Automatica 135 (2022) 110002

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Consensus for clusters of agents with cooperative and antagonistic relationships ☆

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A R T I C L E I N F O

Article history:
Received 7 December 2020
Received in revised form 3 May 2021
Accepted 7 September 2021
Available online 12 November 2021

Keywords:
Social networks
Multi-agent systems
Control of networks
Linear systems
k-partite consensus
Signed graphs

A B S T R A C T

In this paper we address the consensus problem in the context of networked agents whose communication graph splits into clusters: interactions between agents in the same cluster are cooperative, while interactions between agents belonging to different clusters are antagonistic. This problem set-up arises in the context of social networks and opinion dynamics, where reaching a consensus means that the opinions of the agents in the same cluster converge to the same decision, that is typically different for the different clusters. Under the assumption that agents belonging to the same cluster have the same amount of trust (/distrust) to be distributed among their cooperators (/adversaries), we propose a modified version of DeGroot's law. By simply constraining how much agents in each group should be conservative about their own opinions, it is possible to achieve a nontrivial solution by means of a distributed algorithm. The result is then particularized to unweighted complete communication graphs, and subsequently extended to a class of nonlinear multi-agent systems.

1. Introduction

Unmanned air vehicles, sensor networks, opinion formation, mobile robots, and biological systems represent just a few examples of the wide variety of contexts where distributed control, and in particular consensus and synchronization algorithms, have been largely employed (Lin et al., 2004; Ogren et al., 2004; Proskurnikov & Tempo, 2017; Ren et al., 2007). The existence of such a broad area of application stimulated a rich literature. Consensus and synchronization problems for multi-agent networked systems have been addressed under different assumptions on the agents' description, their processing capabilities, the communication structure, the reliability of the communication network, etc.

Most of the literature on consensus has focused on the problem of leading all the agents to a common decision, by assuming that the agents cooperate and that the communication network satisfies some form of connectedness. Recently (see, e.g., Monaco & Ricciardi Celsi, 2019), the case of cooperative multi-agent networks, whose communication graphs are not strongly connected, but admit an almost equitable partition, has been investigated. It was shown that in this set up the standard DeGroot's law (DeGroot, 1974) does not lead to standard consensus, but it allows to achieve "multi-consensus". This means that agents, described as simple integrators and belonging to certain subsets (that do not coincide, in general, with the cells of the partition), asymptotically converge to the same value.

Social networks, however, provide clear evidence that mutual relationships may not always be cooperative, and yet the dynamics of opinion forming may exhibit stable asymptotic patterns (Cisneros-Velarde et al., 2021; Estrada, 2019; Xue et al., 2020). In particular, Altafini (2013) showed that, in a network with cooperative and competitive interactions, the agents' opinions may split into two groups that asymptotically converge to two opposite values. Such *bipartite consensus* is possible, by making use of DeGroot's control law, if the communication network is connected and *structurally balanced*, namely the agents are partitioned into two groups such that intra-group relationships are cooperative and inter-group relationships are antagonistic. This analysis was later extended from the case of simple integrators to the case of homogeneous agents described by an arbitrary state-space model (Valcher & Misra, 2014) (see, also, Bauso et al., 2009; Easley & Kleinberg, 2010), and has been in turn investigated by several other authors under different working conditions Xia et al., 2016. Recently, Meng et al. (2020) showed that if the signed communication graph is quasi-strongly connected, interval bipartite consensus is possible for networks of simple integrators that adopt DeGroot's control law, provided that the set of leaders includes balanced nodes. On the other hand, if the graph

☆ The material in this paper was partially presented at the 59th IEEE Conference on Decision and Control, December 14-18, 2020, Jeju Island, Republic of Korea. This paper was recommended for publication in revised form by Associate Editor Claudio Altafini under the direction of Editor Christos G. Cassandras.

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https://doi.org/10.1016/j.automatica.2021.110002
0005-1098/

is (strongly) connected but not structurally balanced, the only possible long-term scenario of a multi-agent network applying DeGroot's law is the trivial one, in which all agents' states converge to zero (Altafini, 2013) (this phenomenon is known as *neutralization* Liu, Xue et al., 2019).

So, it is clear that in a connected signed network which is not structurally balanced, different strategies need to be adopted in order to guarantee that some form of nontrivial consensus is asymptotically achieved. In the last decade there has been a good number of contributions aiming at enforcing some form of multi-consensus/group consensus in different settings (Liu & Chen, 2011; Qin et al., 2015; Qin & Yu, 2013a, 2013b; Qin et al., 2016; Wu et al., 2009; Xia & Cao, 2011; Yu & Wang, 2010). A common ingredient of all these papers is the assumption that the communication network satisfies the "in-degree balanced condition". This assumption means that interactions within the same group are cooperative, while interactions between agents of different groups have both signs, and every agent has a perfect balance between collaborative relationships and antagonistic relationships in every other group. For instance, in Xia and Cao (2011) the concept of n-cluster synchronization is introduced for diffusively coupled networks. In Qin and Yu (2013a) group consensus for homogeneous multi-agent systems described by a stabilizable pair (A, B) is achieved by means of a state feedback law, under the assumption that the communication graph admits an "acyclic partition". In Qin et al. (2016) group consensus for networked systems is investigated, by assuming that agents are modelled as double integrators. Group consensus can be achieved if the underlying topology for each cluster satisfies certain connectivity assumptions and the intra-cluster weights are sufficiently high. The "in-degree balanced condition" is used also in Han et al. (2015), where it is referred to as "inter-cluster common influence". In Zhan and Li (2017) cluster consensus for systems of single integrators under a pinning control action has been studied. For a recent survey about consensus, including all results achieved for group/cluster consensus, see Qin et al. (2017).

There are contexts, however, in which splitting the agents in groups so that the "in-degree balanced condition" holds, does not seem very realistic and does not lead to any long term stable pattern with a meaningful practical interpretation. Consider, for instance, the case of social networks with antagonistic relationships that arise when modelling soccer fans supporting different clubs, or political activists voting for different parties. In these cases, it is natural to assume that interactions between agents in the same cluster are cooperative, while interactions between agents belonging to different clusters may only be antagonistic. Also, it is reasonable to assume that the dynamics of decision/opinion forming mirrors the group partition of the agents (Altafini, 2012). This set-up represents the natural extension to the case of an arbitrary number of clusters of the one adopted in Altafini (2013) for two groups. For networks clustered in this way, it makes sense to investigate whether a different control strategy, rather than DeGroot's control law, can ensure that all individuals that cooperate (and hence necessarily belong to the same cluster) converge to the same decision/opinion. This seems much closer to the kind of consensus problems arising in the economical, biological, sociological fields (see, e.g., Easley & Kleinberg, 2010; Wasserman & Faust, 1994). Sociological models were, in fact, the primary motivation behind the set-up adopted in Altafini (2013). The proposed extension to an arbitrary number of clusters explored in this paper is in perfect agreement with the perspective adopted in Proskurnikov and Tempo (2017), and in the milestone paper by Davis (1957), where the concept of clustering balance was introduced. It is also worth remarking that, as shown in Cisneros-Velarde and Bullo (2020), clustering balance naturally arises as a stable long term configuration in social networks whose agents try to minimize the cognitive dissonances, by modifying their mutual appraisals.

In detail, in this paper we assume that the communication among agents is modelled by an undirected, signed, weighted, connected graph, and that the agents are partitioned into k clusters, such that intra-cluster interactions may only be nonnegative, while inter-cluster interactions can only be nonpositive. We investigate under what conditions a revised version of the DeGroot's distributed feedback control law can lead the multi-agent system to *k-partite consensus*. To explore this problem we introduce a *homogeneity condition* (in fact, similar to the one adopted in Xia & Cao, 2011) that requires that each agent in a group distributes the same amount of "trust" to the agents in its own group and "distrust" to the agents belonging to adverse groups. This represents an alternative extension of the concept of equitable partition for signed graphs with respect to the one proposed in Liu, Ji et al. (2019).

This work generalizes the preliminary results presented in De Pasquale and Valcher (2020) for the case of multi-agent systems partitioned into three groups. The generalization is not trivial, since it requires to extend the proposed algorithm from three steps to an arbitrary number of steps. Moreover, the assumptions under which k-partite consensus is achieved have been generalized and better clarified. Finally, k-partite consensus is also extended to a special class of nonlinear models.

The rest of the paper is organized as follows. First, some definitions and basic properties in the context of signed graphs are introduced. Section 2 formalizes the k-partite consensus problem for a multi-agent network, whose agents are described as simple integrators and whose communication graph is split into k clusters. Section 3 provides some preliminary results about k-partite consensus. Section 4 provides a complete solution to this problem, under the aforementioned *homogeneity* assumption. As a special case, we address the case of a complete graph in Section 5. In Section 6, k-partite consensus for a class of nonlinear models is studied. Finally, Section 7 concludes the paper.

Notation. Given $k, n \in \mathbb{Z}$, with $k \leq n$, the symbol $[k, n]$ denotes the integer set $\{k, k+1, \dots, n\}$. In the sequel, the (i, j)th entry of a matrix A is denoted by $[A]_{i,j}$, while the ith entry of a vector $\mathbf{v}$ by $[\mathbf{v}]_i$. A matrix A is *nonnegative* (*positive*) if all its entries are nonnegative (positive), namely $[A]_{i,j} \geq 0$ ($[A]_{i,j} > 0$) for every i, j. If so we use the notation $A \geq 0$ ($A > 0$). The same notation holds for vectors.

A symmetric matrix $P \in \mathbb{R}^{n\times n}$ is *positive (semi) definite* if $\mathbf{x}^\top P\mathbf{x} > 0$ ($\mathbf{x}^\top P\mathbf{x} \geq 0$) for every $\mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq 0$, and when so we use the notation $P \succ 0$ ($P \succeq 0$). The notation $A = \mathrm{diag}\{A_1, \dots, A_k\}$ indicates a block diagonal matrix whose diagonal blocks are $A_1, \dots, A_k$. The symbols $\mathbf{0}_n$ and $\mathbf{1}_n$ denote the vectors in $\mathbb{R}^n$ with all entries equal to 0 and to 1, respectively.

For $n \geq 2$, a matrix $A \in \mathbb{R}^{n\times n}$ is *irreducible* (see Minc, 1988) if $I_n + |A| + \cdots + |A|^{n-1} > 0$, where $|A|$ is the matrix whose (i, j)th entry is $|[A]_{i,j}|$. A *Metzler matrix* is a real square matrix, whose off-diagonal entries are nonnegative. If A is an $n\times n$ Metzler matrix, then (Son & Hinrichsen, 1996) it exhibits a real dominant (not necessarily simple) eigenvalue, $\lambda_F(A)$. This means that $\lambda_F(A) > \mathrm{Re}(\lambda), \forall\, \lambda \in \sigma(A), \lambda \neq \lambda_F(A)$. If A is Metzler and irreducible, $\lambda_F(A)$ is necessarily simple.

An *undirected, signed and weighted graph* is a triple (Mohar, 1991) $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{1, \dots, N\} = [1, N]$ is the set of vertices, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of arcs, and $\mathcal{A} \in \mathbb{R}^{N\times N}$ is the *adjacency matrix* of $\mathcal{G}$. An arc (j, i) belongs to $\mathcal{E}$ if and only if $[\mathcal{A}]_{i,j} \neq 0$. As the graph is undirected, (i, j) belongs to $\mathcal{E}$ if and only if $(j, i) \in \mathcal{E}$, or, equivalently, $\mathcal{A}$ is a symmetric matrix. We assume that the graph $\mathcal{G}$ has no self-loops, i.e., $[\mathcal{A}]_{i,i} = 0$ for every $i \in [1, N]$, and arcs in $\mathcal{E}$ have either positive or negative weights, namely the (nonzero) off-diagonal entries of $\mathcal{A}$ are either positive or negative. If all the nonzero weights take values in $\{-1, 1\}$, we call the graph *unweighted*. We say that two vertices i and j are *friends* (*enemies*) if there is a edge with positive (negative) weight connecting them.

A sequence $j_1 \leftrightarrow j_2 \leftrightarrow j_3 \leftrightarrow \ldots \leftrightarrow j_k \leftrightarrow j_{k+1}$ is a *path* of length k connecting j_1 and j_{k+1} provided that $(j_1, j_2), (j_2, j_3), \ldots, (j_k, j_{k+1}) \in \mathcal{E}$. A graph is said to be *connected* if for every pair of vertices $i, j \in [1, N]$ there is a path connecting j and i. This is equivalent to the fact that the adjacency matrix $\mathcal{A}$ is irreducible.

The graph $\mathcal{G}$ is *complete* if, for every $i, j \in \mathcal{V}$, $i \neq j$, there is an edge connecting them, namely $(i, j) \in \mathcal{E}$. Also, $\mathcal{G}$ has a (nontrivial) *clustering* (Davis, 1957) if it has at least one negative edge and the set of vertices $\mathcal{V}$ can be partitioned into say $k \geq 2$ disjoint subsets $\mathcal{V}_1, \ldots, \mathcal{V}_k$ such that for every $i, j \in \mathcal{V}_p$, $p \in [1, k]$, $[\mathcal{A}]_{i,j} \geq 0$, while for every $i \in \mathcal{V}_p, j \in \mathcal{V}_q$, $p, q \in [1, k]$, $p \neq q$, $[\mathcal{A}]_{i,j} \leq 0$.

2. *k*-partite consensus: Problem statement

We consider a multi-agent system consisting of N agents, each of them described as a continuous-time integrator (Altafini, 2013; Meng et al., 2020; Monaco & Ricciardi Celsi, 2019; Olfati-Saber et al., 2007; Olfati-Saber & Murray, 2004; Ren et al., 2007). This simple model has proved to be very effective to test control algorithms in a number of meaningful applications (Zhao & Sun, 2017). The overall system dynamics is described as

$$\dot{\mathbf{x}}(t) = \mathbf{u}(t), \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^N$ and $\mathbf{u} \in \mathbb{R}^N$ are the state and input variables, respectively.

Assumption 1 on the communication structure. [Connectedness and clustering] *The communication among the N agents is described by an undirected, signed and weighted communication graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, with $\mathcal{V} = [1, N]$ and $\mathcal{A} = \mathcal{A}^\top$. The agents i and j have a cooperative relationship if $[\mathcal{A}]_{i,j} > 0$ and an antagonistic one if $[\mathcal{A}]_{i,j} < 0$. The graph $\mathcal{G}$ is connected and the agents are grouped in $k \geq 3$ clusters, $\mathcal{V}_i$, $i \in [1, k]$, with $n_i = |\mathcal{V}_i|$.*

The aim of this paper is to propose an extension to the case of k clusters of the results reported in Altafini (2013) for *structurally balanced graphs*, namely graphs with two clusters, by proposing conditions under which agents in the same cluster $\mathcal{V}_i$, $i \in [1, k]$, reach *consensus*. In other words, we investigate conditions ensuring that the state variables of the agents belonging to the same cluster asymptotically converge to the same value. When dealing with multi-agent systems with cooperative and antagonistic relationships, one can use the DeGroot's type distributed feedback control law (Altafini, 2013; DeGroot, 1974; Ren et al., 2007):

$$u_i(t) = -\sum_{j:(j,i)\in\mathcal{E}} |[\mathcal{A}]_{i,j}| \cdot [x_i(t) - \mathrm{sign}([\mathcal{A}]_{i,j}) x_j(t)],$$

$i \in [1, N]$, with $\mathrm{sign}(\cdot)$ as the sign function, that corresponds, in aggregated form, to

$$\mathbf{u}(t) = -\mathcal{L}\mathbf{x}(t), \tag{2}$$

where $\mathcal{L}$ is the *Laplacian matrix* associated with the adjacency matrix $\mathcal{A}$, defined as in Altafini (2013), Hou et al. (2003), i.e., $[\mathcal{L}]_{i,i} = \sum_{h:(h,i)\in\mathcal{E}} |[\mathcal{A}]_{i,h}|$, and $[\mathcal{L}]_{i,j} = -[\mathcal{A}]_{i,j}$, for $i \neq j$.

As shown in Altafini (2013), however, this control law leads to an autonomous multi-agent system $\dot{\mathbf{x}}(t) = -\mathcal{L}\mathbf{x}(t)$, that achieves a nontrivial consensus only if the underlying communication graph is structurally balanced. This immediately implies that if the agents can be partitioned into $k \geq 3$ clusters, but not into a smaller number of clusters, and the graph is connected, the only possible consensus is the one to the zero value. So, in this paper we investigate how to modify the distributed control law (2), to achieve consensus when the communication graph is connected and signed, but the agents split into $k \geq 3$ disjoint groups.

In the following we will assume that the agents belonging to the cluster $\mathcal{V}_1$ are the first n_1, the agents in the cluster $\mathcal{V}_2$ are the subsequent n_2, …and the agents in the cluster $\mathcal{V}_k$ are the last n_k. Clearly, $n_1 + n_2 + \cdots + n_k = N$. We can always reduce ourselves to this case by means of a relabelling of the nodes/agents. Accordingly, the adjacency matrix of the graph $\mathcal{G}$ is block-partitioned as

$$\mathcal{A} = \begin{bmatrix} \mathcal{A}_{1,1} & \mathcal{A}_{1,2} & \ldots & \mathcal{A}_{1,k} \\ \mathcal{A}_{2,1} & \mathcal{A}_{2,2} & \ldots & \mathcal{A}_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{A}_{k,1} & \mathcal{A}_{k,2} & \ldots & \mathcal{A}_{k,k} \end{bmatrix} \tag{3}$$

with $\mathcal{A}_{i,j} \in \mathbb{R}^{n_i \times n_j}$, $\mathcal{A}_{i,i} = \mathcal{A}_{i,i}^\top \geq 0$, $\forall i \in [1, k]$, $\mathcal{A}_{i,j} \leq 0$ $\forall i \neq j$, $i, j \in [1, k]$, $[\mathcal{A}_{i,i}]_{\ell,\ell} = 0$, $\forall i \in [1, k]$, $\ell \in [1, n_i]$. We consider a distributed control law for the system (1) of the type

$$\mathbf{u}(t) = -\mathcal{L}_{\mathcal{D}}\mathbf{x}(t), \tag{4}$$

where $\mathcal{L}_{\mathcal{D}} \in \mathbb{R}^{N\times N}$ takes the form

$$\begin{cases} \mathcal{L}_{\mathcal{D}} &:= & \mathcal{D} - \mathcal{A}, \\ \mathcal{D} &:= & \mathrm{diag}\{\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_k\} \in \mathbb{R}^{N\times N}, \\ \mathcal{D}_i &:= & \delta_i I_{n_i}, \end{cases} \tag{5}$$

with $\mathcal{A}$ the adjacency matrix of $\mathcal{G}$ and n_i the cardinality of the ith cluster. The overall multi-agent system is hence described as

$$\dot{\mathbf{x}}(t) = -\mathcal{L}_{\mathcal{D}}\mathbf{x}(t), \tag{6}$$

and the aim of this paper is to investigate if it is possible to choose the parameters δ_i so that all the agents reach *k-partite consensus*. This means that for every initial condition $\mathbf{x}(0) \in \mathbb{R}^N$ (except for a set of zero measure in $\mathbb{R}^N$) all the state variables, associated to agents in the same cluster, converge to the same value, namely

$$\lim_{t\to+\infty} \mathbf{x}(t) = [\gamma_1 \mathbf{1}_{n_1}^\top, \gamma_2 \mathbf{1}_{n_2}^\top, \ldots, \gamma_n \mathbf{1}_{n_k}^\top]^\top, \tag{7}$$

for suitable $\gamma_i = \gamma_i(\mathbf{x}(0)) \in \mathbb{R}$, $i \in [1, k]$, not all of them equal to zero.

The diagonal entries δ_i, $i \in [1, k]$, of the matrix $\mathcal{D}$ are henceforth our design parameters. Each δ_i can be seen as the degree of "stubbornness" of the agents of the ith cluster. It quantifies how much individuals in the cluster $\mathcal{V}_i$ are conservative about their own opinions. As it will be clear in the following, the proposed control scheme is not fully distributed, since the agents will not be able to autonomously decide the level of stubbornness to adopt in order to guarantee k-partite consensus. However the proposed modification of the standard control law is minimal, since it only requires the agents to modify the weight that each of them gives to its own opinion. Note that once the diagonal entries of $\mathcal{D}$ have been set, the control algorithm is implemented in a distributed way.

3. *k*-Partite consensus: Preliminary results

As a first step, we present a simple lemma that provides necessary and sufficient conditions for k-partite consensus. The result is elementary and extends the analogous result for consensus of cooperative multi-agent systems. Also, it has similarities with Proposition 6 in Yu and Wang (2010) derived for cooperative networks. For this reason the proof is omitted.

Lemma 1. *If the communication graph $\mathcal{G}$ satisfies Assumption 1, then the closed-loop multi-agent system (6), with $\mathcal{L}_{\mathcal{D}} \in \mathbb{R}^{N\times N}$ as in (5), reaches k-partite consensus if and only if the following conditions hold:*

(C.1) $\mathcal{L}_{\mathcal{D}}$ is a singular positive semidefinite matrix.

(C.2) All vectors in the kernel of $\mathcal{L}_{\mathcal{D}}$ have the following structure: $\mathbf{z} = [\alpha_1 \mathbf{1}_{n_1}^\top, \ldots, \alpha_k \mathbf{1}_{n_k}^\top]^\top$, $\alpha_i \in \mathbb{R}$, $i \in [1, k]$.

We now introduce some additional assumptions on the communication graph that will be used in the following analysis, and comment on their meaning.

Assumption 2 on the communication structure. [Homogeneity of trust/mistrust] *All the agents in a class $\mathcal{V}_i$ have the same constant and pre-fixed amount of trust to distribute among their cooperators and distrust, specific for each class $\mathcal{V}_j, j \neq i$, to distribute among the agents in antagonistic classes. This means that all row sums in the same block take the same value, namely for every $i, j \in [1, k]$, $\mathcal{A}_{i,j}\mathbf{1}_{n_j} = c_{ij}\mathbf{1}_{n_i}$, where $c_{ii} \geq 0$ and $c_{ij} \leq 0$, $\forall i \neq j$.*

Note that even if the adjacency matrix is symmetric, c_{ij} may differ from c_{ji}.

Example 1. Consider the undirected, signed, unweighted, connected and clustered communication graph $\mathcal{G}$, with $k = 3$ clusters of cardinality $n_1 = 2$, $n_2 = 4$, $n_3 = 1$, and adjacency matrix

$$\mathcal{A} = \left[\begin{array}{cc|cccc|c} 0 & 1 & -1 & -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 & -1 & -1 \\ \hline -1 & 0 & 0 & 1 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 & 1 & 0 & -1 \\ \hline -1 & -1 & -1 & -1 & -1 & -1 & 0 \end{array}\right]$$

$\mathcal{G}$ satisfies both Assumption 1 and Assumption 2, and the parameters c_{ij} are $c_{11} = 1, c_{12} = -2, c_{13} = -1, c_{21} = -1, c_{22} = 2, c_{23} = -1, c_{31} = -2, c_{32} = -4, c_{33} = 0$.

Remark 2. Assumption 2 may be regarded as a form of *equitable partition*, a concept originally introduced for undirected, unweighted and unsigned graphs, see Egerstedt et al. (2012). A definition of almost equitable partition for signed communication graphs was given in Liu, Ji et al. (2019) (see Definition 5), based on the Laplacian associated with $\mathcal{A}$. While signed communication graphs satisfying Assumption 2 admit an almost equitable partition according to Liu, Ji et al. (2019), the converse is not necessarily true.

We now present some technical results.

Lemma 3 (*Boyd & Vandenberghe, 2004*).
Let $\Omega = \begin{bmatrix} \Phi & S \\ S^\top & Q \end{bmatrix} \in \mathbb{R}^{n\times n}$, with $\Phi \in \mathbb{R}^{k\times k}$, be a symmetric matrix. If $\Phi = \Phi^\top$ is positive definite and its Schur complement $\mathcal{H} = Q - S^\top R^{-1}S$ is positive (semi)definite, then Ω is positive (semi)definite, and $\sigma(\Omega) = \sigma(\Phi) \cup \sigma(\mathcal{H})$.

Lemma 4, below, follows from basic results about M-matrices (see Berman & Plemmons, 1979), and hence its proof is omitted.

Lemma 4. *Let $D \in \mathbb{R}^{n\times n}$ be a diagonal matrix and let $A \in \mathbb{R}^{n\times n}$ be a symmetric Metzler matrix, then:*

(i) $D - A$ is positive definite if and only if there exists $\mathbf{z} \in \mathbb{R}^n$, $\mathbf{z} > 0$, such that $(D - A)\mathbf{z} > 0$.
(ii) If (i) holds, then $(D - A)^{-1} \geq 0$ and is symmetric.

The proof of Lemma 5 is elementary and hence omitted.

Lemma 5. *Given $\varepsilon > 0$ and matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{m\times n}$, with $A = A^\top$ Metzler, it is always possible to choose a scalar matrix $D = \delta I_n \in \mathbb{R}^{n\times n}$, $\delta > 0$, such that $|[C(D - A)^{-1}B]_{i,j}| < \varepsilon$, $\forall\, i, j \in [1, m]$.*

Assumption 3 on the communication structure. [Close friendship] *There exist $k - 1$ distinct indices $i_1, i_2, \dots, i_{k-1} \in [1, k]$ such that every cluster $\mathcal{V}_h$, $h \in \{i_2, \dots, i_{k-1}\}$, either consists of a single node/agent or for every choice of two distinct agents $i, j \in \mathcal{V}_h$ either one of the following cases applies:*

(a) i and j are *friends* (the edge (i, j) belongs to $\mathcal{E}$ and it has a positive weight);
(b) i and j are *enemies* of two (not necessarily distinct) vertices, say r and s, that belong to the same connected component of $\mathcal{V}_{i_1}$.

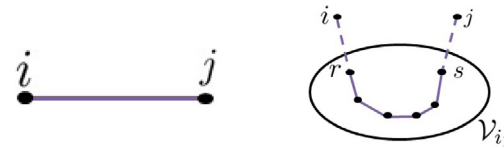

Fig. 1. Graphical representation of conditions a) (on the left) and b) (on the right) in Assumption 3.

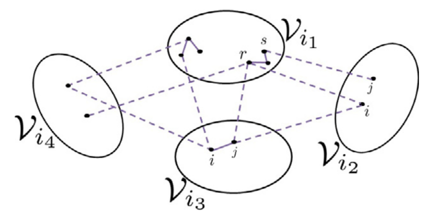

Fig. 2. Graphical representation of Assumption 3.

Remark 6. The idea behind this assumption is that if two agents belong to the same clusters $\mathcal{V}_h$, $h \in \{i_2, i_3, \dots, i_{k-1}\}$, they have a close relationship: they are either friends (case (a)) or they are enemies of agents belonging to the same group of friends in $\mathcal{V}_{i_1}$ (case (b)). Fig. 1 illustrates these two cases for a single pair (i, j). Solid lines represent friendly relationships, while dashed lines antagonistic ones. Fig. 2 provides a graphical representation of this property for the whole graph. The property holds for $\mathcal{V}_{i_2}$ and $\mathcal{V}_{i_3}$ (not for $\mathcal{V}_{i_4}$).

Remark 7. Assumption 3 has important algebraic consequences that we will exploit in the main result. Indeed, we know that for every $h \in \{i_2, i_3, \dots, i_{k-1}\}$ the matrix $\mathcal{A}_{h,h}$ is nonnegative, while the matrices $\mathcal{A}_{h,i_1}$ and $\mathcal{A}_{i_1,h}$ are nonpositive. On the other hand, for every scalar matrix $\mathcal{D}_{i_1}$ such that $\mathcal{D}_{i_1} - \mathcal{A}_{i_1,i_1}$ is positive definite (see Lemma 4), we have $(\mathcal{D}_{i_1} - \mathcal{A}_{i_1,i_1})^{-1} \geq 0$ and therefore $\mathcal{A}_{h,h} + \mathcal{A}_{h,i_1}(\mathcal{D}_{i_1} - \mathcal{A}_{i_1,i_1})^{-1}\mathcal{A}_{i_1,h}$ is a nonnegative matrix. Assumption 3 guarantees that for every $i, j \in \mathcal{V}_h, i \neq j$, either $[\mathcal{A}_{h,h}]_{i,j} > 0$ or there exists $t \in \mathbb{Z}_+$ such that $[\mathcal{A}_{h,i_1}\mathcal{A}^t_{i_1,i_1}\mathcal{A}_{i_1,h}]_{i,j} > 0$. This ensures (see the power series expansion of $(\mathcal{D}_{i_1} - \mathcal{A}_{i_1,i_1})^{-1}$) that

$$[\mathcal{A}_{h,h} + \mathcal{A}_{h,i_1}(\mathcal{D}_{i_1} - \mathcal{A}_{i_1,i_1})^{-1}\mathcal{A}_{i_1,h}]_{i,j} > 0, \ \forall\, i \neq j. \tag{8}$$

By referring to the previous Example 1, it is easy to see that Assumption 3 trivially holds for every choice of $i_1, i_2 \in [1, 3]$, $i_1 \neq i_2$. Note that $\mathcal{V}_3$ consists of a single node, while $\mathcal{V}_1$ and $\mathcal{V}_2$ consist of a single connected component.

4. *k*-partite consensus: Problem solution under the homogeneity constraint

We are now in a position to prove that under the homogeneity constraint imposed by Assumption 2 and the close friendship hypothesis formalized in Assumption 3, we can always find scalar matrices $\mathcal{D}_i = \delta_i I_{n_i}$, $i \in [1, k]$, that lead the multi-agent system to k-partite consensus.

Theorem 8. *Consider the multi-agent system (1), with communication graph $\mathcal{G}$ satisfying Assumptions 1, 2 and 3. There exist $\delta_i \in \mathbb{R}$, $i \in [1,k]$, such that the closed-loop multi-agent system (6), with $\mathcal{L}_\mathcal{D} \in \mathbb{R}^{N\times N}$ as in (5), reaches k-partite consensus, (i.e., (7) holds for suitable $\gamma_i = \gamma_i(\mathbf{x}(0)) \in \mathbb{R}$, $i \in [1,k]$).*

Proof. We assume without loss of generality that Assumption 3 holds for $i_1 = 1$ and $i_h = h+1$ for $h = 2,3,\dots,k-1$. In fact, we can always relabel the clusters, and accordingly permute the blocks of $\mathcal{A}$, so that this condition is satisfied.

By Lemma 1, we need to prove that under the theorem assumptions it is always possible to choose the real parameters $\delta_1, \delta_2, \dots, \delta_k$ so that: (C.1) the matrix $\mathcal{L}_\mathcal{D}$ is singular and positive semidefinite; and (C.2) its kernel is spanned by vectors taking the form $\mathbf{z} = [\alpha_1 \mathbf{1}_{n_1}^\top, \alpha_2 \mathbf{1}_{n_2}^\top, \dots, \alpha_k \mathbf{1}_{n_k}^\top]^\top$, $\alpha_i \in \mathbb{R}$, $i \in [1,k]$.

Condition (C.1). To impose that $\mathcal{L}_\mathcal{D}$ is singular and positive semidefinite, we present an algorithm that recursively makes use of Lemma 3. Specifically, starting from $\mathcal{H}_1 := \mathcal{L}_\mathcal{D} = \mathcal{D} - \mathcal{A}$, we block-partition each matrix $\mathcal{H}_h$, $h \in [1, k-1]$, as follows:

$$\mathcal{H}_h = \left[\begin{array}{c|c} \Phi_h & S_h \\ \hline S_h^\top & Q_h \end{array}\right], \qquad \Phi_h \in \mathbb{R}^{n_h \times n_h}. \tag{9}$$

We choose δ_h so that Φ_h is positive definite and define $\mathcal{H}_{h+1}$ as the Schur complement of Φ_h in $\mathcal{H}_h$, i.e., $\mathcal{H}_{h+1} := Q_h - S_h^\top \Phi_h^{-1} S_h$. At the kth step, the matrix $\mathcal{H}_k = \Phi_k \in \mathbb{R}^{n_k \times n_k}$ is obtained. We choose δ_k so that Φ_k is *positive semidefinite and singular.*

The whole procedure, whose first steps we will now describe in detail, is summarized, at the end, in Algorithm 1. To make the algorithm details clear, it is convenient to introduce some notation.

Set $\mathcal{M}^{(1)} := \mathcal{A}$, so that $\mathcal{H}_1 = \mathcal{D} - \mathcal{M}^{(1)}$. Clearly, $\mathcal{M}^{(1)}$ is block-partitioned according to the block-partitioning of $\mathcal{A}$, which means that $\mathcal{M}^{(1)}_{i,j} = \mathcal{A}_{i,j}$, $i,j \in [1,k]$.

As a first step, we choose δ_1 so that Φ_1 (see (9) for $h=1$) is positive definite. This means that condition

$$\Phi_1 := \mathcal{D}_1 - \mathcal{M}^{(1)}_{1,1} = \delta_1 I_{n_1} - \mathcal{A}_{1,1} \succ 0 \tag{10}$$

holds. We note that if we assume $\delta_1 > m^{(1)}_{1,1} := c_{11} \geq 0$, then $\Phi_1 \mathbf{1}_{n_1} = (\delta_1 I_{n_1} - \mathcal{A}_{1,1})\mathbf{1}_{n_1} > 0$. Therefore, by Lemma 4, part (i), for $D = \delta_1 I_{n_1}$, $A = \mathcal{M}^{(1)}_{1,1} = \mathcal{A}_{1,1}$ (a Metzler matrix) and $\mathbf{z} = \mathbf{1}_{n_1}$, we can claim that $\Phi_1 = D - A$ is positive definite. The Schur complement of Φ_1 in $\mathcal{H}_1$, namely $\mathcal{H}_2$, is given in Eq. (11) in Box I and can be expressed as $\mathcal{H}_2 = \mathrm{diag}\{\mathcal{D}_2, \dots, \mathcal{D}_k\} - \mathcal{M}^{(2)}$, where

$$\mathcal{M}^{(2)} := \begin{bmatrix} \mathcal{M}^{(2)}_{2,2} & \dots & \mathcal{M}^{(2)}_{2,k} \\ \vdots & \ddots & \vdots \\ \mathcal{M}^{(2)}_{k,2} & \dots & \mathcal{M}^{(2)}_{k,k} \end{bmatrix} \in \mathbb{R}^{(N-n_1)\times(N-n_1)},$$

and $\mathcal{M}^{(2)}_{i,j} := \mathcal{M}^{(1)}_{i,j} + \mathcal{M}^{(1)}_{i,1}\Phi_1^{-1}\mathcal{M}^{(1)}_{1,j} = \mathcal{A}_{i,j} + \mathcal{A}_{i,1}(\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}\mathcal{A}_{1,j}$, with $i,j \in [2,k]$.

By Lemma 4, part ii), $\Phi_1^{-1} = (\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}$ is symmetric and nonnegative, and hence $\mathcal{M}^{(2)}_{2,2} := \mathcal{A}_{2,2} + \mathcal{A}_{2,1}\Phi_1^{-1}\mathcal{A}_{1,2}$ is a Metzler matrix.

We now choose δ_2 so that Φ_2, the first block of $\mathcal{H}_2$, is positive definite, namely

$$\Phi_2 := \mathcal{D}_2 - \mathcal{M}^{(2)}_{2,2} = \mathcal{D}_2 - \mathcal{A}_{2,2} - \mathcal{A}_{2,1}\Phi_1^{-1}\mathcal{A}_{1,2} \succ 0. \tag{13}$$

Again, by Lemma 4, part i), for $D = \mathcal{D}_2$ and $A = \mathcal{M}^{(2)}_{2,2}$ (a Metzler matrix) and $\mathbf{z} = \mathbf{1}_{n_2}$, we observe that if we impose

$$\delta_2 > m^{(2)}_{2,2} := c_{22} + c_{12}(\delta_1 - c_{11})^{-1}c_{21}, \tag{14}$$

then

$$\Phi_2 \mathbf{1}_{n_2} = (D - A)\mathbf{1}_{n_2} = (\delta_2 - c_{22})\mathbf{1}_{n_2} - \mathcal{A}_{2,1}\Phi_1^{-1}c_{12}\mathbf{1}_{n_1}$$
$$= (\delta_2 - c_{22})\mathbf{1}_{n_2} - c_{21}(\delta_1 - c_{11})^{-1}c_{12}\mathbf{1}_{n_1} > 0,$$

where we used the fact that $\Phi_1^{-1}\mathbf{1}_{n_1} = (\mathcal{D}_1 - \mathcal{A}_{11})^{-1}\mathbf{1}_{n_1} = (\delta_1 - c_{11})^{-1}\mathbf{1}_{n_1}$. Therefore $\Phi_2 = D - A$ is positive definite, namely (13) holds.

The Schur complement of Φ_2 in $\mathcal{H}_2$ is $\mathcal{H}_3$ (see Eq. (12) given in Box II) and its first block can be expressed as $\Phi_3 := \mathcal{D}_3 - \mathcal{M}^{(3)}_{3,3}$, where

$$\begin{aligned}\mathcal{M}^{(3)}_{3,3} &:= \mathcal{M}^{(2)}_{3,3} + \mathcal{M}^{(2)}_{3,2}\Phi_2^{-1}\mathcal{M}^{(2)}_{2,3} \\ &= \mathcal{A}_{3,3} + \mathcal{A}_{3,1}\Phi_1^{-1}\mathcal{A}_{1,3} + [\mathcal{A}_{3,2} \\ &+ \mathcal{A}_{3,1}\Phi_1^{-1}\mathcal{A}_{1,2}] \cdot \Phi_2^{-1}[\mathcal{A}_{2,3} + \mathcal{A}_{2,1}\Phi_1^{-1}\mathcal{A}_{1,3}].\end{aligned}$$

From Assumption 3 and the properties of $\Phi_1^{-1} = (\mathcal{D}_1 - \mathcal{A}_{1,1})^{-1}$ it follows (see Remark 7) that $\mathcal{M}^{(2)}_{\ell,\ell}$, $\ell \in [3,k]$, (and hence, in particular, $\mathcal{M}^{(2)}_{3,3}$) is a nonnegative matrix whose off-diagonal entries are all positive. On the other hand, by Lemma 5 we can always choose $\delta_2 > 0$ sufficiently large (something that ensures, in particular, that (14) is satisfied) to guarantee that the entries of $\mathcal{M}^{(2)}_{3,2}\Phi_2^{-1}\mathcal{M}^{(2)}_{2,3}$ are arbitrarily small. This ensures that the off-diagonal entries of $\mathcal{M}^{(3)}_{3,3}$ are positive.

If we now choose δ_3 such that

$$\delta_3 > m^{(3)}_{3,3} := c_{33} + \frac{c_{31}c_{13}}{\delta_1 - c_{11}} + \left(c_{32} + \frac{c_{31}c_{12}}{\delta_1 - c_{11}}\right) \cdot$$
$$\cdot \left(\delta_2 - c_{22} - \frac{c_{21}c_{12}}{\delta_1 - c_{11}}\right)^{-1}\left(c_{23} + \frac{c_{21}c_{13}}{\delta_1 - c_{11}}\right), \tag{15}$$

we ensure that Φ_3 satisfies $\Phi_3 \mathbf{1}_{n_3} > 0$. And since $-\Phi_3$ is a (irreducible) Metzler matrix, this proves that Φ_3 is positive definite.

Algorithm 1 Selection of the δ_h, $h = 1,2,\dots,k$.

for $i,j \in [1,k]$ **do** ▷ Initialization
  $\mathcal{M}^{(1)}_{i,j} := \mathcal{A}_{i,j}$
  $m^{(1)}_{i,j} := c_{i,j}$
for $h \in [1, k-1]$ **do** ▷ Recursive Step
  Choose $\delta_h > 0$ so that
  $\delta_h > m^{(h)}_{h,h}$ and
  if $h \geq 2$ **then**
    $\forall\ \ell \in [h+1, k], \forall\ i \neq j$
    $[\mathcal{M}^{(h)}_{\ell,\ell} + \mathcal{M}^{(h)}_{\ell,h}[\delta_h I_{n_k} - \mathcal{M}^{(h)}_{h,h}]^{-1}\mathcal{M}^{(h)}_{h,\ell}]_{ij} > 0$
  Set
   $\mathcal{D}_h := \delta_h I_{n_h}$
   $\Phi_h := \mathcal{D}_h - \mathcal{M}^{(h)}_{h,h}$
   $\phi_h := \delta_h - m^{(h)}_{h,h}$
   $\mathcal{M}^{(h+1)}_{i,j} := \mathcal{M}^{(h)}_{i,j} + \mathcal{M}^{(h)}_{i,h}\Phi_h^{-1}\mathcal{M}^{(h)}_{h,j}$
   $m^{(h+1)}_{i,j} := m^{(h)}_{i,j} + m^{(h)}_{i,h}\phi_h^{-1}m^{(h)}_{h,j} \quad \forall i,j \geq h$
Set ▷ Final Step
  $\delta_k := m^{(k)}_{k,k}$
  $\mathcal{D}_k := \delta_k I_{n_k}$
  $\Phi_k := \mathcal{D}_k - \mathcal{M}^{(k)}_{k,k}$
  $\phi_k := \delta_k - m^{(k)}_{k,k} = 0$

The procedure generalizes as follows (see Algorithm 1). At each step h, ranging from 4 to $k-1$:

- We first determine the expression of $\mathcal{H}_h$ and verify that the off-diagonal entries of $\mathcal{M}^{(h)}_{h,h}$ are positive. If this is not the case, we can increase the values of $\delta_2, \dots, \delta_{h-1}$ (meanwhile respecting all the previous inequalities) to impose

$$\mathcal{H}_2=\left[\begin{array}{c|ccc}\mathcal{D}_2-\mathcal{M}^{(1)}_{2,2} & -\mathcal{M}^{(1)}_{2,3} & \dots & -\mathcal{M}^{(1)}_{2,k}\\ \hline -\mathcal{M}^{(1)}_{3,2} & \mathcal{D}_3-\mathcal{M}^{(1)}_{3,3} & \dots & -\mathcal{M}^{(1)}_{3,k}\\ \vdots & \vdots & \ddots & \vdots\\ -\mathcal{M}^{(1)}_{k,2} & -\mathcal{M}^{(1)}_{k,3} & \dots & \mathcal{D}_k-\mathcal{M}^{(1)}_{k,k}\end{array}\right]-\left[\begin{array}{c}\mathcal{M}^{(1)}_{2,1}\\ \hline \mathcal{M}^{(1)}_{3,1}\\ \vdots\\ \mathcal{M}^{(1)}_{k,1}\end{array}\right]\Phi_1^{-1}\left[\begin{array}{c|ccc}\mathcal{M}^{(1)}_{1,2} & \mathcal{M}^{(1)}_{1,3} & \dots & \mathcal{M}^{(1)}_{1,k}\end{array}\right]=\begin{bmatrix}\Phi_2 & S_2\\ S_2^\top & Q_2\end{bmatrix} \tag{11}$$

Box I.

$$\mathcal{H}_3:=\begin{bmatrix}\mathcal{D}_3-\mathcal{M}^{(2)}_{3,3} & \dots & -\mathcal{M}^{(2)}_{3,k}\\ \vdots & \ddots & \vdots\\ -\mathcal{M}^{(2)}_{k,3} & \dots & \mathcal{D}_k-\mathcal{M}^{(2)}_{k,k}\end{bmatrix}-\begin{bmatrix}\mathcal{M}^{(2)}_{3,2}\\ \vdots\\ \mathcal{M}^{(2)}_{k,2}\end{bmatrix}\Phi_2^{-1}\begin{bmatrix}\mathcal{M}^{(2)}_{2,3} & \dots & \mathcal{M}^{(2)}_{2,k}\end{bmatrix}=\begin{bmatrix}\Phi_3 & S_3\\ S_3^\top & Q_3\end{bmatrix} \tag{12}$$

Box II.

this condition.[1] Indeed, we note that $\mathcal{M}^{(h)}_{h,h}$ is the sum of $\mathcal{M}^{(2)}_{h,h}$ and of other terms that depend on $\Phi_2^{-1},\dots,\Phi_{h-1}^{-1}$. Assumption 3 ensures that $\mathcal{M}^{(2)}_{h,h}$ is a Metzler matrix with positive off-diagonal entries. So, by making use of Lemma 5 we can always increase the coefficients $\delta_2,\dots,\delta_{h-1}$ (in this order) so that all the entries of the terms that depend on $\Phi_2^{-1},\dots,\Phi_{h-1}^{-1}$ become negligible, and hence $\mathcal{M}^{(h)}_{h,h}$ has positive off-diagonal entries. This allows to say that, for every choice of $\mathcal{D}_h=\delta_h I_{n_h}$, the matrix $-\Phi_h$ is (irreducible and) Metzler.
- We choose $\delta_h > m^{(h)}_{h,h}$, where $m^{(h)}_{h,h}\mathbf{1}_{n_h} := \mathcal{M}^{(h)}_{h,h}\mathbf{1}_{n_h}$. This ensures that the irreducible Metzler matrix $-\Phi_h=-\mathcal{D}_h+\mathcal{M}^{(h)}_{h,h}=-\delta_h I_{n_h}+\mathcal{M}^{(h)}_{h,h}$ satisfies $-\Phi_h\mathbf{1}_{n_h}<0$, and hence Φ_h is (symmetric and) positive definite.

By proceeding in this way, we construct all positive definite matrices $\Phi_1,\dots,\Phi_{k-1}$ and at the last step we choose $\delta_k>0$ so that $-\Phi_k\mathbf{1}_{n_k}=0$. Being $-\Phi_k$ an irreducible Metzler matrix, this ensures (see Berman & Plemmons, 1979) that 0 is a simple dominant eigenvalue of $-\Phi_k$. Therefore Φ_k is positive semidefinite and singular, with a simple eigenvalue in 0. Since $\sigma(\mathcal{L}_{\mathcal{D}})=\cup_{h\in[1,k]}\sigma(\Phi_h)$, then $\mathcal{L}_{\mathcal{D}}$ is positive semidefinite with a simple eigenvalue in 0.

Condition (C.2). We want to prove that if the parameters $\delta_1,\delta_2,\dots,\delta_k$ are selected according to Algorithm 1, then $\mathcal{L}_{\mathcal{D}}$ has an eigenvector associated with the 0 eigenvalue with the desired block structure. If this is the case, since we proved that 0 is a simple eigenvalue, all the eigenvectors of $\mathcal{L}_{\mathcal{D}}$ corresponding to 0 have the desired block structure.

We note that $\mathbf{z}=[\alpha_1\mathbf{1}_{n_1}^\top,\alpha_2\mathbf{1}_{n_2}^\top,\dots,\alpha_k\mathbf{1}_{n_k}^\top]^\top$ is an eigenvector of $\mathcal{L}_{\mathcal{D}}$ associated with the zero eigenvalue if and only if $\mathbf{w}=[\alpha_1,\alpha_2,\dots,\alpha_k]^\top$ is an eigenvector of $\mathbb{D}-\mathbb{C}$ corresponding to 0, where $\mathbb{D}:=\mathrm{diag}\{\delta_1,\delta_2,\dots,\delta_k\}$ and $\mathbb{C}:=[c_{ij}]_{i,j\in[1,k]}$. So, we need to simply prove that $\mathbb{D}-\mathbb{C}$ is a singular matrix. If we evaluate first the $(1,1)$ entry of $\mathbb{D}-\mathbb{C}$ and then the $(1,1)$-entry of each of the $k-1$ Schur complements, obtained from $\mathbb{D}-\mathbb{C}$ according to the same algorithm that we used to define the matrices Φ_h, $h\in[1,k-1]$, we obtain the sequence of coefficients $\phi_1,\phi_2,\dots,\phi_k$. By Algorithm 1, the first $k-1$ are positive, while $\phi_k=0$. On the other hand, $\det(\mathbb{D}-\mathbb{C})=\phi_1\phi_2\dots\phi_k$, and hence $\mathbb{D}-\mathbb{C}$ is singular. □

[1] As a matter of fact, Algorithm 1 provides a more efficient procedure to choose the δ_h's so that at each step there is no need to go backward and increase their values.

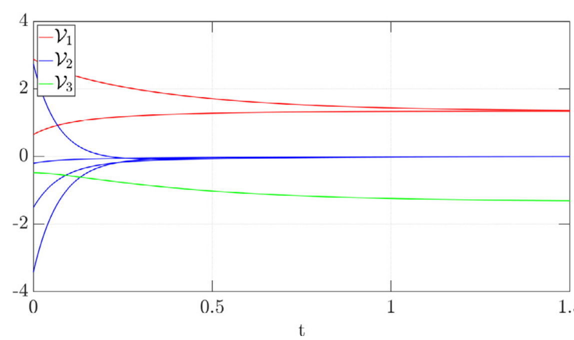

Fig. 3. Graph corresponding to Example 2.

Example 2. Consider, again, Example 1. As previously remarked, the communication graph satisfies Assumptions 1, 2 and 3 for $i_1=1$ and $i_2=3$ (as in the proof). If we apply Algorithm 1 we obtain the constraints

$$\delta_1>1,\qquad \delta_2>2+\frac{2}{\delta_1-1},$$
$$\delta_3=\frac{2}{\delta_1-1}+\frac{\left(-4+\frac{4}{\delta_1-1}\right)\left(-1+\frac{1}{\delta_1-1}\right)}{\left[\delta_2-2-\frac{2}{\delta_1-1}\right]}.$$

If we assume $\delta_1=2$ then, independently of δ_2, one gets $\delta_3=2$. It turns out that for every choice of $\delta_2>4$ the eigenvector corresponding to the zero eigenvalue of $\mathcal{L}_{\mathcal{D}}$ is $\mathbf{z}=[\,1\ 1\,|\ 0\ 0\ 0\ 0\,|\,-1]^\top$.

Fig. 3 shows the state evolution of the system described as in (6), with adjacency matrix as in Example 1, with random initial conditions $\mathbf{x}(0)$ taken as realizations of a gaussian vector with 0 mean and variance $\sigma^2=4$, i.e. $\mathbf{x}(0)\sim\mathcal{N}(0,4)$. The graph shows that tripartite consensus is reached after about 1.5 units of time with regime values $\gamma_1=-1.39$, $\gamma_2=0$, $\gamma_3=1.39$.

5. *k*-partite consensus for multi-agent systems with complete unweighted graph

In this subsection we will focus our attention on multi-agent systems with complete, unweighted and undirected communication graphs, clustered into an arbitrary number k of groups. By resorting to a suitable relabelling of the agents, we can always assume that the adjacency matrix $\mathcal{A}$ is described as in (3) with

$$\mathcal{A}_{i,i}=\mathbf{1}_{n_i}\mathbf{1}_{n_i}^\top-I_{n_i}\ \text{and}\ \mathcal{A}_{i,j}=-\mathbf{1}_{n_i}\mathbf{1}_{n_j}^\top,\ \text{for } i\neq j, \tag{16}$$

n_i being the cardinality of the ith cluster. Also in this case we plan to design a distributed control law for the system (1) of the type (4), with $\mathcal{L}_\mathcal{D} = \mathcal{D} - \mathcal{A}$, and $\mathcal{D} = \mathrm{diag}\{\delta_1 I_{n_1}, \ldots, \delta_k I_{n_k}\} \in \mathbb{R}^{N\times N}$.

Under the previous hypotheses on the adjacency matrix $\mathcal{A}$, Assumptions 1, 2 and 3 are trivially satisfied. So, the existence of a choice of the coefficients $\delta_i, i \in [1,k]$, that ensures k-partite consensus follows from Theorem 8. On the other hand, the particular structure of $\mathcal{A}$ allows to obtain a much simpler proof as well as an explicit expression of (a possible choice of) the δ_i's that cannot be obtained in the general homogeneous case. For this reason we provide here an independent proof of this result.

Theorem 9. *Consider an unweighted and complete communication graph $\mathcal{G}$ split into k clusters, with adjacency matrix $\mathcal{A}$ as in (3), and blocks described as in (16). If we assume $\delta_i = 2n_i - 1, i \in [1,k]$, the closed-loop multi-agent system (6), with $\mathcal{L}_\mathcal{D} \in \mathbb{R}^{N\times N}$ described as in (5), reaches k-partite consensus.*

Proof. By Lemma 1, we need to prove that under the theorem hypotheses and by assuming the parameters $\delta_i = 2n_i - 1, i \in [1,k]$, we can ensure that: (C.1) the matrix $\mathcal{L}_\mathcal{D}$ is singular and positive semidefinite; and (C.2) its kernel is spanned by vectors taking the block form $\mathbf{z} = [\alpha_1 \mathbf{1}_{n_1}^\top, \alpha_2 \mathbf{1}_{n_2}^\top, \ldots, \alpha_k \mathbf{1}_{n_k}^\top]^\top, \alpha_i \in \mathbb{R}, i \in [1,k]$.

Condition (C.2). By assuming $\delta_i = 2n_i - 1, i \in [1,k]$, and by imposing $\mathcal{L}_\mathcal{D}\mathbf{z} = \mathbf{0}_N$, for $\mathbf{z}$ described as above, we obtain the family of equations $\mathbb{N}_k\ [\alpha_1\ \alpha_2\ \ldots\ \alpha_k]^\top = 0$, where $\mathbb{N}_k := \mathbf{1}_k[n_1\ n_2\ \ldots\ n_k]$ is a singular matrix whose kernel coincides with $\ker[n_1\ n_2\ \ldots\ n_k]$. This implies that $\ker\ \mathcal{L}_\mathcal{D}$ includes all the vectors $\mathbf{z} = [\alpha_1 \mathbf{1}_{n_1}^\top, \alpha_2 \mathbf{1}_{n_2}^\top, \ldots, \alpha_k \mathbf{1}_{n_k}^\top]^\top$, with $[\alpha_1, \alpha_2, \ldots, \alpha_k] \in \ker[n_1\ n_2\ \ldots\ n_k]$. To prove that *all* the eigenvectors of $\mathcal{L}_\mathcal{D}$ corresponding to the zero eigenvalue take the form $[\alpha_1 \mathbf{1}_{n_1}^\top, \ldots, \alpha_k \mathbf{1}_{n_k}^\top]^\top, \alpha_i \in \mathbb{R}, i \in [1,k]$, let $\mathbf{w} = [\mathbf{w}_1^\top\ \mathbf{w}_2^\top\ \ldots\ \mathbf{w}_k^\top]^\top$ be any eigenvector of $\mathcal{L}_\mathcal{D}$ corresponding to 0. Then condition $\mathcal{L}_\mathcal{D}\mathbf{w} = \mathbf{0}_N$ implies

$$2n_i \mathbf{w}_i = (\mathbf{1}_{n_i}^\top \mathbf{w}_i)\mathbf{1}_{n_i} - \sum_{j=1, j\neq i}^{k} (\mathbf{1}_{n_j}^\top \mathbf{w}_j)\mathbf{1}_{n_i}, \quad i \in [1,k].$$

This ensures that every $\mathbf{w}_i$ is a scalar multiple of $\mathbf{1}_{n_i}$.

Condition (C.1). We now prove that by assuming $\delta_i = 2n_i - 1, i \in [1,k]$:

(A) the upper diagonal block of $\mathcal{L}_\mathcal{D}$, namely $\Phi := 2n_1 I_{n_1} - \mathbf{1}_{n_1}\mathbf{1}_{n_1}^\top$, is positive definite, and

(B) its Schur complement

$$\mathcal{H} := \begin{bmatrix} \mathcal{H}_{2,2} & \mathcal{H}_{2,3} & \ldots & \mathcal{H}_{2,k} \\ \mathcal{H}_{3,2} & \mathcal{H}_{3,3} & \ldots & \mathcal{H}_{3,k} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{H}_{k,2} & \mathcal{H}_{k,3} & \ldots & \mathcal{H}_{k,k} \end{bmatrix},$$

with $\mathcal{H}_{i,i} = 2n_i I_{n_i} - \mathbf{1}_{n_i}\mathbf{1}_{n_i}^\top - \mathbf{1}_{n_i}\mathbf{1}_{n_1}^\top (2n_1 I_{n_1} - \mathbf{1}_{n_1}\mathbf{1}_{n_1}^\top)^{-1} \mathbf{1}_{n_1}\mathbf{1}_{n_i}^\top$ and $\mathcal{H}_{i,j} = \mathbf{1}_{n_i}\mathbf{1}_{n_j}^\top - \mathbf{1}_{n_i}\mathbf{1}_{n_1}^\top (2n_1 I_{n_1} - \mathbf{1}_{n_1}\mathbf{1}_{n_1}^\top)^{-1} \mathbf{1}_{n_i}\mathbf{1}_{n_j}^\top, i \neq j$, is positive semidefinite and singular.

Therefore, by Lemma 3, $\mathcal{L}_\mathcal{D}$ is positive semidefinite and singular.

By Lemma 4 part i), we can claim that, since $(2n_1 I_{n_1} - \mathbf{1}_{n_1}\mathbf{1}_{n_1}^\top)\mathbf{1}_{n_1} = n_1 \mathbf{1}_{n_1} > 0$, (A) holds.

Now, we observe that, for any vector $\mathbf{z} = [\alpha_1 \mathbf{1}_{n_1}^\top, \alpha_2 \mathbf{1}_{n_2}^\top, \ldots, \alpha_k \mathbf{1}_{n_k}^\top]^\top$, with $[\alpha_1, \alpha_2, \ldots, \alpha_k] \in \ker[n_1\ n_2\ \ldots\ n_k]$, we have $0 = (2n_1 I_{n_1} - \mathbf{1}_{n_1}\mathbf{1}_{n_1}^\top)\alpha_1 \mathbf{1}_{n_1} + \mathbf{1}_{n_1}\alpha_2 n_2 + \cdots + \mathbf{1}_{n_1}\alpha_k n_k$, and hence

$$(2n_1 I_{n_1} - \mathbf{1}_{n_1}\mathbf{1}_{n_1}^\top)^{-1}\mathbf{1}_{n_1} = -\frac{\alpha_1}{(\sum_{i=2}^k \alpha_i n_i)}\mathbf{1}_{n_1} = \frac{1}{n_1}\mathbf{1}_{n_1}.$$

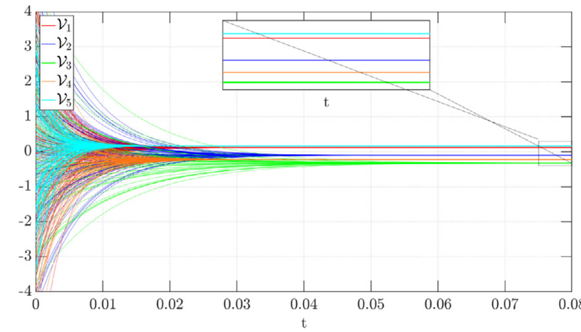

Fig. 4. Graph corresponding to Example 3.

This allows to verify that the matrix $\mathcal{H}$ takes the block diagonal form $\mathcal{H} = \mathrm{diag}\{2n_2 I_{n_2} - 2\mathbf{1}_{n_2}\mathbf{1}_{n_2}^\top, 2n_3 I_{n_3} - 2\mathbf{1}_{n_3}\mathbf{1}_{n_3}^\top, \ldots, 2n_k I_{n_k} - 2\mathbf{1}_{n_k}\mathbf{1}_{n_k}^\top\}$. Each diagonal block $2n_i I_{n_i} - 2\mathbf{1}_{n_i}\mathbf{1}_{n_i}^\top, i \in [2,k]$, is easily seen (by a straightforward extension of Lemma 4) to be positive semidefinite and singular (with 0 as a simple eigenvalue). So, we have shown that $\mathcal{L}_\mathcal{D}$ is positive semidefinite and singular and hence (B) holds. Therefore condition (C.1) holds and k-partite consensus is asymptotically achieved. □

Example 3. Consider the multi-agent system (6), with unweighted and complete communication graph and 5 clusters of size $n_1 = 128, n_2 = 72, n_3 = 44, n_4 = 115, n_5 = 194$. We assume $\delta_i = 2n_i - 1, i \in [1,5]$, and $\mathbf{x}(0) \sim \mathcal{N}(0,4)$. The system reaches 5-partite consensus after about 0.08 units of time, with regime values $\gamma_1 = 0.1193, \gamma_2 = -0.1022, \gamma_3 = -0.3281, \gamma_4 = -0.2236, \gamma_5 = 0.1655$, as illustrated in Fig. 4.

6. k-partite consensus for a class of nonlinear models

In the following, an extension of the k-partite consensus analysis to nonlinear systems is proposed. To this aim, by adopting a set-up similar to the one in Altafini (2013), we consider a multi-agent system described as in (1), with communication graph $\mathcal{G}$ satisfying Assumption 1 and subjected to the feedback law

$$\mathbf{u} = \mathbf{f}(\mathbf{x}), \tag{17}$$

where $\mathbf{f} : \mathbb{R}^N \to \mathbb{R}^N$ is a Lipschitz continuous function satisfying $\mathbf{f}(\mathbf{0}) = \mathbf{0}$.

Assumption 4 on the vector field f: *We assume for $\mathbf{f}$ a distributed additive expression. Specifically, each component $f_i(\mathbf{x}), i \in [1,N]$, of the function $\mathbf{f}$ depends only on the states of the neighbours of the ith agent, namely those entries x_j such that $(j,i) \in \mathcal{E}$. It is expressed as*

$$f_i(\mathbf{x}) = -\sum_{j:(j,i)\in\mathcal{E}} \Big([\mathcal{D}]_i \tilde{h}_i(x_i(t)) - [\mathcal{A}]_{i,j} \tilde{h}_j(x_j(t)) \Big) \tag{18}$$

where $[\mathcal{D}]_i$ is a real number, and the nonlinear function $\tilde{h}_\ell(\cdot)$ is the same for all the agents belonging to the same cluster. So, if we assume that the agents are partitioned into k clusters and ordered in such a way that $\mathcal{A}$ is described as in (3), the vector $\mathbf{x}$ is accordingly partitioned as $\mathbf{x} = \begin{bmatrix} \mathbf{x}_1^\top & \mathbf{x}_2^\top & \ldots & \mathbf{x}_k^\top \end{bmatrix}^\top$, with $\mathbf{x}_i \in \mathbb{R}^{n_i}$ representing the states of the agents belonging to the ith cluster. The function $\mathbf{f}$ can be expressed as the product of the matrix $\mathcal{L}_\mathcal{D}$, given in (5), and of a nonlinear function $\mathbf{h}(\mathbf{x})$:

$$\dot{\mathbf{x}} = -\mathcal{L}_\mathcal{D}\mathbf{h}(\mathbf{x}), \tag{19}$$

with $\mathbf{h}(\mathbf{x}) = [\mathbf{h_1}(\mathbf{x_1})^\top\ \mathbf{h_2}(\mathbf{x_2})^\top\ \ldots\ \mathbf{h_k}(\mathbf{x_k})^\top]^\top$, and $\mathbf{h_i}(\mathbf{x_i}) : \mathbb{R}^{\mathbf{n_i}} \to \mathbb{R}^{\mathbf{n_i}}, i \in [1,k]$, is described as follows

$$\mathbf{h}_i(\mathbf{x}_i) = [h_i(x_{s_i+1})\ h_i(x_{s_i+2})\ \ldots\ h_i(x_{s_i+n_i})]^\top, \tag{20}$$

where $s_1 = 0$, while $s_i = \sum_{j<i} n_j$, $i \in [2, k]$. The scalar functions $h_i(\cdot)$ are assumed to be monotone, bijective functions belonging to the set $\mathcal{R} := \Big\{h : \mathbb{R} \to \mathbb{R} : (h(x_a) - h(x_b))(x_a - x_b) > 0, x_a \neq x_b, h(0) = 0, \int_{x_b}^{x_a}(h(z) - h(x_b))dz \to \infty \text{ as } |x_a - x_b| \to \infty\Big\}$. *Each function $h_i(x_{s_i+m})$, $m \in [1, n_i]$, represents how the opinion of the $(s_i + m)$-th agent in the ith cluster is perceived by its neighbours.*

The following theorem provides sufficient conditions for a networked closed-loop system described as in (19) to reach k-partite consensus that extend those given in Theorem 8. Similarly, the extension of Theorem 9 would be possible.

Theorem 10. *Consider the multi-agent system (1), with communication graph $\mathcal{G}$ satisfying Assumptions 1, 2 and 3, and distributed control law (17) satisfying Assumption 4 and (18). There exist $\delta_i \in \mathbb{R}$, $i \in [1, k]$, such that the closed-loop multi-agent system described as in (19), with the function $\mathbf{h}(\mathbf{x})$ defined as above, and $\mathcal{L}_\mathcal{D} \in \mathbb{R}^{N\times N}$ described as in (5), reaches k-partite consensus.*

Proof. Clearly, the equilibrium points of system (19) are all the vectors $\mathbf{x}^*$ in $\mathbb{R}^N$ such that $\mathbf{0} = \mathcal{L}_\mathcal{D}\mathbf{h}(\mathbf{x}^*)$. We want to show that it is possible to choose the coefficients δ_i, $i \in [1, k]$, so that all the equilibrium points of the system are block partitioned according to the block partitioning of the matrix $\mathcal{L}_\mathcal{D}$, and they are globally simply stable. This ensures that the set of all such equilibrium points is the attractor of every state trajectory (there cannot be limit cycles and the trajectories cannot diverge), and hence the multi-agent system asymptotically reaches k-partite consensus.

We have proved (see Theorem 8) that under Assumptions 1, 2 and 3 it is possible to choose the coefficients $\delta_1, \delta_2, \dots, \delta_k \in \mathbb{R}$ so that $\mathcal{L}_\mathcal{D}$ is a singular positive semidefinite matrix, having 0 as a simple eigenvalue and the corresponding eigenvector takes the form $\mathbf{z} = [\alpha_1\mathbf{1}_{n_1}^\top, \alpha_2\mathbf{1}_{n_2}^\top, \dots \alpha_k\mathbf{1}_{n_k}^\top]^\top$, for suitable $\alpha_i \in \mathbb{R}$, $i \in [1, k]$. This implies that the equilibrium points of the system (19) are the vectors $\mathbf{x}^*$ such that $\mathbf{h}(\mathbf{x}^*) \in \langle \mathbf{z} \rangle$. As the maps h_i belong to $\mathcal{R}$, for every $c \in \mathbb{R}$ such that $c \cdot \alpha_i$ belongs to the image of the corresponding h_i for every $i \in [1, k]$, there exist $\beta_1, \beta_2 \dots \beta_k \in \mathbb{R}$ such that $c \cdot [\alpha_1\mathbf{1}_{n_1}^\top, \dots \alpha_k\mathbf{1}_{n_k}^\top]^\top = \mathbf{h}([\beta_1\mathbf{1}_{n_1}^\top, \dots \beta_k\mathbf{1}_{n_k}^\top]^\top)$.

Suppose, without loss of generality, that this is the case for $c = 1$, set $\mathbf{x}^* := [\beta_1\mathbf{1}_{n_1}^\top, \beta_2\mathbf{1}_{n_2}^\top, \dots, \beta_k\mathbf{1}_{n_k}^\top]^\top$, and consider a suitably modified version of the Lyapunov function $V : \mathbb{R}^N \to \mathbb{R}$ adopted in Altafini (2013):

$$V(\mathbf{x}) = \sum_{i=1}^{k}\sum_{j=s_i+1}^{s_i+n_i}\int_{x_j^*}^{x_j}(h_i(z) - h_i(x_j^*))dz = \\ = \sum_{i=1}^{k}\sum_{j=s_i+1}^{s_i+n_i}\int_{\beta_i}^{x_j}(h_i(z) - \alpha_i)dz, \quad (21)$$

(see Assumption 4 for the definition of s_i) for $\mathbf{x} \neq \mathbf{x}^*$. Also, $V(\mathbf{x})$ is radially unbounded and its derivative is

$$\dot{V}(\mathbf{x}) = \sum_{i=1}^{k}\sum_{j=s_i+1}^{s_i+n_i}(h_i(x_j) - h_i(x_j^*))\dot{x}_j \\ = -(\mathbf{h}(\mathbf{x}) - \mathbf{h}(\mathbf{x}^*))^\top\mathcal{L}_\mathcal{D}\mathbf{h}(\mathbf{x}) = -\mathbf{h}(\mathbf{x})^\top\mathcal{L}_\mathcal{D}\mathbf{h}(\mathbf{x}) \leq 0,$$

where we used the fact that $\mathcal{L}_\mathcal{D} = \mathcal{L}_\mathcal{D}^\top$ and $\mathcal{L}_\mathcal{D}\mathbf{h}(\mathbf{x}^*) = \mathcal{L}_\mathcal{D}\mathbf{z} = 0$, and the last inequality holds since $\mathcal{L}_\mathcal{D}$ is a singular positive semidefinite matrix. This ensures that every equilibrium point $\mathbf{x}^*$ of the system is globally stable and since all such equilibrium points have the required block-structure, k-partite consensus is guaranteed. □

Example 4. Consider the multi-agent system (19), with unweighted and complete communication graph, $\mathbf{h}(\mathbf{x}(t)) = \tanh(\mathbf{x}(t))$, and 4 clusters of size $n_1 = 137, n_2 = 81, n_3 = 53, n_4 = 98$. We have assumed that $\mathbf{x}(0) \sim \mathcal{N}(0, 4)$ and $\delta_i = 2n_i - 1$ for every $i \in [1, 4]$. The system reaches 4-partite consensus after approximately 0.07 time units, with regime values $\gamma_1 = 0.9387, \gamma_2 = 0.9248, \gamma_3 = 2.0188, \gamma_4 = -1.1172$, as illustrated in Figs. 5 and 6.

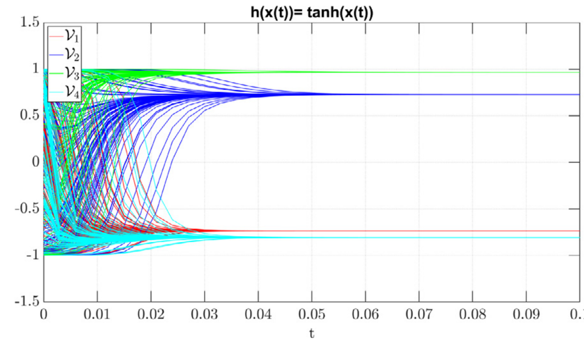

Fig. 5. Graph associated with Example 4: time evolution of $\mathbf{h}(\mathbf{x}(t)) = \tanh(\mathbf{x}(t))$.

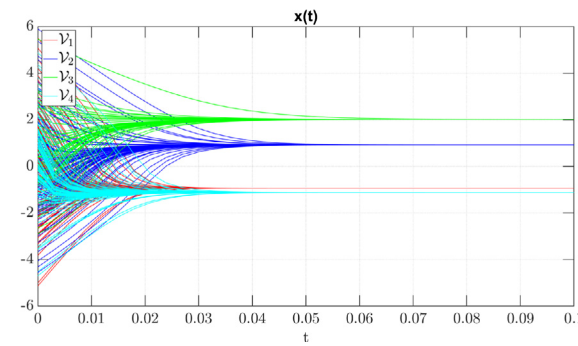

Fig. 6. Graph associated with Example 4: time evolution of $\mathbf{x}(t)$.

7. Conclusions

In this work we addressed the consensus problem for multi-agent systems whose agents split into k groups: agents belonging to the same group cooperate, while those belonging to different ones compete. The proposed algorithm represents a modified version of the classical DeGroot's type of consensus algorithm. The modification pertains how much agents in the same group are conservative of their own opinions in order to guarantee that they converge to a common decision, namely they reach k-partite consensus. We investigated this problem under the assumption that agents in the same cluster have the same amount of trust(/distrust) to be distributed among their friends(/enemies). For the special case of complete, signed, unweighted graphs a simplified solution was proposed. Finally, an extension of the k-partite consensus problem to a nonlinear set-up was investigated.

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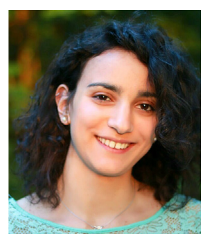

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