Chapter 5

Linear Differential Equations

5.1 Introduction

Linear differential equations occur in the study of many practical problems in science and engineering. Constant coefficient equations arise in the theory of electric circuits, vibrations etc. Variable coefficient equations arise in many areas of physics, electric circuits, mathematical modelling of physical problems etc. Some of the important variable coefficient differential equations are Bessel equation, Legendre equation, Chebyshev equation etc. The solution of constant coefficient equations can be obtained in terms of known standard functions. However, no such solution procedure exists for variable coefficient equations. Often, we attempt their solution in the form of an infinite series. These solutions may sometimes reduce to known standard functions.

A linear ordinary differential equation of order n, is written as

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = r(x)$$

or

$$a_0(x)y^{(n)}(x) + a_1(x)y^{(n-1)}(x) + \dots + a_{n-1}(x)y'(x) + a_n(x)y(x) = r(x) \tag{5.1}$$

where y is the dependent variable and x is the independent variable and $a_0(x) \neq 0$. If $r(x) = 0$, then it is called a *homogeneous equation*, otherwise it is called a *non-homogeneous equation*. For example, a second order homogeneous equation is of the form

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0, \quad a_0(x) \neq 0 \tag{5.2}$$

and a non-homogeneous second order equation is of the form

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = r(x), \quad a_0(x) \neq 0. \tag{5.3}$$

If $a_i(x)$, $i = 0, 1, 2$ are constants then the equations are linear second order constant coefficient equations. A few examples of linear second order equations are

$$y'' + 4y' + 3y = x^2e^x, \tag{5.4}$$

$$y'' + 2y' + y = \sin x, \tag{5.5}$$

$$x^2y'' + xy' + (x^2 - 4)y = 0, \tag{5.6}$$

$$(1 - x^2)y'' - 2xy' + 20y = 0. \tag{5.7}$$

Equations (5.4), (5.5) are constant coefficient second order equations and Eqs. (5.6), (5.7) are variable coefficient second order equations.

5.2 Solutions of Linear Differential Equations

We assume that x in Eq. (5.1) varies on some interval I, where the interval may be open, closed, semi-open or infinite. For example, the differential equation may be valid for all $x \in (0, \infty)$ or $x \in (-\infty, \infty)$. If $y_1(x)$ is a solution of the Eq. (5.1), then it must identically satisfy the equation. Hence, $y_1(x)$ must be continuously differentiable $n - 1$ times and $y_1^{(n)}(x)$ must be continuous on I.

We now state an important result regarding the uniqueness of solutions.

Theorem 5.1 If the functions $a_0(x)$, $a_1(x)$, . . . , $a_n(x)$ and $r(x)$ are continuous over I and $a_0(x) \neq 0$ on I, then there exists a unique solution to the initial value problem

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1} y' + a_n y = r(x) \tag{5.8}$$

$$y(x_0) = c_1,\ y'(x_0) = c_2,\ \ldots,\ y^{(n-1)}(x_0) = c_n \tag{5.9}$$

where $x_0 \in I$, and $c_1, c_2, \ldots, c_n$ are n known constants.

This theorem does not give us a procedure to find the solutions but guarantees that there exists a unique solution if the conditions stated in the theorem are satisfied.

If the conditions of the Theorem 5.1 are satisfied, then the differential Equation (5.8) is said to be *normal* on I (these conditions are both necessary and sufficient for the differential equation to be normal).

A point $x_0 \in I$, for which $a_0(x) \neq 0$, is called an *ordinary point* or a *regular point* of the differential equation.

Example 5.1 Find the intervals on which the following differential equations are normal.

(a) $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$, n an interger.

(b) $x^2y'' + xy' + (n^2 - x^2)y = 0$, n real.

(c) $\sqrt{x}\, y'' + 6xy' + 15y = \ln (x^4 - 256)$.

Solution

(a) Here, $a_0(x) = (1 - x^2)$, $a_1(x) = -2x$, and $a_2(x) = n(n + 1)$. Now, a_0, a_1 and a_2 are continuous everywhere in $(-\infty, \infty)$. Also, $a_0(x) = 1 - x^2 \neq 0$ for all $x \in (-\infty, \infty)$, except at the points $x = -1, 1$. Hence, the differential equation is normal on every subinterval I of the open intervals $(-\infty, -1)$, $(-1, 1)$ and $(1, \infty)$.

(b) Here, $a_0(x) = x^2$, $a_1(x) = x$, $a_2(x) = n^2 - x^2$. We find that a_0, a_1 and a_2 are continuous everywhere in $(-\infty, \infty)$. Also, $a_0(x) = x^2 \neq 0$ for all $x \in (-\infty, \infty)$ except at $x = 0$. Hence, the differential equation is normal on every subinterval I of the open intervals $(-\infty, 0)$, $(0, \infty)$.

(c) Here, $a_0(x) = \sqrt{x}$, $a_1(x) = 6x$, $a_2(x) = 15$, and $r(x) = \ln (x^4 - 256)$. Now, a_0, a_1, a_2 and $r(x)$ are continuous for all x satisfying $x > 4$. Hence, the differential equation is normal on every subinterval I of the open interval $(4, \infty)$.

Remark 1

If the functions $a_0(x)$, $a_1(x)$, . . . , $a_n(x)$ are continuous over I and $a_0(x) \neq 0$ on I, then the only solution of the homogeneous initial value problem

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1} y' + a_n y = 0 \tag{5.10}$$

$$y(x_0) = 0,\ y'(x_0) = 0, \ldots, y^{(n-1)}(x_0) = 0 \tag{5.11}$$

where $x_0 \in I$, is the trivial solution $y \equiv 0$ on I.

Linear combination of functions Let $f_1(x), f_2(x), \ldots, f_n(x)$ be n functions. Then

$$c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x), \text{ where } c_1, c_2, \ldots, c_n \text{ are constants}$$

is called a linear combination of the given functions.

We had earlier defined that a function $y_1(x)$ is a solution of a non-homogeneous or a homogeneous equation, if the equation reduces to an identity when $y_1(x)$ is substituted into it. Let $y_1(x), y_2(x), \ldots, y_m(x)$ be m solutions of the linear homogeneous equation (5.10). Then, we show in the following that the *superposition principle* or *linearity principle* holds.

Theorem 5.2 If $y_1(x), y_2(x), \ldots, y_m(x)$ are m solutions of the linear homogeneous equation (5.10) on I, then a linear combination of the solutions $c_1y_1(x) + c_2y_2(x) + \ldots + c_my_m(x)$, where $c_1, c_2, \ldots, c_m$ are constants is also a solution of Eq. (5.10) on I.

Proof Substituting the linear combination $y = c_1y_1(x) + c_2y_2(x) + \ldots + c_my_m(x)$ into Eq. (5.10), we get

$$a_0(x)\frac{d^n}{dx^n}[c_1y_1(x) + c_2y_2(x) + \ldots + c_my_m(x)]$$

$$+ a_1(x)\frac{d^{n-1}}{dx^{n-1}}[c_1y_1(x) + c_2y_2(x) + \ldots + c_my_m(x)]$$

$$+ \ldots + a_{n-1}(x)\frac{d}{dx}[c_1y_1(x) + c_2y_2(x) + \ldots + c_my_m(x)]$$

$$+ a_n(x)[c_1y_1(x) + c_2y_2(x) + \ldots + c_my_m(x)]$$

$$= c_1\left[a_0(x)\frac{d^ny_1}{dx^n} + a_1(x)\frac{d^{n-1}y_1}{dx^{n-1}} + \ldots + a_{n-1}(x)\frac{dy_1}{dx} + a_n(x)y_1\right]$$

$$+ c_2\left[a_0(x)\frac{d^ny_2}{dx^n} + a_1(x)\frac{d^{n-1}y_2}{dx^{n-1}} + \ldots + a_{n-1}(x)\frac{dy_2}{dx} + a_n(x)y_2\right]$$

$$+ \ldots + c_m\left[a_0(x)\frac{d^ny_m}{dx^n} + a_1(x)\frac{d^{n-1}y_m}{dx^{n-1}} + \ldots + a_{n-1}(x)\frac{dy_m}{dx} + a_n(x)y_m\right]$$

$$= c_1[0] + c_2[0] + \ldots + c_m[0] = 0$$

since $y_1(x), y_2(x), \ldots, y_m(x)$ are solutions of the linear homogeneous equation.

Remark 2

Superposition principle does not hold for a non-homogeneous equation or a nonlinear equation.

Example 5.2 Show that e^{-x}, e^x and their linear combination $c_1e^{-x} + c_2e^x$ are solutions of the homogeneous equation $y'' - y = 0$.

Solution For $y_1 = e^{-x}$, we have $y_1' = -e^{-x}, y_1'' = e^{-x}, y_1'' - y_1 = 0$.

For $y_2 = e^x$, we have $y_2' = e^x, y_2'' = e^x, y_2'' - y_2 = 0$.

Hence, e^{-x} and e^x are solutions of $y'' - y = 0$.

Substituting $y = c_1e^{-x} + c_2e^{x} = c_1 y_1 + c_2 y_2$, we obtain

$$y'' - y = (c_1y_1 + c_2 y_2)'' - (c_1y_1 + c_2y_2) = c_1(y_1'' - y_1) + c_2(y_2'' - y_2) = c_1(0) + c_2(0) = 0.$$

5.2.1 Linear Independence and Dependence

Let $f_1(x), f_2(x), \ldots, f_n(x)$ be n functions. Then, these functions are said to be *linearly independent* on some interval I (where they are defined), if the equation

$$c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x) = 0 \tag{5.12}$$

implies $c_1 = 0 = c_2 = \ldots = c_n$.

These functions are said to be *linearly dependent* on I, if Eq. (5.12) holds for $c_1, c_2, \ldots, c_n$ not all zero. In this case, one or more functions can be expressed as a linear combination of the remaining functions. For example, if $c_1 \neq 0$, then

$$f_1(x) = -\frac{1}{c_1}[c_2 f_2(x) + \ldots + c_n f_n(x)].$$

Conversely, if any function $f_i(x)$ can be expressed as a linear combination of the functions $f_1, f_2, \ldots f_{i-1}, f_{i+1} \ldots, f_n$ then the given set of functions are linearly dependent.

Example 5.3 Show that the functions $f_1(x) = x^2, f_2(x) = x^3, f_3(x) = 6x^2 - x^3$ are linearly dependent on any interval I.

Solution We have $f_3(x) = 6x^2 - x^3 = 6f_1(x) - f_2(x)$. Hence, the given functions are linearly dependent on any interval I.

Example 5.4 Show that the functions $x^2 - 1$, $3x^2$ and $2 - 5x^2$ are linearly dependent.

Solution The given functions are linearly dependent if the equation

$$c_1(x^2 - 1) + c_2(3x^2) + c_3(2 - 5x^2) = 0 \tag{5.13}$$

holds for c_1, c_2, c_3 not all zero. We have from Eq. (5.13)

$$(c_1 + 3c_2 - 5c_3)x^2 + (2c_3 - c_1) = 0, \quad \text{for all } x.$$

We have, $c_1 + 3c_2 - 5c_3 = 0$, and $2c_3 - c_1 = 0$. The solution of these equations is $c_1 = 2c_3, c_2 = c_3$ where c_3 is arbitrary. For example, if $c_3 = 1$, then $c_1 = 2, c_2 = 1$ and $f_3(x) = -2f_1(x) - f_2(x)$. The given functions are linearly dependent.

Example 5.5 Show that the functions x, x^2, x^3 are linearly independent on any interval I.

Solution We have $f_1(x) = x$, $f_2(x) = x^2$, $f_3(x) = x^3$. Substituting in equation (5.12), we get $c_1x + c_2x^2 + c_3x^3 = 0$. For finding the values of the three constants, take three distinct arbitrary points $x_0, x_1, x_2 (\neq 0)$ on I. Hence

$$c_1x_0 + c_2x_0^2 + c_3x_0^3 = 0, c_1x_1 + c_2x_1^2 + c_3x_1^3 = 0, c_1x_2 + c_2x_2^2 + c_3x_2^3 = 0.$$

This system of homogeneous algebraic equations has a non-trivial solution if the determinant of the coefficient matrix vanishes, that is

$$\det = \begin{vmatrix} x_0 & x_0^2 & x_0^3 \\ x_1 & x_1^2 & x_1^3 \\ x_2 & x_2^2 & x_2^3 \end{vmatrix} = 0.$$

Evaluating the determinant, we have

$$\det = (x_1 - x_0)(x_2 - x_0)(x_2 - x_1)x_0x_1x_2.$$

Since x_0, x_1, x_2 are distinct, $\det \neq 0$. Therefore, the only solution is $c_1 = 0, c_2 = 0, c_3 = 0$. Hence, the given functions are linearly independent.

The procedure used in Example 5.5 is lengthy and a difficult one. It is not always possible to examine the linear dependence or independence in this way. A very elegant procedure to test the linear independence or dependence of a given set of functions is the application of *Wronskians.* Let $f_1(x), f_2(x), \ldots, f_n(x)$ be n functions. The Wronskian of these functions is denoted by $W(f_1, f_2, \ldots, f_n)$ and is defined by

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \cdots & \cdots & \cdots & \cdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix} = W(x). \tag{5.14}$$

The Wronskian of the n functions exists if all the functions $f_1, f_2, \ldots, f_n$ are differentiable $n - 1$ times on the interval I. If any one or more functions are not differentiable then the Wronskian does not exist.

We have the following result for testing the linear dependence or independence of the solutions of the linear homogeneous differential equation (5.10).

Theorem 5.3 If the coefficients $a_0(x), a_1(x), \ldots, a_n(x)$ in the linear homogeneous equation

$$a_0y^{(n)} + a_1y^{(n-1)} + \ldots + a_{n-1}y' + a_ny = 0, \quad a_0 \neq 0 \tag{5.15}$$

are continuous on I and $y_1(x), y_2(x), \ldots, y_n(x)$ are n solutions of this equation, then

(i) $W(x) = W(y_1, y_2, \ldots, y_n) \neq 0$ for all $x \in I \Leftrightarrow y_1(x), y_2(x), \ldots, y_n(x)$ are linearly independent on I,

(ii) $W(x_0) = 0$ where $x_0 \in I$ is any fixed point, implies $W(x) = 0$ for all x in I and the functions $y_1(x), y_2(x), \ldots, y_n(x)$ are linearly dependent.

Proof Let $y_1(x), y_2(x), \ldots, y_n(x)$ be linearly dependent on I. By definition, there exist constants $c_1, c_2, \ldots, c_n$ not all zero, such that

$$c_1y_1(x) + c_2y_2(x) + \ldots + c_ny_n(x) = 0, \text{ for all } x \in I. \tag{5.16}$$

Differentiating Eq. (5.16) successively, $n - 1$ times, we get

$$c_1y_1'(x) + c_2y_2'(x) + \ldots + c_ny_n'(x) = 0$$

$$c_1y_1''(x) + c_2y_2''(x) + \ldots + c_ny_n''(x) = 0$$

$$\cdots \quad \cdots$$

$$c_1y_1^{(n-1)}(x) + c_2y_2^{(n-1)}(x) + \ldots + c_ny_n^{(n-1)}(x) = 0. \tag{5.17}$$

Eqs. (5.16), (5.17) form a homogeneous, linear system of algebraic equations. Non-trivial solutions of the system exist if and only if the determinant of the coefficient matrix is zero for all $x \in I$. But this determinant is the Wronskian $W(x)$ of the solutions. Hence, if the solutions are dependent then $W(x) = 0$ for all $x \in I$.

Let now $W(x_0) = 0$ for some fixed point $x_0 \in I$. Then, the system of equations given by (5.16), (5.17) has a nontrivial solution $c_1 = c_1^*, c_2 = c_2^*, \ldots, c_n = c_n^*$ not all zero. Hence, $y^*(x) = c_1^* y_1(x) + c_2^* y_2(x) + \ldots + c_n^* y_n(x)$ is a solution of the linear homogeneous Equation (5.15). Using Eqs. (5.16) and (5.17), we find that $y^*(x)$ also satisfies the initial conditions $y^*(x_0) = 0$, $(y^*)'(x_0) = 0$, $\ldots$, $(y^*)^{(n-1)}(x_0) = 0$. Now, the differential equation (5.15) and these conditions form a homogeneous initial value problem. Hence, $y^*(x) \equiv 0$ is the solution of the initial value problem. Since the solution of the initial value problem is unique, we obtain $y^*(x) = y(x) = 0$, or, for all x, $c_1^* y_1(x) + c_2^* y_2(x) + \ldots + c_n^* y_n(x) = 0$, where not all c_i are zero. Hence, the solutions are dependent. Since x_0 is arbitrary, $W(x_0) = 0$ for some $x_0 \in I$ implies $W(x) = 0$ for all $x \in I$.

We now define the general solution of the homogeneous Eq. (5.15).

Theorem 5.4 If the coefficients $a_0(x), a_1(x), \ldots, a_n(x)$, $a_0(x) \neq 0$, in the linear homogeneous equation (5.15) are continuous on I, then the equation (5.15) has n linearly independent solutions. If $y_1(x)$, $y_2(x), \ldots, y_n(x)$ are n linearly independent solutions, then the general solution is given by $y(x) = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x)$, that is, their linear combination.

The n linearly independent solutions $y_1(x), y_2(x), \ldots, y_n(x)$ are also called the *fundamental solutions* of Eq. (5.15) on I. This set of fundamental solutions forms a *basis* of the nth order linear homogeneous equation.

Example 5.6 Show that the functions x, x^2, x^3 are linearly independent on any interval I, not containing zero (see Example 5.5).

Solution The Wronskian of the functions is

$$W(x) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = x(12x^2 - 6x^2) - (6x^3 - 2x^3) = 2x^3.$$

Therefore, $W(x) \neq 0$ on any interval not containing zero. Hence, the functions are linearly independent in $(-\infty, 0)$, $(0, \infty)$.

Example 5.7 Show that the functions 1, $\sin x$, $\cos x$ are linearly independent.

Solution The Wronskian of the functions is

$$W(x) = \begin{vmatrix} 1 & \sin x & \cos x \\ 0 & \cos x & -\sin x \\ 0 & -\sin x & -\cos x \end{vmatrix} = -1.$$

Hence, the given functions are linearly independent on any interval I.

Example 5.8 Show that e^x, e^{2x}, e^{3x} are the fundamental solutions of $y''' - 6y'' + 11y' - 6y = 0$, on any interval I.

Solution Substituting $y = e^x, e^{2x}, e^{3x}$, we find that they satisfy the differential equation

$$y''' - 6y'' + 11y' - 6y = 0.$$

The Wronskian of these functions is

$$W(x) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = e^{6x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = 2e^{6x} \neq 0.$$

Therefore, the solutions are linearly independent and they form a set of fundamental solutions on any interval I.

Example 5.9 Show that the set of functions $\{x, 1/x\}$ forms a basis of the equation $x^2y'' + xy' - y = 0$. Obtain a particular solution when $y(1) = 1$, $y'(1) = 2$.

Solution We have

$$y_1(x) = x, y_1' = 1, y_1'' = 0, \quad \text{and} \quad x^2y_1'' + xy_1' - y_1 = x - x = 0$$

$$y_2(x) = 1/x, y_2' = -1/x^2, y_2'' = 2/x^3$$

and
$$x^2y_2'' + xy_2' - y_2 = x^2\left(\frac{2}{x^3}\right) + x\left(-\frac{1}{x^2}\right) - \left(\frac{1}{x}\right) = 0.$$

Hence $y_1(x)$ and $y_2(x)$ are solutions of the given equation. The Wronskian is given by

$$W(y_1, y_2) = \begin{vmatrix} x & 1/x \\ 1 & -1/x^2 \end{vmatrix} = -\frac{2}{x} \neq 0, \quad \text{for } x \geq 1.$$

Therefore, the set $\{y_1(x), y_2(x)\}$ forms a basis of the equation. The general solution is

$$y(x) = c_1y_1(x) + c_2y_2(x) = c_1x + \frac{c_2}{x}.$$

Substituting in the given conditions, we get

$$y(1) = 1 = c_1 + c_2, \; y'(1) = 2 = c_1 - c_2.$$

Solving, we obtain $c_1 = 3/2$, $c_2 = -1/2$.

The particular solution is $y(x) = \frac{1}{2}\left(3x - \frac{1}{x}\right)$.

Exercise 5.1

From the following linear differential equations, find the constant coefficient and variable coefficient equations.

1. $y'' - a^2y = 0$.
2. $y' = y/x$.
3. $y''' + 3y'' + 6y' + 12y = x^2$.
4. $x^3y''' + 9x^2y'' + 18xy' + 6y = 0$.
5. $(1 - x)y'' + xy' - y = 0$.
6. $y'' - (1 + x^2)y = 0$.

Find the intervals on which the following differential equations are normal.

7. $y' = 3y/x$.
8. $(1 + x^2)y'' + 2xy' + y = 0$.
9. $x^2y'' - 4xy' + 6y = x$.
10. $y'' + 3y' + \sqrt{x}\, y = \sin x$.
11. $y''' + 9y' + y = \log(x^2 - 9)$.
12. $y'' + |x|y' + y = x \ln x$.

13. $x(1-x)y'' - 3xy' - y = 0$.

14. $y'' + xy' + 6y = \ln \sin (\pi x/4)$.

15. Verify that $y = x^2$ is a solution of $x^2y'' + xy' - 4y = 0$, $x \in (0, \infty)$ and satisfies the conditions $y(0) = 0$, $y'(0) = 0$. Does Theorem 5.1 guarantee the existence and uniqueness of such a solution? Is the Remark 1 applicable in this case?

16. By inspection find a solution of $x^2y'' + xy' - y = 0$, $x \in (-\infty, \infty)$ which satisfies the conditions $y(0) = 0$, $y'(0) = 2$. Does Theorem 5.1 guarantee the existence and uniqueness of such a solution?

17. Show that

$$y_1(x) = x^3 - x^2, \; -3 \le x \le 3, \quad \text{and} \quad y_2(x) = \begin{cases} x^2 - x^3, & -3 \le x \le 0, \\ x^3 - x^2, & 0 \le x \le 3 \end{cases}$$

both satisfy the differential equation $x^2y'' - 4xy' + 6y = 0$ and the conditions $y(2) = 4$, $y'(2) = 8$. But $y_1(x)$, $y_2(x)$ are different. Does this contradict Theorem 5.1?

Verify that the given functions are solutions of the associated differential equation. Verify also that a linear combination of these functions is also a solution.

18. $1, x, e^x$; $y''' - y'' = 0$.

19. e^x, e^{-2x}; $y'' + y' - 2y = 0$.

20. $e^{-x} \cos 2x$, $e^{-x} \sin 2x$; $y'' + 2y' + 5y = 0$.

Examine whether the following functions are linearly independent for $x \in (0, \infty)$.

21. $2x$, $6x + 3$, $3x + 2$.

22. $x^2 - x$, $3x^2 + x + 1$, $9x^2 - x + 2$.

23. $x^2 - 2x$, $3x^2 + x + 2$, $4x^2 - x + 1$.

24. $\sin x$, $\sin 2x$, $\sin 3x$.

25. 1, $\cos x$, $\sin x$.

26. e^x, $\sinh x$, $\cosh x$.

27. x^2, $1/x^2$.

28. $\ln x$, $\ln x^2$, $\ln x^3$.

29. $x - 1$, $x + 1$, $(x - 1)^2$.

30. e^{-x}, $\sinh x$, $\cosh x$.

31. Find the intervals on which the three functions 1, $\cos x$, $\sec x$, $x > 0$ are linearly independent.

32. Determine how many of the given functions are linearly independent on $[0, 1]$.

(i) 1, $1 + x$, x^2, $x(1 - x)$, x; (ii) $1 + x$, $1 - x$, 1, x^2, $1 + x^2$.

33. Show that $y_1(x) = \sin x$, and $y_2(x) = 4 \sin x - 2 \cos x$ are linearly independent solutions of $y'' + y = 0$. Write the solution $y_3(x) = \cos x$ as a linear combination of y_1 and y_2.

34. Let $a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$ be a second order differential equation. Let $a_0(x)$, $a_1(x)$, $a_2(x)$ be continuous and $a_0(x) \ne 0$ on I and $y_1(x)$, $y_2(x)$ be two linearly independent solutions. Show that the Wronskian of $y_1(x)$, $y_2(x)$ satisfies the differential equation $a_0(x)W'(x) + a_1(x)W(x) = 0$. Also, show that the Wronskian is given by

$$W(x) = c\, e^{-\int [a_1(x)/a_0(x)]dx}.$$

(This is called the *Abel's formula*).

35. Show that $\cos at$, $\sin at$ are solutions of the equation $y'' + a^2y = 0$, $a \ne 0$ on any interval. Show that they are independent. Use the result (Abel's formula) given in Problem 34 and find the Wronskian. Are the two Wronskians same?

36. Show that e^{2x} and xe^{2x} are solutions of the equation $y'' - 4y' + 4y = 0$ on any interval. Show that they are independent. Use the result given in problem 34 and find the Wronskian. Are the two Wronskians same?

Show that in the following problems, $\{y_i(x)\}$ forms a set of fundamental solutions (basis) to the corresponding differential equation.

37. $x^{1/4}$, $x^{5/4}$; $16x^2y'' - 8xy' + 5y = 0$, $x > 0$.

38. $e^{2x} \cos 3x$, $e^{2x} \sin 3x$; $2y'' - 8y' + 26y = 0$.
39. $1, x^2$; $x^2y'' - xy' = 0$, $x > 0$.
40. e^x, e^{2x}, e^{-3x}; $y''' - 7y' + 6y = 0$.
41. $e^x, e^x \cos x$, $e^x \sin x$; $y''' - 3y'' + 4y' - 2y = 0$.
42. $e^{2x}, e^{-x} \cos(\sqrt{3}x)$, $e^{-x} \sin(\sqrt{3}x)$; $y''' - 8y = 0$.
43. $\sin(\ln x^2)$, $\cos(\ln x^2)$; $x^2y'' + xy' + 4y = 0$, $x > 0$.
44. Let the coefficients $a_0(x), a_1(x), a_2(x)$ in the equation $a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$ be continuous and $a_0(x) \neq 0$ on I. Let $\{y_1(x), y_2(x)\}$ be the basis (set of fundamental solutions) of the equation. Show that the set $\{u(x), v(x)\}$ such that $u = ay_1(x) + by_2(x)$, $v = cy_1(x) + dy_2(x)$, is also a basis of the equation if $ad - bc \neq 0$. If $y_1(x) = \cosh kx$, $y_2 = \sinh kx$, obtain a simple form of u and v.
45. Let $y_1(x), y_2(x)$ be the linearly independent solutions of the equation $y'' + a(x)y' + b(x)y = 0$ on I. Show that there is no point $x_0 \in I$ at which (i) both $y_1(x), y_2(x)$ vanish, (ii) both $y_1(x), y_2(x)$ take extreme values.
46. Let $\{y_1(x), y_2(x)\}$ be the basis of the equation $y'' + a(x)y' + b(x)y = 0$. Show that the equation can be written as the Wronskian $W(y, y_1, y_2) = 0$.
47. Let $y_1(x)$ be a solution of the homogeneous equation $y'' + a(x)y' + b(x)y = 0$, on the interval $I: \alpha \le x \le \beta$. The coefficients $a(x)$ and $b(x)$ are continuous on I. If the curve $y = y_1(x)$ is tangential to the x-axis at a point x_1 in I, then prove that $y_1(x) \equiv 0$.

Using the problem 46, find a differential equation of the form $y'' + a(x)y' + b(x)y = 0$ for which the following functions are solutions.

48. e^{3x}, e^{-2x}. 49. $e^{-(\alpha + i\omega)x}, e^{-(\alpha - i\omega)x}$. 50. e^{5x}, xe^{5x}.

5.3 Methods for Solution of Linear Equations

In this section, we shall discuss various methods of finding solution of linear equations. We first define the differential operator D.

5.3.1 Differential Operator D

Sometimes, it is convenient to write the given linear differential equation in a simple form using the differential operator $D = d/dx$. We define an operator T as a transformation $T: V \to W$ that transforms a function f in V into another function $T(V)$ in W. Let the operator D be defined, over the set V_1 of all differentiable functions f on I, by $D = d/dx$.

Then, we write
$$Df(x) = Df = (Df)(x) = \frac{df}{dx} = f'. \tag{5.18}$$

We have, for example $D(x^n) = \dfrac{d}{dx}(x^n) = nx^{n-1}$, n constant; $D(\cos x) = -\sin x$, etc.

Let $f(x)$ and $g(x)$ be differentiable functions. Since D is a linear operator, we have
$$D(af + bg) = aDf + bDg, \quad a, b \text{ constants.}$$

We also have for $f \in V_2$, the set of functions having a second derivative on I
$$D(Df) = D(f') = \frac{d}{dx}(f') = f''.$$

We simply write $D(Df) = D(D)f = D^2f$ so that

$$D^3(f) = D(D^2 f) = D(f'') = f''', \ldots, D^k(f) = f^{(k)}.$$

where f is sufficiently differentiable.

We define $D^0 = 1$, so that if 1 is the operator defined by $1(f) = f$, we have $D^0(f) = 1(f) = f$. We, now define the operator L by

$$L = a_0(x)\frac{d^n}{dx^n} + a_1(x)\frac{d^{n-1}}{dx^{n-1}} + \ldots + a_{n-1}(x)\frac{d}{dx} + a_n(x)$$

$$= a_0(x)D^n + a_1(x)D^{n-1} + \ldots + a_{n-1}(x)D + a_n(x) = P(D) \tag{5.19}$$

which is a polynomial in D, so that

$$Ly = a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y$$

$$= a_0(x)D^n y + a_1(x)D^{n-1}y + \ldots + a_{n-1}(x)Dy + a_n(x)y$$

$$= [a_0(x)D^n + a_1(x)D^{n-1} + \ldots + a_{n-1}(x)D + a_n(x)]y = P(D)y. \tag{5.20}$$

For example, the differential equation

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$$

can be written as

$$Ly = (D^2 + 5D + 6)y = 0 \tag{5.21}$$

where the operator L is given by $L = P(D) = D^2 + 5D + 6$.

Similarly, the equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = x^2$$

can be written as

$$Ly = (D^2 + 2D + 2)y = x^2 \tag{5.22}$$

where the operator L is defined by $L = P(D) = D^2 + 2D + 2$.

Suppose $y = e^{mx}$. Then, $D(y) = D(e^{mx}) = me^{mx}$ and $D^2(y) = m^2e^{mx}$. Substituting in Eq. (5.21), we obtain

$$Ly = (D^2 + 5D + 6)y = (m^2 + 5m + 6)y = P(m)y$$

using Eq. (5.19). Therefore,

$$P(D)y = (D^2 + 5D + 6)e^{mx} = P(m)y.$$

In general, substituting $y = e^{mx}$ in the equation (5.20), we get

$$P(D)y = (a_0D^n + a_1D^{n-1} + \ldots + a_{n-1}D + a_n)e^{mx}$$

$$= (a_0m^n + a_1m^{n-1} + \ldots + a_{n-1}m + a_n)e^{mx} = P(m)e^{mx}. \tag{5.23}$$

When a_i, $i = 0, 1, \ldots, n$ are constants, the operator $L = P(D)$ can often be factorised. For example, we have

(i) $D^2 + 5D + 6 = (D + 2)(D + 3)$

(ii) $D^3 - 6D^2 + 11D - 6 = (D - 1)(D - 2)(D - 3)$.

When a_i are functions of x, factorisation is often not possible. For example, $x^2Dy \neq D(x^2y)$, or in general $a(x)Dy \neq D[a(x)y]$, since the right hand side is $D[a(x)y] = a(x)y' + a'(x)y$.

5.3.2 Solution of Second Order Linear Homogeneous Equations with Constant Coefficients

Consider the linear homogeneous second order equation

$$ay'' + by' + cy = 0, \quad a, b, c \text{ are constants.} \tag{5.24}$$

In the operator notation, we write the equation as

$$Ly = P(D)y = aD^2y + bDy + cy = (aD^2 + bD + c)y = 0. \tag{5.25}$$

In the previous chapter, we have shown that the solution of the first order equation $y' + my = 0$ is $y = e^{-mx} + c$; and the solution of the equation $y' - my = 0$ is $y = e^{mx} + c$. Therefore, it is natural to try for a particular solution of the form $y = e^{mx}$, for Eq. (5.25), where m is an unknown constant to be determined. Since $y' = me^{mx}$, $y'' = m^2e^{mx}$, we obtain from Eq. (5.24)

$$(am^2 + bm + c)e^{mx} = 0.$$

Since $e^{mx} \neq 0$, we obtain

$$am^2 + bm + c = 0. \tag{5.26}$$

This is an algebraic equation in m. It is called the *characteristic equation* or the *auxiliary equation* of the linear homogeneous equation (5.24) (we can write the characteristic equation by replacing y'' by m^2, y' by m and y by 1 in Eq. (5.24) implicitly noting that solutions of the form e^{mx} are being determined). The roots of this equation are called the *characteristic roots*. The quadratic equation (5.26) has the roots

$$m = [-b \pm \sqrt{b^2 - 4ac}\,]/2a.$$

We have the following three cases.

(i) The roots are real and distinct, say $m = m_1, m_2$; $m_1 \neq m_2$ if $b^2 - 4ac > 0$.

(ii) The roots are real and equal, say $m = m_1, m_1$ if $b^2 - 4ac = 0$.

(iii) The roots are complex if $b^2 - 4ac < 0$.

To find the complete solution in the above three cases, we proceed as follows.

Real and distinct roots

Let the distinct roots be $m = m_1$ and $m = m_2$. Then, we obtain two solutions of the equation (5.24) as e^{m_1x} and e^{m_2x}. The two solutions are linearly independent on any interval I, since the Wronskian,

$$W(y_1, y_2) = y_1y_2' - y_2y_1' = m_2e^{m_1x}e^{m_2x} - m_1e^{m_1x}e^{m_2x}$$

$$= (m_2 - m_1)e^{(m_1+m_2)x} \neq 0.$$

Hence, the general solution of Eq. (5.24) is

$$y(x) = c_1e^{m_1x} + c_2e^{m_2x}. \tag{5.27}$$

Example 5.10 Find the solution of the differential equation $y'' - y' - 6y = 0$.

Solution Substituting $y = e^{mx}$, we obtain the characteristic equation as

$$m^2 - m - 6 = 0, \quad \text{or} \quad (m - 3)(m + 2) = 0, \quad \text{or} \quad m = -2, 3.$$

The two linearly independent solutions are e^{3x} and e^{-2x}. The general solution is

$$y(x) = Ae^{3x} + Be^{-2x}.$$

Example 5.11 Solve the initial value problem

$$4y'' - 8y' + 3y = 0, \; y(0) = 1, \; y'(0) = 3.$$

Solution Substituting $y = e^{mx}$, we obtain the characteristic equation as

$$4m^2 - 8m + 3 = 0, \quad \text{or} \quad m = 1/2, 3/2.$$

Hence, the linearly independent solutions are $e^{x/2}$ and $e^{(3x)/2}$. The general solution is

$$y(x) = Ae^{(3x)/2} + Be^{x/2}.$$

Substituting the initial conditions, we get

$$y(0) = 1 = A + B, \quad y'(0) = 3 = \frac{3A}{2} + \frac{B}{2}.$$

Solving the above equations, we get $A = 5/2$ and $B = -3/2$. The solution of the initial value problem is $y(x) = [5e^{(3x)/2} - 3e^{x/2}]/2$.

Real and equal roots

Real and equal roots are obtained for the characteristic equation (5.26) when $b^2 - 4ac = 0$. In this case the repeated root is $m = -b/(2a)$. This value gives one solution as $y_1(x) = e^{mx} = e^{-(bx)/(2a)}$. We need to determine another linearly independent solution $y_2(x)$, so that $\{y_1(x), y_2(x)\}$ forms a basis for the equation. The second solution $y_2(x)$ can be determined in a number of ways. We shall show that if m is a repeated root then e^{mx} and xe^{mx} are the two linearly independent solutions.
For $y_2(x) = xe^{mx}$, $m = -b/(2a)$, we have

$$y_2 = xe^{mx}, \; y_2' = (1 + mx)e^{mx}, \; y_2'' = (2 + mx)me^{mx}.$$

Substituting in Eq. (5.24), we get

$$[ma(2 + mx) + b(1 + mx) + cx]e^{mx} = 0$$

or

$$[(2ma + b) + (am^2 + bm + c)x]e^{mx} = 0.$$

Since $2ma + b = -b + b = 0$ and $am^2 + bm + c = 0$, this equation is automatically satisfied. Therefore, xe^{mx} is also a solution. Since e^{mx} and xe^{mx} are linearly independent, they form a set of the two fundamental solutions. Hence, the general solution is

$$y(x) = Ae^{mx} + Bxe^{mx} = (A + Bx)e^{mx}, \quad m = -b/(2a). \tag{5.28}$$

Alternative We can use the following method (*reduction of order*) to find the second linearly independent solution.

Let

$$y_2(x) = u(x)\,y_1(x)$$

where $y_1(x) = e^{mx}$, $m = -b/(2a)$ be a solution of Eq. (5.24). We have

$$y_2' = uy_1' + u'y_1,\ y_2'' = uy_1'' + 2u'y_1' + u''y_1.$$

Substituting in the differential equation, we obtain

$$a(uy_1'' + 2u'y_1' + u''y_1) + b(uy_1' + u'y_1) + cuy_1$$
$$= ay_1u'' + (2ay_1' + by_1)u' + (ay_1'' + by_1' + cy_1)u = 0 \tag{5.29}$$

Since $y_1(x)$ is a solution, we have $ay_1'' + by_1' + cy_1 = 0$.

Also, $$2ay_1' + by_1 = 2a\left(-\frac{b}{2a}\right)e^{-(bx)/(2a)} + be^{-(bx)/(2a)} = 0.$$

Hence, Eq. (5.29) reduces to $ay_1u'' = 0$. Since $a \neq 0$, $y_1 \neq 0$, we get $u'' = 0$, whose solution is $u = c_1x + c_2$. Therefore, $y_2(x) = (c_1x + c_2)y_1(x) = c_1xy_1(x) + c_2y_1(x)$. Since $y_1(x)$ is a solution, the second linearly independent solution is $xy_1(x)$, (note that a linear combination of the two linearly independent solutions is also a solution). The general solution is

$$y(x) = Ay_1(x) + Bxy_1(x) = (A + Bx)e^{mx},\ m = -b/(2a)$$

which is same as Eq. (5.28).

Alternative The second linearly independent solution can be determined by factorising the differential operator and reducing the given second order equation to a first order equation. We have

$$(aD^2 + bD + c)y = a\left[D^2 + \frac{b}{a}D + \frac{c}{a}\right]y = 0, \quad a \neq 0.$$

Since $b^2 - 4ac = 0$ and $m = m_1 = -b/(2a)$ is a repeated root, the operator is factorisable so that we can write the equation as

$$(D - m_1)(D - m_1)y = 0. \tag{5.30}$$

Set $(D - m_1)y = u$. Then, Eq. (5.30) reduces to $(D - m_1)u = 0$ or $u' - m_1u = 0$ whose solution is $u = c_1e^{m_1x}$. Substituting in the equation $(D - m_1)y = u$, we obtain

$$(D - m_1)y = y' - m_1y = u = c_1e^{m_1x}.$$

The integrating factor of this equation is e^{-m_1x}. Therefore, the solution of this equation is

$$ye^{-m_1x} = \int c_1e^{m_1x}e^{-m_1x}dx + c_2 = c_1x + c_2$$

or $$y = (c_1x + c_2)e^{m_1x}$$

which is same as $y_2(x)$ obtained in the previous case.

Example 5.12 Find the solution of the differential equation $4y'' + 4y' + y = 0$.

Solution Assume a solution of the form $y = e^{mx}$. The characteristic equation is given by

$$4m^2 + 4m + 1 = 0, \text{ or } (2m + 1)^2 = 0, \text{ or } m = -1/2, -1/2,$$

which is a repeated root. Hence, the general solution is $y(x) = (A + Bx)e^{-x/2}$.

Example 5.13 Solve the initial value problem

$$y'' + 6y' + 9y = 0,\ y(0) = 2,\ y'(0) = 3,$$

Solution Assume a solution of the form $y = e^{mx}$. The characteristic equation is given by

$$m^2 + 6m + 9 = 0, \text{ or } (m + 3)^2 = 0, \text{ or } m = -3, -3,$$

which is a repeated root. The general solution is $y(x) = (A + Bx)e^{-3x}$. Substituting in the initial conditions, we get

$$y(0) = 2 = A,\ y' = Be^{-3x} - 3(A + Bx)e^{-3x},\ y'(0) = 3 = B - 3A.$$

The solution is $A = 2$, $B = 9$. The solution of the given initial value problem is $y(x) = (2 + 9x)e^{-3x}$.

Example 5.14 Factorising the differential operator and reducing it into first order equations, solve the differential equation $y'' - 4y' - 5y = 0$.

Solution In the operator notation, the differential equation can be written as

$$(D^2 - 4D - 5)y = 0, \quad \text{or} \quad (D - 5)(D + 1)y = 0. \tag{5.31}$$

Set $(D + 1)y = u$. Then, we obtain from Eq. (5.31), $(D - 5)u = 0$. This is a first order equation whose solution is $u = Ae^{5x}$. Hence,

$$(D + 1)y = Ae^{5x}.$$

This is a first order linear equation, whose integrating factor is e^x. Hence, we have the solution as

$$e^x y = \int Ae^{6x}dx + B = \frac{A}{6}e^{6x} + B,$$

or
$$y = \frac{A}{6}e^{5x} + Be^{-x} = Ce^{5x} + Be^{-x}$$

where $C = A/6$ is an arbitrary constant.
We could have written Eq. (5.31) as $(D + 1)(D - 5)y = 0$ and obtain the same answer.

Example 5.15 Factorising the differential operator and reducing it to first order equations, solve the differential equation $4y'' + 12y' + 9y = 0$

Solution In the operator notation, the differential equation can be written as

$$(4D^2 + 12D + 9)y = (2D + 3)^2 y = 0. \tag{5.32}$$

Set $(2D + 3)y = u$. Then, we obtain from Eq. (5.32),

$$(2D + 3)u = 0.$$

The solution of this equation is $u = Ae^{-(3x)/2}$. Therefore,

$$(2D + 3)y = Ae^{-(3x)/2}, \quad \text{or} \quad \left(D + \frac{3}{2}\right)y = \frac{A}{2}e^{-(3x)/2}.$$

This is a linear first order equation whose integrating factor is $e^{(3x)/2}$. The solution is given by

$$ye^{(3x)/2} = \int \frac{A}{2}dx + B = \frac{Ax}{2} + B, \quad \text{or} \quad y = (Cx + B)e^{-(3x)/2},$$

where $C = A/2$.

Complex roots

When $b^2 - 4ac < 0$, then the roots of the characteristic equation (5.26) are complex. We have

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm i\sqrt{4ac - b^2}}{2a} = p \pm iq$$

where $p = -b/(2a)$ and $q = \sqrt{4ac - b^2}/(2a)$. Since the characteristic equation (5.26) has real coefficients, the complex roots occur in conjugate pairs and are of the form $p \pm iq$. Then, the solution of the equation can be written as

$$y(x) = Ae^{(p+iq)x} + Be^{(p-iq)x} = Ae^{px}e^{iqx} + Be^{px}e^{-iqx} = (Ae^{iqx} + Be^{-iqx})e^{px}$$

$$= [A(\cos qx + i \sin qx) + B(\cos qx - i \sin qx)]e^{px}$$

by the Euler formula. Simplifying, we obtain

$$y(x) = [c_1 \cos qx + c_2 \sin qx]e^{px} \tag{5.33}$$

where $c_1 = A + B$ and $c_2 = i(A - B)$. Therefore, the two linearly independent solutions are $y_1 = e^{px} \cos qx$ and $y_2 = e^{px} \sin qx$. The Wronskian is given by

$$W(y_1, y_2) = \begin{vmatrix} e^{px} \cos qx & e^{px} \sin qx \\ e^{px}(p \cos qx - q \sin qx) & e^{px}(p \sin qx + q \cos qx) \end{vmatrix} = q e^{2px} \neq 0$$

showing that $y_1(x)$ and $y_2(x)$ are linearly independent.

Example 5.16 Find the solution of the differential equation $y'' + 2y' + 2y = 0$.

Solution Assume a solution of the form $y = e^{mx}$. The characteristic equation is given by

$$m^2 + 2m + 2 = 0, \quad \text{or} \quad m = \frac{-2 \pm \sqrt{-4}}{2} = -1 \pm i = p \pm iq.$$

The general solution is

$$y(x) = (A \cos qx + B \sin qx)e^{px} = (A \cos x + B \sin x)e^{-x}.$$

Example 5.17 Find the solution of the initial value problem

$$y'' + 4y' + 13y = 0, \ y(0) = 0, \ y'(0) = 1.$$

Solution Assume a solution of the form $y = e^{mx}$. The characteristic equation is given by

$$m^2 + 4m + 13 = 0, \quad \text{or} \quad m = \frac{-4 \pm \sqrt{16 - 52}}{2} = -2 \pm 3i = p \pm iq.$$

The general solution is given by

$$y(x) = [A \cos qx + B \sin qx]e^{px} = [A \cos 3x + B \sin 3x]e^{-2x}.$$

Substituting in the initial conditions, we obtain

$$y(0) = 0 = A,$$

$$y'(x) = Be^{-2x}(3 \cos 3x - 2 \sin 3x), \quad y'(0) = 1 = 3B, \quad \text{or} \quad B = 1/3.$$

The solution of the initial value problem is

$$y(x) = (e^{-2x} \sin 3x)/3.$$

Example 5.18 Find all the non-trivial solutions, if any, of the boundary value problem

$$y'' + \omega^2 y = 0,\ y(0) = 0,\ y(l) = 0.$$

Solution Assume a solution of the form $y = e^{mx}$. The characteristic equation is given by

$$m^2 + \omega^2 = 0, \quad \text{or} \quad m = \pm\, i\omega.$$

The general solution is

$$y(x) = A \cos \omega x + B \sin \omega x. \tag{5.34}$$

Substituting in the boundary conditions, we obtain

$$y(0) = 0 = A,\ y(l) = 0 = B \sin (\omega l).$$

If $B = 0$, then we obtain the trivial solution $y = 0$.

For $B \neq 0$, we get $\sin \omega l = 0 = \sin n\pi$, $n = 1, 2, \ldots$

Therefore, $\omega = n\pi/l$. The general solution is

$$y_n(x) = B_n \sin [(n\pi x)/l],\ n = 1, 2, \ldots$$

where B_i's are arbitrary. There are infinite number of solutions. Since the boundary value problem is homogenous, by the superposition principle, the sum of these solutions is also a solution. Therefore, the general solution is given by

$$y(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right).$$

(The convergence of such an infinite series called the Fourier series, is discussed in chapter 9.)

5.3.3 Method of Reduction of Order for Variable Coefficient Linear Homogeneous Second Order Equations

Suppose that we know one of the solutions of the second order equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0, \quad a_0(x) \neq 0 \text{ on } I. \tag{5.35}$$

Then, we can obtain the second linearly independent solution by the method of reduction of order. Let $y = y_1(x)$ be a non-trivial solution of Eq. (5.35), that is

$$a_0(x)y_1'' + a_1(x)y_1' + a_2(x)y_1 = 0. \tag{5.36}$$

Then, we write the second solution as $y_2(x) = u(x)y_1(x)$. Since $u(x) = y_2(x)/y_1(x)$ is not a constant, y_1 and y_2 are two linearly independent solutions of Eq. (5.35). Now,

$$y_2' = u'y_1 + uy_1', \quad \text{and} \quad y_2'' = u''y_1 + 2u'y_1' + uy_1''.$$

Substituting in Eq. (5.35) and collecting the terms, we get

$$a_0(x)y_1u'' + [2a_0(x)\, y_1' + a_1(x)y_1]u' + [a_0(x)\, y_1'' + a_1(x)\, y_1' + a_2(x)y_1]u = 0.$$

Using Eq. (5.36), we obtain

$$a_0(x)y_1u'' + [2a_0(x)y_1' + a_1(x)y_1]u' = 0.$$

Now, let $v = u'$. Then, we have

$$a_0(x)y_1v' + [2a_0(x)y_1' + a_1(x)y_1]\,v = 0 \tag{5.37}$$

which is a first order equation in v.

Separating the variables, we obtain

$$\frac{v'}{v} = -\frac{(2a_0y_1' + a_1y_1)}{a_0y_1} = -\left[\frac{2y_1'}{y_1} + \frac{a_1}{a_0}\right].$$

Integrating, we obtain

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} y_1 & uy_1 \\ y_1' & uy_1' + y_1u' \end{vmatrix} = \begin{vmatrix} y_1 & uy_1 \\ y_1' & uy_1' + \dfrac{1}{y_1}e^{-\int p dx} \end{vmatrix} = e^{-\int p dx}.$$

where $p(x) = a_1(x)/a_0(x)$. Integrating $u' = v$, we obtain $u = \int v(x)\,dx$. The second linearly independent solution is given by $y_2(x) = u(x)y_1(x)$. It can be verified that the Wronskian of y_1, y_2 is equal to

$$W(y_1, y_2) = e^{-\int p(x)dx} \neq 0$$

showing that $y_1(x)$ and $y_2(x)$ are linearly independent.

Example 5.19 It is known that $1/x$ is a solution of the differential equation $x^2y'' + 4xy' + 2y = 0$. Find the second linearly independent solution and write the general solution.

Solution Write $y_2(x) = u(x)y_1(x) = u(x)/x$. Here, $p(x) = a_1(x)/a_0(x) = 4/x$. Hence,

$$v(x) = \frac{1}{y_1^2}e^{-\int p(x)dx} = x^2e^{-\int (4/x)dx} = x^2\left(\frac{1}{x^4}\right) = \frac{1}{x^2}.$$

$$u(x) = \int v(x)\,dx = \int \frac{dx}{x^2} = -\frac{1}{x}, \text{ and } y_2(x) = u(x)y_1(x) = -\frac{1}{x^2}.$$

The general solution is $y(x) = Ay_1(x) + By_2(x) = \dfrac{A}{x} + \dfrac{B}{x^2}$.

Exercise 5.2

Show that the given set of functions $\{y_1(x), y_2(x)\}$ forms a basis of the equation and hence solve the initial value problem.

1. e^x, e^{4x}, $y'' - 5y' + 4y = 0$, $y(0) = 2$, $y'(0) = 1$.
2. e^{2x}, e^{-2x}, $y'' - 4y = 0$, $y(0) = 1$, $y'(0) = 4$.
3. e^{-3x}, xe^{-3x}, $y'' + 6y' + 9y = 0$, $y(0) = 1$, $y'(0) = 2$.
4. $x^2, 1/x^2$, $x^2y'' + xy' - 4y = 0$, $y(1) = 2$, $y'(1) = 6$.
5. $x, x\ln x$, $x^2y'' - xy' + y = 0$, $y(1) = 3$, $y'(1) = 4$.

Find a general solution of the following differential equations.

6. $y'' - 4y = 0$.
7. $y'' - y' - 2y = 0$.
8. $y'' + y' - 2y = 0$.
9. $y'' - 4y' - 12y = 0$.

10. $y'' + 4y' + y = 0$.

11. $4y'' - 9y' + 2y = 0$.

12. $4y'' + 8y' - 5y = 0$.

13. $y'' + 2y' + y = 0$.

14. $y'' + 2\pi y' + \pi^2 y = 0$.

15. $9y'' - 12y' + 4y = 0$.

16. $4y'' + 4y' + y = 0$.

17. $25y'' - 20y' + 4y = 0$.

18. $y'' + 25y = 0$.

19. $y'' + 4y' + 5y = 0$.

20. $y'' - 2y' + 2y = 0$.

21. $(4D^2 - 4D + 17)y = 0$.

22. $(D^2 - 6D + 18)y = 0$.

23. $(D^2 + 9D)y = 0$.

24. $[D^2 - 2aD + (a^2 + b^2)]y = 0$.

Find a differential equation of the form $ay'' + by' + cy = 0$, for which the following functions are solutions.

25. e^{3x}, e^{-2x}.

26. $e^{x/4}, e^{-(3x)/4}$.

27. $1, e^{-2x}$.

28. e^{2x}, xe^{2x}.

29. e^{-x}, xe^{-x}.

30. e^{-3ix}, e^{3ix}.

31. $e^{-(a+ib)x}, e^{-(a-ib)x}$.

32. $e^{(5+3i)x}, e^{(5-3i)x}$.

Solve the following initial value problems.

33. $y'' - y = 0$, $y(0) = 0$, $y'(0) = 2$.

34. $y'' - y' - 12y = 0$, $y(0) = 4$, $y'(0) = -5$.

35. $y'' + y' - 2y = 0$, $y(0) = 0$, $y'(0) = 3$.

36. $\dfrac{d^2\theta}{dt^2} + g\theta = 0$, g constant, $\theta(0) = a$, constant, $\dfrac{d\theta}{dt}(0) = 0$.

37. $y'' - 4y' + 5y = 0$, $y(0) = 2$, $y'(0) = -1$.

38. $25y'' - 10y' + 2y = 0$, $y(0) = 1$, $y'(0) = 0$.

39. $4y'' + 12y' + 9y = 0$, $y(0) = -1$, $y'(0) = 2$.

40. $9y'' + 6y' + y = 0$, $y(0) = 0$, $y'(0) = 1$.

Solve the following boundary value problems.

41. $y'' + 25y = 0$, $y(0) = 1$, $y(\pi) = -1$.

42. $y'' - 36y = 0$, $y(0) = 2$, $y(1/6) = 1/e$.

43. $y'' + 2y' + 2y = 0$, $y(0) = 1$, $y(\pi/2) = e^{-\pi/2}$.

44. $9y'' - 6y' + y = 0$, $y(1) = e^{1/3}$, $y(2) = 1$.

45. $y'' - 4y' + 3y = 0$, $y(0) = 1$, $y(1) = 0$.

46. Verify that $(D - 2)(D + 3) \sin x = (D + 3)(D - 2) \sin x = (D^2 + D - 6) \sin x$.

47. Show that $x^2 Dy \neq D(x^2 y)$.

48. Find the conditions under which the following equations hold.

   (i) $(D + a)[D + b(x)]f(x) = [D + b(x)][D + a]f(x)$, a constant.

   (ii) $[D + a(x)][D + b(x)]f(x) = [D + b(x)][D + a(x)]f(x)$.

Factorize the operator and find the solution of the following differential equations using the method of reduction of order or by the direct method.

49. $(D^2 + 5D + 4)y = 0$.

50. $(4D^2 + 8D + 3)y = 0$.

51. $(4D^2 + 12D + 9)y = 0$.

52. $(D^2 + 6D + 9)y = 0$.

53. $(D^2 - 4)y = 0$.

54. $(9D^2 + 6D + 1)y = 0$.

55. The displacement $x(t)$ of a particle is governed by the differential equation $\ddot{x} + \dot{x} + bx = c\dot{x}$, $b > 0$. For what values of b and c is the motion of the particle oscillatory?

56. Find all non-trivial solutions of the boundary value problem

$$y'' + \omega^2 y = 0,\ y(0) = 0,\ y(\pi) = 0.$$

57. Find all the non-trivial solutions of the boundary value problem

$$y'' + \omega^2 y = 0,\ y'(0) = 0,\ y'(\pi) = 0.$$

58. Find all non-trivial solutions of the boundary value problem

$$y'' + \omega^2 y = 0,\ y(0) = 0,\ y'(\pi) = 0.$$

59. If $a^2 > 4b$, then show that the solution of the differential equation $y'' + ay' + by = 0$ can be expressed as $y(x) = e^{px}(A \cosh qx + B \sinh qx)$ where $p = -a/2$ and $q = \sqrt{a^2 - 4b}/2$.

60. The motion of a damped mechanical system is governed by the linear differential equation $m\ddot{y} + c\dot{y} + ky = 0$ in which m (mass), k (spring modulus), c (damping factor) are positive constants and dot denotes derivative with respect to time t. Discuss the behaviour of the general solution when $t \to \infty$ in the following three cases: (i) $c^2 > 4mk$ (over damping), (ii) $c^2 < 4mk$ (under damping), (iii) $c^2 = 4mk$ (critical damping). In each case, obtain the solution subject to the initial conditions $y(0) = 0$, $\dot{y}(0) = v_0$.

Find the solution of the following differential equations, if one of its solutions is known.

61. $y'' - y' - 6y = 0$, $y_1 = e^{-2x}$.

62. $y'' + 3y' - 4y = 0$, $y_1 = e^x$.

63. $(x^2 - 1)y'' - 2xy' + 2y = 0$, $y_1 = x$, $x \neq \pm 1$.

64. $x^2y'' + xy' + (x^2 - 1/4)y = 0$, $x > 0$, $y_1 = x^{-1/2} \sin x$.

65. $(x - 2)y'' - xy' + 2y = 0$, $x \neq 2$, $y_1 = e^x$.

5.3.4 Solution of Higher Order Homogeneous Linear Equations with Constant Coefficients

In this section, we shall extend the methods discussed in section 5.3.2, for the solution of higher order linear homogeneous equations with constant coefficients.

Consider the nth order homogeneous linear equation with constant coefficients

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1} y' + a_n y = 0. \tag{5.38}$$

We attempt to find a solution of the form $y = e^{mx}$, as in the case of second order equations. Substituting $y = e^{mx}$, $y^{(k)} = m^k e^{mx}$, $k = 1, 2, \ldots, n$ in Eq. (5.38) and cancelling e^{mx}, we obtain the characteristic equation as

$$a_0 m^n + a_1 m^{n-1} + \ldots + a_{n-1} m + a_n = 0. \tag{5.39}$$

The degree of this algebraic equation is same as the order of the differential equation. This equation has n roots. All the roots may be real and distinct, all or some of the roots may be equal, all or some of the roots may be complex. Consider the following cases.

Real and distinct roots

Let the polynomial equation (5.39) have all real and distinct roots as $m_1, m_2, \ldots m_n$. Then the n solutions

$$y_1(x) = e^{m_1 x},\ y_2(x) = e^{m_2 x},\ \ldots,\ y_n(x) = e^{m_n x} \tag{5.40}$$

are the linearly independent solutions of the differential equation (5.38). Since $m_1 \neq m_2 \neq \ldots \neq m_n$, it can be easily shown that the Wronskian of the solutions $y_1, y_2, \ldots, y_n$ given in Eq. (5.40) does not vanish and therefore they are linearly independent solutions.

Hence, the set of the solutions forms a basis and the general solution is given by

$$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots + c_n e^{m_n x}. \tag{5.41}$$

Example 5.20 Find the general solution of the differential equation

$$y''' - 2y'' - 5y' + 6y = 0.$$

Solution Substituting $y = e^{mx}$, we obtain the characteristic equation as

$$m^3 - 2m^2 - 5m + 6 = 0.$$

The roots of this equation are $m = 1, -2, 3$. Since the roots are real and distinct, the general solution of the equation is given by

$$y(x) = Ae^x + Be^{-2x} + Ce^{3x}.$$

Example 5.21 Solve the differential equation $y''' - y'' - 4y' + 4y = 0$.

Solution Substituting $y = e^{mx}$, we obtain the characteristic equation as

$$m^3 - m^2 - 4m + 4 = 0 \text{ or } (m - 1)(m^2 - 4) = 0.$$

The roots of this equation are $m = 1, -2, 2$ which are real and distinct. The general solution of the equation is given by

$$y(x) = Ae^x + Be^{-2x} + Ce^{2x}.$$

Example 5.22 Solve the differential equation $y^{iv} - 5y'' + 4y = 0$.

Solution Substituting $y = e^{mx}$, we obtain the characteristic equation as.

$$m^4 - 5m^2 + 4 = 0 \quad \text{or} \quad (m^2 - 4)(m^2 - 1) = 0.$$

The roots of this equation are $m = -1, 1, -2, 2$. The general solution is

$$y(x) = Ae^{-x} + Be^x + Ce^{-2x} + De^{2x}.$$

Example 5.23 Solve the differential equation $4y^{iv} - 12y''' - y'' + 27y' - 18y = 0$.

Solution Substituting $y = e^{mx}$, we obtain the characteristic equation as

$$4m^4 - 12m^3 - m^2 + 27m - 18 = 0.$$

We find that $m = 1$ is a root. We write the equation as

$$(m - 1)(4m^3 - 8m^2 - 9m + 18) = 0,\ (m - 1)(m - 2)(4m^2 - 9) = 0.$$

The roots of the characteristic equation are $m = 1, 2, 3/2, -3/2$. The general solution is

$$y(x) = Ae^x + Be^{2x} + Ce^{-3x/2} + De^{3x/2}.$$

Example 5.24 Solve the initial value problem

$$y''' - 6y'' + 11y' - 6y = 0,\ y(0) = 0,\ y'(0) = -4,\ y''(0) = -18.$$

Solution Substituting $y = e^{mx}$, we obtain the characteristic equation as

$$m^3 - 6m^2 + 11m - 6 = 0, \quad \text{or} \quad (m-1)(m-2)(m-3) = 0.$$

The roots of this equation are $m = 1, 2, 3$ and the general solution is

$$y(x) = Ae^x + Be^{2x} + Ce^{3x}.$$

Substituting the initial conditions, we get

$$y(0) = 0 = A + B + C,\ y'(0) = -4 = A + 2B + 3C,\ y''(0) = -18 = A + 4B + 9C.$$

Solving, we obtain $A = 1$, $B = 2$ and $C = -3$. Hence, the particular solution is $y(x) = e^x + 2e^{2x} - 3e^{3x}$.

Real multiple roots

The characteristic equation (5.39) may have some multiple roots. Let r be the multiplicity of the root m_1, that is the root $m = m_1$ is repeated r times. Let the remaining $n - r$ roots be real and distinct. Substituting $m = m_1$ we obtain $y_1(x) = e^{m_1 x}$ as one of the solutions. We shall now show that the remaining $r - 1$ linearly independent solutions corresponding to the multiple root $m = m_1$ are given by

$$xy_1, x^2y_1, \ldots, x^{r-1}y_1.$$

That is, the linearly independent solutions in this case are

$$e^{m_1 x}, xe^{m_1 x}, x^2e^{m_1 x}, \ldots, x^{r-1}e^{m_1 x} \tag{5.42}$$

since the Wronskian of these solutions $W \neq 0$.

If
$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1}y' + a_n y$$

then, substituting $y = e^{mx}$ in this equation, we get

$$L[e^{mx}] = [a_0 m^n + a_1 m^{n-1} + \ldots + a_n]e^{mx}$$
$$= (m - m_1)^r g(m)e^{mx},\ g(m_1) \neq 0 \tag{5.43}$$

since $m = m_1$ is a multiple root of multiplicity r. Consider now m as a parameter. Differentiating Eq. (5.43) with respect to m, we get

$$\frac{d}{dm} L[e^{mx}] = r(m - m_1)^{r-1} g(m)e^{mx} + (m - m_1)^r \frac{d}{dm}[g(m)e^{mx}].$$

Now, L is a linear differentiable operator with respect to the independent variable x. Since m and x are independent, we obtain

$$\frac{d}{dm} L[e^{mx}] = L\left[\frac{d}{dm} e^{mx}\right] = L[xe^{mx}]$$
$$= r(m - m_1)^{r-1} g(m)e^{mx} + (m - m_1)^r \frac{d}{dm}[g(m)e^{mx}]. \tag{5.44}$$

Since the right hand side of Eq. (5.44) vanishes at $m = m_1$, $xe^{m_1 x}$ is also a solution of the differential equation. Differentiating Eq. (5.44) with respect to m, we get

$$\frac{d}{dm} L[xe^{mx}] = L\left[\frac{d}{dm}(xe^{mx})\right] = L[x^2 e^{mx}]$$

$$= r(r-1)(m-m_1)^{r-2} g(m)e^{mx} + 2r(m-m_1)^{r-1} \frac{d}{dm}[g(m)e^{mx}]$$

$$+ (m-m_1)^r \frac{d^2}{dm^2}[g(m)e^{mx}]. \tag{5.45}$$

The right hand side of Eq. (5.45) vanishes at $m = m_1$ again. Hence, $x^2 e^{m_1 x}$ is also a solution. After $r-1$ differentiations, the first term on the right hand side is obtained as $r!(m-m_1)g(m)e^{mx}$ which vanishes for $m = m_1$. The other terms also vanish for $m = m_1$. Therefore, $x^{r-1}e^{m_1 x}$ is also a solution. If we differentiate one more time, that is r times, the first term on the right hand side becomes $r!g(m)e^{mx}$ which does not vanish at $m = m_1$, showing that $x^r e^{m_1 x}$ is not a solution. Hence, we find that $e^{m_1 x}, xe^{m_1 x}, x^2 e^{m_1 x}, \ldots, x^{r-1}e^{m_1 x}$ are the linearly independent solutions corresponding to the multiple root $m = m_1$. For example, if $m = m_1$ is a multiple root of order 3, then $e^{m_1 x}$, $xe^{m_1 x}$ and $x^2 e^{m_1 x}$ are the linearly independent solutions.

Example 5.25 Solve the differential equation $y''' - 3y' - 2y = 0$.

Solution Substituting $y = e^{mx}$, we obtain the characteristic equation as

$$m^3 - 3m - 2 = 0, \quad \text{or} \quad (m+1)(m^2 - m - 2) = 0$$

or $$(m+1)^2 (m-2) = 0, \quad \text{or} \quad m = -1, -1, 2.$$

Corresponding to the double root $m = -1$, the linearly independent solutions are e^{-x} and xe^{-x}. Hence, the general solution is

$$y(x) = Ae^{2x} + (Bx + C)e^{-x}.$$

Example 5.26 Solve the differentiable equation $8y''' - 12y'' + 6y' - y = 0$.

Solution Substituting $y = e^{mx}$, we obtain the characteristic equation as

$$8m^3 - 12m^2 + 6m - 1 = 0, \quad \text{or} \quad (2m-1)^3 = 0, \quad \text{or} \quad m = 1/2, 1/2, 1/2.$$

The general solution is $$y(x) = (A + Bx + Cx^2)e^{x/2}.$$

Example 5.27 Solve the initial value problem

$$y''' + 3y'' - 4y = 0, \; y(0) = 1, \; y'(0) = 0, \; y''(0) = 1/2.$$

Solution Substituting $y = e^{mx}$, we obtain the characteristic equation as

$$m^3 + 3m^2 - 4 = 0, \quad \text{or} \quad (m-1)((m^2 + 4m + 4) = 0, \quad \text{or} \quad (m-1)(m+2)^2 = 0.$$

The roots of this equation are $m = 1, -2, -2$. The general solution is

$$y(x) = Ae^x + (Bx + C)e^{-2x}.$$

Substituting in the initial conditions, we get

$$y(0) = 1 = A + C,$$

$$y'(x) = Ae^x + Be^{-2x} - 2(Bx + C)e^{-2x}, \; y'(0) = 0 = A + B - 2C,$$

$$y''(x) = Ae^x - 4Be^{-2x} + 4(Bx + C)e^{-2x}, \; y''(0) = \frac{1}{2} = A - 4B + 4C.$$

The solution of the system is $A = 1/2$, $B = 1/2$ and $C = 1/2$. The particular solution is

$$y(x) = [e^x + (x+1)e^{-2x}]/2.$$

Simple complex roots

Since the coefficients in the characteristic equation (5.39) are real, complex roots occur in conjugate pairs. That is, if $p + iq$ is a root, then $p - iq$ is also a root. In this case, the linearly independent solutions are given by $e^{px} \cos qx$ and $e^{px} \sin qx$. If the characteristic equation (5.39) has r complex conjugate pairs of roots $p_k \pm i\, q_k$, $k = 1, 2, \ldots r$, then the corresponding linearly independent solutions are $e^{p_1 x} \cos q_1 x$, $e^{p_1 x} \sin q_1 x$, $e^{p_2 x} \cos q_2 x$, $e^{p_2 x} \sin q_2 x, \ldots, e^{p_r x} \cos q_r x$ and $e^{p_r x} \sin q_r x$.

Example 5.28 Solve the differential equation $y^{iv} + 5y'' + 4y = 0$.

Solution Substituting $y = e^{mx}$, we obtain the characteristic equation as

$$m^4 + 5m^2 + 4 = 0, \quad \text{or} \quad (m^2 + 4)(m^2 + 1) = 0.$$

The roots are $m = \pm i, \pm 2i$. The general solution is

$$y(x) = A \cos x + B \sin x + C \cos 2x + D \sin 2x.$$

Example 5.29 Solve the initial value problem

$$y^{iv} + 2y''' + 11y'' + 18y' + 18 = 0, \; y(0) = 2, \; y'(0) = 3, \; y''(0) = -11, \; y'''(0) = -23.$$

Solution Substituting $y = e^{mx}$, we obtain the characteristic equation as

$$m^4 + 2m^3 + 11m^2 + 18m + 18 = 0 \quad \text{or} \quad (m^2 + 9)(m^2 + 2m + 2) = 0.$$

The roots are $m = \pm 3i, -1 \pm i$. The general solution is

$$y(x) = A \cos 3x + B \sin 3x + e^{-x} (C \cos x + D \sin x).$$

Substituting in the initial conditions, we get

$$\begin{aligned}
y(0) &= 2 = A + C,\\
y'(x) &= -3A \sin 3x + 3B \cos 3x + e^{-x} (-C \sin x + D \cos x - C \cos x - D \sin x),\\
y'(0) &= 3 = 3B + D - C,\\
y''(x) &= -9A \cos 3x - 9B \sin 3x\\
&\quad + e^{-x}[-(C + D) \cos x + (C - D) \sin x + (C + D) \sin x + (C - D) \cos x]\\
&= -9A \cos 3x - 9B \sin 3x + 2e^{-x} [C \sin x - D \cos x],\\
y''(0) &= -11 = -9A - 2D,\\
y'''(x) &= 27A \sin 3x - 27B \cos 3x + 2e^{-x} [C \cos x + D \sin x - C \sin x + D \cos x],\\
y'''(0) &= -23 = -27B + 2C + 2D.
\end{aligned}$$

Therefore, we have the system of equations

$$A + C = 2, \;\; 3B - C + D = 3,$$
$$-9A - 2D = -11, \;\; -27B + 2C + 2D = -23.$$

The solution of this system is $A = 1$, $B = 1$, $C = 1$, $D = 1$. The particular solution is

$$y(x) = \cos 3x + \sin 3x + e^{-x} (\cos x + \sin x).$$

Example 5.30 Find the non trivial solutions of the boundary value problem

$$y^{iv} - \omega^4 y = 0, \; y(0) = 0, \; y''(0) = 0, \; y(l) = 0, \; y''(l) = 0.$$

Solution Assume the solution to be of the form $y = e^{mx}$. The characteristic equation is given by

$$m^4 - \omega^4 = 0, \quad \text{or} \quad m^2 = \pm\, \omega^2, \quad \text{or} \quad m = \pm\, \omega, \pm\, i\omega.$$

The general solution is given by

$$y(x) = A_1 e^{\omega x} + B_1 e^{-\omega x} + C \cos \omega x + D \sin \omega x$$
$$= A \cosh \omega x + B \sinh \omega x + C \cos \omega x + D \sin \omega x$$

Substituting in the initial conditions, we get

$$y(0) = A + C = 0.$$

$$y'' = \omega^2 [A \cosh \omega x + B \sinh \omega x - C \cos \omega x - D \sin \omega x];$$

$$y''(0) = \omega^2 (A - C) = 0, \quad \text{or} \quad A - C = 0.$$

Solving the two equations, we get $A = 0$, $C = 0$. We also have

$$y(l) = 0 = B \sinh \omega l + D \sin \omega l, \; y''(l) = 0 = B \sinh \omega l - D \sin \omega l.$$

Adding, we obtain $2B \sinh \omega l = 0$, or $B = 0$. Therefore, we obtain $D \sin \omega l = 0$. Since, we require non-trivial solutions, we have $D \neq 0$. Hence, $\sin \omega l = 0 = \sin n\pi$, $n = 1, 2, \ldots$.

Therefore, $\omega = n\pi/l$, $n = 1, 2, \ldots$

The solution of the boundary value problem is

$$y_n(x) = D_n \sin (n\pi x/l), \; n = 1, 2, \ldots,$$

By superposition principle, the solution can be written as

$$y(x) = \sum_{n=1}^{\infty} D_n \sin (n\pi x/l).$$

Multiple complex roots

This case is a combination of the two earlier cases of real multiple roots and simple complex roots. Now, if $p + iq$ is a multiple root of order m, then $p - iq$ is also a multiple root of order m. For example, if $p_1 + iq_1$ is a double root, then $p_1 - iq_1$ is also a double root. The corresponding linearly independent solutions are

$$e^{p_1 x} \cos q_1 x, \; e^{p_1 x} \sin q_1 x, \; x e^{p_1 x} \cos q_1 x, \; x e^{p_1 x} \sin q_1 x.$$

Example 5.31 Solve the differential equation $y^{iv} + 32y'' + 256y = 0$.

Solution Substituting $y = e^{mx}$, we obtain the characteristic equation as

$$m^4 + 32m^2 + 256 = 0, \text{ or } (m^2 + 16)^2 = 0.$$

The roots of this equation are the double roots $m = \pm\, 4i$. Therefore, the general solution is

$$y(x) = (Ax + B) \cos 4x + (Cx + D) \sin 4x.$$

Exercise 5.3

Find the general solution of the following differential equations.

1. $y''' - 9y' = 0$.
2. $2y''' + y'' - 13y' + 6y = 0$.
3. $3y''' - 2y'' - 3y' + 2y = 0$.
4. $y^{iv} - 13y'' + 36y = 0$.
5. $4y^{iv} - 12y''' + 7y'' + 3y' - 2y = 0$.
6. $y^{iv} + y''' - 4y'' - 4y' = 0$.
7. $8y^{iv} - 6y''' - 7y'' + 6y' - y = 0$.
8. $144y^{iv} - 25y'' + y = 0$.
9. $y''' - 2y'' + y' = 0$.
10. $y''' + 4y'' + 5y' + 2y = 0$.
11. $y''' - 2y'' - 4y' + 8y = 0$.
12. $27y''' - 27y'' + 9y' - y = 0$.
13. $y^{iv} - 11y''' + 35y'' - 25y' = 0$.
14. $y^{iv} - 3y''' + 3y'' - y' = 0$.
15. $4y^{iv} + 4y''' - 3y'' - 2y' + y = 0$.
16. $9y^{iv} - 66y''' + 157y'' - 132y' + 36y = 0$.
17. $y''' + y' = 0$.
18. $y''' - 2y'' + 4y' - 8y = 0$.
19. $y''' + 5y'' + 8y' + 6y = 0$.
20. $y''' - 7y'' + 19y' - 13y = 0$.
21. $y^{iv} + 8y'' - 9y = 0$.
22. $y^{iv} + y''' + 14y'' + 16y' - 32y = 0$.
23. $4y^{iv} + 101y'' + 25y = 0$.
24. $y^{iv} + 2y''' - 9y'' - 10y' + 50y = 0$.
25. $y^{iv} + 50y'' + 625y = 0$.
26. $y^{iv} + 2y'' + y = 0$.

Find a homogeneous linear differential equation with real constant coefficients of lowest order which has the following particular solution.

27. $5 + e^{x} + 2e^{3x}$.
28. $e^{-x} + \cos 5x + 3 \sin 5x$.
29. $xe^{-x} + e^{2x}$.
30. $1 + x + e^{x} - 3e^{3x}$.
31. $x^2e^{2x} + 2e^{-2x}$.
32. $3 \cos 2x + 5 \sinh 3x$.

Solve the following initial value problems.

33. $y''' - 2y'' - 5y' + 6y = 0$, $y(0) = 0$, $y'(0) = 0$, $y''(0) = 1$.
34. $4y''' - 4y'' - 9y' + 9y = 0$, $y(0) = 1$, $y'(0) = 0$, $y''(0) = 0$.
35. $y''' - 5y'' + 7y' - 3y = 0$, $y(0) = 1$, $y'(0) = 0$, $y''(0) = -5$.
36. $y^{iv} - 2y''' - 3y'' + 4y' + 4y = 0$, $y(0) = 3$, $y'(0) = 3$, $y''(0) = 3$, $y'''(0) = 6$.
37. $y^{iv} + y'' = 0$, $y(0) = 1$, $y'(0) = 2$, $y''(0) = -1$, $y'''(0) = -1$.
38. $y''' - y'' + 4y' - 4y = 0$, $y(0) = 0$, $y'(0) = 3$, $y''(0) = -5$.
39. $y''' + y'' - 2y = 0$, $y(0) = 2$, $y'(0) = 2$, $y''(0) = -3$.
40. $y^{iv} - 3y''' = 0$, $y(0) = 2$, $y'(0) = 5$, $y''(0) = 15$, $y'''(0) = 27$.

Find the solution of the following differential equations satisfying the given conditions.

41. $y''' + \pi^2 y' = 0$, $y(0) = 0$, $y(1) = 0$, $y'(0) + y'(1) = 0$.
42. $y''' - 36y' = 0$, $y(0) = 2$, $y'(0) = 12$, $y'(1) = 6 \sinh(6) + 12 \cosh(6)$.
43. $y^{iv} + 13y'' + 36y = 0$, $y(0) = 0$, $y''(0) = 0$, $y(\pi/2) = -1$, $y'(\pi/2) = -4$.
44. $y^{iv} - \omega^4 y = 0$, $\omega \neq 0$, $y(0) = 0$, $y''(0) = 0$, $y(\pi) = 0$, $y''(\pi) = 0$.
45. $y^{iv} + 10y'' + 9y = 0$, $y'(0) = 0$, $y'''(0) = 0$, $y'(\pi/2) = 5$, $y'''(\pi/2) = -53$.

5.4 Solution of Non-Homogeneous Linear Equations

In the previous section, we have discussed methods for finding the general and particular solutions of homogeneous linear equations. In this section, we shall discuss methods for finding the general solution of a non-homogeneous linear equation (see Eq. (5.1)) of the form

$$L[y] = a_0(x)y^{(n)}(x) + a_1(x)y^{(n-1)}(x) + \ldots + a_{n-1}(x)y' + a_n(x)y = r(x),\ a_0(x) \neq 0, \tag{5.46}$$

when the general solution of the corresponding homogeneous linear equation $L[y] = 0$ is known. We present the following theorem.

Theorem 5.5 If $\{y_1(x), y_2(x), \ldots, y_n(x)\}$ is a basis and $c_1y_1(x) + c_2y_2(x) + \ldots + c_ny_n(x)$ is the general solution of the corresponding homogeneous linear equation $L[y] = 0$ and if $y_p(x)$ is any particular solution (a solution not containing any arbitrary constants) of the non-homogeneous equation (5.46), then the general solution of equation (5.46) is given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x) + y_p(x). \tag{5.47}$$

Proof Since $y_p(x)$ is a particular solution, we have

$$L[y_p(x)] = a_0 y_p^{(n)} + a_1 y_p^{(n-1)} + \ldots + a_{n-1} y_p' + a_n y_p = r(x). \tag{5.48}$$

Subtracting Eq. (5.48) from (5.46), we obtain

$$a_0(y^{(n)} - y_p^{(n)}) + a_1(y^{(n-1)} - y_p^{(n-1)}) + \ldots + a_{n-1}(y' - y_p') + a_n(y - y_p) = 0. \tag{5.49}$$

Denote $y - y_p = z$. Then, from Eq. (5.49) we obtain

$$a_0 z^{(n)} + a_1 z^{(n-1)} + \ldots + a_{n-1} z' + a_n z = 0. \tag{5.50}$$

But, this equation is the corresponding homogeneous equation of Eq. (5.46), whose basis is $\{y_1(x), y_2(x), \ldots, y_n(x)\}$. Hence, the general solution of Eq. (5.50) is given by

$$z = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x).$$

Replacing $z = y - y_p$, and taking y_p to the right hand side, we obtain

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x) + y_p(x). \tag{5.51}$$

Since, this solution contains n arbitrary constants, it is the general solution of the Eq. (5.46).

From the above theorem, we conclude that the solution of a non-homogeneous equation consists of the sum of the following two parts.

(i) The general solution of the corresponding homogeneous equation. This solution is called the *complementary function* and is denoted by $y_c(x)$.

(ii) A particular solution of the non-homogeneous equation. This solution is also called a *particular integral* of the non-homogeneous equation and is denoted by $y_p(x)$.

The general solution of the non-homogeneous equation is then written as

$$y(x) = y_c(x) + y_p(x).$$

Now, suppose that the right hand side $r(x)$ is the sum of a number of functions

$$r(x) = r_1(x) + r_2(x) + \ldots + r_m(x). \tag{5.52}$$

Let $y_{p_i}(x)$, $i = 1, 2, \ldots, m$ be any particular solutions, not containing any arbitrary constants, of the equations

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1} y' + a_n y = r_i(x),\ i = 1, 2, \ldots, m. \tag{5.53}$$

Then, $y_{p1} + y_{p2} + \ldots + y_{pm}$ is the particular integral of the equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1} y' + a_n y = r_1(x) + r_2(x) + \ldots + r_m(x) = r(x)$$

and hence of the given non-homogeneous linear equation. This can be proved by summing Eq. (5.53) over i. In other words, if the right hand side of Eq. (5.46) consists of sum of a number of functions, then particular integrals of the Eq. (5.53) can be obtained with respect to each of the functions and the particular integral of Eq. (5.46) is then given by the sum of these particular integrals.

The methods for finding $y_c(x)$ have been discussed in the previous section. In the remaining part of this section, we shall derive methods for finding the particular integral $y_p(x)$ of the non-homogeneous equation.

5.4.1 Method of Variation of Parameters

Consider the second order non-homogeneous linear equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = r(x), \quad a_0(x) \neq 0. \tag{5.54}$$

We shall discuss a general method of solution, called the method of *variation of parameters*, which can always be used to find a particular integral whenever the complementary function of the equation is known. Consider first, the solution of the corresponding homogeneous equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0, \quad a_0(x) \neq 0. \tag{5.55}$$

Using the methods given in the previous section, we can find two linearly independent solutions $y_1(x)$ and $y_2(x)$ of the equation (5.55). The complementary function is given by

$$y_c(x) = A y_1(x) + B y_2(x) \tag{5.56}$$

where A and B are arbitrary constants. The idea behind the method of variation of parameters is to vary the parameters A and B. That is, we assume A and B to be functions of x and determine $A(x)$, $B(x)$ such that

$$y(x) = A(x)y_1(x) + B(x)y_2(x) \tag{5.57}$$

is the general solution of Eq. (5.54). Now, $y(x)$ contains two functions $A(x)$ and $B(x)$ which are to be determined. Therefore, we need two equations to determine them. One equation is obtained by substituting $y(x)$ from Eq. (5.57) in Eq. (5.54). The determination of the second equation is at our disposal. This equation is chosen such that the determination of $A(x)$ and $B(x)$ is simple. Differentiating Eq. (5.57), we obtain

$$y'(x) = A'y_1 + Ay_1' + By_2' + B'y_2 = (A'y_1 + B'y_2) + (Ay_1' + By_2'). \tag{5.58}$$

If we differentiate this equation again, then the equation would contain the second derivatives A'' and B'' of the unknown functions. In order that these derivatives are not used, we set in Eq. (5.58)

$$A'y_1 + B'y_2 = 0. \tag{5.59}$$

which gives us the second equation to determine $A(x)$ and $B(x)$. Now, differentiating $y'(x) = Ay_1' + By_2'$, we obtain

$$y''(x) = Ay_1'' + A'y_1' + By_2'' + B'y_2'. \tag{5.60}$$

Substituting the expressions for $y(x)$, $y'(x)$ and $y''(x)$ in Eq. (5.54), we obtain

$$a_0(x)[Ay_1'' + A'y_1' + By_2'' + B'y_2'] + a_1(x)[Ay_1' + By_2'] + a_2(x)[Ay_1 + By_2] = r(x)$$

or
$$a_0(x)[A'y_1' + B'y_2'] + A[a_0(x)y_1'' + a_1(x)y_1' + a_2(x)y_1]$$
$$+ B[a_0(x)y_2'' + a_1(x)y_2' + a_2(x)y_2] = r(x).$$

Since, $y_1(x)$ and $y_2(x)$ are the solutions of the homogeneous equation (5.55), we obtain
$$a_0(x)[A'y_1' + B'y_2'] = r(x), \text{ or } A'y_1' + B'y_2' = \frac{r(x)}{a_0(x)} = g(x). \tag{5.61}$$

Since $a_0(x) \neq 0$ on the given interval I, $g(x)$ is continuous on I. Solving the equations
$$A'y_1 + B'y_2 = 0$$
$$A'y_1' + B'y_2' = g(x),$$

we obtain
$$A' = -\frac{g(x)y_2}{y_1y_2' - y_2y_1'}, \quad B' = \frac{g(x)y_1}{y_1y_2' - y_2y_1'}. \tag{5.62}$$

We note that the Wronskian $W(y_1, y_2)$ is
$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1' \neq 0$$

since y_1, y_2 are the linearly independent solutions of the homogeneous equation. Hence, we can write Eqs. (5.62) as
$$A' = -\frac{g(x)y_2}{W(x)}, \text{ and } B' = \frac{g(x)y_1}{W(x)}. \tag{5.63}$$

Integrating, we obtain
$$A(x) = -\int \frac{g(x)y_2(x)}{W(x)}\,dx + c_1 \text{ and } B(x) = \int \frac{g(x)y_1(x)}{W(x)}\,dx + c_2. \tag{5.64}$$

Substituting in Eq. (5.57), we obtain the general solution which contains two arbitrary constants. If we do not add the arbitrary constants while carrying out integrations of Eqs. (5.63), then we obtain the particular solution as $y_p(x) = A(x)y_1(x) + B(x)y_2(x)$, which does not contain any arbitrary constants. The general solution is then given by $y(x) = y_c(x) + y_p(x)$.

The method is applicable both for constant coefficient and variable coefficient problems. The method can also be easily extended to equations of any order. At each differentiation step, we set the part containing the derivatives of the unknown functions to zero, until we arrive at the final substitution step. For example, consider the third order equation
$$a_0(x)y''' + a_1(x)y'' + a_2(x)y' + a_3(x)y = r(x), \quad a_0(x) \neq 0. \tag{5.65}$$

The complementary function is
$$y(x) = Ay_1(x) + By_2(x) + Cy_3(x)$$

where y_1, y_2, y_3 are the linearly independent solutions of the corresponding homogeneous equation and A, B, C are arbitrary constants. We assume the solution as
$$y(x) = A(x)y_1(x) + B(x)y_2(x) + C(x)y_3(x). \tag{5.66}$$

Following the procedure discussed earlier, we obtain the required equations for determining $A(x)$, $B(x)$ and $C(x)$ as

$$A'(x)y_1 + B'(x)y_2 + C'(x)y_3 = 0$$

$$A'(x)y_1' + B'(x)y_2' + C'(x)y_3' = 0$$

and
$$A'(x)y_1'' + B'(x)y_2'' + C'(x)y_3'' = \frac{r(x)}{a_0(x)} = g(x). \tag{5.67}$$

The determinant of the coefficient matrix is the Wronskian $W(y_1, y_2, y_3) \neq 0$. We determine $A(x)$, $B(x)$, $C(x)$ and substitute in Eq. (5.66) to obtain the general solution.

Example 5.32 Find the general solution of the equation $y'' + 3y' + 2y = 2e^x$, using the method of variation of parameters.

Solution The corresponding homogeneous equation is $y'' + 3y' + 2y = 0$. The characteristic equation is $m^2 + 3m + 2 = 0$ and its roots are $m = -1, -2$. Hence, the complementary function is

$$y_c(x) = Ay_1(x) + By_2(x) = Ae^{-x} + Be^{-2x}$$

where $y_1(x) = e^{-x}$ and $y_2(x) = e^{-2x}$ are two linearly independent solutions of the homogeneous equation. Assume the general solution as

$$y(x) = A(x)e^{-x} + B(x)e^{-2x}$$

We have $g(x) = r(x)/a_0(x) = 2e^x$.

The Wronskian of $y_1(x)$, $y_2(x)$ is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -e^{-3x}.$$

Using Eq. (5.64), we obtain the solutions for $A(x)$ and $B(x)$ as

$$A(x) = -\int \frac{g(x)y_2(x)}{W}\,dx + c_1 = -\int \frac{2e^x e^{-2x}}{-e^{-3x}}\,dx + c_1 = e^{2x} + c_1$$

$$B(x) = \int \frac{g(x)y_1(x)}{W}\,dx + c_2 = \int \frac{2e^x e^{-x}}{-e^{-3x}}\,dx + c_2 = -\frac{2}{3}e^{3x} + c_2.$$

The general solution is

$$y(x) = A(x)e^{-x} + B(x)e^{-2x}$$

$$= (e^{2x} + c_1)e^{-x} + \left(-\frac{2}{3}e^{3x} + c_2\right)e^{-2x} = c_1e^{-x} + c_2e^{-2x} + \frac{1}{3}e^x.$$

Example 5.33 Find the general solution of the equation $y'' + 16y = 32 \sec 2x$, using the method of variation of parameters.

Solution The characteristic equation of the corresponding homogeneous equation is $m^2 + 16 = 0$. The characteristic roots are $m = \pm 4i$. The complementary function is given by

$$y_c(x) = Ay_1(x) + By_2(x) = A\cos 4x + B\sin 4x$$

where $y_1(x) = \cos 4x$ and $y_2(x) = \sin 4x$ are two linearly independent solutions of the homogeneous equation. By the method of the variation of parameters, we write the general solution as

$$y(x) = A(x) \cos 4x + B(x) \sin 4x.$$

We have $g(x) = r(x)/a_0(x) = 32 \sec 2x$. The Wronskian of y_1, y_2 is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 4x & \sin 4x \\ -4 \sin 4x & 4 \cos 4x \end{vmatrix} = 4.$$

Therefore, from Eq. (5.64), we obtain

$$A(x) = -\int \frac{g(x) y_2(x)}{W} dx + c_1 = -\frac{1}{4} \int 32 \sec 2x \sin 4x \, dx + c_1$$

$$= -16 \int \sin 2x \, dx + c_1 = 8 \cos 2x + c_1.$$

$$B(x) = \int \frac{g(x) y_1(x)}{W} dx + c_2 = \frac{1}{4} \int 32 \sec 2x \cos 4x \, dx + c_2$$

$$= 8 \int \frac{2 \cos^2 2x - 1}{\cos 2x} dx + c_2 = 8 \int (2 \cos 2x - \sec 2x) dx + c_2$$

$$= 8 \sin 2x - 4 \ln | \sec 2x + \tan 2x | + c_2.$$

The general solution is

$$\begin{aligned} y(x) &= A(x) \cos 4x + B(x) \sin 4x \\ &= c_1 \cos 4x + c_2 \sin 4x + 8 \cos 2x \cos 4x + 8 \sin 2x \sin 4x \\ &\quad - 4 \sin 4x \ln | \sec 2x + \tan 2x | \\ &= c_1 \cos 4x + c_2 \sin 4x + 8 \cos 2x - 4 \sin 4x \ln | \sec 2x + \tan 2x |. \end{aligned}$$

Example 5.34 Find the general solution of the equation $y''' - 6y'' + 11y' - 6y = e^{-x}$.

Solution The characteristic equation of the corresponding homogeneous equation is $m^3 - 6m^2 + 11m - 6 = 0$ and its roots are $m = 1, 2, 3$. The complementary function is given by

$$y_c(x) = Ae^x + Be^{2x} + Ce^{3x}.$$

By the method of variation of parameters, we assume the solution as

$$y(x) = A(x)e^x + B(x)e^{2x} + C(x)e^{3x}.$$

We have $$g(x) = r(x)/a_0(x) = e^{-x}.$$

From Eqs. (5.67), the equations for determining $A(x)$, $B(x)$ and $C(x)$ are

$$A'e^x + B'e^{2x} + C'e^{3x} = 0$$

$$A'e^x + 2B'e^{2x} + 3C'e^{3x} = 0$$

$$A'e^x + 4B'e^{2x} + 9C'e^{3x} = e^{-x}.$$

The Wronskian of $y_1 = e^x$, $y_2 = e^{2x}$, $y_3 = e^{3x}$ is given by

$$W(x) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = e^{6x}\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = 2e^{6x}.$$

By the Cramer's rule, we obtain

$$WA' = \begin{vmatrix} 0 & e^{2x} & e^{3x} \\ 0 & 2e^{2x} & 3e^{3x} \\ e^{-x} & 4e^{2x} & 9e^{3x} \end{vmatrix} = e^{4x}, \text{ or } A' = \frac{e^{4x}}{2e^{6x}} = \frac{1}{2}e^{-2x}.$$

Integrating, we get $A = -\frac{1}{4}e^{-2x} + c_1$.

Similarly, we have

$$WB' = \begin{vmatrix} e^x & 0 & e^{3x} \\ e^x & 0 & 3e^{3x} \\ e^x & e^{-x} & 9e^{3x} \end{vmatrix} = -2e^{3x}, \text{ or } B' = -\frac{2e^{3x}}{2e^{6x}} = -e^{-3x}.$$

$$WC' = \begin{vmatrix} e^x & e^{2x} & 0 \\ e^x & 2e^{2x} & 0 \\ e^x & 4e^{2x} & e^{-x} \end{vmatrix} = e^{2x}, \text{ or } C' = \frac{e^{2x}}{2e^{6x}} = \frac{1}{2}e^{-4x}.$$

Integrating, we obtain $B(x) = \frac{1}{3}e^{-3x} + c_2$ and $C(x) = -\frac{1}{8}e^{-4x} + c_3$. The general solution is

$$y(x) = A(x)e^x + B(x)e^{2x} + C(x)e^{3x}$$

$$= \left(-\frac{1}{4}e^{-2x} + c_1\right)e^x + \left(\frac{1}{3}e^{-3x} + c_2\right)e^{2x} + \left(-\frac{1}{8}e^{-4x} + c_3\right)e^{3x}$$

$$= c_1e^x + c_2e^{2x} + c_3e^{3x} - \frac{1}{24}e^{-x}.$$

Example 5.35 It is given that $y_1 = x$ and $y_2 = 1/x$ are two linearly independent solutions of the associated homogeneous equation of $x^2y'' + xy' - y = x$, $x \neq 0$. Find a particular integral and the general solution of the equation.

Solution By the method of variation of parameters, we write

$$y(x) = A(x)x + B(x)\left(\frac{1}{x}\right).$$

The Wronskian of $y_1(x) = x$ and $y_2(x) = 1/x$ is given by

$$W = \begin{vmatrix} x & 1/x \\ 1 & -1/x^2 \end{vmatrix} = -\frac{2}{x}, \quad x \neq 0.$$

We have $g(x) = r(x)/a_0(x) = 1/x$. Using Eq. (5.64), we obtain

$$A(x) = -\int \frac{g(x)y_2(x)}{W}\,dx = -\int \frac{1}{x^2}\left(-\frac{x}{2}\right)dx = \frac{1}{2}\ln|x| + c_1.$$

$$B(x) = \int \frac{g(x)y_1(x)}{W}\,dx = \int \frac{1}{x}\left(-\frac{x^2}{2}\right)dx = -\frac{1}{4}x^2 + c_2.$$

The particular integral is

$$y_p(x) = A(x)x + B(x)\left(\frac{1}{x}\right) = \frac{x}{2}\ln|x| - \frac{x}{4}.$$

The general solution is

$$y(x) = y_c(x) + y_p(x) = c_1 x + \frac{1}{x}c_2 + \frac{x}{2}\ln|x| - \frac{x}{4}$$

or

$$y(x) = c_1^* x + \frac{1}{x}c_2 + \frac{x}{2}\ln|x|, \text{ where } c_1^* = c_1 - \frac{1}{4}.$$

Exercise 5.4

Find the general solution of the following differential equations, using the method of variation of parameters.

1. $y'' - 2y' - 3y = e^x$.
2. $y'' - 4y' + 4y = e^{-2x}$.
3. $y'' + 4y = \cos x$.
4. $y'' + y = \sec x$.
5. $y'' + y = \operatorname{cosec} x$.
6. $y'' + y = \tan x$.
7. $y'' - 4y' + 3y = e^x$.
8. $y'' + 4y = \sec 2x$.
9. $y'' + 4y = \cos 2x$.
10. $y'' + 4y' + 4y = e^{-2x}\sin x$.
11. $y'' + 6y' + 9y = e^{-3x}/x$.
12. $y'' + 2y' + 2y = e^{-x}\cos x$.

In the following problems, using the method of variation of parameters and the given linearly independent solutions, find a particular integral and the general solution.

13. $x^2y'' + xy' - y = x^3$, $y_1 = x$, $y_2 = 1/x$.
14. $x^2y'' + xy' - 4y = x^2\ln|x|$, $y_1 = x^2$, $y_2 = 1/x^2$.
15. $x^2y'' - xy' + y = 1/x^4$, $y_1 = x$, $y_2 = x\ln|x|$.
16. $x^2y'' - 2xy' + 2y = x^3 + x$, $y_1 = x$, $y_2 = x^2$.
17. $y'' + 4y' + 8y = 16\,e^{-2x}\operatorname{cosec}^2 2x$, $y_1 = e^{-2x}\cos 2x$, $y_2 = e^{-2x}\sin 2x$.
18. $y''' + 4y' = \sec 2x$, $y_1 = 1$, $y_2 = \cos 2x$, $y_3 = \sin 2x$.
19. $y''' - 6y'' + 12y' - 8y = e^{2x}/x$, $y_1 = e^{2x}$, $y_2 = xe^{2x}$, $y_3 = x^2e^{2x}$.
20. Show that the general solution of the equation $y'' + k^2y = g(x)$, where $k \neq 0$ and $g(x)$ is continuous on I, can always be written as

$$y(x) = A\cos kx + B\sin kx + \frac{1}{k}\int_0^x \sin k(x-t)g(t)\,dt.$$

5.4.2 Method of Undetermined Coefficients

In the previous section, we have discussed the method of variation of parameters for finding the solution of the differential equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1}y' + a_n y = r(x)$$

where $a_0, a_1, \ldots, a_n$ are constants. In the cases when the right hand side $r(x)$ is of a special form containing exponentials, polynomials, cosine and sine functions, sums or products of these functions, then the particular integral can be easily obtained by the method of undetermined coefficients. The basic idea behind this approach is as follows.

If $r(x)$ is of exponential form e^{mx}, then its derivatives also contain exponentials e^{mx} only, that is, if $r(x) = pe^{mx}$, p constant, then we can choose the particular integral as $y_p(x) = ce^{mx}$, c constant and determine c by substituting $y_p(x)$ in the given equation and comparing both sides of the equation. That is, the equation is identically satisfied.

If $r(x)$ is a cosine or a sine function, cos mx or sin mx, then their derivatives contain the terms cos mx and sin mx. In other words, if $r(x) = p \cos mx$ or $p \sin mx$, p constant, then we can choose the particular integral as $y_p(x) = c_1 \cos mx + c_2 \sin mx$. The constants c_1, c_2 are determined by substituting $y_p(x)$ in the given equation and comparing both sides of the equation.

If $r(x)$ is of the form x^m, then its derivatives contain the terms $x^m, x^{m-1}, \ldots, x, 1$. Hence, when $r(x) = px^m$, p constant then we can choose the particular integral as

$$y_p(x) = c_0 x^m + c_1 x^{m-1} + \ldots + c_{m-1} x + c_m \tag{5.68}$$

where $c_0, c_1, \ldots, c_m$ are constants.

If $r(x)$ is of the forms $e^{ax} \cos bx$ or $e^{ax} \sin bx$ then their derivatives contain the terms $e^{ax} \cos bx$ and $e^{ax} \sin bx$. Hence, when $r(x) = e^{ax} \cos bx$ or $e^{ax} \sin bx$, then we can choose the particular integral as

$$y_p(x) = e^{ax}(c_1 \cos bx + c_2 \sin bx). \tag{5.69}$$

However, if any term in the choice of the particular integral is also a solution of the corresponding homogeneous equation, that is, a term in the complementary function, then we multiply this term by x or by x^m (if the term in the complementary function corresponds to a multiple root of multiplicity m). If $r(x)$ is the sum of a number of functions, then the contribution with respect to each of the terms is included in the choice of the particular integral.

Example 5.36 Using the method of undetermined coefficients find the general solution of the differential equation $y'' + y = 32x^3$.

Solution The characteristic equation of the homogeneous equation is $m^2 + 1 = 0$ and its roots are $m = \pm i$. The complementary function is $y_c(x) = A \cos x + B \sin x$.
Since $r(x) = 32x^3$, we choose the particular integral as

$$y_p(x) = c_1 x^3 + c_2 x^2 + c_3 x + c_4.$$

Substituting in the given equation, we get

$$(6c_1 x + 2c_2) + (c_1 x^3 + c_2 x^2 + c_3 x + c_4) = 32x^3.$$

Comparing the coefficients of various powers of x, we get

$$c_1 = 32,\ c_2 = 0,\ 6c_1 + c_3 = 0,\ 2c_2 + c_4 = 0.$$

The solution of the system is $c_1 = 32$, $c_2 = 0$, $c_3 = -192$, $c_4 = 0$. Therefore, $y_p(x) = 32x^3 - 192x$.
The general solution is

$$y(x) = A \cos x + B \sin x + 32x\,(x^2 - 6).$$

Example 5.37 Find the general solution of the differential equation $y'' - 2y' - 3y = 6e^{-x} - 8e^{x}$.

Solution The characteristic equation of the homogeneous equation is $m^2 - 2m - 3 = 0$ and its roots are $m = -1, 3$.

The complementary function is $y_c(x) = Ae^{-x} + Be^{3x}$.

We note that e^{-x} appears both as a term in $y_c(x)$ (due to the simple root $m = -1$) and the right hand side $r(x)$. The term e^x appears only in $r(x)$. Hence, we choose the particular integral as

$$y_p(x) = c_1 x e^{-x} + c_2 e^x.$$

We have $y_p'(x) = c_1(1 - x)e^{-x} + c_2 e^x, \; y_p''(x) = -c_1(2 - x)e^{-x} + c_2 e^x.$

Substituting in the given equation, we get

$$c_1[-(2 - x) - 2(1 - x) - 3x]e^{-x} + c_2[1 - 2 - 3]e^x = 6x^{-x} - 8e^x$$

or

$$-4c_1 e^{-x} - 4c_2 e^x = 6e^{-x} - 8e^x.$$

Comparing the coefficients of e^{-x} and e^x, we get $c_1 = -3/2$, $c_2 = 2$. The general solution is

$$y(x) = Ae^{-x} + Be^{3x} - \frac{3}{2}xe^{-x} + 2e^x.$$

Example 5.38 Find the general solution of the equation $y'' + 9y = \cos 3x$.

Solution The characteristic equation of the homogeneous equation is $m^2 + 9 = 0$ and its roots are $m = \pm 3i$. The complementary function is

$$y_c(x) = A\cos 3x + B\sin 3x.$$

We note that $\cos 3x$ appears as a term in $y_c(x)$ and the right hand side $r(x)$. Hence, we choose the particular integral as

$$y_p(x) = x(c_1 \cos 3x + c_2 \sin 3x).$$

We have $y_p'(x) = c_1 \cos 3x + c_2 \sin 3x + 3x(-c_1 \sin 3x + c_2 \cos 3x)$

$$y_p''(x) = 6(-c_1 \sin 3x + c_2 \cos 3x) + 9x(-c_1 \cos 3x - c_2 \sin 3x).$$

Substituting in the given equation, we get

$$y_p'' + 9y_p = \sin 3x\,[-6c_1 - 9xc_2 + 9xc_2] + \cos 3x\,[6c_2 - 9xc_1 + 9xc_1] = \cos 3x$$

or

$$-6c_1 \sin 3x + 6c_2 \cos 3x = \cos 3x.$$

Comparing both sides, we get $c_1 = 0$ and $c_2 = 1/6$. The particular integral is $y_p(x) = (x \sin 3x)/6$. The general solution is

$$y(x) = A\cos 3x + B\sin 3x + \frac{1}{6}x\sin 3x.$$

Example 5.39 Find the general solution of the equation $y'' + 4y' + 4y = 12e^{-2x}$.

Solution The characteristic equation of the homogeneous equation is $m^2 + 4m + 4 = (m + 2)^2 = 0$ and its roots are $m = -2, -2$.

The complementary function is $y_c(x) = (Ax + B)e^{-2x}$.

We note that e^{-2x} and xe^{-2x} are present in the complementary function (due to the double root

$m = -2$) and e^{-2x} is also a term on the right hand side $r(x)$. Therefore, we choose the particular integral as

$$y_p(x) = c_1 x^2 e^{-2x}.$$

We have
$$y_p'(x) = c_1[2x - 2x^2]e^{-2x},\quad y_p''(x) = c_1[2 - 8x + 4x^2]e^{-2x}.$$

Substituting in the given equation, we get

$$y_p'' + 4y_p' + 4y_p = c_1[(2 - 8x + 4x^2) + 4(2x - 2x^2) + 4x^2]e^{-2x} = 12e^{-2x}$$

or
$$2c_1 e^{-2x} = 12e^{-2x}.$$

Comparing both sides, we get $c_1 = 6$. Therefore, the particular integral is $y_p(x) = 6x^2e^{-2x}$. The general solution is

$$y(x) = (Ax + B)e^{-2x} + 6x^2e^{-2x} = (Ax + B + 6x^2)e^{-2x}.$$

Example 5.40 Find the general solution of the equation $y'' - 4y' + 13y = 12e^{2x} \sin 3x$.

Solution The characteristic equation of the homogeneous equation is $m^2 - 4m + 13 = 0$. The roots of this equation are

$$m = \frac{4 \pm \sqrt{16 - 52}}{2} = 2 \pm 3i.$$

The complementary function is $y_c(x) = e^{2x}(A \cos 3x + B \sin 3x)$.

We note that $e^{2x} \sin 3x$ appears both in the complementary function and the right hand side $r(x)$. Therefore, we choose

$$y_p(x) = xe^{2x}(c_1 \cos 3x + c_2 \sin 3x).$$

We have

$$y_p'(x) = (1 + 2x)e^{2x}(c_1 \cos 3x + c_2 \sin 3x) + 3xe^{2x}(-c_1 \sin 3x + c_2 \cos 3x)$$

$$y_p''(x) = (4 + 4x)e^{2x}(c_1 \cos 3x + c_2 \sin 3x)$$
$$+ 6(1 + 2x)e^{2x}(-c_1 \sin 3x + c_2 \cos 3x) + 9xe^{2x}(-c_1 \cos 3x - c_2 \sin 3x).$$

Substituting in the given equation, we get

$$y_p'' - 4y_p' + 13y_p = e^{2x} \cos 3x\,[c_1(4 + 4x) + 6c_2(1 + 2x) - 9c_1x - 4c_1(1 + 2x)$$
$$- 12xc_2 + 13c_1x] + e^{2x} \sin 3x\,[c_2(4 + 4x) - 6c_1(1 + 2x) - 9c_2x$$
$$- 4c_2(1 + 2x) + 12c_1x + 13xc_2] = 12e^{2x} \sin 3x$$

or
$$6c_2 e^{2x} \cos 3x - 6c_1 e^{2x} \sin 3x = 12\, e^{2x} \sin 3x.$$

Comparing both sides, we get $c_1 = -2$ and $c_2 = 0$. Therefore, the particular integral is $y_p(x) = -2xe^{2x} \cos 3x$. The general solution is

$$y(x) = e^{2x}[A \cos 3x + B \sin 3x - 2x \cos 3x].$$

Example 5.41 Find the general solution of the differential equation $y''' - 2y'' - 5y' + 6y = 18e^x$.

Solution The characteristic equation of the homogeneous equation is

$$m^3 - 2m^2 - 5m + 6 = (m - 1)(m + 2)(m - 3) = 0, \text{ or } m = 1, -2, 3.$$

The complementary function is $y_c(x) = Ae^x + Be^{-2x} + Ce^{3x}$.

Choose the particular integral as $y_p(x) = c_1 x e^x$.

We have $\quad y_p' = c_1 (1 + x)e^x, y_p'' = c_1 (2 + x)e^x, y_p''' = c_1 (3 + x)e^x.$

Substituting in the given equation, we get

$$y_p''' - 2y_p'' - 5y_p' + 6y_p = c_1 e^x [(3 + x) - 2(2 + x) - 5(1 + x) + 6x]$$
$$= -6c_1 e^x = 18e^x.$$

Comparing both sides, we get $c_1 = -3$. Hence, the particular integral is $y_p = -3xe^x$. The general solution is

$$y(x) = Ae^x + Be^{-2x} + Ce^{3x} - 3xe^x.$$

Example 5.42 Find the general solution of the differential equation

$$y''' - 6y'' + 12y' - 8y = 12e^{2x} + 27e^{-x}.$$

Solution The characteristic equation of the homogeneous equation is

$$m^3 - 6m^2 + 12m - 8 = (m - 2)^3 = 0, \text{ or } m = 2, 2, 2.$$

The complementary function is $y_c(x) = (Ax^2 + Bx + C)e^{2x}$. Note that $m = 2$ is a triple root and e^{2x} is contained in a term in $r(x)$. Therefore, we choose the particular integral as

$$y_p(x) = c_1 x^3 e^{2x} + c_2 e^{-x}.$$

We have $\quad y_p' = c_1 (3x^2 + 2x^3)e^{2x} - c_2 e^{-x}, y_p'' = c_1 (6x + 12x^2 + 4x^3)e^{2x} + c_2 e^{-x},$

$$y_p''' = c_1 (6 + 36x + 36x^2 + 8x^3)e^{2x} - c_2 e^{-x}.$$

Substituting in the given equation, we get

$$y_p''' - 6y_p'' + 12y_p' - 8y_p = c_1 e^{2x} [(6 + 36x + 36x^2 + 8x^3) - 6(6x + 12x^2 + 4x^3)$$
$$+ 12(3x^2 + 2x^3) - 8x^3] + c_2 e^{-x} [-1 - 6 - 12 - 8]$$
$$= 6c_1 e^{2x} - 27c_2 e^{-x} = 12e^{2x} + 27e^{-x}.$$

Comparing both sides, we get $c_1 = 2$ and $c_2 = -1$. Therefore, the particular integral is $y_p(x) = 2x^3 e^{2x} - e^{-x}$. The general solution is

$$y(x) = (Ax^2 + Bx + C)e^{2x} + 2x^3 e^{2x} - e^{-x}.$$

Exercise 5.5

Find the general solution of the following differential equations by the method of undetermined coefficients.

1. $y'' - 3y' - 10y = 1 + x^2$.
2. $2y'' - y' - 3y = x^3 + x + 1$.

3. $4y'' - y = e^x + e^{3x}$.
4. $3y'' + 2y' - y = e^{-2x} + x$.
5. $y'' + 6y' + 8y = e^{-3x} + e^x$.
6. $y'' + 4y' + 3y = 6e^{-x}$.
7. $2y'' + 3y' - 2y = 5e^{-2x} + e^x$.
8. $y'' - y' - 6y = 5e^{-2x} + 10e^{3x}$.
9. $3y'' + 5y' - 2y = 14e^{x/3}$.
10. $y'' + 3y' + 2y = \cos x + \sin x$.
11. $y'' + y' - 6y = 39 \cos 3x$.
12. $y'' + 4y' - 5y = 34 \cos 2x - 2 \sin 2x$.
13. $y'' + 25y = 50 \cos 5x + 30 \sin 5x$.
14. $y'' + 16y = 16 \sin 4x$.
15. $y'' - 4y' + 4y = 8e^{2x} + e^{3x}$.
16. $4y'' - 4y' + y = 6e^{x/2}$.
17. $y'' + 6y' + 9y = 26e^{-3x} + 5e^{2x}$.
18. $y'' + y = e^x \sin x$.
19. $y'' + 2y' + 10y = e^{-x} \sin 3x$.
20. $y'' - 4y' + 5y = 16e^{2x} \cos x$.
21. $y'' - 6y' + 13y = 6e^{3x} \sin x \cos x$.
22. $y'' + 4y' + 4y = 6e^{-2x} \cos^2 x$.
23. $y'' + 3y' + 2y = 12e^{-x} \cdot \sin^3 x$.
24. $y'' - 4y' + 3y = 4 \cosh 3x$.
25. $y''' + 4y'' - y' - 4y = 18e^{-x}$.
26. $y''' + 3y'' - 4y = 12e^{-2x} + 9e^x$.
27. $y''' - 9y'' + 27y' - 27y = 36e^{3x}$.
28. $y''' - y'' + y' - y = 6 \cos 2x$.
29. $y''' - 2y'' + 4y' - 8y = 8(x^2 + \cos 2x)$.
30. $y^{iv} - 256y = 128 \cos 4x$.
31. $y^{iv} - y = x^4 + 1$.
32. $y^{iv} + 3y''' + 3y'' + y' = 2x + 4$.
33. $y^{iv} - 3y'' - 4y = 60e^{2x}$.
34. $y^{iv} + 6y''' + 12y'' + 8y' = 60e^{-2x}$.
35. $y^{iv} - 16y'' = 8x + 16$.

5.4.3 Solution of Euler-Cauchy Equation

In the previous sections, we have discussed methods for finding the solution of the constant coefficient differential equations. Closed form solutions do not exist, in general, for the variable coefficient linear equations. However, for the *Euler-Cauchy equation*

$$a_0 x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \ldots + a_{n-1} x y' + a_n y = r(x), \quad x \neq 0 \tag{5.70}$$

where $a_0, a_1, \ldots, a_n$ are constants, closed form solutions can be obtained by using one of the following two procedures.

We shall illustrate these procedures using the second order equation

$$a_0 x^2 y'' + a_1 x y' + a_2 y = r(x), \quad a_0 \neq 0, x \neq 0. \tag{5.71}$$

Consider first, the corresponding homogeneous equation

$$a_0 x^2 y'' + a_1 x y' + a_2 y = 0. \tag{5.72}$$

We attempt to find a solution of the form $y = x^m$. We have $y' = mx^{m-1}$ and $y'' = m(m-1)x^{m-2}$. Substituting in Eq. (5.72), we get

$$[a_0 m(m-1) + a_1 m + a_2] x^m = 0. \tag{5.73}$$

Cancelling x^m, we get

$$a_0 m(m-1) + a_1 m + a_2 = a_0 m^2 + (a_1 - a_0)m + a_2 = 0 \tag{5.74}$$

which is called the *auxiliary* equation corresponding to the Eq. (5.72). Equation (5.74) has two roots $m = m_1, m_2$, which may be real and distinct, real and equal or complex conjugates. In these cases, we obtain the following solutions.

Real and distinct roots

If the roots m_1 and m_2 are real and distinct, then the two linearly independent solutions are

$$y_1(x) = x^{m_1} \quad \text{and} \quad y_2(x) = x^{m_2}. \tag{5.75}$$

The general solution is given by

$$y(x) = Ax^{m_1} + Bx^{m_2} \tag{5.76}$$

where A, B are arbitrary constants.

Example 5.43 Find the solution of the differential equation $x^2y'' + 2xy' - 2y = 0$.

Solution Here, $a_0 = 1$, $a_1 = 2$ and $a_2 = -2$. The auxiliary equation is

$$a_0m(m-1) + a_1m + a_2 = m^2 + m - 2 = 0, \quad \text{or} \quad (m+2)(m-1) = 0.$$

The roots of this equation are $m = 1, -2$. Hence, the two linearly independent solutions are

$$y_1(x) = x, \quad \text{and} \quad y_2(x) = x^{-2}.$$

The general solution is $y(x) = Ax + (B/x^2)$.

Example 5.44 Find the solution of the differential equation $2x^2y'' + xy' - 6y = 0$.

Solution Here, $a_0 = 2$, $a_1 = 1$, and $a_2 = -6$. The auxiliary equation is

$$a_0m(m-1) + a_1m + a_2 = 2m^2 - m - 6 = 0, \quad \text{or} \quad (m-2)(2m+3) = 0.$$

The roots of this equation are $m = 2, -3/2$. The two linearly independent solutions are

$$y_1(x) = x^2, \quad \text{and} \quad y_2(x) = x^{-3/2}.$$

The general solution is $y(x) = Ax^2 + \dfrac{B}{x\sqrt{x}}$.

Real and equal roots

Let the roots of the auxiliary equation be real and equal, that is, $m = m_1$ is a double root. Then $m_1 = (a_0 - a_1)/(2a_0)$. Since, the discriminant of Eq. (5.74) vanishes in this case, we can also write $m_1^2 = a_2/a_0$ (product of roots). Then, $y_1(x) = x^{m_1}$ is one of the linearly independent solutions. The second linearly independent solution can now be obtained by the method of reduction of order (see section 5.3.3). Write $y_2(x) = u(x)y_1(x)$. We have

$$y_2' = uy_1' + u'y_1, \; y_2'' = uy_1'' + 2u_1'y_1' + u''y_1.$$

Substituting in Eq. (5.72) and simplifying, we get

$$a_0x^2(uy_1'' + 2u'y_1' + u''y_1) + a_1x(uy_1' + u'y_1) + a_2uy_1 = 0$$

or
$$a_0y_1x^2u'' + xu'(2a_0xy_1' + a_1y_1) + u(a_0x^2y_1'' + a_1xy_1' + a_2y_1) = 0. \tag{5.77}$$

Since $y_1(x)$ is a solution of Eq. (5.72), the third term in Eq. (5.77) vanishes. Further, since $y_1(x) = x^{m_1}$ where $m_1 = (a_0 - a_1)/(2a_0)$, we obtain

$$2a_0xy_1' + a_1y_1 = (2a_0m_1 + a_1)x^{m_1} = a_0x^{m_1} = a_0y_1.$$

Therefore, Eq. (5.77) simplifies to

$$a_0y_1x^2u'' + a_0xu'y_1 = (xu'' + u')a_0xy_1 = 0$$

Since $x \neq 0$, $y_1 \neq 0$, $a_0 \neq 0$, we get $xu'' + u' = 0$. Separating the variables, we get

$$\frac{u''}{u'} = -\frac{1}{x}$$

Integrating, we get for $x > 0$

$$\ln|u'| = -\ln x, \quad \text{or} \quad u' = \frac{1}{x}.$$

Integrating again, we get $u = \ln x$.

Therefore, $y_2 = uy_1 = y_1 \ln x$. Since $y_2/y_1 = \ln x$, is not a constant, the two solutions y_1, y_2 are linearly independent. The general solution in this case is

$$y(x) = Ay_1 + By_2 = (A + B\ln x)y_1 = (A + B\ln x)x^{m_1} \tag{5.78}$$

where $m_1 = (a_0 - a_1)/(2a_0)$.

Example 5.45 Find the solution of the differential equation $4x^2y'' + y = 0$.

Solution Here, $a_0 = 4$, $a_1 = 0$, $a_2 = 1$. The auxiliary equation is

$$a_0m(m-1) + a_1m + a_2 = 4m^2 - 4m + 1 = 0, \quad \text{or} \quad (2m-1)^2 = 0.$$

The equation has the double root $m = 1/2$. The general solution is (from Eq. (5.78))

$$y(x) = (A + B\ln x)\, x^{1/2}, \quad x > 0.$$

Complex roots

Let the roots of the auxiliary equation (5.74) be a complex conjugate pair, $m = p \pm iq$. Then the solutions are given by

$$x^m = x^{p \pm iq} = x^p\, x^{\pm iq} = x^p(e^{\ln x})^{\pm iq}$$

$$= x^p e^{\pm iq\ln x} = x^p[\cos(q\ln x) \pm i\sin(q\ln x)], \quad x > 0.$$

Therefore, we can take the two linearly independent solutions as

$$y_1(x) = x^p\cos(q\ln x), \quad \text{and} \quad y_2(x) = x^p\sin(q\ln x). \tag{5.79}$$

Example 5.46 Find the general solution of the equation $4x^2y'' + 8xy' + 17y = 0$.

Solution Here, $a_0 = 4$, $a_1 = 8$ and $a_2 = 17$. The auxiliary equation is

$$a_0m(m-1) + a_1m + a_2 = 4m^2 + 4m + 17 = 0.$$

The roots of this equation are $m = \dfrac{-4 \pm \sqrt{16 - 272}}{8} = \dfrac{-4 \pm 16i}{8} = -\dfrac{1}{2} \pm 2i = p \pm iq.$

The general solution is (from Eq. (5.79))

$$y(x) = Ax^{-1/2}\cos(2\ln x) + Bx^{-1/2}\sin(2\ln x).$$

The method considered here is easily applicable for the homogeneous equations. However, for non-homogeneous equations, finding a particular solution is difficult.

We now discuss a method which can be applied for the solution of general Euler-Cauchy equation given by Eq. (5.71). For $x > 0$, we change the independent variable to

$$x = e^t, \quad \text{or} \quad t = \ln x, \quad x > 0. \tag{5.80}$$

The case $x < 0$ can also be considered by writing the transformation as

$$|x| = e^t, \quad \text{or} \quad t = \ln |x|. \tag{5.81}$$

This transformation always reduces the Euler-Cauchy equation into a linear equation with constant coefficients. The solution of this equation can tnen be obtained using the methods discussed in the previous sections. Finally, the solution of the given equation, in terms of the original variable x, is obtained by replacing t by $\ln x$.

When $x = e^t$, $t = \ln x$, we have

$$\frac{d}{dx} = \frac{d}{dt} \cdot \frac{dt}{dx} = \frac{1}{x}\frac{d}{dt}, \quad \text{or} \quad x\frac{d}{dx} = \frac{d}{dt} \tag{5.82}$$

$$\frac{d^2}{dx^2} = -\frac{1}{x^2}\frac{d}{dt} + \frac{1}{x}\frac{d}{dt}\left(\frac{d}{dt}\right)\frac{dt}{dx} = -\frac{1}{x^2}\frac{d}{dt} + \frac{1}{x^2}\frac{d^2}{dt^2}$$

or

$$x^2\frac{d^2}{dx^2} = \frac{d^2}{dt^2} - \frac{d}{dt} = \frac{d}{dt}\left(\frac{d}{dt} - 1\right). \tag{5.83}$$

In operator notation, set $D = d/dx$, $D^2 = d^2/dx^2$, $\theta = d/dt$, $\theta^2 = d^2/dt^2$ etc. Then, Eqs. (5.82), (5.83) can be written as

$$xD = \theta, \quad x^2D^2 = \theta^2 - \theta = \theta(\theta - 1) \tag{5.84}$$

or

$$xDy = \theta y, \quad x^2D^2y = \theta(\theta - 1)y. \tag{5.85}$$

By induction, we can prove that

$$x^n D^n y = \theta(\theta - 1) \ldots [\theta - (n - 1)]y. \tag{5.86}$$

Substituting in the non-homogeneous second order linear equation (5.71), we obtain the reduced equation as

$$a_0\,\theta(\theta - 1)\,y + a_1\,\theta y + a_2\,y = a_0\,\theta^2 y + (a_1 - a_0)\,\theta\,y + a_2 y = r(e^t). \tag{5.87}$$

This is a linear equation with constant coefficients. The methods described in the previous sections can be applied to find its solution.

Example 5.47 Find the general solution of the equation $2x^2y'' + 3xy' - 3y = x^3$.

Solution Using the transformation $x = e^t$, we get (using Eqs. (5.82) and 5.83))

$$2\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + 3\frac{dy}{dt} - 3y = e^{3t}, \quad \text{or} \quad 2\frac{d^2y}{dt} + \frac{dy}{dt} - 3y = e^{3t}. \tag{5.88}$$

This is a linear, constant coefficient equation. Substituting, $y = e^{mt}$, the characteristic equation of the corresponding homogeneous equation is obtained as

$$2m^2 + m - 3 = 0, \quad \text{or} \quad (m - 1)(2m + 3) = 0, \quad \text{or} \quad m = 1, -3/2.$$

The complementary function is $y_c(t) = Ae^t + Be^{-3t/2}$.

Let the particular integral be written as $y_p = ce^{3t}$. Substituting in Eq. (5.88), we obtain

$$(18 + 3 - 3)ce^{3t} = e^{3t}, \quad \text{or} \quad c = 1/18.$$

The particular integral is $y_p = e^{3t}/18$.

The general solution is $y(t) = Ae^{t} + Be^{-3t/2} + \frac{1}{18}e^{3t}$.

Substituting $e^t = x$, we get the general solution as

$$y(x) = Ax + \frac{B}{x\sqrt{x}} + \frac{x^3}{18}.$$

Example 5.48 Find the general solution of the equation $x^2y'' + 5xy' + 3y = \ln x, \quad x > 0$.

Solution Using the transformation $x = e^t$, we obtain

$$\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + 5\frac{dy}{dt} + 3y = \ln(e^t), \quad \text{or} \quad \frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = t. \tag{5.89}$$

The characteristic equation of the corresponding homogeneous equation is

$$m^2 + 4m + 3 = 0, \quad \text{or} \quad (m + 1)(m + 3) = 0, \quad \text{or} \quad m = -1, -3.$$

The complementary function is $y_c(t) = Ae^{-t} + Be^{-3t}$.

Let the particular integral be written as $y_p = c_1t + c_2$. Substituting in Eq. (5.89), we get

$$4c_1 + 3(c_1t + c_2) = t.$$

Comparing the coefficients of t and the constant terms on both sides, we obtain $3c_1 = 1$ and $4c_1 + 3c_2 = 0$. The solution is $c_1 = 1/3$, $c_2 = -4/9$.

The particular integral is $y_p = \frac{t}{3} - \frac{4}{9}$.

The general solution of the given equation is

$$y(t) = Ae^{-t} + Be^{-3t} + \frac{t}{3} - \frac{4}{9}.$$

Substituting $e^t = x$, we get the general solution as

$$y(x) = \frac{A}{x} + \frac{B}{x^3} + \frac{1}{3}\ln x - \frac{4}{9}.$$

Example 5.49 Find the general solution of the equation $x^2y'' - 5xy' + 13y = 30x^2$.

Solution Using the transformation $x = e^t$, we obtain

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 5\frac{dy}{dt} + 13y = 30e^{2t}, \quad \text{or} \quad \frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 13y = 30e^{2t}. \tag{5.90}$$

The characteristic equation of the corresponding homogeneous equation is

$$m^2 - 6m + 13 = 0, \quad \text{or} \quad m = \frac{6 \pm \sqrt{36 - 52}}{2} = \frac{6 \pm 4i}{2} = 3 \pm 2i.$$

The complementary function is $y_c(t) = e^{3t}(A\cos 2t + B\sin 2t)$.

Let the particular integral be written as $y_p = ce^{2t}$. Substituting in equation (5.90), we obtain

$$(4 - 12 + 13)ce^{2t} = 30e^{2t}, \quad \text{or} \quad c = 6.$$

The particular integral is $y_p = 6e^{2t}$.

The general solution is $y(t) = e^{3t}(A \cos 2t + B \sin 2t) + 6e^{2t}$.

Substituting $e^t = x$, we get

$$y(x) = x^3[A \cos (2 \ln x) + B \sin (2 \ln x)] + 6x^2.$$

Example 5.50 Find the general solution of the equation

$$x^3y''' + 5x^2y'' + 5xy' + y = x^2 + \ln x, \quad x > 0.$$

Solution Using the transformation $x = e^t$, we get (in operator notation)

$$[\theta(\theta - 1)(\theta - 2) + 5\theta(\theta - 1) + 5\theta + 1]\, y = e^{2t} + t$$

or
$$[\theta^3 - 3\theta^2 + 2\theta + 5\theta^2 - 5\theta + 5\theta + 1]\, y = e^{2t} + t$$

or
$$[\theta^3 + 2\theta^2 + 2\theta + 1]\, y = e^{2t} + t \tag{5.91}$$

where $\theta = d/dt$.

The characteristic equation of the corresponding homogeneous equation is

$$m^3 + 2m^2 + 2m + 1 = 0, \quad \text{or} \quad (m + 1)(m^2 + m + 1) = 0.$$

Its roots are $m = -1, \dfrac{-1 \pm i\sqrt{3}}{2}$.

The complementary function is

$$y_c(t) = Ae^{-t} + [B \cos (\sqrt{3}t/2) + C \sin (\sqrt{3}t/2)]e^{-t/2}.$$

Let the particular integral be written as $y_p = c_1e^{2t} + c_2t + c_3$.

Then, $\quad y_p' = 2c_1e^{2t} + c_2,\ y_p'' = 4c_1e^{2t},\ y_p''' = 8c_1e^{2t}$.

Substituting in Eq. (5.91), we obtain

$$(8 + 8 + 4 + 1)c_1e^{2t} + c_2t + 2c_2 + c_3 = e^{2t} + t, \quad \text{or} \quad 21c_1e^{2t} + c_2t + 2c_2 + c_3 = e^{2t} + t,$$

Comparing both sides, we get $c_1 = 1/21$, $c_2 = 1$, $c_3 = -2$.

The particular integral is $y_p = \dfrac{1}{21}e^{2t} + t - 2$.

The general solution is

$$y(t) = Ae^{-t} + [B \cos (\sqrt{3}t/2) + C \sin (\sqrt{3}t/2)]e^{-t/2} + \frac{1}{21}e^{2t} + t - 2.$$

Substituting $e^t = x$, we get

$$y(x) = \frac{A}{x} + \frac{1}{\sqrt{x}}[B \cos (\sqrt{3} \ln x/2) + C \sin (\sqrt{3} \ln x/2)] + \left(\frac{x^2}{21} + \ln x - 2\right).$$

Example 5.51 Find the general solution of the equation

$$x^3y''' - 3xy' + 3y = 16x + 9x^2 \ln x, \quad x > 0.$$

Solution Using the transformation $x = e^t$, we get (in operator notation)

$$[\theta(\theta - 1)(\theta - 2) - 3\theta + 3]y = 16e^t + 9te^{2t}$$

or
$$(\theta^3 - 3\theta^2 - \theta + 3)y = 16e^t + 9te^{2t} \tag{5.92}$$

where $\theta = d/dt$. The characteristic equation of the corresponding homogeneous equation is

$$m^3 - 3m^2 - m + 3 = 0, \quad \text{or} \quad (m-1)(m+1)(m-3) = 0, \quad \text{or} \quad m = \pm 1, 3.$$

The complementary function is given by $y_c(t) = Ae^t + Be^{-t} + Ce^{3t}$. Note that e^t, which is one of the linearly independent solutions, also appears as a term on the right hand side of Eq. (5.92). Hence, by the method of undetermined parameters, we write the particular solution as

$$y_p = (c_1 t + c_2)e^{2t} + c_3 t e^t$$

We have

$$y_p' = (c_1 + 2c_1 t + 2c_2)e^{2t} + (1+t)c_3 e^t$$

$$y_p'' = (4c_1 + 4c_1 t + 4c_2)e^{2t} + (2+t)c_3 e^t$$

$$y_p''' = (12c_1 + 8c_1 t + 8c_2)e^{2t} + (3+t)c_3 e^t.$$

Substituting in Eq. (5.92), we obtain

$$[(12c_1 + 8c_1 t + 8c_2) - 3(4c_1 + 4c_1 t + 4c_2) - (c_1 + 2c_1 t + 2c_2) + 3(c_1 t + c_2)]e^{2t}$$
$$+ [(3+t)c_3 - 3(2+t)c_3 - (1+t)c_3 + 3c_3 t]e^t = 16e^t + 9te^{2t}$$

or

$$-(c_1 + 3c_1 t + 3c_2)e^{2t} - 4c_3 e^t = 16e^t + 9te^{2t}.$$

Comparing both sides, we obtain $c_1 + 3c_2 = 0$, $-3c_1 = 9$, $-4c_3 = 16$. The solution is $c_1 = -3$, $c_2 = 1$ and $c_3 = -4$.

The particular integral is $y_p(t) = (1-3t)e^{2t} - 4te^t$.

The general solution is $y(t) = Ae^t + Be^{-t} + Ce^{3t} + (1-3t)e^{2t} - 4te^t$.

Substituting $x = e^t$, we obtain the general solution as

$$y(x) = Ax + \frac{B}{x} + Cx^3 + (1 - 3\ln x)x^2 - 4x\ln x.$$

Exercise 5.6

Find the general solution of the following homogeneous differential equations (Assume $x > 0$ in Problems 1 to 20).

1. $x^2y'' + xy' - 4y = 0.$
2. $x^2y'' + 4xy' + 2y = 0.$
3. $x^2y'' + xy' - y = 0.$
4. $9x^2y'' + 15xy' + y = 0.$
5. $4x^2y'' + 16xy' + 9y = 0.$
6. $2x^2y'' + 2xy' + y = 0.$
7. $x^2y'' + 3xy' + y = 0.$
8. $x^2y'' - xy' + 5y = 0.$
9. $x^2y'' + 3xy' + 10y = 0.$
10. $9x^2y'' + 3xy' + 10y = 0.$
11. $x^3y''' + 2x^2y'' = 0.$
12. $x^3y''' + xy' - y = 0.$
13. $x^3y''' + 4x^2y'' + 2xy' - 2y = 0.$
14. $x^3y''' + 9x^2y'' + 18xy' + 6y = 0.$
15. $x^3y''' - 2xy' + 4y = 0.$
16. $x^3y''' + 3x^2y'' + 14xy' + 34y = 0.$
17. $x^4y^{iv} + 3x^3y''' = 0.$
18. $x^4y^{iv} + 6x^3y''' + 4x^2y'' - 2xy' - 4y = 0.$
19. $4x^4y^{iv} + 24x^3y''' + 43x^2y'' + 19xy' - 4y = 0.$
20. $x^4y^{iv} + 6x^3y''' + 5x^2y'' - xy' + y = 0.$

Find the general solution of the following differential equations (Assume $x > 0$ in Problems 21 to 40).

21. $x^2y'' - 2y = 2x + 6.$
22. $x^2y'' - 3xy' + 3y = 2 + 3\ln x,$

23. $x^2y'' + 2xy' - 2y = 6x - 14$.
24. $x^2y'' + 2xy' - 6y = 15x^2$.
25. $x^2y'' + 2xy' = \cos(\ln x)$.
26. $x^2y'' + 5xy' - 5y = 24x \ln x$.
27. $4x^2y'' + y = 25 \sin(\ln x)$.
28. $x^2y'' - 3xy' + 4y = x^3$.
29. $4x^2y'' + 16xy' + 9y = 19 \cos(\ln x) + 22 \sin(\ln x)$.
30. $x^2y'' + 2xy' - 2y = x \sin(\ln x)$.
31. $x^2y'' - 2xy' - 4y = 6x^2 + 4 \ln x$.
32. $x^3y''' + 8x^2y'' + 5xy' - 5y = 42x^2$.
33. $x^3y''' + 6x^2y'' - 12y = 12/x^2$.
34. $x^3y''' - 3x^2y'' + 7xy' - 8y = 3x^3 + 8x$.
35. $4x^3y''' + 12x^2y'' + xy' + y = 50 \sin(\ln x)$.
36. $(3x + 1)^2y'' + (3x + 1)y' + y = 6x$.
37. $(x + 2)^3y''' + (x + 2)^2y'' + (x + 2)y' - y = 24x^2$.
38. $x^4y^{iv} + 6x^3y''' + 2x^2y'' - 4xy' + 4y = 10/x^3$.
39. $4x^4y^{iv} + 16x^3y''' - x^2y'' + 9xy' - 9y = 14x^2 + 1$.
40. $x^4y^{iv} + 6x^3y''' + 12x^2y'' + 6xy' + 4y = 2/x^2$.

Find the solutions of the following differential equations, which satisfy the given conditions.

41. $2x^2y'' + 3xy' - y = x$, $y(1) = 1$, $y(4) = 41/16$.
42. $4x^2y'' + y = \ln x$, $x > 0$, $y(1) = 0$, $y(e) = 5$.
43. $x^2y'' - 3xy' + 3y = 5x^2 - x$, $y(1) = 1$, $y'(1) = 3/2$.
44. $x^2y'' - xy' + 2y = 6$, $y(1) = 1$, $y'(1) = 2$.
45. $x^2y'' + 3xy' + 10y = 9x^2$, $y(1) = 5/2$, $y'(1) = 8$.

5.5 Operator Methods for Finding Particular Integrals

In section 5.3.1, we have introduced the differential operator D, where $D = d/dx$. For example, we can write

$$L(y) = a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = (aD^2 + bD + c)y = F(D)y.$$

Since D is a differential operator, its inverse D^{-1} defines the integral operator, such that $D^{-1}Df(x) = f(x)$.

In this section, we develop symbolic short cut methods for finding a particular integral of a linear non-homogeneous equation with constant coefficients.

Consider the linear non-homogeneous equation with constant coefficients

$$L(y) = a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_{n-1}\frac{dy}{dx} + a_ny = r(x) \tag{5.93}$$

or

$$L(y) = (a_0D^n + a_1D^{n-1} + \ldots + a_{n-1}D + a_n)y = F(D)y = r(x) \tag{5.94}$$

where $F(D) = a_0D^n + a_1D^{n-1} + \ldots + a_{n-1}D + a_n$, and $a_0, a_1, \ldots, a_n$ are constants. From Eq. (5.94), we write the particular integral as

$$y_p(x) = [F(D)]^{-1}r(x). \tag{5.95}$$

In the following, we develop methods for finding $[F(D)]^{-1}r(x)$ for particular cases of $r(x)$.

5.5.1 Case $r(x) = e^{\alpha x}$.

When $y = e^{\alpha x}$, we have

$$F(D)y = (a_0 D^n + a_1 D^{n-1} + \ldots + a_{n-1} D + a_n)e^{\alpha x}$$
$$= (a_0 \alpha^n + a_1 \alpha^{n-1} + \ldots + a_{n-1}\alpha + a_n)e^{\alpha x} = F(\alpha)e^{\alpha x}.$$

Case $F(\alpha) \neq 0$

We may now symbolically write this equation as

$$y = [F(D)]^{-1} F(\alpha) e^{\alpha x} = F(\alpha)[F(D)]^{-1} e^{\alpha x}$$

since $F(\alpha)$ is a constant. We can further write

$$\frac{1}{F(\alpha)} y = [F(D)]^{-1} e^{\alpha x}, \quad \text{or} \quad [F(D)]^{-1} e^{\alpha x} = \frac{1}{F(\alpha)} e^{\alpha x}.$$

Hence, if $r(x) = e^{\alpha x}$, we obtain

$$y_P(x) = [F(D)]^{-1} e^{\alpha x} = \frac{1}{F(\alpha)} e^{\alpha x}, \quad F(\alpha) \neq 0. \tag{5.96}$$

We can verify that this result is true. Operating with $F(D)$ on both sides, we get

$$F(D) y_P(x) = F(D) \cdot \frac{1}{F(\alpha)} e^{\alpha x} = \frac{1}{F(\alpha)} F(D) e^{\alpha x}$$
$$= \frac{1}{F(\alpha)} F(\alpha) e^{\alpha x} = e^{\alpha x}.$$

Example 5.52 Find the general solution of the differential equation $y'' - 2y' - 3y = 3e^{2x}$.

Solution In operator notation, the given equation is $(D^2 - 2D - 3)y = 3e^{2x}$. The characteristic equation of the corresponding homogeneous equation is

$$(m^2 - 2m - 3) = (m - 3)(m + 1) = 0. \text{ Its roots are } m = -1, 3.$$

The complementary function is given by $y_c(x) = Ae^{-x} + Be^{3x}$.

We have $F(D) = D^2 - 2D - 3$. The particular integral is

$$y_p(x) = [F(D)]^{-1} r(x) = [D^2 - 2D - 3]^{-1} (3e^{2x}) = \frac{3e^{2x}}{F(2)} = \frac{3}{-3} e^{2x} = -e^{2x}.$$

The general solution is

$$y(x) = y_c(x) + y_p(x) = Ae^{-x} + Be^{3x} - e^{2x}.$$

Example 5.53 Find the general solution of the equation $y''' - 2y'' - 5y' + 6y = 4e^{-x} - e^{2x}$.

Solution The given equation in operator notation is

$$F(D)y = (D^3 - 2D^2 - 5D + 6)y = 4e^{-x} - e^{2x}, \quad \text{where} \quad F(D) = D^3 - 2D^2 - 5D + 6.$$

The characteristic equation of the corresponding homogeneous equation is

$$m^3 - 2m^2 - 5m + 6 = 0, \quad \text{or} \quad (m - 1)(m + 2)(m - 3) = 0.$$

The roots of this equation are $m = 1, -2, 3$. The complementary function is

$$y_c(x) = Ae^{x} + Be^{-2x} + Ce^{3x}.$$

The particular integral is

$$y_p(x) = [F(D)]^{-1}(4e^{-x} - e^{2x})$$
$$= [F(D)]^{-1}(4e^{-x}) - [F(D)]^{-1}e^{2x}$$
$$= \frac{4}{F(-1)} e^{-x} - \frac{1}{F(2)} e^{2x} = \frac{e^{-x}}{2} + \frac{e^{2x}}{4}.$$

The general solution is

$$y(x) = y_c(x) + y_p(x) = Ae^x + Be^{-2x} + Ce^{3x} + \frac{e^{-x}}{2} + \frac{e^{2x}}{4}.$$

Before we discuss the case when $F(\alpha) = 0$, let us derive the following result

$$F(D)[g(x)e^{\alpha x}] = e^{\alpha x} F(D + \alpha)\, g(x). \tag{5.97}$$

By Leibniz theorem, we have

$$D^n[e^{\alpha x}g(x)] = (D^n e^{\alpha x})g + {}^nC_1(D^{n-1}e^{\alpha x})(Dg) + \ldots + e^{\alpha x}(D^n g)$$
$$= \alpha^n e^{\alpha x}g + {}^nC_1\alpha^{n-1}e^{\alpha x}Dg + {}^nC_2\alpha^{n-2}e^{\alpha x}(D^2g) + \ldots + e^{\alpha x}(D^n g)$$
$$= e^{\alpha x}[D^n g + {}^nC_1(D^{n-1}g)\alpha + {}^nC_2(D^{n-2}g)\alpha^2 + \ldots + \alpha^n g]$$

(by reversing the order of terms and using the result ${}^nC_r = {}^nC_{n-r}$)

$$= e^{\alpha x}[D^n + {}^nC_1\alpha D^{n-1} + {}^nC_2\alpha^2 D^{n-2} + \ldots + \alpha^n]g$$
$$= e^{\alpha x}[D + \alpha]^n g.$$

Substituting the expressions for $n = 1, 2, \ldots$ on the left hand side of Eq. (5.97), we obtain

$$F(D)[e^{\alpha x}g(x)] = [a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \ldots + a_n](e^{\alpha x}g)$$
$$= a_0D^n(e^{\alpha x}g) + a_1D^{n-1}(e^{\alpha x}g) + \ldots + a_n(e^{\alpha x}g)$$
$$= a_0\, e^{\alpha x}(D + \alpha)^n g + a_1e^{\alpha x}(D + \alpha)^{n-1}g + \ldots + a_n\, e^{\alpha x}g$$
$$= e^{\alpha x}[a_0(D + \alpha)^n + a_1(D + \alpha)^{n-1} + \ldots + a_n]g$$
$$= e^{\alpha x}F(D + \alpha)g.$$

Case $F(\alpha) = 0$.

Let us now consider the case $F(\alpha) = 0$. From the theory of polynomial equations, $(D - \alpha)$ is a factor of $F(D)$. If $F'(\alpha)$ also vanishes, then $(D - \alpha)^2$ is a factor of $F(D)$. If $F(\alpha) = 0 = F'(\alpha) = \ldots = F^{(r-1)}(\alpha)$, $F^r(\alpha) \neq 0$, then $(D - \alpha)^r$ is a factor of $F(D)$. Then, we can write

$$F(D) = (D - \alpha)^r G(D), \quad G(\alpha) \neq 0. \tag{5.98}$$

Let us now write the particular integral of $F(D)y = e^{\alpha x}$ as

$$y_p(x) = [F(D)]^{-1}e^{\alpha x} = [(D - \alpha)^r G(D)]^{-1} e^{\alpha x} = [(D - \alpha)^r]^{-1}[G(D)]^{-1}e^{\alpha x}$$
$$= [(D - \alpha)^r]^{-1}[G(\alpha)]^{-1}e^{\alpha x} \qquad \text{(since } G(\alpha) \neq 0 \text{ and using Eq. (5.96))}$$
$$= \frac{1}{G(\alpha)} [(D - \alpha)^r]^{-1}[e^{\alpha x} \cdot 1] = \frac{1}{G(\alpha)} e^{\alpha x} [(D + \alpha - \alpha)^r]^{-1}[1]$$

(using Eq. (5.97))

$$= \frac{1}{G(\alpha)} e^{\alpha x} [D^r]^{-1}[1] = \frac{1}{G(\alpha)} e^{\alpha x} [D^{-r}][1]$$

$$= \frac{1}{G(\alpha)} e^{\alpha x} \frac{x^r}{r!} = \frac{x^r}{r!} \frac{e^{\alpha x}}{G(\alpha)} \tag{5.99}$$

since D^{-r} represents integration r times.

Therefore, when $F(\alpha) = 0$, $F'(\alpha) \neq 0$, we have $F(D) = (D - \alpha)G(D)$ and the particular integral of $F(D)y = e^{\alpha x}$ is given by

$$y_p(x) = \frac{x}{1!} \frac{e^{\alpha x}}{G(\alpha)}. \tag{5.100}$$

Generalization to the case $r(x) = e^{\alpha x}h(x)$.

Irrespective of whether $F(\alpha)$ vanishes or does not vanish, the above result can be extended to the case $r(x) = e^{\alpha x}h(x)$. We have the particular integral in this case as

$$y_p(x) = [F(D)]^{-1}[e^{\alpha x}h(x)] = e^{\alpha x}[F(D + \alpha)]^{-1}h(x) \tag{5.101}$$

using Eq. (5.97). Now, $[F(D + \alpha)]^{-1}h(x)$ can be evaluated when $h(x)$ is of some particular forms.

Example 5.54 Find the general solution of the equation $y'' + y' - 6y = 5e^{-3x}$.

Solution The equation in operator notation is $(D^2 + D - 6)y = 5e^{-3x}$, where

$$F(D) = D^2 + D - 6 = (D + 3)(D - 2).$$

The characteristic equation of the corresponding homogeneous equation $(D^2 + D - 6)y = 0$ is

$$m^2 + m - 6 = 0, \quad \text{or} \quad (m + 3)(m - 2) = 0, \quad \text{or} \quad m = 2, -3.$$

The complementary function is $y_c(x) = Ae^{2x} + Be^{-3x}$.

Now, $F(m) = m^2 + m - 6$, $F(-3) = 0$ and $F'(-3) = -5 \neq 0$. Therefore,

$$y_p(x) = [(D + 3)(D - 2)]^{-1}(5e^{-3x}) = 5(D + 3)^{-1}[(D - 2)^{-1}e^{-3x}]$$

$$= 5(D + 3)^{-1}(-5)^{-1}e^{-3x} = -(D + 3)^{-1}[e^{-3x} \cdot 1] \qquad \text{(using Eq. (5.96))}$$

$$= -e^{-3x}(D - 3 + 3)^{-1} \cdot 1 = -e^{-3x} D^{-1}(1) = -x\,e^{-3x}. \qquad \text{(using Eq. (5.99))}$$

We might have also used the formula (5.100) directly where $G(D) = D - 2$. The general solution is

$$y(x) = y_c(x) + y_p(x) = Ae^{2x} + Be^{-3x} - x\,e^{-3x}.$$

Example 5.55 Find the general solution of the equation $4y'' - 4y' + y = e^{x/2}$.

Solution The characteristic equation of the corresponding homogeneous equation is

$$4m^2 - 4m + 1 = 0, \quad \text{or} \quad (2m - 1)^2 = 0. \text{ Its roots are } m = 1/2, 1/2.$$

The complementary function is $y_c(x) = (A + Bx)\,e^{x/2}$. We have

$$F(D) = 4D^2 - 4D + 1 = (2D - 1)^2, \text{ where } F(1/2) = 0, \text{ and } F'(1/2) = 0.$$

The particular integral is

$$y_p(x) = (2D - 1)^{-2}(e^{x/2} \cdot 1) = e^{x/2}\left[2\left(D + \frac{1}{2}\right) - 1\right]^{-2}(1)$$

$$= \frac{1}{4} e^{x/2} D^{-2}(1) = \frac{x^2}{8} e^{x/2}.$$

The general solution is $y(x) = (A + Bx)e^{x/2} + (x^2e^{x/2})/8$.

Example 5.56 Find the general solution of the equation $9y''' + 3y'' - 5y' + y = 42e^x + 64e^{x/3}$

Solution The characteristic equation of the corresponding homogeneous equation $9y''' + 3y'' - 5y' + y = 0$ is

$$9m^3 + 3m^2 - 5m + 1 = 0, \quad \text{or} \quad (m + 1)(3m - 1)^2 = 0.$$

The roots of this equation are $m = -1, 1/3, 1/3$. The complementary function is

$$y_c(x) = Ae^{-x} + (Bx + C)e^{x/3}$$

We have $F(D) = 9D^3 + 3D^2 - 5D + 1 = (D + 1)(3D - 1)^2$ and $F(1/3) = 0, F'(1/3) = 0$.

The particular integral is

$$y_p(x) = [(D + 1)(3D - 1)^2]^{-1}(42e^x + 64e^{x/3})$$

$$= [(D + 1)(3D - 1)^2]^{-1}(42e^x) + [(D + 1)(3D - 1)^2]^{-1}(64e^{x/3}).$$

Since $F(1) \neq 0$ and $F(1/3) = 0$, we obtain

$$y_p(x) = [(1 + 1)(3 - 1)^2]^{-1}(42e^x) + (3D - 1)^{-2}[(D + 1)^{-1}(64e^{x/3})]$$

$$= \frac{21}{4}e^x + (3D - 1)^{-2}\left[\frac{64}{(4/3)}e^{x/3}\right]$$

$$= \frac{21}{4}e^x + 48e^{x/3}\left[3\left(D + \frac{1}{3}\right) - 1\right]^{-2}(1)$$

$$= \frac{21}{4}e^x + \frac{48}{9}e^{x/3}D^{-2}(1) = \frac{21}{4}e^x + \frac{8}{3}x^2e^{x/3}.$$

The general solution is

$$y(x) = Ae^{-x} + (Bx + C)e^{x/3} + \frac{21}{4}e^x + \frac{8}{3}x^2e^{x/3}.$$

Example 5.57 Find the general solution of the equation $16y'' + 8y' + y = 48xe^{-x/4}$.

Solution The characteristic equation of the corresponding homogeneous equation is

$$16m^2 + 8m + 1 = 0, \quad \text{or} \quad (4m + 1)^2 = 0. \text{ Its roots are } m = -1/4, -1/4.$$

The complementary function is $y_c(x) = (Ax + B)e^{-x/4}$.

We have $F(D) = 16D^2 + 8D + 1 = (4D + 1)^2$ where $F(-1/4) = 0$, and $F'(-1/4) = 0$.

The particular integral is

$$y_P(x) = (4D + 1)^{-2}(48xe^{-x/4}) = 48e^{-x/4}\left[4\left(D - \frac{1}{4}\right) + 1\right]^{-2}x$$

$$= 48e^{-x/4}(4D)^{-2}(x) = 3e^{-x/4}D^{-2}(x) = \frac{1}{2}x^3e^{-x/4}.$$

The general solution is $y(x) = (Ax + B)e^{-x/4} + \frac{1}{2}x^3e^{-x/4}$.

5.5.2 Case $r(x) = \cos \alpha x$ or $\sin \alpha x$.

Consider first, the case when $F(D)$ contains even powers of D.
When $f(x) = \cos \alpha x$, we have

$$D^2 f = -\alpha^2 \cos \alpha x,\ D^4 f = (-\alpha^2)^2 \cos \alpha x,\ D^6 f = (-\alpha^2)^3 \cos \alpha x, \ldots$$

Let
$$F(D^2)y = [a_0(D^2)^n + a_1(D^2)^{n-1} + a_2(D^2)^{n-2} + \ldots + a_n]y.$$

Now, let $y = \cos \alpha x$, then

$$F(D^2) \cos \alpha x = [a_0(D^2)^n + a_1(D^2)^{n-1} + \ldots + a_n] \cos \alpha x$$
$$= [a_0(-\alpha^2)^n + a_1(-\alpha^2)^{n-1} + \ldots + a_n] \cos \alpha x = F(-\alpha^2) \cos \alpha x \quad (5.102)$$

Case $F(-\alpha^2) \neq 0$

From Eq. (5.102), we symbolically write

$$\cos \alpha x = [F(D^2)]^{-1}[F(-\alpha^2) \cos \alpha x] = F(-\alpha^2)[F(D^2)]^{-1} \cos \alpha x$$

or
$$[F(D^2)]^{-1} \cos \alpha x = \frac{\cos \alpha x}{F(-\alpha^2)}.$$

Therefore, the particular integral of the equation $F(D^2)y = \cos \alpha x$ is given by

$$y_p(x) = [F(D^2)]^{-1} \cos \alpha x = \frac{\cos \alpha x}{F(-\alpha^2)},\quad F(-\alpha^2) \neq 0. \quad (5.103)$$

It is easy to show that similar formula holds when $r(x) = \sin \alpha x$. That is, if $F(D^2)y = \sin \alpha x$, then

$$y_p(x) = [F(D^2)]^{-1} \sin \alpha x = \frac{\sin \alpha x}{F(-\alpha^2)}. \quad (5.104)$$

When odd powers of D also exist in $F(D)$, we can follow the same procedure to obtain $y_p(x)$. Let $F(D) = F_1(D^2) + F_2(D)$, where $F_2(D)$ contains odd powers of D. Then

$$[F_1(D^2) + F_2(D)] \cos \alpha x = [F_1(-\alpha^2) + F_2(D)] \cos \alpha x$$

Since $F(D)$ has constant coefficients, we obtain

$$[F_1(D^2) + F_2(D)]^{-1} \cos \alpha x = [F_1(-\alpha^2) + F_2(D)]^{-1} \cos \alpha x, \quad (5.105)$$

We now simplify $F_1(-\alpha^2) + F_2(D)$ and multiply it by $F_3(D)[F_3(D)]^{-1}$, where $F_3(D)$ contains odd powers of D, such that $[F_1(-\alpha^2) + F_2(D)]F_3(D)$ contains only even powers of D. Formula (5.105) is applied and the procedure is repeated to obtain $y_p(x)$. We illustrate this technique through examples.

Example 5.58 Find the general solution of the equation $y'' + 4y = 6 \cos x$.

Solution It is easy to verify that the complementary function is

$$y_c(x) = A \cos 2x + B \sin 2x.$$

We have $F(D^2) = D^2 + 4$ and $r(x) = 6 \cos x$, that is $\alpha = 1$. Since $F(-\alpha^2) = -\alpha^2 + 4$ and $F(-1) = -1 + 4 = 3 \neq 0$, we have

$$y_p(x) = [(D^2 + 4)]^{-1}(6 \cos x) = \frac{6 \cos x}{F(-\alpha^2)} = \frac{6 \cos x}{F(-1)} = 2 \cos x.$$

The general solution is $y(x) = A \cos 2x + B \sin 2x + 2 \cos x$.

Example 5.59 Find the general solution of the equation $2y'' + y' - y = 16 \cos 2x$.

Solution The characteristic equation of the corresponding homogeneous equation is

$$2m^2 + m - 1 = 0, \quad \text{or} \quad (m + 1)(2m - 1) = 0.$$

Its roots are $m = -1, 1/2$. The complementary function is

$$y_c(x) = Ae^{-x} + Be^{x/2}.$$

We have $F(D) = 2D^2 + D - 1$, $r(x) = 16 \cos 2x$. Therefore, $\alpha = 2$. Using Eq. (5.105), we get

$$\begin{aligned} y_p(x) &= [(2D^2 + D - 1)]^{-1}(16 \cos 2x) = 16[2(-4) + D - 1]^{-1} \cos 2x \\ &= 16(D - 9)]^{-1} \cos 2x = 16(D + 9)[(D + 9)(D - 9)]^{-1} \cos 2x \\ &= 16(D + 9)[(D^2 - 81)]^{-1} \cos 2x = -\frac{16}{85}(D + 9) \cos 2x \\ &= -\frac{16}{85}(9 \cos 2x - 2 \sin 2x) \end{aligned}$$

The general solution is

$$y(x) = Ae^{-x} + Be^{x/2} - \frac{16}{85}(9 \cos 2x - 2 \sin 2x).$$

Example 5.60 Find the general solution of the equation $y'' - 5y' + 4y = 65 \sin 2x$.

Solution The characteristic equation of the corresponding homogeneous equation is

$$m^2 - 5m + 4 = 0, \quad \text{or} \quad (m - 1)(m - 4) = 0. \text{ Its roots are } m = 1, 4.$$

The complementary function is $y_c(x) = Ae^x + Be^{4x}$.

We have $F(D) = D^2 - 5D + 4$, $r(x) = 65 \sin 2x$. Therefore $\alpha = 2$. Using Eq. (5.105) we get the particular integral as

$$\begin{aligned} y_p(x) &= (D^2 - 5D + 4)^{-1}(65 \sin 2x) = 65[-4 - 5D + 4]^{-1}(\sin 2x) \\ &= -\frac{65}{5} D^{-1}(\sin 2x) = \frac{13}{2} \cos 2x. \end{aligned}$$

since integral of $\sin 2x$ is $(-\cos 2x)/2$. The general solution is

$$y(x) = Ae^x + Be^{4x} + \frac{13}{2} \cos 2x.$$

Example 5.61 Find the general solution of the equation $y''' - y'' + 4y' - 4y = \sin 3x$.

Solution The characteristic equation of the homogeneous equation is

$$m^3 - m^2 + 4m - 4 = 0, \quad \text{or} \quad (m - 1)(m^2 + 4) = 0. \text{ Its roots are } m = 1, \pm 2i.$$

The complementary function is $y_c(x) = Ae^x + B \cos 2x + C \sin 2x$.

We have $F(D) = D^3 - D^2 + 4D - 4 = (D - 1)(D^2 + 4)$, $r(x) = \sin 3x$, $\alpha = 3$.

Using Eq. (5.105) we get the particular integral as

$$y_p(x) = [(D - 1)(D^2 + 4)]^{-1}(\sin 3x) = [(D - 1)(-9 + 4)]^{-1} \sin 3x$$

$$= -\frac{1}{5}(D+1)(D+1)^{-1}(D-1)^{-1}\sin 3x = -\frac{1}{5}(D+1)(D^2-1)^{-1}\sin 3x$$

$$= -\frac{1}{5}(D+1)(-9-1)^{-1}\sin 3x = \frac{1}{50}(D+1)\sin 3x = \frac{1}{50}(\sin 3x + 3\cos 3x).$$

The general solution is

$$y(x) = Ae^x + B\cos 2x + C\sin 2x + (3\cos 3x + \sin 3x)/50.$$

Remark 3

Note that the above results hold good when $r(x)$ is also of the form $\cos(\alpha x + a)$ or $\sin(\alpha x + b)$.

Case $F(-\alpha^2) = 0$.

When $F(-\alpha^2) = 0$, we write $\cos\alpha x = \text{Re}(e^{i\alpha x})$ and $\sin\alpha x = \text{Im}(e^{i\alpha x})$ and apply the formula (5.97). We shall illustrate this technique through the following examples.

Example 5.62 Find the general solution of the equation $y'' + y = 6\sin x$.

Solution The complementary function is $y_c(x) = A\cos x + B\sin x$.

We have $F(D^2) = D^2 + 1$, $r(x) = 6\sin x$. Therefore $\alpha = 1$ and $F(-\alpha^2) = F(-1) = 0$.

We write the particular integral as

$$y_p(x) = (D^2+1)^{-1}(6\sin x) = \text{Im}\,(D^2+1)^{-1}(6e^{ix})$$

$$= 6\,\text{Im}\,\{e^{ix}[(D+i)^2+1]^{-1}(1)\} = 6\,\text{Im}\,\{e^{ix}[D^2+2iD]^{-1}(1)\}$$

$$= 6\,\text{Im}\,\{e^{ix}D^{-1}[(D+2i)^{-1}](1)\} = 6\,\text{Im}\,\{e^{ix}D^{-1}(0+2i)^{-1}(1)\} \quad (\because 1 = e^{0x})$$

$$= 3\,\text{Im}\left\{\frac{1}{i}e^{ix}x\right\} = 3x\,\text{Im}\,\{-i(\cos x + i\sin x)\} = -3x\cos x.$$

The general solution is $y(x) = A\cos x + B\sin x - 3x\cos x$.

Example 5.63 Find the general solution of the equation $y'' - 4y' + 13y = 18e^{2x}\sin 3x$.

Solution The characteristic equation of the homogeneous equation is

$$m^2 - 4m + 13 = 0. \text{ Its roots are } m = \frac{4 \pm \sqrt{16-52}}{2} = 2 \pm 3i.$$

The complementary function is $y_c(x) = e^{2x}(A\cos 3x + B\sin 3x)$.

We have $F(D) = D^2 - 4D + 13$ and $r(x) = 18e^{2x}\sin 3x$.

We write the particular integral as

$$y_p(x) = 18[D^2 - 4D + 13]^{-1}(e^{2x}\sin 3x)$$

$$= 18e^{2x}[(D+2)^2 - 4(D+2) + 13]^{-1}(\sin 3x) \qquad \text{(using Eq. (5.97))}$$

$$= 18e^{2x}[D^2+9]^{-1}(\sin 3x) = 18e^{2x}\{\text{Im}\,(D^2+9)^{-1}(e^{3ix})\}$$

$$= 18e^{2x}\{\text{Im}\,e^{3ix}[(D+3i)^2+9]^{-1}(1)\}$$

$$= 18e^{2x}\{\text{Im}\,e^{3ix}[D^2+6iD]^{-1}(1)\}$$

$$= 18e^{2x}\{\text{Im } e^{3ix}D^{-1}(D + 6i)^{-1}(1)\}$$

$$= 18e^{2x}\{\text{Im } e^{3ix}D^{-1}(0 + 6i)^{-1}(1)\} = 18e^{2x}\left\{\text{Im } \frac{1}{6i}xe^{3ix}\right\}$$

$$= 3xe^{2x}\text{ Im }\{-i(\cos 3x + i\sin 3x)\} = -3xe^{2x}\cos 3x.$$

The general solution is

$$y(x) = e^{2x}(A\cos 3x + B\sin 3x) - 3xe^{2x}\cos 3x.$$

5.5.3 Case $r(x) = x^{\alpha}$, $\alpha > 0$ and Integer.

The particular integral of $F(D)y = x^{\alpha}$, is

$$y_p(x) = [F(D)]^{-1}x^{\alpha}$$

Symbolically, we expand the operator $[F(D)]^{-1}$ as an infinite series in ascending powers of D and operate on x^{α}.

Example 5.64 Find the general solution of the equation $y'' + 16y = 64x^2$.

Solution The complementary function is $y_c(x) = A\cos 4x + B\sin 4x$.
The particular integral is

$$y_p(x) = (D^2 + 16)^{-1}(64x^2) = \frac{64}{16}\left[1 + \frac{D^2}{16}\right]^{-1}(x^2)$$

$$= 4\left[1 - \frac{D^2}{16} + \frac{D^4}{256} - \ldots\right]x^2 = 4\left[x^2 - \frac{1}{8}\right]$$

The general solution is $y(x) = A\cos 4x + B\sin 4x + 4x^2 - (1/2)$.

Example 5.65 Find the general solution of the equation $y'' + 4y' + 3y = x\sin 2x$.

Solution The characteristic equation of the corresponding homogeneous equation is

$m^2 + 4m + 3 = 0$, or $(m + 1)(m + 3) = 0$. Its roots are $m = -1, -3$.

The complementary function is $y_c(x) = Ae^{-x} + Be^{-3x}$.

The particular integral is

$$y_p(x) = [D^2 + 4D + 3]^{-1}(\text{Im } xe^{2ix}) = \text{Im }\{e^{2ix}[(D + 2i)^2 + 4(D + 2i) + 3]^{-1}(x)\}$$

$$= \text{Im }\{e^{2ix}[D^2 + 4(1 + i)D + (8i - 1)]^{-1}(x)\}$$

$$= \text{Im }\left\{\frac{e^{2ix}}{8i - 1}\left[1 + \frac{4(1 + i)D}{8i - 1} + \frac{D^2}{8i - 1}\right]^{-1}(x)\right\}$$

$$= \text{Im }\left\{\frac{e^{2ix}}{8i - 1}\left[1 - \frac{4(1 + i)D}{8i - 1} + \ldots\right](x)\right\}$$

$$= \text{Im }\left\{\frac{(8i + 1)}{(-65)}e^{2ix}\left[x - \frac{4(8i + 1)(1 + i)}{(-65)}\right]\right\}$$

$$= \text{Im}\left\{-\frac{1}{65}(8i+1)(\cos 2x + i\sin 2x)\left[x + \frac{4}{65}(9i-7)\right]\right\}$$

$$= \text{Im}\left\{-\frac{1}{65}[(\cos 2x - 8\sin 2x) + i(\sin 2x + 8\cos 2x)]\left[\left(x - \frac{28}{65}\right) + \frac{36}{65}i\right]\right\}$$

$$= -\frac{1}{4225}[65x(8\cos 2x + \sin 2x) - 28(8\cos 2x + \sin 2x) + 36(\cos 2x - 8\sin 2x)]$$

$$= -\frac{1}{4225}[65x(8\cos 2x + \sin 2x) - 188\cos 2x - 316\sin 2x].$$

The general solution is

$$y(x) = Ae^{-x} + Be^{-3x} - \frac{1}{4225}[65x(8\cos 2x + \sin 2x) - 188\cos 2x - 316\sin 2x].$$

Example 5.66 Find the general solution of the equation $y^{iv} + 3y'' = 108x^2$.

Solution The characteristic equation of the homogeneous equation is

$m^4 + 3m^2 = 0$, or $m^2(m^2 + 3) = 0$. Its roots are $m = 0, 0, \pm\sqrt{3}i$.

The complementary function is $y_c(x) = A + Bx + (C\cos\sqrt{3}x + D\sin\sqrt{3}x)$.
We have $F(D) = D^4 + 3D^2 = D^2(D^2 + 3)$. The particular integral is given by

$$y_p(x) = 108[D^2(D^2+3)]^{-1}(x^2) = 108[D^{-2}]\frac{1}{3}\left[1 + \frac{D^2}{3}\right]^{-1}(x^2)$$

$$= 36[D^{-2}]\left[1 - \frac{D^2}{3} + \frac{D^4}{9} - \ldots\right](x^2) = 36D^{-2}\left[x^2 - \frac{2}{3}\right]$$

$$= 36\left[\frac{x^4}{12} - \frac{x^2}{3}\right] = 3x^4 - 12x^2.$$

The general solution is $y(x) = A + Bx + (C\cos\sqrt{3}x + D\sin\sqrt{3}x) + 3x^4 - 12x^2$.

Exercise 5.7

Find the general solution of the following differential equations.

1. $(D^2 + 5D + 4)y = 18e^{2x}$.
2. $(D^2 - 1)y = 8e^{3x}$.
3. $(D^2 - 3D - 4)y = e^x + 6e^{5x}$.
4. $(D^2 + D + 2)y = e^{x/2}$.
5. $(D^2 + 3D + 3)y = 7e^x$.
6. $(D^2 - 2D + 1)y = 5e^{4x} + 4e^{2x}$.
7. $(9D^2 - 6D + 1)y = 4e^{-x}$.
8. $(D^2 - 6D + 9)y = 14e^{3x}$.
9. $(D^2 + D - 6)y = e^{2x}$.
10. $(2D^2 - 3D - 2)y = xe^{-x/2}$.
11. $(D^2 - 1)y = 6xe^x$.
12. $(4D^2 + 9D + 2)y = xe^{-2x}$.
13. $(9D^2 + 6D + 1)y = e^{-x/3}$.
14. $(2D^2 + 7D - 4)y = xe^{-4x}$.
15. $(D^3 + 2D^2 - 5D - 6)y = 4e^x$.
16. $(2D^3 + 3D^2 - 3D - 2)y = 10e^{2x}$.
17. $(D^3 - 2D^2 - D + 2)y = e^{3x}$.
18. $(D^3 - 6D^2 + 12D - 8)y = 18e^{2x}$.

19. $(2D^3 - 3D^2 + 1)y = 16e^x$.
20. $(D^3 + 3D^2 - 4D - 12)y = 12xe^{-2x}$.
21. $(D^2 + 16)y = \cos 2x$.
22. $(2D^2 - 5D + 3)y = \sin x$.
23. $(3D^2 - 7D + 2)y = \sin x + \cos x$.
24. $(2D^2 - 7D + 3)y = \sin 2x$.
25. $(D^2 + D + 1)y = 16 \cos x$.
26. $(8D^2 - 12D + 5)y = 16 \sin x$.
27. $(D^2 + 9)y = \sin 3x$.
28. $(D^2 + 3)y = \cos \sqrt{3}x$.
29. $(D^2 + 2D + 5)y = e^{-x} \cos 2x$.
30. $(D^2 - 4D + 5)y = 24e^{2x} \sin x$.
31. $(D^2 - 6D + 13)y = 28e^{3x} \sin 2x$.
32. $(D^2 - 2D + 10)y = 16e^x \cos 3x + 24e^x \sin 3x$.
33. $(D^3 - 3D^2 + D - 3)y = 6 \cos x$.
34. $(D^3 - D^2 + 9D - 9)y = 30 \cos 3x$.
35. $(D^3 - 4D^2 + 9D - 10)y = 24e^x \sin 2x$.
36. $(4D^3 - 12D^2 + 13D - 10)y = 16e^{x/2} \cos x$.
37. $(D^4 + 5D^2 + 4)y = 16 \sin x + 64 \cos 2x$.
38. $(D^2 + 25)y = 9x^3 + 4x^2$.
39. $(D^2 + 6D + 9)y = 4x^2 - 1$.
40. $(D^2 - 2D - 3)y = 2x^2 + 6x$.
41. $(D^2 - 5D + 6)y = x \cos 2x$.
42. $(D^2 + D - 2)y = x^2 \sin x$.
43. $(D^2 - D - 6)y = xe^{-2x}$.
44. $(D^2 + 7D + 12)y = e^x \sin 2x$.
45. $(D^2 + 4D + 3)y = e^{2x} \cos x$.
46. $(D^2 + 3D + 4)y = e^x \cos(\sqrt{7}x/2)$.
47. $(D^2 + 3D + 2)y = xe^x \sin x$.
48. $(D^2 + 9)y = xe^{2x} \cos x$.
49. $(4D^2 + 8D + 3)y = xe^{-x/2} \cos x$.
50. $(D^4 + 3D^2 + 2)y = 16x^2 \cos x$.
51. If $(2D - 1)y = e^{3x}$, then prove that $(D - 3)(2D - 1)y = 0$. Find the general solution of the second equation and substituting in the first equation obtain the general solution of the first order equation.
52. If $F(D)y = (D - m)y = r(x)$, then show that the particular integral can be written as

$$y_p(x) = e^{mx} \int e^{-mx} r(x)\, dx.$$

53. Show that $y = \frac{1}{n} \int_a^x r(t) \sin n(x - t)\, dt$ is the solution of the equation $y'' + n^2 y = r(x)$.
54. If u is a function of x, then show that

$$F(D)xu = xF(D)u + F'(D)u$$

where $F(D) = a_0 D^n + a_1 D^{n-1} + \ldots + a_n$, and a_i are constants.
55. Let a given differential equation be of the form $F(D)y = r(x) = xu(x)$. Then, using the result in problem 54 prove that the particular integral $y(x)$ can be written as

$$y(x) = [F(D)]^{-1} xu(x) = x[F(D)]^{-1} u(x) - [F'(D)\{F(D)\}^{-2}]u(x).$$

56. The particular integral of the equation $F(D)y = e^{mx}$ is

$$y_p(x) = \frac{x}{1!} \frac{e^{mx}}{G(m)}, \quad \text{where} \quad F(D) = (D - m)G(D),\ G(m) \neq 0,$$

$$y_p(x) = \frac{x^2}{2!} \frac{e^{mx}}{G(m)}, \quad \text{where} \quad F(D) = (D - m)^2 G(D),\ G(m) \neq 0, \text{ etc.}$$

Show that these particular integrals can be written as

$$[F(D)]^{-1} e^{mx} = x\left[\frac{1}{F'(m)}\right]e^{mx}, \quad F(m) = 0,\ F'(m) \neq 0$$

$$[F(D)]^{-1} e^{mx} = x^2\left[\frac{1}{F''(m)}\right]e^{mx}, \quad F(m) = 0,\ F'(m) = 0,\ F''(m) \neq 0, \text{ etc.}$$

Use these formulas to evaluate the particular integral in Problems 8, 9, 13 and 19 of this exercise.

57. If $F(D)$ can be factorised into n distinct factors $F(D) = (D - m_1)(D - m_2) \ldots (D - m_n)$, then show that the particular integral of $F(D)y = r(x)$, can be written as

$$y_p(x) = A_1 e^{m_1 x}\int e^{-m_1 x} r(x)\,dx + A_2 e^{m_2 x}\int e^{-m_2 x} r(x)\,dx + \ldots + A_n e^{m_n x}\int e^{-m_n x} r(x)\,dx$$

Use this formula to evaluate the particular integral in problem 40 of this exercise.

58. The forced oscillations of a mechanical system with periodic input are governed by the non-homogeneous equation

$$m\ddot{y} + c\dot{y} + ky = F_0 \cos \omega t,$$

where $m > 0$, $c > 0$ and $k > 0$. Obtain its general solution when (i) $c \neq 0$ (forced damped oscillations), (ii) $c = 0$ (forced undamped oscillations).

5.6 Simultaneous Linear Equations

In the previous sections, we have discussed the solution of a single linear differential equation, in which y is the dependent variable and x is the independent variable. In this section, we consider the solution of a system of two linear first order equations in two dependent variables y_1 and y_2 and one independent variable t. We shall restrict ourselves to the solution of constant coefficient equations. For example, the equations

(i) $6\dfrac{dy_1}{dt} + 5\dfrac{dy_2}{dt} + 3y_1 + y_2 = 0, \quad \dfrac{dy_2}{dt} - 5y_1 + 3y_2 = e^t$

(ii) $3\dfrac{dy_1}{dt} + 2y_1 + y_2 = e^{-t}, \quad \dfrac{dy_1}{dt} + \dfrac{dy_2}{dt} - 2y_1 + 3y_2 = t$

are two systems of linear, constant coefficient first order equations. These two systems can respectively be written in operator form as

(i) $(6D + 3)y_1 + (5D + 1)y_2 = 0,$ and (ii) $(3D + 2)y_1 + y_2 = e^{-t},$

$\quad -5y_1 + (D + 3)y_2 = e^t, \qquad (D - 2)y_1 + (D + 3)y_2 = t$

where $D = d/dt$.

The solution of such systems can be obtained by eliminating one of the variables and solving the resulting linear, second order equation for the second variable. Sometimes, elimination of one of the variables may also produce a first order equation for the second variable.

We illustrate the method of obtaining the solution through the following examples.

Example 5.67 Find the solution of the system of equations

$$\frac{dy_1}{dt} + 2\frac{dy_2}{dt} - 2y_1 - y_2 = e^{2t} \tag{5.106}$$

$$\frac{dy_2}{dt} + y_1 - 2y_2 = 0. \tag{5.107}$$

Solution

Method 1

Eliminate one of the dependent variables directly. Differentiating Eq. (5.107) with respect to t, we get

$$y_2'' + y_1' - 2y_2' = 0$$

where dash denotes differentiation with respect to t. Substituting for y_1' from Eq. (5.106), we obtain

$$y_2'' - 2y_2' + 2y_1 + y_2 + e^{2t} - 2y_2' = 0, \quad \text{or} \quad y_2'' - 4y_2' + 2y_1 + y_2 = -e^{2t}.$$

Substituting for y_1 from Eq. (5.107), we get

$$y_2'' - 4y_2' + 4y_2 - 2y_2' + y_2 = -e^{2t}, \quad \text{or} \quad y_2'' - 6y_2' + 5y_2 = -e^{2t} \tag{5.108}$$

which is a second order equation in the variable y_2.

The complementary function is $(y_2)_c = Ae^t + Be^{5t}$. The particular integral is

$$(y_2)_p = (D^2 - 6D + 5)^{-1}(-e^{2t}) = \frac{1}{3}e^{2t}.$$

The general solution is $y_2(t) = Ae^t + Be^{5t} + \frac{1}{3}e^{2t}$.

From Eq. (5.107), we obtain

$$y_1 = 2y_2 - y_2' = 2\left(Ae^t + Be^{5t} + \frac{1}{3}e^{2t}\right) - \left(Ae^t + 5Be^{5t} + \frac{2}{3}e^{2t}\right)$$

$$= Ae^t - 3Be^{5t}.$$

This procedure can be very cumbersome in general.

Method 2

We eliminate one of the dependent variables after writing the equations in operator notation. We have

$$(D-2)y_1 + (2D-1)y_2 = e^{2t} \tag{5.109}$$

$$y_1 + (D-2)y_2 = 0. \tag{5.110}$$

Operating with $(D - 2)$ on equation (5.110), we get $(D-2)y_1 + (D-2)^2 y_2 = 0$.

Subtracting Eq. (5.109) from this equation, we get

$$[(D-2)^2 - (2D-1)]y_2 = -e^{2t}, \quad \text{or} \quad (D^2 - 6D + 5)y = -e^{2t}$$

which is the same as Eq. (5.108). The remaining solution procedure is same as in method 1.

Method 3

In this method, we find the equations governing y_1 and y_2 using the determinants, by considering (symbolically) the given equations as algebraic equations. Solving the equations (5.109) and (5.110) by Cramer's rule we obtain

$$\begin{vmatrix} D-2 & 2D-1 \\ 1 & D-2 \end{vmatrix} y_1 = \begin{vmatrix} e^{2t} & 2D-1 \\ 0 & D-2 \end{vmatrix} \text{ and } \begin{vmatrix} D-2 & 2D-1 \\ 1 & D-2 \end{vmatrix} y_2 = \begin{vmatrix} D-2 & e^{2t} \\ 1 & 0 \end{vmatrix}$$

or
$$[(D-2)^2 - (2D-1)]y_1 = (D-2)e^{2t}, \text{ or } (D^2 - 6D + 5)y_1 = 2e^{2t} - 2e^{2t} = 0 \quad (5.111)$$

and
$$(D^2 - 6D + 5)y_2 = -e^{2t}. \quad (5.112)$$

(Note the order of evaluation of the determinants on the right hand side. Otherwise, the method does not make any sense). We would have obtained the first equation if we had eliminated y_2 in method 1 or method 2. Care must be taken to properly choose the arbitrary constants. If we solve Eqs. (5.111) and (5.112), we obtain

$$y_1(t) = Ae^t + Be^{5t} \quad \text{and} \quad y_2(t) = C^*e^t + D^*e^{5t} + \frac{1}{3}e^{2t}.$$

These solutions should satisfy the given equation. Substituting in either of the equations, say (5.110), we get

$$(Ae^t + Be^{5t}) + \left(C^*e^t + 5D^*e^{5t} + \frac{2}{3}e^{2t} - 2C^*e^t - 2D^*e^{5t} - \frac{2}{3}e^{2t}\right) = 0$$

or
$$(A - C^*)e^t + (B + 3D^*)e^{5t} = 0.$$

Since this equation is to be identically satisfied, we get

$$A - C^* = 0, \quad \text{and} \quad B + 3D^* = 0, \quad \text{or} \quad A = C^* \quad \text{and} \quad B = -3D^*.$$

We obtain the general solution as

$$y_1(t) = C^*e^t - 3D^*e^{5t}, \; y_2(t) = C^*e^t + D^*e^{5t} + \frac{1}{3}e^{2t}$$

which is same as the solution obtained earlier.

Example 5.68 Solve the system of equations

$$(2D - 4)y_1 + (3D + 5)y_2 = 3t + 2$$
$$(D - 2)y_1 + (D + 1)y_2 = t.$$

Solution Multiply the second equation by 2 and subtract from the first equation. We obtain

$$(D + 3)y_2 = t + 2$$

which is a linear first order equation in y_2.

The integrating factor is e^{3t}. The solution is

$$e^{3t}y_2 = \int (t + 2)e^{3t}\,dt + A = \frac{1}{3}(t + 2)e^{3t} - \frac{e^{3t}}{9} + A$$

or
$$y_2 = Ae^{-3t} + \frac{1}{9}(3t + 5).$$

Substituting in the second equation, we get

$$(D-2)y_1 + \left[-3Ae^{-3t} + \frac{1}{3} + Ae^{-3t} + \frac{1}{9}(3t+5)\right] = t$$

$$(D-2)y_1 = 2Ae^{-3t} + t - \frac{1}{9}(3t+8) = 2Ae^{-3t} + \frac{1}{9}(6t-8)$$

which is a linear first order equation in y_1.

The integrating factor of this equation is e^{-2t}. The solution is

$$e^{-2t}y_1 = \int [2Ae^{-5t} + \frac{1}{9}(6t-8)e^{-2t}]dt + B$$

$$= -\frac{2}{5}Ae^{-5t} + \frac{1}{9}\left[(6t-8)\frac{e^{-2t}}{(-2)} + \frac{1}{2}\frac{(6e^{-2t})}{(-2)}\right] + B$$

$$= -\frac{2}{5}Ae^{-5t} - \frac{1}{18}(6t-5)e^{-2t} + B$$

or $$y_1 = Be^{2t} - \frac{2}{5}Ae^{-3t} - \frac{1}{18}(6t-5).$$

Example 5.69 Find the solution of the system of equations

$$(3D+1)y_1 + 3Dy_2 = 3t+1$$

$$(D-3)y_1 + Dy_2 = 2t.$$

Solution Multiply the second equation by 3 and substract from the first equation. We obtain

$$10y_1 = (3t+1) - 6t = 1 - 3t, \quad \text{or} \quad y_1 = (1-3t)/10.$$

Substituting in the second equation, we get

$$\frac{1}{10}[-3-3+9t] + Dy_2 = 2t, \quad \text{or} \quad Dy_2 = 2t - \frac{1}{10}(9t-6) = \frac{1}{10}(11t+6)$$

which is a linear first order equation in y_2.

Integrating, we obtain $$y_2 = \frac{11}{20}t^2 + \frac{6}{10}t + A.$$

Note that the system has only one arbitrary constant as the eliminant is a first order equation. This can also be verified by writing the determinant of the coefficient matrix, which is

$$\begin{vmatrix} 3D+1 & 3D \\ D-3 & D \end{vmatrix} = 10D$$

which is of first order only.

5.6.1 Solution of First Order Systems by Matrix Method

The method presented in this section uses some concepts of matrix theory. We discuss the application of this method for solving a 2×2 system which can be generalized to an $n \times n$ system.

Homogeneous systems

Consider a linear homogeneous constant coefficient 2×2 system of the form

$$y_1' = a_{11}y_1 + a_{12}y_2 \tag{5.113}$$

$$y_2' = a_{21}y_1 + a_{22}y_2 \tag{5.114}$$

where y_1, y_2 are the dependent variables, t is the independent variable and $a_{11}, a_{12}, a_{21}, a_{22}$ are constants. Denote

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

In matrix notation, we can write Eqs. (5.113), (5.114) as

$$\mathbf{y}' = \mathbf{A}\mathbf{y}. \tag{5.115}$$

Note that a higher order, constant coefficient homogeneous equation can be reduced to this form. For example, consider the second order equation $y'' + ay' + by = 0$. Denote $y = y_1$ and write

$$y_1' = y_2, \quad (y_2 = y')$$

$$y_2' = -ay' - by = -ay_2 - by_1, \quad (y_2' = y'')$$

or
$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \text{or} \quad \mathbf{y}' = \mathbf{A}\mathbf{y}. \tag{5.116}$$

Therefore, the equation $y'' + ay' + by = 0$ is equivalent to the first order system given by Eq. (5.116).

We know that the scalar equation $y' = my$ has the solution $y = ce^{mt}$. Therefore, for the solution of Eq. (5.115), we examine a solution of the form

$$\mathbf{y} = e^{\lambda t}\mathbf{x} \tag{5.117}$$

where $\mathbf{x} = [x_1, x_2]^T$, or equivalently $y_1 = e^{\lambda t}x_1$, and $y_2 = e^{\lambda t}x_2$.

Substituting Eq. (5.117) into Eq. (5.115), we obtain

$$\lambda e^{\lambda t}\mathbf{x} = \mathbf{A}e^{\lambda t}\mathbf{x} = e^{\lambda t}\mathbf{A}\mathbf{x}.$$

Cancelling $e^{\lambda t}$, we get

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \tag{5.118}$$

which is an algebraic eigenvalue problem. The eigenvalues are the roots of the characteristic equation

$$|\mathbf{A} - \lambda\mathbf{I}| = 0, \quad \text{or} \quad \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0$$

or
$$\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0.$$

Note that the coefficient of λ is equal to – (trace of **A**) and the constant term is $|\mathbf{A}|$. The roots of this equation, that is $\lambda = \lambda_1, \lambda_2$ are called the eigenvalues of **A**. The eigenvalues may be real and distinct, real and equal or a complex conjugate pair. We assume that the system has the complete set of eigenvectors, that is, in the present case the system has two linearly independent eigenvectors. Let the eigenvectors be denoted by $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$. (If $\lambda_1 \neq \lambda_2$, then linear independence of $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}$ is guaranteed). Therefore, we obtain the two linearly independent solutions as

$$\mathbf{y}_1^* = e^{\lambda_1 t}\mathbf{x}^{(1)}, \quad \mathbf{y}_2^* = e^{\lambda_2 t}\mathbf{x}^{(2)}. \tag{5.119}$$

The general solution is

$$\mathbf{y}(t) = A_1\mathbf{y}_1^* + B_1\mathbf{y}_2^* = A_1e^{\lambda_1 t}\mathbf{x}^{(1)} + B_1e^{\lambda_2 t}\mathbf{x}^{(2)}. \tag{5.120}$$

Let $\mathbf{x}^{(1)} = [x_{11}, x_{12}]^T$ and $\mathbf{x}^{(2)} = [x_{21}, x_{22}]^T$. Componentwise, we can write the solution $\mathbf{y}(t)$ as

$$y_1(t) = A_1e^{\lambda_1 t}x_{11} + B_1e^{\lambda_2 t}x_{21}, \quad y_2(t) = A_1e^{\lambda_1 t}x_{12} + B_1e^{\lambda_2 t}x_{22}. \tag{5.121}$$

The method can be extended to an $n \times n$ system of linear (constant coefficient) first order equations.

Example 5.70 Find the general solution of the homogeneous linear system

$$y_1' = -2y_1 + y_2, \quad y_2' = y_1 - 2y_2.$$

Solution In matrix notation, the given system can be written as

$$\mathbf{y}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\mathbf{y} = \mathbf{A}\mathbf{y}.$$

where $\mathbf{y} = [y_1, y_2]^T$. Substituting $\mathbf{y} = e^{\lambda t}\mathbf{x}$ and cancelling $e^{\lambda t}$, we obtain the eigenvalue problem $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$, that is

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{5.122}$$

The characteristic equation of $\mathbf{A}$ is given by

$$\begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = 0, \quad \text{or} \quad (2+\lambda)^2 - 1 = 0, \quad \text{or} \quad \lambda^2 + 4\lambda + 3 = 0.$$

The roots of this equation are $\lambda = -1, -3$.

For $\lambda = -1$, we get from Eq. (5.122)

$$-x_1 + x_2 = 0 \quad \text{and} \quad x_1 - x_2 = 0.$$

The solution is $x_1 = x_2$, so that we can take $\mathbf{x}^{(1)} = [1 \quad 1]^T$

For $\lambda = -3$, we get from Eq. (5.122), $x_1 + x_2 = 0$, so that we can take $\mathbf{x}^{(2)} = [1 \quad -1]^T$.

These two vectors, $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$ are linearly independent. The general solution of the system is

$$\mathbf{y} = A_1e^{-t}\mathbf{x}^{(1)} + B_1e^{-3t}\mathbf{x}^{(2)}$$

or

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = A_1e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + B_1e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Componentwise we can write the solution as

$$y_1 = A_1e^{-t} + B_1e^{-3t}, \quad y_2 = A_1e^{-t} - B_1e^{-3t}.$$

Example 5.71 Find the general solution of the linear homogeneous system

$$y_1' = -ay_1 + ay_2, \quad y_2' = -ay_1 - ay_2, \quad a \neq 0.$$

Solution In matrix notation, the given system can be written as

$$\mathbf{y}' = \begin{bmatrix} -a & a \\ -a & -a \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{A}\,\mathbf{y}$$

Substituting $\mathbf{y} = e^{\lambda t}\mathbf{x}$ and cancelling $e^{\lambda t}$, we obtain the eigenvalue problem $\mathbf{Ax} = \lambda\mathbf{x}$, that is

$$\begin{bmatrix} -a & a \\ -a & -a \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{5.123}$$

The characteristic equation of $\mathbf{A}$ is

$$\begin{vmatrix} -a-\lambda & a \\ -a & -a-\lambda \end{vmatrix} = 0, \text{ or } (a+\lambda)^2 + a^2 = 0 \text{ and its roots are } \lambda = -a \pm ia = -a(1 \mp i).$$

For $\lambda = -a(1+i)$, we get from Eq. (5.123)

$$aix_1 + ax_2 = 0, \quad -ax_1 + aix_2 = 0.$$

The solution is $x_2 = -ix_1$. We can take $\mathbf{x}^{(1)}$ as $\mathbf{x}^{(1)} = [1 \quad -i]^T$.

For $\lambda = -a(1-i)$, we get from Eq. (5.123)

$$-aix_1 + ax_2 = 0, \quad -ax_1 - aix_2 = 0.$$

The solution is $x_2 = ix_1$. We can take $\mathbf{x}^{(2)}$ as $\mathbf{x}^{(2)} = [1 \quad i]^T$.

Therefore, the general solution is

$$\mathbf{y} = A_1 e^{-a(1+i)t}\mathbf{x}^{(1)} + B_1 e^{-a(1-i)t}\mathbf{x}^{(2)}$$

$$= A_1 e^{-at}(\cos at - i\sin at)\begin{bmatrix} 1 \\ -i \end{bmatrix} + B_1 e^{-at}(\cos at + i\sin at)\begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$= A_1 e^{-at}\begin{bmatrix} \cos at - i\sin at \\ -\sin at - i\cos at \end{bmatrix} + B_1 e^{-at}\begin{bmatrix} \cos at + i\sin at \\ -\sin at + i\cos at \end{bmatrix}$$

$$= (A_1 + B_1)e^{-at}\begin{bmatrix} \cos at \\ -\sin at \end{bmatrix} + (B_1 - A_1)ie^{-at}\begin{bmatrix} \sin at \\ \cos at \end{bmatrix} = C^* e^{-at}\begin{bmatrix} \cos at \\ -\sin at \end{bmatrix} + D^* e^{-at}\begin{bmatrix} \sin at \\ \cos at \end{bmatrix}$$

where $C^* = A_1 + B_1$, and $D^* = (B_1 - A_1)i$. Componentwise, the solution is

$$y_1 = e^{-at}(C^* \cos at + D^* \sin at), \quad y_2 = e^{-at}(-C^* \sin at + D^* \cos at).$$

Non-homogeneous systems

Consider now a non-homogeneous, linear constant coefficient system of equations of the form

$$y_1' = a_{11}y_1 + a_{12}y_2 + h_1(t) \tag{5.124}$$

$$y_2' = a_{21}y_1 + a_{22}y_2 + h_2(t) \tag{5.125}$$

or

$$\mathbf{y}' = \mathbf{Ay} + \mathbf{h} \tag{5.126}$$

where $\mathbf{A}$, $\mathbf{y}$ are as defined earlier and $\mathbf{h} = [h_1(t) \quad h_2(t)]^T$. The solution of the system is $\mathbf{y} = \mathbf{y}_c + \mathbf{y}_p$ where $\mathbf{y}_c$ is the complementary function and $\mathbf{y}_p$ is the particular integral. The complementary function $\mathbf{y}_c$ is the solution of the homogeneous equation which can be obtained by the methods described above. The particular integral $\mathbf{y}_p$ can be obtained by the method of undetermined coefficients, if the components of $\mathbf{h}(t)$ are simple functions like a polynomial, exponential, sine or cosine function. However, in general, we can use the *diagonalisation method* to find $\mathbf{y}_p$. We now illustrate both these methods.

5.6.2 Method of Undetermined Coefficients to Find the Particular Integral

Since the method is straightforward we illustrate it through examples.

Examples 5.72 Find the general solution of the linear system of equations $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{h}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}, \mathbf{h} = \begin{bmatrix} 6t^2 + 12t + 9 \\ 4t^2 + 3t + 6 \end{bmatrix}, \text{ and } \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Solution Consider the corresponding homogeneous equation $\mathbf{y}' = \mathbf{A}\mathbf{y}$. Substituting $\mathbf{y} = e^{\lambda t}\mathbf{x}$ and cancelling $e^{\lambda t}$, we obtain the eigenvalue problem $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$, that is

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

The characteristic equation of $\mathbf{A}$ is

$$\begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0, \text{ or } \lambda^2 - 4\lambda - 5 = 0, \text{ and its roots are } \lambda = -1, 5.$$

For $\lambda = -1$, $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ gives the equations

$$2x_1 + 2x_2 = 0, \quad \text{and} \quad 4x_1 + 4x_2 = 0,$$

whose solution is $x_1 = -x_2$. The eigenvector can be taken as $\mathbf{x}^{(1)} = [1 \ \ -1]^T$.

For $\lambda = 5$, $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ gives the equations

$$-4x_1 + 2x_2 = 0, \quad \text{and} \quad 4x_1 - 2x_2 = 0$$

whose solution is $x_2 = 2x_1$. The eigenvector can be taken as $\mathbf{x}^{(2)} = [1 \ \ 2]^T$.

The complementary function is given by

$$\mathbf{y}_c = c_1 e^{-t}\mathbf{x}^{(1)} + c_2 e^{5t}\mathbf{x}^{(2)}.$$

Since the elements of $\mathbf{h}$ are polynomials, we write the particular integral as

$$\mathbf{y}_p = \mathbf{d}t^2 + \mathbf{e}t + \mathbf{f}.$$

Differentiating, we have $\mathbf{y}_p' = 2\mathbf{d}t + \mathbf{e}$.

Substituting in the differential equation, we obtain

$$2\mathbf{d}t + \mathbf{e} = \mathbf{A}(\mathbf{d}t^2 + \mathbf{e}t + \mathbf{f}) + \mathbf{h}.$$

Comparing the coefficients of t^2, t and the constant term, we obtain

$$\mathbf{A}\mathbf{d} + \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \mathbf{0}, \ \mathbf{A}\mathbf{e} + \begin{bmatrix} 12 \\ 3 \end{bmatrix} = 2\mathbf{d}, \ \text{and} \ \mathbf{A}\mathbf{f} + \begin{bmatrix} 9 \\ 6 \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

Consider now the first system. We have the equations

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} + \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or $\quad d_1 + 2d_2 = -6,\quad 4d_1 + 3d_2 = -4$, whose solution is $d_1 = 2$, and $d_2 = -4$.

The second system gives the equations

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} 12 \\ 3 \end{bmatrix} = 2\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = 2\begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

or $\quad e_1 + 2e_2 = -8,\ 4e_1 + 3e_2 = -11$, whose solution is $e_1 = 2/5$, and $e_2 = -21/5$.

The third system gives the equations

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + \begin{bmatrix} 9 \\ 6 \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 2/5 \\ -21/5 \end{bmatrix}$$

or $\quad f_1 + 2f_2 = -43/5,\ 4f_1 + 3f_2 = -51/5$, whose solution is $f_1 = 27/25$, and $f_2 = -121/25$.

The general solution of the given system is

$$\mathbf{y} = \mathbf{y}_c + \mathbf{y}_p = c_1 e^{-t}\mathbf{x}^{(1)} + c_2 e^{5t}\mathbf{x}^{(2)} + \mathbf{d}t^2 + \mathbf{e}t + \mathbf{f}$$

$$= c_1 e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{5t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + t^2\begin{bmatrix} 2 \\ -4 \end{bmatrix} + \frac{t}{5}\begin{bmatrix} 2 \\ -21 \end{bmatrix} + \frac{1}{25}\begin{bmatrix} 27 \\ -121 \end{bmatrix}.$$

Componentwise, the solution is

$$y_1 = c_1 e^{-t} + c_2 e^{5t} + 2t^2 + (2t/5) + (27/25),$$

$$y_2 = -c_1 e^{-t} + 2c_2 e^{5t} - 4t^2 - (21t/5) - (121/25).$$

Example 5.73 Find the general solution of the linear system of equations

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{h} = \begin{bmatrix} 5 & -7 \\ 2 & -4 \end{bmatrix}\mathbf{y} - \begin{bmatrix} 2 \\ 4 \end{bmatrix} e^t.$$

Solution The eigenvalue problem corresponding to the homogeneous equation is $\mathbf{Ax} = \lambda\mathbf{x}$, that is

$$\begin{bmatrix} 5 & -7 \\ 2 & -4 \end{bmatrix}\mathbf{x} = \lambda\mathbf{x}.$$

The characteristic equation is given by

$$\begin{vmatrix} 5-\lambda & -7 \\ 2 & -4-\lambda \end{vmatrix} = 0, \quad \text{or} \quad \lambda^2 - \lambda - 6 = 0, \text{ and its roots are } \lambda = 3, -2.$$

For $\lambda = 3$, $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ gives the equations

$$2x_1 - 7x_2 = 0, \quad \text{and} \quad 2x_1 - 7x_2 = 0$$

whose solution is $x_2 = 2x_1/7$. The eigenvector can be taken as $\mathbf{x}^{(1)} = [7 \quad 2]^T$.

For $\lambda = -2$, $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ gives the equations

$$7x_1 - 7x_2 = 0, \quad 2x_1 - 2x_2 = 0$$

whose solution is $x_1 = x_2$. The eigenvector can be taken as $\mathbf{x}^{(2)} = [1 \quad 1]^T$.

The complementary function is given by

$$\mathbf{y}_c = c_1 e^{3t}\mathbf{x}^{(1)} + c_2 e^{-2t}\mathbf{x}^{(2)}.$$

As in the scalar case, we write the particular integral as $\mathbf{y}_p = \mathbf{d}\, e^t$. We have, $\mathbf{y}_p' = \mathbf{d}e^t$. Substituting in the given equation, we have

$$\mathbf{d}\, e^t = \mathbf{A}\,\mathbf{d}\, e^t - \begin{bmatrix} 2 \\ 4 \end{bmatrix} e^t.$$

Cancelling e^t, we obtain

$$\begin{bmatrix} 5 & -7 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} - \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

or

$$4d_1 - 7d_2 = 2, \quad \text{and} \quad 2d_1 - 5d_2 = 4.$$

The solution of these equations is $d_1 = -3$ and $d_2 = -2$, or $\mathbf{d} = [-3 \quad -2]^T$.

The general solution of the system is

$$\mathbf{y} = \mathbf{y}_c + \mathbf{y}_p = c_1 e^{3t}\mathbf{x}^{(1)} + c_2 e^{-2t}\mathbf{x}^{(2)} + \mathbf{d}\, e^t$$

$$= c_1 e^{3t}\begin{bmatrix} 7 \\ 2 \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^t.$$

Example 5.74 Find the general solution of the linear system of equations

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{h} = \begin{bmatrix} -1 & 3 \\ 2 & -2 \end{bmatrix}\mathbf{y} + \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{-4t}.$$

Solution The eigenvalue problem corresponding to the homogeneous equation is $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$, that is

$$\begin{bmatrix} -1 & 3 \\ 2 & -2 \end{bmatrix}\mathbf{x} = \lambda\,\mathbf{x}.$$

The characteristic equation is

$$\begin{vmatrix} -1-\lambda & 3 \\ 2 & -2-\lambda \end{vmatrix} = 0, \quad \text{or} \quad \lambda^2 + 3\lambda - 4 = 0, \quad \text{or} \quad \lambda = 1, -4.$$

For $\lambda = 1$, $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ gives the equations

$$-2x_1 + 3x_2 = 0, \quad \text{and} \quad 2x_1 - 3x_2 = 0$$

whose solution is $2x_1 = 3x_2$. The eigenvector can be taken as $\mathbf{x}^{(1)} = [3 \quad 2]^T$.

For $\lambda = -4$, $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ gives the equations

$$3x_1 + 3x_2 = 0, \quad 2x_1 + 2x_2 = 0$$

whose solution is $x_1 = -x_2$. The eigenvector can be taken as $\mathbf{x}^{(2)} = [1 \quad -1]^T$.

The complementary function is given by

$$\mathbf{y}_c = c_1 e^t \mathbf{x}^{(1)} + c_2 e^{-4t} \mathbf{x}^{(2)}.$$

We note that e^{-4t} occurs both in $\mathbf{y}_c$ and on the right hand side. Hence, as in the scalar case, we write the particular integral as

$$\mathbf{y}_p = (\mathbf{d}t + \mathbf{e})e^{-4t}$$

We have

$$\mathbf{y}_p' = (\mathbf{d} - 4\mathbf{d}t - 4\mathbf{e})e^{-4t}.$$

Substituting in the given equation, we obtain

$$(-4\mathbf{d}t + \mathbf{d} - 4\mathbf{e})e^{-4t} = \mathbf{A}(\mathbf{d}t + \mathbf{e})e^{-4t} + \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{-4t}.$$

Comparing the coefficients of te^{-4t}, we get

$$-4\mathbf{d} = \mathbf{A}\mathbf{d}.$$

Hence, $\mathbf{d}$ is the eigenvector corresponding to the eigenvalue $\lambda = -4$. Hence, $\mathbf{d} = p\mathbf{x}^{(2)}$, where p is a constant. Comparing the coefficients of e^{-4t}, we obtain

$$\mathbf{d} - 4\mathbf{e} = \mathbf{A}\mathbf{e} + \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

or

$$\begin{bmatrix} -1 & 3 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + 4\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = p\begin{bmatrix} 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

or

$$3e_1 + 3e_2 = p - 4, \quad 2e_1 + 2e_2 = -(p + 1).$$

Therefore, $2p - 8 = -3(p + 1)$, or $p = 1$. Hence, $\mathbf{d} = \mathbf{x}^{(2)}$. With this value, we get $e_1 + e_2 = -1$. We can choose $e_1 = -1$, $e_2 = 0$, so that $\mathbf{e} = [-1 \quad 0]^T$. The general solution of the given system is

$$\begin{aligned} \mathbf{y} = \mathbf{y}_c + \mathbf{y}_p &= c_1 e^t \mathbf{x}^{(1)} + c_2 e^{-4t} \mathbf{x}^{(2)} + (\mathbf{d}t + \mathbf{e})e^{-4t} \\ &= c_1 e^t \mathbf{x}^{(1)} + (c_2 \mathbf{x}^{(2)} + \mathbf{e})e^{-4t} + \mathbf{d}te^{-4t} \\ &= c_1 e^t \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \left\{ c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\} e^{-4t} + te^{-4t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \end{aligned}$$

5.6.3 Method of Diagonalisation to Find the Particular Integral

Let the matrix $\mathbf{A}$ of the non-homogeneous system $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{h}$, have a complete system of eigenvectors. That is, if $\mathbf{A}$ is an $n \times n$ matrix, then there exist n linearly independent eigenvectors $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots \mathbf{x}^{(n)}$. If the eigenvalues λ_i of $\mathbf{A}$ are distinct, then linear independence of the eigenvectors is guaranteed. Let $\mathbf{x}$ denote the matrix of eigenvectors

$$\mathbf{x} = [\mathbf{x}^{(1)}, \mathbf{x}^{(2)} \ldots, \mathbf{x}^{(n)}]. \tag{5.127}$$

Since $\mathbf{x}^{(i)}$, $i = 1, 2, \ldots, n$ are linearly independent, $\mathbf{x}$ is non-singular and $\mathbf{x}^{-1}$ exists. Premultiplying $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{h}$ by $\mathbf{x}^{-1}$, we get

$$\mathbf{x}^{-1}\mathbf{y}' = \mathbf{x}^{-1}\mathbf{A}\mathbf{y} + \mathbf{x}^{-1}\mathbf{h}. \tag{5.128}$$

Let $\mathbf{y} = \mathbf{x}\mathbf{u}$. We have, $\mathbf{y}' = \mathbf{x}\mathbf{u}'$. Substituting in Eq. (5.128), we get

$$\mathbf{x}^{-1}\mathbf{x}\mathbf{u}' = \mathbf{x}^{-1}\mathbf{A}\mathbf{x}\mathbf{u} + \mathbf{x}^{-1}\mathbf{h}. \tag{5.129}$$

We know from matrix theory that the matrix $\mathbf{x}$ (of eigenvectors) diagonalises the matrix $\mathbf{A}$, that is $\mathbf{x}^{-1}\mathbf{A}\mathbf{x} = \mathbf{D}$, where $\mathbf{D}$ is a diagonal matrix with its diagonal entries as the eigenvalues of $\mathbf{A}$. Therefore, we obtain from Eq. (5.129)

$$\mathbf{u}' = \mathbf{D}\mathbf{u} + \mathbf{g}, \quad \text{where} \quad \mathbf{g} = \mathbf{x}^{-1}\mathbf{h}. \tag{5.130}$$

Now, Eq. (5.130) is of the form

$$\begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix}. \tag{5.131}$$

Therefore, Eq. (5.130) degenerates into n independent scalar equations

$$u_j' = \lambda_j u_j + g_j, \quad j = 1, 2, \ldots, n. \tag{5.132}$$

The method of finding the solution of these first order equations was discussed in the previous chapter. The solution of these equations can be written as

$$u_j = c_j e^{\lambda_j t} + e^{\lambda_j t}\left[\int g_j e^{-\lambda_j t}\, dt\right]. \tag{5.133}$$

The solution of the given non-homogeneous system is then obtained from the equation $\mathbf{y} = \mathbf{x}\mathbf{u}$.

Example 5.75 Find the general solution of the linear system of equations

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{h} = \begin{bmatrix} 5 & -7 \\ 2 & -4 \end{bmatrix}\mathbf{y} - \begin{bmatrix} 2 \\ 4 \end{bmatrix} e^t$$

(see Example 5.73)

Solution From Example 5.73, we have

$$\mathbf{x} = [\mathbf{x}^{(1)} \ \ \mathbf{x}^{(2)}] = \begin{bmatrix} 7 & 1 \\ 2 & 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{x}^{-1} = \frac{1}{5}\begin{bmatrix} 1 & -1 \\ -2 & 7 \end{bmatrix}.$$

Now, $$\mathbf{g} = \mathbf{x}^{-1}\mathbf{h} = -\frac{1}{5}\begin{bmatrix} 1 & -1 \\ -2 & 7 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} e^t = -\frac{1}{5}\begin{bmatrix} -2 \\ 24 \end{bmatrix} e^t.$$

The eigenvalues of $\mathbf{A}$ are $\lambda_1 = 3$ and $\lambda_2 = -2$. Eqs. (5.132) become

$$u_1' = \lambda_1 u_1 + g_1 = 3u_1 + \frac{2}{5} e^t$$

$$u_2' = \lambda_2 u_2 + g_2 = -2u_2 - \frac{24}{5} e^t.$$

The solutions of these equations are $u_1 = c_1 e^{3t} - \frac{1}{5} e^t$, $u_2 = c_2 e^{-2t} - \frac{8}{5} e^t$, respectively. Hence

$$\mathbf{y} = \mathbf{x}\mathbf{u} = \begin{bmatrix} 7 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} c_1e^{3t} - (1/5)e^t \\ c_2e^{-2t} - (8/5)e^t \end{bmatrix} = \begin{bmatrix} 7c_1e^{3t} + c_2e^{-2t} - 3e^t \\ 2c_1e^{3t} + c_2e^{-2t} - 2e^t \end{bmatrix}$$

which is same as the solution obtained in Example 5.73.

Exercise 5.8

Find the solution of the following systems of equations using the elimination method.

1. $y_1' = 2y_1 + y_2,\ y_2' = y_1 + 2y_2.$
2. $y_1' = y_1 + y_2,\ y_2' = 9y_1 + y_2.$
3. $y_1' = y_2,\ y_2' = -9y_1.$
4. $y_1' = 2y_1 + y_2,\ y_2' = -18y_1 - 7y_2.$
5. $y_1' + y_2 = 4\sin t,\ y_2' + y_1 = 8\cos t.$
6. $y_1' + y_1 + 3y_2 = 4e^{-t},\ y_2' + 4y_1 - 3y_2 = 8t.$
7. $y_1' + 3y_1 + y_2 = 6e^t,\ y_2' - 5y_1 - 3y_2 = 3e^{-t}.$
8. $y_1' + 4y_1 - 5y_2 = 16\sin t,\ y_2' + 5y_1 - 4y_2 = e^t.$
9. $y_1' + 3y_1 - 5y_2 = 64\sin 4t,\ y_2' + 5y_1 - 3y_2 = 12\cos 2t.$
10. $y_1' + y_1 - 3y_2 = 6e^{-t},\ y_2' + 2y_1 - 4y_2 = 12e^t.$

Find the solution of the following systems of equations.

11. $(D-2)y_1 + (D+3)y_2 = t+4,\ (3D+5)y_1 + (2D+6)y_2 = 5t+3.$
12. $(2D+3)y_1 + (D-1)y_2 = e^t,\ (4D+6)y_1 + 3Dy_2 = e^{-t}.$
13. $(D-1)y_1 + 2Dy_2 = t^2+1,\ (3D+5)y_1 + 6Dy_2 = t+3.$
14. $Dy_1 + (3D-1)y_2 = e^{-t},\ 3Dy_1 + (11D-1)y_2 = 2(e^t + e^{-t}).$
15. $(D+3)y_1 + (3D+23)y_2 = e^{-2t},\ (D+2)y_1 + (4D+14)y_2 = e^{2t}.$
16. $(2D+3)y_1 + (D+5)y_2 = t^2,\ (8D+14)y_1 + (11D+28)y_2 = t+3.$
17. $(2D+1)y_1 + (D+1)y_2 = t,\ (D+2)y_1 + (3D+2)y_2 = 2t+1.$
18. $(D-1)y_1 - (D+1)y_2 = t,\ (D+1)y_1 + (2D+1)y_2 = e^t.$

Using the matrix method, find the solution of the systems of equations $\mathbf{y}' = \mathbf{A}\mathbf{y}$, where $\mathbf{y} = [y_1\ \ y_2]^T$ and

19. $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}.$
20. $\mathbf{A} = \begin{bmatrix} 5 & -3 \\ 2 & -2 \end{bmatrix}.$
21. $\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 9 & -3 \end{bmatrix}.$
22. $\mathbf{A} = \begin{bmatrix} 2 & -4 \\ 2 & -2 \end{bmatrix}.$
23. $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$
24. $\mathbf{A} = \begin{bmatrix} 1/2 & -1 \\ 1 & 1/2 \end{bmatrix}.$

Using the matrix method and the method of undetermined parameters, find the solution of the non-homogeneous system of equations $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{h}$, where $\mathbf{y} = [y_1\ \ y_2]^T$, and

25. $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix},\ \mathbf{h} = \begin{bmatrix} 3t+1 \\ 2t+5 \end{bmatrix}.$
26. $\mathbf{A} = \begin{bmatrix} 1 & -3/2 \\ 1/2 & -1 \end{bmatrix},\ \mathbf{h} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}e^{2t}.$
27. $\mathbf{A} = \begin{bmatrix} 3 & -6 \\ 1 & -4 \end{bmatrix},\ \mathbf{h} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}e^{2t}.$
28. $\mathbf{A} = \begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix},\ \mathbf{h} = \begin{bmatrix} 24 \\ 18 \end{bmatrix}e^t.$

29. $A = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$, $h = \begin{bmatrix} 25 \\ 13 \end{bmatrix} e^{5t}$.

30. In Problems 25 and 26, use the method of diagonalisation to find the solution of the systems.

5.7 Answers and Hints

Exercise 5.1

1. Constant coeff.
2. Variable coeff.
3. Constant coeff.
4. Variable coeff.
5. Variable coeff.
6. Variable coeff.
7. Any subinterval on $(-\infty, 0)$, $(0, \infty)$.
8. Any subinterval on $(-\infty, \infty)$.
9. Any subinterval on $(-\infty, 0)$, $(0, \infty)$.
10. Any subinterval on $[0, \infty)$.
11. Any subinterval on $(3, \infty)$.
12. Any subinterval on $(0, \infty)$.
13. Any subinterval on $(-\infty, 0)$, $(0, 1)$, $(1, \infty)$.
14. $4m < x < 4(m + 1)$, $m = 0, 2, 4, \ldots$.
15. No, because the equation is not normal on any interval containing $x = 0$, Remark 1 is also not applicable.
16. $2x$. No, because the equation is not normal on any interval containing $x = 0$.
17. No, because $x = 0$ at which the equation is not normal is included in the interval $[-3, 3]$, even though the conditions are specified at $x = 2$.
21. $6x + 3 = (3/4)(2x) + (3/2)(3x + 2)$, linearly dependent.
22. Dependent, $9x^2 - x + 2 = 3(x^2 - x) + 2(3x^2 + x + 1)$.
23. Independent, no linear combination can be found, alternately $W = 14$.
24. $W = -16 \sin^6 x$, linearly indpendent.
25. $W = 1$, linearly independent.
26. Dependent, $W = 0$, $x \in I$. Alternately, $\cosh x = e^x - \sinh x$.
27. Linearly independent, $W = -4/x$.
28. Dependent, $W = 0$.
29. Linearly independent, $W = -4$.
30. Dependent, $\sinh x = \cosh x - e^{-x}$.
31. $W = -2 \tan^3 x$, linearly independent on $(0, \pi/2)$, $\left((2n-1)\dfrac{\pi}{2}, (2n+1)\dfrac{\pi}{2}\right)$, $n = 1, 2, \ldots$.
32. (i) Three, (ii) Three.
33. $W(y_1, y_2) = 2$, $y_3 = 2y_1 - y_2/2$.
34. $y_i'' = -(a_1/a_0)y_i' - (a_2/a_0)y_i$, $W(x) = y_1 y_2' - y_2 y_1'$. Differentiating $W(x)$ and substituting for y_i'', we obtain $a_0 W'(x) + a_1 W(x) = 0$. Finding the integrating factor we obtain the solution as given. The value of c depends on y_1, y_2.
35. Substitution shows that $\cos at$, $\sin at$ are solutions. $W = a \neq 0$. y_1, y_2 are linearly independent on any interval I. Using the Abel's formula we get $W = c$, where c can be taken as a. Yes.
36. Substitution shows that e^{2x} and xe^{2x} are solutions of the equation. $W = e^{4x} \neq 0$, y_1, y_2 are linearly independent on any interval I. Using Abel's formula we get $W = ce^{4x}$ which is same as the earlier value when $c = 1$.
37. Normal in $(0, \infty)$, $W = x^{1/2}$. $\{y_1, y_2\}$ forms a basis.
38. Normal in any I, $W = 3e^{4x}$. $\{y_1, y_2\}$ forms a basis.
39. Normal in $(0, \infty)$, $W = 2x$. $\{y_1, y_2\}$ forms a basis.

40. Normal in $(-\infty, \infty)$, $W = 20$. $\{y_1, y_2, y_3\}$ forms a basis.
41. Normal in $(-\infty, \infty)$, $W = e^{3x}$. $\{y_1, y_2, y_3\}$ forms a basis.
42. Normal in $(-\infty, \infty)$, $W = 12\sqrt{3}$. $\{y_1, y_2, y_3\}$ forms a basis.
43. Normal in $(0, \infty)$, $W = -2/x$. $\{y_1, y_2\}$ forms a basis.
44. $W(u, v) = (ad - bc)(y_1 y_2' - y_2 y_1')$. Since $y_1 y_2' - y_2 y_1' \neq 0$, $W(u, v) \neq 0$ if $ad - bc \neq 0$, (the determinant of the coefficient matrix of the transformation). Take $a = 1$, $b = 1$, $c = 1$, $d = -1$, $ad - bc = -2$, $u = e^{kx}$, $v = e^{-kx}$.
45. $W(y_1, y_2) \neq 0$. If for $x_0 \in I$, either $y_1(x_0)$, $y_2(x_0)$ vanish or $y_1'(x_0), y_2'(x_0)$ vanish, then $W(y_1, y_2) = 0$.
46. Simplify $W(y, y_1, y_2)$ and substitute $y_i'' = -(ay_i' + by_i)$, $i = 1, 2$. We obtain
$$W(y, y_1, y_2) = (y'' + ay' + by)(y_1 y_2' - y_2 y_1') = 0.$$
47. At the given point $y_1(x_1) = y'(x_1) = 0$. Therefore, $y_1 \equiv 0$.
48. The differential equation is $W(y, y_1, y_2) = 0$, where $y_1 = e^{3x}$, $y_2 = e^{-2x}$, $y'' - y' - 6y = 0$.
49. $y'' + 2\alpha y' + (\alpha^2 + \omega^2)y = 0$.
50. $y'' - 10y' + 25y = 0$.

Exercise 5.2

1. $(7e^x - e^{4x})/3$.
2. $(3e^{2x} - e^{-2x})/2$.
3. $(1 + 5x)e^{-3x}$.
4. $\frac{1}{2}(5x^2 - (1/x^2))$.
5. $(3 + \ln x)x$.
6. $Ae^{2x} + Be^{-2x}$.
7. $Ae^{2x} + Be^{-x}$.
8. $Ae^{x} + Be^{-2x}$.
9. $Ae^{6x} + Be^{-2x}$.
10. $Ae^{m_1 x} + Be^{m_2 x}$, $m_1 = -2 + \sqrt{3}, m_2 = -2 - \sqrt{3}$.
11. $Ae^{2x} + Be^{x/4}$.
12. $Ae^{x/2} + Be^{-(5x)/2}$.
13. $(A + Bx)e^{-x}$.
14. $(A + Bx)e^{-\pi x}$.
15. $(A + Bx)e^{(2x)/3}$.
16. $(A + Bx)e^{-x/2}$.
17. $(A + Bx)e^{(2x)/5}$.
18. $A \cos 5x + B \sin 5x$.
19. $(A \cos x + B \sin x)e^{-2x}$.
20. $e^x(A \cos x + B \sin x)$.
21. $e^{x/2}(A \cos 2x + B \sin 2x)$.
22. $e^{3x}(A \cos 3x + B \sin 3x)$.
23. $A + Be^{-9x}$.
24. $e^{ax}(A \cos bx + B \sin bx)$.
25. $m = 3, -2$, ch. equation is $m^2 - m - 6 = 0$, diff. equation is $y'' - y' - 6y = 0$.
26. $m = 1/4, -3/4$, ch. equation is $16m^2 + 8m - 3 = 0$, diff. equation is $16y'' + 8y' - 3y = 0$.
27. $m = 0, -2$, ch. equation is $m(m + 2) = 0$, diff. equation is $y'' + 2y' = 0$.
28. $m = 2, 2$, ch. equation is $(m - 2)^2 = 0$, diff. equation is $y'' - 4y' + 4y = 0$.
29. $m = -1, -1$, ch. equation is $(m + 1)^2 = 0$, diff. equation is $y'' + 2y' + y = 0$.
30. $y'' + 9y = 0$.
31. $y'' + 2ay' + (a^2 + b^2)y = 0$.
32. $y'' - 10y' + 34y = 0$.
33. $e^x - e^{-x}$.
34. $e^{4x} + 3e^{-3x}$.
35. $e^x - e^{-2x}$.

36. $a \cos \sqrt{g}\, t$.

37. $e^{2x}(2 \cos x - 5 \sin x)$.

38. $e^{x/5}[\cos (x/5) - \sin (x/5)]$.

39. $((x/2) - 1)\, e^{-(3x)/2}$.

40. $xe^{-x/3}$.

41. $\cos 5x + B \sin 5x$, B arbitrary.

42. $[(2e^2 - 1)e^{-6x} - e^{6x}]/(e^2 - 1)$.

43. $e^{-x}(\cos x + \sin x)$.

44. $(Ax + B)e^{x/3}$, $A = e^{-2/3} - 1$, $B = 2 - e^{-2/3}$.

45. $(e^{x+2} - e^{3x})/(e^2 - 1)$.

48. (i) b = constant, (ii) $a(x) = b(x)$.

49. $(D + 4)(D + 1)y = 0$, set $(D + 1)y = v$ and $(D + 4)v = 0$; $v = A_1e^{-4x}$, $y = Ae^{-4x} + Be^{-x}$.

50. $(2D + 1)(2D + 3)y = 0$, set $(2D + 3)y = v$ and $(2D + 1)v = 0$, $v = A_1e^{-x/2}$, $y = Ae^{-x/2} + Be^{(-3x)/2}$.

51. $(2D + 3)(2D + 3)y = 0$, set $(2D + 3)y = v$, $(2D + 3)v = 0$, $v = A_1e^{-(3x)/2}$, $y = (Ax + B)e^{-(3x)/2}$.

52. $(D + 3)(D + 3)y = 0$, set $(D + 3)y = v$, $(D + 3)v = 0$, $v = A_1e^{-3x}$, $y = (Ax + B)e^{-3x}$.

53. $(D + 2)(D - 2)y = 0$, set $(D - 2)y = v$, $(D + 2)v = 0$, $v = A_1e^{-2x}$, $y = Ae^{-2x} + Be^{2x}$.

54. $(3D + 1)(3D + 1)y = 0$, set $(3D + 1)y = v$, $(3D + 1)v = 0$, $v = A_1e^{-x/3}$, $y = (Ax + B)e^{-x/3}$.

55. For oscillatory solutions, the discriminant of the characteristic equation should be less than zero. $|1 - c| < 2\sqrt{b}$, $1 - 2\sqrt{b} < c < 1 + 2\sqrt{b}$.

56. $\omega = n$, $y(x) = B_n \sin nx$, B_n arbitrary.

57. $y_n(x) = A_n \cos nx$, A_n arbitrary $y(x) = \sum_{n=1}^{\infty} y_n(x)$.

58. $y_n(x) = B_n \sin [(2n + 1)x/2]$, B_n arbitrary $y(x) = \sum_{n=1}^{\infty} y_n(x)$.

59. $y(x) = e^{px}(A'e^{qx} + B'e^{-qx}) = e^{px}[A \cosh qx + B \sinh qx]$.

60. (i) For $c^2 > 4mk$, both the characteristic roots $-p \pm q$ where $p = c/(2m)$ and $q = \sqrt{c^2 - 4mk}/(2m)$, are negative and $q < p$. Therefore, the solution $y(t) = e^{-pt}(Ae^{qt} + Be^{-qt}) \to 0$ as $t \to \infty$, that is, there exists a t_0 such that for $t > t_0$ the system is in equilibrium. $y = [av_0 e^{-pt} \sinh qt]/q$.

(ii) For $c^2 < 4mk$, the characteristic roots are $-p \pm iq$, where $p = c/(2m)$ and $q = \sqrt{4mk - c^2}/(2m)$ are complex. The solutions are oscillatory in this case. The solution is $y(t) = e^{-pt}(A \cos qt + B \sin qt)$. The oscillations are damped and they decay as $t \to \infty$. $y = (e^{-pt}v_0 \sin qt)/q$.

(iii) For $c^2 = 4mk$, the characteristic roots are repeated roots $-p$. The solution is $y(t) = (A + Bt)e^{-pt}$. $y = v_0 te^{-pt}$.

61. $Ae^{3x} + Be^{-2x}$.

62. $Ae^{x} + Be^{-4x}$.

63. $u = x + 1/x$, $y_2 = 1 + x^2$, $Ax + B(1 + x^2)$.

64. $u = -\cot x$, $y_2 = -x^{-1/2} \cos x$, $x^{-1/2}(A \cos x + B \sin x)$.

65. $u = -e^{-x}(x^2 - 2x + 2)$, $y_2 = -(x^2 - 2x + 2)$, $Ae^{x} + B(x^2 - 2x + 2)$.

Exercise 5.3

1. $A + Be^{3x} + Ce^{-3x}$.

2. $Ae^{x/2} + Be^{2x} + Ce^{-3x}$.

3. $Ae^{x} + Be^{-x} + Ce^{2x/3}$.

4. $Ae^{2x} + Be^{-2x} + Ce^{3x} + De^{-3x}$.

5. $Ae^{x} + Be^{2x} + Ce^{-x/2} + De^{x/2}$.

6. $A + Be^{2x} + Ce^{-2x} + De^{-x}$.

7. $Ae^{x/4} + Be^{x/2} + Ce^{x} + De^{-x}$.

8. $Ae^{x/3} + Be^{-x/3} + Ce^{x/4} + De^{-x/4}$.

9. $A + (Bx + C)e^x$.
10. $Ae^{-2x} + (Bx + C)e^{-x}$.
11. $Ae^{-2x} + (Bx + C)e^{2x}$.
12. $(A + Bx + Cx^2)e^{x/3}$.
13. $A + Be^x + (Cx + D)e^{5x}$.
14. $A + (Bx^2 + Cx + D)e^x$.
15. $(Ax + B)e^{-x} + (Cx + D)e^{x/2}$.
16. $(Ax + B)e^{3x} + (Cx + D)e^{2x/3}$.
17. $A + B\cos x + C\sin x$.
18. $Ae^{2x} + B\cos 2x + C\sin 2x$.
19. $Ae^{-3x} + e^{-x}(B\cos x + C\sin x)$.
20. $Ae^x + e^{3x}(B\cos 2x + C\sin 2x)$.
21. $Ae^x + Be^{-x} + C\cos 3x + D\sin 3x$.
22. $Ae^x + Be^{-2x} + C\cos 4x + D\sin 4x$.
23. $A\cos 5x + B\sin 5x + C\cos(x/2) + D\sin(x/2)$.
24. $e^{2x}(A\cos x + B\sin x) + e^{-3x}(C\cos x + D\sin x)$.
25. $(A + Bx)\cos 5x + (C + Dx)\sin 5x$.
26. $(A + Bx)\cos x + (C + Dx)\sin x$.
27. $m = 0, 1, 3,\ y''' - 4y'' + 3y' = 0$.
28. $m = -1, \pm 5i,\ y''' + y'' + 25y' + 25y = 0$.
29. $m = -1, -1, 2,\ y''' - 3y' - 2y = 0$.
30. $m = 0, 0, 1, 3,\ y^{iv} - 4y''' + 3y'' = 0$.
31. $m = 2, 2, 2, -2,\ y^{iv} - 4y''' + 16y' - 16y = 0$.
32. $m = \pm 3, \pm 2i,\ y^{iv} - 5y'' - 36y = 0$.
33. $(3e^{3x} + 2e^{-2x} - 5e^x)/30$.
34. $(9e^x - 5e^{3x/2} + e^{-3x/2})/5$.
35. $(2 + x)e^x - e^{3x}$.
36. $(1 + x)e^{-x} + (2 - x)e^{2x}$.
37. $x + \cos x + \sin x$.
38. $\cos 2x + 2\sin 2x - e^x$.
39. $e^x + e^{-x}(\cos x + 2\sin x)$.
40. $1 + 2x + 3x^2 + e^{3x}$.
41. $A\sin \pi x$, A arbitrary.
42. $1 + 2\sinh 6x + \cosh 6x$.
43. $2\sin 2x + \sin 3x$.
44. $D_n \sin nx$, $\Sigma D_n \sin nx$.
45. $2\cos 3x + \cos x$.

Exercise 5.4

1. $A(x) = -e^{2x}/8$, $B(x) = -e^{-2x}/8$, $y = c_1e^{-x} + c_2e^{3x} - (e^x/4)$.
2. $A(x) = -e^{-4x}/4$, $B(x) = (4x + 1)e^{-4x}/16$, $y = (c_1x + c_2)e^{2x} + e^{-2x}/16$.
3. $A(x) = \cos^3 x/3$, $B(x) = (\sin 3x + 3\sin x)/12$, $y_p = (\cos x)/3$, $y = c_1\cos 2x + c_2\sin 2x + y_p$.
4. $A(x) = \ln|\cos x|$, $B(x) = x$, $y_p = \cos x \ln|\cos x| + x\sin x$, $y = c_1\cos x + c_2\sin x + y_p$.
5. $A(x) = -x$, $B(x) = \ln|\sin x|$, $y_p = \sin x \ln|\sin x| - x\cos x$, $y = c_1\cos x + c_2\sin x + y_p$.
6. $A(x) = \sin x - \ln|\sec x + \tan x|$, $B(x) = -\cos x$, $y_p = -\cos x \ln|\sec x + \tan x|$,
   $y = c_1\cos x + c_2\sin x + y_p$.
7. $A(x) = -x/2$, $B(x) = -e^{-2x}/4$, $y(x) = c_1e^x + c_2e^{3x} - (xe^x)/2$.
8. $A(x) = \frac{1}{4}\ln|\cos 2x|$, $B(x) = x/2$, $y_p = \frac{1}{4}\cos 2x \ln|\cos 2x| + \frac{1}{2}x\sin 2x$.
   $y(x) = c_1\cos 2x + c_2\sin 2x + y_p$.
9. $A(x) = (\cos 4x)/16$, $B(x) = (4x + \sin 4x)/16$, $y_p = (\cos 2x + 4x\sin 2x)/16$.
   $y(x) = c_1\cos 2x + c_2\sin 2x + (x\sin 2x)/4$.
10. $A(x) = \sin x + x\cos x$, $B(x) = -\cos x$, $y_p = -e^{-2x}\sin x$, $y(x) = (c_1x + c_2)e^{-2x} + y_p$.
11. $A(x) = -x$, $B(x) = \ln|x|$, $y_p = x[\ln|x| - 1]e^{-3x}$, $y(x) = (c_1x + c_2)e^{-3x} + y_p$.

12. $A(x) = (\cos 2x)/4$, $B(x) = (2x + \sin 2x)/4$, $y(x) = c_1e^{-x}\cos x + c_2e^{-x}\sin x + (xe^{-x}\sin x)/2$.
13. $g(x) = x$, $A(x) = x^2/4$, $B(x) = -x^4/8$, $y_p = x^3/8$, $y(x) = c_1x + (c_2/x) + y_p$.
14. $g(x) = \ln|x|$, $A(x) = [\ln|x|]^2/8$, $B(x) = -x^4[4\ln|x| - 1]/64$.
$y_p = x^2[8(\ln|x|)^2 - 4\ln|x| + 1]/64$, $y(x) = c_1x^2 + c_2/x^2 + y_p$.
15. $g(x) = 1/x^6$, $A(x) = [1 + 5\ln|x|]/(25x^5)$, $B(x) = -1/(5x^5)$,
$y_p = 1/(25x^4)$, $y(x) = c_1x + c_2x\ln|x| + y_p$.
16. $g(x) = x + (1/x)$, $A(x) = -[(x^2/2) + \ln|x|]$, $B(x) = x - (1/x)$,
$y_p = (x^3/2) - x(1 + \ln|x|)$, $y(x) = c_1x + c_2x^2 + y_p$.
17. $g(x) = 16e^{-2x}\operatorname{cosec}^2 2x$, $A(x) = 4\ln|\operatorname{cosec} 2x + \cot 2x|$, $B(x) = -4/\sin 2x$.
$y_p = 4e^{-2x}\cos 2x\ln|\operatorname{cosec} 2x + \cot 2x| - 4e^{-2x}$, $y(x) = e^{-2x}(c_1\cos 2x + c_2\sin 2x) + y_p$.
18. $A(x) = (\ln|\sec 2x + \tan 2x|)/8$, $B(x) = -x/4$, $C(x) = (\ln|\cos 2x|)/8$,
$y(x) = c_1 + c_2\cos 2x + c_3\sin 2x - (x\cos 2x)/4 + (\sin 2x\ln|\cos 2x|)/8 + (\ln|\sec 2x + \tan 2x|)/8$.
19. $A(x) = x^2/4$, $B(x) = -x$, $C(x) = (\ln|x|)/2$,
$y(x) = (c_1 + c_2x + c_3x^2)e^{2x} + (x^2\ln|x|e^{2x})/2$.
20. $y_p = \dfrac{1}{k}\displaystyle\int_0^x g(t)[\sin kx\cos kt - \cos kx\sin kt]\,dt = \dfrac{1}{k}\int_0^x g(t)\sin[k(x-t)]\,dt$.

Exercise 5.5

1. $y_p = -(50x^2 - 30x + 69)/500$, $y_c = Ae^{-2x} + Be^{5x}$.
2. $y_p = (20 - 51x + 9x^2 - 9x^3)/27$, $y_c = Ae^{-x} + Be^{3x/2}$.
3. $y_p = (35e^x + 3e^{3x})/105$, $y_c = Ae^{x/2} + Be^{-x/2}$.
4. $y_p = (e^{-2x} - 7x - 14)/7$, $y_c = Ae^{-x} + Be^{x/3}$.
5. $y_p = -e^{-3x} + e^x/15$, $y_c = Ae^{-2x} + Be^{-4x}$.
6. $y_p = 3xe^{-x}$, $y_c = Ae^{-x} + Be^{-3x}$.
7. $y_p = -xe^{-2x} + e^x/3$, $y_c = Ae^{-2x} + Be^{x/2}$.
8. $y_p = 2xe^{3x} - xe^{-2x}$, $y_c = Ae^{-2x} + Be^{3x}$.
9. $y_p = 2xe^{x/3}$, $y_c = Ae^{-2x} + Be^{x/3}$.
10. $y_p = (2\sin x - \cos x)/5$, $y_c = Ae^{-x} + Be^{-2x}$.
11. $y_p = (\sin 3x - 5\cos 3x)/2$, $y_c = Ae^{2x} + Be^{-3x}$.
12. $y_p = 2(\sin 2x - \cos 2x)$, $y_c = Ae^x + Be^{-5x}$.
13. $y_p = x(-3\cos 5x + 5\sin 5x)$, $y_c = A\cos 5x + B\sin 5x$.
14. $y_p = -2x\cos 4x$, $y_c = A\cos 4x + B\sin 4x$.
15. $y_p = 4x^2e^{2x} + e^{3x}$, $y_c = (Ax + B)e^{2x}$.
16. $y_p = 3x^2e^{(x/2)}/4$, $y_c = (Ax + B)e^{x/2}$.
17. $y_p = 13x^2e^{-3x} + e^{2x}/5$, $y_c = (Ax + B)e^{-3x}$.
18. $y_p = e^x(\sin x - 2\cos x)/5$, $y_c = A\cos x + B\sin x$.
19. $y_p = -(xe^{-x}\cos 3x)/6$, $y_c = e^{-x}(A\cos 3x + B\sin 3x)$.
20. $y_p = 8xe^{2x}\sin x$, $y_c = e^{2x}(A\cos x + B\sin x)$.

21. $y_p = -3xe^{3x}\cos 2x/4$, $y_c = e^{3x}(A\cos 2x + B\sin 2x)$.
22. $r(x) = 3e^{-2x}(1+\cos 2x)$, $y_p = e^{-2x}(c_1x^2 + c_2\cos 2x + c_3\sin 2x) = [3e^{-2x}(2x^2 - \cos 2x)]/4$, $y_c = (Ax+B)e^{-2x}$.
23. $r(x) = 3e^{-x}(3\sin x - \sin 3x)$, $y_p = e^{-x}[-45(\cos x + \sin x) + (\cos 3x + 3\sin 3x)]/10$, $y_c = Ae^{-x} + Be^{-2x}$.
24. $r(x) = 2(e^{3x} + e^{-3x})$, $y_p = (e^{-3x} + 12xe^{3x})/12$, $y_c = Ae^{x} + Be^{3x}$.
25. $y_p = -3xe^{-x}$, $y_c = Ae^{x} + Be^{-x} + Ce^{-4x}$.
26. $y_p = xe^{x} - 2x^2e^{-2x}$, $y_c = (Ax+B)e^{-2x} + Ce^{x}$.
27. $y_p = 6x^3e^{3x}$, $y_c = (Ax^2 + Bx + C)e^{3x}$.
28. $y_p = 2(\cos 2x - 2\sin 2x)/5$, $y_c = Ae^{x} + B\cos x + C\sin x$.
29. $y_p = -[2(x^2+x) + x(\cos 2x + \sin 2x)]/2$, $y_c = Ae^{2x} + B\cos 2x + C\sin 2x$.
30. $y_p = -x\sin 4x/2$, $y_c = Ae^{4x} + Be^{-4x} + C\cos 4x + D\sin 4x$.
31. $y_p = -(x^4 + 25)$, $y_c = Ae^{x} + Be^{-x} + C\cos x + D\sin x$.
32. $y_p = x^2 - 2x$, $y_c = A + (Bx^2 + Cx + D)e^{-x}$.
33. $y_p = 3xe^{2x}$, $y_c = Ae^{2x} + Be^{-2x} + C\cos x + D\sin x$.
34. $y_p = -5x^3e^{-2x}$, $y_c = A + (Bx^2 + Cx + D)e^{-2x}$.
35. $y_p = -(x^3 + 6x^2)/12$, $y_c = Ax + B + Ce^{4x} + De^{-4x}$.

Exercise 5.6

1. $y = Ax^2 + B/x^2$.
2. $y = (A/x) + (B/x^2)$.
3. $y = Ax + B/x$.
4. $y = (A + B\ln x)x^{-1/3}$.
5. $y = (A + B\ln x)x^{-3/2}$.
6. $y = A\cos(\ln x/\sqrt{2}) + B\sin(\ln x/\sqrt{2})$.
7. $y = (A + B\ln x)/x$.
8. $y = x[A\cos(2\ln x) + B\sin(2\ln x)]$.
9. $y = x^{-1}[A\cos(3\ln x) + B\sin(3\ln x)]$.
10. $y = x^{1/3}[A\cos(\ln x) + B\sin(\ln x)]$.
11. $y = A + Bx + C\ln x$.
12. $y = [A + B\ln x + C\ln^2 x]x$.
13. $y = Ax + x^{-1}[B\cos(\ln x) + C\sin(\ln x)]$.
14. $y = (A/x) + (B/x^2) + (C/x^3)$.
15. $y = (A/x) + (B + C\ln x)x^2$.
16. $y = (A/x^2) + x[B\cos(4\ln x) + C\sin(4\ln x)]$.
17. $y = A + Bx + Cx^2 + D\ln x$.
18. $y = Ax^2 + (B/x^2) + C\cos(\ln x) + D\sin(\ln x)$.
19. $y = A\sqrt{x} + (B/\sqrt{x}) + C\cos(2\ln x) + D\sin(2\ln x)$.
20. $y = (A + B\ln x)x + (C + D\ln x)/x$.
21. $y = Ax^2 + (B/x) - x - 3$.
22. $y = Ax + Bx^3 + \ln x + 2$.
23. $y = Ax + (B/x^2) + 2x\ln x + 7$.
24. $y = Ax^2 + (B/x^3) + 3x^2\ln x$.
25. $y = A + (B/x) + [\sin(\ln x) - \cos(\ln x)]/2$.
26. $y = Ax + (B/x^5) + 2x(3\ln^2 x - \ln x)/3$.
27. $y = (A + B\ln x)x^{1/2} + 4\cos(\ln x) - 3\sin(\ln x)$.
28. $y = (A + B\ln x)x^2 + x^3$.
29. $y = (A + B\ln x)x^{-3/2} + 2\sin(\ln x) - \cos(\ln x)$.
30. $y = Ax + (B/x^2) - x[3\cos(\ln x) + \sin(\ln x)]/10$.
31. $y = (A/x) + Bx^4 - x^2 - \ln x + 3/4$.
32. $y = Ax + (B/x) + (C/x^5) + 2x^2$.

33. $y = Ax^2 + (B/x^2) + (C/x^3) - (3 \ln x)/x^2$.

34. $y = (A + B \ln x + C \ln^2 x)x^2 + 3x^3 - 8x$.

35. $y = (A + B \ln x)x^{1/2} + (C/x) + \sin(\ln x) + 7 \cos(\ln x)$.

36. Set $3x + 1 = z$, $y = [A + B \ln(3x+1)](3x+1)^{1/3} + \frac{3}{2}(x-1)$.

37. Set $x + 2 = z$, $y = A(x+2) + (x+2)^{1/2}[B \cos t + C \sin t] + 8(x+2)^2 - 96(x+2)\ln(x+2) - 96$, where $t = \sqrt{3}\ln(x+2)/2$.

38. $y = Ax + (B/x) + Cx^2 + (D/x^2) + 1/(4x^3)$.

39. $y = Ax^{3/2} + Bx^{-3/2} + (C + D \ln x)x + 2x^2 - 1/9$.

40. $y = A \cos(\ln x) + B \sin(\ln x) + C \cos(2 \ln x) + D \sin(2 \ln x) + 1/(20x^2)$.

41. $y = \frac{1}{4}\left(\sqrt{x} + \frac{1}{x}\right) + \frac{x}{2}$.

42. $y = 4(\ln x - 1)\sqrt{x} + \ln x + 4$.

43. $y = [7x - 10x^2 + 5x^3 + x \ln x]/2$.

44. $y = x[4 \sin(\ln x) - 2 \cos(\ln x)] + 3$.

45. $y = \frac{1}{x}[2 \cos(3 \ln x) + 3 \sin(3 \ln x) + \frac{x^2}{2}$.

Exercise 5.7

1. $Ae^{-x} + Be^{-4x} + e^{2x}$.

2. $Ae^{x} + Be^{-x} + e^{3x}$.

3. $Ae^{-x} + Be^{4x} + e^{5x} - (e^{x})/6$.

4. $e^{-x/2}[A \cos(\sqrt{7}x/2) + B \sin(\sqrt{7}x/2)] + \frac{4}{11}e^{x/2}$.

5. $e^{-3x/2}[A \cos(\sqrt{3}x/2) + B \sin(\sqrt{3}x/2)] + e^{x}$.

6. $(A + Bx)e^{x} + 4e^{2x} + (5e^{4x})/9$.

7. $(A + Bx)e^{x/3} + (e^{-x})/4$.

8. $(A + Bx)e^{3x} + 7x^2e^{3x}$.

9. $Ae^{2x} + Be^{-3x} + (xe^{2x})/5$.

10. $Ae^{2x} + Be^{-x/2} - e^{-x/2}(4x + 5x^2)/50$.

11. $Ae^{x} + Be^{-x} + [3e^{x}(x^2 - x)]/2$.

12. $Ae^{-2x} + Be^{-x/4} - \frac{1}{98}(7x^2 + 8x)e^{-2x}$.

13. $(A + Bx)e^{-x/3} + (x^2e^{-x/3})/18$.

14. $Ae^{x/2} + Be^{-4x} - e^{-4x}(9x^2 + 4x)/162$.

15. $Ae^{-x} + Be^{2x} + Ce^{-3x} - (e^{x})/2$.

16. $Ae^{x} + Be^{-2x} + Ce^{-x/2} + (e^{2x})/2$.

17. $Ae^{x} + Be^{-x} + Ce^{2x} + (e^{3x})/8$.

18. $(A + Bx + Cx^2)e^{2x} + 3x^3e^{2x}$.

19. $(A + Bx)e^{x} + Ce^{-x/2} + (8x^2e^{x})/3$.

20. $Ae^{2x} + Be^{-2x} + Ce^{-3x} - 3e^{-2x}(2x^2 - 3x)/4$.

21. $A \cos 4x + B \sin 4x + (\cos 2x)/12$.

22. $Ae^{x} + Be^{3x/2} + (\sin x + 5 \cos x)/26$.

23. $Ae^{2x} + Be^{x/3} + (3 \cos x - 4 \sin x)/25$.

24. $Ae^{3x} + Be^{x/2} + (14 \cos 2x - 5 \sin 2x)/221$.

25. $e^{-x/2}[A \cos(\sqrt{3}x/2) + B \sin(\sqrt{3}x/2)] + 16 \sin x$.

26. $e^{3x/4}[A \cos(x/4) + B \sin(x/4)] + 16(4 \cos x - \sin x)/51$.

27. $A \cos 3x + B \sin 3x - (x \cos 3x)/6$.

28. $A \cos(\sqrt{3}x) + B \sin(\sqrt{3}x) + (x \sin\sqrt{3}x)/(2\sqrt{3})$.

29. $e^{-x}(A\cos 2x + B\sin 2x) + (xe^{-x}\sin 2x)/4$.
30. $e^{2x}(A\cos x + B\sin x) - 12\,x\cos x\,e^{2x}$.
31. $e^{3x}(A\cos 2x + B\sin 2x) - 7x\cos 2x\,e^{3x}$.
32. $e^{x}[A\cos 3x + B\sin 3x + x\,(8\sin 3x - 12\cos 3x)/3]$.
33. $Ae^{3x} + B\cos x + C\sin x - 3x(\cos x + 3\sin x)/10$.
34. $Ae^{x} + B\cos 3x + C\sin 3x - x(3\cos 3x + \sin 3x)/2$.
35. $Ae^{2x} + e^{x}(B\cos 2x + C\sin 2x) - 6xe^{x}(2\sin 2x - \cos 2x)/5$.
36. $Ae^{2x} + e^{x/2}(B\cos x + C\sin x) - 4xe^{x/2}(2\cos x + 3\sin x)/13$.
37. $A\cos x + B\sin x + C^*\cos 2x + D^*\sin 2x - 8x(\cos x + 2\sin 2x)/3$.
38. $A\cos 5x + B\sin 5x + (225x^3 + 100x^2 - 54x - 8)/625$.
39. $(A + Bx)e^{-3x} + (12x^2 - 16x + 5)/27$.
40. $Ae^{-x} + Be^{3x} - (18x^2 + 30x - 8)/27$.
41. $Ae^{2x} + Be^{3x} + [(52x + 25)(\cos 2x - 5\sin 2x) - 21(5\cos 2x + \sin 2x)]/2704$.
42. $Ae^{x} + Be^{-2x} - [(25x^2 + 5x - 9)(3\sin x + \cos x) + (35x + 12)(3\cos x - \sin x)]/250$.
43. $Ae^{3x} + Be^{-2x} - e^{-2x}(5x^2 + 2x)/50$.
44. $Ae^{-3x} + Be^{-4x} + e^{x}(8\sin 2x - 9\cos 2x)/290$.
45. $Ae^{-x} + Be^{-3x} + e^{2x}(7\cos x + 4\sin x)/130$.
46. $e^{-3x/2}[A\cos p + B\sin p] + 4e^{x}(25\cos p + 10\sqrt{7}\sin p)/1325,\ p = \sqrt{7}x/2$.
47. Write $xe^{x}\sin x = \text{Im}\,[xe^{(1+i)x}]$, $Ae^{-x} + Be^{-2x} + e^{x}[5(1 - x)\cos x + (5x - 2)\sin x]/50$.
48. Write $xe^{2x}\cos x = \text{Re}\,[xe^{(2+i)x}]$, $A\cos 3x + B\sin 3x + e^{2x}[(30x - 11)\cos x + (10x - 2)\sin x]/400$.
49. $Ae^{-x/2} + Be^{-3x/2} - e^{-x/2}[(x - 2)\cos x - (x + 1)\sin x]/8$.
50. $A\cos x + B\sin x + C^*\cos\sqrt{2}x + D^*\sin\sqrt{2}x - 4[9x^2\cos x - (2x^3 - 51x)\sin x]/3$.
51. $y = Ae^{x/2} + Be^{3x},\quad B = 1/5$.
52. $\int e^{-mx}r(x)\,dx = \int e^{-mx}(D - m)y\,dx = e^{-mx}y$, or $y = e^{mx}\int e^{-mx}r(x)\,dx$.
53. Use the result

$$\frac{d}{dx}\int_a^b f(x,t)\,dt = f(x,b)\,\frac{db}{dx} - f(x,a)\,\frac{da}{dx} + \int_a^b \frac{\partial f}{\partial x}\,dt$$

$$\frac{dy}{dx} = \int_a^x r(t)\cos n(x-t)\,dt,\quad \frac{d^2y}{dx^2} = r(x) - n\int_a^b r(t)\sin n(x-t)\,dt = r(x) - n^2 y.$$

54. $D^m(x\,u) = xD^m u + mD^{m-1}u = xD^m u + \left[\frac{d}{dD}D^m\right]u \quad m = 1, 2, \ldots$

$$F(D)(x\,u) = x[a_0D^n + a_1D^{n-1} + \ldots + a_n]u + \frac{d}{dD}[a_0D^n + a_1D^{n-1} + \ldots + a_n]u$$
$$= xF(D)u + F'(D)u.$$

55. $F(D)(xv) = x\,F(D)v + F'(D)v$. Let $F(D)v = u$.

$F(D)[x\{F(D)\}^{-1}u] = xF(D)[F(D)]^{-1}u + F'(D)[F(D)]^{-1}u = xu + F'(D)[F(D)]^{-1}u$

$xu = F(D)[x\{F(D)\}^{-1}u] - F'(D)[F(D)]^{-1}u$

$[F(D)]^{-1}(xu) = x[F(D)]^{-1}u - F'(D)[F(D)]^{-2}u.$

56. When $F(m) = 0$, $F'(m) \neq 0$, $F(D) = (D - m)G(D)$ and $F'(m) = G(m)$.

$$[F(D)]^{-1} e^{mx} = \frac{xe^{mx}}{G(m)} = x\left(\frac{1}{F'(m)}\right)e^{mx},$$

When $F(m) = 0$, $F'(m) = 0$, $F''(m) \neq 0$, $F(D) = (D - m)^2G(D)$ and $F''(m) = 2G(m)$.

$$[F(D)]^{-1} e^{mx} = \frac{x^2}{2!}\frac{e^{mx}}{G(m)} = x^2\left(\frac{1}{F''(m)}\right)e^{mx}.$$

Problem 8: $F(3) = 0$, $F'(3) = 0$, $F''(3) = 2$, $y_p = 7x^2e^{3x}$.

Problem 9: $F(2) = 0$, $F'(2) = 5$, $y_p = (xe^{2x})/5$.

Problem 13: $F(-1/3) = 0$, $F'(-1/3) = 0$, $F''(-1/3) = 18$, $y_p = (x^2e^{-x/3})/18$.

Problem 19: $F(1) = 0$, $F'(1) = 0$, $F''(1) = 6$, $y_p = (8x^2e^x)/3$.

57. Note that $[F(D)]^{-1}$ can be written as

$$[F(D)]^{-1} = A_1(D - m_1)^{-1} + A_2(D - m_2)^{-1} + \ldots + A_n(D - m_n)^{-1}$$

$$\left(\text{equivalent to writing in partial fractions of } \frac{1}{F(m)} \text{ as } \frac{1}{F(m)} = \frac{A_1}{m - m_1} + \frac{A_2}{m - m_2} + \ldots + \frac{A_n}{m - m_n}\right)$$

Now apply the solution of Problem 52.

Problem 40: $y_p = \frac{1}{4}e^{3x}\int e^{-3x}(2x^2 + 6x)\,dx - \frac{1}{4}e^{-x}\int e^{x}(2x^2 + 6x)\,dx$

$= -\frac{1}{27}(18x^2 + 30x - 8).$

58. For forced damped oscillations, $c^2 < 4mk$, $y_c = e^{-ct/(2m)}.[A \cos dt + B \sin dt]$

$d = \sqrt{4mk - c^2}/(2m)$.

For forced undamped oscillations, $y_c = A \cos(\sqrt{k/m}\, t) + B \sin(\sqrt{k/m}\, t)$.

$c \neq 0$, $y_p = F_0\,[(k - m\omega^2) \cos \omega t + c\omega \sin \omega t]/[(k - m\omega^2)^2 + c^2\omega^2]$

$c = 0$, $y_p = F_0 \cos \omega t/(k - m\omega^2)$.

Exercise 5.8

1. $y_1'' - 4y_1' + 3y_1 = 0$, $y_1 = Ae^t + Be^{3t}$, $y_2 = Be^{3t} - Ae^t$.

2. $y_1'' - 2y_1' - 8y_1 = 0$, $y_1 = Ae^{-2t} + Be^{4t}$, $y_2 = 3Be^{4t} - 3Ae^{-2t}$.

3. $y_1 = A \cos 3t + B \sin 3t$, $y_2 = 3(B \cos 3t - A \sin 3t)$.

4. $y_1 = Ae^{-t} + Be^{-4t}$, $y_2 = -(3Ae^{-t} + 6Be^{-4t})$.

5. $y_1 = Ae^t + Be^{-t} + 2 \cos t$, $y_2 = Be^{-t} - Ae^t + 6 \sin t$.

6. $y_1 = Ae^{5t} + Be^{-3t} + 4e^{-t}/3 + 8(15t - 2)/75$.

$3y_2 = 4e^{-t} - 6Ae^{5t} + 2Be^{-3t} - 8(15t + 13)/75$.

7. $y_1 = Ae^{2t} + Be^{-2t} + 4e^t + e^{-t}$, $y_2 = -(5Ae^{2t} + Be^{-2t} + 10e^t + 2e^{-t})$.

8. $y_1 = A\cos 3t + B\sin 3t + (1/2)e^t - 8\sin t + 2\cos t.$
$5y_2 = (4B - 3A)\sin 3t + (3B + 4A)\cos 3t + (5/2)e^t - 50\sin t.$
9. $y_1 = A\cos 4t + B\sin 4t + 5\cos 2t + 32t\sin 4t + 24t\cos 4t.$
$5y_2 = (3B - 4A)\sin 4t + (4B + 3A)\cos 4t - 10\sin 2t + 15\cos 2t - 32\sin 4t + (24 + 200t)\cos 4t.$
10. $y_1 = Ae^t + Be^{2t} - 5e^{-t} - 36te^t$, $3y_2 = 2Ae^t + 3Be^{2t} - 6e^{-t} - 36(1 + 2t)e^t$.
11. $y_1 = Ae^{-9t} + (9t - 16)/27$, $y_2 = -(11/6)Ae^{-9t} + (1/81)(52 + 45t) + Be^{-3t}$.
12. $y_1 = (1/5)e^t + 2e^{-t} - 3Ae^{-2t} + Be^{-3t/2}$, $y_2 = Ae^{-2t} + e^{-t} - (2/3)e^t$.
13. $y_1 = (t - 3t^2)/8$, $y_2 = A + (10t^3 + 21t^2 + 42t)/96$.
14. $y_1 = [(4t - 1 - 8A)e^{-t} - 2e^t]/2$, $y_2 = [(2A - t)e^{-t} + e^t]/2$.
15. $y_1 = (29/6)e^{2t} - 5Ae^{4t} - 10Be^{-t} + e^{-2t}$, $y_2 = Ae^{4t} + Be^{-t} - (5/6)e^{2t}$.
16. $y_1 = -(9/4)Ae^{-t/2} + 3Be^{-2t} + (1413 - 738t + 168t^2)/84$.
$y_2 = Ae^{-t/2} + Be^{-2t} - (207 - 114t + 28t^2)/28$.
17. $y_1 = -(1 + A) + (1/3)Be^{-4t/5}$, $y_2 = A + Be^{-4t/5} + t$.
18. $y_1 = [e^t - 2A - Be^{-t/3} - 6 - 2t + t^2]/2$, $y_2 = A + Be^{-t/3} + 2t - (t^2/2)$.
19. $\mathbf{y} = Ce^{2t}\mathbf{x}^{(1)} + De^{3t}\mathbf{x}^{(2)}$, $\mathbf{x}^{(1)} = [-2, 1]^T$, $\mathbf{x}^{(2)} = [-1, 1]^T$.
20. $\mathbf{y} = Ce^{-t}\mathbf{x}^{(1)} + De^{4t}\mathbf{x}^{(2)}$, $\mathbf{x}^{(1)} = [1, 2]^T$, $\mathbf{x}^{(2)} = [3, 1]^T$.
21. $\mathbf{y} = Ce^{3it}\mathbf{x}^{(1)} + De^{-3it}\mathbf{x}^{(2)}$, $\mathbf{x}^{(1)} = [1 + i, 3]^T$, $\mathbf{x}^{(2)} = [1 - i, 3]^T$, or as
$\mathbf{y} = C\mathbf{z}^{(1)} + D\mathbf{z}^{(2)}$, $\mathbf{z}^{(1)} = [\cos 3t - \sin 3t,\ 3\cos 3t]^T$, $\mathbf{z}^{(2)} = [\cos 3t + \sin 3t,\ 3\sin 3t]$.
22. $\mathbf{y} = A_1e^{2it}\mathbf{x}^{(1)} + B_1e^{-2it}\mathbf{x}^{(2)}$, $\mathbf{x}^{(1)} = [2, 1 - i]^T$, $\mathbf{x}^{(2)} = [2, 1 + i]^T$, or as
$\mathbf{y} = C\mathbf{z}^{(1)} + D\mathbf{z}^{(2)}$, $\mathbf{z}^{(1)} = [2\cos 2t,\ \cos 2t + \sin 2t]^T$, $\mathbf{z}^{(2)} = [2\sin 2t,\ \sin 2t - \cos 2t]^T$.
23. $\mathbf{y} = Ce^t\mathbf{z}^{(1)} + De^t\mathbf{z}^{(2)}$, $\mathbf{z}^{(1)} = [\cos t,\ \sin t]^T$, $\mathbf{z}^{(2)} = [\sin t,\ -\cos t]^T$.
24. $\mathbf{y} = e^{t/2}(C\mathbf{z}^{(1)} + D\mathbf{z}^{(2)})$, $\mathbf{z}^{(1)}$, $\mathbf{z}^{(2)}$ as in problem 23.
25. $\mathbf{y} = c_1e^{2t}\mathbf{x}^{(1)} + c_2e^{-3t}\mathbf{x}^{(2)} - \mathbf{d}t - \mathbf{e}$, $\mathbf{x}^{(1)} = [1, 1]^T$, $\mathbf{x}^{(2)} = [1, -4]^T$, $\mathbf{d} = [4/3, 5/3]^T$, $\mathbf{e} = [17/9, 4/9]^T$.
26. $\mathbf{y} = c_1e^{t/2}\mathbf{x}^{(1)} + c_2e^{-t/2}\mathbf{x}^{(2)} + \mathbf{d}e^{2t}$, $\mathbf{x}^{(1)} = [3, 1]^T$, $\mathbf{x}^{(2)} = [1, 1]^T$, $\mathbf{d} = [-12/15, 28/15]$.
27. $\mathbf{y} = c_1e^{2t}\mathbf{x}^{(1)} + c_2e^{-3t}\mathbf{x}^{(2)} + (\mathbf{d}t + \mathbf{e})e^{2t}$, $\mathbf{x}^{(1)} = [6, 1]^T$, $\mathbf{x}^{(2)} = [1, 1]^T$, $\mathbf{d} = [-6/5, -1/5]^T$, $\mathbf{e} = [-16/5, 0]^T$.
28. $\mathbf{y} = c_1e^t\mathbf{x}^{(1)} + c_2e^{3t}\mathbf{x}^{(2)} + (\mathbf{d}t + \mathbf{e})e^t$, $\mathbf{x}^{(1)} = [1, 1]^T$, $\mathbf{x}^{(2)} = [2, 1]^T$, $\mathbf{d} = [12, 12]^T$, $\mathbf{e} = [-3, 0]^T$.
29. $\mathbf{y} = c_1e^{-t}\mathbf{x}^{(1)} + c_2e^{5t}\mathbf{x}^{(2)} + (\mathbf{d}t + \mathbf{e})e^{5t}$, $\mathbf{x}^{(1)} = [1, -2]^T$, $\mathbf{x}^{(2)} = [1, 1]^T$, $\mathbf{d} = [21, 21]^T$, $\mathbf{e} = [2, 0]^T$.
30. *Problem* 25: $\mathbf{x} = \begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix}$, $\mathbf{g} = \mathbf{x}^{-1}\mathbf{h} = \dfrac{1}{5}\begin{bmatrix} 14t + 9 \\ t - 4 \end{bmatrix}$, $\lambda_1 = 2, \lambda_2 = -3$,

$u_j' = \lambda_j u_j + g_j$, $u_1 = c_1e^{2t} - (7t + 8)/5$, $u_2 = c_2e^{-3t} + (3t - 13)/45$,

$$\mathbf{y} = \mathbf{x}\,\mathbf{u} = \begin{bmatrix} c_1e^{2t} + c_2e^{-3t} - (12t + 17)/9 \\ c_1e^{2t} - 4c_2e^{-3t} - (15t + 4)/9 \end{bmatrix}$$

Problem 26: $\mathbf{x} = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$, $\mathbf{g} = \mathbf{x}^{-1}\mathbf{h} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}e^{2t}$, $\lambda_1 = 1/2, \lambda_2 = -1/2$.

$u_j' = \lambda_j u_j + g_j$, $u_1 = c_1e^{t/2} - (4/3)e^{2t}$, $u_2 = c_2e^{-t/2} + (16/5)e^{2t}$.

$$\mathbf{y} = \mathbf{x}\,\mathbf{u} = \begin{bmatrix} 3c_1e^{t/2} + c_2e^{-t/2} - (4/5)e^{2t} \\ c_1e^{t/2} + c_2e^{-t/2} + (28/15)e^{2t} \end{bmatrix}.$$