Integral Calculus

Multiple Integrals

The integral $\iint\limits_R f(x,y)\,dx\,dy$ or $\iiint\limits_T f(x,y,z)\,dx\,dy\,dz$, where f is continuous at all points inside and on the boundary of the region R or T, is called multiple integrals.

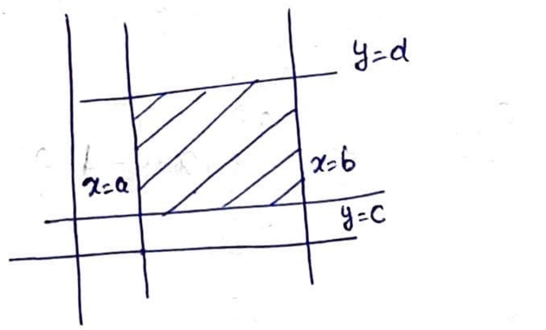
Double Integrals

$$I = \iint\limits_R f(x,y)\,dx\,dy \text{ or } \iint\limits_R f(x,y)\,dy\,dx \text{ or } \iint\limits_R f(x,y)\,dA.$$

Evaluation of double Integral

Case 1 Double integrals over rectangles (Limits of both of variables are constant)

If $f(x,y)$ is defined on a planar region (Rectangular Region) given by $R = \{(x,y): a \le x \le b,\ c \le y \le d\}$



The double integral of $f(x,y)$ over R is given by

$$\int_{y=c}^{y=d}\int_{x=a}^{x=b} f(x,y)\,dx\,dy = \int_{x=a}^{b}\int_{y=c}^{d} f(x,y)\,dy\,dx$$

Note: 1. When limits of both the variables are constant.
$R = \{(x,y): a \le x \le b,\ c \le y \le d\}$, the order of integration does not make any difference.

2. In case of constant limits,

$$\int_{y=c}^{d}\int_{x=a}^{b} f(x)\,g(y)\,dx\,dy = \int_{y=c}^{d} g(y)\,dy \int_{x=a}^{b} f(x)\,dx.$$

Q→ The value of integral $\int_{y=0}^{1}\int_{x=0}^{2} e^x \, dx \, dy$ is.

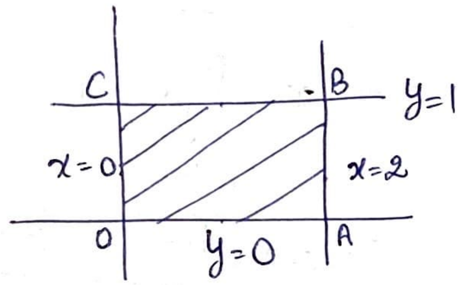
(a) e^2-1 (b) $e-1$ (c) e^2+1 (d) e^2.

Sol: $\int_{y=0}^{1}\left[\int_{x=0}^{2} e^x \, dx\right] dy = \int_{0}^{1} [e^x]_0^2 \, dy = \int_0^1 (e^2-1)\, dy$

$= [y(e^2-1)]_0^1$

$= e^2-1.$

or $\int_{x=0}^{2} e^x \, dx \int_0^1 dy = e^2-1.$

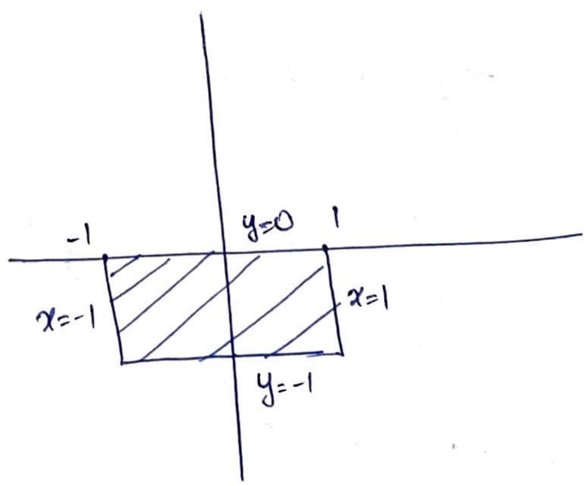


Q→ Sketch the region of integration and evaluate the double integral

$$I = \iint_R (x+y+1)\, dx\, dy \quad \text{over } R = \{(x,y): -1 \le x \le 1, -1 \le y \le 0\}$$

Sol→ $I = \int_{-1}^{0}\int_{-1}^{1} (x+y+1)\, dx\, dy$

$= \int_{-1}^{0} \left[\frac{x^2}{2} + xy + x\right]_{-1}^{1} dy$



$= \int_{-1}^{0} \left(\frac{1}{2} + y + 1 - \frac{1}{2} + y + 1\right) dy$

$= \int_{-1}^{0} (2y+2)\, dy = \left[\frac{2y^2}{2} + 2y\right]_{-1}^{0}$

$= 0 - 1 + 2(0+1)$

$= 1.$

$$\therefore I = \int_{-1}^{1}\int_{-1}^{0} (x+y+1)\, dy\, dx = \int_{-1}^{1} \left[xy + \frac{y^2}{2} + y\right]_{-1}^{0} dx \qquad (1)$$

$$= \int_{-1}^{1} 0 - \left(-x + \frac{1}{2} - 1\right) dx$$

$$= \int_{-1}^{1} \left(x + \frac{1}{2}\right) dx = \left[\frac{x^2}{2} + \frac{x}{2}\right]_{-1}^{1}$$

$$= \frac{1}{2} + \frac{1}{2} - \frac{1}{2} + \frac{1}{2}$$

$$= 1.$$

<u>Case 2</u>: Let the region R be expressed in the form

$$R = \{ (x,y) : a \le x \le b,\ g_1(x) \le y \le g_2(x) \},$$

where $g_1(x)$ and $g_2(x)$ are integrable functions, such that $g_1(x) \le g_2(x)$ for all x in $[a,b]$.

[limits of x are constant and limits of y are variable]

$$I = \int_{a}^{b} \left[\int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\, dy \right] dx.$$

<u>Case 3</u> Let the region R be defined as

$$R = \{ (x,y) : h_1(y) \le x \le h_2(y),\ c \le y \le d \},$$

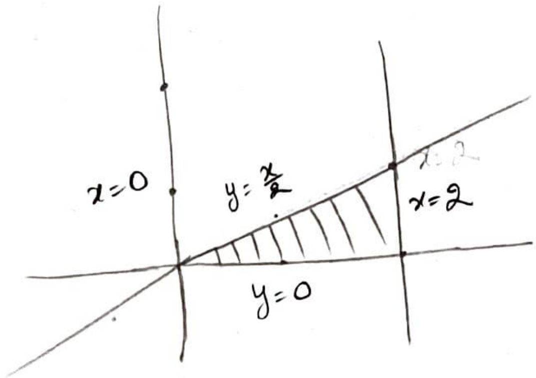
where $h_1(y)$ and $h_2(y)$ are integrable functions, such that $h_1(y) \le h_2(y)$ for all y in $[c,d]$.

[limits of y are constant and limits of x are variable.]

$$I = \int_{y=c}^{d} \left[\int_{x=h_1(y)}^{x=h_2(y)} f(x,y)\, dx \right] dy.$$

Q:- Evaluate $\displaystyle\int_{x=0}^{2}\int_{y=0}^{x/2} e^{x^2}\,dy\,dx$

Sol:-



$$\int_0^2\left[\int_0^{x/2} e^{x^2}\,dy\right]dx$$

$$= \int_0^2 \left[y\, e^{x^2}\right]_0^{x/2} dx$$

$$= \int_0^2 \frac{x}{2} e^{x^2}\,dx = \frac{1}{2}\int_0^2 x e^{x^2}\,dx$$

Let $t = \cancel{e}\, x^2$

$dt = 2x\,dx$

$\Rightarrow x\,dx = \frac{dt}{2}$

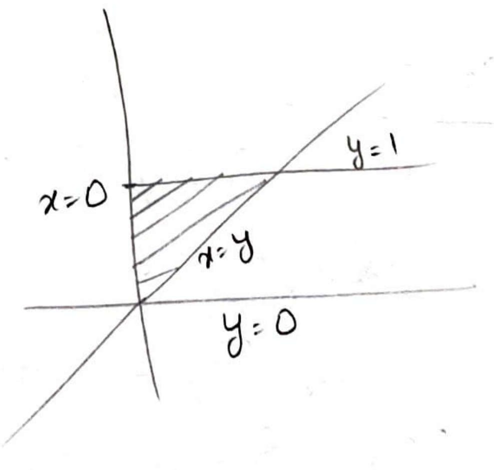
when $x=0$, $t=0$

$x=2$, $t=4$

$$= \frac{1}{4}\int_0^4 e^t\,dt$$

$$= \frac{1}{4}\left[e^t\right]_0^4 = \frac{1}{4}\left[e^4 - 1\right]$$

Q:- Evaluate $\displaystyle\int_{y=0}^{1}\int_{x=0}^{y}(x^2+y^2)\,dx\,dy$



Sol:- $\displaystyle\int_0^1\left[\frac{x^3}{3}+xy^2\right]_0^y dy$

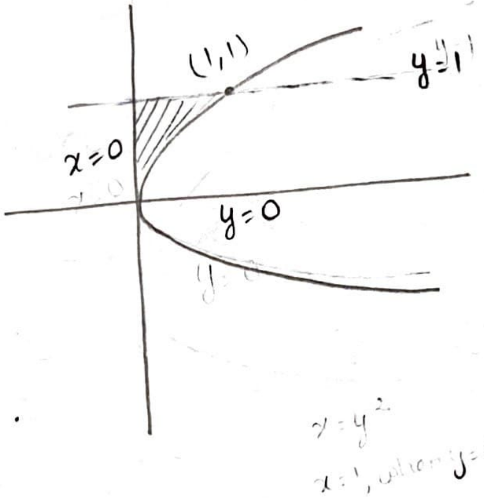
$$= \int_0^1 \left(\frac{y^3}{3} + y^3\right)dy$$

$$= \frac{4}{3}\int_0^1 y^3\,dy = \frac{4}{3}\left[\frac{y^4}{4}\right]_0^1 = \frac{1}{3}.$$

Evaluate $\iint_R f(x,y)\,dx\,dy$,

where $f(x,y) = 3y^3 e^{xy}$, $R: 0 \le x \le y^2;\ 0 \le y \le 1$.

Sol:- $\displaystyle\int_0^1 \int_0^{y^2} 3y^3 e^{xy}\,dx\,dy$



$$= \int_0^1 \left[3y^3 \frac{e^{xy}}{y}\right]_0^{y^2} dy$$

$$= \int_0^1 \left[3y^2 e^{xy}\right]_0^{y^2} dy = \int_0^1 3y^2(e^{y^3} - 1)\,dy$$

$$= 3\int_0^1 y^2 e^{y^3}\,dy - 3\int_0^1 y^2\,dy$$

$$= \left[\int_0^1 e^t\,dt\right] - 3\left[\frac{y^3}{3}\right]_0^1$$

Let $y^3 = t$
$3y^2\,dy = dt$
when $y=0, t=0$
$y=1, t=1$

$$= \left[e^t\right]_0^1 - 1$$

$$= e - 1 - 1$$

$$= e - 2.$$

Properties of double Integral

① If $f(x,y)$ and $g(x,y)$ are integrable functions, then

$$\iint_R \left[f(x,y) \pm g(x,y)\right] dx\,dy = \iint_R f(x,y)\,dx\,dy \pm \iint_R g(x,y)\,dx\,dy.$$

2. $\iint_R k f(x,y)\,dx\,dy = k \iint_R f(x,y)\,dx\,dy$, where k is any real constant.

3. When $f(x,y)$ is integrable, then $|f(x,y)|$ is also integrable,

$$\left| \iint_R f(x,y)\,dx\,dy \right| \le \iint_R |f(x,y)|\,dx\,dy.$$

4. If $0 < f(x,y) \le g(x,y)$ for all (x,y) in R, then

$$\iint_R f(x,y)\,dx\,dy \le \iint_R g(x,y)\,dx\,dy.$$

5. If $f(x,y) \ge 0$ for all (x,y) in R, then

$$\iint_R f(x,y)\,dx\,dy \ge 0.$$

Applications of double Integrals

1. If $f(x,y) = 1$, then $\iint_R dx\,dy$ gives the area A of region R.

Ex:- If R is the rectangle bounded by lines $x=a$, $x=b$, $y=c$, $y=d$, then

$$A = \int_c^d \int_a^b dx\,dy = (b-a)(d-c)$$

gives the area of rectangle.

If $z = f(x,y)$ is a surface, then

$$\iint_R z\,dx\,dy \text{ or } \iint_R f(x,y)\,dx\,dy$$

gives the volume of the region beneath the surface $z = f(x,y)$ and above the xy plane.

~~Ex~~

Change of order of integration

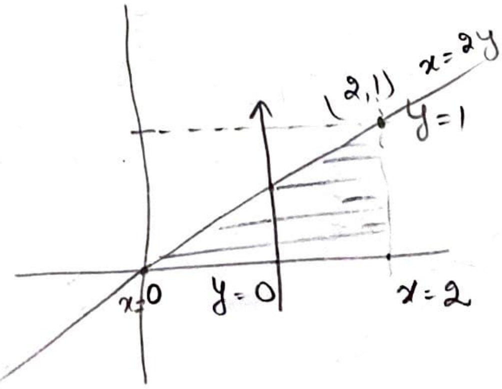
$$a \le x \le b,\ g_1(x) \le y \le g_2(x) \xrightarrow{\text{change of order}} c \le y \le d,\ h_1(y) \le x \le h_2(y).$$

i.e.
$$\int_{x=a}^{b}\left[\int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\,dy\right]dx \xrightarrow{\text{change of order}} \int_{y=c}^{d}\left[\int_{x=h_1(y)}^{h_2(y)} f(x,y)\,dx\right]dy$$

Q-1 Evaluate the double integral $\iint_R e^{x^2}\,dx\,dy$, where the region R is given by

R: $2y \le x \le 2,\ 0 \le y \le 1.$

Sol:
$$I = \int_0^1 \int_{2y}^{2} e^{x^2}\,dx\,dy$$



Since, the integral cannot be evaluated by integrating first w.r.t x. So, we try to evaluate it by integrating it first w.r.t y.

Note: To find the limits of y, consider an imaginary line entering the region R vertically in the increasing direction of y. Check where it enters R and leaves R. That will give lower limit (enter R) and upper limit (exit R) of y.

After the change of order of integration, we have

$$I = \int_{x=0}^{x=2} \int_{y=0}^{y=\frac{x}{2}} e^{x^2}\, dy\, dx$$

$$= \int_0^2 \left[y e^{x^2}\right]_0^{\frac{x}{2}} dx$$

$$= \int_0^2 \frac{x}{2} e^{x^2} dx$$

$$= \frac{1}{2}\int_0^2 x e^{x^2} dx$$

Let $x^2 = t$

$2x\,dx = dt$

$x\,dx = \frac{dt}{2}$

When $x = 0,\ t = 0$

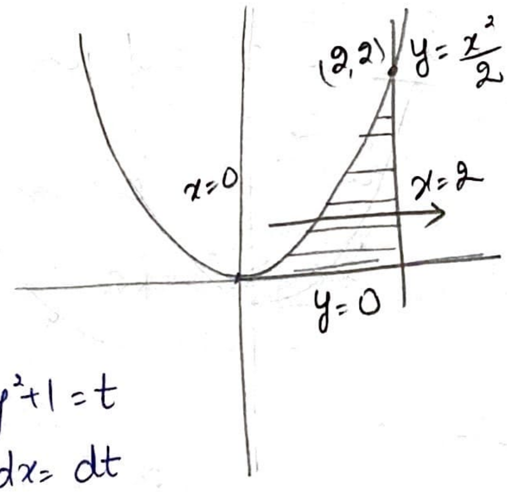
$x = 2,\ t = 4$

$$= \frac{1}{2}\int_0^4 e^t \frac{dt}{2}$$

$$= \frac{1}{4}\left[e^4 - 1\right].$$

Q: Evaluate $\displaystyle\int_{x=0}^{2} \int_{y=0}^{x^2/2} \frac{x}{\sqrt{x^2+y^2+1}}\, dy\, dx.$

Sol:



$$\int_0^2 \int^{2} \frac{x}{\sqrt{x^2+y^2+1}}\, dx\, dy$$

$\cancel{y=}$ $x=\sqrt{2y}$

$$= \int_0^2 \int_{(y+1)^2}^{y^2+5} \frac{dt/2}{\sqrt{t}}\, dy$$

$x^2+y^2+1=t$

$2x\,dx = dt$

when $x=\sqrt{2y}$

$t = 2y+y^2+1$

$=(y+1)^2$

when $x=2$, $t = \cancel{t}\ y^2+5$

$$= \frac{1}{2}\int_0^2 \left[\frac{t^{1/2}}{1/2}\right]_{(y+1)^2}^{y^2+5} dy$$

$$= \frac{2}{2}\int_0^2 \left[\sqrt{y^2+5} - (y+1)\right] dy$$

$$= \int_0^2 \sqrt{y^2+(\sqrt{5})^2}\, dy - \int_0^2 (y+1)\, dy$$

$$\left[\int \sqrt{a^2+x^2}\, dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\log\left|x+\sqrt{a^2+x^2}\right| + c\right.$$

$$= \left[\frac{y}{2}\sqrt{y^2+5} + \frac{5}{2}\log\left|y+\sqrt{y^2+5}\right|\right]_0^2$$

$$- \left[\frac{y^2}{2}+y\right]_0^2$$

$$= 3 + \frac{5}{2}\log|2+3) - \frac{5}{2}\log|\sqrt{5}| - 4$$

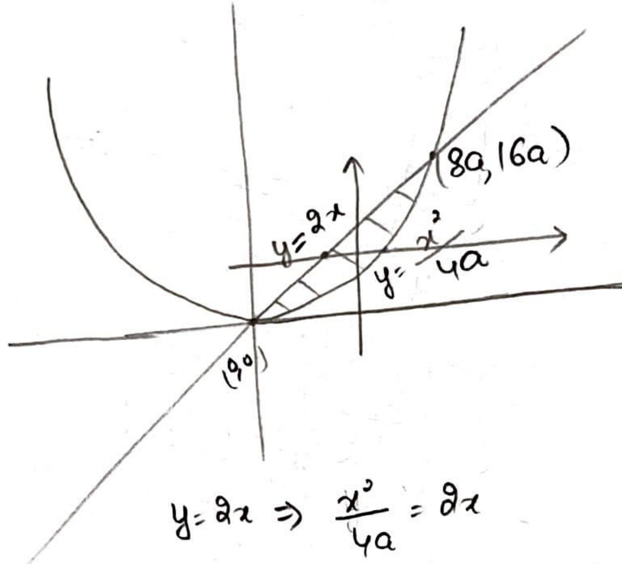
$$= \frac{5}{2}\log 5 - \frac{5}{2}\log\sqrt{5} - 1$$

$$= \frac{5}{2}\log 5 - \frac{5}{4}\log 5 - 1$$

$$= \frac{5}{4}\log 5 - 1.$$

Q:- Evaluate the double integral $\iint_R xy\,dxdy$, where R is the region bounded by the x-axis, the line $y=2x$ and the parabola $y=\frac{x^2}{4a}$.

Sol :-



$$\iint_R xy\,dxdy$$

$$= \int_0^{16a}\int_{y/2}^{\sqrt{4ay}} xy\,dx\,dy \text{ or } \int_0^{8a}\int_{\frac{x^2}{4a}}^{2x} xy\,dy\,dx$$

$$= \int_0^{16a} y\left[\frac{x^2}{2}\right]_{y/2}^{\sqrt{4ay}} dy \text{ or } \int_0^{8a} x\left[\frac{y^2}{2}\right]_{\frac{x^2}{4a}}^{2x} dx$$

$$= \int_0^{16a} \frac{y}{2}\left[4ay - \frac{y^2}{4}\right] dy \text{ or } \int_0^{8a} \frac{x}{2}\left[4x^2 - \frac{x^4}{16a^2}\right] dx$$

$$= \frac{1}{2}\int_0^{16a}\left[4ay^2 - \frac{y^3}{4}\right] dy \text{ or } \frac{1}{2}\int_0^{8a}\left(4x^3 - \frac{x^5}{16a^2}\right) dx$$

$$= \frac{1}{2}\left[\frac{4ay^3}{3} - \frac{y^4}{16}\right]_0^{16a} \text{ or } \frac{1}{2}\left[\frac{4x^4}{4} - \frac{x^6}{16a^2(6)}\right]_0^{8a}$$

$$= \frac{4a\,(16a)^3}{6} - \frac{(16a)^4}{\cancel{8}\,2\cdot 16} \text{ or } \frac{1}{2}\left[(8a)^4 - \frac{(8a)^6}{16\times 6\,a^2}\right]$$

$y=2x \Rightarrow \frac{x^2}{4a} = 2x$

$\Rightarrow x^2 = 8ax$

$\Rightarrow x^2 - 8ax = 0$

$\Rightarrow x(x-8a) = 0$

$\Rightarrow x=0,\ x=8a$

$y=0,\ y=16a$

$$= \frac{4a}{3}(2048)a^3 - 2048a^4 \quad \text{or} \quad 2048a^4 - \frac{8192}{2.3}a^4$$

$$= \frac{8192}{3}a^4 - 2048a^4 \quad \text{or} \quad 2048a^4 - \frac{4096}{3}a^4$$

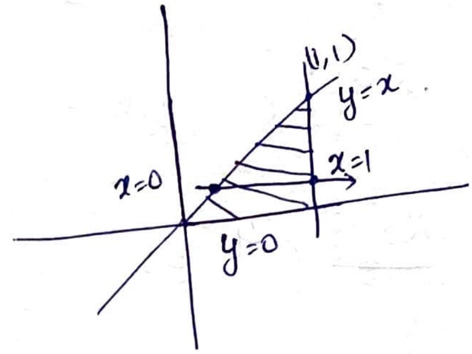
$$= \frac{2048}{3}a^4 \quad \text{or} \quad \frac{2048}{3}a^4.$$

<u>Note</u> :- ① To find the limits of x, Consider an imaginary line entering the region R horizontally in the increasing direction of x. Check where it enters R (lower limit) and leaves R (upper limit).

② To find the limits of y, Consider an imaginary line entering the region R vertically in the increasing direction of y. Check where it enters R (lower limit) and leaves R (upper limit).

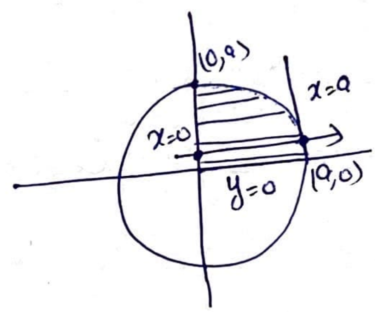
Change the order of integration.

(1) $$I = \int_0^1 \int_0^x f(x,y)\,dy\,dx$$

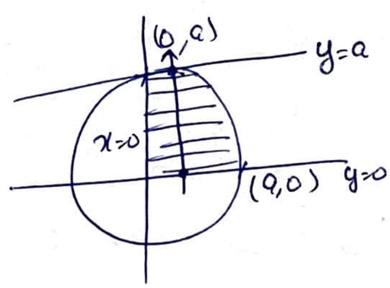


$$= \int_0^1 \int_y^1 f(x,y)\,dx\,dy$$

(2) $$I = \int_0^a \int_0^{\sqrt{a^2-x^2}} f(x,y)\,dy\,dx$$



$$= \int_0^a \int_0^{\sqrt{a^2-y^2}} f(x,y)\,dx\,dy$$



Area using double integrals

If $f(x,y)=1$, then the integral

$$A = \iint_R dx\,dy$$ is called the area of the region R.

Q: Find the area bounded by the curves $y=x^2, y=4-x^2$.

Sol:

$y=x^2, y=4-x^2$

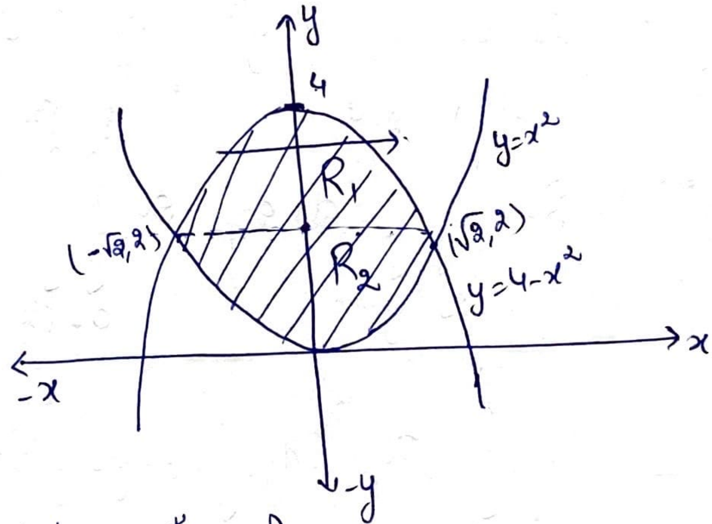
$\Rightarrow 4-x^2 = x^2$

$\Rightarrow 2x^2 = 4$

$\Rightarrow x^2 = 2$

$\Rightarrow x = \sqrt{2}, -\sqrt{2}$

$y = 2, 2$



So, $(\sqrt{2}, 2)$ and $(-\sqrt{2}, 2)$ are points of intersection of parabolas.

$$A = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{x^2}^{4-x^2} 1.\, dy\, dx \quad \text{or} \quad \int_{2}^{4} \int_{-\sqrt{4-y}}^{\sqrt{4-y}} dx\, dy + \int_{0}^{2} \int_{-\sqrt{y}}^{\sqrt{y}} dx\, dy$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} [y]_{x^2}^{4-x^2} dx = \int_{-\sqrt{2}}^{\sqrt{2}} (4-x^2-x^2)\, dx = \left[4x - \frac{2x^3}{3}\right]_{-\sqrt{2}}^{\sqrt{2}}$$

$$= 4\sqrt{2} - \frac{4\sqrt{2}}{3} + 4\sqrt{2} - \frac{4\sqrt{2}}{3}$$

$$= \frac{16\sqrt{2}}{3}.$$

Q:- Find the area bounded by the curves $x = y^2$, $x + y - 2 = 0$.

Sol:-

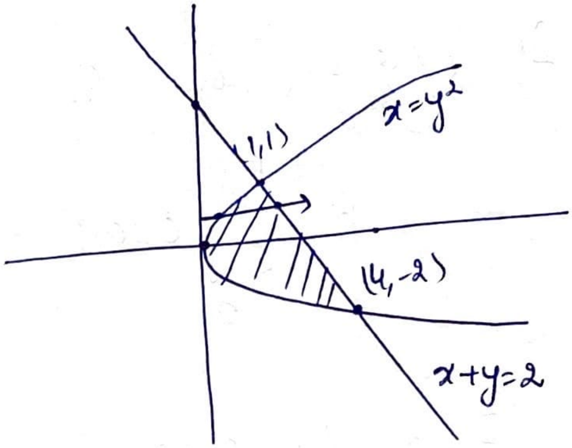
$x = y^2$; $x + y - 2 = 0$

$y^2 + y - 2 = 0$

$(y-1)(y+2) = 0$

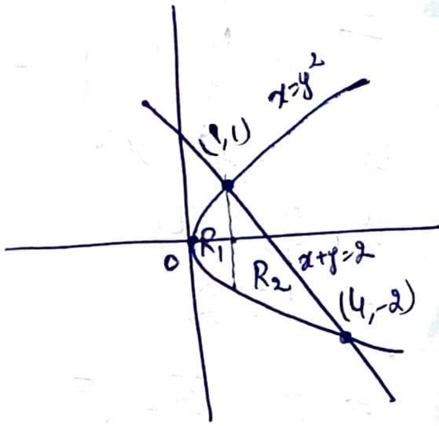
$y = 1, -2$

$x = 1, 4$



So, (1,1) and (4,-2) are the points of intersection of the curves $x = y^2$; $x + y - 2 = 0$.

$$A = \int_{-2}^{1}\int_{y^2}^{2-y} dx\,dy \quad \text{or} \quad \int_{0}^{1}\int_{-\sqrt{x}}^{\sqrt{x}} dy\,dx + \int_{1}^{4}\int_{-\sqrt{x}}^{2-x} dy\,dx$$



$$= \int_{-2}^{1} [x]_{y^2}^{2-y} dy = \int_{-2}^{1} (2 - y - y^2)\,dy = \left[2y - \frac{y^2}{2} - \frac{y^3}{3}\right]_{-2}^{1}$$

$$= 2 - \frac{1}{2} - \frac{1}{3} + 4 + \cancel{\frac{4}{2}}^{\,2} - \frac{8}{3}$$

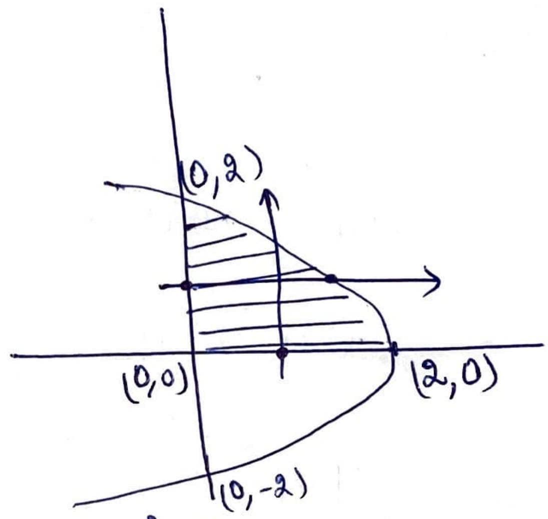
$$= 8 - \cancel{\frac{9}{3}}^{\,3} - \frac{1}{2}$$

$$= 5 - \frac{1}{2} = \frac{9}{2}.$$

Q: Find the area bounded by the curves $y^2 = 4 - 2x,\ x \geq 0,\ y \geq 0$

Sol:

$$A = \int_0^2 \int_0^{2-\frac{y^2}{2}} dx\, dy$$



$$= \int_0^2 \left(2 - \frac{y^2}{2}\right) dy$$

$$y^2 = 4 - 2x$$

$$2x = 4 - y^2$$

$$x = 2 - \frac{y^2}{2}$$

$$= \left[2y - \frac{y^3}{6}\right]_0^2 = 4 - \frac{8}{6} = 4 - \frac{4}{3}$$

$$= \frac{8}{3}.$$

or

$$A = \int_0^2 \int_0^{\sqrt{4-2x}} dy\, dx = \frac{8}{3}.$$

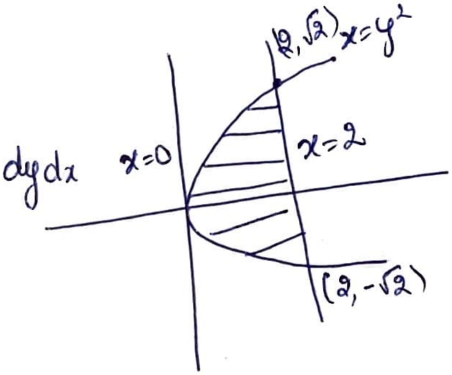
Volume by double Integrals

$I = \iint_R f(x,y)\, dA$ represents volume of the region R bounded below by the surface $z = f(x,y)$ and above by the region R (in xy plane).

Q: Find the volume of the solid which is below the plane $z = 2x+3$ and above the xy-plane and bounded by $y^2 = x$, $x=0$, $x=2$.

Sol:

$$\text{Volume} = \int_{-\sqrt{2}}^{\sqrt{2}}\int_{y^2}^{2} (2x+3)\, dx\, dy \quad \text{or} \quad \int_{0}^{2}\int_{-\sqrt{x}}^{\sqrt{x}} (2x+3)\, dy\, dx$$



$$= 2\int_0^{\sqrt{2}} \left[x^2 + 3x\right]_{y^2}^{2} dy$$

$$= 2\int_0^{\sqrt{2}} (10 - y^4 - 3y^2)\, dy = 2\left[10y - \frac{y^5}{5} - \frac{3y^3}{3}\right]_0^{\sqrt{2}}$$

$$= 2\left[10\sqrt{2} - \frac{4\sqrt{2}}{5} - 2\sqrt{2}\right]$$

$$= 2\left(\frac{36\sqrt{2}}{5}\right) = \frac{72\sqrt{2}}{5}.$$

Triple Integral

Let $f(x,y,z)$ be a Cont function defined over a closed and bounded region T in R^3. Then, triple integral of f over T is

$$I = \iiint_T f(x,y,z)\,dV$$

$$= \int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y,z)\,dx\,dy\,dz$$

dxdydz
~~dxdydz~~ dxdydz
dydxdz
dydzdx
dzdxdy
dzdydx

Application of triple Integral

If $f(x,y,z)=1$, then the triple integral

$$V = \iiint_T dx\,dy\,dz$$

gives the volume of the region T.

Evaluation of Triple Integrals

1. If the limits of all three variables are constant, then order of integration does not make any difference.

2. We evaluate the triple integral by three successive integrations. If the region R can be described by

$$R = \left\{ \begin{array}{l} a \le x \le b, \\ g_1(x) \le y \le g_2(x) \\ h_1(x,y) \le z \le h_2(x,y) \end{array} \right\}.$$

then we evaluate the triple integral as

$$\int_a^b \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=h_1(x,y)}^{z=h_2(x,y)} f(x,y,z)\,dz\,dy\,dx.$$

Q: Evaluate $\int_0^a \int_0^b \int_0^c x^2y^2z^2\,dx\,dy\,dz.$

Sol:

$$I = \int_0^a \int_0^b y^2z^2\left(\frac{x^3}{3}\right)_0^c dy\,dz$$

$$= \int_0^a \int_0^b y^2z^2\left(\frac{c^3}{3}\right) dy\,dz$$

$$= \int_0^a \frac{c^3}{3}z^2\left(\frac{b^3}{3}\right)dz = \frac{a^3b^3c^3}{3.9} = \frac{a^3b^3c^3}{27}$$

Q: If $T: 0\le x\le 1,\ 0\le y\le x^2,\ 0\le z\le x+y.$

Evaluate $\iiint_T (2x-y-z)\,dx\,dy\,dz.$

Sol:

$$I = \int_0^1 \int_0^{x^2} \int_0^{x+y} (2x-y-z)\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^{x^2} \left[2xz - yz - \frac{z^2}{2}\right]_0^{x+y} dy\,dx$$

$$= \int_0^1 \int_0^{x^2} \left[2x(x+y) - y(x+y) - \frac{(x+y)^2}{2}\right] dy\,dx$$

$$= \int_0^1 \int_0^{x^2} \left[2x^2 + 2xy - xy - y^2 - \frac{x^2}{2} - \frac{y^2}{2} - \frac{\cancel{2}xy}{\cancel{2}}\right] dy\,dx$$

$$= \int_0^1 \int_0^{x^2} \left(\frac{3x^2}{2} - \frac{3y^2}{2} \right) dy\, dx$$

$$= \int_0^1 \left[\frac{3}{2} x^2 y - \frac{\cancel{3} y^3}{2 \cdot \cancel{3}} \right]_0^{x^2} dx$$

$$= \int_0^1 \left(\frac{3}{2} x^4 - \frac{x^6}{2} \right) dx.$$

$$= \frac{3}{2} \left[\frac{x^5}{5} \right]_0^1 - \frac{1}{2} \left[\frac{x^7}{7} \right]_0^1$$

$$= \frac{3}{2 \cdot 5} - \frac{1}{2 \cdot 7} = \frac{1}{2} \left[\frac{3}{5} - \frac{1}{7} \right] = \frac{1}{2} \left[\frac{21-5}{35} \right] = \frac{1}{2} \left[\frac{16}{35} \right] = \frac{8}{35}.$$

$$I = \frac{8}{35}.$$

Q: Evaluate $\int_0^1 \int_0^x \int_0^{x+y} dz\, dy\, dx$ (Volume)

Sol: $I = \int_0^1 \int_0^x \int_0^{x+y} dz\, dy\, dx$

$$= \int_0^1 \int_0^x (x+y)\, dy\, dx = \int_0^1 \left[xy + \frac{y^2}{2} \right]_0^x dx$$

$$= \int_0^1 \left(x^2 + \frac{x^2}{2} \right) dx = \frac{3}{2} \int_0^1 x^2 dx$$

$$= \frac{3}{2} \left(\frac{x^3}{3} \right)_0^1$$

$$= \frac{1}{2}.$$

Q: The value of integral $\int_0^1\int_0^1\int_0^1 dz\,dy\,dx$ is (Volume)

(a) 3 (b) 2 (c) 1 (d) $\frac{1}{3}$.

Sol: $I = \int_0^1\int_0^1 1\,dy\,dx = \int_0^1 1\,dx = 1.$

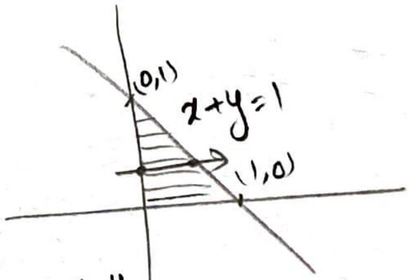
Q: Evaluate $\iiint_R \frac{dx\,dy\,dz}{(x+y+z+1)^3}$.

R is bounded by $x=0,\ y=0,\ z=0,\ x+y+z=1$.

Sol: $x+y+z=1 \Rightarrow z=1-x-y,\ z=0$

let $z=0,\ x+y=1 \Rightarrow y=1-x,\ y=0$

Let $y,z=0,\ x=1,\ x=0$



$$I = \int_0^1\int_0^{1-x}\int_0^{1-x-y} (x+y+z+1)^{-3}\,dz\,dy\,dx \qquad \int_0^1\int_0^{1-y}\int_0^{1-x-y} (x+y+z+1)^{-3}\,dz\,dx\,dy$$

$$= \int_0^1\int_0^{1-x} \left[\frac{(x+y+z+1)^{-2}}{-2}\right]_0^{1-x-y} dy\,dx$$

$$= -\frac{1}{2}\int_0^1\int_0^{1-x} \left[\cancel{x}+\cancel{y}+1-\cancel{x}-\cancel{y}+1 - (x+y+1)^{-2}\right] dy\,dx$$

$$= -\frac{1}{2}\int_0^1\int_0^{1-x} \left[2^{-2} - (x+y+1)^{-2}\right] dy\,dx$$

$$= \frac{-1}{2}\int_0^1\int_0^{1-x}\left[\frac{1}{2^2} - (x+y+1)^{-2}\right]dy\,dx$$

$$= \frac{-1}{2}\int_0^1 \left[\frac{1}{4}y - \frac{(x+y+1)^{-1}}{-1}\right]_0^{1-x} dx$$

$$= \frac{-1}{2}\int_0^1\left[\frac{1-x}{4} + (\cancel{x}+1-\cancel{x}+1)^{-1} - (1+x)^{-1}\right]dx$$

$$= \frac{-1}{2}\int_0^1\left(\frac{1-x}{4} + \frac{1}{2} - \frac{1}{x+1}\right)dx$$

$$= \frac{-1}{2}\left[\frac{1}{4}\left(x - \frac{x^2}{2}\right) + \frac{1}{2}x - \log(x+1)\right]_0^1$$

$$= \frac{-1}{2}\left[\frac{1}{4}\cdot\frac{1}{2} + \frac{1}{2} - \log 2\right]$$

$$= \frac{-1}{2}\left[\frac{1}{8} + \frac{1}{2} - \log 2\right]$$

$$= \frac{-1}{16}\left[5 - 8\log 2\right] = \frac{8\log 2 - 5}{16}.$$

Volume using triple integrals

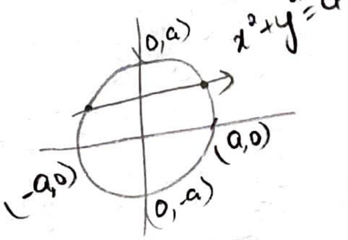
Q: Find the volume of the solid enclosed between the surfaces $x^2+y^2=a^2$ and $x^2+z^2=a^2$.

Sol:

$x^2+z^2=a^2 \Rightarrow z^2=a^2-x^2 \Rightarrow z=\pm\sqrt{a^2-x^2}$.

$x^2+y^2=a^2 \Rightarrow y=\pm\sqrt{a^2-x^2}$

$x=\pm a$.



$$V=\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dz\,dy\,dx$$

or $$\int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dz\,dx\,dy$$

$$=\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}[z]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\,dy\,dx$$

$$=\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 2\sqrt{a^2-x^2}\,dy\,dx$$

$$=2\int_{-a}^{a}\sqrt{a^2-x^2}\left(2\sqrt{a^2-x^2}\right)dx$$

$$=4\int_{-a}^{a}(a^2-x^2)\,dx=4\left[a^2x-\frac{x^3}{3}\right]_{-a}^{a}$$

$$=4\left[a^3-\frac{a^3}{3}+a^3-\frac{a^3}{3}\right]$$

$$=8\left(\frac{2a^3}{3}\right)$$

$$=\frac{16a^3}{3}.$$

Change of variables in integrals

In a double integral (x-y plane to u-v plane)

Let $x = \phi(u,v)$, $y = \psi(u,v)$.

$$\iint_{R_{xy}} f(x,y)\,dx\,dy = \iint_{R'_{uv}} f[\phi(u,v), \psi(u,v)]\,|J|\,du\,dv.$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Particular case

To change cartesian coordinates (x,y) to polar coordinates (r,θ).

$x = r\cos\theta$, $y = r\sin\theta$.

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

$$\therefore \iint_{R_{xy}} f(x,y)\,dx\,dy = \iint_{R'_{r\theta}} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

① Evaluate $\iint_R (x^2+y^2)\,dx\,dy$, $R: 0 \le y \le \sqrt{1-x^2}$, $0 \le x \le 1$.

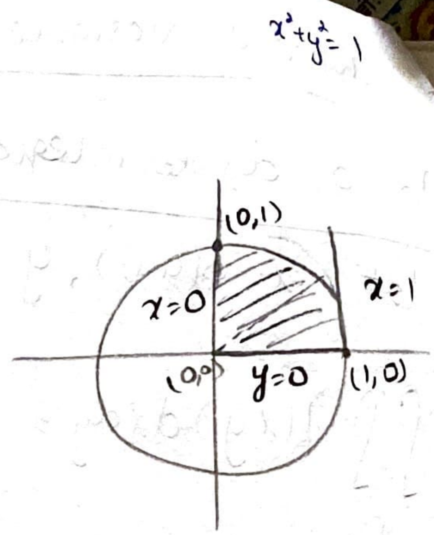
Sol:

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx = \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_0^{\sqrt{1-x^2}} dx$$

$$= \int_0^1 \left[x^2\sqrt{1-x^2} + \frac{(1-x^2)^{3/2}}{3}\right] dx$$

Let $x = r\cos\theta,\ y = r\sin\theta$

$x^2+y^2 = r^2$.



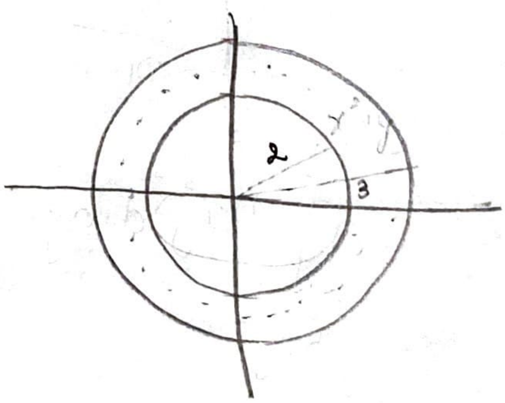
$$\iint_R f(x,y)\,dx\,dy = \iint_R f(r,\theta)\,|J|\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^1 r^2 \cdot r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left(\frac{r^4}{4}\right)_0^1 d\theta = \frac{1}{4}\int_0^{\pi/2} d\theta = \frac{1}{4}\cdot\frac{\pi}{2} = \frac{\pi}{8}.$$

Q: Evaluate the integral $\iint_R \sqrt{x^2+y^2}\,dx\,dy$ by changing to polar co-ordinates, where R is the region in the xy plane bounded by the circles $x^2+y^2=4$ and $x^2+y^2=9$.

Sol:



$$I = \iint_R \sqrt{x^2+y^2}\,dx\,dy$$

$$= \int_0^{2\pi}\int_2^3 r\cdot r\,dr\,d\theta = \int_2^3 r^2\,dr\int_0^{2\pi} d\theta$$

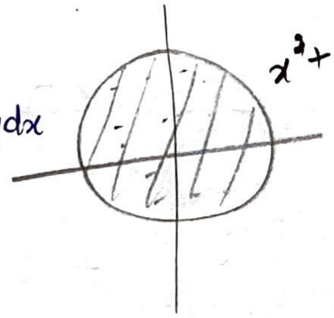
$$= \left[\frac{r^3}{3}\right]_2^3 \cdot 2\pi$$

$$= \frac{(27-8)(2\pi)}{3} = \frac{38\pi}{3}.$$

(2) Evaluate the integral $\iint_R (a^2-x^2-y^2)\,dxdy$, where R is the region $x^2+y^2 \le a^2$.

Sol:

$$I = \int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} (a^2-x^2-y^2)\,dxdy \quad \text{or} \quad \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (a^2-x^2-y^2)\,dydx$$



$x^2+y^2=a^2$

Changing to polar co-ordinates:

$x = r\cos\theta,\ y = r\sin\theta,\ J = r$

$$I = \int_0^{2\pi}\int_0^{a} (a^2-r^2)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{a} (a^2 r - r^3)\,dr = 2\pi\left[\frac{a^2 r^2}{2} - \frac{r^4}{4}\right]_0^a$$

$$= 2\pi\left[\frac{a^4}{2} - \frac{a^4}{4}\right]$$

$$= 2\pi\,\frac{\cancel{2}a^4}{4}$$

$$= \frac{\cancel{2}a^4\pi}{4}$$

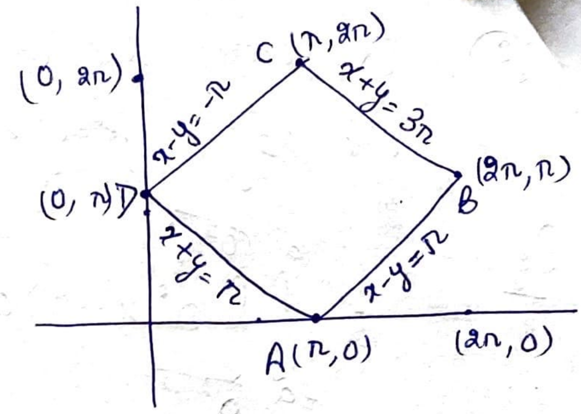
$$= \frac{\pi a^4}{2}.$$

(3) Evaluate the integral $\iint_R (x-y)^2 \cos^2(x+y)\,dxdy$, where R is the rhombus with successive vertices at $(\pi,0)$, $(2\pi,\pi)$, $(\pi,2\pi)$, $(0,\pi)$.

Sol :

Substitute $x-y=u$
$x+y=v$

$\Rightarrow -\pi \le u \le \pi, \ \pi \le v \le 3\pi.$



$x-y=u$
$x+y=v$ $\Rightarrow x = \dfrac{u+v}{2}$

$y = \dfrac{v-u}{2}$

Eq of AB : $y-y_1 = m(x-x_1)$
$y-0 = m(x-\pi)$

$m = \dfrac{y_2-y_1}{x_2-x_1} = \dfrac{\pi}{\pi} = 1$

$\Rightarrow y = x-\pi \Rightarrow \cancel{x+y=\pi}$

$\Rightarrow \boxed{x-y=\pi}$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix}$$

$$= \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2}.$$

Eq of BC : $y-\pi = m(x-2\pi)$

$m = \dfrac{\pi}{-\pi} = -1$

$y-\pi = -1(x-2\pi)$
$= -x+2\pi$

$\boxed{x+y=3\pi}$

$$\therefore \iint_R (x-y)^2 \cos^2(x+y)\, dx\, dy$$

$$= \int_{v=\pi}^{3\pi} \int_{u=-\pi}^{\pi} u^2 \cos^2 v \ \frac{1}{2}\, du\, dv$$

Eq of CD : $y-2\pi = m(x-\pi)$

$m = \dfrac{\pi}{\pi} = 1$

$y-2\pi = x-\pi$

$\Rightarrow \boxed{x-y=-\pi}$

$$= \frac{1}{2} \int_{\pi}^{3\pi} \cos^2 v\, dv \int_{-\pi}^{\pi} u^2\, du$$

Eq of AD : $y-\pi = m(x-0)$

$m = \dfrac{\pi}{-\pi} = -1$

$y-\pi = -x$

$\Rightarrow \boxed{x+y=+\pi}$

$$= \frac{1}{2} \int_{\pi}^{3\pi} \left(\frac{1+\cos 2v}{2}\right) dv \left[\frac{u^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{12}\left[v + \frac{\sin 2v}{2}\right]_{\pi}^{3\pi} (\pi^3 + \pi^3)$$

$$= \frac{2\pi^3}{12}\left[3\pi + 0 - \pi - 0\right]$$

$$= \frac{\pi^3}{6}\left[2\pi\right] = \frac{\pi^4}{3}.$$

Q: The transformations $x+y=u$, $y=uv$ transform the area element $dy\,dx$ into $|J|\,du\,dv$, where $|J|$ is equal to

(a) 1 (b) u (c) -1 (d) none of these.

Sol: $J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

$x = u - uv$

$y = uv$

$J = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u - uv + uv = u.$

(b) is correct.

Q: The area bounded by the circle $r=4$ is

(i) 16π (ii) 17π (iii) 18π (iv) 19π.

Sol: $\text{Area} = \int_0^{2\pi}\int_0^4 1\,r\,dr\,d\theta = \left[\frac{r^2}{2}\right]_0^4 (2\pi)$

$= \frac{1}{\cancel{2}}(16)\cancel{2}\pi = 16\pi.$ (i)

Q: The formula of area in polar co-ordinates is

(i) $\iint d\theta\,dr$ (ii) $\iint r^2\,d\theta\,dr$ (iii) $\iint r\,d\theta\,dr$ (iv) $\iint \frac{1}{r}\,d\theta\,dr$.

Sol: (iii)

Change of variables

Triple Integrals

Define $x = \phi(u,v,w)$, $y = \psi(u,v,w)$, $z = \varphi(u,v,w)$.

Then
$$\iiint_T f(x,y,z)\,dx\,dy\,dz = \iiint_{T^*} f[\phi(u,v,w), \psi(u,v,w), \varphi(u,v,w)]\,|J|\,du\,dv\,dw$$

$$= \iiint_{T^*} g(u,v,w)\,|J|\,du\,dv\,dw,$$

where
$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Particular cases

Changing Cartesian coordinates to cylindrical coordinates

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \cos\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.$$

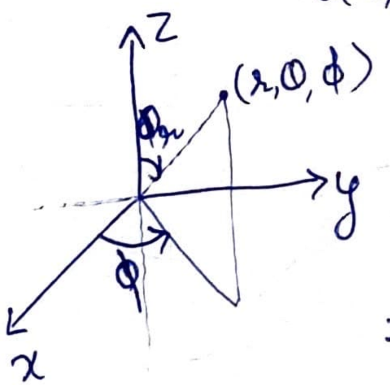
$$\iiint_T f(x,y,z)\,dx\,dy\,dz = \iiint_{T^*} g(r,\theta,z)\,r\,dr\,d\theta\,dz.$$

Changing cartesian coordinates to spherical coordinates

$\cancel{x = r\sin\phi\cos\theta,\ y =}$

$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta,\ 0 \le \theta \le \pi,$
$0 \le \phi \le 2\pi.$

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$



$$= r\sin\theta \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & \cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r\sin\theta\left[-\sin\phi(-r\sin^2\theta\sin\phi - r\cos^2\theta\sin\phi) - \cos\phi(-r\sin^2\theta\cos\phi - r\cos^2\theta\cos\phi)\right]$$

$$= r\sin\theta\left[+r\sin^2\phi(\sin^2\theta+\cos^2\theta)+r\cos^2\phi(\sin^2\theta+\cos^2\theta)\right]$$

$$= r\sin\theta\left[r\sin^2\phi+r\cos^2\phi\right]$$

$$J = r^2\sin\theta.$$

$$\iiint_T f(x,y,z)\,dx\,dy\,dz = \iiint_{T^*} g(r,\theta,\phi)\,r^2\sin\theta\,dr\,d\theta\,d\phi.$$