Sr. No.

Note

Dated

Section - B

Vectors

Using vector algebra, find the area of a parallelogram / triangle. Also derive the area analytically and verify the same.

A Product

Teacher's Signature :

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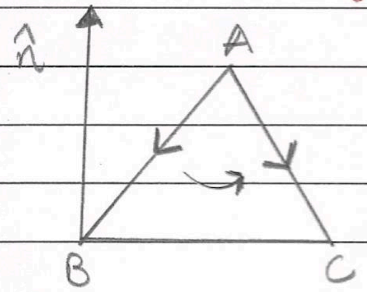
Vector Area of a Triangle



Let Δ be the area of triangle ABC. Let $\hat{n}$ be a unit vector perperpendicular to plane of the triangle considered in anticlockwise direction from $\vec{AB}$ to $\vec{AC}$.

Now, $\vec{AB}$, $\vec{AC}$, $\hat{n}$ form a vector triad in a right handed system. Then vector area of the triangle is $\Delta \hat{n}$

$$\Delta = \text{area of triangle } ABC$$
$$= \frac{1}{2} bc \sin A = \frac{1}{2} cb \sin A$$
$$= \frac{1}{2} |\vec{AB}| \, |\vec{AC}| \sin A$$

$\therefore$ vector area of triangle ABC

$$= \Delta \hat{n} = \frac{1}{2} |\vec{AB}| \, |\vec{AC}| \sin A \; \hat{n}$$
$$= \frac{1}{2} (\vec{AB} \times \vec{AC})$$

If vertices are represented by their position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ respectively.

Then,

Vector area of ΔABC

$$= \frac{1}{2} (\vec{AB} \times \vec{AC})$$

$$= \frac{1}{2} (\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})$$

$$= \frac{1}{2} [\vec{b} \times \vec{c} - \vec{b} \times \vec{a} - \vec{a} \times \vec{c}]$$

$$\boxed{\Delta = \frac{1}{2} [\vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b}]}$$

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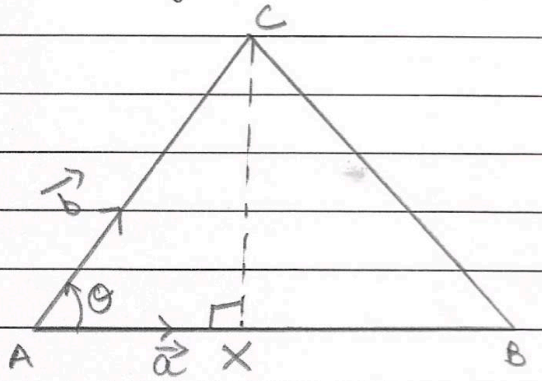
Verification

If vectors $\vec{a}$ and $\vec{b}$ represent two adjacent sides of a triangle, then its area $= \frac{1}{2}|\vec{a}\times\vec{b}|$



Let vector $\vec{a}$ and $\vec{b}$ represent two adjacent sides of the triangle ABC, from C draw a CX perpendi-cular to AB and let $\angle CAX = \theta$

From right angled ΔCAX

$$\sin\theta = \frac{CX}{AC} = \frac{CX}{|\vec{b}|} = \frac{CX}{|\vec{b}|}$$

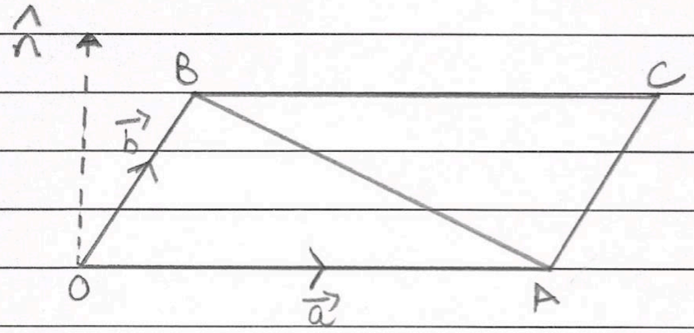
$$\Rightarrow CX = |\vec{b}|\sin\theta$$

Area of Triangle ABC $= \frac{1}{2}(\text{base}) \times \text{height}$

$$= \frac{1}{2} AB \cdot CX$$

$$= \frac{1}{2}|\vec{a}|\,|\vec{b}|\sin\theta = \frac{1}{2}|\vec{a}\times\vec{b}|$$

Vector Area of a Parallelogram



Let OACB be a parallelogram

Let $\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$

Let $\hat{n}$ be the unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ so that $\vec{a}, \vec{b}, \hat{n}$ form a vector triad in a right hand system

$\therefore$ Area of triangle $OAB = \frac{1}{2} |\vec{OA} \times \vec{OB}|$

$$= \frac{1}{2} |\vec{a} \times \vec{b}|$$

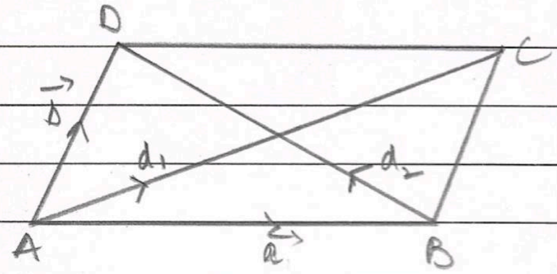
$\therefore$ Area of parallelogram OACB =

2 × Area of triangle OAB

$$= 2 \times \frac{1}{2} |\vec{a} \times \vec{b}| = |\vec{a} \times \vec{b}|$$

$$\boxed{\text{Area of parallelogram} = |\vec{a} \times \vec{b}|}$$

If the diagnols of the parallelogram are given, then,



Let $\vec{AB} = \vec{a}$, $\vec{AD} = \vec{b}$, $\vec{AC} = \vec{d_1}$, $\vec{BD} = \vec{d_2}$

Since, diagnols of a parallelogram bisect each other,

$$\vec{AO} = \frac{1}{2}\vec{d_1} \quad , \quad \vec{BO} = \frac{1}{2}\vec{d_2}$$

Now, by triangle law of vector addition

$$\vec{a} = \vec{AB} = \vec{AO} + \vec{OB} = \frac{1}{2}(\vec{d_1} - \vec{d_2})$$

$$\vec{b} = \vec{AD} = \vec{AO} + \vec{OD} = \frac{1}{2}(\vec{d_1} + \vec{d_2})$$

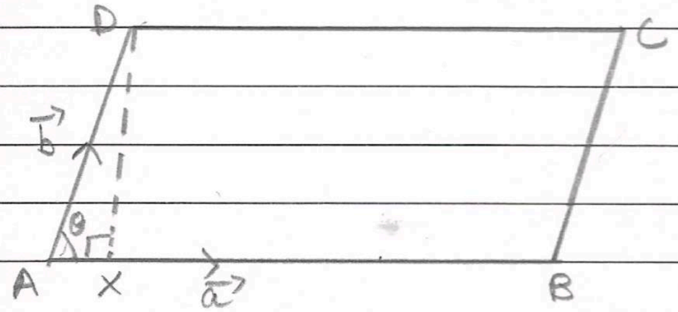
Area of parallelogram ABCD $= |\vec{a} \times \vec{b}|$

$$= \left| \frac{1}{2}(\vec{d_1} - \vec{d_2}) \times \frac{1}{2}(\vec{d_1} + \vec{d_2}) \right|$$

$$\boxed{\text{Area} = \frac{1}{2}|\vec{d_1} \times \vec{d_2}|}$$

Verification

If vector $\vec{a}$ and $\vec{b}$ represent two adjacent sides of a parallelogram, then its area $= |\vec{a} \times \vec{b}|$



Let vectors $\vec{a}$ and $\vec{b}$ represent two adjacent sides AB and AD of the parallelogram ABCD. From D, draw DX perpendicular to AB and let $\angle DAX = \theta$

From right angled $\triangle DAX$

$$\sin\theta = \cancel{DAX}\ \frac{DX}{AD} = \frac{DX}{|\vec{b}|} \Rightarrow DX = |\vec{b}| \sin\theta$$

Area of parallogram = Base × height

$$= AB \cdot DX$$

$$= |\vec{a}|\,|\vec{b}| \sin\theta$$

$$= |\vec{a} \times \vec{b}|$$