Normal Distribution

Normal Distribution

A continuous random variable X having the bell-shaped distribution of Figure is called a **normal random variable**. The mathematical equation for the probability distribution of the normal variable depends on the two parameters μ and σ, its mean and standard deviation, respectively. Hence, we denote the values of the density of X by $n(x; \mu, \sigma)$.

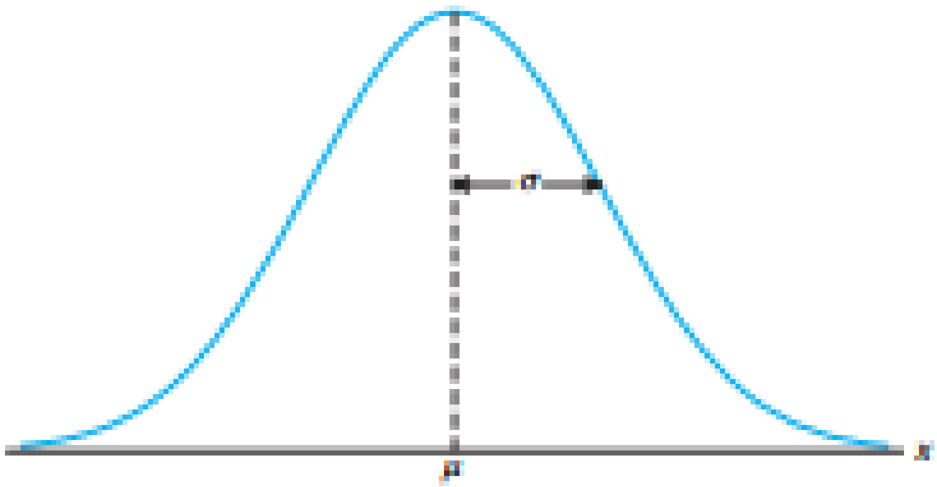

Figure : The normal curve.

Probability density function of normal distribution

The density of the normal random variable X, with mean μ and variance σ^2, is

$$n(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}, \qquad -\infty < x < \infty,$$

where $\pi = 3.14159\ldots$ and $e = 2.71828\ldots$.

Properties of Normal distribution

1. The mode, which is the point on the horizontal axis where the curve is a maximum, occurs at $x = \mu$.
2. The curve is symmetric about a vertical axis through the mean μ.
3. The curve has its points of inflection at $x = \mu \pm \sigma$; it is concave downward if $\mu - \sigma < X < \mu + \sigma$ and is concave upward otherwise.
4. The normal curve approaches the horizontal axis asymptotically as we proceed in either direction away from the mean.
5. The total area under the curve and above the horizontal axis is equal to 1.
6. The mean and variance of $n(x; \mu, \sigma)$ are μ and σ^2, respectively. Hence, the standard deviation is σ.

Standard Normal variate

The distribution of a normal random variable with mean 0 and variance 1 is called a **standard normal distribution**.

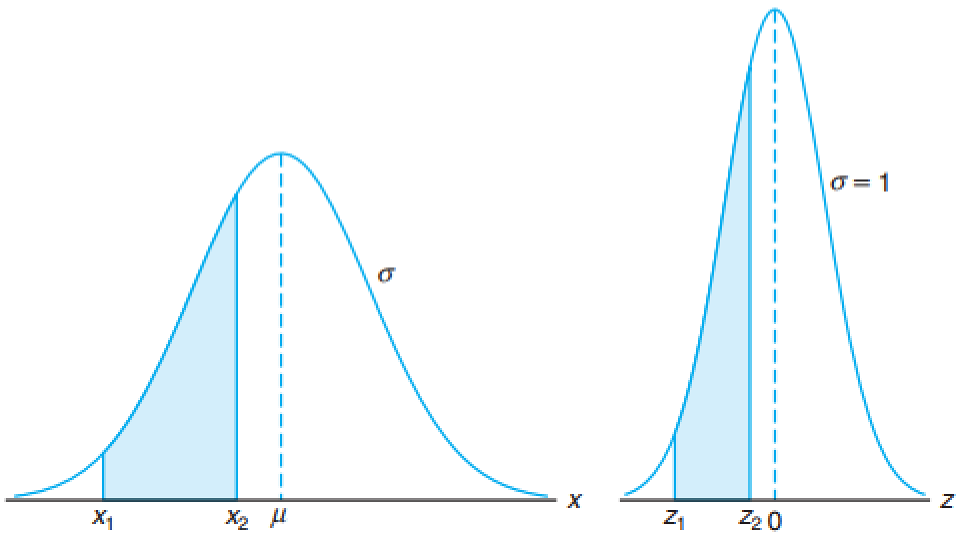

Figure : The original and transformed normal distributions.

6.5 Given a standard normal distribution, find the area under the curve that lies

(a) to the left of $z = -1.39$;

(b) to the right of $z = 1.96$;

(c) between $z = -2.16$ and $z = -0.65$;

(d) to the left of $z = 1.43$;

(e) to the right of $z = -0.89$;

(f) between $z = -0.48$ and $z = 1.74$.

6.6 Find the value of z if the area under a standard normal curve

(a) to the right of z is 0.3622;

(b) to the left of z is 0.1131;

(c) between 0 and z, with $z > 0$, is 0.4838;

(d) between $-z$ and z, with $z > 0$, is 0.9500

6.7 Given a standard normal distribution, find the value of k such that

(a) $P(Z > k) = 0.2946$;

(b) $P(Z < k) = 0.0427$;

(c) $P(-0.93 < Z < k) = 0.7235$.

6.8 Given a normal distribution with $\mu = 30$ and $\sigma = 6$, find

(a) the normal curve area to the right of $x = 17$;
(b) the normal curve area to the left of $x = 22$;
(c) the normal curve area between $x = 32$ and $x = 41$;
(d) the value of x that has 80% of the normal curve area to the left;
(e) the two values of x that contain the middle 75% of the normal curve area

6.9 Given the normally distributed variable X with mean 18 and standard deviation 2.5, find

(a) $P\ (X < 15)$;

(b) the value of k such that $P\ (X < k) = 0.2236$;

(c) the value of k such that $P\ (X > k) = 0.1814$;

(d) $P\ (17 < X < 21)$

6.11 A soft-drink machine is regulated so that it discharges an average of 200 milliliters per cup. If the
amount of drink is normally distributed with a standard deviation equal to 15 milliliters,

(a) what fraction of the cups will contain more than 224 milliliters?

(b) what is the probability that a cup contains between 191 and 209 milliliters?

(c) how many cups will probably overflow if 230-milliliter cups are used for the next 1000 drinks?

(d) below what value do we get the smallest 25% of the drinks?

6.13 A research scientist reports that mice will live an average of 40 months when their diets are sharply restricted and then enriched with vitamins and proteins. Assuming that the lifetimes of such mice are normally distributed with a standard deviation of 6.3 months, find the probability that a given mouse will live

(a) more than 32 months;

(b) less than 28 months;

(c) between 37 and 49 months

6.18: The heights of 1000 students are normally distributed with a mean of 174.5 centimeters and a standard deviation of 6.9 centimeters. Assuming that the
heights are recorded to the nearest half-centimeter,
how many of these students would you expect to have heights
(a) less than 160.0 centimeters?
(b) between 171.5 and 182.0 centimeters inclusive?
(c) equal to 175.0 centimeters?
(d) greater than or equal to 188.0 centimeters?

6.21 The tensile strength of a certain metal component is normally distributed with a mean of 10,000 kilograms per square centimeter and a standard deviation of 100 kilograms per square centimeter. Measurements are recorded to the nearest 50 kilograms per square centimeter.

(a) What proportion of these components exceed 10,150 kilograms per square centimeter in tensile strength?

(b) If specifications require that all components have tensile strength between 9800 and 10,200 kilograms per square centimeter inclusive, what proportion of pieces would we expect to scrap?

6.22 If a set of observations is normally distributed, what percent of these differ from the mean by

(a) more than 1.3σ?

(b) less than 0.52σ?

Normal Approximation to the Binomial

If X is a binomial random variable with mean $\mu = np$ and variance $\sigma^2 = npq$, then the limiting form of the distribution of

$$Z = \frac{X - np}{\sqrt{npq}},$$

as $n \to \infty$, is the standard normal distribution $n(z; 0, 1)$.

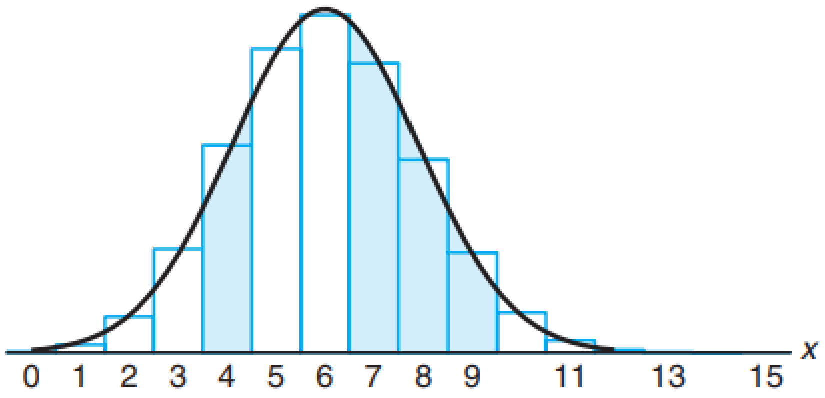

Figure 6.23: Normal approximation of $b(x; 15, 0.4)$ and $\sum_{x=7}^{9} b(x; 15, 0.4)$.

if we seek the area under the normal curve to the left of, say, x, it is more accurate to use $x + 0.5$. This is a correction to accommodate the fact that a discrete distribution is being approximated by a continuous distribution. The correction **+0.5** is called a **continuity correction**.

6.27 The probability that a patient recovers from a delicate heart operation is 0.9. Of the next 100 patients having this operation, what is the probability that

(a) between 84 and 95 inclusive survive?

(b) fewer than 86 survive?

6.29 If 20% of the residents in a U.S. city prefer a white telephone over any other color available, what is the probability that among the next 1000 telephones installed in that city

(a) between 170 and 185 inclusive will be white?

(b) at least 210 but not more than 225 will be white?

Moment generating function (m.g.f) of Normal Distribution

- The moment generating function of normal distribution ;

$$M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$