Function having finite points of discontinuity

$f(x)$

Continuous function

$f(x) = x\sin x,\ 0 < x < 2\pi$

$f(x) = x,\ -\pi < x < \pi$

discontinuous function

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ \sin x & 0 < x < \pi \end{cases}$$

Point of discontinuous $= 0$.

$f(x) = \frac{\sin x}{2}$.

If $f(x)$ has discontinuous at $x = c$, then

$$\boxed{f(x) = \frac{f(c^-) + f(c^+)}{2}}$$

How to check if ~~a~~ function is even or odd, when function has discontinuities.

① $$f(x) = \begin{cases} -x, & -\pi \le x \le 0 \\ x, & 0 \le x \le \pi \end{cases}$$

$f(x) = -x,\ -\pi \le x \le 0$

$f(-x) = x,\ -\pi \le -x \le 0$

$\pi \ge x \ge 0$

$f(-x) = x,\ 0 \le x \le \pi$

$= f(x)$

$\Rightarrow f(-x) = f(x)$

$f(x) = x,\ 0 \le x \le \pi$

$f(-x) = -x,\ -\pi \le x \le 0.$

$f(-x) = f(x)$

$\Rightarrow f(x)$ is an even function.

(2) $f(x) = \begin{cases} x^2; & -1 \le x \le 0 \\ -x^2, & 0 \le x \le 1 \end{cases}$

$f(x) = x^2, \ -1 \le x \le 0$

$f(-x) = x^2; \ -1 \le -x \le 0$

$f(-x) = x^2, \ 0 \le x \le 1$

$= -f(x)$

$f(-x) = -f(x)$

$f(x) = -x^2, \ 0 \le x \le 1$

$f(-x) = -x^2; \ 0 \le -x \le 1$

$= -x^2; \ -1 \le x \le 0$

$= -f(x)$

$f(-x) = -f(x)$

$\Rightarrow f(x)$ is an odd function.

Ex: $f(t) = \begin{cases} -1, & -\pi < t \le -\frac{\pi}{2} \\ 0, & -\frac{\pi}{2} < t < \frac{\pi}{2} \\ 1, & \frac{\pi}{2} < t < \pi \end{cases}$

$f(t) = -1, \ -\pi < t < -\frac{\pi}{2}$

$f(-t) = -1, \ -\pi < -t < -\frac{\pi}{2}$

$= -1, \ \frac{\pi}{2} < t < \pi$

$= -f(t)$

$f(t) = -f(t)$

$f(t) = 0, \ -\frac{\pi}{2} < t < \frac{\pi}{2}$

$f(-t) = 0, \ -\frac{\pi}{2} < -t < \frac{\pi}{2}$

$= 0, \ -\frac{\pi}{2} < t < \frac{\pi}{2}$

$= -f(t)$

$f(t) = -f(t)$

$f(t) = 1, \ +\frac{\pi}{2} < t < \pi$

$f(-t) = 1, \ +\frac{\pi}{2} < -t < \pi$

$= 1, \ -\frac{\pi}{2} > t > -\pi$

$= 1, \ -\pi < t < -\frac{\pi}{2}$

$= -f(t)$

$f(t) = -f(t)$

$\Rightarrow f(t)$ is an odd function.

$\Rightarrow a_0 = a_n = 0.$

Ex 9.1

(12) $f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ \sin x, & 0 \le x \le \pi. \end{cases}$

~~Sol~~: (31) Use the result to show that

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \frac{1}{7.9} + \cdots$$

Sol:- $f(x) = \begin{cases} 0 & -\pi \le x \le 0 \\ \sin x & 0 \le x \le \pi \end{cases}$ in $[-\pi, \pi]$

The fourier expansion for $f(x)$ in $[-\pi,\pi]$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right]$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi} \sin x\,dx$$

$$= 0 + \frac{1}{\pi}\left[-\cos x\right]_0^{\pi} = \frac{-1}{\pi}\left[\cos\pi - \cos 0\right]$$

$$= \frac{-1}{\pi}\left[(-1)^{n} - 1\right] = \frac{2}{\pi}$$

$$\boxed{a_0 = \frac{2}{\pi}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 0\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} \sin x\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \cos nx\sin x\,dx$$

$$\boxed{2\cos A\sin B = \sin(A+B) - \sin(A-B)}$$

$$= \frac{1}{2\pi}\int_{0}^{\pi}[\sin(n+1)x - \sin(n-1)x]\,dx$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{-1}{n+1}\left(\cos(n+1)\pi\right) + \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{2\pi}\left[\frac{-1}{n+1}(-1)^{n+1} + \frac{1}{n-1}(-1)^{n-1} + \frac{n-1-n-1}{n^2-1}\right]$$

$$= \frac{1}{2\pi}\left[\frac{(-1)^n(-1)^2}{n+1} - \frac{(-1)^n}{n-1} + \frac{(-2)}{n^2-1}\right]$$

$$= \frac{1}{2\pi}\left[(-1)^n\left(\frac{n-1-n-1}{n^2-1}\right) - \frac{2}{n^2-1}\right] = \frac{-2}{2\pi}\left[\frac{(-1)^n+1}{n^2-1}\right]$$

$$a_n = \frac{-1}{\pi}\left(\frac{(-1)^n+1}{n^2-1}\right)$$, case of failure at $n=1$.

$$a_n = \frac{-1}{\pi}\left(\frac{(-1)^n+1}{n^2-1}\right)$$, $n>1$ or $n\geq 2$.

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin x\cos x\,dx = \frac{1}{2\pi}\int_0^{\pi} \sin 2x\,dx$$

$$= \frac{1}{2\pi}\left[\frac{-\cos 2x}{2}\right]_0^{\pi}$$

$$= \frac{-1}{4\pi}(1-1) = 0$$

$$\boxed{a_1 = 0}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin x\sin nx\,dx = \frac{1}{2\pi}\int_0^{\pi} 2\sin nx\sin x\,dx$$

$$= \frac{1}{2\pi}\int_0^{\pi}\left[\cos(n-1)x - \cos(n+1)x\right]dx$$

$$\boxed{2\sin A\sin B = \cos(A-B) - \cos(A+B)}$$

$$= \frac{1}{2\pi}\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]_0^\pi$$

$$= \frac{1}{2\pi}(0)$$

$$\boxed{b_n = 0}$$

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin x \, dx$$

$$= \frac{1}{\pi}\int_0^\pi \sin^2 x \, dx = \frac{1}{\pi}\int_0^\pi \left(\frac{1-\cos 2x}{2}\right) dx$$

$$= \frac{1}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_0^\pi$$

$$= \frac{1}{2\pi}\left[\pi - 0\right] = \frac{1}{2}$$

$$\boxed{b_1 = \frac{1}{2}}$$

$$f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx + b_1 \sin x + \sum_{n=2}^{\infty} b_n \cancel{\cos n} \sin nx$$

$$= \frac{1}{\pi} + 0 + \sum_{n=2}^{\infty}\left(\frac{-1}{\pi}\right)\left(\frac{(-1)^n+1}{n^2-1}\right)\cos nx + \frac{1}{2}\sin x + 0$$

$$f(x) = \frac{1}{\pi} - \frac{1}{\pi}\sum_{n=2}^{\infty}\left(\frac{(-1)^n+1}{n^2-1}\right)\cos nx + \frac{\sin x}{2}$$

$$\boxed{f(x) = \frac{1}{\pi} + \frac{\sin x}{2} - \frac{1}{\pi}\sum_{n=2}^{\infty}\left(\frac{(-1)^n+1}{n^2-1}\right)\cos nx}$$

$$f(x) = \frac{1}{\pi} + \frac{\sin x}{2} - \frac{1}{\pi}\left[\frac{(-1)^2+1}{2^2-1}\cos 2x + \frac{(-1)^3+1}{3^2-1}\cos 3x\right.$$

$$\left. + \frac{(-1)^4+1}{4^2-1}\cos 4x + \frac{(-1)^5+1}{5^2-1}\cos 5x + \cdots\right]$$

$$= \frac{1}{\pi} + \frac{\sin x}{2} - \frac{1}{\pi}\left[\frac{2}{3}\cos 2x + 0 + \frac{2}{15}\cos 4x + 0 + \frac{2}{35}\cos 6x + \cdots\right]$$

$$= \frac{1}{\pi} + \frac{\sin x}{2} - \frac{2}{\pi}\left[\frac{\cos 2x}{1\cdot 3} + \frac{\cos 4x}{3\cdot 5} + \frac{\cos 6x}{5\cdot 7} + \cdots\right]$$

Put $x = \frac{\pi}{2}$

$$f\left(\frac{\pi}{2}\right) = \frac{1}{\pi} + \frac{1}{2} - \frac{2}{\pi}\left[\frac{-1}{1\cdot 3} + \frac{1}{3\cdot 5} - \frac{1}{5\cdot 7} + \cdots\right]$$

$$\Rightarrow 1 = \frac{1}{\pi} + \frac{1}{2} + \frac{2}{\pi}\left[\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} \cdots\right]$$

$$\Rightarrow 1 - \frac{1}{2} - \frac{1}{\pi} = \frac{2}{\pi}\left[\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} \cdots\right]$$

$$\Rightarrow \left(\frac{1}{2} - \frac{1}{\pi}\right)\frac{\pi}{2} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} \cdots$$

$$\Rightarrow \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - - = \frac{\pi - 2}{2\cancel{\pi}} \cdot \frac{\cancel{\pi}}{2}$$

$$= \frac{\pi}{4} - \frac{1}{2}$$

$$\Rightarrow \boxed{\frac{1}{2} + \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - - = \frac{\pi}{4}}$$

Ex Find the fourier series for the function

$$f(t) = \begin{cases} -1 & \text{for } -\pi < t < -\pi/2 \\ 0 & \text{for } -\pi/2 < t < \pi/2 \\ 1 & \text{for } \frac{\pi}{2} < t < \pi \end{cases}$$

Sol: $$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\, dt$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{-\pi/2} (-1)\, dt + \int_{-\pi/2}^{\pi/2} 0\, dt + \int_{\pi/2}^{\pi} 1\, dt \right]$$

$$= \frac{1}{\pi} \left[[-t]_{-\pi}^{-\pi/2} + 0 + [t]_{\pi/2}^{\pi} \right]$$

$$= \frac{1}{\pi} \left[-\left(-\frac{\pi}{2} + \pi\right) + \pi - \frac{\pi}{2} \right] = \frac{1}{\pi} \left[\frac{\pi}{2} - \pi + \pi - \frac{\pi}{2} \right]$$

$$\boxed{a_0 = 0}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2} -\cos nt\,dt + \int_{-\pi/2}^{\pi/2} 0\,dt + \int_{\pi/2}^{\pi}\cos nt\,dt\right]$$

$$= \frac{1}{\pi}\left[\left(-\frac{\sin nt}{n}\right)_{-\pi}^{-\pi/2} + \left(\frac{\sin nt}{n}\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{-1}{n}\left(-\sin\frac{n\pi}{2} + \sin n\pi\right) + \frac{1}{n}\left(\sin n\pi - \sin\frac{n\pi}{2}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{-1}{n}\left(-\sin\frac{n\pi}{2} + 0\right) + \frac{1}{n}\left(0 - \sin\frac{n\pi}{2}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n}\sin\frac{n\pi}{2} - \frac{1}{n}\sin\frac{n\pi}{2}\right]$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{-\pi/2} -\sin nt\,dt + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 0\,dt + \frac{1}{\pi}\int_{\pi/2}^{\pi}\sin nt\,dt$$

$$= \frac{1}{\pi}\left[\frac{\cos nt}{n}\right]_{-\pi}^{-\pi/2} + \frac{1}{\pi}\left[-\frac{\cos nt}{n}\right]_{\pi/2}^{\pi}$$

$$= \frac{1}{n\pi}\left[\cos\frac{n\pi}{2} - \cos n\pi\right] - \frac{1}{n\pi}\left[\cos n\pi - \cos\frac{n\pi}{2}\right]$$

$$= \frac{1}{n\pi}\left[\cos\frac{n\pi}{2} - (-1)^n - (-1)^n + \cos\frac{n\pi}{2}\right]$$

$$\boxed{b_n = \frac{2}{n\pi}\left[\cos\left(\frac{n\pi}{2}\right) - (-1)^n\right]}$$

$$f(t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[\cos\left(\frac{n\pi}{2}\right) - (-1)^n\right] \sin nt$$

$$\boxed{f(t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{\cos(n\pi/2) - (-1)^n}{n}\right) \sin nt.}$$