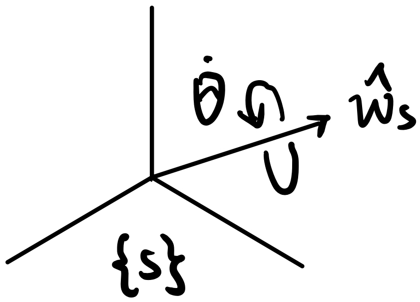

$\hat{w}_s$, rotation axis

$\dot{\theta}$, rate of rotation

w, angular velocity

$w = \hat{w}\dot{\theta}$

$$\left.\begin{aligned} \dot{\hat{x}} &= w \times \hat{x} \\ \dot{\hat{y}} &= w \times \hat{y} \\ \dot{\hat{z}} &= w \times \hat{z} \end{aligned}\right\}$$

Rate of direction change

$\hat{x}, \hat{y}, \hat{z}$ all determined by current position

For R be the R_{sb} representation, r_1, r_2, r_3 are its columns, representing its $\hat{x}, \hat{y}, \hat{z}$ axis in fixed frame $\{s\}$, therefore we have:

$$\dot{R} = [\dot{r}_1, \dot{r}_2, \dot{r}_3] = [w_s \times r_1, w_s \times r_2, w_s \times r_3] = w_s \times R$$
$$= \hat{w}_s \dot{\theta} \times R$$

$a \times b = [a]\, b$, where $[a]$. bracket form of a, equals as:

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad [a] = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

$[a]$ is a skew-symmetric matrix, with property:

$$[a] = -[a]^T \in so(3)$$

$$R[w]R^T = [Rw] \qquad \dot{R} = w \times R = [w]R$$

$$[w] = \dot{R}R^{-1}$$

By sub-script cancellation Rule:

$$w_b = R_{sb}^{-1} w_s = R_{sb}^T w_s$$

$$[w_b] = [R_{sb}^T w_s]$$

$$= R_{sb}^T [w_s] R_{sb}$$

$$= R_{sb}^T [\dot{R}_{sb} R_{sb}^{-1}] R_{sb}$$

$$= R_{sb}^T \dot{R}_{sb} = R_{sb}^{-1} \dot{R}_{sb}$$

$$\dot{R}_{sb} R_{sb}^{-1} = [w_s] \qquad R_{sb}^{-1} \dot{R}_{sb} = [w_b]$$

Exponential Representation for Rotation:

$R = \hat{w}\theta$, $\hat{w}$ is rotation axis

θ is the total angle

$\hat{w}$ tells rotation center, and θ tells how much need to go

$\dot{x} = ax(t)$

$x = e^{at} x_0$

$e^{at} = 1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \cdots$

$\dot{x} = Ax(t)$

$x = e^{At} x_0$

$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots$

Proposition 3.10. *The linear differential equation $\dot{x}(t) = Ax(t)$ with initial condition $x(0) = x_0$, where $A \in \mathbb{R}^{n\times n}$ is constant and $x(t) \in \mathbb{R}^n$, has solution*

$$x(t) = e^{At} x_0 \tag{3.47}$$

where

$$e^{At} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots . \tag{3.48}$$

The matrix exponential e^{At} further satisifies the following properties:

(a) $d(e^{At})/dt = Ae^{At} = e^{At}A$.

(b) *If $A = PDP^{-1}$ for some $D \in \mathbb{R}^{n\times n}$ and invertible $P \in \mathbb{R}^{n\times n}$ then $e^{At} = Pe^{Dt}P^{-1}$.*

(c) *If $AB = BA$ then $e^A e^B = e^{A+B}$.*

(d) $(e^A)^{-1} = e^{-A}$.

Now consider case with $\dot{\theta} = 1$ rad/s

$\dot{p} = w \times p = [w]p = [\hat{w}]p$

$\Downarrow$

$p(t) = e^{[\hat{w}]t} p(0)$

$p(\theta) = e^{[\hat{w}]\theta} p(0)$

we know for $[\hat{w}]$, also for all skew-symmetric matrix, $[\hat{w}] = -[\hat{w}]^T$

therefore $[\hat{\omega}]^3 = [\hat{\omega}](-[\hat{\omega}]^T)([\hat{\omega}])$

$$[\hat{\omega}]^3 = -[\hat{\omega}]$$

$$[\hat{\omega}]^4 = -[\hat{\omega}]^2$$

$$e^{[\hat{\omega}]\theta} = I + [\hat{\omega}]\theta + [\hat{\omega}]^2\frac{\theta^2}{2!} + [\hat{\omega}]^3\frac{\theta^3}{3!} + [\hat{\omega}]^4\frac{\theta^4}{4!} + \cdots$$

$$= I + [\hat{\omega}]\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) + [\hat{\omega}]^2\left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \cdots\right)$$

$$\begin{cases} \sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots \\ \cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} \cdots \end{cases}$$

$so(3) \Rightarrow SO(3)$

$$\underbrace{e^{[\hat{\omega}]\theta} = I + \sin\theta[\hat{\omega}] + (1-\cos\theta)[\hat{\omega}]^2}_{\text{Rodrigues' Formula, Matrix Exponentiation}}$$

$e^{[\hat{\omega}]\theta}$ is $Rot(\hat{\omega}, \theta)$

Example 3.12. The frame {b} in Figure 3.12 is obtained by rotation from an initial orientation aligned with the fixed frame {s} about a unit axis $\hat{\omega}_1 = (0, 0.866, 0.5)$ by an angle $\theta_1 = 30° = 0.524$ rad. The rotation matrix representation of {b} can be calculated as

$$\begin{aligned} R &= e^{[\hat{\omega}_1]\theta_1} \\ &= I + \sin\theta_1[\hat{\omega}_1] + (1-\cos\theta_1)[\hat{\omega}_1]^2 \\ &= I + 0.5\begin{bmatrix} 0 & -0.5 & 0.866 \\ 0.5 & 0 & 0 \\ -0.866 & 0 & 0 \end{bmatrix} + 0.134\begin{bmatrix} 0 & -0.5 & 0.866 \\ 0.5 & 0 & 0 \\ -0.866 & 0 & 0 \end{bmatrix}^2 \\ &= \begin{bmatrix} 0.866 & -0.250 & 0.433 \\ 0.250 & 0.967 & 0.058 \\ -0.433 & 0.058 & 0.899 \end{bmatrix}. \end{aligned}$$

$\exp: [\hat{\omega}]\theta \in so(3) \to R \in SO(3)$

$\log: R \in SO(3) \rightarrow [\hat{w}]\theta \in so(3)$

① When $\theta \neq$ any integer multiply of "π"

By expanding $\hat{w}$, $\hat{w}^2$ to $[\hat{w}]$, $[\hat{w}]^2$, multiplying $\sin\theta_1$ and $(1-\cos\theta_1)$ respectively. Then add $I + \sin\theta_1[\hat{w}] + (1-\cos\theta_1)[\hat{w}]^2$

$$\begin{bmatrix} c_\theta + \hat{\omega}_1^2(1-c_\theta) & \hat{\omega}_1\hat{\omega}_2(1-c_\theta) - \hat{\omega}_3 s_\theta & \hat{\omega}_1\hat{\omega}_3(1-c_\theta) + \hat{\omega}_2 s_\theta \\ \hat{\omega}_1\hat{\omega}_2(1-c_\theta) + \hat{\omega}_3 s_\theta & c_\theta + \hat{\omega}_2^2(1-c_\theta) & \hat{\omega}_2\hat{\omega}_3(1-c_\theta) - \hat{\omega}_1 s_\theta \\ \hat{\omega}_1\hat{\omega}_3(1-c_\theta) - \hat{\omega}_2 s_\theta & \hat{\omega}_2\hat{\omega}_3(1-c_\theta) + \hat{\omega}_1 s_\theta & c_\theta + \hat{\omega}_3^2(1-c_\theta) \end{bmatrix}$$

$C_\theta = \cos(\theta)$ $\quad S = \sin(\theta)$ $\quad \hat{w} = \begin{bmatrix} \hat{w}_1 \\ \hat{w}_2 \\ \hat{w}_3 \end{bmatrix}$

in matrix r, we have:

$$\begin{cases} r_{32} - r_{23} = 2\hat{w}_1 \sin\theta \\ r_{13} - r_{31} = 2\hat{w}_2 \sin\theta \\ r_{21} - r_{12} = 2\hat{w}_3 \sin\theta \end{cases}$$

$\Downarrow$ (when $\sin\theta \neq 0$)

$$\begin{cases} \hat{w}_1 = \frac{1}{2\sin\theta}(r_{32} - r_{23}) \\ \hat{w}_2 = \frac{1}{2\sin\theta}(r_{13} - r_{31}) \\ \hat{w}_3 = \frac{1}{2\sin\theta}(r_{21} - r_{12}) \end{cases}$$

$$[\hat{w}] = \begin{bmatrix} 0 & -\hat{w}_3 & \hat{w}_2 \\ \hat{w}_3 & 0 & -\hat{w}_1 \\ -\hat{w}_2 & \hat{w}_1 & 0 \end{bmatrix} = \frac{1}{2\sin\theta}(R - R^T)$$

$$\operatorname{tr} R = 1 + 2\cos\theta$$

② When θ is the integer multiply of π

Assume $\theta = k\pi$, we have:

$\Rightarrow$ when k is an even number:

$$e^{[\hat{w}]\theta} = I + \sin\theta[\hat{w}] + (1-\cos\theta)[\hat{w}]^2$$
$$= I$$

All $[\hat{w}]$ ends with "I", therefore $[\hat{w}]$ is undefined under this case

$\Rightarrow$ when k is an odd number:

$$e^{[\hat{w}]\theta} = I + 2[\hat{w}]^2 \qquad \operatorname{tr} R = 1 + 2\cos\theta = -1$$

$$\hat{w}_i = \pm\sqrt{\frac{r_{ii}+1}{2}}, \quad i = 1, 2, 3$$

or

$$\{ 2\hat{w}\,\hat{w} = r$$

$$\begin{cases} 2\hat{w}_1\hat{w}_2 = r_{12} \\ 2\hat{w}_2\hat{w}_3 = r_{23} \\ 2\hat{w}_1\hat{w}_3 = r_{13} \end{cases}$$

$\rightarrow$ Implies R must be symmetric

Summary for log algorithm:

(a) if $R = I$ ($\theta = 2k\pi$), $\hat{w}$ undefined, θ regard as 0

(b) if $\operatorname{tr} R = -1$, $\theta = \pi$, and $\hat{w}$:

$$\hat{w} = \frac{1}{\sqrt{2(1+r_{33})}} \begin{bmatrix} r_{13} \\ r_{23} \\ 1+r_{33} \end{bmatrix}$$

or

$$\hat{w} = \frac{1}{\sqrt{2(1+r_{22})}} \begin{bmatrix} r_{12} \\ 1+r_{22} \\ r_{32} \end{bmatrix}$$

or

$$\hat{w} = \frac{1}{\sqrt{2(1+r_{11})}} \begin{bmatrix} 1+r_{11} \\ r_{12} \\ r_{13} \end{bmatrix}$$

(c) Otherwise, θ is not integer multiple of π:

$$\theta = \cos^{-1}\left(\frac{1}{2}(\operatorname{tr} R - 1)\right) \in [0, \pi)$$

$$[\hat{w}] = \frac{1}{2\sin\theta}(R - R^T)$$