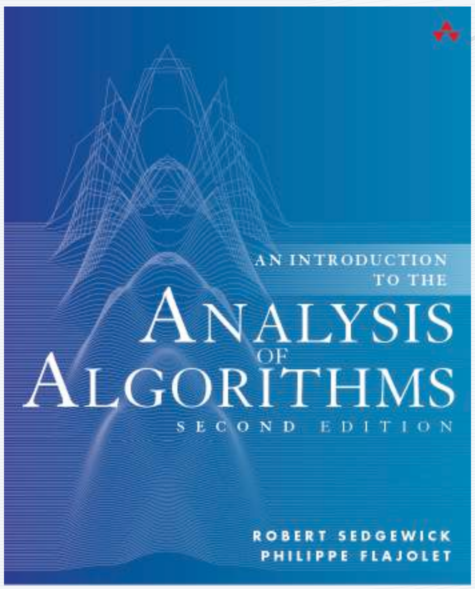
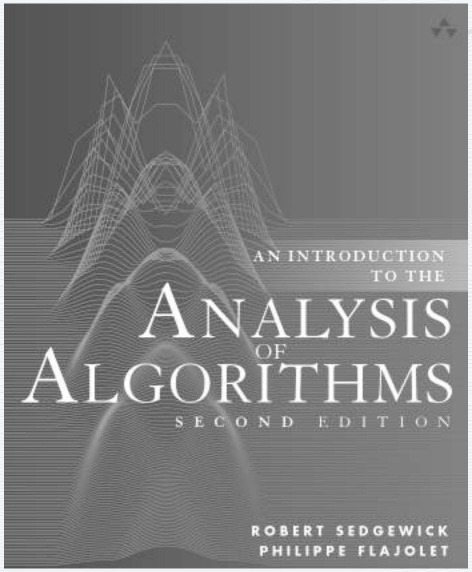


http://aofa.cs.princeton.edu

3. Generating Functions



http://aofa.cs.princeton.edu

3. Generating Functions

- **OGFs**
- Solving recurrences
- Catalan numbers
- EGFs
- Counting with GFs

Ordinary generating functions

Definition.

$$A(z) = \sum_{k\geq 0} a_k z^k$$ is the ordinary generating function (OGF)

of the sequence $a_0, a_1, a_2, \ldots, a_k, \ldots$

Notation. $[z^N]A(z)$ is "the coefficient of z^N in $A(z)$"

sequence	OGF
1, 1, 1, 1, 1, ...	$\sum_{N\geq 0} z^N = \frac{1}{1-z}$
1, 1/2, 1/6, 1/24, ...	$\sum_{N\geq 0} \frac{z^N}{N!} = e^z$

$\longleftarrow [z^N]e^z = 1/N!$

Significance. Can represent an entire sequence with a single function.

Operations on OGFs: Scaling

If $\quad A(z) = \sum_{k\geq 0} a_k z^k \quad$ is the OGF of $\quad a_0, a_1, a_2, \ldots, a_k, \ldots$

then $\quad A(cz) = \sum_{k\geq 0} a_k c^k z^k \quad$ is the OGF of $\quad a_0, ca_1, c^2 a_2, c^3 a_3, \ldots$

sequence	OGF
1, 1, 1, 1, 1, ...	$\sum_{N\geq 0} z^N = \frac{1}{1-z}$
1, 2, 4, 8, 16, 32, ...	$\sum_{N\geq 0} 2^N z^N = \frac{1}{1-2z}$

$[z^N]\frac{1}{1-2z} = 2^N$

Operations on OGFs: Addition

If $A(z) = \sum_{k\geq 0} a_k z^k$ is the OGF of $a_0, a_1, a_2, \ldots, a_k, \ldots$

and $B(z) = \sum_{k\geq 0} b_k z^k$ is the OGF of $b_0, b_1, b_2, \ldots, b_k, \ldots$

then $A(z) + B(z)$ is the OGF of $a_0 + b_0, a_1 + b_1, a_2 + b_2, \ldots, a_k + b_k \ldots$

Example:

sequence	OGF
`1, 1, 1, 1, 1, ...`	$\sum_{N\geq 0} z^N = \frac{1}{1-z}$
`1, 2, 4, 8, 16, 32, ...`	$\sum_{N\geq 0} 2^N z^N = \frac{1}{1-2z}$
`0, 1, 3, 7, 15, 31, ...`	$\frac{1}{1-2z} - \frac{1}{1-z}$

Operations on OGFs: Differentiation

If $A(z) = \sum_{k\geq 0} a_k z^k$ is the OGF of $a_0, a_1, a_2, \ldots, a_k, \ldots$

then $zA'(z) = \sum_{k\geq 1} k a_k z^k$ is the OGF of $0, a_1, 2a_2, 3a_3, \ldots, ka_k, \ldots$

OGF	sequence
$\frac{1}{1-z} = \sum_{N\geq 0} z^N$	`1, 1, 1, 1, 1, ...`
$\frac{z}{(1-z)^2} = \sum_{N\geq 1} Nz^N$	`0, 1, 2, 3, 4, 5, ...`
$\frac{z^2}{(1-z)^3} = \sum_{N\geq 2} \binom{N}{2} z^N$	`0, 0, 1, 3, 6, 10, ...`
$\frac{z^M}{(1-z)^{M+1}} = \sum_{N\geq M} \binom{N}{M} z^N$	`0, ..., 1, M+1, (M+2)(M+1)/2, ...`
$\frac{1}{(1-z)^{M+1}} = \sum_{N\geq 0} \binom{N+M}{M} z^N$	`1, M+1, (M+2)(M+1)/2, ...`

Operations on OGFs: Integration

If $A(z) = \sum_{k\geq 0} a_k z^k$ is the OGF of $a_0, a_1, a_2, \ldots, a_k, \ldots$

then $\int_0^z A(t)dt = \sum_{n\geq 1} \frac{a_{n-1}}{n} z^n$ is the OGF of $0, a_0, \frac{a_1}{2}, \frac{a_2}{3}, \ldots, \frac{a_{k-1}}{k}, \ldots$

Example:

OGF	sequence
$\frac{1}{1-z} = \sum_{N\geq 0} z^N$	1, 1, 1, 1, 1, ...
$\ln \frac{1}{1-z} = \sum_{N\geq 1} \frac{z^N}{N}$	0, 1, 1/2, 1/3, 1/4, 1/5, ...

Operations on OGFs: Partial sum

If $A(z) = \sum_{k\geq 0} a_k z^k$ is the OGF of $a_0, a_1, a_2, \ldots, a_k, \ldots$

then $\frac{1}{1-z}A(z)$ is the OGF of $a_0, a_0 + a_1, a_0 + a_1 + a_2, \ldots$

Proof.

$$\frac{1}{1-z}A(z) = \sum_{k\geq 0} z^k \sum_{n\geq 0} a_n z^n$$

Distribute

$$= \sum_{k\geq 0} \sum_{n\geq 0} a_n z^{n+k}$$

Change n to $n-k$

$$= \sum_{k\geq 0} \sum_{n\geq k} a_{n-k} z^n$$

Switch order of summation.

$$= \sum_{n\geq 0} \Big(\sum_{0\leq k\leq n} a_{n-k} \Big) z^n$$

Change k to $n-k$

$$= \sum_{n\geq 0} \Big(\sum_{0\leq k\leq n} a_k \Big) z^n$$

Operations on OGFs: Partial sum

If $A(z) = \sum_{k\geq 0} a_k z^k$ is the OGF of $a_0, a_1, a_2, \ldots, a_k, \ldots$

then $\frac{1}{1-z}A(z)$ is the OGF of $a_0, a_0 + a_1, a_0 + a_1 + a_2, \ldots$

Example:

OGF	sequence
$\frac{1}{1-z} = \sum_{N\geq 0} z^N$	1, 1, 1, 1, 1, ...
$\ln\frac{1}{1-z} = \sum_{N\geq 1} \frac{z^N}{N}$	0, 1, 1/2, 1/3, 1/4, 1/5, ...
$\frac{1}{1-z}\ln\frac{1}{1-z} = \sum_{N\geq 1} H_N z^N$	1, 1 + 1/2, 1 + 1/2 + 1/3, ...

Operations on OGFs: Convolution

If $A(z) = \sum_{k\geq 0} a_k z^k$ is the OGF of $a_0, a_1, a_2, \ldots, a_k, \ldots$

and $B(z) = \sum_{k\geq 0} b_k z^k$ is the OGF of $b_0, b_1, b_2, \ldots, b_k, \ldots$

then $A(z)B(z)$ is the OGF of $a_0 b_0, a_1 b_0 + a_1 b_0, \ldots, \sum_{0\leq k\leq n} a_k b_{n-k}, \ldots$

Proof.

$$A(z)B(z) = \sum_{k\geq 0} a_k z^k \sum_{n\geq 0} b_n z^n$$

Distribute

$$= \sum_{k\geq 0} \sum_{n\geq 0} a_k b_n z^{n+k}$$

Change n to $n-k$

$$= \sum_{k\geq 0} \sum_{n\geq k} a_k b_{n-k} z^n$$

Switch order of summation.

$$= \sum_{n\geq 0} \Big(\sum_{0\leq k\leq n} a_k b_{n-k} \Big) z^n$$

Operations on OGFs: Convolution

If $\quad A(z) = \sum_{k\geq 0} a_k z^k$ is the OGF of $a_0, a_1, a_2, \ldots, a_k, \ldots$

and $B(z) = \sum_{k\geq 0} b_k z^k$ is the OGF of $b_0, b_1, b_2, \ldots, b_k, \ldots$

then $A(z)B(z)$ is the OGF of $a_0 b_0, a_1 b_0 + a_1 b_0, \ldots, \sum_{0\leq k\leq n} a_k b_{n-k}, \ldots$

Example:

OGF	sequence
$\frac{1}{1-z} = \sum_{N\geq 0} z^N$	1, 1, 1, 1, 1, ...
$\frac{1}{(1-z)^2} = \sum_{N\geq 0} (N+1) z^N$	1, 2, 3, 4, 5, ...

Expanding a GF (summary)

The process of expressing an unknown GF as a power series (finding the coefficients) is known as expanding the GF.

Techniques we have been using:

1. Taylor theorem:

$$f(z) = f(0) + f'(0)z + \frac{f''(0)}{2!}z^2 + \frac{f'''(0)}{3!}z^3 + \frac{f''''(0)}{4!}z^4 + \ldots.$$

Example.

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \ldots$$

2. Reduce to known GFs.

Example.

$$[z^N]\frac{1}{(1-z)}\ln\frac{1}{1-z} = \sum_{1\le k\le N}\frac{1}{k} = H_N \longleftarrow$$

Integrate $\frac{1}{1-z}$ to get $\ln\frac{1}{1-z}$

then convolve $\frac{1}{1-z}$ with $\ln\frac{1}{1-z}$

In-class exercise

Exercise 3.4 Prove that $\displaystyle\sum_{1\le k\le N} H_k = (N+1)(H_{N+1}-1)$

Operations on OGFs: Partial sum

If $A(z)=\sum_{k\ge 0} a_k z^k$ is the OGF of $a_0, a_1, a_2, \ldots, a_k, \ldots$

then $\dfrac{1}{1-z}A(z)$ is the OGF of $a_0, a_0+a_1, a_0+a_1+a_2, \ldots$

Example:	OGF	sequence
	$\dfrac{1}{1-z}=\sum_{N\ge 0} z^N$	1, 1, 1, 1, 1, ...
	$\ln\dfrac{1}{1-z}=\sum_{N\ge 1}\dfrac{z^N}{N}$	0, 1, 1/2, 1/3, 1/4, 1/5, ...
	$\dfrac{1}{1-z}\ln\dfrac{1}{1-z}=\sum_{N\ge 1} H_N z^N$	1, 1 + 1/2, 1 + 1/2 + 1/3, ...

1. Find GF for LHS (convolve $\dfrac{1}{1-z}$ with $\dfrac{1}{1-z}\ln\dfrac{1}{1-z}$)

$$\frac{1}{(1-z)^2}\ln\frac{1}{1-z}$$

2. Expand GF to find RHS coefficients (convolve $\ln\dfrac{1}{1-z}$ with $\dfrac{1}{(1-z)^2}$)

$$[z^N]\frac{1}{(1-z)^2}\ln\frac{1}{1-z} = \sum_{1\le k\le N}\frac{1}{k}(N+1-k)$$

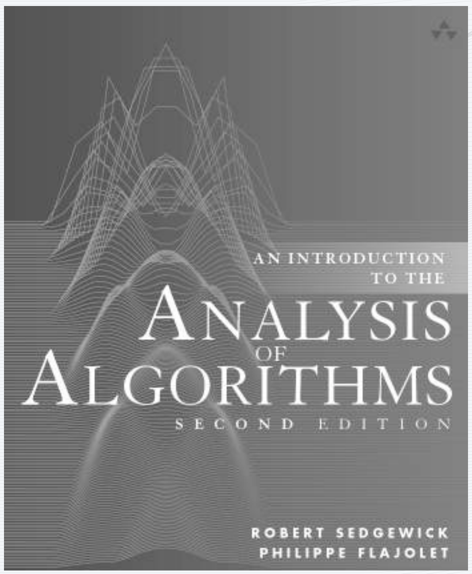
3. Do some math

$$= (N+1)H_N - N$$

$$= (N+1)\Big(H_{N+1} - \frac{1}{N+1}\Big) - N$$

$$= (N+1)(H_{N+1} - 1)$$

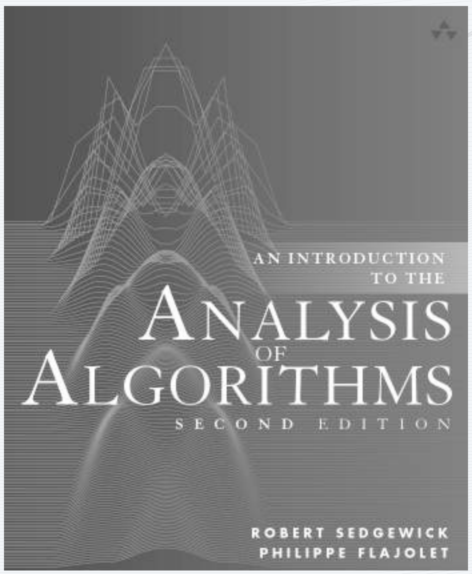
ANALYTIC COMBINATORICS
PART ONE



http://aofa.cs.princeton.edu

3. Generating Functions

- **OGFs**
- Solving recurrences
- Catalan numbers
- EGFs
- Counting with GFs



http://aofa.cs.princeton.edu

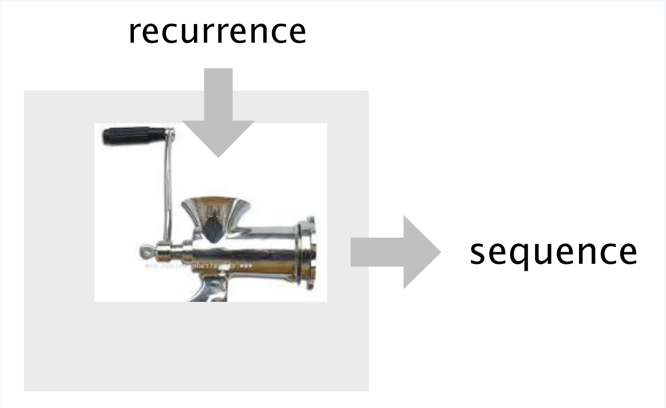
3. Generating Functions

- OGFs
- **Solving recurrences**
- Catalan numbers
- EGFs
- Counting with GFs

Solving recurrences with OGFs

General procedure:

- Make recurrence valid for all n.
- Multiply both sides of the recurrence by z^n and sum on n.
- Evaluate the sums to derive an equation satisfied by the OGF.
- Solve the equation to derive an explicit formula for the OGF.
  (Use the initial conditions!)
- Expand the OGF to find coefficients.



Solving recurrences with GFs

For linear recurrences with constant coefficients, the GF equation is a polynomial, so the general procedure is a algorithm.

Example 4 from previous lecture.

$$a_n = 5a_{n-1} - 6a_{n-2} \quad \text{for } n \geq 2 \text{ with } a_0 = 0 \text{ and } a_1 = 1$$

Make recurrence valid for all n.

$$a_n = 5a_{n-1} - 6a_{n-2} + \delta_{n1}$$

Multiply by z^n and sum on n.

$$A(z) = 5zA(z) - 6z^2A(z) + z$$

Solve.

$$A(z) = \frac{z}{1 - 5z + 6z^2}$$

Use partial fractions:
solution must be of the form

$$A(z) = \frac{c_0}{1-3z} + \frac{c_1}{1-2z}$$

Solve for coefficients.

$$c_0 + c_1 = 0$$
$$2c_0 + 3c_1 = -1$$

Solution is $c_0 = 1$ and $c_1 = -1$

$$A(z) = \frac{1}{1-3z} - \frac{1}{1-2z}$$

Expand.

$$a_n = 3^n - 2^n$$

Solving linear recurrences with GFs

For linear recurrences with constant coefficients, the GF equation is a polynomial, so the general procedure is a algorithm.

Example with multiple roots.

$$a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3} \quad \text{for } n \geq 3 \text{ with } a_0 = 0, a_1 = 1 \text{ and } a_2 = 4$$

Make recurrence valid for all n.

$$a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3} + \delta_{n1} - \delta_{n2}$$

Multiply by z^n and sum on n.

$$A(z) = 5zA(z) - 8z^2A(z) + 4z^3A(z) + z - z^2$$

Solve.

$$A(z) = \frac{z - z^2}{1 - 5z + 8z^2 - 4z^3}$$

Simplify.

$$A(z) = \frac{z(1-z)}{(1-z)(1-2z)^2} = \frac{z}{(1-2z)^2}$$

Expand.

$$a_n = n2^{n-1}$$

multiplicity 3 gives terms of the form $n^2\beta^n$, etc.

Solving linear recurrences with GFs

For linear recurrences with constant coefficients, the GF equation is a polynomial, so the general procedure is a algorithm.

Example with complex roots.

$$a_n = 2a_{n-1} - a_{n-2} + 2a_{n-3} \quad \text{for } n \geq 3 \text{ with } a_0 = 1, a_1 = 0 \text{ and } a_2 = -1$$

Make recurrence valid for all n.

$$a_n = 2a_{n-1} - a_{n-2} + 2a_{n-3} + \delta_{n0} - 2\delta_{n1}$$

Multiply by z^n and sum on n.

$$A(z) = 2zA(z) - z^2A(z) + 2z^3A(z) + 1 - 2z$$

Solve.

$$A(z) = \frac{1-2z}{1-2z+z^2-2z^3}$$

Simplify.

$$A(z) = \frac{1-2z}{(1-2z)(1+z^2)} = \frac{1}{(1+z^2)}$$

Use partial fractions.

$$A(z) = \frac{1}{2}\left(\frac{1}{1-iz} + \frac{1}{1+iz}\right)$$

Expand.

$$a_n = \frac{1}{2}(i^n + (-i)^n) = \frac{1}{2}i^n(1+(-1)^n)$$

`1, 0, -1, 0, 1, 0, -1, 0, 1...`

Solving linear recurrences with GFs (summary)

Solution to $a_n = x_1 a_{n-1} + x_2 a_{n-2} + \ldots + x_t a_{n-t}$

is a linear combination of t terms.

$z^t - x_1 z^{t-1} - x_2 z^{t-2} - \ldots - x_t z^0$

Suppose the roots of the polynomial $1 - x_1 z + x_2 z^2 + \ldots + x_t z^t$

are β_1, β_2,..., β_r where the multiplicity of β_i is m_i so $m_1 + m_2 + \ldots + m_r = t$

Solution is

$$\sum_{0 \le j < m_1} c_{1j} n^j \beta_1^n + \sum_{0 \le j < m_2} c_{2j} n^j \beta_2^n + \ldots + \sum_{0 \le j < m_r} c_{rj} n^j \beta_r^n$$

$\leftarrow$ t terms

The t constants c_{ij} are determined from the initial conditions.

Note: complex roots (and −1) introduce periodic behavior.

Solving the Quicksort recurrence with OGFs

$$C_N = N + 1 + \frac{2}{N}\sum_{1\le k\le N} C_{k-1}$$

Multiply both sides by *N*.

$$NC_N = N(N+1) + 2\sum_{1\le k\le N} C_{k-1}$$

Multiply by z^N and sum.

$$\sum_{N\ge 1} NC_N z^N = \sum_{N\ge 1} N(N+1)z^N + 2\sum_{N\ge 1}\sum_{1\le k\le N} C_{k-1}z^N$$

Evaluate sums to get an *ordinary differential equation*

$$C'(z) = \frac{2}{(1-z)^3} + 2\frac{C(z)}{1-z}$$

homogeneous equation

$\rho'(z) = 2\rho(z)/(1-z)$

solution (integration factor)

$\rho(z) = 1/(1-z)^2$

Solve the ODE.

$$((1-z)^2C(z))' = (1-z)^2C'(z) - 2(1-z)C(z)$$

$$= (1-z)^2\left(C'(z) - 2\frac{C(z)}{1-z}\right) = \frac{2}{1-z}$$

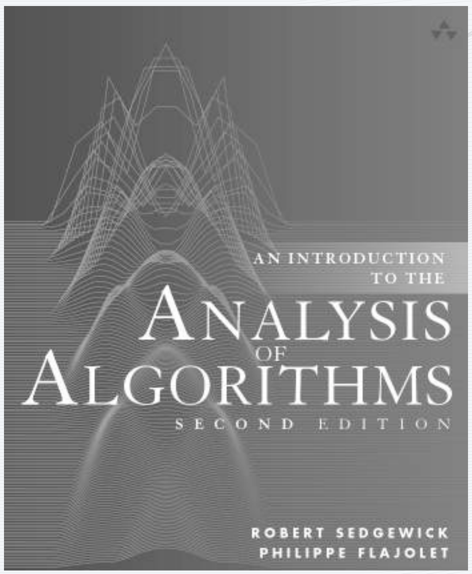
Integrate.

$$C(z) = \frac{2}{(1-z)^2}\ln\frac{1}{1-z}$$

Expand.

$$C_N = [z^N]\frac{2}{(1-z)^2}\ln\frac{1}{1-z} = 2(N+1)(H_{N+1}-1)$$

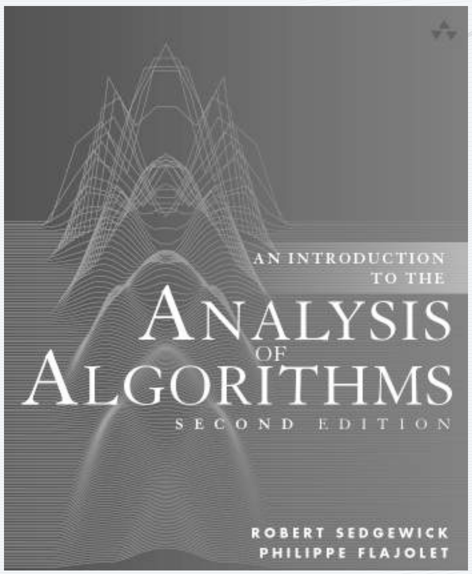
ANALYTIC COMBINATORICS
PART ONE



http://aofa.cs.princeton.edu

3. Generating Functions

- OGFs
- **Solving recurrences**
- Catalan numbers
- EGFs
- Counting with GFs



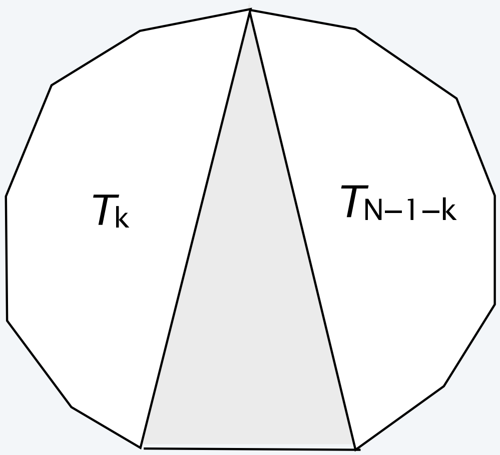
http://aofa.cs.princeton.edu

3. Generating Functions

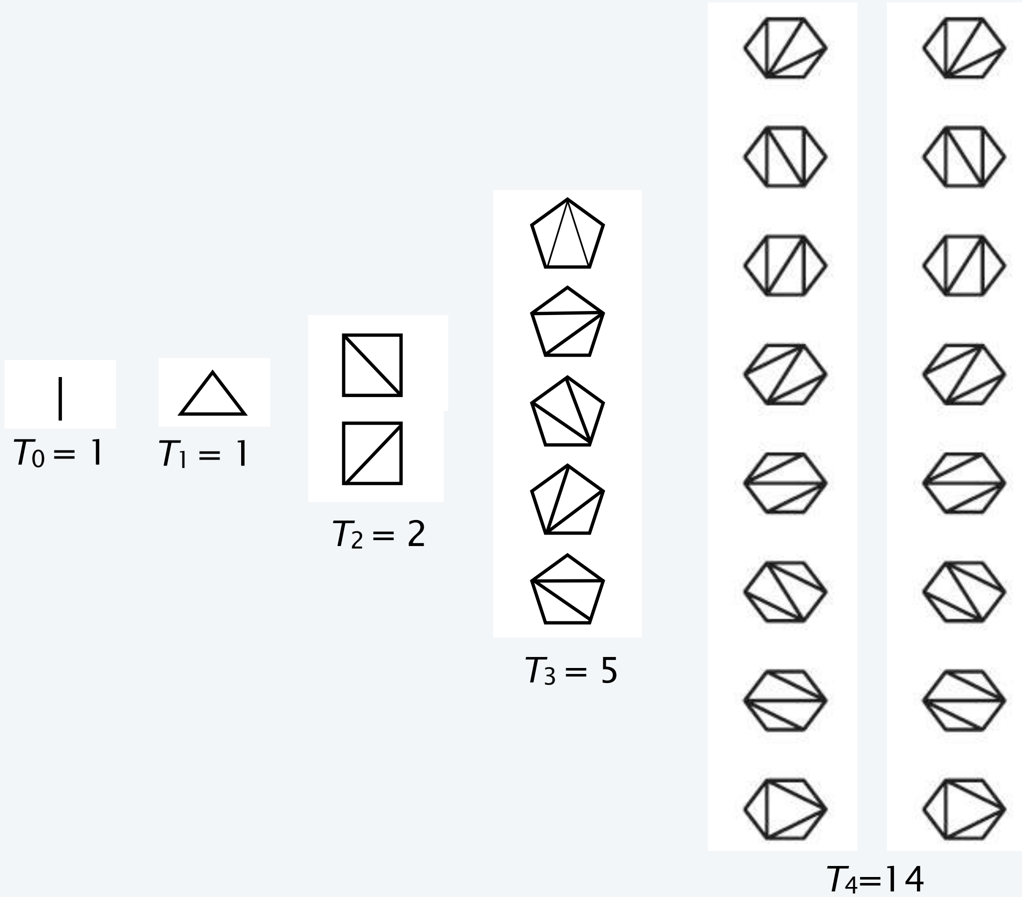
- OGFs
- Solving recurrences
- Catalan numbers
- EGFs
- Counting with GFs

Catalan numbers

How many triangulations of an (N+2)-gon?

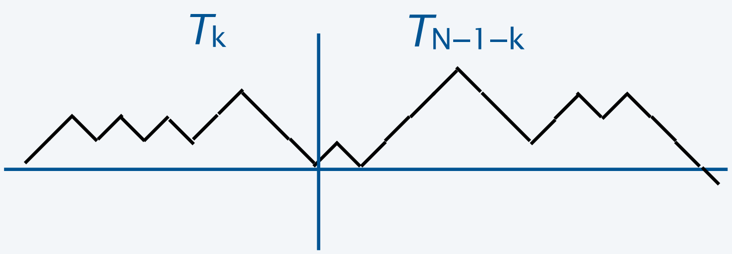


$$T_N = \sum_{0 \leq k < N} T_k T_{N-1-k} + \delta_{N0}$$



Catalan numbers

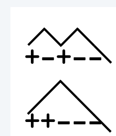
How many gambler's ruin sequences with N wins?



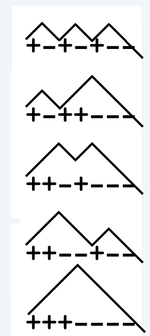
$$T_N = \sum_{0 \leq k < N} T_k T_{N-1-k} + \delta_{N0}$$

$T_0 = 1$

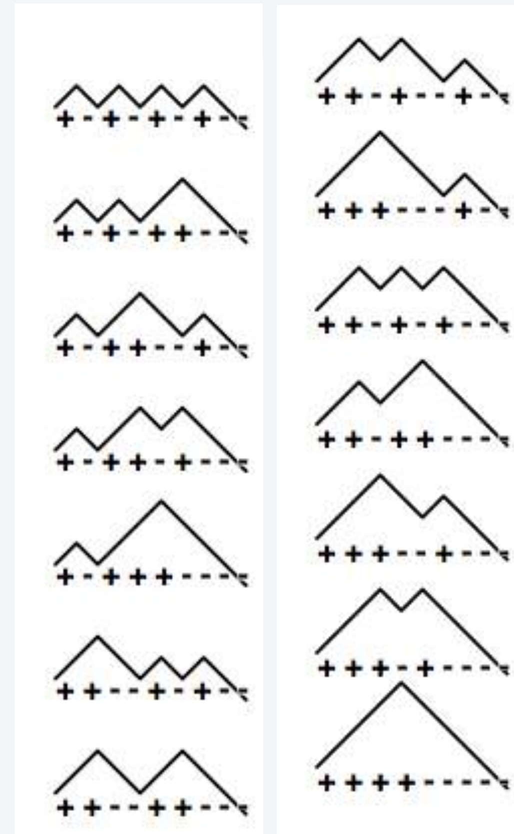
$T_1 = 1$



$T_2 = 2$



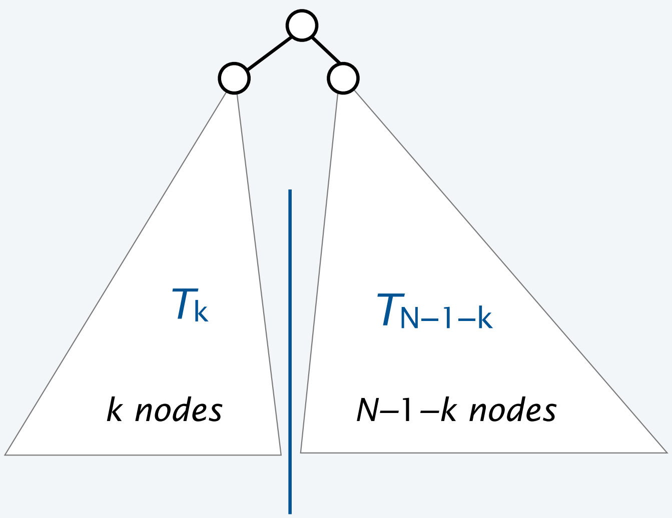
$T_3 = 5$



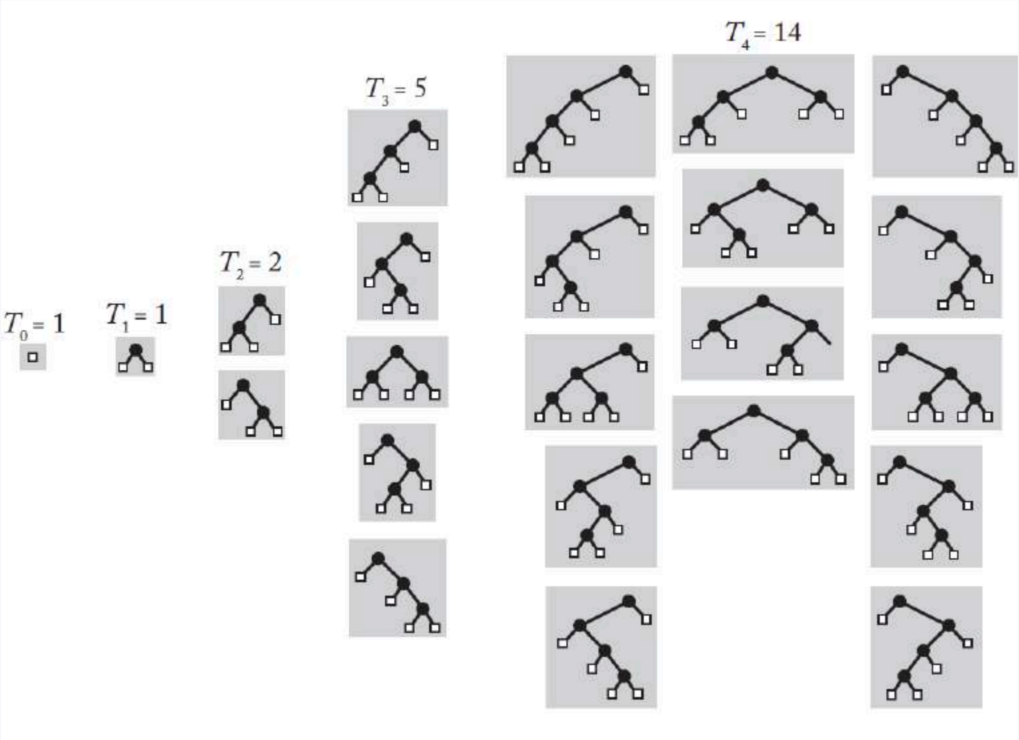
$T_4 = 14$

Catalan numbers

How many binary trees with N nodes?

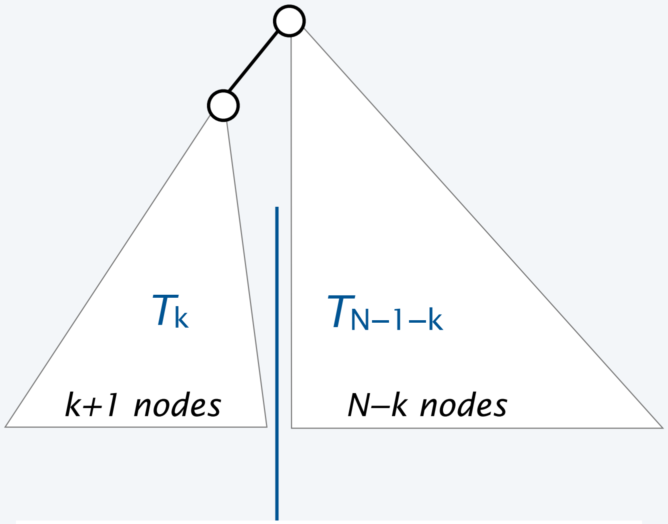


$$T_N = \sum_{0 \le k < N} T_k T_{N-1-k} + \delta_{N0}$$

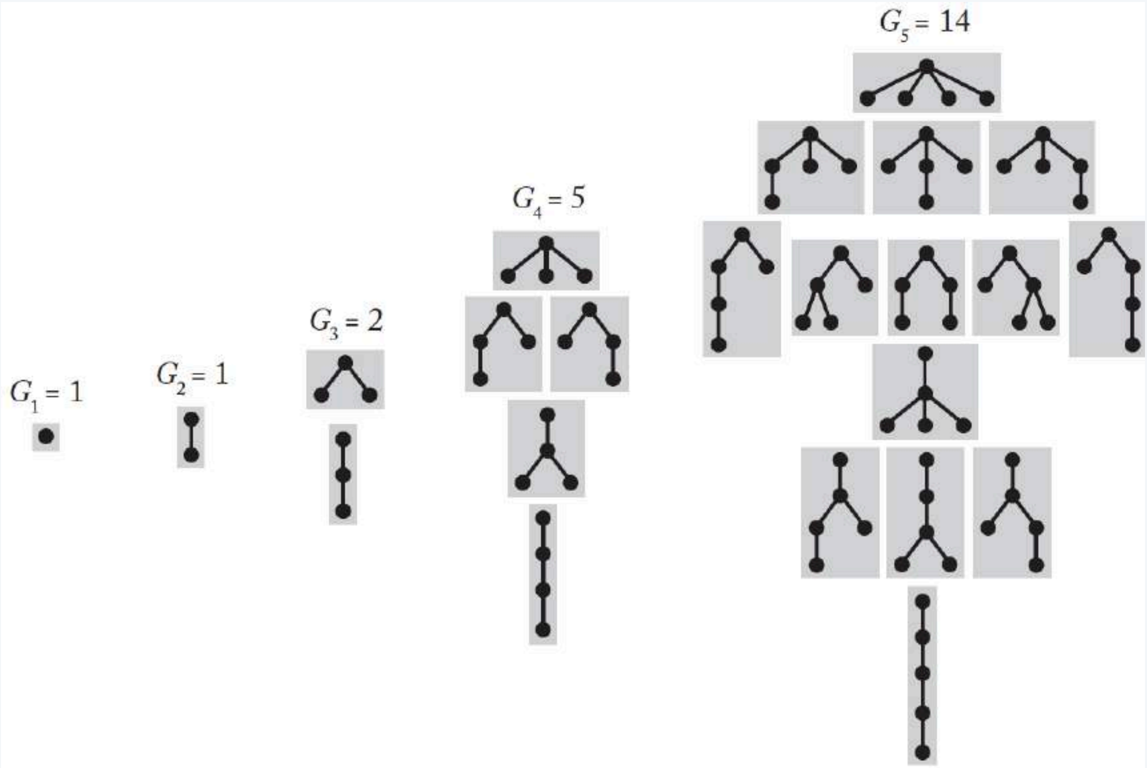


Catalan numbers

How many trees with N+1 nodes?



$$T_N = \sum_{0 \le k < N} T_k T_{N-1-k} + \delta_{N0}$$



Solving the Catalan recurrence with GFs

Recurrence that holds for all *N*.

$$T_N = \sum_{0 \le k < N} T_k T_{N-1-k} + \delta_{N0}$$

Multiply by z^N and sum.

$$T(z) \equiv \sum_{N \ge 0} T_N z^N = \sum_{N \ge 0} \sum_{0 \le k < N} T_k T_{N-1-k} z^N + 1$$

Switch order of summation

$$T(z) = 1 + \sum_{k \ge 0} \sum_{N > k} T_k T_{N-1-k} z^N$$

Change *N* to *N*+*k*+1

$$T(z) = 1 + \sum_{k \ge 0} \sum_{N \ge 0} T_k T_N z^{N+k+1}$$

Distribute.

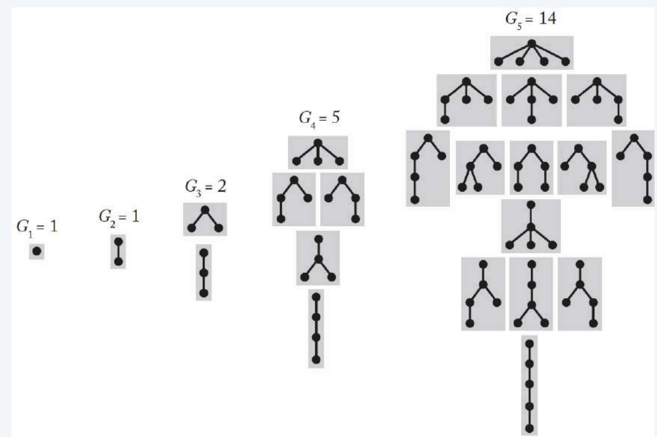
$$T(z) = 1 + z\Big(\sum_{k \ge 0} T_k z^k\Big)\Big(\sum_{N \ge 0} T_N z^N\Big)$$

$$T(z) = 1 + zT(z)^2$$

convolution
(backwards)

Common-sense rule for working with GFs

It is always worthwhile to check your math with your computer.



Known from initial values:

$$T(z) = 1 + z + 2z^2 + 5z^3 + 14z^4 + \ldots$$

Check:

$$T(z) = 1 + zT(z)^2$$

```
sage: ZP.<z> = ZZ[]
sage: 1 + z*(1+z+2*z^2+5*z^3+14*z^4)*(1+z+2*z^2+5*z^3+14*z^4)
196*z^9 + 140*z^8 + 81*z^7 + 48*z^6 + 42*z^5 + 14*z^4 + 5*z^3 + 2*z^2 + z + 1
```

not valid because
z^5 and beyond
missing in factors

✓

Solving the Catalan recurrence with GFs (continued)

Functional GF equation.

$$T(z) = 1 + zT(z)^2$$

Solve with quadratic formula.

$$zT(z) = \frac{1}{2}(1 \pm \sqrt{1-4z})$$

Expand via binomial theorem.

$$zT(z) = -\frac{1}{2}\sum_{N\geq 1}\binom{\frac{1}{2}}{N}(-4z)^N$$

Set coefficients equal

$$T_N = -\frac{1}{2}\binom{\frac{1}{2}}{N+1}(-4)^{N+1}$$

Expand via definition.

$$= -\frac{1}{2}\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\ldots(\frac{1}{2}-N)(-4)^{N+1}}{(N+1)!}$$

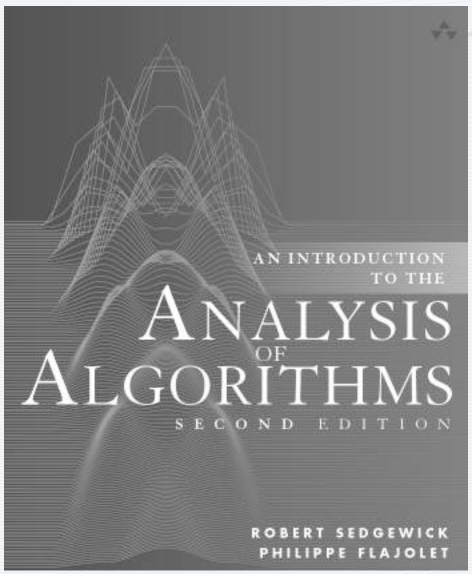
Distribute $(-2)^N$ among factors.

$$= \frac{1\cdot 3\cdot 5\cdots(2N-1)\cdot 2^N}{(N+1)!}$$

Substitute (2/1)(4/2)(6/3)... for 2^N.

$$= \frac{1}{N+1}\,\frac{1\cdot 3\cdot 5\cdots(2N-1)}{N!}\,\frac{2\cdot 4\cdot 6\cdots 2N}{1\cdot 2\cdot 3\,\cdots\, N}$$

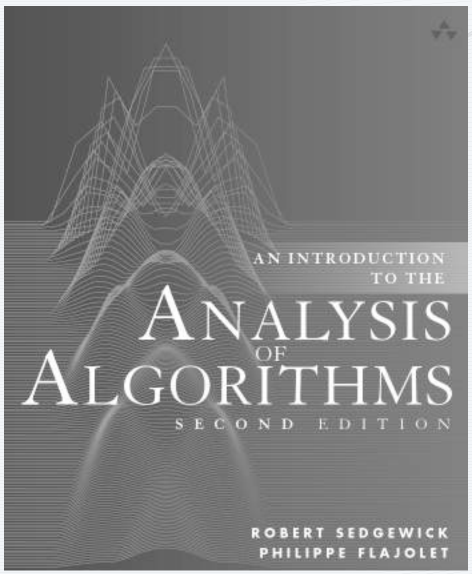
$$= \frac{1}{N+1}\binom{2N}{N}$$



http://aofa.cs.princeton.edu

3. Generating Functions

- OGFs
- Solving recurrences
- **Catalan numbers**
- EGFs
- Counting with GFs



http://aofa.cs.princeton.edu

3. Generating Functions

- OGFs
- Solving recurrences
- Catalan numbers
- **EGFs**
- Counting with GFs

Exponential generating functions (EGFs)

Definition.

$$A(z) = \sum_{k\geq 0} a_k \frac{z^k}{k!}$$ is the exponential generating function (EGF)

of the sequence $a_0, a_1, a_2, \ldots, a_k, \ldots$

sequence	EGF
1, 1, 1, 1, 1, ...	$\sum_{N\geq 0} \frac{z^N}{N!} = e^z$
1, 2, 4, 8, 16, 32, ...	$\sum_{N\geq 0} 2^N \frac{z^N}{N!} = e^{2z}$
1, 1, 2, 6, 24, 120 ...	$\sum_{N\geq 0} N! \frac{z^N}{N!} = \frac{1}{1-z}$

Operations on EGFs: Binomial convolution

If $A(z) = \sum_{k\geq 0} a_k \frac{z^k}{k!}$ is the EGF of $a_0, a_1, a_2, \ldots, a_k, \ldots$

and $B(z) = \sum_{k\geq 0} b_k \frac{z^k}{k!}$ is the EGF of $b_0, b_1, b_2, \ldots, b_k, \ldots$

then $A(z)B(z)$ is the EGF of $a_0b_0, a_0b_1 + a_1b_0, \ldots, \binom{n}{k} a_k b_{n-k}, \ldots$

Σ $0 \leq k \leq n$

Proof. $$A(z)B(z) = \sum_{k\geq 0} a_k \frac{z^k}{k!} \sum_{n\geq 0} b_n \frac{z^n}{n!}$$

Distribute. $$= \sum_{k\geq 0}\sum_{n\geq 0} \frac{a_k}{k!}\frac{b_n}{n!} z^{n+k}$$

Change *n* to *n–k* $$= \sum_{k\geq 0}\sum_{n\geq k} \frac{a_k}{k!}\frac{b_{n-k}}{(n-k)!} z^n$$

Multiply and divide by *n*! $$= \sum_{k\geq 0}\sum_{n\geq k} \binom{n}{k} a_k b_{n-k} \frac{z^n}{n!}$$

Switch order of summation. $$= \sum_{n\geq 0}\Big(\sum_{0\leq k\leq n} \binom{n}{k} a_k b_{n-k}\Big) \frac{z^n}{n!}$$

Solving recurrences with EGFs

Choice of EGF vs. OGF is typically dictated naturally from the problem.

Example. $$f_n = \sum_k \binom{n}{k} \frac{f_k}{2^k}$$

Multiply by $z^n/n!$ and sum on n. $$f(z) = \sum_{n\geq 0} \sum_k \binom{n}{k} \frac{f_k}{2^k} \frac{z^n}{n!}$$

Switch order of summation. $$f(z) = \sum_{k\geq 0} \sum_{n\geq k} \binom{n}{k} \frac{f_k}{2^k} \frac{z^n}{n!}$$

Change n to $n+k$. $$f(z) = \sum_{k\geq 0} \sum_{n\geq 0} \binom{n+k}{k} \frac{f_k}{2^k} \frac{z^{n+k}}{(n+k)!}$$

Simplify. $$f(z) = \sum_{k\geq 0} \sum_{n\geq 0} f_k \frac{(z/2)^k}{k!} \frac{z^n}{n!}$$

Distribute. $$f(z) = \Big(\sum_{k\geq 0} f_k \frac{(z/2)^k}{k!}\Big)\Big(\sum_{n\geq 0} \frac{z^n}{n!}\Big)$$

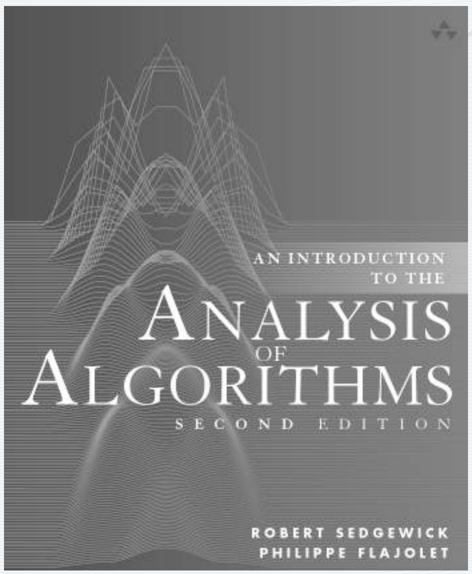
binomial convolution (backwards)

Evaluate and telescope. $$f(z) = e^z f(z/2) = e^{z+z/2+z/4+z/8+\ldots} = e^{2z}$$

convergence not assured

Expand. $$f_n = 2^n$$

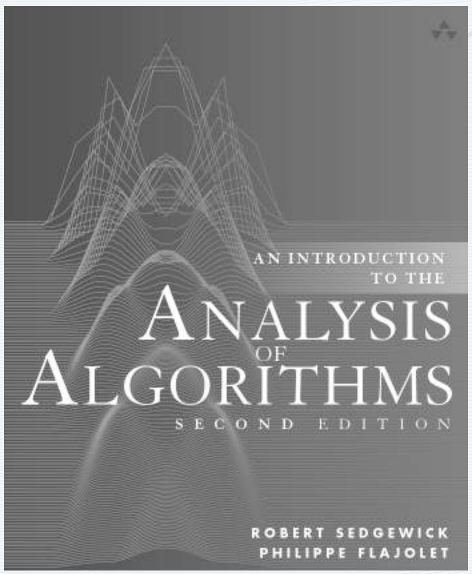
Check. $$2^n = \sum_k \binom{n}{k} \frac{2^k}{2^k}$$



http://aofa.cs.princeton.edu

3. Generating Functions

- OGFs
- Solving recurrences
- Catalan numbers
- **EGFs**
- Counting with GFs



http://aofa.cs.princeton.edu

3. Generating Functions

- OGFs
- Solving recurrences
- Catalan numbers
- EGFs
- **Counting with GFs**

Counting with generating functions

An alternative (combinatorial) view of GFs

- Define a *class* of combinatorial obects with associated *size* function.
- GF is sum over all members of the class.

Example.

T ≡ set of all binary trees

$|t|$ ≡ number of *internal* nodes in $t \in T$

T_N ≡ number of $t \in T$ with $|t| = N$

$$T(z) \equiv \sum_{t \in T} z^{|t|} = \sum_{N \geq 0} T_N z^N$$

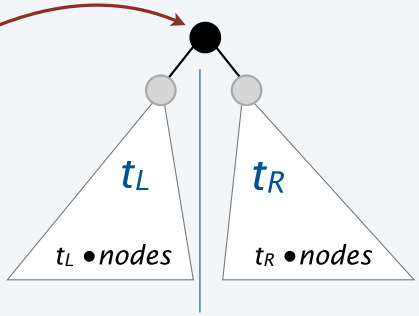
Decompose from definition

$$T(z) = 1 + \sum_{t_L \in T} \sum_{t_R \in T} z^{|t_L|+|t_R|+1}$$

Distribute

$$= 1 + z\Big(\sum_{t_L \in T} z^{|t_L|}\Big)\Big(\sum_{t_R \in T} z^{|t_R|}\Big)$$

$$= 1 + zT(z)^2$$



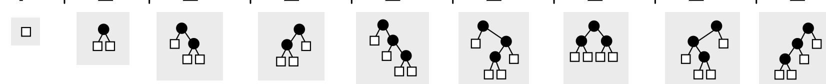
Combinatorial view of Catalan GF

Each term z^N in the GF corresponds to an object of size N.

Collect all the terms with the same exponent to expose counts.

Each term $z^i z^j$ in a product corresponds to an object of size $i + j$.

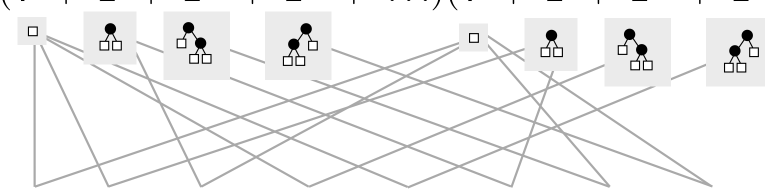
$$T(z) = 1 + z + z^2 + z^2 + z^3 + z^3 + z^3 + z^3 + z^3 + \ldots$$



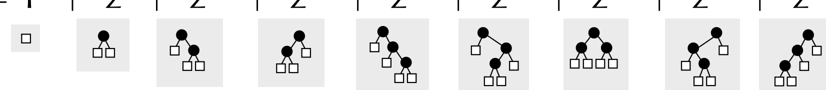
$$= 1 + z + 2z^2 + 5z^3 + \ldots$$

$$T(z) = 1 + zT(z)^2$$

$$= 1 + z(1 + z + z^2 + z^2 + \ldots)(1 + z + z^2 + z^2 + \ldots)$$



$$= 1 + z + z^2 + z^2 + z^3 + z^3 + z^3 + z^3 + z^3 + \ldots$$



$z \times 1 \times 1$ $z \times 1 \times z$ $z \times z \times 1$ $z \times 1 \times z^2$ $z \times 1 \times z^2$ $z \times z \times z$ $z \times z^2 \times 1$ $z \times z^2 \times 1$

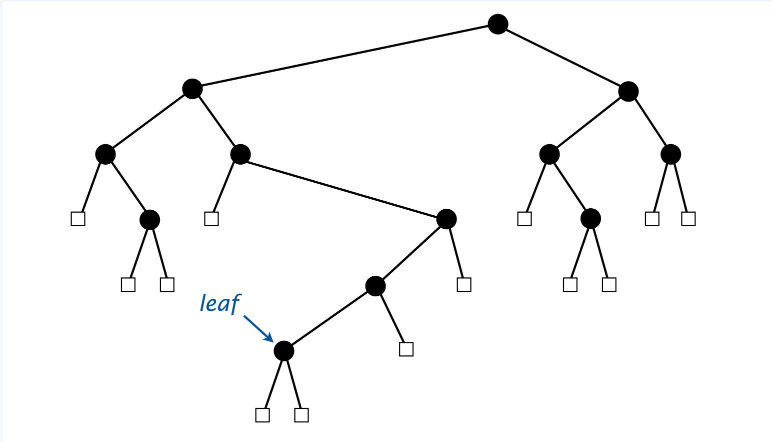
Values of parameters ("costs")

are often the object of study in the analysis of algorithms.

How many 1 bits in a random bitstring? (Easy)

0111010010001000111010100000101000O

How many leaves in a random binary tree? (Not so easy)



Computing expected costs by counting

An alternative (combinatorial) view of probability

- Define a *class* of combinatorial obects.
- Model: All objects of size N are equally likely

$$\mathcal{P} \equiv \text{ set of all objects in the class}$$

$$|p| \equiv \text{ size of } p \in \mathcal{P}$$

$$P_N \equiv \text{ number of } p \in \mathcal{P} \text{ with } |p| = N$$

$$\text{cost}(p) \equiv \text{ cost associated with } p$$

$$P_{Nk} \equiv \text{ number of } p \in \mathcal{P} \text{ with } |p| = N \text{ and cost(p) = k}$$

Expected cost of an object of size N

$$C_N \equiv \sum_{k\geq 0} k\frac{P_{Nk}}{P_N}$$

P_{Nk}/P_N is the probability that the cost of on object of size N is k

$$= \frac{\sum_{k\geq 0} kP_{Nk}}{P_N}$$

"cumulated cost"

Def. *Cumulated cost* is total cost of all objects of a given size.

Expected cost is cumulated cost divided by number of objects.

Counting with generating functions: cumulative costs

An alternative (combinatorial) view of GFs

- Define a *class* of combinatorial obects.
- Model: All objects of size N are equally likely
- GF is sum over all members of the class.

$$\begin{aligned} \mathcal{P} &\equiv \text{ set of all objects in the class} \\ |p| &\equiv \text{ size of } p \in \mathcal{P} \\ P_N &\equiv \text{ number of } p \in \mathcal{P} \text{ with } |p| = N \\ \text{cost}(p) &\equiv \text{ cost associated with } p \end{aligned}$$

Counting GF

$$P(z) \equiv \sum_{p \in \mathcal{P}} z^{|p|} = \sum_{N \geq 0} P_N z^N$$

Cumulative cost GF

$$C(z) \equiv \sum_{p \in \mathcal{P}} \text{cost}(p) z^{|p|} = \sum_{N \geq 0} \sum_{k \geq 0} k P_{Nk} z^N$$

Average cost

$$[z^N]C(z)/[z^N]P(z)$$

Bottom line: Reduces computing expectation to GF counting

Warmup: How many 1 bits in a random bitstring?

B is the set of all bitstrings.
$|b|$ is the number of bits in b.
ones(b) is the number of 1 bits in b.
B_N is the # of bitstrings of size N (2^N).
C_N is the total number of 1 bits in all bitstrings of size N.

Counting GF.

$$B(z) = \sum_{b \in B} z^{|b|} = \sum_{N \geq 0} 2^N z^N = \frac{1}{1 - 2z}$$

Cumulative cost GF.

$$\begin{aligned} C(z) &= \sum_{b \in B} \text{ones}(b) z^{|b|} \\ &= \sum_{b' \in B} \big(1 + 2 \cdot \text{ones}(b')\big) z^{|b'|+1} \\ &= zB(z) + 2zC(z) \\ &= \frac{z}{(1 - 2z)^2} \end{aligned}$$

0	b'
1	b'

$$\frac{2z}{(1 - 2z)^2} = \sum_{N \geq 1} N(2z)^N$$

Average # 1 bits in a random bitstring of length N.

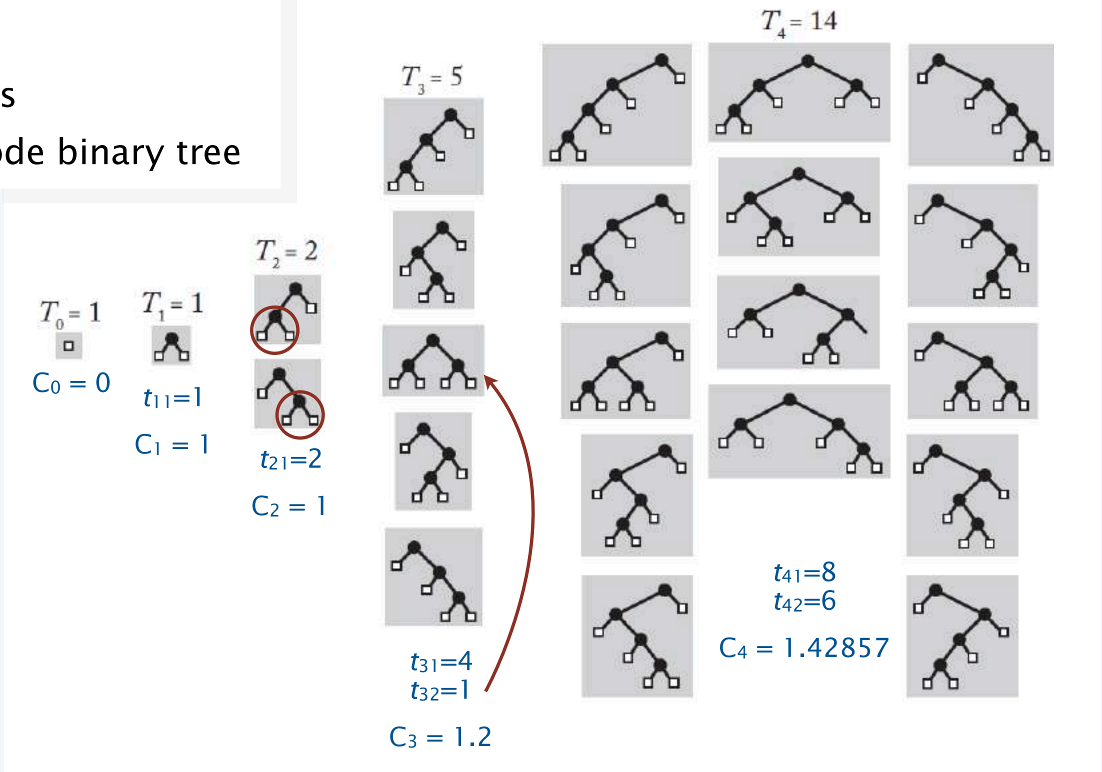
$$\frac{[z^N]C(z)}{[z^N]B(z)} = \frac{N2^{N-1}}{2^N} = \frac{N}{2}$$ ✓

Leaves in binary trees

are internal nodes whose children are both external.

Definitions:

- T_N is the # of binary trees with N nodes.
- t_{Nk} is the # of N-node binary trees with k leaves
- C_N is the average # of leaves in a random N-node binary tree



Q. How many leaves in a random binary tree?

How many leaves in a random binary tree?

T is the set of all binary trees.

$|t|$ is the number of internal nodes in t.

leaves(t) is the number of leaves in t.

T_N is the # of binary trees of size N (Catalan).

C_N is the total number of leaves in all binary trees of size N.

Counting GF.

$$T(z) = \sum_{t \in T} z^{|t|} = \sum_{N \geq 0} T_N z^N = \sum_{N \geq 0} \frac{1}{N+1} \binom{2N}{N} z^N$$

Cumulative cost GF.

$$C(z) = \sum_{t \in T} \text{leaves}(t) z^{|t|}$$

Average # leaves in a random N-node binary tree.

$$\frac{[z^N]C(z)}{[z^N]T(z)} = \frac{[z^N]C(z)}{T_N}$$

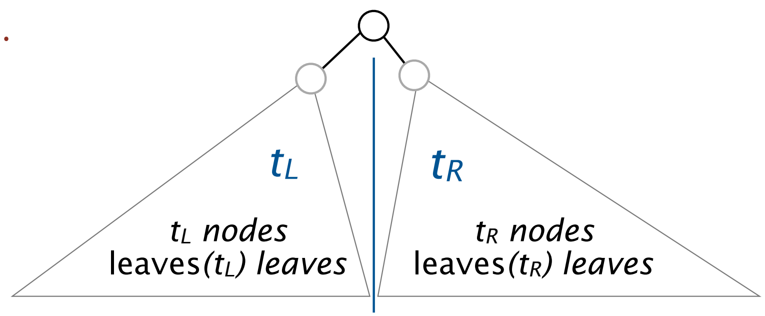
Next: Derive a functional equation for the CGF.

CGF functional equation for leaves in binary trees

CGF.

$$C(z) = \sum_{t \in T} \text{leaves}(t) z^{|t|}$$

Decompose from definition.



$$C(z) = z + \sum_{t_L \in T} \sum_{t_R \in T} \big(\text{leaves}(t_L) + \text{leaves}(t_R)\big) z^{|t_L| + |t_R| + 1}$$

$$= z + z \sum_{t_L \in T} \text{leaves}(t_L) z^{|t_L|} \sum_{t_R \in T} z^{|t_R|} + z \sum_{t_L \in T} z^{|t_L|} \sum_{t_R \in T} \text{leaves}(t_R) z^{|t_R|}$$

$$= z + 2zC(z)T(z)$$

How many leaves in a random binary tree?

CGF. $$C(z) = \sum_{t \in T} \text{leaves}(t) z^{|t|}$$

Decompose from definition. $$C(z) = z + \sum_{t_L \in T} \sum_{t_R \in T} (\text{leaves}(t_L) + \text{leaves}(t_R)) z^{|t_L|+|t_R|+1}$$
$$= z + 2zC(z)T(z)$$

Compute number of trees T_N.

Catalan numbers

$$T(z) = zT(z)^2 - z$$
$$= \frac{1}{2z}(1 - \sqrt{1-4z})$$

$$T_N = [z^N]\frac{1}{2z}(1 - \sqrt{1-4z}) = \frac{1}{N+1}\binom{2N}{N}$$

Compute cumulated cost C_N.

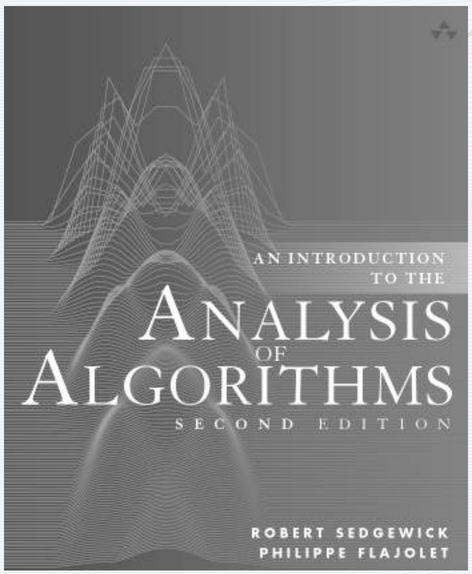
$$C(z) = z + 2zT(z)C(z)$$
$$= \frac{z}{1 - 2zT(z)} = \frac{z}{\sqrt{1-4z}}$$

$$C_N = [z^N]\frac{z}{\sqrt{1-4z}} = \binom{2N-2}{N-1}$$

Compute *average number of leaves.*

$$C_N/T_N = \frac{\binom{2N-2}{N-1}}{\frac{1}{N+1}\binom{2N}{N}} = \frac{(N+1)\cdot N \cdot N}{2N(2N-1)} \sim N/4$$

ANALYTIC COMBINATORICS
PART ONE



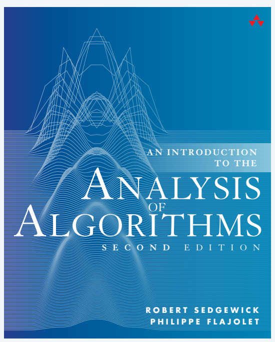
http://aofa.cs.princeton.edu

3. Generating Functions

- OGFs
- Solving recurrences
- Catalan numbers
- EGFs
- **Counting with GFs**

Exercise 3.20

Solve a linear recurrence. Initial conditions matter.



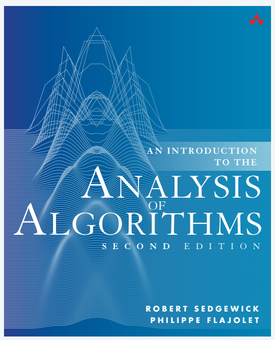
Exercise 3.20 Solve the recurrence

$$a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3} \qquad \text{for } n > 2 \text{ with } a_0 = a_1 = 0 \text{ and } a_2 = 1.$$

Solve the same recurrence with the initial condition on a_1 changed to $a_1 = 1$.

Exercise 3.28

The art of expanding GFs.



Exercise 3.28 Find an expression for

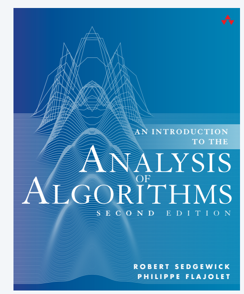
$$[z^n]\frac{1}{\sqrt{1-z}}\ln\frac{1}{1-z}.$$

(*Hint*: Expand $(1-z)^{-\alpha}$ and differentiate with respect to α.)

Assignments for next lecture

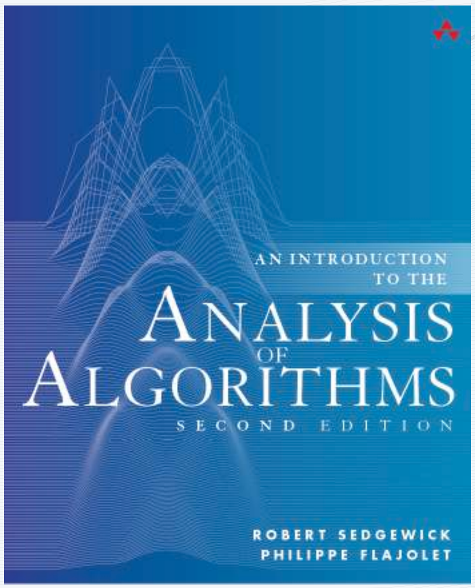
1. Use a symbolic mathematics system to check initial values for $C(z) = z + 2C(z)T(z)$.

2. Read pages 89-147 in text.



3. Write up solutions to Exercises 3.20 and 3.28.

A N A L Y T I C   C O M B I N A T O R I C S
P A R T   O N E



http://aofa.cs.princeton.edu

3. Generating Functions