Lecture 19:
Continuous Random Variables - Part III

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Continuous random variables

- The median of a continuous r.v. X with a CDF $F_X(x)$ is the value x such that $F_X(x) = 0.5$.
- The r.v. is equally likely to fall above or below the median value.
  50% - 50%
- Example continued: (g) Find the median of the battery failure time.

We need to solve $F_X(x) = 0.5$.

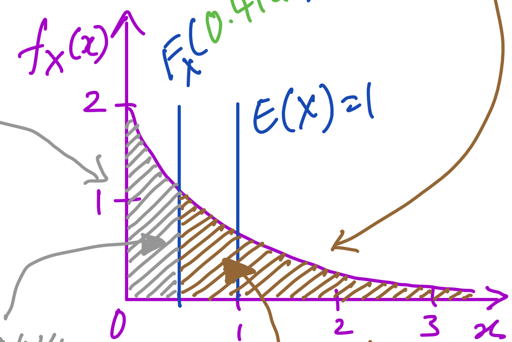
$$\Rightarrow 1 - \frac{1}{(x+1)^2} = \frac{1}{2}, \quad x \geq 0$$

$$\Rightarrow \frac{1}{2} = \frac{1}{(x+1)^2}$$

$$\Rightarrow (x+1)^2 = 2$$

$$\Rightarrow x + 1 = \sqrt{2} \quad (\because x \geq 0)$$

$$\Rightarrow x = \sqrt{2} - 1 \approx 0.414. \quad \left(\because \int_{0.414}^{\infty} f_X(x)\,dx = 0.5 = 1 - \int_{0}^{0.414} f_X(x)\,dx = 0.5\right)$$



$$F_X(0.414) = \int_0^{0.414} f_X(x)\,dx = 0.5$$

$$\int_{0.414}^{\infty} f_X(x)\,dx = 0.5$$

Continuous random variables

Median for general R.V. X:
x s.t. $\underbrace{P(X \le x) \geq .5}$ & $\underbrace{P(X \geq x) \geq .5}$
where X is cont., discrete or mixed.

- Similar to variance for discrete r.v.s, the variance for continuous r.v. X is

$$\mathrm{Var}(X) = E[(X - E(X))^2] = E(X^2) - E(X)^2.$$

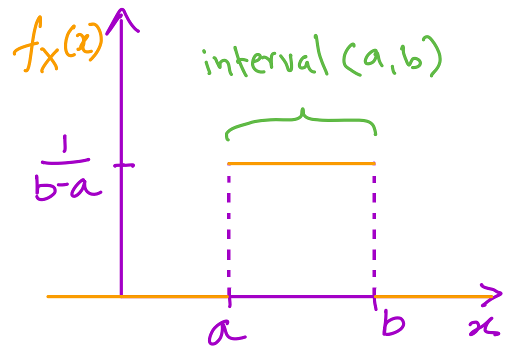
- Caution: Expectation and Variance are defined only if the integral is absolutely convergent (advance).
- In this course, we will mainly focus on examples such that expectation and variance are defined.

- Let X be a R.V. with $p_X(2^n) = \frac{1}{2^n}$, $n = 1, 2, 3, \ldots$ and 0 otherwise.
- What is $E(X)$?
- A continuous dist. with undefined $E(\cdot)$ and $\mathrm{Var}(\cdot)$: Cauchy dist.
- Pareto dist: for some range of parameters $E(\cdot)$ defined but $\mathrm{Var}(\cdot)$ not

Continuous random variables

- Now we will discuss three important continuous distributions: Uniform, Normal (also called Gaussian) and exponential.
- A continuous r.v. X is said to have the Uniform distribution on the interval (a, b) if its PDF is

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a < x < b; \\ 0, & \text{elsewhere.} \end{cases}$$



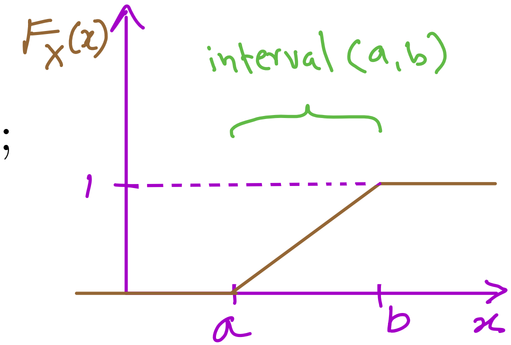
- Notation: $X \sim \text{Unif}(a, b)$ means that X is uniformly distributed in the interval (a, b).
- This is a valid PDF: the area under the curve is just the area of a rectangle with width $b - a$ and height $1/(b - a)$.

(1) $f_X(x) \geq 0$, (2) $\int_{-\infty}^{\infty} f_X(x)\,dx = \int_a^b \frac{1}{b-a}\,dx = \frac{x}{b-a}\Big|_a^b = \frac{b-a}{b-a} = 1.$

Continuous random variables

- The CDF is the accumulated area under the PDF:

$$F_X(x) = \begin{cases} 0, & x \leq a; \\ \frac{x-a}{b-a}, & a < x < b; \\ 1 & x \geq b. \end{cases}$$



- For $x \leq a$, $F_X(x) = 0$ since $f_X(x) = 0$ for $x \leq a$.

- For $a < x < b$, $F_X(x) = \int_a^x \frac{1}{b-a} dt = \frac{t}{b-a}\Big|_a^x = \frac{x-a}{b-a}$

- For $x \geq b$, $F_X(x) = \int_a^b f_X(x)\, dx + \int_b^x \underbrace{f_X(t)}_{0 \text{ for } x \geq b} dt$

$$= \frac{b-a}{b-a} + 0 = 1$$

Continuous random variables

Note that F_X is not differentiable at a and b.

- Example: Find the expectation and variance of $X \sim \text{Unif}(a, b)$.

$$E(X) = \int_a^b x \cdot \frac{1}{b-a} \, dx$$

$$= \frac{x^2}{2(b-a)}\Big|_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{a+b}{2}.$$

$$E(X^2) = \int_a^b x^2 \frac{1}{b-a} \, dx$$

$$= \frac{b^3-a^3}{3(b-a)}$$

$$= \frac{(b-a)(b^2+ab+a^2)}{3(b-a)}$$

difference of 2 powers:

$$a^n - b^n$$

$$= (a-b) \sum_{i=1}^{n-1} a^{n-i-1} b^i$$

Proof: H.W.

Continuous random variables

$$= \frac{a^2+ab+b^2}{3}$$

$$Var(X) = E(X^2) - E(X)^2$$

$$= \frac{a^2+ab+b^2}{3} - \frac{(a+b)^2}{2^2}$$

$$= \frac{a^2+ab+b^2}{3} - \frac{a^2+2ab+b^2}{4}$$

$$= \frac{a^2-2ab+b^2}{12}$$

$$= \frac{(a-b)^2}{12}$$

Continuous random variables

- Example: When pearl oysters are opened, pearls of various sizes are found. Suppose that each oyster contains a pearl with a diameter in mm that has a $U_{nif}(0, 10)$ distribution. denote as r.v. D
  (a) Find mean and variance of the diameter of a pearl.

  $a = 0, b = 10$

  $E(D) = a+b/2$
  $= 5.$

  $Var(D) = (a-b)^2/12$
  $= \frac{100}{12} = \frac{25}{3} \approx 8.33.$

  (b) If only the pearls with diameter at least 4 have commercial value, what is the probability that a randomly chosen oyster contains a pearl of commercial value?

  Pr(oyster has a pearl of commercial value)

  = Pr(its pearl has diameter $\geq$ 4 mm)

  $= P(D \geq 4) = 1 - F_D(4) = 1 - \frac{4-0}{10-0} = 0.6$