Unit III: Special Distributions

- **Binomial Distribution**

1. The experiment consists of repeated trials.
2. Each trial results in two exhaustive and mutually disjoint outcomes, termed as success and failure.
3. The number of trials "n" is finite.
4. The trials are independent of each other.
5. The probability of success is constant for each trial.

X, The **number of successes** in Bernoulli trials is called a **Binomial random variable**. The probability distribution of X is called **Binomial distribution**. It is denoted as

$$b(x; n, p)$$

A Bernoulli trial can result in a success with probability p and a failure with probability $q = 1 - p$. Then the probability distribution of the binomial random variable X, the number of successes in n independent trials, is

$$b(x; n, p) = \binom{n}{x} p^x q^{n-x}, \quad x = 0, 1, 2, \ldots, n.$$

$$B(r; n, p) = \sum_{x=0}^{r} b(x; n, p)$$

Q1.A and B play a game in which their chances of winning are in the ratio 3:5. Find A's chance of winning the game at least three games out of the five games played.

Q2.

5.2 Twelve people are given two identical speakers, which they are asked to listen to for differences, if any. Suppose that these people answer simply by guessing. Find the probability that three people claim to have heard a difference between the two speakers.

Q3. In 256 sets of 12 tosses of a fair coin, in how many cases may one expect eight heads and four tails?

Q4.

5.16 Suppose that airplane engines operate independently and fail with probability equal to 0.4. Assuming that a plane makes a safe flight if at least one-half of its engines run, determine whether a 4-engine plane or a 2-engine plane has the higher probability for a successful flight.

Mean and Variance of Binomial distribution

$$E(X) = \mu = np, \qquad Var(X) = \sigma^2 = npq$$

Proof has been discussed in class.

- **For Binomial distribution, Variance is less than Mean.**

Q5. In testing a certain kind of truck tire over rugged terrain, it is found that 40% of the trucks fail to complete the test run without a blowout.

(a) How many of the 15 trucks would you expect to have blowouts?
(b) What is the variance of the number of blowouts experienced by the 15 trucks?

Q6. Which of the following can be possible values of mean and variance for Binomial distribution?

(A) Mean=5, Variance=7

(B) Mean=3, Variance= -2

(C) Mean=7, Variance=5

(D) Mean=Variance=5

Q7. The mean and variance of Binomial distribution is given as 4 and 4/3 respectively. Find the parameters of distribution.

- **Negative Binomial and Geometric Distribution**

Geometric Distribution

1. The properties of experiment is same as those mentioned for Binomial experiment.
2. The trials will be repeated until first success occur.
3. Instead of probability of x successes in n trials, where n is fixed for Binomial. Now we are interested in probability that first success occur in x trials.

If repeated independent trials can result in a success with probability p and a failure with probability $q = 1 - p$, then the probability distribution of the random variable X, the number of the trial on which the first success occurs, is

$$g(x; p) = pq^{x-1}, \quad x = 1, 2, 3, \ldots.$$

Q8.

For a certain manufacturing process, it is known that, on the average, 1 in every 100 items is defective. What is the probability that the fifth item inspected is the first defective item found?

Q9.

5.51 Three people toss a fair coin and the odd one pays for coffee. If the coins all turn up the same, they are tossed again. Find the probability that fewer than 4 tosses are needed.

Mean and Variance of Geometric distribution

The mean and variance of a random variable following the geometric distribution are

$$\mu = \frac{1}{p} \text{ and } \sigma^2 = \frac{1-p}{p^2}.$$

Variance can also be written as $\frac{q}{p^2}$

Proof has been discussed in class.

Negative Binomial Distribution

1. The properties of experiment is same as those mentioned for Binomial experiment.
2. The trials will be repeated until a fixed number of successes occur.
3. Instead of probability of x successes in n trials, where n is fixed for Binomial. Now we are interested in probability that k successes occur in x trials or we say k^{th} success occur in x^{th} trial for Negative Binomial.

If repeated independent trials can result in a success with probability p and a failure with probability $q = 1 - p$, then the probability distribution of the random variable X, the number of the trial on which the kth success occurs, is

$$b^*(x; k, p) = \binom{x-1}{k-1} p^k q^{x-k}, \quad x = k, k+1, k+2, \ldots.$$

Q10.

In an NBA (National Basketball Association) championship series, the team that wins four games out of seven is the winner. Suppose that teams A and B face each other in the championship games and that team A has probability 0.55 of winning a game over team B.

(a) What is the probability that team A will win the series in 6 games?

(b) What is the probability that team A will win the series?

Q11. Find the probability that a person flipping a coin gets

(a) 3 heads in 7 trials.

(b) 3^{rd} head on 7^{th} trial.

(c) First head on 4^{th} trial.

***Geometric Distribution is a special case of Negative Binomial Distribution.**

Mean and Variance of Negative Binomial distribution

$$E(X) = \mu = \frac{k}{p}, \quad Var(X) = \sigma^2 = \frac{kq}{p^2}$$

Negative Binomial Distribution

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$$E(X) = \mu = \sum_{x=k}^{\infty} x \; {}^{x-1}C_{k-1} \, p^{k} q^{x-k}$$

$$\sum_{x=k}^{\infty} x \frac{(x-1)!}{(k-1)!\,(x-k)!} p^{k} q^{x-k}$$

$$\sum_{x=k}^{\infty} \frac{x!}{(k-1)!\,(x-k)!} p^{k} q^{x-k}$$

$$\sum_{x=k}^{\infty} k \frac{x!}{k!\,(x-k)!} p^{k} q^{x-k}$$

$$= \sum_{x=k}^{\infty} {}^{x}C_{k} \, p^{k} q^{x-k} . k$$

$$= kp^{k} \sum_{x=k}^{\infty} {}^{x}C_{k} \, q^{x-k}$$

$$= kp^{k} \left[1 + (k+1)q + \frac{(k+2)(k+1)}{2!} q^{2} + \frac{(k+3)(k+2)(k+1)}{3!} q^{3} + \ldots \right]$$

$$= kp^{k} \left[1 + (-k-1)(-q) + \frac{(-k-1)(-k-1-1)}{2!} (-q)^{2} + \frac{(-k-1)(-k-1-1)(-k-1-2)}{3!} (-q)^{3} + \ldots \right]$$

$$kp^k (1-q)^{-k-1} \quad \text{———} \quad (*)$$

$$E(X) = \mu = \frac{kp^k}{p^{k+1}} = \frac{k}{p}$$

$$E(X^2) = \sum_{x=k}^{\infty} x^2 \binom{x-1}{k-1} p^k q^{x-k}$$

$$= p^k \cdot q^{-k} \sum_{x=k}^{\infty} x^2 \frac{(x-1)!}{(k-1)!\,(x-k)!} q^x$$

$$= p^k \cdot q^{-k} \sum_{x=k}^{\infty} x \cdot k \frac{x!}{k!\,(x-k)!} q^x$$

$$= \left(\frac{p}{q}\right)^k \cdot k \sum_{x=k}^{\infty} \binom{x}{k} x q^x$$

$$= \left(\frac{p}{q}\right)^k \cdot k \sum_{x=k}^{\infty} \binom{x}{k} x q^{x-1} \cdot q$$

$$= \frac{p^k}{q^{k-1}} k \sum_{x=k}^{\infty} \binom{x}{k} \frac{d}{dq}(q^x)$$

$$= \frac{p^k \cdot k}{q^{k-1}} \frac{d}{dq}\left[\sum \binom{x}{k} q^x\right] \Rightarrow \frac{p^k}{q^{k-1}} \cdot k \frac{d}{dq}\left[q^k \sum_{x=k}^{\infty} \binom{x}{k} q^{x-k}\right]$$

from $(*)$

$$= \frac{p^k}{q^{k-1}} \cdot k \frac{d}{dq}\left[q^k \cdot (1-q)^{-k-1}\right] = \frac{p^k k}{q^{k-1}} \frac{(1-q)^{k+1} \cdot kq^{k-1} + q^k \cdot (k+1)(1-q)^k}{(1-q)^{2k+2}}$$

$$E(X^2) = \frac{p^k k}{q^{k-1}} \cdot \frac{(1-q)kq^{k-1} + q^k(k+1)}{(1-q)^{k+2}}$$

$$\frac{p^k \cdot k}{q^{k-1}} \; \frac{kq^{k-1} - \cancel{kq^k} + \cancel{kq^k} + q^k}{p^{k+2}}$$

$$= \frac{k \cdot \cancel{q^{k-1}}(k+q)}{p^2 \cancel{q^{k-1}}} = \frac{k(k+q)}{p^2}$$

$$Var(X) = E(X^2) - (E(X))^2$$

$$= \frac{k(k+q)}{p^2} - \frac{k^2}{p^2}$$

$$= \frac{kq}{p^2}$$

- **Poisson distribution and the Poisson process**

Poisson distribution occurs when there are events which do not occur as outcomes of a definite number of trials of an experiment, but which occur at random points of time and space.

The probability distribution of the Poisson random variable X, representing number of outcomes occurring in a given time interval or specified region denoted by t, is

$$p(x;\lambda t) = \frac{e^{-\lambda t}\,(\lambda t)^x}{x!} \quad , x = 0, 1, 2, \ldots$$

where λ is the average number of outcomes occurring per unit time interval or specified region.

- **Mean and Variance of Poisson distribution**

Proof has been discussed in class.

Both the mean and the variance of the Poisson distribution $p(x;\lambda t)$ are λt.

Mean=Variance for Poisson distribution.

$$p(x;\mu) = \frac{e^{-\mu}\,\mu^x}{x!} \quad , x = 0, 1, 2, \ldots$$

Q12.

5.53 An inventory study determines that, on average, demands for a particular item at a warehouse are made 5 times per day. What is the probability that on a given day this item is requested

(a) more than 5 times?

(b) not at all?

Q13.

5.72 Potholes on a highway can be a serious problem, and are in constant need of repair. With a particular type of terrain and make of concrete, past experience suggests that there are, on the average, 2 potholes per mile after a certain amount of usage. It is assumed that the Poisson process applies to the random variable "number of potholes."

(a) What is the probability that no more than one pothole will appear in a section of 1 mile?

(b) What is the probability that no more than 4 potholes will occur in a given section of 5 miles?

(c) What is the mean and variance of the random variable representing number of potholes appearing in a section of 3 miles?

Theorem 5.5: Let X be a binomial random variable with probability distribution $b(x; n, p)$. When $n \to \infty$, $p \to 0$, and $np \overset{n\to\infty}{\longrightarrow} \mu$ remains constant,

$$b(x; n, p) \overset{n\to\infty}{\longrightarrow} p(x; \mu).$$

Proof has been discussed in class.

Q14. Suppose that, on average, 1 person in 1000 makes a numerical error in preparing his or her income tax return. Find mean and variance of the random variable representing the number of persons among 10000 who make an error in preparing their income tax returns.

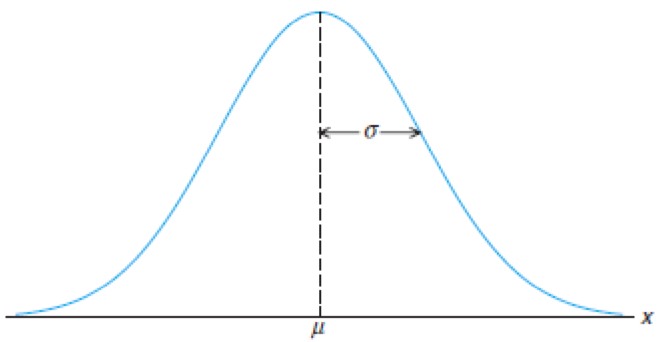
Q15. An insurance company insures 4000 people against loss of both eyes in a car accident. Based on previous data, the rates were computed on the assumption that on the average 10 persons in 1,00, 000 will have car accident each year that result in the type of injury. What is the probability that more than 3 of the insured will collect on their policy in a given year?

Continuous Probability Distributions

- **Normal Distribution**

A continuous random variable X having the bell shaped distribution is called a normal random variable. The p.d.f. of normal probability curve is given as

$$n(x;\ \mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad , \ -\infty < x < \infty$$



- The curve is bell shaped and symmetrical about the line $x = \mu$.

- Mean, Median and Mode coincide.

- Maximum probability occurs at $x = \mu$ and maximum probability is given by $(x) = \frac{1}{\sigma\sqrt{2\pi}}$.
- The curve has its points of inflection at $x = \mu \pm \sigma$; it is concave downward if $\mu - \sigma < X < \mu + \sigma$ and is concave upwards otherwise.
- The normal curve approaches the horizontal axis asymptotically as we proceed away from mean in either direction.
- $n(x;\ \mu,\ \sigma)$ with mean μ and variance σ^2 and $Z = \frac{X-\mu}{\sigma}$ is standard normal variate with $E(Z) = 0,\ V(Z) = 1$.
- p.d.f. of standard normal variate is $\varphi(Z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \quad , \ -\infty < z < \infty$
- Maximum probability occurs at $Z = 0$ for standard normal variate.

STEPS TO SOLVE QUESTION ON NORMAL DISTRIBUTION

(i) Convert Normal Variate X to standard normal variate using $Z = \frac{X-\mu}{\sigma}$.

(ii) Convert the probability interval in the table form $P(Z \leq z_1)$.

- $P(Z \leq z_0) = \Phi(z_0)$

- $P(z > z_0) = 1 - \Phi(z_0)$
- $P(z_1 \leq Z \leq z_2) = \Phi(z_2) - \Phi(z_1)$

(iii) Use the table value at z_1.

Find

(i) $P(\mathrm{z} \leq -1) =$

(ii) $P(z \geq -1) =$

(iii) $P(z \leq 1) =$

(iv) $P(-1 \leq z \leq 1) =$

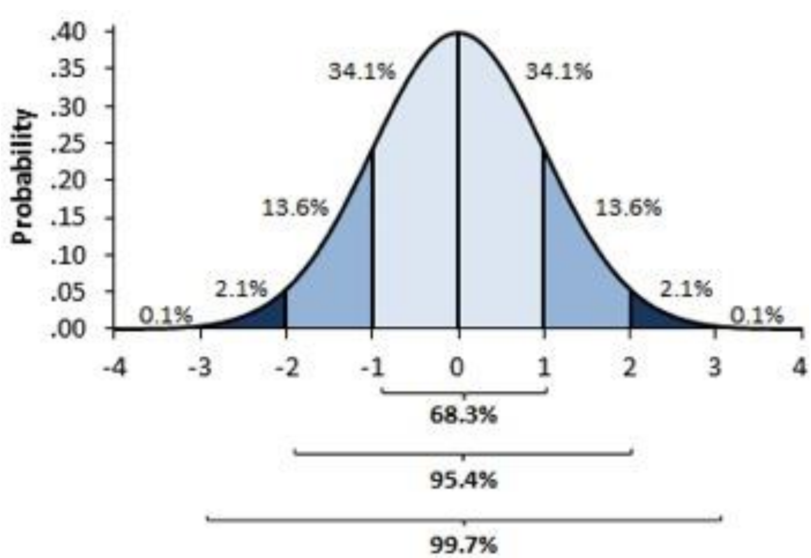
(v) $P(-1.645 \leq z \leq 1.645) =$

(vi) $P(-1.96 \leq z \leq 1.96) =$

(vii) $P(-2 \leq z \leq 2) =$

(viii) $P(-2.58 \leq z \leq 2.58) =$

(ix) $P(-3 \leq z \leq 3) =$



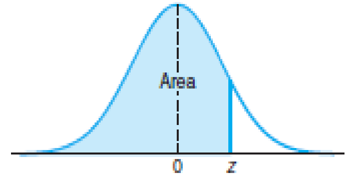


Table A.3 Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
−3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
−3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
−3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

Table A.3 (continued) Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Q16.

6.8 Given a normal distribution with $\mu = 30$ and $\sigma = 6$, find

(a) the normal curve area to the right of $x = 17$;

(b) the normal curve area to the left of $x = 22$;

(c) the normal curve area between $x = 32$ and $x = 41$;

(d) the value of x that has 80% of the normal curve area to the left;

(e) the two values of x that contain the middle 75% of the normal curve area.

Given

$$P(Z \le 0.845) = 0.80, P(Z \le -1.15) = 0.125, P(Z \le 1.15) = 0.875,$$
$$P(Z \le -2.167) = 0.0152, P(Z \le -1.33) = 0.0918,$$
$$P(Z \le 1.83) = 0.9664, P(Z \le 0.33) = 0.6293,$$

Q17.

6.11 A soft-drink machine is regulated so that it discharges an average of 200 milliliters per cup. If the amount of drink is normally distributed with a standard deviation equal to 15 milliliters,

(a) what fraction of the cups will contain more than 224 milliliters?

(b) what is the probability that a cup contains between 191 and 209 milliliters?

(c) how many cups will probably overflow if 230-milliliter cups are used for the next 1000 drinks?

(d) below what value do we get the smallest 25% of the drinks?

Given

$$P(Z \le 0.6) = 0.7257, P(Z \le -0.6) = 0.2743, P(Z \le 2) = 0.9772,$$
$$P(Z \le 1.6) = 0.9452, P(Z \le -0.675) = 0.25$$

Q18. A normal distribution has mean 25 and variance 25. Find the limits which include 90% of the area under the curve.

Q19. In a university examination of a particular year, 60% of the students failed when mean of the marks was 50% and s.d. 5%. University decided to relax the conditions of passing by lowering the pass marks, to show its result 70%. Find the minimum marks for a student to pass, supposing the marks to be normally distributed and no change in the performance of students take place.

Given: $P(0 \leq z \leq 0.525) = 0.20, P(0 \leq z \leq 0.845) = 0.30$

Q20. The height measurements of 600 adult males are arranged in ascending order and it is observed that 180th and 450th entries are 64.2" and 67.8" respectively. Assuming that the sample of heights drawn from a normal population, estimate the mean and s.d. of the distribution.

Given: $P(0 \leq z \leq 0.525) = 0.20, P(0 \leq z \leq 0.675) = 0.25,$
$$P(0 \leq z \leq 0.845) = 0.30$$

Normal approximation to Binomial distribution

If X is a binomial random variable with mean $\mu = np$ and variance $\sigma^2 = npq$, then the limiting form of the distribution of

$$Z = \frac{X - np}{\sqrt{npq}},$$

as $n \to \infty$, is the standard normal distribution $n(z; 0, 1)$.

Normal Approximation to the Binomial Distribution

Let X be a binomial random variable with parameters n and p. For large n, X has approximately a normal distribution with $\mu = np$ and $\sigma^2 = npq = np(1-p)$ and

$$\begin{aligned} P(X \leq x) &= \sum_{k=0}^{x} b(k; n, p) \\ &\approx \text{area under normal curve to the left of } x + 0.5 \\ &= P\left(Z \leq \frac{x + 0.5 - np}{\sqrt{npq}}\right), \end{aligned}$$

and the approximation will be good if np and $n(1-p)$ are greater than or equal to 5.

If $np, nq \geq 5$

- $P(X \leq x) = P\left(Z \leq \frac{x+0.5-np}{\sqrt{npq}}\right)$ — (a)

- $P(X < x) = P\left(Z \leq \frac{x-0.5-np}{\sqrt{npq}}\right)$ — (b)

Here, x is not included, so $P(X<x) = P(X \leq x-1)$ for discrete

So using result (a)

$$P(X<x) = P(X \leq x-1) = P\left(Z \leq \frac{x-1+0.5-np}{\sqrt{npq}}\right)$$

$$P(X<x) = P\left(Z \leq \frac{x-0.5-np}{\sqrt{npq}}\right)$$ which takes (b) form

- $P(X \geq x) = P\left(Z \geq \frac{x-0.5-np}{\sqrt{npq}}\right)$ — (c)

- $P(X > x) = P\left(Z \geq \frac{x+0.5-np}{\sqrt{npq}}\right)$ — (d)

Q21.

6.27 The probability that a patient recovers from a delicate heart operation is 0.9. Of the next 100 patients having this operation, what is the probability that

(a) between 84 and 95 inclusive survive?

(b) fewer than 86 survive?

(c) exactly 85 survive?

Q2)

(a) X = no. of patients out of 100 who are able to recover for operation

which is clearly a binomial distribution.

$$n = 100, \quad p = 0.9, \quad q = 0.1$$

(a) $$P(84 \leq X \leq 95) = \sum_{r=84}^{95} {}^{100}C_r (0.9)^r (0.1)^{100-r}$$

Observing above, we can decide that it ~~requires~~ involves lengthy calculation.

So, lets check if it can be approximated with Normal distribution

$$np = 100(0.9) = 90 \\ nq = 100(0.1) = 10 \Bigg\} \geq 5$$

So, we can approximate using Normal distribution

$P(84 \leq X \leq 95)$

Use
$X \geq 84$ — (c)

$X \leq 95$ — (a)

$$\simeq P(84 - 0.5 \leq X \leq 95 + 0.5)$$

$$= P\left(\frac{83.5 - np}{\sqrt{npq}} \leq Z \leq \frac{95.5 - np}{\sqrt{npq}}\right)$$

$np = 90, \quad npq = 9, \quad \sqrt{npq} = 3$

$$P(84 \leq X \leq 95) \simeq P\left(\frac{83.5-90}{3} \leq Z \leq \frac{95.5-90}{3}\right)$$

$$= P(-2.167 \leq Z \leq 1.83)$$

$$= P(Z \leq 1.83) - P(Z \leq -2.167)$$

$$= 0.9664 - 0.0152$$

$$= 0.95112$$

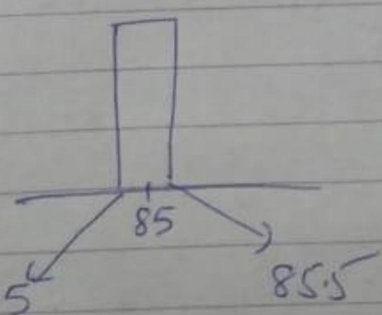
one. (b). $P(X<86)$ using (b) formula

$$\simeq P(X \leq 85.5) = P\left(Z \leq \frac{85.5-90}{3}\right)$$

$$= P(Z \leq -1.5)$$

$$= 0.0668$$

(c) $P(X=85)$



$P(84.5 \leq X \leq 85.5)$

$$P\left(\frac{84.5-90}{3} \leq Z \leq \frac{85.5-90}{3}\right)$$

$$P(-1.83 \leq Z \leq -1.5) = P(Z \leq -1.5) - P(Z \leq -1.83)$$

$$= 0.0668 - 0.0336 = 0.0332$$

Theorem 6.2: The mean and variance of $n(x;\mu,\sigma)$ are μ and σ^2, respectively. Hence, the standard deviation is σ.

***Proof*:** To evaluate the mean, we first calculate

$$E(X-\mu) = \int_{-\infty}^{\infty} \frac{x-\mu}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx.$$

Setting $z = (x-\mu)/\sigma$ and $dx = \sigma\, dz$, we obtain

$$E(X-\mu) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z e^{-\frac{1}{2}z^2} dz = 0,$$

since the integrand above is an odd function of z. Using Theorem 4.5 on page 128, we conclude that

$$E(X) = \mu.$$

The variance of the normal distribution is given by

$$E[(X-\mu)^2] = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} (x-\mu)^2 e^{-\frac{1}{2}[(x-\mu)/\sigma]^2}\, dx.$$

Again setting $z = (x-\mu)/\sigma$ and $dx = \sigma\, dz$, we obtain

$$E[(X-\mu)^2] = \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-\frac{z^2}{2}}\, dz.$$

Integrating by parts with $u = z$ and $dv = ze^{-z^2/2}\, dz$ so that $du = dz$ and $v = -e^{-z^2/2}$, we find that

$$E[(X-\mu)^2] = \frac{\sigma^2}{\sqrt{2\pi}} \left(-ze^{-z^2/2} \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-z^2/2}\, dz \right) = \sigma^2(0+1) = \sigma^2.$$

Exponential Distribution

- The time between events in a Poisson process is an exponential distribution.
- Poisson — Events per single unit of time; Exponential — Time per single event

Poisson	Exponential
Events per single unit of time	Time per single event

- The time required for first event to occur is Exponential distribution.
- The mean of the exponential distribution is the parameter β, the reciprocal of the parameter in the Poisson distribution.
- β is called mean time between events.

Exponential Distribution The continuous random variable X has an **exponential distribution**, with parameter β, if its density function is given by

$$f(x;\beta) = \begin{cases} \frac{1}{\beta} e^{-x/\beta}, & x > 0, \\ 0, & \text{elsewhere}, \end{cases}$$

where $\beta > 0$.

The mean and variance of the exponential distribution are

$$\mu = \beta \text{ and } \sigma^2 = \beta^2.$$

$$\mu = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_0^{\infty} \frac{x}{\beta} e^{-x/\beta} dx = \frac{1}{\beta}\left[\left[x \frac{e^{-x/\beta}}{-1/\beta} \right]_0^{\infty} - \int_0^{\infty} 1 \cdot \frac{e^{-x/\beta}}{-1/\beta} dx \right]$$

$$= \frac{1}{\beta}\left[(0-0) + \beta \int_0^{\infty} e^{-x/\beta} dx \right] \qquad \boxed{e^{-\infty} = 0}$$

$$= \left[\frac{e^{-x/\beta}}{-1/\beta} \right]_0^{\infty} = -\beta(0-1) = \beta$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^{\infty} x^2 \frac{1}{\beta} e^{-x/\beta}\,dx$$

Apply By Parts thea

$$= \frac{1}{\beta}\left[x^2 \cdot \frac{e^{-x/\beta}}{-1/\beta} - 2x \frac{e^{-x/\beta}}{\left(-\frac{1}{\beta}\right)^2} + 2 \frac{e^{-x/\beta}}{\left(-\frac{1}{\beta}\right)^3} \right]_0^{\infty}$$

$$= \frac{1}{\beta}\left[-\beta x^2 e^{-x/\beta} - 2x\beta^2 e^{-x/\beta} - 2\beta^3 e^{-x/\beta} \right]_0^{\infty}$$

$$= \frac{1}{\beta}\left[(0-0-0) - (0-0-2\beta^3) \right] = 2\beta^2$$

$$Var(X) = E(X^2) - (E(X))^2 = 2\beta^2 - \beta^2 = \beta^2$$

Q22.

6.45 The length of time for one individual to be served at a cafeteria is a random variable having an exponential distribution with a mean of 4 minutes. What is the probability that a person is served in less than 3 minutes on at least 4 of the next 6 days?

Q23.

6.46 The life, in years, of a certain type of electrical switch has an exponential distribution with an average life $\beta = 2$. If 100 of these switches are installed in different systems, what is the probability that at most 30 fail during the first year?

Gamma Distribution

The **gamma function** is defined by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx, \qquad \text{for } \alpha > 0.$$

- $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1), \qquad \alpha > 1$
- $\Gamma(n) = (n-1)!$
- $\Gamma(1) = 1$
- $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Gamma Distribution The continuous random variable X has a **gamma distribution**, with parameters α and β, if its density function is given by

$$f(x;\alpha,\beta) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, & x > 0, \\ 0, & \text{elsewhere,} \end{cases}$$

where $\alpha > 0$ and $\beta > 0$.

- The special gamma distribution for which $\alpha = 1$ is called the exponential distribution.
- The time until the occurrence of a Poisson event is random variable described by exponential distribution, whereas the time until a specified number of Poisson events occur is a random variable described by gamma distribution.
- The specific number of events is the parameter α in the gamma distribution.
- α is called shape parameter and β is called scale parameter.
- If $X_i \sim Exp(\beta)$, then $Y = \sum_{i=1}^{n} X_i \sim$gamma distribution.

The mean and variance of the gamma distribution are

$$\mu = \alpha\beta \text{ and } \sigma^2 = \alpha\beta^2.$$

$$\mu = \int_0^\infty x \cdot \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} dx$$

$$= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty x^\alpha e^{-x/\beta} dx$$

$$\frac{x}{\beta} = y \ , \ dx = \beta dy$$

$$= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty y^\alpha \beta^\alpha \cdot e^{-y} \cdot \beta dy$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \cdot \frac{\beta^\alpha \cdot \beta}{\cancel{\beta}} \int_0^\infty y^\alpha e^{-y} dy$$

$$\frac{1}{\Gamma(\alpha)} \cdot \beta \boxed{\int_0^\infty y^\alpha e^{-y} dy} = \Gamma(\alpha+1)$$

$$= \frac{1}{\Gamma(\alpha)} \beta \ \Gamma(\alpha+1)$$

$$= \frac{1}{\Gamma(\alpha)} \beta \cdot \alpha \Gamma(\alpha)$$

$$\mu = \alpha\beta$$

$$E(X^2) = \int_0^\infty x^2 \cdot \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} \, dx$$

$$= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty x^{\alpha+1} e^{-x/\beta} \, dx$$

$\frac{x}{\beta} = y$, $x = y\beta$

$dx = \beta \, dy$

$$= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty y^{\alpha+1} \cdot \beta^{\alpha+1} \cdot e^{-y} \cdot \beta \, dy$$

$$= \frac{\beta^{\alpha+2}}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty y^{\alpha+1} e^{-y} \, dy$$

$$= \frac{\beta^2}{\Gamma(\alpha)} \cdot \Gamma(\alpha+2)$$

$$= \frac{\beta^2}{\Gamma(\alpha)} \cdot (\alpha+1)\alpha\Gamma(\alpha)$$

$$= \beta^2 \alpha (\alpha+1) = \beta^2(\alpha^2+\alpha)$$

$$Var(X) = E(X^2) - (E(X))^2$$

$$= \beta^2(\alpha^2+\alpha) - (\beta\alpha)^2 = \alpha\beta^2$$

Q24. On Saturday morning, customers arrive at bakery according to Poisson process at an average rate of 15 per hour.

(a) What is the probability that it takes less than 10 minutes for the first 3 customers to arrive?

(b) What is the average amount of time that will elapse before 3 customers arrive in the bakery?

(c) What is the probability that exactly 15 customers arrive in an hour?

Q24 Y = No. of customers per hour — Poisson Process

Average no. of customers per hour = 15

$\mu = \lambda t = 15$

(a) X: Time for first three customers to arrive

Gamma distribution with $\alpha = 3$

$\beta = \frac{1}{15}$ hr or $\frac{60}{15}$ minutes = 4 minutes

$$P(X < 10) = \int_0^{10} \frac{x^{3-1} e^{-x/4}}{4^3 \, \Gamma(3)} dx$$

Here given minutes, so $\beta = 4$

$$= \int_0^{10} \frac{x^2 e^{-x/4}}{128} dx$$

$$= \frac{1}{128}\left[x^2 \left(\frac{e^{-x/4}}{-1/4}\right) - 2x\left(\frac{e^{-x/4}}{1/16}\right) + 2\left(\frac{e^{-x/4}}{-1/64}\right)\right]_0^{10}$$

$$= \frac{1}{128}\left[e^{-\frac{5}{2}}\left(-400 - 320 - 128\right) - \left(0 - 0 - 128\right)\right]$$

$$\frac{1}{128}\left[128 - 848 e^{-5/2}\right] = 0.4562$$

(b) Average amount of time for first 3 customers

↓ ↓ ↓

Mean of gamma distrib

$= \alpha\beta$

$= 3(4)$

$= 12$ minutes

(c) P(exactly 15 customers arrive in an hour)

↓

Here it is not time, its about no of customers per hour

So it is not gamma distribution rather Poisson distribution is applicable

$$P(Y=15) = \frac{e^{-15}(15)^{15}}{15!}$$

$\because P(Y=y) = \frac{e^{-\mu}\mu^{y}}{y!}$

using (*)

$= 0.1024$

Q25.

Suppose that telephone calls arriving at a particular switchboard follow a Poisson process with an average of 5 calls coming per minute. What is the probability that up to a minute will elapse by the time 2 calls have come in to the switchboard?

Solution.

The Poisson process applies, with time until 2 Poisson events following a gamma distribution with $\beta = 1/5$ and $\alpha = 2$. Denote by X the time in minutes that transpires before 2 calls come. The required probability is given by

$$P(X \leq 1) = \int_0^1 \frac{1}{\beta^2} x e^{-x/\beta}\, dx = 25 \int_0^1 x e^{-5x}\, dx = 1 - e^{-5}(1 + 5) = 0.96.$$