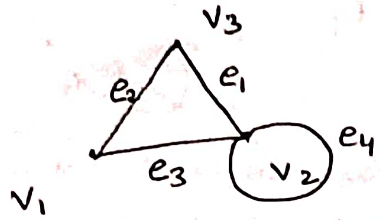
Graphs

A graph is consist of V, a non-empty set of vertices (or nodes) and E, a set of edges. Each edge has either one or two vertices associated with it, called its end points.

Eg:

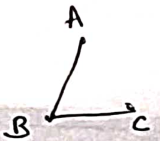


$V = \{V_1, V_2, V_3\}$
$E = \{e_1, e_2, e_3, e_4\}$

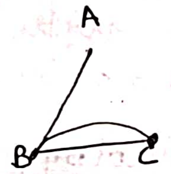
Self loop:- An edge having same vertices as its end points.

Simple graph:- A simple graph is a graph that does not have more than one edge between any two vertices and no edge starts and ends at the same vertex. In other words, a simple graph is a graph without loops and multiple edges.

Eg:



Simple graph



not a simple graph

Infinite and finite graph:-

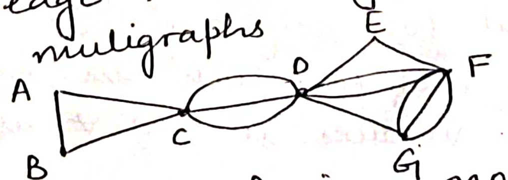
The set of vertices V of a graph may be infinite. A graph with an infinite vertex is called an infinite graph.

Eg: Telecommunication of the whole world.

- A graph with a finite vertex set is called a finite graph.

Multiple edges and multigraphs:-

A computer network may contain multiple links b/w data centres. The (graph) model, in which we have more than one edge containing the same pairs of vertices are called muligraphs

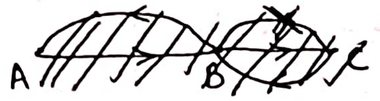


- Multi edges & Multigraphs

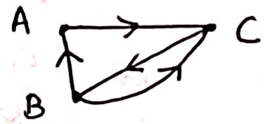
Pseudograph: A pseudograph is a non-simple graph in which both ~~graph~~ loops and multiple edges are allowed.

Null graph: A graph whose edge set is empty. In other words, a graph with vertices without edges.

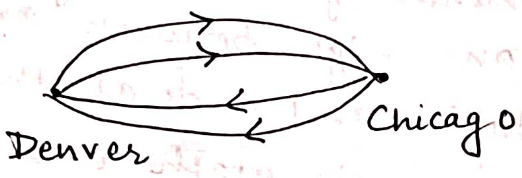
Directed graph :- A directed graph (or diagraph) (V, E) consists of a non-empty set of vertices V and a set of directed edges (or arcs) E. Each directed edge is associated with an ordered pair of vertices. The directed edge associated with the ordered pair (u, v) is said to start at u and end at v.



Eg:



- Directed graph that may have multiple directed edges from a vertex to a second vertex. Such graphs are called directed multigraphs.



Mixed graph

- A graph with both directed and undirected edges is called a mixed graph.

Graph terminology

Type	Edges	Multiple Edges allowed?	Loops Allowed?
Simple graph	Undirected	✗	✗
Multigraph	"	✓	✗
Pseudograph	"	✓	✓
Simple directed	Directed	✗	✗
Directed graph	"	✓	✓
Mixed graph	Both	✓	✓

Basic terminology related to vertices and edges

Def^n 1 Two vertices u and v in an undirected graph G are called adjacent (or neighbours) in G if u and v are endpoints of an edge e of G. Such an edge e is called incident with the vertices u and v is said to connect u and v.

Def^n 2 The degree of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex.

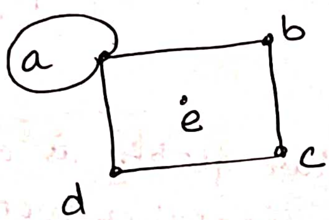
The degree of the vertex v is denoted by $\deg(v)$.

$$\boxed{\text{Total degree of vertex} = \text{Edges incident} + 2\cdot(\text{No. of self loops})}$$

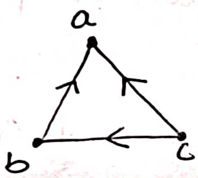
* In a directed graph, each vertex has an indegree and an outdegree.

- Indegree of a graph :- Indegree of vertex V is the number of edges which are coming into the vertex V, denoted by $\deg^{-}(v)$
- Outdegree of a graph : Outdegree of vertex V is the number of edges which are going out from the vertex V, denoted by $\deg^{+}(v)$

Eg :-



$\deg(a) = 4$
$\deg(b) = 2$
$\deg(c) = 2$
$\deg(d) = 2$
$\deg(e) = 0$



$\deg(a) = \text{Indegree}(a) + \text{Outdegree}(a)$
$= 2 + 0 = 2$

$\deg(b) = 1 + 1$
$= 2$

$\deg(c) = 0 + 2$

Def 3 :- A vertex of degree zero is called isolated

Def 4 : A vertex is pendant if and only if it has degree one.

Handshake Theorem :- Let $G = (V, E)$ be an undirected graph with m edges. Then

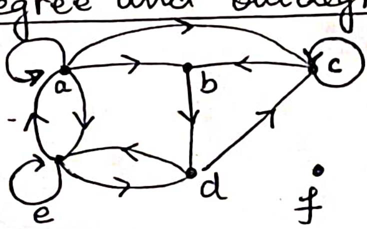
$$2m = \sum_{v \in V} \deg(v) = \text{Total degree of a graph}$$

Theorem :- Number of odd degree vertices are always even in undirected graph.

Proof $\quad \Sigma$ odd degree $+ \Sigma$ even degree $= 2m$

Σ odd degree $= 2m - \Sigma$ even degree

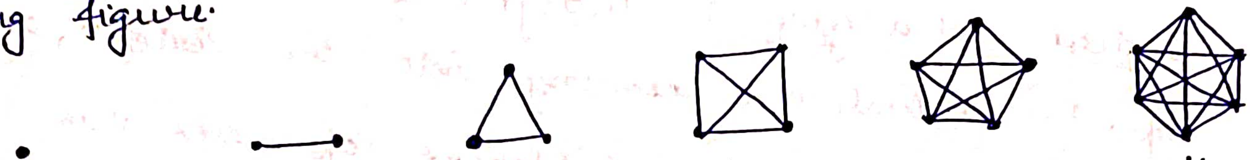
Indegree and Outdegree of each vertex in the graph G :-



Vertex	Indegree	Outdegree
a	2	4
b	2	1
c	3	2
d	2	2
e	3	3
f	0	0

Some special types of graphs

1. **Complete graph** :- A complete graph on n vertices, denoted by K_n, is a simple graph that contains exactly one edge between each pair of distinct vertices.
The graph K_n for $n = 1, 2, 3, 4, 5, 6$ are displayed in following figure.

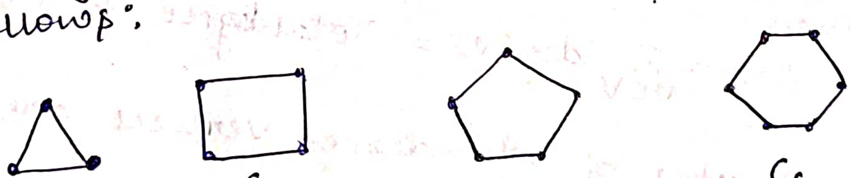


K_1 K_2 K_3 K_4 K_6

- Total number of edges in a complete graph of N vertices $= \frac{n(n-1)}{2}$.

2. **Cycles** :- A cycle graph or circular graph is a graph that consists of a single cycle, or in other words, some no. of vertices (at least 3, if the graph is simple) connected in a closed chain.
   - The cycle graph with n vertices is called C_n.
   - The number of vertices in C_n equals to the number of edges, and every vertex has degree 2 i.e., every vertex has exactly two edges incident with it.

Eg: The cycle C_3, C_4, C_5 and C_6 are displayed as follows:



C_3 C_4 C_5 C_6

3. **Regular graph** :- A graph is called regular graph if degree of each vertex is equal. A graph is called K regular if degree of each vertex in the graph is K.

2 Regular 3 regular

Properties of Regular - Graphs :-

- ★ A complete graph with N vertices is (N-1) regular
- ★ For a K Regular graph, if K is odd, then the number of vertices of the graph must be even. (In complete graph)
- ★ Cycle C_n is always 2 Regular
- ★ Number of edges of a K Regular graph with N vertices $= \frac{N*K}{2}$.

Proof: Let the number of edges of a K Regular graph with N vertices be E.

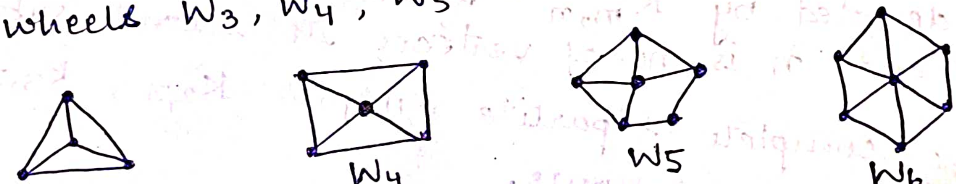
From Handshaking theorem, we know,
sum of degree of all the vertices = 2*E

$$N* E = 2*E$$

$$\text{or } E = \frac{N*K}{2}$$

4. **Wheel :-** A wheel graph is a graph formed by connecting a single universal vertex to all vertices of a cycle.

- ★ We obtain a wheel W_n when we add an additional vertex to a cycle C_n, for $n \geq 3$, and connect this new vertex to each of the n vertices in C_n, by new edges.
- ★ The wheels W_3, W_4, W_5, and W_6 are shown as follows



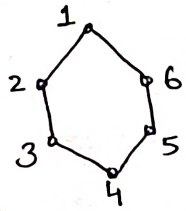
W_3 $\quad$ W_4 $\quad$ W_5 $\quad$ W_6

5. **Bipartite graphs.** A simple graph G is called bipartite if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects in V_1 and a vertex in V_2 and no edge in G connects either two vertices in V_1 or two vertices in V_2.

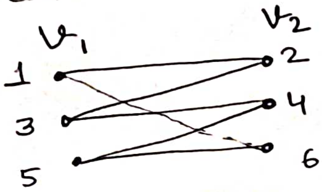
∴ K_3 is not Bipartite.

Q: C_6 is bipartite

Ans: C_6 is



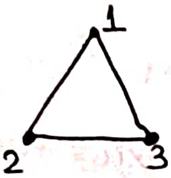
break into two vertices

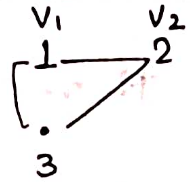


[1 is connected with 2, so it will be in opposite set of vertices]

∴ C_6 is bipartite

K_3 -

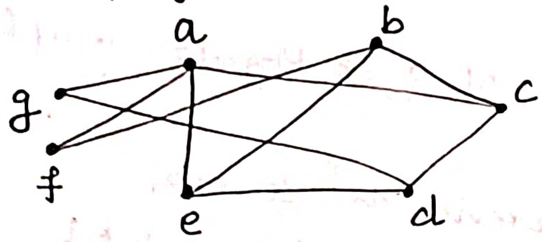




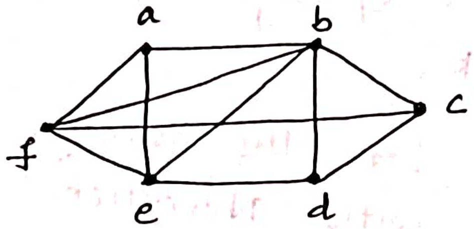
1 is also connected with 3 but 1 and 3 are in same set of vertices

$\therefore$ K_3 is not bipartite.

Q:- Are the graphs G and H displayed as follows bipartite?

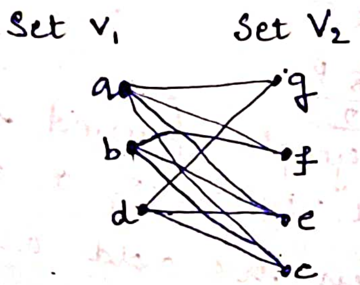


G

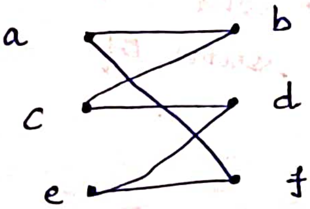


H

Solution:-



$\therefore$ G is bipartite



} in same set

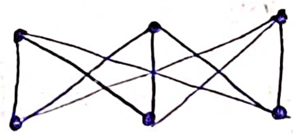
$\therefore$ H is not bipartite

Complete bipartite :- is a special type of bipartite graph where every vertex of one set is connected to every vertex of other set.

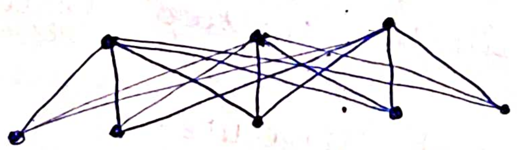
- It is denoted by $K_{m,n}$ where m is no. of vertices in ~~one~~ first ~~set~~ subset of V and n is no. of vertices in second subset of V.

Eg:- The complete bipartite graphs $K_{2,3}$, $K_{3,3}$, $K_{3,5}$ and K_{5} are shown as follows.

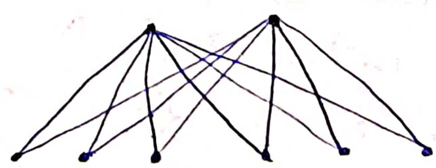
$K_{2,3}$



$K_{3,3}$



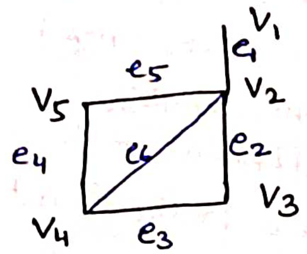
$K_{3,5}$



$K_{2,6}$

Connectivity:-

(1) ~~Paths~~ :
Path / Walk: A walk is a finite or infinite sequence of edges which joins a sequence of vertices.
For ex: $v_1 e_1 v_2 \dots e_{n-1} v_n$ where v_1 – initial vertex & v_n – terminal vertex



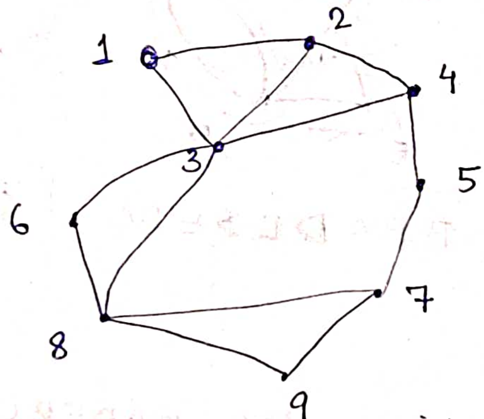
$v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_5$

Note: Vertices and Edges can be repeated
- Walk can be open or closed

Open Walk:- A walk is said to be an open walk if the starting and ending vertices are different.

Closed Walk:- A walk is said to be a closed walk if the starting and ending vertices are identical. That is, if a walk starts and ends at the same vertex.

(2) **Trail:-** Trail is an open walk in which no edge is repeated.
* Vertex can be repeated
* Trail is replaced by the term simple path.



Here $1 \to 3 \to 8 \to 6 \to 3 \to 2$ is trail.

Also $1 \to 3 \to 8 \to 6 \to 3 \to 2 \to 1$ will be a closed trail.

(3) **Circuit:-** A path with same starting and ending points. A circuit in a graph is also called as cycle in a graph.

* Circuit is a closed trail.

equivalent terms:
- Circuit
- Cycle
- Closed walk | Closed Path