Lecture 11:
Discrete Random Variables - Part V

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Functions of r.v.s

- **Definition (function of 2 r.v.s):** Given an experiment with sample space S, if X and Y are r.v.s that map $s \in S$ to $X(s)$ and $Y(s)$ respectively, then $g(X,Y)$ is the r.v. that maps s to $g(X(s), Y(s))$.
- Two fair dice are rolled. X is the number shown on 1st die and Y is the number shown on 2nd die. Find the PMF of $Z = \max(X, Y)$.

Ex:

$p_Z(1) = P(\max(X,Y)=1) = P(\{X=1, Y=1\}) = 1/36$

$p_Z(2) = P(\max(X,Y)=2) = P(\{X=1,Y=2\} \cup \{X=2,Y=1\} \cup \{X=2,Y=2\}) = 3/36$

$p_Z(3) = P(\{X=1,Y=3\} \cup \{X=2,Y=3\} \cup \{X=3,Y=3\} \cup \{X=3,Y=2\} \cup \{X=3,Y=1\}) = 5/36$

- Similarly, verify that (Homework):

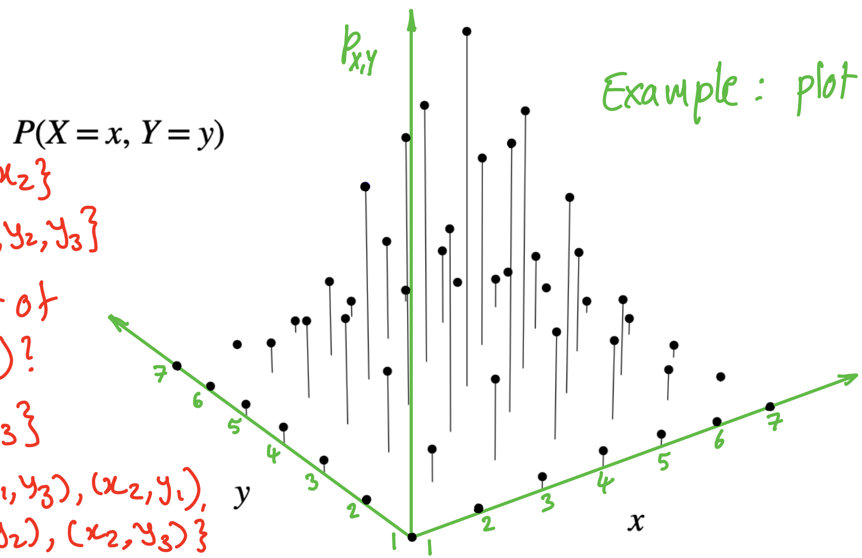
$p_Z(4) = 7/36, \quad p_Z(5) = 9/36, \quad p_Z(6) = 11/36.$

Joint distributions

- Recall: The distribution of a discrete random variable can be described by its PMF and also by CDF.
- The joint distribution of two (or more) discrete r.v.s can be described by their joint PMF and also by joint CDF.
- The joint PMF of discrete r.v.s X and Y is the function $p_{X,Y} : \mathbb{R} \times \mathbb{R} \to [0,1]$ given by

$$p_{X,Y}(x,y) = P(X = x, Y = y).$$

Example: plot of joint PMF.



Example:
alphabet of $X = \{x_1, x_2\}$
alphabet of $Y = \{y_1, y_2, y_3\}$
What is the alphabet of the joint r.v. (X,Y)?
$\{x_1, x_2\} \times \{y_1, y_2, y_3\}$
$= \{(x_1,y_1), (x_1,y_2), (x_1,y_3), (x_2,y_1), (x_2,y_2), (x_2,y_3)\}$

Joint distributions

- Example (Two Bernoulli r.v.s X and Y):

The joint PMF can be completely specified by 4 values:

These are two alterative ways to express joint PMF.

$$\begin{cases} P((X,Y)=(1,1)),\ P((X,Y)=(0,1)),\ P((X,Y)=(1,0)),\ P((X,Y)=(0,0)) \\ P(X=1,Y=1),\ P(X=0,Y=1),\ P(X=1,Y=0),\ P(X=0,Y=0) \end{cases}$$

Let $= \frac{5}{100}$, $= \frac{3}{100}$, $= \frac{20}{100}$, $= \frac{72}{100}$.

Then the joint PMF can be expressed in the table form:

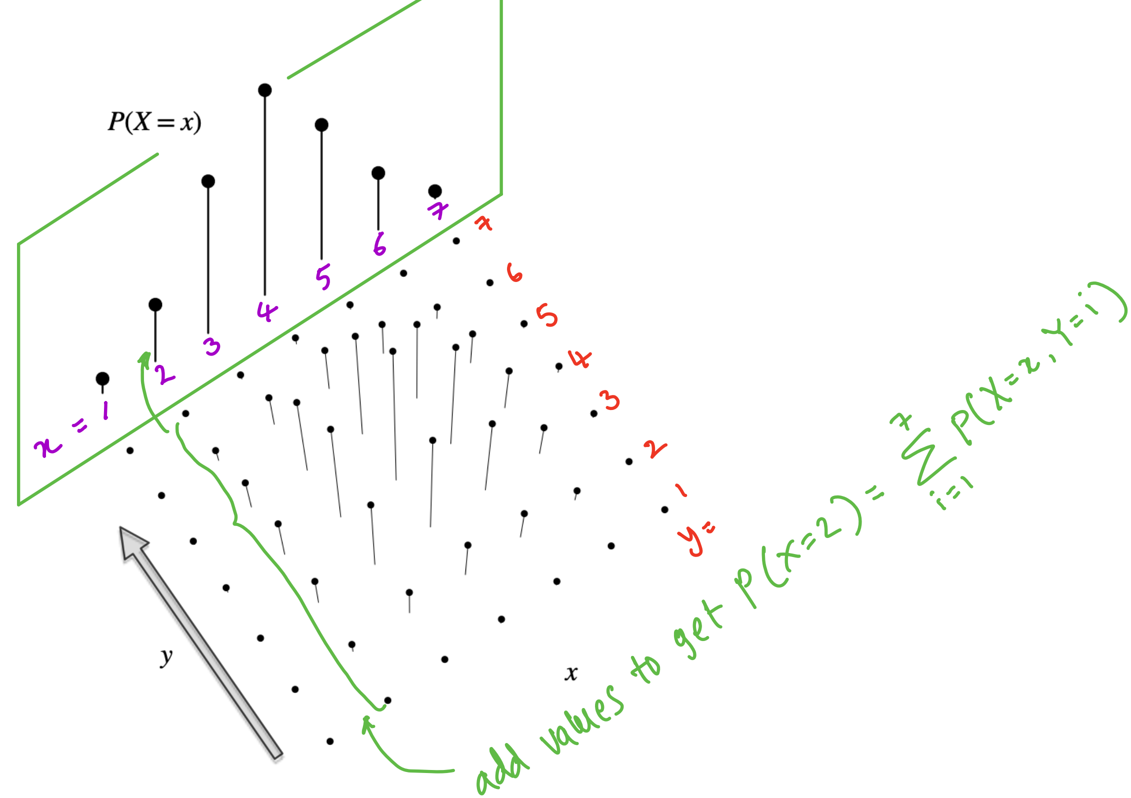
$X=$ \ $Y=$	1	0
1	0.05	0.2
0	0.03	0.72

$$p_{X,Y}(x,y) = P(X=x, Y=y)$$

Joint distributions

- For discrete r.v.s X and Y, the marginal PMF of X is

$$P(X = x) = \sum_y P(X = x, Y = y).$$



Joint distributions

- Example (Two Bernoulli r.v.s X and Y):

p_X

$X = \backslash Y =$	1	0
1	0.05	0.2
0	0.03	0.72

$p_X(1) = 0.25$ sum

$p_X(0) = 0.75$

$p_{X,Y}(x,y) = P(X=x, Y=y)$

sum

$0.08 \quad 0.92$

$\| \quad \|$

$p_Y(1) \quad p_Y(0)$

p_Y

Joint distributions

- The joint CDF of r.v.s X and Y is the function $F_{X,Y} : \mathbb{R} \times \mathbb{R} \to [0,1]$ given by

$$F_{X,Y}(x,y) = P(X \leq x, Y \leq y).$$

- Example (Two Bernoulli r.v.s X and Y):

$$F_{X,Y}(0,0) = P(X \leq 0, Y \leq 0) = 0.72$$

$$F_{X,Y}(0,1) = \underbrace{P(X \leq 0, Y \leq 1)}_{P(\{X=0, Y=0\} \cup \{X=0, Y=1\})} = 0.75$$

$$F_{X,Y}(1,0) = \underbrace{P(X \leq 1, Y \leq 0)}_{P(\{X=0, Y=0\} \cup \{X=1, Y=0\})} = 0.92$$

Similarly, $F_{X,Y}(1,1) = 1$.

Note that, $F_{X,Y}(x,y) = 0$ for all $x < 0$ or $y < 0$.

$F_{X,Y}(x,y) = 1$ for all $x \geq 1$ and $y \geq 1$.

Independent r.v.s

- Recall (independent events): The events A and B are independent if $P(A \cap B) = P(A)P(B)$.
- Random variables X and Y are said to be independent if

$$P(X \leq x, Y \leq y) = P(X \leq x)P(Y \leq y),$$

for all $x, y \in \mathbb{R}$.
- For discrete r.v.s, this is equivalent to

$$P(X = x, Y = y) = P(X = x)P(Y = y).$$

Independent r.v.s

For discrete r.v.s, ① is equivalent to
$P(X_1 = x_1, \dots, X_n = x_n) = P(X_1 = x_1) \cdot \ldots \cdot P(X_n = x_n)$

- Random variables $X_1, \dots, X_n$ are independent if

$$P(X_1 \le x_1, \dots, X_n \le x_n) = P(X_1 \le x_1) \dots P(X_n \le x_n), \quad \text{— ①}$$

  for all $x_1, \dots, x_n \in \mathbb{R}$.
- Recall: For n events to be independent, 2^n (one equality for each subset $J \subseteq \{1, 2, \dots, n\}$) equalities must be satisfied. } Lecture 6
- But for n r.v.s to be independent only one equality must be satisfied. In fact, this one equality implies all other equalities for subsets of $\{1, 2, \dots, n\}$.

For example, assume that
$$P(X=x, Y=y, Z=z) = P(X=x) \cdot P(Y=y) \cdot P(Z=z).$$

Then,
$$\sum_z P(X=x, Y=y, Z=z) = \sum_z P(X=x) \cdot P(Y=y) \cdot P(Z=z)$$
$$\Rightarrow P(X=x, Y=y) = P(X=x) \cdot P(Y=y).$$

i.e., X and Y are also pairwise independent.

Independent r.v.s

- Example (Two Bernoulli r.v.s X and Y): Are they independent?

p_X :

$p_X(1) = 0.25$

$p_X(0) = 0.75$

$X =$ \ $Y =$	1	0
1	0.05	0.2
0	0.03	0.72

$p_{X,Y}(x,y)$
$= P(X=x, Y=y)$

p_Y : $p_Y(1) = 0.08$, $p_Y(0) = 0.92$

– Note that, $P(X=1, Y=1) = 0.05$

$P(X=1) \cdot P(Y=1) = 0.25 \cdot 0.08 = 0.02$

$\Rightarrow P(X=1, Y=1) \neq P(X=1) \cdot P(Y=1)$

$\Rightarrow$ X and Y are not independent.

Independent r.v.s

- The random variables that are independent and have the same distribution are called independent and identically distributed, or i.i.d.

Note

- Source of figures: reference books