Assignment - 6

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Q-115 (Dec 2017) Suppose $X_1, \ldots, X_n$ are i.i.d. random vectors from $N_p(0, \Sigma)$. Let $l \in R^p$, $E(\Sigma_{i=1}^n l^t X_i X_i^t l) = c$ and $E(\Sigma_{i=1}^n X_i X_i^t) = A$. Which of the following statements are necessarily true?

1. $c = l^t l$

2. $l^t(\Sigma_{i=1}^n X_i X_i^t)l$ follows a chi-squared distribution

3. $l^t(\Sigma_{i=1}^{n_1} X_i X_i^t)l$ and $l^t(\Sigma_{i=n_1+1}^n X_i X_i^t)l$ are independently distributed for $1 \leq n_1 \leq n-1$.

4. $A = \Sigma$

Solution Here, $X_1, \ldots, X_n$ are i.i.d. random vectors from $N_p(0, \Sigma)$.

So, $\Sigma_{i=1}^n (X_i X_i^t) \sim W_p(\Sigma, n)$ which is a wishart distribution with degree of freedom n and scale matrix Σ.

$W_p(\Sigma, n)$ is a probability distribution on the set of pxp symmetric non-negative definite random matrices. it is the sum of zero mean (multivariate) normal random variables squared.

now,

$$\Sigma_{i=1}^n l^t X_i X_i^t l = \Sigma_{i=1}^n (l^t X_i)(l^t X_i)^t \tag{1}$$

Let's take $y_i = l^t X_i$ and here $X_i \sim N(0, \Sigma)$ then $y_i \sim N(0, l^t \Sigma l)$ So,

$$\Sigma_{i=1}^n l^t X_i X_i^t l = \Sigma_{i=1}^n y_i y_i^t \tag{2}$$

from eq 1,2 we can say that

$$\Sigma_{i=1}^n l^t X_i X_i^t l \sim (l^t \Sigma l) \chi_n^2 \tag{3}$$

Hence option 2 is correct because $l^t(\Sigma_{i=1}^{n_1} X_i X_i^t)l$ follows non-central chi square distribution

Option 3 is correct because we know that functions of independent random variables are independent and here $l^t(\Sigma_{i=1}^{n_1} X_i X_i^t)l$ and $l^t(\Sigma_{i=n_1+1}^n X_i X_i^t)l$ are functions of X_i which are i.i.d in nature and resultant functions are also independent.

For option 1,4 first We'll find $E(\Sigma_{i=1}^n X_i X_i^t)$

$$\begin{aligned} E(\Sigma_{i=1}^n X_i X_i^t) &= \Sigma_{i=1}^n E(X_i X_i^t) \\ &= nE(X_i X_i^t) \end{aligned} \tag{4}$$

Where the equality follows from the linearity of the expectation and that X_i are identically distributed. but for any random vector Y, $Cov(Y) = E(YY^t) - E(Y)E(Y^t)$. Hence from eq 4

$$\begin{aligned} nE(X_i X_i^t) &= n(Cov(X_i X_i^t) - E(X_i)E(X_i^t)) \\ &= nCov(X_i X_i^t) \\ &= n\Sigma \end{aligned} \tag{5}$$

becuase $E(X_i) = 0$ and $Cov(X_i X_i^t) = \Sigma$. Hence Option 4 is incorrect Nor here from,

$$E(\Sigma_{i=1}^n X_i X_i^t) = n\Sigma \tag{6}$$

So from eq 6,

$$\begin{aligned} l^t E(\Sigma_{i=1}^n X_i X_i^t)l &= nl^t \Sigma l \\ &= c \end{aligned} \tag{7}$$

Hence, Option 1 is also Incorrect.

So here, option 2,3 are correct and option 1,4 are incorrect