31. Find the extreme value of $a^3x^2 + b^3y^2 + c^3z^2$ such that $x^{-1} + y^{-1} + z^{-1} = 1$, where $a > 0$, $b > 0$, $c > 0$.
32. Find the extreme value of $x^p + y^p + z^p$ on the surface $x^q + y^q + z^q = 1$, where $0 < p < q$, $x > 0$, $y > 0$, $z > 0$.
33. Find the extreme value of $x^3 + 8y^3 + 64z^3$, when $xyz = 1$.
34. Find the dimensions of a rectangular parallelopiped of maximum volume with edges parallel to the coordinate axes that can be inscribed in the ellipsoid $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$.
35. Divide a number into three parts such that the product of the first, square of the second and cube of the third is maximum.
36. Find the dimensions of a rectangular parallelopiped of fixed total edge length with maximum surface area.
37. Find the dimensions of a rectangular parallelopiped of greatest volume having constant surface area S.
38. A rectangular box without top is to have a given volume. How should the box be made so as to use the least material.
39. Find the dimensions of a right circular cone of fixed lateral area with minimum volume.
40. A tent is to be made in the form of a right circular cylinder surmounted by a cone. Find the ratios of the height H of the cylinder and the height h of the conical part to the radius r of the base, if the volume V of the tent is maximum for a given surface area S of the tent.
41. Find the maximum value of xyz under the constraints $x^2 + z^2 = 1$ and $y - x = 0$.
42. Find the extreme value of $x^2 + 2xy + z^2$ under the constraints $2x + y = 0$ and $x + y + z = 1$.
43. Find the extreme value of $x^2 + y^2 + z^2 + xy + xz + yz$ under the constraints $x + y + z = 1$ and $x + 2y + 3z = 3$.
44. Find the points on the ellipse obtained by the intersection of the plane $x + z = 1$ and the ellipsoid $x^2 + y^2 + 2z^2 = 1$ which are nearest and farthest from the origin.
45. Find the smallest and the largest distance between the points P and Q such that P lies on the plane $x + y + z = 2a$ and Q lies on the sphere $x^2 + y^2 + z^2 = a^2$, where a is any constant.

2.6 Multiple Integrals

In the previous chapter, we studied methods for evaluating the definite integral $\int_a^b f(x)dx$, where the integrand $f(x)$ is piecewise continuous on the interval $[a, b]$. In this section, we shall discuss methods for evaluating the double and triple integrals, that is integrals of the forms

$$\iint_R f(x, y)dx\,dy \quad \text{and} \quad \iiint_T f(x, y, z)dx\,dy\,dz.$$

We assume that the integrand f is continuous at all points inside and on the boundary of the region R or T. These integrals are called *multiple integrals*. The multiple integral over $\mathbb{R}^n$ is written as

$$\iint_R \cdots \int f(x_1, x_2, \ldots, x_n)dx_1 dx_2 \ldots dx_n.$$

2.6.1 Double Integrals

Let $f(x, y)$ be a continuous function in a simply connected, closed and bounded region R in a two dimensional space $\mathbb{R}^2$, bounded by a simple closed curve C (Fig. 2.5).

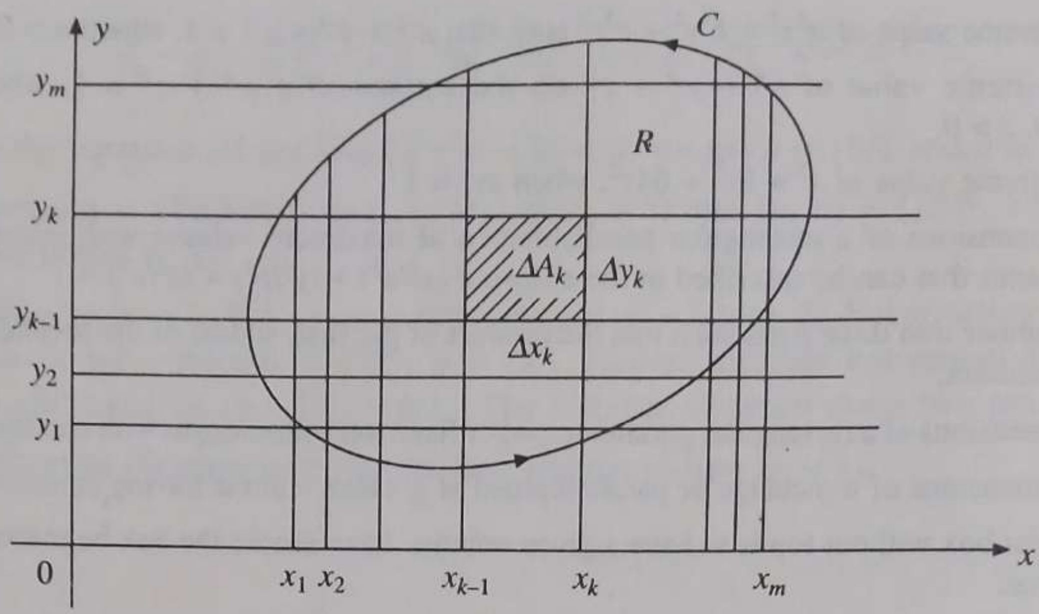

Fig. 2.5. Region *R* for double integral.

Subdivide the region R by drawing lines $x = x_k$, $y = y_k$, $k = 1, 2, \ldots, m$, parallel to the coordinate axes. Number the rectangles which are inside R from 1 to n. In each such rectangle, take an arbitrary point, say (ξ_k, η_k) in the kth rectangle and form the sum

$$J_n = \sum_{k=1}^{n} f(\xi_k, \eta_k)\Delta A_k$$

where $\Delta A_k = \Delta x_k\, \Delta y_k$ is the area of the kth rectangle and $d_k = \sqrt{(\Delta x_k)^2 + (\Delta y_x)^2}$ is the length of the diagonal of this rectangle. The maximum length of the diagonal, that is max d_k of the subdivisions is also called the *norm* of the subdivision. For different values of n, say $n_1, n_2, \ldots, n_m, \ldots$, we obtain a sequence of sums $J_{n_1}, J_{n_2}, \ldots, J_{n_m}, \ldots$. Let $n \to \infty$, such that the length of the largest diagonal $d_k \to 0$. If $\lim_{n\to\infty} J_n$ exists, independent of the choice of the subdivision and the point (ξ_k, η_k), then we say that $f(x, y)$ is integrable over R. This limit is called the *double integral* of $f(x, y)$ over R and is denoted by

$$J = \iint_R f(x, y)\,dx\,dy. \tag{2.78}$$

Evaluation of double integrals by two successive integrations

A double integral can be evaluated by two successive integrations. We evaluate it with respect to one variable (treating the other variable as constant) and reduce it to an integral of one variable. Thus, there are two possible ways to evaluate a double integral, which are the following:

$$J = \iint_R f(x, y)\,dy\,dx = \iint_R [f(x, y)\,dy]\,dx:$$ first integrate with respect to y and then integrate with respect to x.

or $$J = \iint_R f(x, y)\,dx\,dy = \iint_R [f(x, y)\,dx]\,dy:$$ first integrate with respect to x and then integrate with respect to y.

Let f be a continuous function over R. We consider the following cases.

Case 1 Let the region R be expressed in the form

$$R = \{(x, y) : \phi(x) \le y \le \psi(x),\ a \le x \le b\} \tag{2.79}$$

where $\phi(x)$ and $\psi(x)$ are integrable functions, such that $\phi(x) \le \psi(x)$ for all x in $[a, b]$. We write (Fig. 2.6)

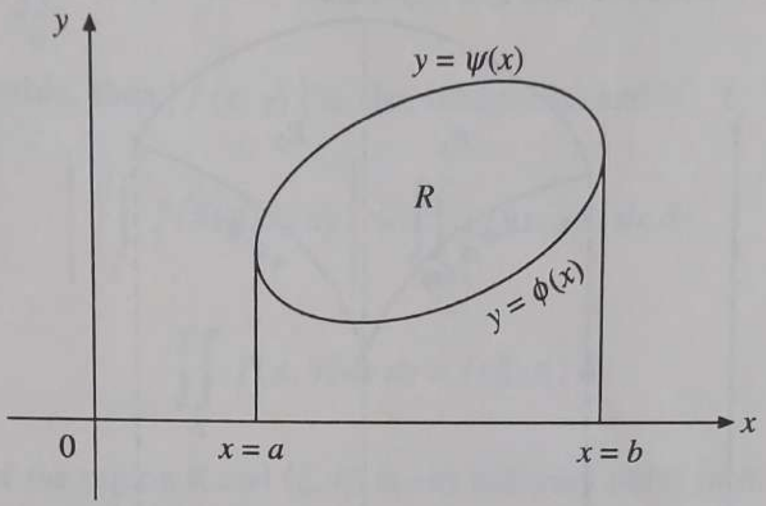

Fig. 2.6. Region of integration.

$$J = \int_{x=a}^{b} \left[\int_{y=\phi(x)}^{\psi(x)} f(x, y)dy \right] dx. \tag{2.80}$$

While evaluating the inner integral, x is treated as constant.

Case 2 Let the region R be expressed in the form

$$R = \{(x, y) : g(y) \le x \le h(y),\ c \le y \le d\} \tag{2.81}$$

where $g(y)$ and $h(y)$ are integrable functions, such that $g(y) \le h(y)$ for all y in $[c, d]$. We write (Fig. 2.7)

$$J = \int_{y=c}^{d} \left[\int_{x=g(y)}^{h(y)} f(x, y)dx \right] dy. \tag{2.82}$$

While evaluating the inner integral, y is treated as constant.

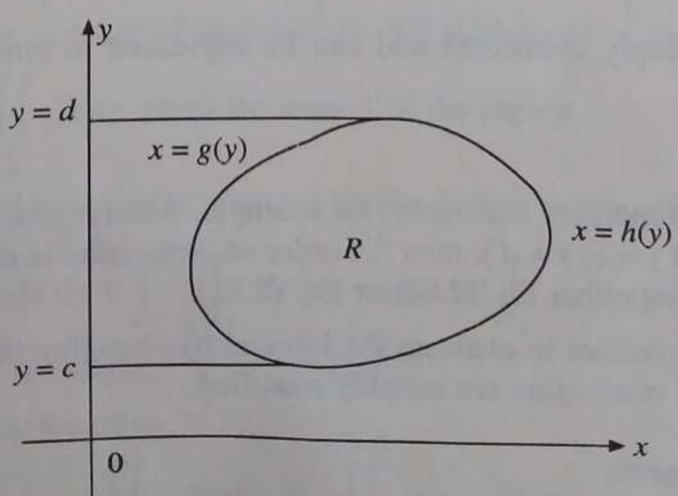

Fig. 2.7. Region of integration.

Often, the region R may be such that it cannot be represented in either of the forms given Eqs. (2.79) or (2.81). In such cases, the region R can be subdivided such that each of these can expressed in either of the forms given in Eqs. (2.79) or (2.81). For example, R may be expressed shown in Fig. 2.8 and we write $R = R_1 \cup R_2$ where R_1, R_2 have no common interior points.

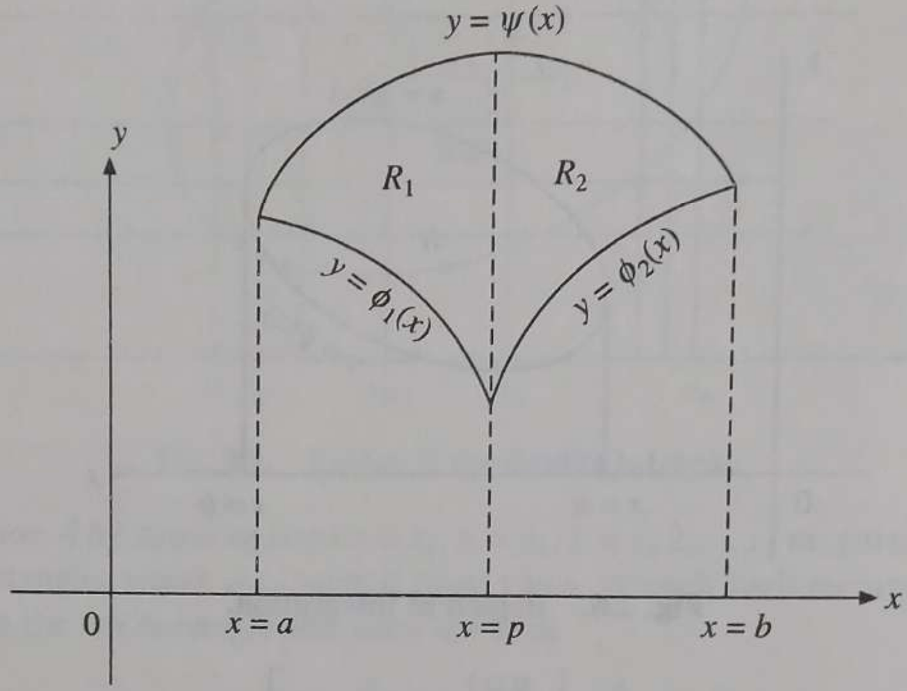

Fig. 2.8. Region of integration.

Then, we have

$$\iint_R f(x, y)dy\,dx = \iint_{R_1} f(x, y)dy\,dx + \iint_{R_2} f(x, y)dy\,dx$$

$$= \int_a^p \left[\int_{\phi_1(x)}^{\psi(x)} f(x, y)\,dy\right] dx + \int_p^b \left[\int_{\phi_2(x)}^{\psi(x)} f(x, y)dy\right] dx. \quad (2.83)$$

In the general case, the region R may be subdivided into a number of parts so that

$$\iint_R f(x, y)dy\,dx = \sum_{i=1}^{m} \left[\iint_{R_i} f(x, y)\,dy\,dx\right] \quad (2.84)$$

where each region R_i is simply connected and can be expressed in either of the forms given in Eqs. (2.79) or (2.81).

Remark 11

(a) If the limits of integration are constants (for example, when R is a rectangle bounded by the lines $x = a$, $x = b$ and $y = c$, $y = d$), then the order of integration is not important. The integral can be evaluated using either Eq. (2.80) or Eq. (2.82).

(b) Sometimes, it is convenient to evaluate the integral by changing the order of integration. In such cases, limits of integration are suitably modified.

Properties of double integrals

1. If $f(x, y)$ and $g(x, y)$ are integrable functions, then

$$\iint_R [f(x, y) \pm g(x, y)]dx\,dy = \iint_R f(x, y)dx\,dy \pm \iint_R g(x, y)dx\,dy.$$

2. $\iint_R k f(x, y)dx\,dy = k\iint_R f(x, y)dx\,dy$, where k is any real constant.

3. When $f(x, y)$ is integrable, then $|f(x, y)|$ is also integrable, and

$$\left|\iint_R f(x, y)dx\,dy\right| \le \iint_R |f(x, y)|\,dx\,dy. \tag{2.85}$$

4.

$$\iint_R f(x, y)dx\,dy = f(\xi, \eta)\,A \tag{2.86}$$

where A is the area of the region R and (ξ, η) is any arbitrary point in R. This result is called the *mean value theorem* of the double integrals.

If $m \le f(x, y) \le M$ for all (x, y) in R, then

$$m\,A \le \iint_R f(x, y)dx\,dy \le M A. \tag{2.87}$$

5. If $0 < f(x, y) \le g(x, y)$ for all (x, y) in R, then

$$\iint_R f(x, y)dx\,dy \le \iint_R g(x, y)dx\,dy. \tag{2.88}$$

6. If $f(x, y) \ge 0$ for all (x, y) in R, then

$$\iint_R f(x, y)dx\,dy \ge 0. \tag{2.89}$$

Application of double integrals

Double integrals have large number of applications. We state some of them.

1. If $f(x, y) = 1$, then $\iint_R dx\,dy$ gives the *area A* of the region R.

   For example, if R is the rectangle bounded by the lines $x = a$, $x = b$, $y = c$ and $y = d$, then

$$A = \int_c^d \int_a^b dx\,dy = \int_c^d \left[\int_a^b dx\right]dy = (b-a)\int_c^d dy = (b-a)(d-c)$$

   gives the area of the rectangle.

2. If $z = f(x, y)$ is a surface, then

$$\iint_R z\,dx\,dy \quad \text{or} \quad \iint_R f(x, y)dx\,dy$$

gives the *volume* of the region beneath the surface $z = f(x, y)$ and above the x-y plane.

For example, if $z = \sqrt{a^2 - x^2 - y^2}$ and $R: x^2 + y^2 \le a^2$, then

$$V = \iint_R \sqrt{a^2 - x^2 - y^2}\, dx\, dy$$

gives the volume of the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \ge 0$.

3. Let $f(x, y) = \rho(x, y)$ be a density function (mass per unit area) of a distribution of mass in the x-y plane. Then

$$M = \iint_R f(x, y)\, dx\, dy \tag{2.90}$$

give the total *mass* of R.

4. Let $f(x, y) = \rho(x, y)$ be a density function. Then

$$\bar{x} = \frac{1}{M} \iint_R x f(x, y)\, dx\, dy, \ \bar{y} = \frac{1}{M} \iint_R y f(x, y)\, dx\, dy \tag{2.91}$$

give the coordinates of the *centre of gravity* $(\bar{x}, \bar{y})$ of the mass M in R.

5. Let $f(x, y) = \rho(x, y)$ be a density function. Then

$$I_x = \iint_R y^2 f(x, y)\, dx\, dy \ \text{ and } \ I_y = \iint_R x^2 f(x, y)\, dx\, dy \tag{2.92}$$

give the *moments of inertia* of the mass in R about the x-axis and the y-axis respectively, whereas $I_0 = I_x + I_y$ is called the moment of inertia of the mass in R about the origin. Similarly,

$$I_y = \iint_R (x - a)^2 f(x, y)\, dx\, dy \ \text{ and } \ I_x = \iint_R (y - b)^2 f(x, y)\, dx\, dy \tag{2.93}$$

give the moment of inertia of the mass in R about the lines $x = a$ and $y = b$ respectively.

6. $\dfrac{1}{A} \iint_R f(x, y)\, dx\, dy$ gives the *average value* of $f(x, y)$ over R, where A is the area of the region R.

Example 2.41 Evaluate the double integral $\iint_R xy\, dx\, dy$, where R is the region bounded by the x-axis, the line $y = 2x$ and the parabola $y = x^2/(4a)$.

Solution The points of intersection of the curves $y = 2x$ and $y = x^2/(4a)$ are $(0, 0)$ and $(8a, 16a)$. The region

$$R = \{(x, y): (x^2/4a) \le y \le 2x,\ 0 \le x \le 8a\}$$

is given in Fig. 2.9.

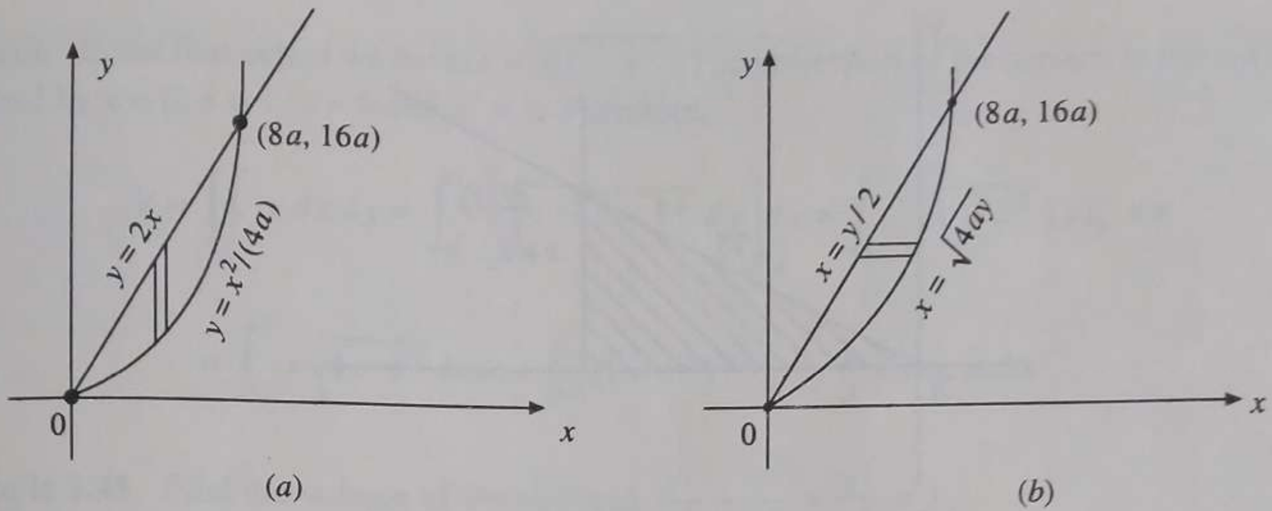

Fig. 2.9. Region in Example 2.41.

We evaluate the double integral as

$$I = \iint_R xy\,dx\,dy = \int_0^{8a}\left[\int_{x^2/(4a)}^{2x} x\,y\,dy\right]dx = \int_0^{8a}\left[\frac{xy^2}{2}\right]_{x^2/(4a)}^{2x} dx$$

$$= \int_0^{8a} \frac{x}{2}\left(4x^2 - \frac{x^4}{16a^2}\right)dx = \left[\frac{x^4}{2} - \frac{x^6}{192a^2}\right]_0^{8a} = 4096\left[\frac{1}{2} - \frac{64}{192}\right]a^4 = \frac{2048}{3}a^4.$$

Alternative We can evaluate the integral as

$$I = \iint_R xy\,dx\,dy = \int_0^{16a}\left[\int_{y/2}^{\sqrt{4ay}} xy\,dx\right]dy = \int_0^{16a}\left[\frac{1}{2}yx^2\right]_{y/2}^{\sqrt{4ay}} dy$$

$$= \frac{1}{2}\int_0^{16a} y\left(4ay - \frac{y^2}{4}\right)dy = \frac{1}{2}\left[\frac{4ay^3}{3} - \frac{y^4}{16}\right]_0^{16a} = \frac{4096\,a^3}{2}\left[\frac{4a}{3} - \frac{16a}{16}\right] = \frac{2048}{3}a^4.$$

Example 2.42 Evaluate the double integral $\iint_R e^{x^2}\,dx\,dy$, where the region R is given by $R : 2y \le x \le 2$ and $0 \le y \le 1$.

Solution The integral cannot be evaluated by integrating first with respect to x. We try to evaluate it by integrating it first with respect to y. The region of integration is given in Fig. 2.10. We have

$$I = \int_0^2\left[\int_0^{x/2} e^{x^2}\,dy\right]dx = \int_0^2\left[y\,e^{x^2}\right]_0^{x/2} dx$$

$$= \frac{1}{2}\int_0^2 x\,e^{x^2}\,dx = \left[\frac{1}{4}e^{x^2}\right]_0^2 = \frac{1}{4}(e^4 - 1).$$

Example 2.43 Evaluate the integral $\int_0^2\int_0^{y^2/2} \frac{y}{\sqrt{x^2+y^2+1}}\,dx\,dy.$

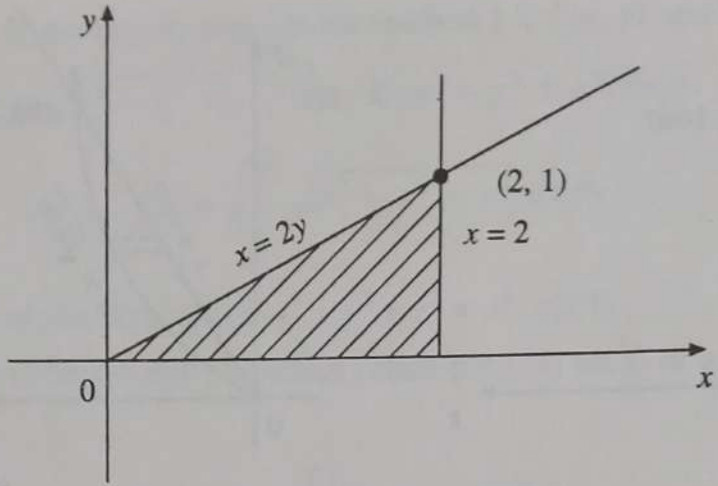

Fig. 2.10. Region in Example 2.42.

Solution Because of the form of the integrand, it would be easier to integrate it first with respect to y. The point of intersection of the line $y = 2$ and the curve $y^2 = 2x$ is (2, 2). The region of integration is given in Fig. 2.11.

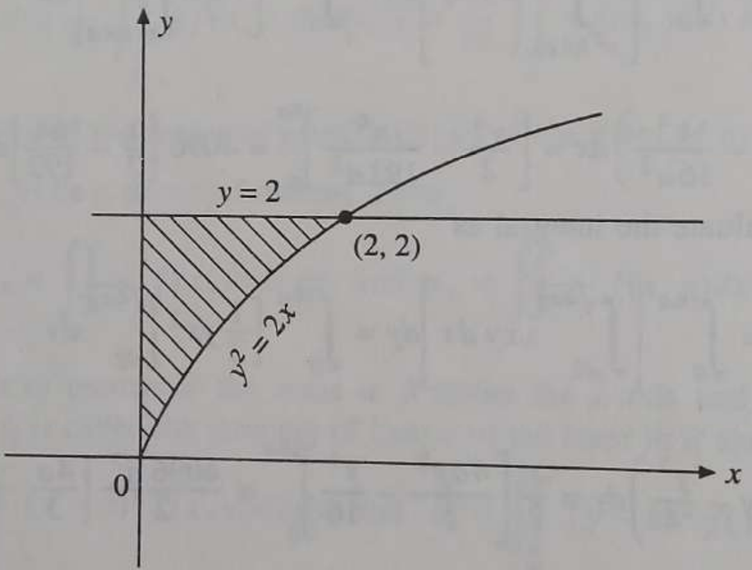

Fig. 2.11. Region in Example 2.43.

The given region of integration $0 \le y \le 2$ and $0 \le x \le y^2/2$ can also be written as $0 \le x \le 2$ and $\sqrt{2x} \le y \le 2$. Hence, we obtain

$$I = \int_0^2 \left[\int_{\sqrt{2x}}^2 \frac{y}{\sqrt{x^2+y^2+1}}\,dy\right] dx = \int_0^2 \left[\sqrt{x^2+y^2+1}\right]_{\sqrt{2x}}^2 dx = \int_0^2 \left[\sqrt{x^2+5} - (x+1)\right] dx$$

$$= \left[\frac{x\sqrt{x^2+5}}{2} + \frac{5}{2}\ln\left(x+\sqrt{x^2+5}\right) - \frac{1}{2}(x+1)^2\right]_0^2$$

$$= 3 + \frac{5}{2}(\ln 5 - \ln\sqrt{5}) - \frac{1}{2}(9-1) = \frac{5}{4}\ln 5 - 1.$$

Example 2.44 The cylinder $x^2 + z^2 = 1$ is cut by the planes $y = 0$, $z = 0$ and $x = y$. Find the volume of the region in the first octant.

Solution In the first octant we have $z = \sqrt{1-x^2}$. The projection of the surface in the x-y plane is bounded by $x = 0$, $x = 1$, $y = 0$ and $y = x$. Therefore,

$$V = \iint_R z\,dx\,dy = \int_0^1 \left[\int_0^x \sqrt{1-x^2}\,dy\right]dx = \int_0^1 \sqrt{1-x^2}\,[y]_0^x\,dx$$

$$= \int_0^1 x\sqrt{1-x^2}\,dx = -\frac{1}{3}\left[(1-x^2)^{3/2}\right]_0^1 = \frac{1}{3} \text{ cubic units.}$$

Example 2.45 Find the volume of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

Solution We have volume = 8 (volume in the first octant). The projection of the surface $z = c\sqrt{1 - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}}$ in the x-y plane is the region in the first quadrant of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Therefore,

$$V = 8\int_0^a \left[\int_0^{b\sqrt{1-x^2/a^2}} c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\,dy\right]dx = 8c\int_0^a \left[\int_0^{bk} \sqrt{k^2 - \frac{y^2}{b^2}}\,dy\right]dx$$

where $k^2 = 1 - (x^2/a^2)$. Setting $y = b\,k\sin\theta$, we obtain

$$V = 8c\int_0^a \left[\int_0^{\pi/2} \sqrt{k^2 - k^2\sin^2\theta}\,(bk\cos\theta)\,d\theta\right]dx = 8bc\int_0^a \left[\int_0^{\pi/2} k^2\cos^2\theta\,d\theta\right]dx$$

$$= 4bc\left(\frac{\pi}{2}\right)\int_0^a \left(1 - \frac{x^2}{a^2}\right)dx = \frac{2\pi bc}{a^2}\int_0^a (a^2 - x^2)\,dx$$

$$= \frac{2\pi bc}{a^2}\left[a^2x - \frac{x^3}{3}\right]_0^a = \frac{4\pi abc}{3} \text{ cubic units.}$$

Example 2.46 Find the centre of gravity of a plate whose density $\rho(x, y)$ is constant and is bounded by the curves $y = x^2$ and $y = x + 2$. Also, find the moments of inertia about the axes.

Solution The mass of the plate is given by (see Eq. 2.90)

$$M = \iint_R \rho(x, y)\,dx\,dy = k\iint_R dx\,dy \qquad (\rho(x, y) = k \text{ constant}).$$

The boundary of the plate is given in Fig. 2.12. The line $y = x + 2$ intersects the parabola $y = x^2$ at the points $(-1, 1)$ and $(2, 4)$. The limits of integration can be written as $-1 \le x \le 2$, $x^2 \le y \le x + 2$. Therefore,

$$M = k\int_{-1}^2 \left[\int_{x^2}^{x+2} dy\right]dx = k\int_{-1}^2 (x + 2 - x^2)\,dx$$

$$= k\left[-\frac{x^3}{3} + \frac{x^2}{2} + 2x\right]_{-1}^{2} = k\left(-\frac{9}{3} + \frac{3}{2} + 6\right) = \frac{9}{2}k.$$

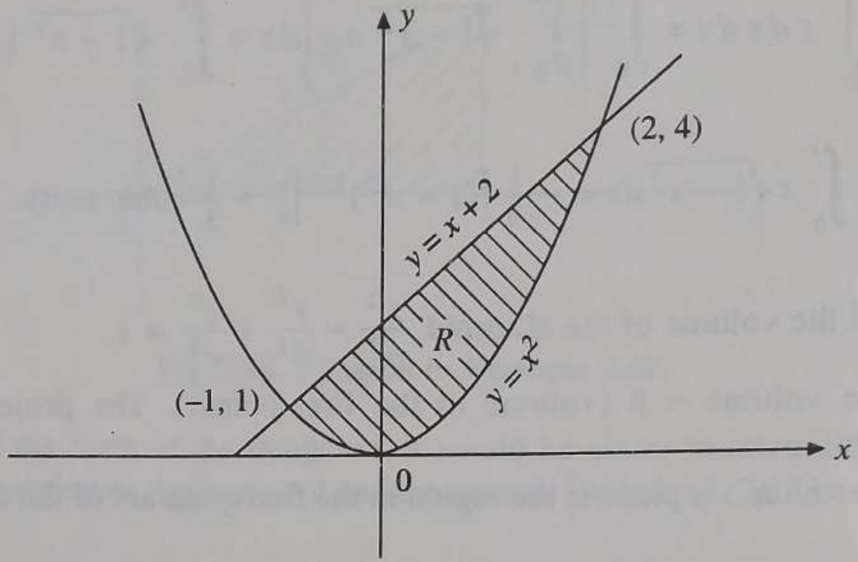

Fig. 2.12. Region in Example 2.46.

The centre of gravity $(\bar{x}, \bar{y})$ is given by (see Eq. 2.91)

$$\bar{x} = \frac{1}{M}\iint_R x\,\rho(x, y)\,dx\,dy = \frac{2}{9}\int_{-1}^{2}\left[\int_{x^2}^{x+2} dy\right] x\,dx$$

$$= \frac{2}{9}\int_{-1}^{2} x(x + 2 - x^2)\,dx = \frac{2}{9}\left[\frac{x^3}{3} + x^2 - \frac{x^4}{4}\right]_{-1}^{2} = \frac{1}{2}.$$

$$\bar{y} = \frac{1}{M}\iint_R y\,\rho(x, y)\,dx\,dy = \frac{2}{9}\int_{-1}^{2}\left[\int_{x^2}^{x+2} y\,dy\right]dx = \frac{2}{9}\int_{-1}^{2}\left[\frac{y^2}{2}\right]_{x^2}^{x+2} dx$$

$$= \frac{1}{9}\int_{-1}^{2}[(x + 2)^2 - x^4]\,dx = \frac{1}{9}\left[\frac{(x + 2)^3}{3} - \frac{x^5}{5}\right]_{-1}^{2}$$

$$= \frac{1}{9}\left[\frac{1}{3}(64 - 1) - \frac{1}{5}(32 + 1)\right] = \frac{1}{9}\left[21 - \frac{33}{5}\right] = \frac{8}{5}.$$

Therefore, the centre of gravity is located at (1/2, 8/5).

Moment of inertia about the x-axis is given by (see Eq. 2.92)

$$I_x = \iint_R y^2\rho(x, y)\,dx\,dy = k\int_{-1}^{2}\left[\int_{x^2}^{x+2} y^2\,dy\right]dx = k\int_{-1}^{2}\left[\frac{y^3}{3}\right]_{x^2}^{x+2} dx$$

$$= \frac{k}{3}\int_{-1}^{2}[(x + 2)^3 - x^6]\,dx = \frac{k}{3}\left[\frac{(x + 2)^4}{4} - \frac{x^7}{7}\right]_{-1}^{2}$$

$$= \frac{k}{3}\left(\frac{255}{4} - \frac{129}{7}\right) = \frac{423}{28}k.$$

Moment of inertia about the y-axis is given by (see Eq. 2.92)

$$I_y = \iint_R x^2 \rho(x, y) dx\, dy = k \int_{-1}^{2} \left[\int_{x^2}^{x+2} dy \right] x^2 dx = k \int_{-1}^{2} x^2 (x + 2 - x^2) dx$$

$$= k \left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{x^5}{5} \right]_{-1}^{2} = k \left[\frac{15}{4} + 6 - \frac{33}{5} \right] = \frac{63}{20} k.$$

2.6.2 Triple Integrals

Let $f(x, y, z)$ be a continuous function defined over a closed and bounded region T in $\mathbb{R}^3$. Divide the region T into a number of parallelopipeds by drawing planes parallel to the coordinate planes. Number the parallelopipeds inside T from 1 to n and form the sum

$$J_n = \sum_{k=1}^{n} f(x_k, y_k, z_k) \Delta V_k$$

where (x_k, y_k, z_k) is an arbitrary point in the kth parallelopiped and ΔV_k is its volume. For different values of n, say $n_1, n_2, \ldots, n_m, \ldots$, we obtain a sequence of sums $J_{n_1}, J_{n_2}, \ldots, J_{n_m}, \ldots$ The length of the diagonal of the kth parallelopiped is $d_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2 + (\Delta z_k)^2}$. Let $n \to \infty$ such that $\max d_k \to 0$. If $\lim_{n\to\infty} J_n$ exists, independent of the choice of the subdivision and the point (x_k, y_k, z_k), then we say that $f(x, y, z)$ is integrable over T. This limit is called the *triple integral* of $f(x, y, z)$ over T and is denoted by

$$J = \iiint_T f(x, y, z)\, dx\, dy\, dz. \tag{2.94}$$

Triple integrals satisfy properties similar to double integrals.

Application of triple integrals

1. If $f(x, y, z) = 1$, then the triple integral

$$V = \iiint_T dx\, dy\, dz \tag{2.95}$$

gives the volume of the region T.

2. If $f(x, y, z) = \rho(x, y, z)$ is the density of a mass, then the triple integral

$$M = \iiint_T f(x, y, z) dx\, dy\, dz \tag{2.96}$$

gives the *mass* of the solid.

3.

$$\bar{x} = \frac{1}{M} \iiint_T x f(x, y, z) dx\, dy\, dz, \quad \bar{y} = \frac{1}{M} \iiint_T y f(x, y, z) dx\, dy\, dz,$$

$$\bar{z} = \frac{1}{M} \iiint_T z f(x, y, z) dx\, dy\, dz \tag{2.97}$$

give the coordinates of the *centre of mass* (or the *centre of gravity*) of the solid of mass M in T, where $f(x, y, z) = \rho(x, y, z)$ is the density function.

4. $$I_x = \iiint_T (y^2 + z^2) f(x, y, z)\, dx\, dy\, dz, \quad I_y = \iiint_T (x^2 + z^2) f(x, y, z)\, dx\, dy\, dz,$$

$$I_z = \iiint_T (x^2 + y^2) f(x, y, z)\, dx\, dy\, dz \tag{2.98}$$

give the *moments of inertia* of the mass in T about the x-axis, y-axis and z-axis respectively where $f(x, y, z) = \rho(x, y, z)$ is the density function.

Evaluation of triple integrals

We evaluate the triple integral by three successive integrations. If the region T can be described by

$$x_1 \le x \le x_2, \quad y_1(x) \le y \le y_2(x), \quad z_1(x, y) \le z \le z_2(x, y)$$

then we evaluate the triple integral as

$$\int_{x_1}^{x_2} \int_{y_1(x)}^{y_2(x)} \int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)\, dz\, dy\, dx = \int_{x_1}^{x_2} \left[\int_{y_1(x)}^{y_2(x)} \left[\int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)\, dz \right] dy \right] dx \tag{2.99}$$

We note that there are six possible ways in which a triple integral can be evaluated (order of variables of integration). We choose the one which is simple to use.

Example 2.47 Evaluate the triple integral $\iiint_T y\, dx\, dy\, dz$, where T is the region bounded by the surfaces $x = y^2$, $x = y + 2$, $4z = x^2 + y^2$ and $z = y + 3$.

Solution The variable z varies from $(x^2 + y^2)/4$ to $y + 3$. The projection of T on the x-y plane is the region bounded by the curves $x = y^2$ and $x = y + 2$. These curves intersect at the points $(1, -1)$ and $(4, 2)$. Also, $y^2 \le y + 2$ for $-1 \le y \le 2$. Hence, the required region can be written as

$$-1 \le y \le 2, \quad y^2 \le x < y + 2 \quad \text{and} \quad [(x^2 + y^2)/4] \le z \le y + 3.$$

Therefore, we can evaluate the triple integral as

$$J = \int_{-1}^{2} \left[\int_{y^2}^{y+2} \left[\int_{(x^2+y^2)/4}^{(y+3)} y\, dz \right] dx \right] dy = \int_{-1}^{2} \left[\int_{y^2}^{y+2} y \left\{ y + 3 - \frac{x^2 + y^2}{4} \right\} dx \right] dy$$

$$= \int_{-1}^{2} \left[\left(y^2 + 3y - \frac{y^3}{4} \right) x - \frac{x^3 y}{12} \right]_{y^2}^{y+2} dy$$

$$= \int_{-1}^{2} \left[\left(y^2 + 3y - \frac{y^3}{4} \right) (y + 2 - y^2) - \frac{1}{12} y \{(y + 2)^3 - y^6\} \right] dy$$

$$= \int_{-1}^{2} \left[\frac{y^7}{12} + \frac{y^5}{4} - \frac{4y^4}{3} - 3y^3 + 4y^2 + \frac{16y}{3} \right] dy$$

$$= \left[\frac{y^8}{96} + \frac{y^6}{24} - \frac{4y^5}{15} - \frac{3y^4}{4} + \frac{4y^3}{3} + \frac{8y^2}{3}\right]_{-1}^{2} = \frac{837}{160}.$$

Example 2.48 Evaluate the integral $\iiint_T z\,dx\,dy\,dz$, where T is the region bounded by the cone $z^2 = x^2\tan^2\alpha + y^2\tan^2\beta$ and the planes $z = 0$ to $z = h$ in the first octant.

Solution The required region can be written as

$$0 \le z \le \sqrt{x^2\tan^2\alpha + y^2\tan^2\beta},\ 0 \le y \le (\sqrt{h^2 - x^2\tan^2\alpha})\cot\beta,\ 0 \le x \le h\cot\alpha$$

Therefore,

$$J = \int_0^{h\cot\alpha}\left[\int_0^{(\sqrt{h^2-x^2\tan^2\alpha})\cot\beta} \frac{1}{2}(x^2\tan^2\alpha + y^2\tan^2\beta)dy\right]dx$$

$$= \frac{1}{2}\int_0^{h\cot\alpha}\left[x^2(h^2 - x^2\tan^2\alpha)^{1/2}\tan^2\alpha + \frac{1}{3}(h^2 - x^2\tan^2\alpha)^{3/2}\right]\cot\beta\,dx.$$

Substituting $x\tan\alpha = h\sin\theta$, we obtain

$$J = \frac{\cot\beta}{2}\int_0^{\pi/2}\left[h^2\sin^2\theta\,(h\cos\theta) + \frac{1}{3}(h^3\cos^3\theta)\right]h\cot\alpha\cos\theta\,d\theta$$

$$= \frac{1}{2}h^4\cot\beta\cot\alpha\left[\int_0^{\pi/2}(\sin^2\theta\cos^2\theta + \frac{1}{3}\cos^4\theta)d\theta\right]$$

$$= \frac{1}{2}h^4\cot\beta\cot\alpha\left[\int_0^{\pi/2}(\sin^2\theta - \sin^4\theta + \frac{1}{3}\cos^4\theta)d\theta\right]$$

$$= \frac{1}{2}h^4\cot\beta\cot\alpha\left[\frac{\pi}{4} - \frac{3\pi}{16} + \frac{\pi}{16}\right] = \frac{h^4\pi}{16}\cot\alpha\cot\beta.$$

Example 2.49 Find the volume of the solid in the first octant bounded by the paraboloid $z = 36 - 4x^2 - 9y^2$.

Solution We have

$$V = \iiint_T dz\,dy\,dx.$$

The projection of the paraboloid (in the first octant) in the x-y plane is the region in the first quadrant of the ellipse $4x^2 + 9y^2 = 36$.

Therefore, the region T is given by

$$0 \le z \le 36 - 4x^2 - 9y^2,\quad 0 \le y \le \frac{1}{3}\sqrt{36 - 4x^2},\quad 0 \le x \le 3.$$

Hence,

$$V = \int_0^3 \left[\int_0^{(2\sqrt{9-x^2}/3)} (36 - 4x^2 - 9y^2)dy \right] dx$$

$$= \int_0^3 [4(9 - x^2)y - 3y^3]_0^{(2\sqrt{9-x^2}/3)} dx$$

$$= \int_0^3 \left[\frac{8}{3}(9 - x^2)^{3/2} - \frac{8}{9}(9 - x^2)^{3/2} \right] dx = \frac{16}{9} \int_0^3 (9 - x^2)^{3/2} dx.$$

Substituting $x = 3 \sin \theta$, we obtain

$$V = \frac{16}{9} \int_0^{\pi/2} (27 \cos^3\theta)(3 \cos \theta) d\theta = 144 \int_0^{\pi/2} \cos^4\theta \, d\theta$$

$$= 144 \left(\frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \right) = 27\pi \text{ cubic units.}$$

Example 2.50 Find the volume of the solid enclosed between the surfaces $x^2 + y^2 = a^2$ an $x^2 + z^2 = a^2$.

Solution We have the region as

$$-\sqrt{a^2 - x^2} \le z \le \sqrt{a^2 - x^2}, \quad -\sqrt{a^2 - x^2} \le y \le \sqrt{a^2 - x^2}, \quad -a \le x \le a.$$

Therefore,

$$V = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dz\, dy\, dx = 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2}\, dy\, dx$$

$$= 8 \int_0^a (a^2 - x^2) dx = 8 \left(a^2 x - \frac{x^3}{3} \right)_0^a = \frac{16a^3}{3} \text{ cubic units.}$$

2.6.3 Change of Variables in Integrals

In the case of definite integrals $\int_a^b f(x)dx$ of one variable, we have seen that the evaluation of the integral is often simplified by using some substitution and thus changing the variable of integration. Similarly, the double and triple integrals can be evaluated by using some substitutions and changing the variables of integration.

Double integrals

Let the variables x, y defined in a region R of the x-y plane be transformed as

$$x = x(u, v), \; y = y(u, v). \tag{2.100}$$

We assume that the functions $x(u, v)$, $y(u, v)$ are defined and have continuous partial derivatives in the region R^* of interest in the u-v plane. We also assume that the inverse functions $u = u(x, y)$, $v = v(x, y)$ are defined and are continuous in the region of interest in the x-y plane, so that the mapping is one-to-one. Since the function $f(x, y)$ is continuous in R, the function $f[x(u, v), y(u, v)]$ is also continuous in R^*. Then, the double integral transforms as

$$\iint\limits_{R} f(x, y)dx\,dy = \iint\limits_{R^*} f[x(u, v), y(u, v)]\,|J|\,du\,dv = \iint\limits_{R^*} F(u, v)du\,dv \qquad (2.101)$$

where
$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$$

is the *Jacobian* of the variables of transformation.

For example, if we change the cartesian coordinates to *polar* coordinates, we have

$$x = r\cos\theta,\ \ y = r\sin\theta,\ \ r_1 \le r \le r_2,\ \ \theta_1 \le \theta \le \theta_2$$

$$J = \begin{vmatrix} \partial x/\partial r & \partial x/\partial \theta \\ \partial y/\partial r & \partial y/\partial \theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r \qquad (2.102)$$

Therefore,

$$\iint\limits_{R} f(x, y)dx\,dy = \iint\limits_{R^*} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta = \iint\limits_{R^*} F(r, \theta)\,r\,dr\,d\theta$$

where R^* is the region corresponding to R is the r-θ plane.

Triple integrals

Analogous to double integrals, we define x, y, z as functions of three new variables

$$x = x(u, v, w),\ y = y(u, v, w),\ z = z(u, v, w). \qquad (2.103)$$

Then,

$$\iiint\limits_{T} f(x, y, z)dx\,dy\,dz = \iiint\limits_{T^*} f[x(u, v, w), y(u, v, w), z(u, v, w)]\,|J|\,du\,dv\,dw \qquad (2.104)$$

where
$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix}$$

is the Jacobian of the variables of transformation.

For example, if we change the cartesian coordinates to *cylindrical* coordinates, we have

$$x = r\cos\theta,\ y = r\sin\theta,\ z = z$$

$$J = \begin{vmatrix} \partial x/\partial r & \partial x/\partial \theta & \partial x/\partial z \\ \partial y/\partial r & \partial y/\partial \theta & \partial y/\partial z \\ \partial z/\partial r & \partial z/\partial \theta & \partial z/\partial z \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r \qquad (2.105)$$

and
$$\iiint\limits_{T} f(x, y, z)dx\,dy\,dz = \iiint\limits_{T^*} f(r\cos\theta, r\sin\theta, z)\,r\,dr\,d\theta\,dz$$

If we change the cartesian coordinates to *spherical* coordinates, we have (Fig. 2.13)

$$x = r\sin\phi\cos\theta,\ y = r\sin\phi\sin\theta,\ z = r\cos\phi,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi$$

$$J = \begin{vmatrix} \partial x/\partial r & \partial x/\partial \phi & \partial x/\partial \theta \\ \partial y/\partial r & \partial y/\partial \phi & \partial y/\partial \theta \\ \partial z/\partial r & \partial z/\partial \phi & \partial z/\partial \theta \end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & r\cos\phi\cos\theta & -r\sin\phi\sin\theta \\ \sin\phi\sin\theta & r\cos\phi\sin\theta & r\sin\phi\cos\theta \\ \cos\phi & -r\sin\phi & 0 \end{vmatrix}$$

$$= \cos\phi[r^2\sin\phi\cos\phi\cos^2\theta + r^2\sin\phi\cos\phi\sin^2\theta] + r\sin\phi[r\sin^2\phi\cos^2\theta + r\sin^2\phi\sin^2\theta]$$

$$= r^2[\sin\phi\cos^2\phi + \sin^3\phi] = r^2\sin\phi \qquad (2.1$$

and
$$\iiint_T f(x, y, z)\,dx\,dy\,dz = \iiint_{T^*} F(r, \theta, \phi)\, r^2\sin\phi\, dr\, d\theta\, d\phi.$$

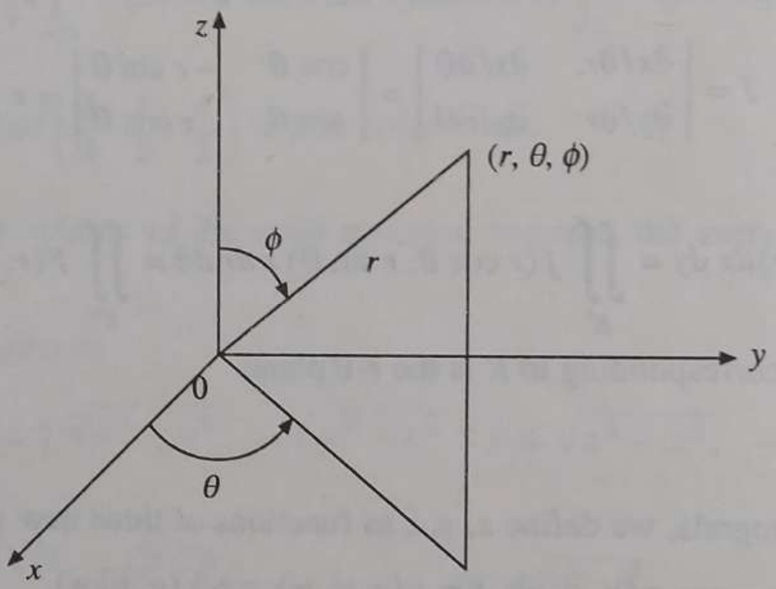

Fig. 2.13. Spherical coordinates.

Example 2.51 Evaluate the integral $\iint_R (a^2 - x^2 - y^2)\,dx\,dy$, where R is the region $x^2 + y^2 \le a$

Solution We can evaluate the integral directly by writing it as

$$I = \int_{-a}^{a}\left[\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (a^2 - x^2 - y^2)\,dy\right]dx.$$

However, it is easier to evaluate, if we change to polar coordinates. Transforming cartesian coordinate to polar coordinates, we have (see Eq. 2.102)

$$x = r\cos\theta,\ y = r\sin\theta,\ J = r.$$

Therefore,

$$I = \int_0^a \int_0^{2\pi} (a^2 - r^2)\, r\, dr\, d\theta = \int_0^a \left[\int_0^{2\pi} d\theta\right](a^2 r - r^3)\,dr$$

$$= 2\pi \int_0^a (a^2 r - r^3) dr = 2\pi \left(\frac{a^2 r^2}{2} - \frac{r^4}{4} \right)_0^a = \frac{\pi a^4}{2}.$$

*

Example 2.52 Evaluate the integral $\iint\limits_R (x-y)^2 \cos^2(x+y) dx\, dy$, where R is the rhombus with successive vertices at $(\pi, 0)$, $(2\pi, \pi)$, $(\pi, 2\pi)$ and $(0, \pi)$.

Solution The region R is given in Fig. 2.14. The equations of the sides AB, BC, CD and DA are respectively

$$x - y = \pi, \; x + y = 3\pi, \; x - y = -\pi \quad \text{and} \quad x + y = \pi.$$

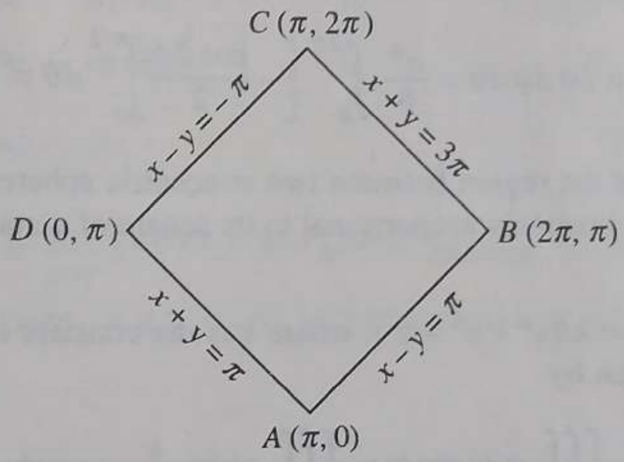

Fig. 2.14. Region in Example 2.52.

Substitute $y - x = u$ and $y + x = v$. Then, $-\pi \le u \le \pi$ and $\pi \le v \le 3\pi$. We obtain

$$x = (v-u)/2, \; y = (v+u)/2$$

and
$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{vmatrix} = -\frac{1}{2}, \; |J| = \frac{1}{2}.$$

Therefore,

$$I = \iint\limits_R (x-y)^2 \cos^2(x+y) dx\, dy = \frac{1}{2} \int_{\pi}^{3\pi} \int_{-\pi}^{\pi} u^2 \cos^2 v \, du\, dv$$

$$= \frac{\pi^3}{3} \int_{\pi}^{3\pi} \cos^2 v \, dv = \frac{\pi^3}{6} \int_{\pi}^{3\pi} (1 + \cos 2v)\, dv = \frac{\pi^4}{3}.$$

Example 2.53 Evaluate the integral $\iint\limits_R \sqrt{x^2 + y^2}\, dx\, dy$ by changing to polar coordinates, where R is the region in the x-y plane bounded by the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.

Solution Using $x = r\cos\theta$, $y = r\sin\theta$, we get $dx\, dy = r\, dr\, d\theta$, and

$$I = \int_0^{2\pi} \int_2^3 r(r\, dr\, d\theta) = \int_0^{2\pi} \left[\frac{r^3}{3} \right]_2^3 d\theta = \frac{19}{3} \int_0^{2\pi} d\theta = \frac{38\pi}{3}.$$

Example 2.54 Evaluate the integral $\iiint_T z\,dx\,dy\,dz$, where T is the hemisphere of radius $x^2 + y^2 + z^2 = a^2$, $z \geq 0$.

Solution Changing to spherical coordinates

$$x = r \sin\phi \cos\theta,\ y = r\sin\phi\sin\theta,\ z = r\cos\phi,\ 0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \pi/2,$$

we obtain $dx\,dy\,dz = r^2 \sin\phi\,dr\,d\phi\,d\theta$ (see Eq. 2.106). Therefore,

$$I = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a (r\cos\phi)\,r^2\sin\phi\,dr\,d\phi\,d\theta = \frac{a^4}{4}\int_0^{2\pi}\int_0^{\pi/2}\sin\phi\cos\phi\,d\phi\,d\theta$$

$$= \frac{a^4}{8}\int_0^{2\pi}\int_0^{\pi/2}\sin 2\phi\,d\phi\,d\theta = \frac{a^4}{8}\int_0^{2\pi}\left[-\frac{\cos 2\phi}{2}\right]_0^{\pi/2} d\theta = \frac{a^4}{8}\int_0^{2\pi} d\theta = \frac{\pi a^4}{4}.$$

Example 2.55 A solid fills the region between two concentric spheres of radii a and b, $0 < a <$ The density at each point is inversely proportional to its square of distance from the origin. Find t total mass.

Solution The density is $\rho = k/(x^2 + y^2 + z^2)$, where k is the constant of proportionality. Therefor the mass of the solid is given by

$$M = \iiint_T \rho\,dx\,dy\,dz = \iiint_T \frac{k}{x^2 + y^2 + z^2}\,dx\,dy\,dz$$

where $a^2 < x^2 + y^2 + z^2 < b^2$. Changing to spherical coordinates, we obtain

$$x = r\sin\phi\cos\theta,\ y = r\sin\phi\sin\theta,\ z = r\cos\phi,\ x^2 + y^2 + z^2 = r^2,\ a \leq r \leq b,$$

$$dx\,dy\,dz = r^2\sin\phi\,dr\,d\theta\,d\phi,\ 0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \pi.$$

Therefore,

$$M = k\int_0^{2\pi}\int_0^{\pi}\int_a^b \frac{r^2\sin\phi}{r^2}\,dr\,d\phi\,d\theta = k(b-a)\int_0^{2\pi}\int_0^{\pi}\sin\phi\,d\phi\,d\theta$$

$$= k(b-a)\int_0^{2\pi}[-\cos\phi]_0^{\pi}\,d\theta = 2k(b-a)\int_0^{2\pi} d\theta = 4\pi k(b-a).$$

2.6.4 Dirichlet Integrals

Let T be a closed region in the first octant in $\mathbb{R}^3$, bounded by the surface $(x/a)^p + (y/b)^q + (z/c)^r = 1$ and the coordinate planes. Then, an integral of the form

$$I = \iiint_T x^{\alpha-1} y^{\beta-1} z^{\gamma-1}\,dxdydz \tag{2.107}$$

is called a Dirichlet integral, where all the constants α, β, γ, a, b, c and p, q, r are assumed to be positive.

We now show that

$$I = \iiint_T x^{\alpha-1} y^{\beta-1} z^{\gamma-1} dx\,dy\,dz = \frac{a^\alpha b^\beta c^\gamma}{pqr} \frac{\Gamma(\alpha/p)\Gamma(\beta/q)\Gamma(\gamma/r)}{\Gamma\left(\dfrac{\alpha}{p}+\dfrac{\beta}{q}+\dfrac{\gamma}{r}+1\right)}. \tag{2.108}$$

Let $\left(\dfrac{x}{a}\right)^p = u, \left(\dfrac{y}{b}\right)^q = v, \left(\dfrac{z}{c}\right)^r = w$, or $x = au^{1/p}$, $y = bv^{1/q}$, $z = cw^{1/r}$.

The Jacobian of the transformation is given by

$$J = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix} = \begin{vmatrix} (a/p)u^{(1/p)-1} & 0 & 0 \\ 0 & (b/q)v^{(1/q)-1} & 0 \\ 0 & 0 & (c/r)w^{(1/r)-1} \end{vmatrix}$$

$$= \frac{abc}{pqr} u^{(1/p)-1} v^{(1/q)-1} w^{(1/r)-1}$$

and $dxdydz = |J|\,dudvdw = \dfrac{abc}{pqr} u^{(1/p)-1} v^{(1/q)-1} w^{(1/r)-1}\,dudvdw$.

Now, $x \ge 0$, $y \ge 0$, $z \ge 0$ gives $u \ge 0$, $v \ge 0$, $w \ge 0$ respectively.

Hence, we obtain

$$I = \iiint_R \left[au^{(1/p)}\right]^{\alpha-1} \left[bv^{(1/q)}\right]^{\beta-1} \left[cw^{(1/r)-1}\right] \frac{abc}{pqr} u^{(1/p)-1} v^{(1/q)-1} w^{(1/r)-1}\,dudvdw$$

$$= \frac{a^\alpha b^\beta c^\gamma}{pqr} \iiint_R u^{(\alpha/p)-1} v^{(\beta/q)-1} w^{(\gamma/r)-1} dudvdw$$

where R is the region in the uvw-space bounded by the plane $u + v + w = 1$ and the uv, vw and uw coordinate planes, (Fig. 2.15), that is, R is defined by

$$0 \le w \le 1 - u - v, \quad 0 \le v \le 1 - u, \quad 0 \le u \le 1.$$

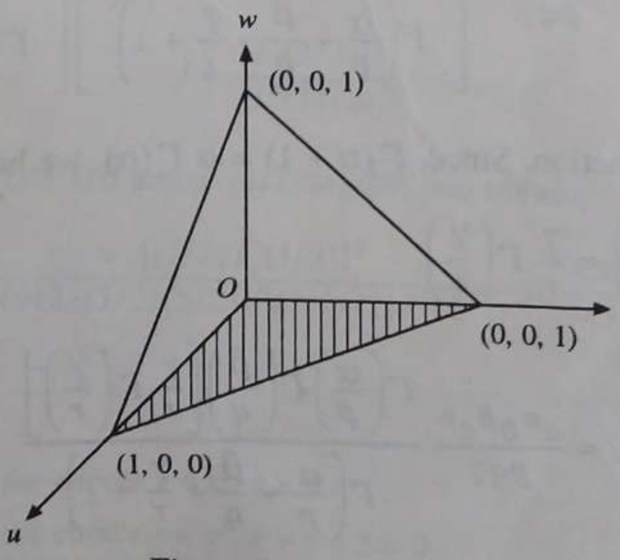

Fig. 2.15. Dirichlet integral.

Therefore, we get

$$I = \frac{a^\alpha b^\beta c^\gamma}{pqr} \int_{u=0}^{1} \int_{v=0}^{1-u} \int_{w=0}^{1-u-v} u^{(\alpha/p)-1} v^{(\beta/q)-1} w^{(\gamma/r)-1} du dv dw$$

$$= \frac{a^\alpha b^\beta c^\gamma}{pqr} \int_{u=0}^{1} \int_{v=0}^{1-u} u^{(\alpha/p)-1} v^{(\beta/q)-1} \left[\frac{w^{(\gamma/r)}}{(\gamma/r)} \right]_0^{1-u-v} du dv$$

$$= \frac{a^\alpha b^\beta c^\gamma}{pq\gamma} \int_{u=0}^{1} \int_{v=0}^{1-u} u^{(\alpha/p)-1} v^{(\beta/q)-1} (1-u-v)^{(\gamma/r)} du dv$$

Substituting $v = (1-u)\,t$, $dv = (1-u)dt$, we obtain

$$I = \frac{a^\alpha b^\beta c^\gamma}{pq\gamma} \int_{u=0}^{1} \int_{t=0}^{1} u^{(\alpha/p)-1} (1-u)^{[(\beta/q)+(\gamma/r)]} t^{(\beta/q)-1} (1-t)^{(\gamma/r)} du dt.$$

Since the limits are constants, we can write

$$I = \frac{a^\alpha b^\beta c^\gamma}{pq\gamma} \left[\int_0^1 u^{(\alpha/p)-1} (1-u)^{[(\beta/q)+(\gamma/r)]} du \right] \left[\int_0^1 t^{(\beta/q)-1} (1-t)^{(\gamma/r)} dt \right]$$

Using the definition of Beta function

$$\int_0^1 x^{m-1} (1-x)^{n-1} dx = \beta(m, n)$$

we obtain

$$I = \frac{a^\alpha b^\beta c^\gamma}{pq\gamma} \beta\left(\frac{\alpha}{p}, \frac{\beta}{q} + \frac{\gamma}{r} + 1 \right) \beta\left(\frac{\beta}{q}, \frac{\gamma}{r} + 1 \right)$$

$$= \frac{a^\alpha b^\beta c^\gamma}{pq\gamma} \left[\frac{\Gamma\left(\frac{\alpha}{p}\right) \Gamma\left(\frac{\beta}{q} + \frac{\gamma}{r} + 1\right)}{\Gamma\left(\frac{\alpha}{p} + \frac{\beta}{q} + \frac{\gamma}{r} + 1\right)} \right] \left[\frac{\Gamma\left(\frac{\beta}{q}\right) \Gamma\left(\frac{\gamma}{r} + 1\right)}{\Gamma\left(\frac{\beta}{q} + \frac{\gamma}{r} + 1\right)} \right]$$

where $\Gamma(x)$ is the Gamma function. Since, $\Gamma(\alpha + 1) = \alpha\, \Gamma(\alpha)$, we have

$$\Gamma\left(\frac{\gamma}{r} + 1\right) = \frac{\gamma}{r} \Gamma\left(\frac{\gamma}{r}\right)$$

Hence,

$$I = \frac{a^\alpha b^\beta c^\gamma}{pq\gamma} \frac{\Gamma\left(\frac{\alpha}{p}\right) \Gamma\left(\frac{\beta}{q}\right) \left[\frac{\gamma}{r} \Gamma\left(\frac{\gamma}{r}\right) \right]}{\Gamma\left(\frac{\alpha}{p} + \frac{\beta}{q} + \frac{\gamma}{r} + 1\right)}$$

$$= \frac{a^{\alpha} b^{\beta} c^{\gamma}}{pqr} \frac{\Gamma(\alpha/p)\Gamma(\beta/q)\Gamma(\gamma/r)}{\Gamma\left(\frac{\alpha}{p} + \frac{\beta}{q} + \frac{\gamma}{r} + 1\right)}$$

which is the required result.

Example 2.56 Evaluate the Dirichlet integral

$$I = \iiint_T x^3 y^3 z^3 dx dy dz$$

where T is the region in the first octant bounded by the sphere $x^2 + y^2 + z^2 = 1$ and the coordinate planes.

Solution Comparing the given integral with Eq. (2.107), we get

$$\alpha = \beta = \gamma = 4, p = q = r = 2, a = b = c = 1.$$

Substituting in Eq. (2.108), we obtain

$$I = \frac{1}{8} \frac{[\Gamma(2)]^3}{\Gamma(7)} = \frac{1}{8(6!)} = \frac{1}{5760}$$

since $\Gamma(n + 1) = n!$, when n is an integer.

Example 2.57 Evaluate the Dirichlet integral

$$I = \iiint_T x^{1/2} y^{1/2} z^{1/2} dx dy dz$$

where T is the region in the first octant bounded by the plane $x + y + z = 1$ and the coordinate planes.

Solution Comparing the given integral with Eq. (2.107), we get

$$\alpha = \beta = \gamma = 3/2, p = q = r = 1, a = b = c = 1.$$

Substituting in Eq. (2.108), we obtain

$$I = \frac{[\Gamma(3/2)]^3}{\Gamma(11/2)}.$$

Using the results, $\Gamma(\alpha + 1) = \alpha\,\Gamma(\alpha)$ and $\Gamma(1/2) = \sqrt{\pi}$, we obtain

$$I = \frac{[(1/2)\Gamma(1/2)]^3}{(9/2)(7/2)(5/2)(3/2)(1/2)\Gamma(1/2)} = \frac{4\pi}{945}.$$

Exercises 2.5

1. Find the area bounded by the curves $y = x^2$, $y = 4 - x^2$.
2. Find the area bounded by the curves $x = y^2$, $x + y - 2 = 0$.
3. Find the area bounded by the curves $y^2 = 4 - 2x$, $x \geq 0$, $y \geq 0$.

4. Find the area bounded by the curves $x^2 = y^3$, $x = y$.
5. By changing to polar coordinates, find the area bounded by the curves $x^2 + y^2 = 2y$, $x^2 + y^2 = 4y$, x and $x = 0$.

Change the order of integration and evaluate the following double integrals.

6. $\displaystyle\int_{y=0}^{1}\int_{x=y}^{\sqrt{2-y^2}} \frac{y\,dx\,dy}{\sqrt{x^2+y^2}}$.

7. $\displaystyle\int_{y=0}^{1}\int_{x=0}^{y+4} \frac{2y+1}{x+1}\,dx\,dy$.

8. $\displaystyle\int_{y=0}^{1}\int_{x=y}^{y^{1/3}} e^{x^2}\,dx\,dy$.

9. $\displaystyle\int_{x=0}^{2}\int_{y=0}^{x^2/2} \frac{x}{\sqrt{x^2+y^2+1}}\,dy\,dx$.

10. $\displaystyle\int_{x=0}^{1}\int_{y=0}^{1-x} e^{y/(x+y)}\,dy\,dx$ (use the substitution $x + y = u$ and $y = u\,v$).

11. Find the volume of the solid which is below the plane $z = 2x + 3$ and above the x-y plane and bound by $y^2 = x$, $x = 0$ and $x = 2$.
12. Find the volume of the solid which is below the plane $z = x + 3y$ and above the ellipse $25x^2 + 16y^2 = 4($ $x \geq 0$, $y \geq 0$.
13. Find the volume of the solid which is bounded by the cylinder $x^2 + y^2 = 1$ and the planes $y + z =$ and $z = 0$.
14. Find the volume of the solid which is bounded by the paraboloid $z = 9 - x^2 - 4y^2$ and the coordina planes $x \geq 0$, $y \geq 0$, $z \geq 0$.
15. Find the volume of the solid which is enclosed between the cylinders $x^2 + y^2 = 2ay$ and $z^2 = 2ay$.
16. Find the volume of the solid which is bounded by the surfaces $2z = x^2 + y^2$ and $z = x$.
17. Find the volume of the solid which is bounded by the surfaces $z = 0$, $3z = x^2 + y^2$ and the cylinde $x^2 + y^2 = 9$.
18. Find the volume of the solid which is in the first octant bounded by the cylinders $x^2 + y^2 = a^2$ an $y^2 + z^2 = a^2$.
19. Find the volume of the solid which is bounded by the paraboloid $4z = x^2 + y^2$, the cone $z^2 = x^2 + y$ and the cylinder $x^2 + y^2 = 2x$.
20. Find the volume of the solid which is common to the right circular cylinders $x^2 + z^2 = 1$, $y^2 + z^2 = 1$ and $x^2 + y^2 = 1$.
21. Find the volume of the solid which is above the cone $z^2 = x^2 + y^2$ and inside the sphere $x^2 + y^2 + (z - a)^2 = a^2$.
22. Find the volume of the solid which is below the surface $z = 4x^2 + 9y^2$ and above the square with vertices at (0, 0), (2, 0), (2, 2) and (0, 2).
23. Find the volume of the solid which is bounded by the paraboloids $z = x^2 + y^2$ and $z = 4 - 3(x^2 + y^2)$.
24. Find the volume of the solid which is bounded by $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{a}$ and the coordinate planes.
25. Find the volume of the solid which is contained between the cone $z^2 = 2(x^2 + y^2)$ and the hyperboloid $z^2 = x^2 + y^2 + a^2$.
26. Find the volume of the region under the cone $z = 3r$ and over the rose petal with boundary $r = \sin 4\theta$, $0 \leq \theta \leq \pi/4$.
27. Find the volume of the portion of the unit sphere which lies inside the right circular cone having its vertex at the origin and making an angle α with the positive z-axis.

28. Find the volume of the region under the plane $z = 1 + 3x + 2y$, $z \geq 0$ and above the region bounded by $x = 1$, $x = 2$, $y = x^2$, and $y = 2x^2$.

29. Find the volume of the portion of the sphere $x^2 + y^2 + z^2 \leq 2ay$ between the planes $y = 0$ and $y = a$.

30. Find the moment of inertia about the axes, of the circular lamina $x^2 + y^2 \leq a^2$, when the density function is $\rho = \sqrt{x^2 + y^2}$.

31. Find the total mass and the centre of gravity of the region bounded by $x^{2/3} + y^{2/3} = a^{2/3}$, $x \geq 0$, $y \geq 0$, when the density is constant k.

32. Show that $I = \iint_R \dfrac{dx\,dy}{(x^2 + y^2)^p}$, p integer, R: $x^2 + y^2 \geq 1$ converges for $p > 1$.

Hence, evaluate the integral.

Evaluate the following integrals (change the variables if necessary) in the given region.

33. $\iint_R (x^2 + y^2)\,dx\,dy$, boundary of R: triangle with vertices (0, 0), (1, 0), (1, 1).

34. $\iint_R x^2\,dx\,dy$, boundary of R : $y = x^2$, $y = x + 2$.

35. $\iint_R (x^2 + y^2)\,dx\,dy$, R: $0 \leq y \leq \sqrt{1 - x^2}$, $0 \leq x \leq 1$.

36. $\iint_R \sqrt{1 - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}}\,dx\,dy$, boundary of R : $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

37. $\iint_R e^{2(x^2 + y^2)}\,dx\,dy$, R : $x^2 + y^2 \geq 4$, $x^2 + y^2 \leq 25$, $y = x$, $x \geq 0$, $y \geq 0$.

38. $\iint_R x^3 y^3\,dx\,dy$, R : $x^2 + y^2 \leq 1$, $x \geq 0$, $y \geq 0$.

39. $\iint_R xy\,dx\,dy$, R : $\sqrt{x} + \sqrt{y} = \sqrt{a}$, $x \geq 0$, $y \geq 0$.

40. $\iint_R (1 - x^2 - y^2)\,dx\,dy$, boundary of R : the square with vertices $(\pm 1, 0)$, $(0, \pm 1)$

(change coordinates : $x - y = u$, $x + y = v$).

41. $\iint_R (x + y)^2\,dx\,dy$, boundary of R : parallelogram with sides $x + y = 1$, $x + y = 4$, $x - 2y = -2$, $x - 2y = 1$, (change coordinates: $x + y = u$, $x - 2y = v$).

42. $\iint_R (4 - 3x^2 - y^2)\,dx\,dy$, boundary of R : $x = 0$, $y = 0$, $x + y - 2 = 0$.

43. $\iint_R xy\, dx\, dy$, region (in polar coordinates) $R: r = \sin 2\theta,\ 0 \le \theta \le \pi/2$.

44. $\iiint_T x^2 y^2 z\, dx\, dy\, dz$, $T: x^2 + y^2 \le 1,\ 0 \le z \le 1$.

45. $\iiint_T \dfrac{dx\, dy\, dz}{(x+y+z+1)^3}$, boundary of $T: x = 0,\ y = 0,\ z = 0,\ x + y + z = 1$.

46. $\iiint_T (x + 3y - 2z)dx\, dy\, dz$, $T: 0 \le y \le x^2,\ 0 \le z \le x + y,\ 0 \le x \le 1$.

47. $\iiint_T x\, dx\, dy\, dz$, boundary of $T: y = x^2,\ y = x + 2,\ 4z = x^2 + y^2,\ z = x + 3$.

48. $\iiint_T (2x - y - z)dx\, dy\, dz$, $T: 0 \le x \le 1,\ 0 \le y \le x^2,\ 0 \le z \le x + y$.

49. $\iiint_T \dfrac{dx\, dy\, dz}{(x^2+y^2+z^2)^{3/2}}$, boundary of $T: x^2 + y^2 + z^2 = a^2,\ x^2 + y^2 + z^2 = b^2,\ a > b$.

50. $\iiint_T z\, dx\, dy\, dz$, boundary of $T: z^2 = x^2 + y^2,\ x^2 + y^2 + z^2 = 1$.

51. $\iiint_T \sqrt{1 - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2}}\, dx\, dy\, dz$, boundary of $T: \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

52. $\iiint_T \sqrt{x^2 + y^2 + z^2}\, dx\, dy\, dz$, $T: x^2 + y^2 + z^2 \le y$.

53. $\iiint_T (x^2 + y^2)dx\, dy\, dz$, boundary of T: the plane $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$ and the coordinate planes.

54. $\iiint_T (y^2 + z^2)dx\, dy\, dz$, boundary of $T: y^2 + z^2 \le a^2,\ 0 \le x \le h$.

55. $\iiint_T x^2 y\, dx\, dy\, dz$, $T: x^2 + y^2 \le 1,\ 0 \le z \le 1$.

Evaluate the following Dirichlet integrals.

56. $\iiint_T xyz\, dx\, dy\, dz$, T: Region bounded by $x + y + z = 2$ and the coordinate planes.

57. $\iiint_T xy^2z^3\, dx\, dy\, dz$, T: Region bounded by $x + y + z = 1$ and the coordinate planes.

58. $\iiint_T \sqrt{xyz}\, dx\, dy\, dz$, T: Region bounded by $x^3 + y^3 + z^3 = 8$ and the coordinate planes.

59. $\iiint_T xy^{1/2}z\, dx\, dy\, dz$, T: Region bounded by $x + y^3 + z^4 = 1$.

60. $\iiint_T x^2y\, dx\, dy\, dz$, T: Region bounded by $\dfrac{x^2}{1} + \dfrac{y^2}{4} + \dfrac{z^2}{9} = 1$.

2.7 Answers and Hints

Exercise 2.1

1. $|f(x, y) - 1| = |(x-1)^2 + (y-1)^2 + 2(x-1) + 2(y-1)|$
$$< |x-1|^2 + |y-1|^2 + 2|x-1| + 2|y-1| < \varepsilon$$
   (i) if $|x-1| < \delta$, $|y-1| < \delta$ is used, we get $2\delta^2 + 4\delta < \varepsilon$ or $\delta < [\sqrt{(\varepsilon+2)/2} - 1]$
   (ii) if $\delta^2 < \delta$ is used, we get $\delta < \varepsilon/6$
   (iii) if $(x-1)^2 + (y-1)^2 < \delta^2$ and $|x-1| < \delta$, $|y-1| < \delta$ is used, we get $\delta < \sqrt{\varepsilon+4} - 2$.
2. $|f(x, y) - 7| = |(x-2)^2 - (y-1)^2 + 6(x-2) - 2(y-1)|$
$$< |x-2|^2 + |y-1|^2 + 6|x-2| + 2|y-1| < \varepsilon.$$
   (i) if $|x-2| < \delta$, $|y-1| < \delta$ is used, we get $2\delta^2 + 8\delta < \varepsilon$, or $\delta < \sqrt{(\varepsilon+8)/2} - 2$.
   (ii) if $\delta^2 < \delta$ is used, we get $\delta < \varepsilon/10$.
   (iii) if $(x-2)^2 + (y-1)^2 < \delta^2$ and $|x-2| < \delta$, $|y-1| < \delta$ is used, we get $\delta < \sqrt{\varepsilon+16} - 4$.
3. $\left|\dfrac{x+y}{x^2+y^2+1}\right| < |x+y| < |x| + |y| < 2\sqrt{x^2+y^2} < \varepsilon$. Take $\delta < \varepsilon/2$.
4. Let $x = r\cos\theta$, $y = r\sin\theta$. Therefore
$$\left|\frac{x^3+y^3}{x^2+y^2}\right| < |r(\cos^3\theta + \sin^3\theta)| < 2r < \varepsilon. \text{ Take } \delta < \varepsilon/2.$$
5. $|f(x, y) - 0| < |x| + |y| < 2\sqrt{x^2+y^2} < \varepsilon$. Take $\delta < \varepsilon/2$.
6. $|f(x, y) - 0| < x^2 + y^2 < \varepsilon$. Take $\delta < \sqrt{\varepsilon}$.
7. Choose the path $y = mx$. Limit does not exist.
8. Factorize and cancel $x - y$; 1.
9. $[1 + (x/y)]^y = [[1 + (x/y)]^{y/x}]^x$; e^{α}.
10. 0.
11. 1/2.
12. 1.
13. Limit does not exist.
14. Limit does not exist.

15. Let $x = r\cos\theta$, $y = r\sin\theta$; $\frac{1}{r}\left(\frac{\cos^2\theta}{\cos^3\theta+\sin^3\theta}\right) \to \infty$ as $r \to 0$. Limit does not exist.

16. Choose the path $y = mx^2$. Limit does not exist.

17. Choose the path $z = x^2$, $y = mx$. Limit does not exist.

18. Choose the path $y = mx$, $z = mx$. Limit does not exist.

19. Choose the path $z = \sqrt{x}$, $y = mx$. Limit does not exist.

20. Choose the path $z = 0$, $y = mx$. Limit does not exist.

21. Choose the path $y = mx$. Discontinuous.

22. Limit is 0 for $x > 0$ and 1 for $x < 0$. Discontinuous.

23. Discontinuous.

24. Choose the path $y = mx$. Discontinuous.

25. Choose the path $y = mx$. Discontinuous.

26. Cancel $(x - y)$. Discontinuous.

27. Let $x = r\cos\theta$, $y = r\sin\theta$. Continuous.

28. Choose the path $y^2 = mx$. Discontinuous.

29. Since $x^2 + y^2 \ge 2|x||y|$, we have $\frac{1}{\sqrt{x^2+y^2}} \le \frac{1}{\sqrt{2|xy|}}$. Therefore, $|f(x, y)| \le \frac{\left|\sin\sqrt{|xy|} - \sqrt{|xy|}\right|}{\sqrt{2}\sqrt{|xy|}}$.
Continuous.

30. Since $2 \le 3 + \sin x \le 4$, we have $[1/(3 + \sin x)] \le 1/2$. Therefore, $|f(x, y)| \le [(2x^2 + y^2)/2] \le x^2 + y^2$. Continuous.

31. The function is not defined along the path $y = -x$. Discontinuous.

32. $\left|\frac{x^5 - y^5}{x^2 + y^2}\right| \le \frac{|x|^5 + |y|^5}{x^2 + y^2} \le \frac{(x^2 + y^2)^{5/2} + (x^2 + y^2)^{5/2}}{x^2 + y^2}$. Continuous.

33. Function is unbounded in any neighborhood of $x = -1$. Discontinuous.

34. Since $|x|$, $|y|$, $|z|$ are all $\le \sqrt{x^2 + y^2 + z^2}$, $|f| \le \sqrt{x^2 + y^2 + z^2}$. Continuous.

35. The function is unbounded along $x = \sqrt{3}z$. Discontinuous.

Exercise 2.2

1. $f_x(0, 0) = 0$, $f_y(0, 0) = 0$. For $(x, y) \ne (0, 0)$, find f_x, f_y and choose the path $y = mx$. The limits do not exist as $(x, y) \to (0, 0)$.

2. $f(x, y)$ is unbounded as $(x, y) \to (0, 0)$, for example along $x = y$; $f_x(0, 0) = 1$, $f_y(0, 0) = -1$.

3. $f_x(0, 0) = 0$, $f_y(0, 0) = -1$, $f_x(0, y) = 0$, $f_y(x, 0) = 1$.

4. $f_x(0, 0) = 1$, $f_y(0, 0) = 1$, $dz = \Delta x + \Delta y$, $\lim\limits_{\Delta\rho\to 0} [(\Delta z - dz)/\Delta\rho]$ does not exist.

5. $f_x(0, 0) = 0 = f_y(0, 0)$, $dz = 0$. $\lim\limits_{\Delta\rho\to 0} [(\Delta z - dz)/\Delta\rho] = 0$.

No contradiction since continuity of f_x, f_y is only a sufficient condition.

In problems **6** to **15**, f_x, f_y and f_z are given in that order at the given point.

6. $-2, 2$.

7. $1/2, -1/3$.

8. $6e^{1/2}, 4e^{1/2}$.

9. $49/(85)^{3/2}, -42/(85)^{3/2}$.

10. $-1/10, -1/10$.

11. $f(x, y) = 2 \ln [\sqrt{x^2 + y^2} - x] - 2 \ln y, -2/5, 3/10$.

12. $-2/27, -1/27, -2/27$.

13. $e, -2e, e$.

14. 5, 3, 0.

15. $1/7, 3/35, 4/35$.

16. 0.

17. 2.

18. $e^x[\sin(y + 2z) + \{(4t^3 - 1)/t^2\} \cos(y + 2z)]$.

19. $2(y + z)t + (x + z)(t + 1)e^t + (x + y)(1 - t)e^{-t}$.

20. $(\pi/2) - (2/\pi)$.

23. Set $s = x - y$, $v = y - z$, $w = z - x$.

31. $-[yx^{y-1} + y^x \ln y]/[xy^{x-1} + x^y \ln x]$.

32. $y/[x + 3y^2(x^2 + y^2)]$.

33. $\left(\frac{\partial z}{\partial x}\right)_y = -\left(\frac{\partial f/\partial x}{\partial f/\partial z}\right) = -\frac{y(\sin xy) + z(\sin xz)}{y(\sin yz) + x(\sin xz)}, \left(\frac{\partial z}{\partial y}\right)_x = -\left(\frac{\partial f/\partial y}{\partial f/\partial z}\right) = -\frac{x(\sin xy) + z(\sin yz)}{y(\sin yz) + x(\sin xz)}$.

34. $\left(\frac{\partial z}{\partial x}\right)_y = -\left(\frac{\partial f/\partial x}{\partial f/\partial z}\right) = -\frac{3x^2 + 3y + 3z}{3x + 2z}, \left(\frac{\partial z}{\partial y}\right)_x = -\left(\frac{\partial f/\partial y}{\partial f/\partial z}\right) = -\frac{3x - 4y}{3x + 2z}$.

35. Let $u = z/y$, $v = x/y$; then $f(u, v) = 0$; x.

36. 499.6.

37. 4.02.

38. $\frac{1}{2\sqrt{2}}\left[1 + \frac{\pi}{180}(2\sqrt{3} + 1)\right]$.

39. 1.81.

40. $\frac{1}{720}[180 + \pi(6 - \sqrt{3})] \approx 0 \cdot 2686$.

41. 5.01.

42. $V = \pi r^2 h/3$, $dV/dt = 85\pi/72 \approx 3.71$ ft^3/hr.

43. $S = 2(xy + xz + yz)$, max. absolute error = 2880 in^2, max. relative error = 0.0766 in, percentage error ≈ 7.66%.

44. $A = \frac{1}{2} xy \sin \alpha$, percentage error ≈ 13.7%.

45. $V = abc$, percentage error = 3%.

46. $V = \pi r^2 h$, percentage error ≈ 9.2%.

47. 121.6 watts.

48. 2.92%.

49. 29.33%.

50. Lateral length $l = \sqrt{r^2 + h^2}$, lateral area $= \pi r l$, $dr = r/100$, $dh = h/100$, $dl = \sqrt{(dr)^2 + (dh)^2} = 1/20$, percentage error = 2%.

Exercise 2.3

1. At (1, 1): $f_{xx} = -1/2, f_{xy} = 0, f_{yy} = 1/2$.
2. At (2, 3): $f_{xxx} = 0, f_{xxy} = 0, f_{xyy} = -1/9, f_{yyy} = 4/27$.
3. At (1, 2): $f_{xx} = 0, f_{xy} = 1, f_{yy} = -3/4$.
4. At $(1, \pi/2)$: $f_{xxx} = e \ln(\pi/2), f_{xxy} = (2e/\pi) + 1, f_{xyy} = -4e/\pi^2, f_{yyy} = 16e/\pi^3$.
5. At $(\pi/2, 1)$: $f_{xx} = -e, f_{xy} = \pi e/2, f_{yy} = -\pi^2 e/4$.
6. At (1, –1, 1): $f_{xx} = -1/2, f_{xy} = -1/4, f_{xz} = -1/4, f_{yy} = 0, f_{yz} = -1/4, f_{zz} = 0$.
7. At (–1, 1, –1): $f_{xx} = 6e^3, f_{xy} = -4e^3, f_{xz} = 4e^3, f_{yy} = 6e^3, f_{yz} = -4e^3, f_{zz} = 6e^3$.
8. At $(1, \pi/2, \pi/2)$: $f_{xx} = -\pi^2/2, f_{xy} = -\pi/2, f_{xz} = -\pi/2, f_{yy} = -[1 + (\pi^2 S/4)], f_{yz} = -[(\pi^2 S/4) - c]$, $f_{zz} = -[1 + (\pi^2 S/4)]$, $S = \sin(\pi^2/4)$, $c = \cos(\pi^2/4)$.
9. At (1, 2, 3) : $f_{xx} = 6, f_{xy} = -1/4, f_{xz} = -1, f_{yy} = 1/4, f_{yz} = -1/9, f_{zz} = 4/27$.
10. $f_{xy} = f \ln(ex) \ln(ey)$.

11. $f_x(0, 0) = 0, f_y(0, 0) = 0, f_x(0, y) = 0, f_y(x, 0) = x, f_{xy}(0, 0) = 1, f_{yx}(0, 0) = 0.$

12. $f_{xy}(x, y) = f_{yx}(x, y) = x^{y-1}(1 + y \ln x).$

13. $f_{xy}(x, y) = f_{yx}(x, y) = -y/(x^2 + y^2)^{3/2}.$

14. $(1 + xy)(\cos z) e^{xy}.$

15. $4(1 + 2y^2)\, z\, e^{x+y^2}.$

16. For $t = 0$, we get $x = 0$, $y = 0$, $dz/dt = -2$.

17. $\partial x/\partial u = 3u/x$, $\partial y/\partial u = 5u/y$; $\partial^2 x/\partial u^2 = 3(x^2 - 3u^2)/x^3$, $\partial^2 y/\partial u^2 = 5(y^2 - 5u^2)/y^3$.

18. For $x = 1$, $y = -1$, $z = 2$, we get $u = 1, v = 2$; $(\partial u/\partial x)_{y,z} = 5/3$; $(\partial v/\partial y)_{x,z} = 1/6$.

19. $\dfrac{dy}{dx} = -\sqrt{\dfrac{1-y^2}{1-x^2}}, \dfrac{d^2y}{dx^2} = -\dfrac{c}{(1-x^2)^{3/2}}.$

20. $dy/dx = (e - 1)/(e + 1)$, $d^2y/dx^2 \doteq 2(e^2 + 1)/(e + 1)^3$.

21. $\dfrac{\partial z}{\partial x} = u^v (v/u)^{1/2} \ln(eu), \quad \dfrac{\partial^2 z}{\partial x^2} = u^{v-1}[1 + v(\ln eu)^2].$

26. $\alpha = 3\beta$ or $\alpha = 4\beta$ and $\beta \neq 0$ arbitrary.

27. Note that $u_x^2 + u_y^2 = v_x^2 + v_y^2 = 1/(x^2 + y^2)^2$, $u_{xx} + u_{yy} = 0 = v_{xx} + v_{yy}$. We have

$$z_{xx} + z_{yy} = f_u(u_{xx} + u_{yy}) + f_v(v_{xx} + v_{yy}) + f_{uu}(u_x^2 + u_y^2) + f_{vv}(v_x^2 + v_y^2).$$

28. Use $x^2 + y^2 = r^2$, $\theta = \tan^{-1}(y/x)$ and differentiate.

29. $\sin u = (x^2 + y^2)/(x + y)$ is a homogeneous function of degree 1.

30. $e^u = [\sqrt{x^2 - y^2}/x]$ is a homogeneous function of degree 0.

31. u is a homogeneous function of degree 1.

32. u is a homogeneous function of degree 1.

33. $w = \tan u$ is a homogeneous function of degree 2.

34. $f(x, y) = 6 - 5(x - 2) + 3(y - 2) + (x - 2)^2 + 3(y - 2)^2.$

35. $f(x, y) \approx -2 - 2(x - 1) - (y - 1)$; $B = 4$; $|E| \le 0.08$.

36. $f(x, y) \approx (x - 1) + y$; $B = 4.6912$; $|E| \le 0.0938$.

37. $f(x, y) \approx 2 + [(x - 1) + 3(y - 1)] + \frac{1}{2}[-(x - 1)^2 + 6(x - 1)(y - 1) + (y - 1)^2]$; $B = 5.1$; $|E| \le 0.0029$.

38. $f(x, y) \approx 2 + \frac{1}{4}[(x - 1) + (y - 3)] - \frac{1}{64}[(x - 1)^2 + 2(x - 1)(y - 3) + (y - 3)^2]$; $B = 0.0142$, $|E| \le 0.64 \times 10^{-4}$.

39. $f(x, y) \approx 1 + (2x + y) + \frac{1}{2}(2x + y)^2 + \frac{1}{6}(2x + y)^3$; $B \doteq 23.87$; $|E| \le 0.008$.

40. $f(x, y) \approx (x + 2y) - \frac{1}{6}(x + 2y)^3$; $B = 16[\sin(0.3)] = 4.7283$; $|E| \le 0.315 \times 10^{-3}$.

41. $f(x, y) \approx \frac{1}{2} + \frac{1}{2}\left[\left(x - \frac{\pi}{4}\right) + \left(y - \frac{\pi}{4}\right)\right] - \frac{1}{4}\left[\left(x - \frac{\pi}{4}\right)^2 - 2\left(x - \frac{\pi}{4}\right)\left(y - \frac{\pi}{4}\right) + \left(y - \frac{\pi}{4}\right)^2\right]$; $B = 1$; $|E| \le 0.0013$.

42. $f(x, y, z) \approx 3 + \frac{2}{3}[(x - 2) + (y - 2) + (z - 1)]$; $B = 0.3872$; $|E| \le 0.017$.

43. $f(x, y, z) \approx 3 + \frac{3}{4}(x - 1) + \frac{5}{12}(y - 3) + \frac{2}{3}\left(z - \frac{3}{2}\right)$; $B = 0.3985$; $|E| \le 0.0179$.

44. $f(x, y, z) \approx x + y + xz + yz$; $B = 1.11$; $|E| \le 0.005$.

45. $f(x, y, z) \approx 1 + x + \frac{1}{2}\left[x^2 - \frac{\pi^2}{4}(y-1)^2 - \left(z - \frac{\pi}{2}\right)^2 - \pi(y-1)\left(z - \frac{\pi}{2}\right)\right]$; $|B| = 7.0817$; $|E| \le 0.0319$.

Exercises 2.4

1. minimum value 9 at (3, 1).
2. maximum value a at (0, 0).
3. minimum value 0 at (0, 0) if $|b| < 1$.
4. minimum value $(3)^{4/3}$ at $(3^{-1/3}, 3^{-1/3})$
5. minimum value $5(2)^{-2/5}$ at $(\pm 2^{3/10}, 2^{-1/5})$.
6. minimum value $-3/2$ at $(\pi/3, 2\pi/3)$.
7. The matrix **A** or the matrix **B** = – **A** is not positive definite. The function has no relative minimum or maximum.
8. The matrix **B** = – **A** is positive definite and $f_{xx}, f_{yy}, f_{zz} < 0$ at (0, 0, 0). Maximum value is 0.
9. **A** is positive definite and $f_{xx}, f_{yy}, f_{zz} > 0$ at (–1, –1, –1), (–1, 1, 1), (1, –1, 1), (1, 1, –1). Minimum value is –1 at all these points.
10. **B** = – **A** is positive definite and $f_{xx}, f_{yy}, f_{zz} < 0$ at $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$. Maximum value is (log 3) –1.
11. No relative maximum and minimum. Absolute minimum value – 3 at (0, 1). Absolute maximum value 3/2 at $(\pm\sqrt{3}/2, -1/2)$.
12. No relative maximum and minimum. Absolute maximum value 1/2 at $(1/\sqrt{2}, 1/\sqrt{2})$ and $(-1/\sqrt{2}, -1/\sqrt{2})$. Absolute minimum – 1/2 at $(-1/\sqrt{2}, 1/\sqrt{2})$ and $(1/\sqrt{2}, -1/\sqrt{2})$.
13. No relative maximum and minimum. Absolute maximum value $\sqrt{13}$ at $(9/\sqrt{13}, 4/\sqrt{13})$. Absolute minimum value $-\sqrt{13}$ at $(-9/\sqrt{13}, -4/\sqrt{13})$.
14. Relative minimum value 3/4 at (1/4, 0). Minimum value 3/2 on the boundary at $(1/2, \pm 1/\sqrt{2})$. Absolute minimum value 3/4 at (1/4, 0).
15. Absolute minimum value 1/2 at (1/2, 1/2). Absolute maximum value 5 at (2, 2).
16. Absolute minimum value – 93/18 at (1/6, 2/3). Absolute maximum value – 4 at (0, 0).
17. Absolute minimum value – 1/27 at (1/3, 1/3). Absolute maximum value 7 at (1, 2).
18. Absolute minimum value – 23/2 at (2, –3/2). Absolute maximum value 37 at (0, – 4).
19. Absolute minimum value – 3/2 at $(2\pi/3, 2\pi/3)$. Absolute maximum value 3 at (0, 0).
20. Absolute maximum value 1 at (0, 0), $(0, \pi)$, $(\pi, 0)$ and (π, π). Absolute minimum value –1/8 at $(\pi/3, \pi/3)$, $(2\pi/3, 2\pi/3)$.
21. $F = f(x, y) + \lambda\phi(x, y) \Rightarrow f_x + \lambda\phi_x = 0$ and $f_y + \lambda\phi_y = 0$. Eliminate λ.
22. $\lambda = -1/2$, $(x, y) = (1, 1/2)$; maximum value is 1/2; minimum value is 0.
23. $\lambda = \sqrt{5}/2$, $(x, y) = (-1/\sqrt{5}, -2/\sqrt{5})$, minimum value is $-\sqrt{5}$;
    $\lambda = -\sqrt{5}/2$, $(x, y) = (1/\sqrt{5}, 2/\sqrt{5})$, maximum value is $\sqrt{5}$.
24. Maximum value $(3\sqrt{3} - \pi)/3$ at $(\sqrt{3}/2, \pi/3)$. Minimum value $-(3\sqrt{3} + 5\pi)/3$ at $(-\sqrt{3}/2, 5\pi/3)$.
25. Extreme value is $\sqrt{2}$.
26. The points (4, – 4), (– 4, 4) are farthest, $d^2 = 32$. The points $(4/\sqrt{3}, 4/\sqrt{3})$, $(-4/\sqrt{3}, -4/\sqrt{3})$ are nearest, $d^2 = 32/3$.

27. Rectangle must be a square.
28. Triangle must be an equilateral triangle.
29. The point is $(AD/t, BD/t, CD/t)$, $t = A^2 + B^2 + C^2$.
30. Extreme value is $a^3/27$ at $(a/3, a/3, a/3)$.
31. Extreme value is $(a + b + c)^3$ at $(t/a, t/b, t/c)$, $t = a + b + c$.
32. Extreme value is $3^{(q-p)/q}$ at (t, t, t), $t = 3^{-1/q}$.
33. Extreme value is 24 at $(2, 1, 1/2)$.
34. Maximise $V = 8xyz$ such that $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$. We get $(x, y, z) = (2a/\sqrt{3}, 2b/\sqrt{3}, 2c/\sqrt{3})$.
35. Maximise xy^2z^3 such that $x + y + z = a$, a constant. We get $x = a/6$, $y = a/3$, $z = a/2$.
36. Maximise $2(xy + xz + yz)$ such that $4(x + y + z) = a$, a constant. We get $x = y = z = a/12$, that is the parallelopiped is a cube.
37. Maximise $V = xyz$ such that $xy + xz + yz = S/2$, we get $x = y = z = \sqrt{S/6}$.
38. Minimise $S = xy + 2xz + 2yz$ such that $xyz = a$. We get $x = y = (2a)^{1/3}$ and $z = x/2$.
39. Maximise $V = \pi r^2 h/3$ such that $\pi r l = a$ where $l = \sqrt{r^2 + h^2}$. We get $h = \sqrt{2}r$.
40. Maximise $V = \pi r^2 H + (\pi r^2 h)/3$ such that $2\pi r H + \pi r l = S$, $l = \sqrt{r^2 + h^2}$. We get $h/r = 2/\sqrt{5}$ and $H/r = 1/\sqrt{5}$.
41. Maximum value is $2/(3\sqrt{3})$ at $(\pm 2/\sqrt{3}, \pm 2/\sqrt{3}, 1/\sqrt{3})$.
42. Extreme value is 3/2 at $(1/2, -1, 3/2)$.
43. Extreme value is 11/12 at $(-1/6, 1/3, 5/6)$.
44. Farthest point $(1, 0, 0)$, $d = 1$; nearest point $(1/3, 0, 2/3)$ $d = \sqrt{5}/3$.
45. The coordinates of the points P and Q are obtained as $(2a/3, 2a/3, 2a/3)$ and $(\pm a/\sqrt{3}, \pm a/\sqrt{3}, \pm a/\sqrt{3})$. Shortest distance : $d^2 = a^2(7 - 4\sqrt{3})/3$; Largest distance : $d^2 = a^2(7 + 4\sqrt{3})/3$.

Exercise 2.5

1. Curves intersect at $x = \pm\sqrt{2}$, $y = 2$; $16\sqrt{2}/3$.
2. Curves intersect at $(1, 1)$ and $(4, -2)$; 9/2.
3. 8/3.
4. Curves intersect at $(0, 0)$ and $(1, 1)$; 1/10.
5. R: $\pi/4 \le \theta \le \pi/2$, $2\sin\theta \le r \le 4\sin\theta$; $3(\pi + 2)/4$.
6. $I = \int_{x=0}^{1}\int_{y=0}^{x} f(x, y)\,dy\,dx + \int_{x=1}^{\sqrt{2}}\int_{0}^{\sqrt{2-x^2}} f(x, y)\,dy\,dx$, where $f(x, y) = \dfrac{y}{\sqrt{x^2 + y^2}}$; $(2 - \sqrt{2})/2$.
7. $I = \int_{x=0}^{4}\int_{y=0}^{1} f(x, y)\,dy\,dx + \int_{x=4}^{5}\int_{y=x-4}^{1} f(x, y)\,dy\,dx$, where $f(x, y) = \dfrac{2y + 1}{x + 1}$; $20\ln(5) - 18\ln(6) + (7/2)$.
8. $I = \int_{x=0}^{1}\int_{y=x^3}^{x} e^{x^2}\,dy\,dx = (e - 2)/2$.
9. $I = \int_{y=0}^{2}\int_{x=\sqrt{2y}}^{2} \dfrac{x}{\sqrt{x^2 + y^2 + 1}}\,dx\,dy = \dfrac{1}{4}(5\ln 5 - 4)$.
10. $I = \int_0^1\int_0^1 ue^{v}\,du\,dv = \dfrac{1}{2}(e - 1)$.
11. $14\sqrt{2}/5$.

12. 380/3. **13.** π. **14.** $81\pi/16$.

15. $128\,a^3/15$. **16.** $\pi/4$. **17.** $27\pi/2$.

18. $2a^3/3$. **19.** $(256 - 27\pi)/72$. **20.** $8(2 - \sqrt{2})$.

21. πa^3. **22.** 208/3. **23.** 2π.

24. $a^3/90$. **25.** $4\pi a^3(\sqrt{2}-1)/3$. **26.** 1/3.

27. $2\pi(1 - \cos\alpha)/3$. **28.** 1931/60. **29.** $2\pi a^3/3$.

30. $I_y = a^5\pi/5 = I_x$. **31.** $M = 3\pi\, k\, a^2/32,\ \bar{x} = \bar{y} = 8ka^3/(105\, M)$.

32. Evaluate the integral over $1 \le x^2 + y^2 \le a^2$ and take the limit as $a \to \infty$. $I = \pi/(p - 1)$.

33. 1/3. **34.** 63/20. **35.** $\pi/8$.

36. $2\pi ab/3$. **37.** $(e^{50} - e^8)\,\pi/16$. **38.** 1/96.

39. $a^4/280$. **40.** 4/3. **41.** 21.

42. 8/3. **43.** 1/15. **44.** $\pi/48$.

45. $(8 \ln 2 - 5)/16$. **46.** 11/42. **47.** 837/160.

48. 8/35. **49.** $4\pi \ln(a/b)$. **50.** $\pi/8$.

51. $\pi^2 abc/4$. **52.** $\pi/10$. **53.** $abc(a^2 + b^2)/60$.

54. $\pi h a^4/2$. **55.** 0.

In problems **56** to **60** compare the given integral with Eq. (2.107).

56. $\alpha = \beta = \gamma = 2,\ a = b = c = 2,\ p = q = r = 1;\ I = 4/45$.

57. $\alpha = 2,\ \beta = 3,\ \gamma = 4,\ a = b = c = 1,\ p = q = r = 1,\ I = 12/9!$.

58. $\alpha = \beta = \gamma = 3/2,\ a = b = c = 1,\ p = q = r = 3,\ I = 64\sqrt{2}\ \pi/81$.

59. $\alpha = 2,\ \beta = 3/2,\ \gamma = 2,\ a = b = c = 1,\ p = 1,\ q = 3,\ r = 4,\ I = \pi/288$.

60. $\alpha = 3,\ \beta = 2,\ \gamma = 1,\ a = 1,\ b = 2,\ c = 3,\ p = q = r = 2,\ I = \pi/8$.