Assignment 3

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Q 51 (June 2018) Consider a Markov Chain having state space $S = 1, 2, 3, 4$ with a transation probablity matrix $P = (p_{i,j})$ given by

$$\mathbf{P} = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{Vmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{Vmatrix} \end{array}$$

Then

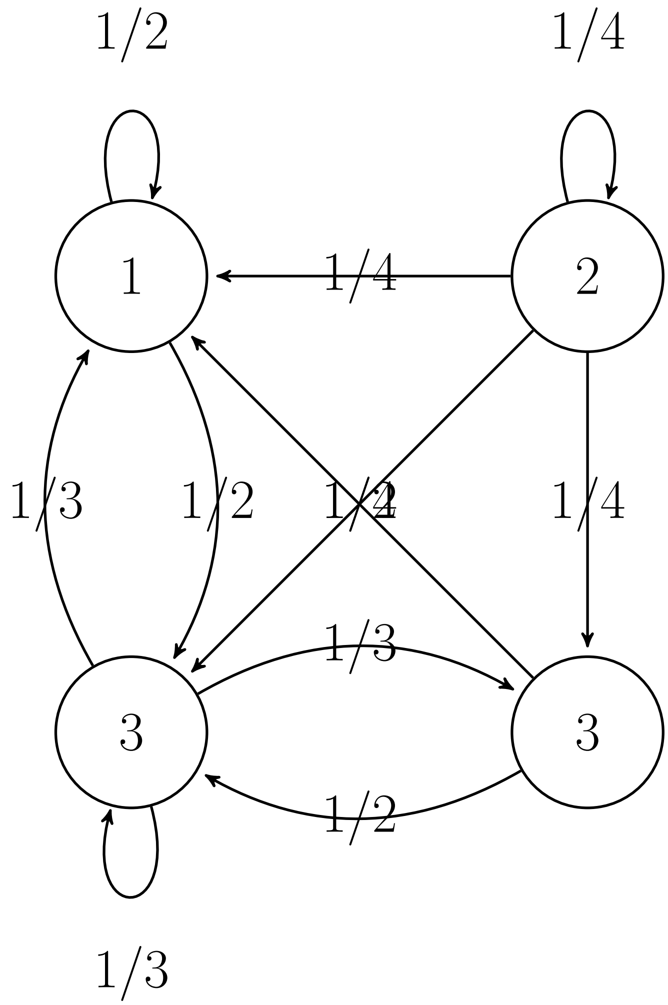
1. $\lim_{n\to\infty} p_{2,2}^{(n)} = 0, \qquad \sum_{n=0}^{\infty} p_{2,2}^{(n)} = \infty$
2. $\lim_{n\to\infty} p_{2,2}^{(n)} = 0, \qquad \sum_{n=0}^{\infty} p_{2,2}^{(n)} < \infty$
3. $\lim_{n\to\infty} p_{2,2}^{(n)} = 1, \qquad \sum_{n=0}^{\infty} p_{2,2}^{(n)} = \infty$
4. $\lim_{n\to\infty} p_{2,2}^{(n)} = 1, \qquad \sum_{n=0}^{\infty} p_{2,2}^{(n)} < \infty$

Solution A markov chain having state space $S = 1, 2, 3, 4$ and transation probablity matrix $P = (p_{i,j})$ is

$$\mathbf{P} = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{Vmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{Vmatrix} \end{array}$$

Limiting Probabilities: $\lim_{n\to\infty} p_{i,j}^{(n)}$ is a probablity of coming to j from i in n steps, where $n \to \infty$.

Let's look at state transition diagram



By looking at state 2 in above diagram we can say that it is a transient state. and

$$p_{2,2}^{(1)} = \frac{1}{4}$$
$$p_{2,2}^{(2)} = \frac{1}{4} * \frac{1}{4} = \left(\frac{1}{4}\right)^2$$
$$p_{2,2}^{(3)} = \left(\frac{1}{4}\right)^3$$

and so on. For $n \to \infty$

$$\begin{aligned} p_{2,2}^{(n)} &= \left(\frac{1}{4}\right)^n \\ &= \left(\frac{1}{4}\right)^\infty \\ &= 0 \end{aligned} \tag{1}$$

Hence, $\lim_{n\to\infty} p_{2,2}^{(n)} = 0$

Now for second part $\sum_{n=0}^{\infty} p_{2,2}^{(n)}$

$$\begin{aligned} \sum_{n=0}^{\infty} p_{2,2}^{(n)} &= p_{2,2}^{(1)} + p_{2,2}^{(2)} + p_{2,2}^{(3)} + \dots \\ &= \left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \dots \\ &= \frac{\frac{1}{4}}{1-\frac{1}{4}} \\ &= \frac{\frac{1}{4}}{\frac{3}{4}} \\ &= \frac{1}{3} \\ &< \infty \end{aligned} \tag{2}$$

Means, $\sum_{n=0}^{\infty} p_{2,2}^{(n)} = \frac{1}{3} < \infty$

Here we have $\lim_{n\to\infty} p_{2,2}^{(n)} = 0$ and $\sum_{n=0}^{\infty} p_{2,2}^{(n)} = \frac{1}{3} < \infty$

Hence, Option 2 is the correct answer