Lecture 16:
Expectation and Variance - Part III

Satyajit Thakor
IIT Mandi

Properties of variance

- $\text{Var}(X) \geq 0.$ $\text{Var}(X) = E[(X-\mu_X)^2]$

$$= \sum_x \underbrace{(x-\mu_X)^2}_{\text{square of a real no.}} \underbrace{p_X(x)}_{\text{by axioms of probability}}$$

$$\geq 0 \quad (\because (x-\mu_X)^2 \geq 0,\ p_X(x) \geq 0 \text{ for all } x)$$

- $\text{Var}(c) = 0$ for any constant c.

Let X be an r.v. s.t. $P(X=c)=1$. $\Rightarrow E(X) = \sum_x x\, P(X=x)$

$$= c \cdot P(X=c)$$

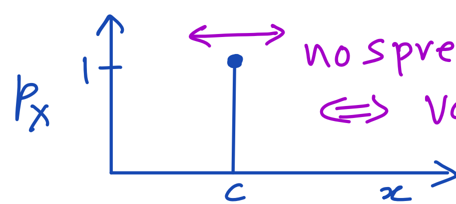
$$= c.$$

$\Downarrow$

Then, for r.v. $(X-c)^2$, $P((X-c)^2=0)=1$.

$$\Rightarrow E[(X-c)^2] = 0 \cdot 1 = 0$$

Note: $\text{Var}(c)=0$ means that when the r.v. X is a constant c (i.e., $P(X=c)=1$) then the spread of PMF is zero:



no spread of PMF $\Leftrightarrow \text{Var}(X)=0$

Properties of variance

shifting X by c

- $\text{Var}(\underbrace{X + c}_{\text{shifting } X \text{ by } c}) = \text{Var}(X)$ for any constant c.

$E(X+c) = \mu_X + c.$

$$\text{Var}(X+c) = E[(X+c-\underbrace{E(X+c)}_{\mu_X + c})^2]$$
$$= E[(X+c-\mu_X-c)^2] = E[(X-\mu_X)^2]$$
$$= \text{Var}(X).$$

- Note: spread does not change by "shifting".

PMF of X $= \text{Bin}(70, 0.1)$

PMF of $X+25$

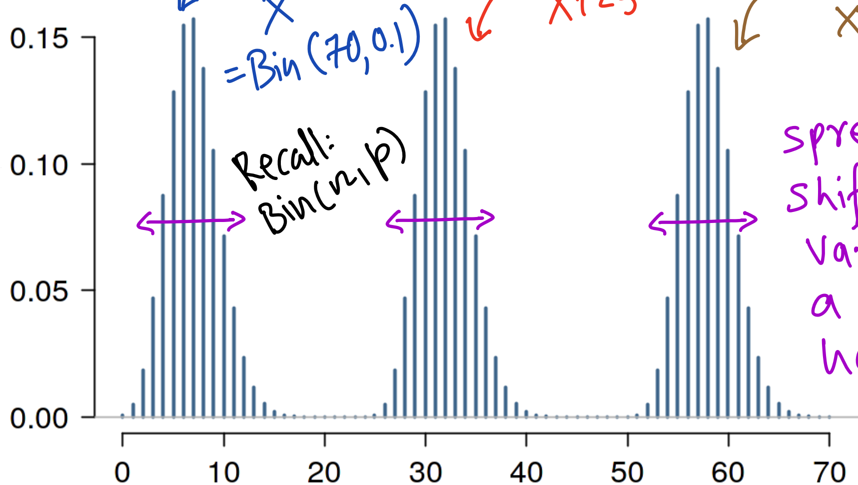
PMF of $X+51$

Recall: $\text{Bin}(n,p)$

p_X

p_{X+25}

p_{X+51}



spread does not change by shifting, i.e., if X has variance Var(X) then a shifted r.v. $X+c$ also has the $\text{var}(X+c) = \text{Var}(X)$

Properties of variance

- $\text{Var}(cX) = c^2\text{Var}(X)$ for any constant c.

$$\begin{aligned} Var(cX) &= E[(cX - E(cX))^2] \\ &= E[c^2(X-\mu_X)^2] \\ &= c^2 E[(X-\mu_X)^2] \quad \because \text{linearity of } E(\cdot). \\ &= c^2 Var(X). \end{aligned}$$

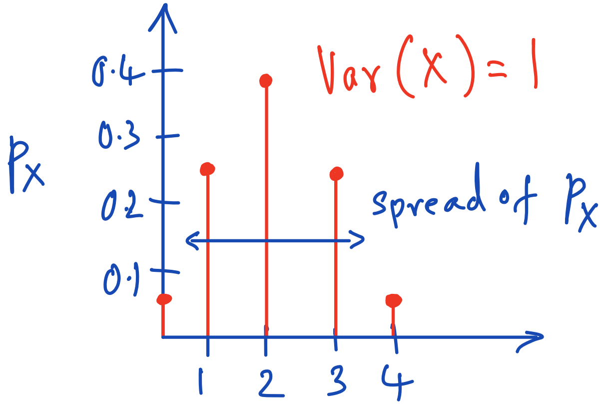
- That is, if we scale X by c then the variance Var(cX) of the resulting r.v. cX is Var(X) scalled by c^2.

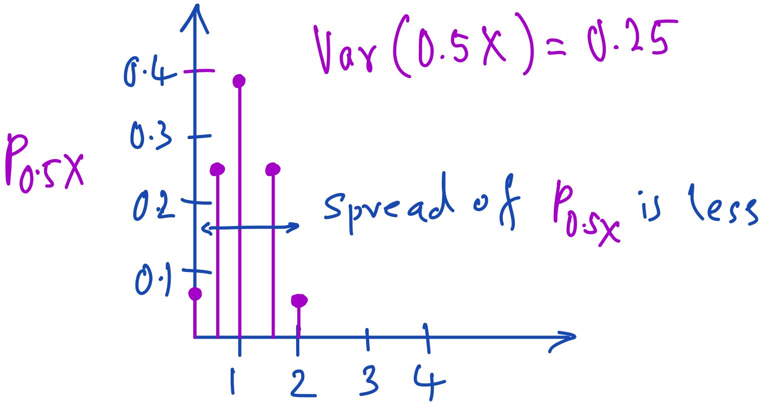
- So, depending on the value of c, the spread for cX either reduces or increases (if c=1 then the spread remains the same.)

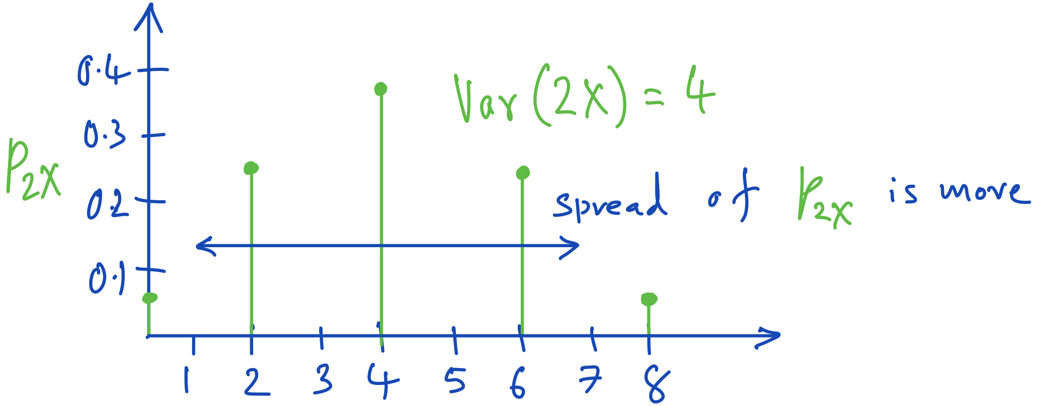
- The next example demonstrates this.

Properties of variance

- Example: Let $X \sim \text{Bin}(4, .5)$. Plot PMFs of $X, 2X, .5X$ and compare their variance.







Functions of independent random variables

- **Theorem:** Suppose that X and Y are independent r.v.s. Then $W = g(X)$ and $Z = h(Y)$ are also independent.

Proof:

$$p_{W,Z}(w,z) = P(W=w, Z=z) = P\left(\left[\bigcup_{x:\, g(x)=w} \{X=x\}\right] \cap \left[\bigcup_{y:\, h(y)=z} \{Y=y\}\right]\right)$$

$$= P\left(\bigcup_{\substack{x:\, g(x)=w \\ y:\, h(y)=z}} \{X=x, Y=y\}\right)$$

$$= \sum_{x:\, g(x)=w} \sum_{y:\, h(y)=z} p_{XY}(x,y)$$

$\Downarrow$ since $p_{XY}(x,y) = p_X(x) \cdot p_Y(y)$

$$= \sum_{x:\, g(x)=w} p_X(x) \sum_{y:\, h(y)=z} p_Y(y) = p_W(w) \cdot p_Z(z).$$

Note: You may skip the proof (will not be asked in exam). But remember the theorem statement. The proof is just to satisfy your curiosity.

Properties of variance

- If X and Y are independent, then

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y).$$

$$\begin{aligned}
Var(X+Y) &= E\left[\left(X+Y-E(X+Y)\right)^2\right] \\
&= E\left[(X+Y-\mu_X-\mu_Y)^2\right] \quad \because \text{linearity of } E(\cdot) \\
&= E\left[(X-\mu_X)^2 + (Y-\mu_Y)^2 + 2(X-\mu_X)(Y-\mu_Y)\right] \\
&= Var(X) + Var(Y) + 2E\left[\underbrace{(X-\mu_X)(Y-\mu_Y)}_{\text{functions of independent r.v.s are independent}}\right] \\
&\Downarrow \\
&= Var(X) + Var(Y) + 2E(X-\mu_X)\cdot E(Y-\mu_Y) \\
&= Var(X) + Var(Y) + 2\underbrace{(\mu_X-\mu_X)}_{0}\cdot\underbrace{(\mu_Y-\mu_Y)}_{0} \\
&= Var(X) + Var(Y)
\end{aligned}$$

Example

- Let X be a random variable with the PMF $p_X(-3) = 1/6, p_X(6) = 1/2, p_X(9) = 1/3$.
  (a) Find $\mu_{g(X)} = E(g(X))$, where $g(X) = (2X+1)^2$.
  (b) Find $\mathrm{SD}(g(X))$.

(a) $\mu_{g(x)} = E[(2X+1)^2]$

$$= \frac{25}{6} + \frac{169}{2} + \frac{361}{3} = 209.$$

(b) $\sigma^2_{g(x)} = E[((2X+1)^2 - 209)^2]$

$$= \sum_x [(2x+1)^2 - 209] \cdot p_X(x)$$

$$= (25-209)^2 \cdot \frac{1}{6} + (169-209)^2 \cdot \frac{1}{2} + (361-209)^2 \cdot \frac{1}{3}$$

$$= 14144 \implies \mathrm{SD}(g(X)) = \sqrt{\sigma^2_{g(x)}} = 118.9.$$