Assignment - 4

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Q-57 (Dec 2018) If $X \sim N_p(0, I)$ and A_{pxp} is an idempotent matrix with $rank(A) = k < p$, then which of the following statements is correct?

1. $\frac{X'AX}{X'X} \sim \frac{k}{p} F_{k,p}$
2. $\frac{X'AX}{X'X} \sim \frac{k}{p-k} F_{k,p-k}$
3. $\frac{X'AX}{X'X} \sim Beta\left(\frac{k}{2}, \frac{p}{2}\right)$
4. $\frac{X'AX}{X'X} \sim Beta\left(\frac{k}{2}, \frac{p-k}{2}\right)$

Solution Here, X has multivariate normal distribution with p variables and A is an idempotent matrix with rank k and and dimension pxp, where $k < p$

Hence, $X'X = \chi_p^2$ (chi squared distribution with p degree of freedom)
Here, A is an idempotent matrix hence we can decompose it like bellow.

$$A = VDV' \tag{1}$$

A is an idempotent matrix means $A^2 = AA = A$ hence from eq 1 we can say that

$$\begin{aligned}(VDV')(VDV') &= A \\ \therefore VD^2V' &= VDV' \\ \therefore D^2 &= D \\ \therefore D &= I\end{aligned} \tag{2}$$

Let's take Z = V'X then $Z \sim N_k(0, I)$ because Z is a linear combination of normal random variables so it also follows normal distribution.

Now,

$$\begin{aligned}X'AX &= X'VDV'X \\ &= Z'DZ \quad (where\ Z\ is\ P'X) \\ &= Z'Z \quad (because\ D\ is\ Identity) \\ &\sim \chi_k^2\end{aligned} \tag{3}$$

Here, in Z'Z chi square distribution is having only k degrees of freedom because in rank(A) = k hence, D will have k entries with 1 and Now we already know that $X'X \sim \chi_p^2$ so

$$\begin{aligned}\frac{X'AX}{X'X} &\sim \frac{\chi_k^2}{\chi_p^2} \\ &\sim \frac{\frac{\chi_k^2}{k}}{\frac{\chi_p^2}{p}} * \frac{k}{p} \\ &\sim F(k, p) * \frac{k}{p}\end{aligned} \tag{4}$$

- Which is Option 1