For a 2R robot, its end-effector can be:

$$\begin{cases} x_1 = L_1\cos\theta_1 + L_2\cos(\theta_1+\theta_2) \\ x_2 = L_1\sin\theta_1 + L_2\sin(\theta_1+\theta_2) \end{cases}$$

$$\begin{cases} \dot{x}_1 = -L_1\dot{\theta}_1\sin\theta_1 - L_2(\dot{\theta}_1+\dot{\theta}_2)\sin(\theta_1+\theta_2) \\ \dot{x}_2 = L_1\dot{\theta}_1\cos\theta_1 + L_2(\dot{\theta}_1+\dot{\theta}_2)\cos(\theta_1+\theta_2) \end{cases}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -L_1\sin\theta_1 - L_2\sin(\theta_1+\theta_2) & -L_2\sin(\theta_1+\theta_2) \\ L_1\cos\theta_1 + L_2\cos(\theta_1+\theta_2) & L_2\cos(\theta_1+\theta_2) \end{bmatrix}}_{\Downarrow \text{ Jacobian Matrix}} \underbrace{\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}}_{\Downarrow\ dq/dt}$$

$$V_{tip} = J_1(\theta)\dot{\theta}_1 + J_2(\theta)\dot{\theta}_2$$

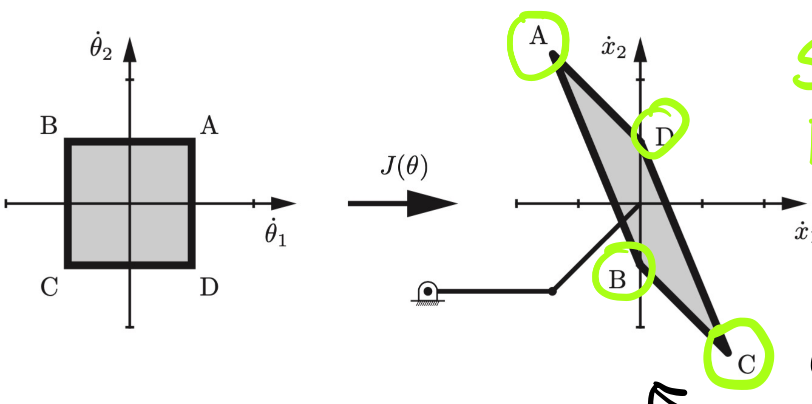
$\Rightarrow J(\theta) \Rightarrow$ Jacobian only has relation with variable θ

has no relation with velocity $\dot{\theta}$

$\Rightarrow \theta$ joint config is dicided, Jacobian matrix is fixed,

but $\dot{\theta}$ is still undicided.

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\text{undicided}} = \underbrace{J(\theta)}_{\text{Fixed}} \underbrace{\dot{\theta}}_{\text{undicided}} \Rightarrow \text{Map } \underbrace{\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}}_{\text{Joints}} \text{ to } \underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\text{End Effector}}$$



Same Joints Config

Different velocity Config

Another Joint corresponds to another shape of $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$

For 2R robot, if $\theta_2 = 0^\circ$ or 180°, two links are co-linear. In this case:

- Jacobian loses one rank
- corresponding velocity goes to "0"

This case called singularity, $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$ collapse to a line

---

⇒ Define the longest and shortest ellipsoids axis to be the l_{max} and l_{min}.

⇒ More l_{max}/l_{min} close to 1, the more ellipsoid shapes like a circle, which means more stable, more away from being a singularity

- F_{tip}, wrench at end-effector
- V_{tip}, twist at end-effector
- τ, joint torque
- $\dot{\theta}$, joint velocity

For conservation of the power, we have:

$$F_{tip}^T V_{tip} = \tau^T \dot{\theta}$$

$$F_{tip}^T J(\theta) \dot{\theta} = \tau^T \dot{\theta}$$

for all possible $\dot{\theta}$, we have

$$F_{tip}^T J(\theta) = \tau^T$$

$$\tau = J(\theta)^T F_{tip}$$

we have relations:

$$\begin{cases} \mathcal{V}_{tip} = J(\theta)\dot{\theta} \\ \tau = J(\theta)^T F_{tip} \end{cases}$$

if $J(\theta)$ is invertible, square $\Rightarrow F_{tip} = J^{-T}(\theta)\,\tau$

When it's easy to generate velocity at a given direction, it would be difficult to generate force at in that direction.

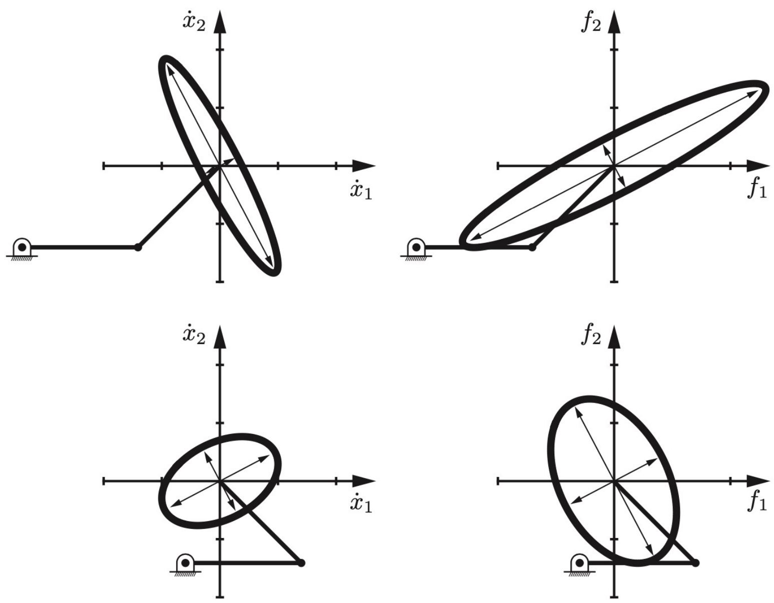


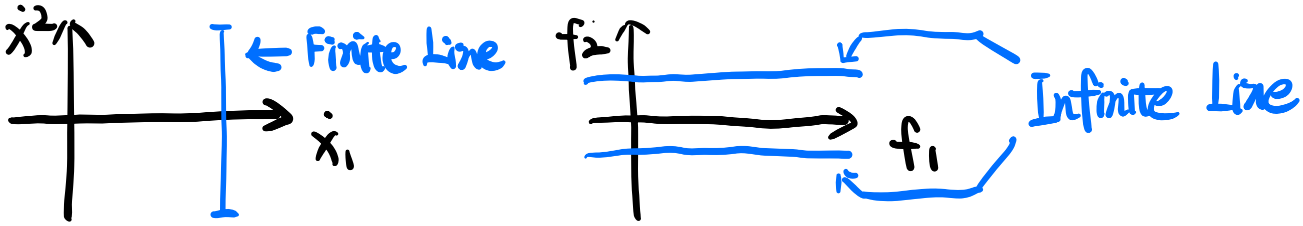
Figure 5.6: Left-hand column: Manipulability ellipsoids at two different arm configurations. Right-hand column: The force ellipsoids for the same two arm configurations.

When robot is in singularity

① Velocity ellipsoid collapse to a line

② Force ellipsoid becomes infinitely large

Exp: For a 2R robot, what if $\theta_2 = 0°$ or $180°$:



① Can't move any further in x, therefore no velocity in x-axis can be generated

② Not feasible to have infite velocity, y-axis velocity also has boundary

① A distance close to "0", force needs to be infitely large to match some τ

② y-axis is somehow available due to finite distance

The calculation of each colume of Jacobian is pretty similar to that of Screw Axis, S. The difference is Jacobian focuses on current config instead of Home Config.

$$V_s = J_s(\theta)\,\dot{\theta}$$

Each colume of J_s is Twist of that joint represent in frame {s}

$$V_B = J_b(\theta)\,\dot{\theta}$$

$$J_{s_i}(\theta) = Ad_{e^{[S_1]\theta_1}\cdots e^{[S_{i-1}]\theta_{i-1}}}[S_i]$$

↳ Influence brought by joints between frame {s} and joint i

$$J_{B_i}(\theta) = Ad_{e^{-[B_n]\theta_n}\cdots e^{-[B_{i+1}]\theta_{i+1}}}[B_i]$$

↳ Influence brought by joints between freme {b} and joint i

$$[V_s] = \dot{T}_{sb}T_{sb}^{-1} \qquad V_s = Ad_{T_{sb}}(V_b)$$

$$[V_b] = T_{sb}^{-1}\dot{T}_{sb} \qquad V_b = Ad_{T_{bs}}(V_s)$$

$$\Downarrow$$

$$J_b = [Ad_{T_{bs}}]J_s(\theta)$$

$$J_s = [Ad_{T_{sb}}]J_b(\theta)$$

When robot has joint number $n > 6$, this robot

① When robot has joint number $n > 6$, this robot is redundant. It has dof inside even if we immoblize its end-effector

End-effector in 3-D only has 6-dimensions, for 7R robot, it can have more than 1 solution to reach the desired config

② if $n \le 6$, and $\text{rank}(J) = 6$, the robot can't move once we immoblize its end-effector

③ if $n < 6$, no matter what v we choose, robot loses its ability generating force at certain direction, 6-N

$\downarrow$

Null Space of J^T

$$\text{Null}(J^T(\theta)) = \{F \mid J^T(\theta)F = 0\}$$

In direction of Null space, $\begin{cases} \text{can't generate force} \\ \text{can resist arbitrary force} \end{cases}$