Arbitrage through Linear Programming

We are given:

- Two securities: **1** and **2**, and three scenarios: **1, 2, 3**. Prices:

	today	**Scen. 1**	**Scen. 2**	**Scen. 3**
1	10	-7	19	-9
2	8	6	-8	8

Is arbitrage possible?

- We **take a position** today, we **close it tomorrow**
- What is today's value?
  The sum of the position values, using today's data!
- What is tomorrow's value?
  The sum of the position values, using tomorrow's data!
- **Arbitrage:** < 0 today, and ≥ 0 tomorrow in *every* scenario
- (But we need a riskless security to make this notion more complete)

	today	**Scen. 1**	**Scen. 2**	**Scen. 3**
1	10	-7	19	-9
2	8	6	-8	8

Example: Short 1 security 1, and **Short 1** security 2

Today: value -18

Tomorrow's values:

	today	Scen. 1	Scen. 2	Scen. 3
1	10	-7	19	-9
2	8	6	-8	8

Example: Short 1 security 1, and **Short 1** security 2

Today: value -18

Tomorrow's values:

Scen. 1	Scen. 2	Scen. 3
1	-11	1

	today	**Scen. 1**	**Scen. 2**	**Scen. 3**
1	10	-7	19	-9
2	8	6	-8	8

Example: **Long 1** security 1, and **Short 2** security 2

So we will have:

Today	**Scen. 1**	**Scen. 2**	**Scen. 3**
-6	-19	35	-25

	today	**Scen. 1**	**Scen. 2**	**Scen. 3**
1	10	-7	19	-9
2	8	6	-8	8

We also have **cash:** interest rate $= 8\%$ in every scenario

	today	**Scen. 1**	**Scen. 2**	**Scen. 3**
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We also have **cash:** interest rate = 8% in every scenario

How about **+5.5** cash, **−2** security 1, **1** security 2?

	today	**Scen. 1**	**Scen. 2**	**Scen. 3**
1	10	-7	19	-9
2	8	6	-8	8

We also have **cash:** interest rate = 8% in every scenario

How about **+5.5** cash, **−2** security 1, **1** security 2?

today	**Scen. 1**	**Scen. 2**	**Scen. 3**
-6.5	25.94	-40.06	31.94

E.g. (Scen. 1): $\mathbf{5.5 * 1.08 + (-7) * (-2) + 6 * 1 \ = \ 25.94}$

General case

We are given:

- n securities: $S_1, \ldots, S_n$, plus cash: this is security S_0
- A collection of K *scenarios* for what the price of each security will be tomorrow (or next month, etc.):

  In scenario $\boldsymbol{i}$, the price of security $\boldsymbol{j}$ will be $\boldsymbol{\pi_{ij}}$. $(\mathbf{1} \leq \boldsymbol{i} \leq \boldsymbol{K},\ \mathbf{1} \leq \boldsymbol{j} \leq \boldsymbol{n})$

  Security 0: be $\boldsymbol{\pi_{i0} = 1 + r}$, where $\boldsymbol{r}$ = risk-free interest rate $(\mathbf{1} \leq \boldsymbol{i} \leq \boldsymbol{K})$

  Scenario 0: today's prices (for cash, price = $\mathbf{1}$)

- How do we investigate the existence of arbitrage?

Use Linear Programming

- Use variables $\boldsymbol{x_j}$ = position we take in security $\boldsymbol{j}$ (incl. cash)

So today's value of our positions is:

$$x_0 \ + \ \Sigma_{j=1}^{n} \pi_{0j} \ x_j$$

And in scenario $\boldsymbol{i}$, tomorrow's value of our positions will be:

$$(1 + r)x_0 \ + \ \Sigma_{j=1}^{n} \pi_{ij} \ x_j$$

Two kinds of arbitrage:

> **Type A**. Today's position $< \mathbf{0}$, and in tomorrow's $\geq \mathbf{0}$ in *every* scenario

> **Type B**. **Zero** cash flow today, tomorrow's position $\leq \mathbf{0}$ in every scenario and $< \mathbf{0}$ in at least one scenario

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Two kinds of arbitrage:

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Exercise. Convince yourself that (because of the riskless security, cash) this is equivalent to the standard notion of arbitrage

$\boldsymbol{x_j}$ = position in security $\boldsymbol{j}$

Today's value: $\boldsymbol{x_0 + \Sigma_{j=1}^{n} \pi_{0j}\, x_j}$

Tomorrow's in scenario $\boldsymbol{i}$: $\boldsymbol{(1+r)x_0 + \Sigma_{j=1}^{n} \pi_{ij}\, x_j}$

Consider the linear program

$$\boldsymbol{V^* = \textbf{Minimize}\ x_0 + \Sigma_{j=1}^{n} \pi_{0j}\, x_j}$$

Subject to:

$$\boldsymbol{(1+r)x_0 + \Sigma_{j=1}^{n} \pi_{ij}\, x_j \geq 0} \quad \text{for every scenario } \boldsymbol{i \geq 1}$$

$\boldsymbol{x_j}$ unrestricted in sign, for every $\boldsymbol{j}$

$\boldsymbol{x_j}$ = position in security $\boldsymbol{j}$

Today's value: $\boldsymbol{x_0 + \Sigma_{j=1}^n \pi_{0j}\, x_j}$

Tomorrow's in scenario $\boldsymbol{i}$: $\boldsymbol{(1+r)x_0 + \Sigma_{j=1}^n \pi_{ij}\, x_j}$

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Subject to:

$\boldsymbol{(1+r)x_0 + \Sigma_{j=1}^n \pi_{ij}\, x_j \geq 0}$ for every scenario $\boldsymbol{i \geq 1}$

$\boldsymbol{x_j}$ unrestricted in sign, for every $\boldsymbol{j}$

Type A: Today's cash flow > 0, and in tomorrow's ≤ 0 in *every* scenario

This happens if, and only if, $\boldsymbol{V^* < 0}$

Why?

$\boldsymbol{x_j}$ = position in security $\boldsymbol{j}$

Today's value: $\boldsymbol{x_0 + \Sigma_{j=1}^n \pi_{0j}\, x_j}$

Tomorrow's in scenario $\boldsymbol{i}$: $\boldsymbol{(1+r)x_0 + \Sigma_{j=1}^n \pi_{ij}\, x_j}$

$\boldsymbol{V^*}$ **=Minimize** $\boldsymbol{x_0 + \Sigma_{j=1}^n \pi_{0j}\, x_j}$

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$\boldsymbol{x_j}$ unrestricted in sign, for every $\boldsymbol{j}$

What if $\boldsymbol{V^* = 0}$?

$\boldsymbol{x_j}$ = position in security $\boldsymbol{j}$

Today's value: $\boldsymbol{x_0 \;+\; \Sigma_{j=1}^{n} \pi_{0j}\, x_j}$

Tomorrow's in scenario $\boldsymbol{i}$: $\boldsymbol{(1+r)x_0 \;+\; \Sigma_{j=1}^{n} \pi_{ij}\, x_j}$

$\boldsymbol{V^*}$ =**Minimize** $\boldsymbol{x_0 \;+\; \Sigma_{j=1}^{n} \pi_{0j}\, x_j}$

Subject to:

$\boldsymbol{(1+r)x_0 \;+\; \Sigma_{j=1}^{n} \pi_{ij}\, x_j \;\geq\; 0}$ for every scenario $\boldsymbol{i \geq 1}$

$\boldsymbol{x_j}$ unrestricted in sign, for every $\boldsymbol{j}$

What if $\boldsymbol{V^* = 0}$?

Type B: can we find a vector $\boldsymbol{x}$, such that

$\boldsymbol{(1+r)x_0 \;+\; \Sigma_{j=1}^{n} \pi_{ij}\, x_j \;\geq\; 0}$ for every scenario $\boldsymbol{i \geq 1}$

$\boldsymbol{(1+r)x_0 \;+\; \Sigma_{j=1}^{n} \pi_{ij}\, x_j \;>\; 0}$ for *at least one* scenario $\boldsymbol{i \geq 1}$

$\boldsymbol{x_0 \;+\; \Sigma_{j=1}^{n} \pi_{0j}\, x_j \;=\; 0}$?

$\boldsymbol{V^*}$ =Minimize $\boldsymbol{x_0 \; + \; \Sigma_{j=1}^{n} \pi_{0j} \, x_j}$

Subject to:

$\boldsymbol{(1+r)x_0 \; + \; \Sigma_{j=1}^{n} \pi_{ij} \, x_j \; \geq \; 0}$ for every scenario $\boldsymbol{i \geq 1}$

$\boldsymbol{x_j}$ unrestricted in sign, for every $\boldsymbol{j}$

If no Type A or Type B arbitrage exist, then ... $\boldsymbol{V^* = 0}$ and every optimal solution to the LP satisfies:

$\boldsymbol{(1+r)x_0 \; + \; \Sigma_{j=1}^{n} \pi_{ij} \, x_j \; = \; 0}$ for every scenario $\boldsymbol{i \geq 1}$

$$\boldsymbol{V^* = \text{Minimize} \;\; x_0 \; + \; \Sigma_{j=1}^{n} \pi_{0j} \; x_j}$$

Subject to:

$$\boldsymbol{(1+r)x_0 \; + \; \Sigma_{j=1}^{n} \pi_{ij} \; x_j \;\; \geq \;\; 0} \quad \text{for every scenario} \;\; \boldsymbol{i \geq 1}$$

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Dual LP:

$$\boldsymbol{V^* = \text{Maximize} \;\; \Sigma_{i=1}^{K} \; P_i}$$

Subject to:

$$\boldsymbol{(1+r)\Sigma_{i=1}^{K} P_i \;\; = \;\; 1}$$

$$\boldsymbol{\Sigma_{i=1}^{K} \pi_{ij} P_i \;\; = \;\; \pi_{0j}} \quad \text{for every security} \;\; \boldsymbol{j \geq 1}$$

$$\boldsymbol{P_i \geq 0} \quad \text{for every scenario} \;\; \boldsymbol{i \geq 1}$$

$\boldsymbol{V^*} =$**Minimize** $\boldsymbol{x_0 \; + \; \Sigma_{j=1}^{n} \pi_{0j} \; x_j}$

Subject to:

$$\boldsymbol{(1+r)x_0 \; + \; \Sigma_{j=1}^{n} \pi_{ij} \; x_j \; \geq \; 0} \quad \text{for every scenario} \;\; \boldsymbol{i \geq 1}$$

$\boldsymbol{x_j}$ unrestricted in sign, for every $\boldsymbol{j}$

If no Type A or Type B arbitrage exist, then ... $\boldsymbol{V^* = 0}$ and every optimal solution to the LP satisfies:

$$\boldsymbol{(1+r)x_0 \; + \; \Sigma_{j=1}^{n} \pi_{ij} \; x_j \; = \; 0} \quad \text{for every scenario} \;\; \boldsymbol{i \geq 1}$$

dual: $\boldsymbol{V^*} =$**Maximize** $\boldsymbol{\Sigma_{i=1}^{K} \; P_i}$
Subject to:

$$\boldsymbol{(1+r)\Sigma_{i=1}^{K} P_i \; = \; 1}$$

$$\boldsymbol{\Sigma_{i=1}^{K} \pi_{ij} P_i \; = \; \pi_{0j}} \quad \text{for every security} \;\; \boldsymbol{j \geq 1}$$

$$\boldsymbol{P_i \geq 0} \quad \text{for every scenario} \;\; \boldsymbol{i \geq 1}$$

...And an *appropriate* optimal dual solution gives us risk-neutral probabilities