Homogenous Transformation Matrix (SE(3))

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \begin{cases} \text{for 3-dimensional space, } T \text{ is a } 4\times4 \text{ matrix} \\ \text{for 2-dimensional space, } T \text{ is a } 3\times3 \text{ matrix} \end{cases}$$

exp: for 2-D, $T = \begin{bmatrix} \cos\theta & -\sin\theta & p_1 \\ \sin\theta & \cos\theta & p_2 \\ 0 & 0 & 1 \end{bmatrix}$

★ Important: $T^{-1} \neq T^T$ $\quad (T_1 T_2)^{-1} = T_2^{-1} T_1^{-1}$

$$T^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix}$$

T^T doesn't have anything special

For $T_1 \in SE(3)$, $T_2 \in SE(3)$, $T_3 \in SE(3)$ $\begin{cases} T_1 T_2 \in SE(3) \\ (T_1 T_2) T_3 = T_1 (T_2 T_3) \\ T_1 T_2 \neq T_2 T_1 \end{cases}$

$$T \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}$$ $\Rightarrow$ for a vector v_b in frame b

$$\begin{bmatrix} v_s \\ 1 \end{bmatrix} = T_{sb} \begin{bmatrix} v_b \\ 1 \end{bmatrix} = \begin{bmatrix} R_{sb} & p_{sb} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_b \\ 1 \end{bmatrix}$$

Proposition: Given $T = (R, p) \in SE(3)$ and $x, y \in \mathbb{R}^3$

① $\|Tx - Ty\| = \|x - y\|$ $\quad \|\cdot\|$ means norm

② $\langle Tx - Ty, Ty - Tz \rangle = \langle x - y, y - z \rangle$

$\langle x, x \rangle$ means inner product

Three uses of SE(3):

(1) Reepresent the configuration

(2) Change the frame repbresentation to another frame

(3) A operation of moving a vector

$T_{ab} = T_{ba}^{-1} \qquad T_{ab}T_{bc} = T_{ac} \qquad T_{ab}v_b = v_a$

★ When we treat T as a displacement operation, huge difference between pre-multiply and post-multiply

Assume $T = \mathrm{Trans}(p)\mathrm{Rot}(\hat{\omega}, \theta)$, $T_{sb} \in SE(3)$

① $T T_{sb} = \mathrm{Trans}(p)\mathrm{Rot}(\hat{\omega}, \theta) T_{sb}$

Rotate first, then transpose, all operation treats $\{s\}$ as main frame

② $T_{sb} T = T_{sb}\mathrm{Trans}(p)\mathrm{Rot}(\hat{\omega}, \theta)$

Translate first, then rotate, all operation treats $\{b\}$ as main frame

窍门是先以T_{sb}为中心，T_{sb}在后方，就往前推并以T_{sb}前一个sub-script的frame为基准

Just similar to $R^{-1}\dot{R} = [w]$, $T^{-1}\dot{T} = \begin{bmatrix} [w_b] & v_b \\ 0 & 0 \end{bmatrix}$

$R^{-1}\dot{R}$ represent angular velocity, $T^{-1}\dot{T}$ represent the combination of angular and linear velocity

A better way to represent the $\begin{bmatrix} [w_b] & v_b \\ 0 & 0 \end{bmatrix}$ is to introduce twist $V = S\dot{\theta}$

$\begin{cases} S\text{, unit screw axis, } \begin{bmatrix} w \\ v \end{bmatrix} \\ \dot{\theta}\text{, rate of change} \end{cases}$

$$\mathcal{V} = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} \in \mathbb{R}^6 \qquad [\mathcal{V}] = \begin{bmatrix} [\omega_s] & v_s \\ 0 & 0 \end{bmatrix} \in se(3)$$

Just like $[\omega_b] = R_{sb}^{-1}[\omega_s]R_{sb}$

$$\Downarrow$$

$$[\mathcal{V}_b] = T_{sb}^{-1}[\mathcal{V}_s]T_{sb}$$

$$[\mathcal{V}_s] = T_{sb}[\mathcal{V}_b]T_{sb}^{-1} = \begin{bmatrix} R[\omega_b]R^T & -R[\omega_b]R^Tp + Rv_b \\ 0 & 0 \end{bmatrix}$$

$$\Downarrow$$

Like $R[\omega]R^T = [R\omega]$, $[\omega]p = -[p]\omega$, we have

$$\underbrace{\begin{bmatrix} \omega_s \\ v_s \end{bmatrix}}_{\substack{\mathcal{V}_s \\ 6\times 1}} = \underbrace{\begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix}}_{\substack{\text{Adjoint Matrix} \\ 6\times 6}} \underbrace{\begin{bmatrix} \omega_b \\ v_b \end{bmatrix}}_{\substack{\mathcal{V}_b \\ 6\times 1}}$$

For a $T = (R, p) \in SE(3)$, its adjoint matrix is

$$[Ad_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix}.$$

$$\mathcal{V}_s = [Ad_{T_{sb}}]\mathcal{V}_b = Ad_{T_{sb}}(\mathcal{V}_b)$$

For $T_1, T_2 \in SE(3)$ and $\mathcal{V} = (\omega, v)$, we have:

$$\begin{cases} Ad_{T_1}(Ad_{T_2}(\mathcal{V})) = [Ad_{T_1}][Ad_{T_2}]\mathcal{V} = Ad_{T_1T_2}(\mathcal{V}) = [Ad_{T_1T_2}]\mathcal{V} \\ Ad_T^{-1}(Ad_T(\mathcal{V})) = Ad_{T^{-1}T}(\mathcal{V}) = Ad_I(\mathcal{V}) \end{cases}$$

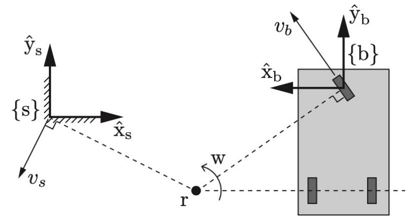

Figure 3.18: The twist corresponding to the instantaneous motion of the chassis of a three-wheeled vehicle can be visualized as an angular velocity w about the point r.

point r is in (2, −1, 0) in {s} and (2, −1.4, 0) in {b}

w = 2 rad/s about an axis out of the page at the point r in the plane. Inspecting the figure, we can write r as $r_s = (2, -1, 0)$ or $r_b = (2, -1.4, 0)$, w as $\omega_s = (0, 0, 2)$ or $\omega_b = (0, 0, -2)$, and T_{sb} as

$$T_{sb} = \begin{bmatrix} R_{sb} & p_{sb} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0.4 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$v_s = w \times (-r) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \times 2$$

$$= \begin{matrix} i & j & k \\ 0 & 0 & 1 \\ -2 & 1 & 0 \end{matrix} = \begin{bmatrix} -2 \\ -4 \\ 0 \end{bmatrix}$$

$$v_b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ -1.4 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.8 \\ 4 \\ 0 \end{bmatrix}$$

From the figure and simple geometry, we get

$$v_s = \omega_s \times (-r_s) = r_s \times \omega_s = (-2, -4, 0),$$
$$v_b = \omega_b \times (-r_b) = r_b \times \omega_b = (2.8, 4, 0),$$

and thus obtain the twists $\mathcal{V}_s$ and $\mathcal{V}_b$:

$$\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ -2 \\ -4 \\ 0 \end{bmatrix}, \quad \mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2 \\ 2.8 \\ 4 \\ 0 \end{bmatrix}.$$

$\Leftarrow$ Twist in {s} and {b}

Representation of twist using screw axis

① $\{q, \hat{s}, h\}$, q is a point on $\hat{s}$ axis

h is ratio of linear velocity to rotation velocity

$$V = \begin{bmatrix} w \\ v \end{bmatrix} = \begin{bmatrix} \hat{s}\dot{\theta} \\ -\hat{s}\dot{\theta} \times q + h\hat{s}\dot{\theta} \end{bmatrix}$$

② Screw Axis, S can be regarded as normalized version of Twist, $S = V / \|w\| = V / \dot{\theta}$

If $w = 0$, means no rotational motion, only linear

h = linear velocity / rotational velocity, goes to ∞

$S_w = 0$, $\|S_v\| = 1$

If $w \neq 0$, $\|S_w\| = 1$, $\|S_v\| = 1$

Same as twist $\mathcal{V}$, $[S] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix}$

$S_a = [Ad_{T_{ab}}] S_b$

☆ If a joint has a pitch property, means it provides extra linear velocity in direction of screw rotation axis.

① Calculate S_ω and S_v as before, use $v = -\omega \times r$ get the linear velocity given by rotation

② Treat pitch as an extra component with linear velocity only in direction of screw rotation direction

③ Add together

Exp: $S_\omega = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $p = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$, $\theta = 1$, pitch $= 3$

① $S_v = -S_\omega \times r = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\mathcal{V}_1 = \begin{bmatrix} S_\omega \\ S_v \end{bmatrix} \times \theta = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

② Due to pitch, $\mathcal{V}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 3 \end{bmatrix}$

③ $\mathcal{V} = \mathcal{V}_1 + \mathcal{V}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 3 \end{bmatrix}$