# Lecture 3: Counting - Part II & Axiomatic Probability

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# Counting

- **Example:** A 5-card hand is dealt from a standard, well-shuffled 52-card deck. The hand is called a **full house** in poker if it consists of three cards of some rank and two cards of another rank, e.g., three 7's and two 10's (in any order). What is the probability of a full house.

club diamond heart spade

♣ ♦ ♥ ♠

4 suits

Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10,

Jack, Queen, King.

13 ranks

# Counting

- $|S| = \binom{52}{5}$
- 13 choices for what rank we have of 3 cards for a full house.
- Fixing some rank $i$, there are $\binom{4}{3}$ ways to choose which 3 cards of rank $i$ we have.
- 12 choices of what rank we have of 2 cards for a full house. Fixing some rank $j$, there are $\binom{4}{2}$ ways to choose 2 cards for a full house.

$$\Rightarrow P(\text{full house}) = \frac{13\binom{4}{3}12\binom{4}{2}}{\binom{52}{5}} = \frac{3744}{2598960}$$

$$\approx 0.00144.$$

# Counting

- The factorial function $n!$ grows extremely quickly as $n$ grows. A famous, useful approximation for factorials is Stirling's formula:

$$\boxed{n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.}$$

- The ratio of the two sides converges to 1 as $n \to \infty$, i.e., the error in approximation reduces as $n$ grows.
- Example (approximating the number of permutations): Suppose that we want to compute the number of tuples of 20 objects by choose objects from a set of 70 objects, i.e., $n!/(n-k)!$ $= 70!/50!$. The approximation from Stirling's formula is

$$\frac{70!}{50!} \approx \frac{\sqrt{140\pi}(70/e)^{70}}{\sqrt{100\pi}(50/e)^{50}} = 3.940 \times 10^{35}.$$

The exact calculation yields $3.938 \times 10^{35}$.

# Counting

Example :

- Suppose that 20 members of an organization are to be divided into three committees A, B, and C in such a way that each of the committees A and B is to have eight members and committee C is to have four members. Determine the number of different ways in which members can be assigned to these committees. (we have discussed a similar problem in the previous lecture)

- $\binom{20}{8}$ ways to choose members for committee A.
- $\binom{12}{8}$ " " " " " B.
- $\binom{4}{4}=1$ " " " " " C.

$\Rightarrow$ The total no. of ways $= \binom{20}{8} \cdot \binom{12}{8} \cdot 1 = \frac{20!}{12!\,8!} \cdot \frac{12!}{4!\,8!}$

$= \frac{20!}{8!\,8!\,4!} = 62355150$

# Counting

- Multinomial coefficient: Suppose that $n$ distinct elements are to be divided into $k$ different groups ($k \geq 2$) in such a way that, for $j = 1, \ldots, k$, the $j$th group contains exactly $n_j$ elements, where

$$n_1 + n_2 + \ldots + n_k = n.$$

Then, the number of different ways in which the $n$ elements can be divided into the $k$ groups is

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-\ldots-n_{k-2}}{n_{k-1}}$$

$$= \frac{n!}{n_1!n_2!\ldots n_k!}.$$

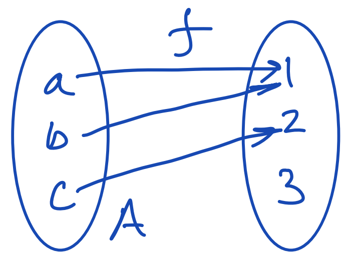
$$\frac{n!}{\cancel{(n-n_1)!}\,n_1!} \cdot \frac{\cancel{(n-n_1)!}}{\cancel{(n-n_1-n_2)!}\,n_2!} \cdot \frac{\cancel{(n-n_1-n_2)!}}{(n-n_1-n_2-n_3)!\,n_3!} \cdot \cdots \cdot \frac{\left(n-\sum_{i=1}^{k-2} n_i\right)!}{n_k!\ n_{k-1}!}$$

# Axiomatic probability

- Previously, we discussed naive probability. Now we will discuss the most general defnition of probability.
- The definition is axiomatic, i.e., defined by a set of rules.
- First, question: what is probability (mathematically)?: A function
- What is a function?

A function $f$ from a set $A$ to a set $B$ maps each element of $A$ to some element of $B$.

$$\underbrace{f}_{\text{function}} : \underbrace{A}_{\text{domain}} \longrightarrow \underbrace{B}_{\text{codomain}}$$




$f(a) = 1$
$f(b) = 1$
$f(c) = 2$

Recall: range (image) $\triangleq \{ b \in B : b = f(a) \text{ for some } a \in A \}$

$\underbrace{\text{injective}}_{f(a) \neq f(b),\ \forall a, b \in A,\ a \neq b}$ $\underbrace{\text{surjective}}_{\text{range} = \text{co-domain}}$ $\underbrace{\text{bijective}}_{\text{injective \& surjective}}$

# Axiomatic probability

- A probability space consists of a sample space $S$ and a probability function $P$ which takes an event $A \subseteq S$ as input and returns $P(A)$, a real number between 0 and 1, as output, i.e.,

Probability function

$$P : \underbrace{\{A \subseteq S : A \text{ is an event}\}}_{\text{set of all events}} \longrightarrow \underbrace{[0, 1]}_{\triangleq \{x \in \mathbb{R} : 0 \leq x \leq 1\}}.$$

i.e., set of reals $x$ s.t. $0 \leq x \leq 1$.

The function $P$ must satisfy the following axioms:

**Axiom 1:** $P(\emptyset) = 0, P(S) = 1.$

**Axiom 2:** If $A_1, A_2, \ldots$ are disjoint events, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

- Events are disjoint means that they are mutually exclusive, i.e., $A_i \cap A_j = \emptyset$ for all $i \neq j$.

# Properties of probability function

- Several properties of the probability function can be derived by its axiomatic definition.
- **Property 1:** $P(A^c) = 1 - P(A)$ (Recall: A similar property was discussed for naive probability)

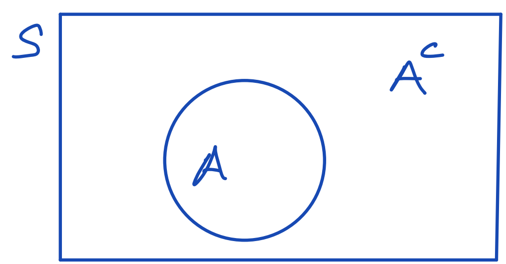
Proof:

$$P(S) = P(A \cup A^c)$$

$$= P(A) + P(A^c) \quad (\because \text{Axiom } 2)$$

$$1 = P(A) + P(A^c) \quad (\because \text{Axiom } 1)$$

$$\Rightarrow P(A^c) = 1 - P(A)$$



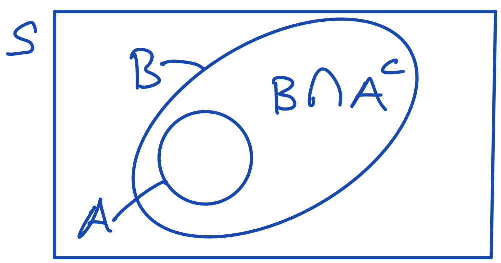
# Properties of probability function

- **Property 2:** If $A \subseteq B$ then $P(A) \leq P(B)$.

$$P(B) = P(A \cup \underbrace{(B \cap A^c)}_{\text{also denoted } B \setminus A \text{ called set subtraction}})$$

$$= P(A) + P(B \cap A^c) \quad (\because \text{Axiom 2})$$

$$P(B) \geq P(A) + 0 \quad (\because \text{Axiom 1})$$



# Properties of probability function

- **Property 3:** $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

$$P(A\cup B) = P(A \cup (B \cap A^c))$$

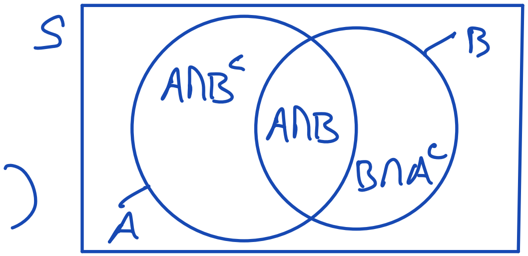
$$= P(A) + P(B \cap A^c) \quad (\because \text{Axiom } 2) \text{ — } ①$$

$$P(B) = P((A\cap B) \cup (A^c \cap B))$$

$$= P(A \cap B) + P(A^c \cap B) \quad (\because \text{Axiom } 2) \text{ — } ②$$

From ① & ②,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



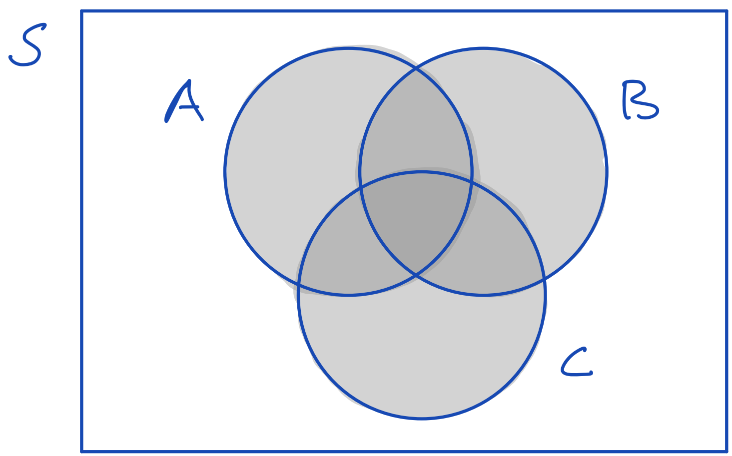

- **Property 3** is the inclusion-exclusion formula (IEF) for the probability function (for the case of 2 events).
- Recall (IEF for cardinality): $|A \cup B| = |A| + |B| - |A \cap B|$.
- Note that, cardinality too is a function (like probability).

# Properties of probability function

- IEF for 3 events:

$$\begin{aligned} P(A \cup B \cup C) =& P(A) + P(B) + P(C) \\ & - P(A \cap B) - P(A \cap C) - P(B \cap C) \\ & + P(A \cap B \cap C). \end{aligned}$$

Proof: Home work (obtain an algebraic proof)



# Properties of probability function

- Generalization: The IEF holds for the probability function involving $n$ events:
- Theorem: For events $A_1, \ldots, A_n$,

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \ldots + (-1)^{n-1} P(A_1 \cap \ldots \cap A_n)$$

$$= \sum_{k=1}^{n} \left( (-1)^{k-1} \sum_{I \subseteq \{1,2,\ldots,n\}:|I|=k} P\left(\cap_{i \in I} A_i\right) \right)$$

- Can be proved using mathematical induction.