Lecture 2: Counting - Part I

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Naive probability (cont.)

- What is the naive probability of the event A^c?

$$P(A^c) = \frac{|A^c|}{|S|} = \frac{|S| - |A|}{|S|} = 1 - \frac{|A|}{|S|} = 1 - P(A).$$

application

- We will see later that this is true even for axiomatic (generalized) definition of probability.
- Assume we want to find $P(A)$. Sometimes, it is easier to find $P(A^c)$ (i.e., finding $|A^c|$ for naive probability). Then, from $P(A^c)$ we can compute $P(A)$.

Counting

- Calculating the naive probability of an event A involves *counting* the number of outcomes in A and the number of outcomes in the sample space S.
- Theorem (Multiplication rule). Suppose that Experiment A has a possible outcomes, and for each of those outcomes Experiment B has b possible outcomes. Then the compound experiment has ab possible outcomes.
- Why?
  Let A be the sample space for Experiment A and B be the sample space for Experiment B. Then, there are $|A \times B| = ab$ outcomes of the compound experiment consisting sub-experiments A and B, where

$$A \times B = \{(i, j) : i \in A, j \in B\}.$$

Ex: Rolling two dice, tossing a coin and rolling a dice.

Counting

- Example: Suppose that 10 people are running a race. Assume that ties are not possible and that all 10 will complete the race, so there will be well-defined first place, second place, and third place winners. How many possibilities are there for the first, second, and third place winners?

  - A person cannot take two or more spaces.
  - 10 possibilities for the first place
  - 9 possibilities for the second place
  - 8 possibilities for the third place

  order does not matter.

  Total: $10 \cdot 9 \cdot 8 = 720$.

- Example: How many $\underbrace{\text{squares}}_{\text{smallest (not including other squares)}}$ are there in an 8×8 chessboard?

  - A square can be identified uniquely by row number and column number.
  - 8 rows and 8 columns $\Rightarrow$ 64 squares.

Counting

- Example: A set with n elements has 2^n subsets. How?

– To form a subset, for each element, we can either choose it or exclude it. i.e., two possibilities for each element.

– There are n total elements.

$\Rightarrow$ the total no. of subsets are $\underbrace{2 \times \cdots \times 2}_{n} = 2^n$.

– E.g., For $S = \{1, 2, 3\}$, the subsets are

$\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}$.

Counting

follows from the multiplication rule

- Theorem (Sampling with replacement): Consider n objects and making k choices from them, one at a time with replacement (i.e., choosing a certain object does not preclude it from being chosen again).
  Then there are n^k possible outcomes (here order matters).

- Ex: Rolling a dice K times

  $\Rightarrow$ 6^k possible outcomes.

  flipping a coin K times

  $\Rightarrow$ 2^k possible outcomes

Counting

follows from the multiplication rule

- Theorem (Sampling without replacement). Consider n objects and making k choices from them, one at a time without replacement (i.e., choosing a certain object precludes it from being chosen again).
  Then there are $n(n-1)\cdots(n-k+1)$ possible outcomes for $1 \leq k \leq n$, and 0 possibilities for $k > n$.
- By convention, $n(n-1)\cdots(n-k+1) = n$ for $k = 1$.
- Also, note that for $k = n$, $n(n-1)\cdots(n-k+1) = n!$. This is the number of ways n objects can be permuted. A permutation of n objects is an arrangement of them in some order, e.g., $(3,5,1,2,4)$ is a permutation of the objects in $\{1,2,3,4,5\}$.

- Ex : Refer the "race" problem with 10 participants.

Counting

- Example (Birthday problem). There are k people in a room. Assume each person's birthday is equally likely to be any of the 365 days of the year (we exclude February 29), and that people's birthdays are independent (we will define independence formally later, but intuitively it means that knowing some people's birthdays gives us no information about other people's birthdays; this would not hold if, e.g., we knew that two of the people were twins).
  What is the probability that at least one pair of people in the group have the same birthday?

- There are 365^k ways to assign birthdays to k people, i.e., $|S| = 365^k$.
- We are interested in no. of ways to assign b'days such that there are at least two people sharing a b'day.

Counting

- It is easier to find the size of the complement event, i.e., no. of ways to assign b'days to k people s.t. no two people share a b'day.

- Apply sampling without replacement:

$$365 \cdot 364 \cdot \cdots \cdot (365-k+1), \quad k \leq 365.$$

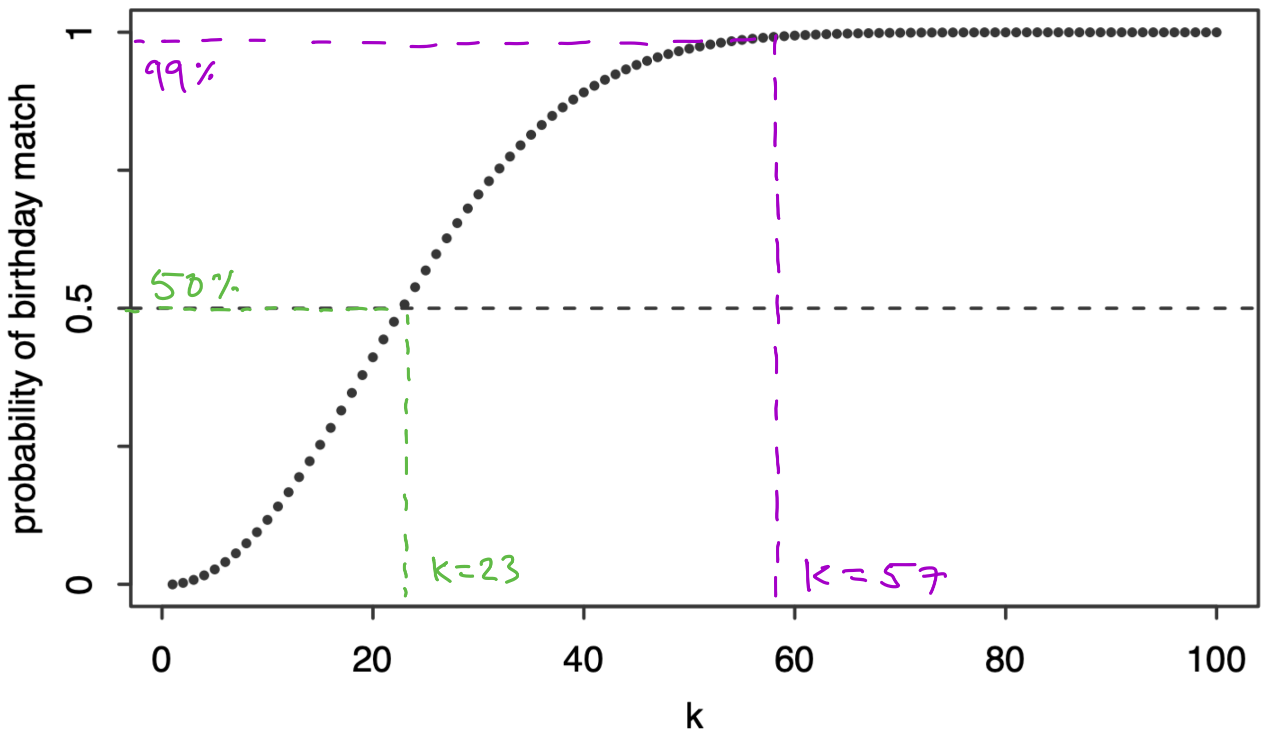
$$\Rightarrow P(\text{no b'day match}) = \frac{365 \cdot 364 \cdot \cdots \cdot (365-k+1)}{365^k}$$

$$\Rightarrow P(\text{at least 1 b'day match}) = 1 - \frac{365 \cdot 364 \cdot \cdots \cdot (365-k+1)}{365^k}.$$

Counting

Note that for $k \geq 366$,

$P(\text{at least 1 b'day match}) = 1$.



Counting

- Binomial coefficient $\binom{n}{k}$, read as "n choose k", is the number of subsets of size k for a set of size n.
- Theorem:

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{(n-k)!k!}, \quad \text{for } k \le n \quad \text{— 1}$$

$$= 0, \quad \text{for } k > n. \quad \text{— 2}$$

Proof: There are $n(n-1)\cdots(n-k+1)$ ways to make ordered choices of k elements without replacement.

- This overcounts each subset of interest by $k!$ (no. of permutations)
- Hence, 1 follows.
- For 2, no subset exists of size $k > n$.

Counting

- Example: Consider a club with n people.
- How many ways to choose a president, vice president, and treasurer?

$$n \cdot (n-1)(n-2)$$

- How many ways to choose 3 officers without predetermined titles?

$$\frac{n(n-1)(n-2)}{3!}$$

Counting

- Theorem: $$\binom{n}{k} = \binom{n}{n-k}$$

Proof 1 : $\binom{n}{k} = \frac{n!}{(n-k)!\,k!} = \frac{n!}{k!\,(n-k)!} = \binom{n}{n-k}$
(Algebraic)

Proof 2 : - choosing a committee of size k from n people is the same as choosing (n-k) people not on the committee.
(Intuitive)

(specifying who is on the committee also determines who is not on the committee)

Counting

- How many ways are there to permute the letters in the word LALALAAA?

– Out of 8 positions, choose 5 positions where "A" goes.

– OR: choose 3 positions where "L" goes.

$$\Rightarrow \binom{8}{5} = \binom{8}{3} = \frac{8 \cdot 7 \cdot 6}{3!} = 56.$$

- Preliminary reading: Blitzstein and Hwang, *Introduction to Probability* (2015), CRC Press.

Note

- Source of figures: reference books