence is a symmetric relation: if A is independent of B, then B is independent of A.
- Independence is completely different from disjointness. Disjoint events can be independent only if $P(A) = 0$ or $P(B) = 0$.

$P(A) > 0, \quad P(B) > 0, \quad A \cap B = \emptyset$

$\Rightarrow P(A \cap B) = 0 \quad \& \quad P(A) \cdot P(B) > 0.$

Independence

- A card is selected at random from an ordinary deck of 52 playing cards. E is the event that the selected card is an ace and F is the event that it is a spade. Are E and F independent?

♠

$P(E) = 4/52$ $\qquad$ $P(F) = 13/52$

$$\left.\begin{aligned} P(E \cap F) &= 1/52 \\ P(E)\cdot P(F) &= \frac{4}{52}\cdot\frac{13}{52} = 1/52 \end{aligned}\right\} \Rightarrow \text{E \& F are independent.}$$

- Suppose that we toss 2 fair dice. Let E denote the event that the sum of the dice is 6 and F denote the event that the first die equals 4. Are E and F independent?

$P(E) = 5/36$ $\quad$ $(1+5,\ 2+4,\ 3+3,\ 4+2,\ 5+1)$

$P(F) = 1/6$

$P(E \cap F) = P(\text{the outcome is } (4,2))$

$= 1/36 \Rightarrow$ E & F are not independent.

(They are dependent)

Independence

a "small" theorem

- Lemma: If A and B are independent, then A and B^c are independent.

Proof:- If $P(A) = 0$ then, A is independent of any event.

- Now assume that $P(A) > 0$. Then,

$$P(B^c \mid A) = 1 - P(B \mid A) \qquad \because 1 = P(S \mid A) = P(B \cup B^c \mid A) = P(B \mid A) + P(B^c \mid A)$$

$$= 1 - P(B)$$

$$= P(B^c)$$

$$P(A \cap B) = P(A) \cdot P(B) \Rightarrow P(A \cap B^c) = P(A) \cdot P(B^c \mid A)$$

$$= P(A) \cdot P(B^c)$$

- Homework: Show that if A and B are independent, then A^c and B are independent, and A^c and B^c are independent.

Independence

- Events A, B, and C are said to be independent if all of the following equalities hold:

$$
\begin{aligned}
P(A \cap B) &= P(A)P(B), \\
P(A \cap C) &= P(A)P(C), \\
P(B \cap C) &= P(B)P(C), \\
P(A \cap B \cap C) &= P(A)P(B)P(C).
\end{aligned}
$$

- If the first three conditions hold, then A, B, and C are called pairwise independent.

Independence

- Pairwise independence doesn't imply independence.
  Example: Consider two fair, independent coin tosses, and let A be the event that the first is Heads, B the event that the second is Heads, and C the event that both tosses have the same result.

  $P(A) = P(B) = 1/2$, $P(C) = 1/2$. $P(A \cap B) = 1/4$

  $\underbrace{P(A \cap C)} = P(C \mid A) \cdot P(A) = 1/4$. Similarly, $P(B \cap C) = 1/4$

  – Now, $\underbrace{P(A \cap B \cap C)} = P(C \mid A \cap B) \cdot P(A) \cdot P(B)$
  $= 1 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \neq P(A) \cdot P(B) \cdot P(C) = 1/8$

  or find no. of ways $A \cap C$, $A \cap B \cap C$ can occur.

- $P(A \cap B \cap C) = P(A)P(B)P(C)$ does not imply pairwise independence.
  Example: In the "dice" example, assume that $G = \emptyset$. Then $P(E \cap F \cap G) = P(E)P(F)P(G) = 0$. However, E and F are not independent.

Independence

- Events $A_1, A_2, \ldots, A_n$ are independent if

$$P\left(\bigcap_{j\in J} A_j\right) = \prod_{j\in J} P(A_j) \quad \text{for all } J \subseteq \{1, 2, \ldots, n\}.$$

- Events $A_1, A_2, \ldots, A_n$ are pairwise independent if

$$P(A_i \cap A_j) = P(A_i)P(A_j) \quad \text{for all } \{i, j\} \subseteq \{1, 2, \ldots, n\}, i \neq j.$$

- Events A and B are conditionally independent given E if

$$P(A \cap B|E) = P(A|E)P(B|E).$$

- $P(A \cap B) = P(A)P(B)$ does not imply

$$P(A \cap B|E) = P(A|E)P(B|E).$$

Independence

i.e., hungry or tired.

- A certain baby cries if and only if she is hungry, tired, or both. Let C be the event that the baby is crying, H be the event that she is hungry, and T be the event that she is tired. Let $P(C) = c$, $P(H) = h$, and $P(T) = t$, where none of c, h, t are equal to 0 or 1. Let H and T be independent.
  (a) Find c, in terms of h and t.
  (b) Are H and T conditionally independent given C?

(a) $P(C) = P(H \cup T)$

$= P(H) + P(T) - P(H \cap T)$ ($\because$ IEF)

$= h + t - ht$

($\because$ H & T are independent)

Independence

(b) Check: $P(H \cap T \mid C) = P(H \mid C) \cdot P(T \mid C)$

$$P(H \mid C) = \frac{\overbrace{P(C \mid H)}^{1} \cdot P(H)}{P(C)} = h/c$$

$$P(T \mid C) = \frac{\overbrace{P(C \mid T)}^{1} \cdot P(T)}{P(C)} = t/c$$

$$P(H \cap T \mid C) = \frac{\overbrace{P(C \mid H, T)}^{1} \cdot P(H, T)}{P(C)} = \frac{ht}{c}$$

$$\Rightarrow P(H, T \mid C) = \frac{ht}{c} \neq P(H \mid C) \cdot P(T \mid C) = \frac{ht}{c^2}$$

i.e., H, T are independent but not conditionally independent.