cannot be improved.

Convergence Concepts and Central Limit Theorem

Let us consider the sequence of RVs $X_1, X_2, X_3, \cdots X_n, \cdots$ or random sequence. The concept of convergence of random sequences is essential in the study of random signals. A few definitions and criteria that are used for determining the convergence of random sequences are given below.

(1) *Convergence everywhere and almost everywhere*

If $\{X_n\}$ is a sequence of RVs and X is a RV such that $\left[\lim_{n \to \infty} (X_n) = X\right]$ i.e., $X_n \to X$ as $n \to \infty$, then the sequence $\{X_n\}$ is said to converge to X everywhere.

If $P\{X_n \to X\} = 1$ as $n \to \infty$, then the sequence $\{X_n\}$ is said to converge to X almost everywhere.

(2) *Convergence in probability or stochastic convergence*

If $P\{|X_n - X| > \varepsilon\} \to 0$ as $n \to \infty$, then the sequence $\{X_n\}$ is said to converge to X in probability or stochastically.

As a particular case of this kind of convergence we have the following result, known as *Bernoulli's law of large numbers*.

If X represents the number of successes out of n Bernoulli's trials with probability of success p (in each trial), then $\{X/n\}$ converges in probability to p.

i.e., $$P\left\{\left|\frac{X}{n} - p\right| > \varepsilon\right\} \to 0 \text{ as } n \to \infty$$

(3) Convergence in the mean square sense

If $E\{|X_n - X|^2\} \to 0$ as $n \to \infty$, then the sequence $\{X_n\}$ is said to converge to X in the mean square sense.

(4) Convergence in distribution

If $F_n(x)$ and $F(x)$ are the distribution functions of X_n and X respectively such that $F_n(x) \to F(x)$ as $n \to \infty$ for every point of continuity of $F(x)$, then the sequence $\{X_n\}$ is said to converge to X in distribution.

Closely associated to the concept of convergence in distribution is a remarkable result known as central limit theorem, which is given below without proof.

Central Limit Theorem (Liapounoff's Form)

If $X_1, X_2, \cdots X_n, \cdots$, be a sequence of independent RVs with $E(X_i) = \mu_i$ and $\text{Var}(X_i) = \sigma_i^2, i = 1, 2, \cdots$, and if $S_n = X_1 + X_2 + \cdots X_n$, then under certain general conditions, S_n follows a normal distribution with mean $\mu = \sum_{i=1}^{n} \mu_i$ and variance $\sigma^2 = \sum_{i=1}^{n} \sigma_i^2$ as n tends to infinity.

Central Limit Theorem (Lindeberg–Levy's Form)

If $X_1, X_2, \cdots, X_n, \cdots$, be a sequence of independent identically distributed RVs with $E(X_i) = \mu$ and $\text{Var}(X_i) = \sigma^2$, $i = 1, 2, \cdots$, and if $S_n = X_1 + X_2 + \cdots + X_n$, then under certain general conditions, S_n follows a normal distribution with mean $n\mu$ and variance $n\sigma^2$ as n tends to infinity.

Corollary

If $\overline{X} = \frac{1}{n}(X_1 + X_2 + \cdots + X_n)$, then $E(\overline{X}) = \mu$ and $\text{Var}(\overline{X}) = \frac{1}{n^2}(n\sigma^2) = \frac{\sigma^2}{n}$

$\therefore \overline{X}$ follows $N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$ as $n \to \infty$

Worked Examples 4(F)

Example 1 *The lifetime of a certain brand of an electric bulb may be considered a RV with mean 1200 h and standard deviation 250 h. Find the probability, using central limit theorem, that the average lifetime of 60 bulbs exceeds 1250 h.*

Solution Let X_i represent the lifetime of the bulb.

$$E(X_i) = 1200 \text{ and Var } (X_i) = 250^2$$

Let $\overline{X}$ denote the mean lifetime of 60 bulbs.

By corollory of Lindeberg-Levy form of CLT

$$\overline{X} \text{ follows } N\left(1200, \frac{250}{\sqrt{60}}\right)$$

$$P(\overline{X} > 1250) = P\left(\frac{\overline{X}-1200}{\frac{250}{\sqrt{60}}} > \frac{1250-1200}{\frac{250}{\sqrt{60}}}\right)$$

$$= P\left(z > \frac{\sqrt{60}}{5}\right)$$

$$= P(z > 1.55),$$

where z is the standard normal variable

$$= 0.0606$$

(from the table of areas under normal curve)

Example 2 *A distribution with unknown mean μ has variance equal to 1.5. Use central limit theorem to find how large a sample should be taken from the distribution in order that the probability will be at least 0.95 that the sample mean will be within 0.5 of the population mean.* (MKU – Apr. 97)

Solution Let n be the size of the sample, a typical member of which is X_i.

Given: $E(X_i) = \mu$ and $\text{Var}(X_i) = 1.5$.

Let $\overline{X}$ denote the sample mean.

By corollary under CLT,

$$\overline{X} \text{ follows } N\left(\mu, \frac{\sqrt{1.5}}{\sqrt{n}}\right)$$

We have to find n such that

$$P\{\mu - 0.5 < \overline{X} < \mu + 0.5\} \geq 0.95$$

i.e., $$P\{-0.5 < \overline{X} - \mu < 0.5\} \geq 0.95$$

i.e., $$P\{|\overline{X} - \mu| < 0.5\} \geq 0.95$$

i.e., $$P\left\{\frac{|\overline{X}-\mu|}{\sqrt{\frac{1.5}{n}}} < \frac{0.5}{\sqrt{\frac{1.5}{n}}}\right\} \geq 0.95$$

i.e., $$P\{|z| < 0.4082\sqrt{n}\} \geq 0.95$$

where z is the standard normal variable.

The least value of n is obtained from

$$P\{|z| < 0.4082\sqrt{n}\} = 0.95$$

From the table of areas under normal curve

$$P\{|z| < 1.96\} = 0.95$$

Therefore, least n is given by $0.4082\sqrt{n} = 1.96$, i.e., least $n = 24$.

Therefore, the size of the sample must be at least 24.

Example 3 *If $X_1, X_2, \cdots X_n$ are Poisson variates with parameter $\lambda = 2$, use the central limit theorem to estimate $P(120 \le S_n \le 160)$, where $S_n = X_1 + X_2 + \cdots X_n$ and $n = 75$.* (MU — Apr. 96)

Solution $E(X_i) = \lambda = 2$ and $\text{Var}(X_i) = \lambda = 2$

By CLT, S_n follows $N(n\mu, \sigma\sqrt{n})$

i.e., S_n follows $N(150, \sqrt{150})$

$$P[120 \le S_n \le 160] = P\left\{\frac{-30}{\sqrt{150}} \le \frac{S_n - 150}{\sqrt{150}} \le \frac{10}{\sqrt{150}}\right\}$$

$$= P\{-2.45 \le z \le 0.85\}$$

where z is the standard normal variable.

$$= 0.4927 + 0.2939, \text{ (from the normal tables)}$$

$$= 0.7866$$

Example 4 *Using the central limit theorem, show that, for large n,*

$$\frac{c^n}{\lfloor n-1}\, x^{n-1}\, e^{-cx} \doteq \frac{c}{\sqrt{2\pi n}}\, e^{-(cx-n)^2/2n}, x > 0$$

(BDU — Apr. 96)

Solution Let $X_1, X_2, \cdots, X_n$ be independent RVs each of which is exponentially distri-buted with parameter c.

i.e., let the pdf of $X_i = c\, e^{-cx}, x > 0$

The characteristic function of X_i is given by

$$\phi_{Xi}(\omega) = (1 - i\omega/c)^{-1}$$

(refer to the Note under Worked Example 7 of the characteric function section).

By Property 4 of CFs, since $X_1, X_2, \cdots, X_n$ are independent RVs,

$$\phi_{(X1 + X2 + \cdots + Xn)}(\omega) = [\phi_{xi}(\omega)]^n$$

$$= \left(1 - \frac{i\omega}{c}\right)^{-n}$$

= CF of Erlang distribution

(refer to Worked Example 7 of Section 4(B))

Therefore, when n is finite, $(X_1 + X_2 + \cdots + X_n)$ follows the Erlang distribution whose pdf is given by

$$\frac{c^n}{\lfloor n-1}\, x^{n-1}\, e^{-cx}, x > 0 \qquad (1)$$

When n tends to infinity, $(X_1 + X_2 + \cdots + X_n)$ follows a normal distribution with mean $nE(X_i) = \frac{n}{c}$ and variance $n\text{Var}(X_i) = \frac{n}{c^2}$ (Central limit theorem), i.e., when $n \to \infty$, $(X_1 + X_2 + \cdots + X_n)$ follows a normal distribution whose pdf is given by

$$\frac{1}{\sqrt{\frac{n}{c^2}}\sqrt{2\pi}} \exp\left\{-\left(x - \frac{n}{c}\right)^2 \Big/ \frac{2n}{c^2}\right\}$$

$$= \frac{c}{\sqrt{2n\pi}} \exp\{-(cx - n)^2/2n\} \qquad (2)$$

From (1) and (2), the required result follows.

Example 5 *Verify central limit theorem for the independent random variables X_k, where for each k, $P\{X_k = \pm 1\} = \frac{1}{2}$.* (MKU – Apr. 96)

Solution $E(X_k) = 1 \times 1/2 + (-1) \times 1/2 = 0$

$$\text{Var}(X_k) = 1^2 \times 1/2 + (-1)^2 \times 1/2 = 1$$

Consider $$Y_n = \frac{1}{\sqrt{n}}(X_1 + X_2 + \cdots + X_n)$$

$$E(Y_n) = 0 \text{ and } \text{Var}(Y_n) = \frac{1}{n} \times n = 1$$

Now $$\phi_{Xk}(\omega) = E\{e^{i\omega xk}\}$$

$$= e^{i\omega} \times 1/2 + e^{i\omega(-1)} \times \frac{1}{2}$$

$$= \cos\omega$$

$\therefore$ $$\phi_{Yn}(\omega) = \phi_{\frac{1}{\sqrt{n}}(X_1 + X_2 + \ldots + X_n)}(\omega)$$

$$= \left\{\phi_{X_k/\sqrt{n}}(\omega)\right\}^n \quad \text{(since } X_1, X_2, \cdots, X_n \text{ are independent)}$$

$$= \left[\cos\left(\frac{\omega}{\sqrt{n}}\right)\right]^n \quad [\text{since } \phi_{aX+b}(\omega) = e^{ib\omega}\phi_X(a\omega)]$$

$$= \left[1 - \frac{\omega^2}{2n} + \text{terms involving } \frac{1}{n^2} \text{ and higher powers of } \frac{1}{n}\right]^n$$

$$= \left(1 - \frac{\omega^2}{2n}\right)^n + \text{terms involving } \frac{1}{n} \text{ and higher powers of } \frac{1}{n}$$

$$\lim_{n\to\infty} \phi_{yn}(\omega) = e^{\frac{-1}{2}\omega^2}$$

$\therefore$ which is the characteristic function of $N(0, 1)$.

$$Y_n \to N(0, 1), \text{ as } n \to \infty$$

$\therefore$ Therefore, CLT holds good for the sequence $\{X_k\}$.

Example 6 *Show that the central limit theorem holds good for the sequence $\{X_k\}$, if*

$$P\{X_k = \pm k^{\alpha}\} = \frac{1}{2} \times k^{-2\alpha},\ P\{X_k = 0\} = 1 - k^{-2\alpha},\ \alpha < \frac{1}{2}.$$ (BDU — Nov. 96)

Note *Liapounoff's form of CLT holds good for a sequence $\{X_k\}$, if*

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} E\{|X_k - \mu_k|^3\}}{\left\{\sum_{k=1}^{n} Var(X_k)\right\}^{\frac{3}{2}}} = 0 \quad \textit{Condition is assumed.}$$

Solution We have to verify whether this condition is satisfied by the given $\{X_k\}$.

Now $\mu_k = E(X_k) = \frac{1}{2} \times k^{-\alpha} - \frac{1}{2} \times k^{-\alpha} = 0$

$$E(X_k^2) = \frac{1}{2} \times k^{2\alpha} \times k^{-2\alpha} + \frac{1}{2} k^{2\alpha} \times k^{-2\alpha} = 1$$

$\therefore$ $\text{Var}(X_k) = 1$

$$E\{|X_k - \mu_k|^3\} = E\{|X_k|^3\}$$

$$= k^{3\alpha} \times \frac{1}{2} k^{-2\alpha} + k^{3\alpha} \times \frac{1}{2} k^{-2\alpha}$$

$$= k^{\alpha}$$

$\therefore$ $\sum_{k=1}^{n} E\{|X_k - \mu_k|^3\} = 1^{\alpha} + 2^{\alpha} + \cdots + n^{\alpha} < n \times n^{\alpha}$ (since each term $\le n\alpha$)

$$\sum_{k=1}^{n} \text{Var}(X_k) = n$$

$$\therefore \lim_{n\to\infty} \frac{\sum_{k=1}^{n} E\{|X_k - \mu_k|^3\}}{\left\{\sum_{k=1}^{n} \text{Var}(X_k)\right\}^{\frac{3}{2}}}$$

$$= \lim_{n\to\infty}\left(\frac{1^\alpha + 2^\alpha + \cdots + n^\alpha}{n^{3/2}}\right) < \lim_{n\to\infty}\frac{n \times n^\alpha}{n^{3/2}}$$

$$= \lim_{n\to\infty}\frac{1}{n^{\frac{1}{2}-\alpha}}$$

$$= 0 \text{ (since } (\alpha < 1/2)$$

i.e., the necessary condition is satisfied. Therefore, CLT holds good for the sequence $\{X_k\}$.

Exercise 4(F)

Part A (Short answer questions)

1. What is the difference between convergence everywhere and almost everywhere of a random sequence $\{X_n\}$?
2. Define stochastic convergence of a random sequence $\{X_n\}$.
3. State Bernoulli's law of large numbers.
4. Define convergence of a random sequence $\{X_n\}$ in the mean square sense.
5. Define convergence in distribution of a random sequence $\{X_n\}$.
6. State the Liapounoff's form of CLT.
7. State the Lindeberg–Levy's form of CLT.
8. What is the importance of CLT?

Part B

9. A random sample of size 100 is taken from a population whose mean is 60 and variance is 400. Using CLT, with what probability can we assert that the mean of the sample will not differ from $\mu = 60$ by more than 4? (MKU — Nov. 96)
10. The guaranteed average life of a certain type of electric light bulb is 1000 h with a standard deviation of 125 h. It is decided to sample the output so as to ensure that 90% of the bulbs do not fall short of the guaranteed average by more than 2.5%. Use CLT to find the minimum sample size.
11. If X_i, $i = 1, 2, \cdots, 50$, are independent RVs, each having a Poisson distribution with parameter $\lambda = 0.03$ and $S_n = X_1 + X_2 + \cdots X_n$, evaluate $P(S_n \geq 3)$, using CLT. Compare your answer with the exact value of the probability.
12. If V_i, $i = 1, 2, \cdots, 20$, are independent noise voltages received in an 'adder' and V is the sum of the voltages received, find the probability that the total incoming voltage V exceeds 105, using CLT. Assume that each of the random variables V_i is uniformly distributed over (0, 10).
13. 30 electronic devices $D_1, D_2, \cdots, D_{30}$ are used in the following manner. As soon as D_1 fails, D_2 becomes operative. When D_2 fails, D_3 becomes