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Unit-I → Propositional logic & Proofs

Proposition :- A declarative statement having answer as True (Yes/1) or False (No/0) and no other option.

- Composite / Compound Proposition :- A statement which is formed by clubbing or merging 2 or more propositions is called composite / compound proposition.

- Primitive Proposition :- A proposition which cannot be further sub-divided is called primitive proposition.
↳ lowercase notations

Operations :-

1. Negation :- Let p be the proposition, then the negation of p is represented as $\sim p$, $\neg p$ or $\bar{p}$, which is "It is not the case that of p".
Example,
p: Atleast 10 inches of rainfall today in LPU.
$\sim p$: Atmost 10 inches of rain fell today in LPU.

2. Conjunction :- Let p & q be the propositions, then the conjunction of p & q is the proposition "p and q". $(p \wedge q)$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example,

p: today is wednesday

q: It is sunny today.

$p \wedge q$: Today is wednesday and it is sunny.

3. **Disjunction ($p \vee q$)** :- Let q & p be the propositions, then the disjunction of p & q is the proposition "p or q".

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

4. **Conditional statement ($p \to q$)** :- Let p & q be the two propositions. The conditional statement $p \to q$ is the proposition "if p then q".

The conditional statement is false when p is true & q is false, otherwise it will be true.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

p: hypothesis

q: conclusion

Representation:-

p implies q, a necessary condition for p is q, q if p, if p then q, p only if q, q when p, q whenever p.

5. **Bi-conditional statement** :- Let p & q be the 2 propositions, then the bi-conditional statement $p \leftrightarrow q$ is the proposition "p iff q".
[Bi-implication statement]
The biconditional statement is true whenever p & q have same symbols, otherwise false.

p: conclusion
q: hypothesis

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

#Q. Let p = You can take the flight.
q = You buy a ticket.
↳ Biconditional Statement = $p \leftrightarrow q$ = You can take the flight if and only if you buy a ticket.

Impt. for MCQs.

6. **Converse of a statement** :- $q \rightarrow p$

7. **Contrapositive** :- $\sim q \rightarrow \sim p$

8. **Inverse** :- $\sim p \rightarrow \sim q$

#Q. The home team wins whenever it is raining.
↳ p: Hometeam wins
q: It is raining

Converse : $q \rightarrow p$ = When it is raining hometeam wins.
Contrapositive : $\sim q \rightarrow \sim p$ = When it is not raining, hometeam
Inverse : $\sim p \rightarrow \sim q$ = Hometeam loses whenever it is not raining.

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Order of Precedence :-

1. $\neg$ negation
2. $\wedge$ Conjunction
3. $\vee$ Disjunction
4. $\rightarrow$ Conditional
5. $\leftrightarrow$ Biconditional

Q. Draw the truth table for $(p \vee \sim q) \rightarrow (p \wedge q) \mp y$.

↳

p	q	$\sim q$ $(p \vee \sim q) \rightarrow (p \wedge q)$			
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F
		$\sim q$	$p \vee \sim q$	$p \wedge q$	$x \rightarrow y$

Tautology :- If all the entries in the truth table's last column are true then it is called tautology.

Contradictions :- If all the entries are false.

Contigency :- mixed entries in last column.

...on pages.

Logically Equivalent :- $P(p,q) \equiv Q(p,q)$

If 2 expressions have the same last column entries.

★ Biconditional of logically equivalent expressions is tautology.

Q. $\sim(p \vee q)$ & $(\sim p \wedge \sim q)$.

p	q	$\sim q$	$\sim p$	$p \vee q$	$\sim(p \vee q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F
T	F	T	F	T	F	F
F	T	F	T	T	F	F
F	F	T	T	F	T	T
					x	y

$$\sim(p \vee q) \equiv (\sim p \wedge \sim q)$$

$$x \equiv y$$

$x \leftrightarrow y$
T
T
T
T

} Tautology

p	q	r	$\sim p$	$q \to r$	$\sim p \to (q \to r)$	$\cancel{q \to}(p \vee r)$	$q \to (p \vee r)$
T	T	T	F	T	T	T	T
T	T	F	F	F	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	F	F
F	F	T	T	T	T	T	T
F	F	F	T	T	T	F	T
					x		y

$x \equiv y$

$\sim p \to (q \to r) \equiv q \to (p \vee r)$ Logically equivalent

#Q. $(p \wedge q) \to r$ & $(p \to r) \to (q \to r)$

p	q	r	$p \wedge q$	$(p \wedge q) \to r$	$p \to r$	$q \to r$	$(p \to r) \to (q \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	T	T
T	F	F	F	T	F	T	T
F	T	T	F	T	T	T	T
F	T	F	F	T	T	F	F
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T
				x			y

$(p \wedge q) \to r \not\equiv (p \to r) \to (q \to r)$

not logically equivalent

Conditional Statement

1. $p \rightarrow q \equiv \sim p \vee q$
2. $p \rightarrow q \equiv \sim q \rightarrow \sim p$
3. $p \vee q \equiv \sim p \rightarrow q$
4. $p \wedge q \equiv \sim(p \rightarrow \sim q)$
5. ★ $\sim(p \rightarrow q) \equiv p \wedge \sim q$

Biconditional Statement

1. $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
2. $p \leftrightarrow q \equiv \sim p \leftrightarrow \sim q$
3. $p \leftrightarrow q \equiv (p \wedge q) \vee (\sim p \wedge \sim q)$
4. $\sim(p \leftrightarrow q) \equiv p \leftrightarrow \sim q$

General Rules

↳ **Identity Rules**

$p \wedge T \equiv p$
$p \vee F \equiv p$

↳ **Double-negation**

$\sim(\neg p) = p$

↳ **Idempotent**

$p \vee p \equiv p$
$p \wedge p \equiv p$

↳ **Domination law**

$p \vee T \equiv T$
$p \wedge F \equiv F$

↳ $p \vee \sim p \equiv T$
$p \wedge \sim p \equiv F$

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★ <u>Exclusive Or (XOR)</u> :- Let p & q be the 2 propositions, then the 'exclusive or' of p & q is represented by '$p \oplus q$' & it is true when either of p & q is True or if p & q have opposite symbols.

[contradiction of biconditional]

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

#Q. $(p \vee q) \wedge (\sim p \vee r) \rightarrow (q \vee r) = x$

$\sim p$	$\sim p \vee r$	p	q	r	$p \vee q$	$p \vee r$	$q \vee r$	$(p \vee q) \wedge (\sim p \vee r)$	x
F	T	T	T	T	T	T	T	T	T
F	F	T	T	F	T	T	T	F	T
F	T	T	F	T	T	T	T	T	T
F	F	T	F	F	T	T	F	F	T
T	T	F	T	T	T	T	T	T	T
T	T	F	T	F	T	F	T	T	T
T	T	F	F	T	F	T	T	F	T
T	T	F	F	F	F	F	F	F	T

Hence, <u>Tautology</u>.

Distributive Property
Associative Property

De-Morgan's Law :-

[I] $\sim(p \wedge q) \equiv \sim p \vee \sim q$

[II] $\sim(p \vee q) \equiv \sim p \wedge \sim q$

Absorption Law :-

[I] $p \vee (p \wedge \overset{p}{\cancel{q}}) \equiv p$

[II] $p \wedge (p \vee p) \equiv p$

Example, (1) If it snows tonight, then I'll stay at home.
p : It snows tonight.
q : I'll stay at home.

$\cancel{\sim(p \wedge q) \equiv}$

$\sim(p \rightarrow q) \equiv p \wedge \sim q$: ~~If~~ It snows tonight, ~~then~~ and I'll not stay at home.

(2) When I stay up late, it is necessary that I'll sleep until noon.

$\sim(p \rightarrow q) \equiv p \wedge \sim q$: I stay up late and I'll not sleep until noon.

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Proof of $\sim(p \to q) \equiv p \wedge \sim q$

$LHS = \sim(\sim p \vee q)$

$\equiv \sim(\sim p) \wedge (\sim q)$

$= p \wedge \sim q$

$\underline{\underline{LHS = RHS}}$.

#Q. $\sim(p \vee (\sim p \wedge q))$ & $\sim p \wedge \sim q$:- Prove that these are logically equal using laws.

↳ $LHS = \cancel{\sim((p \vee \sim p) \wedge (p \vee q))}$

$= \sim(p \vee (\sim p \wedge q))$

$= \sim p \wedge \sim(\sim p \wedge q)$

$= (\sim p) \wedge [(\sim\sim p) \vee (\sim q)]$

$= (\sim p \wedge p) \vee (\sim p \wedge \sim q)$

$= F \vee (\sim p \wedge \sim q)$

$= \sim p \wedge \sim q$

$= RHS$

$\underline{\underline{LHS = RHS}}$.

#Q. Show that $(p \wedge q) \to (p \vee q)$ is a tautology using laws.

↳ $(p \wedge q) \to (p \vee q)$

$\cancel{= \sim p \vee q}$

$= (\sim(p \wedge q)) \vee (p \vee q)$

$= (\sim p \vee \sim q) \vee (p \vee q)$

$= (\sim p \vee p) \vee (\sim q \vee q)$

$= T \vee T$

$= \boxed{T}$

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Predicators & Quantifiers

Predicators :- The statement ~~p(x)~~ $P(x)$ is called the propositional function P at x.

Let ~~p~~ $P(x)$ defined as "x is greater than 3" then x is called as a variable and the condition greater than 3 is called a predicator.

Eg, Let $P(x) = x > 3$
what are the true values of $P(4)$ & $P(2)$?
$P(4) : 4 > 3$ True
$P(2) : 2 > 3$ False

$\forall x \in R \rightarrow$ quantifiers

03/02/23 Revision $\boxed{\text{Laws}}$

Conditional Statement:

1. $p \rightarrow q \equiv \sim p \vee q$
2. $p \rightarrow q \equiv \sim q \rightarrow \sim p$
3. $p \vee q \equiv \sim p \rightarrow q$
4. $p \wedge q \equiv \sim (p \rightarrow \sim q)$

* 5. $\sim (p \rightarrow q) \equiv p \wedge \sim q$

Biconditional statement:

6. $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
7. $p \leftrightarrow q \equiv \sim p \leftrightarrow \sim q$
8. $p \leftrightarrow q \equiv (p \wedge q) \vee (\sim p \wedge \sim q)$
9. $\sim (p \leftrightarrow q) \equiv p \leftrightarrow \sim q$

General Rules

10. $p \wedge T \equiv p$
11. $p \vee F \equiv p$
12. $\sim (\sim p) \equiv p$
13. $p \vee T \equiv T$
14. $p \wedge F \equiv F$
15. $p \vee p \equiv p$
16. $p \wedge p \equiv p$
17. $p \vee \sim p \equiv T$
18. $p \wedge \sim p \equiv F$

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Eg, ~~let Q(x,y) :~~ $Q(x,y) : x = y+3$

what is the truth values of propositon $Q(1,2)$, $Q(3,0)$?

↳ $Q(1,2)$: False

$Q(3,0)$: True

Quantifiers :- Quantification expresses the extent to which predicator is true over a range of elements.

The words like ~~all~~ all, some, many, none, few are used in quantifications.

⇒ Universal Quantifiers :- Universal quantification of ~~p~~ $P(x)$ is the statement "$P(x)\ \forall x$ in the Domain". $[\forall x\, P(x)]$

Example, Let $P(x) : x+1 > x \quad \forall x \in R$.

↳ $\forall x\, P(x)$ is ~~Fa~~ True.

⇒ Existential Quantifiers :- The existential quantification of $P(x)$ is the proposition "There exists an element x in the domain such that $P(x)$ is true". $[\exists x\, P(x)]$

Eg,

★ Let $Q(x) : x = x+1, \forall x \in R$.

$\forall x\, P(x) =$ False

$\exists x\, P(x) =$ ~~True~~ False

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> De-Morgan's law for Quantifiers :-

Negation	Equivalent statement	When is negation true?	When false?
① $\sim \exists x P(x)$	$\forall x \sim P(x)$	For every x, $P(x)$ is false	There is an x for which $P(x)$ is true
② $\sim \forall x P(x)$	$\exists x \sim P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

* Negation of universal quantifier is equal to the there exists with negation of proposition, according to De-morgan's law & vice-versa.

Eg, 1. There is an honest politician.
↳ All the politicians are dishonest.

2. All Americans eat cheeseburger.
↳ There are some Americans who donot eat cheeseburger.

> Proofs

⇒ **Direct proofs**: In direct proof of a conditional statement $p \to q$ is constructed when the first step is the assumption of p is true followed by the subsequent steps & leading to the conclusion.

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> **Proofs** $p \to q \equiv$ True

p	q	$p \to q$
T	T	T

⇒ Direct Proofs ✓

⇒ Example,

(1) Prove the theorem "if n is an order number then n^2 is also an odd number".

Sol. p : n is an odd no. or is true.
q : n^2 is an odd no.

Let 'p' is an odd no.

$$n = 2k+1$$
$$n^2 = (2k+1)^2$$
$$= 4k^2+4k+1$$
$$= 2(2k^2+2k)+1$$
$$n^2 = \boxed{2m+1} \to \text{odd no.}$$

(2) Prove that "If m & n are perfect square then m×n is also a perfect square".

Sol. p : m & n are perfect squares.
q : m×n is a perfect square.

Let 'p' is true, i.e. m & n are perfect squares

$$m = a^2 \ \& \ n = b^2$$
$$m \times n = \boxed{(ab)^2} = \text{perfect square.}$$

⇒ **Proof by Contraposition :-** $\boxed{p \to q \equiv \sim q \to \sim p}$

Proofs by contraposition make the use of the fact that the conditional statement $p \to q$ is logically equivalent to its contrapositive i.e. $\sim q \to \sim p$.

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Example,

(1) Prove that if n is an integer & $3n+2$ is an odd number then n is an odd number.

Sol.

p: n is an integer & $3n+2$ is an odd number
q: n is an odd number.

$\sim q \rightarrow \sim p$

Let 'n' is an even no.

$\Rightarrow n = 2k$

$3n+2 = 6k+2 = 2(3k+1) = \boxed{2m}$ even no.

$\Rightarrow \sim p$ is true

Hence, n is an odd no.

$\Rightarrow$ **Proof by contradiction :-**

Example, Prove that $\sqrt{2}$ is an irrational number

Sol. Let $\sqrt{2}$ is rational no.

$\sqrt{2} = \frac{p}{q}$; $p \neq q$, $q \neq 0$

squaring both sides,

$2 = \frac{p^2}{q^2}$

$p^2 = 2q^2 \Rightarrow p^2$ is even no.

$\Rightarrow p$ is also an even no.

so, $p = 2k$

$(2k)^2 = 2q^2$

$4k^2 = 2q^2$

$\Rightarrow q^2 = 2k^2 \Rightarrow q^2$ is even no.

$\Rightarrow q$ is also an even no.

so, $q = 2m$

Hence, $\sqrt{2}$ is an irrational no.

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⇒ Vacuous Proof & Trivial Proof :-

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

□ → Vacuous Proof

{~~} → Trivial Proof

Consider the conditional statement $p \to q$.
Then in case of vacuous proof p should be false, then $p \to q$ will be $\boxed{\text{true}}$ no matter the q is true / false,
whereas in case of trivial proof q should be true for $p \to q$ to be true no matter whatever is p.

Example, ① Consider "if $n > 1$, then $n^2 > n$". Show that the proposition $p(0)$ is true ~~for~~ $\forall\ n \in \mathbb{Z}$.

Sol.
$p : n > 1$
$q : n^2 > n$
$0 \not> 1$
$p \equiv$ false
$p(0)$ is true $\forall\ n \in Z$ because p $0 > 1$ is false with the help of vacuous proof.

② Let $p(n)$ be the "if a & b are +ve integers with $a \geq b$ then $a^n \geq b^n$". Show that $p(0)$ is true $\forall\ n \in Z$.

Sol.
$p :$ a & b are $+Z$ & $a \geq b$
$q : a^n \geq b^n$
$q(0) \to 1 \geq 1$ True
$q(0)$ is true $\forall\ n \in Z$ because $q(0)$ $1 \geq 1$ is true with the help of trivial proof.

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(3) Prove the proposition $p(0)$ is true, when $p(n)$ is the propositional function 'If n is a positive integer greater than 1, then $n^2 > n$.' ✓ Done

(H.W) (4) $\cancel{p(1)}$ $P(1)$ is true
p : n is positive integer
q : $n^2 \geq n$

(5) Prove $p(1)$ is true if $p(n)$ is 'If a & b are positive real numbers then $(a+b)^n \geq a^n + b^n$'.

~~(6) Proof~~

(4) Sol.

Let $p(1)$ is true
$1 > 0$ true
$\Rightarrow p$ is true
$1^2 \geq 1$ true
$\Rightarrow q$ is true
Hence, $P(1)$ is true.

(5) Sol. p : a & b are positive real numbers.
q : $(a+b)^n \geq a^n + b^n$
for $P(1)$
Let p : a & b are positive real numbers.
q : $(a+b)^1 \geq a^1 + b^1$
$a+b \geq a+b$ true
Hence, $P(1)$ is true.

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⇒ Proofs of Equivalence :- [Bi-conditional]

To prove a theorem that is a biconditional statement ($p \leftrightarrow q$ true) we need to prove $p \to q$ is true & $q \to p$ is true.
↳ using other proofs
Direct, Contra-positive, Vacuous & Trivial

Example,
Prove the theorem if n is a +Z & n is odd no. ~~then~~ iff n^2 is an odd number.

sol
p : n is a positive integer & odd no.
q : n^2 is an odd number.

$p \to q$
$q \to p$

★ for 3 propositions,
$p_1 \to p_2$
$p_2 \to p_3$
$p_3 \to p_1$

Let p is an odd no.

$$n = 2k+1$$
$$(n)^2 = (2k+1)^2$$
$$n^2 = 4k^2 + 4k + 1$$
$$n^2 = \cancel{4k}\ 2m+1$$
$\Rightarrow n^2$ is an odd no.

$\Rightarrow p \to q$ is true.

[using contrapositive]

Let q is true, i.e, n^2 is an ~~odd no~~ even no.
$$n^2 = (2k)^2$$
$\Rightarrow n =$ even no.
$\Rightarrow q \to p$ is true.

$\Rightarrow p \leftrightarrow q$ is true.