ESE2014 Lab 3

Introduction

This report introduces Discrete Fourier Transform, Inverse Discrete Fourier Transform, Discrete-time Fourier Transforms and Inverse Discrete-Time Fourier Transforms. We learn how to calculator DFT, IDFT or DTFT, IDTFT of a function both theoretically and using Matlab.

DFT or DFS is purely Discrete. That is it converts a finite set of sequences in time to periodic discrete-frequency representation.
DFT of a discrete-time sequence is defined as the following equation.

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi kn}{N}}$$

Here, k represents frequency-domain, n represents time-domain ordinals and N is the length of the sequence.

e^-j2 Π/N can be solved using Euler's formula:

e^-j2 Π/N = cos(2 Π/N) - j sin(2 Π/N)

DFT can be calculated in Softwares using FFT algorithms.

The IDFT or IDFS converts discrete-frequency samples to the same number of discrete-time samples.

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k]e^{+j\frac{2\pi kn}{N}}$$

For example, DFT of the equations x1 = [4,1,-1,1] and X2 =[2,0,0,0,-1,0,0,0] are calculated below.

ESE2014 Lab 3

1a) $x(n) = \{4, 1, -1, 1\}$; $N = 4$

w.k.t, $X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}nk}$

Here, $e^{-j\frac{2\pi}{N}} = e^{-j\frac{\cancel{2}\pi}{\cancel{4}_2}}$

$= \cos\left(\frac{\pi}{2}\right) - j\sin\left(\frac{\pi}{2}\right)$

$= -j$

$$X(0) = \sum_{n=0}^{3} x(n)(-j)^{n.0}$$

$$= x(0) + x(1) + x(2) + x(3)$$

$$\underline{X(0) = 5}$$

$$X(1) = \sum_{n=0}^{3} x(n)(-j)^{n.1}$$

$$= x(0)(-j)^0 + x(1)(-j)^1 + x(2)(-j)^2 + x(3)(-j)^3$$

$$= 4 + 1(-j) + (-1)(-1) + 1(-j)(-1)$$

$$= 4 - j + 1 + j$$

$$\underline{X(1) = 5}$$

$$X(2) = \sum_{n=0}^{3} x(n)(-j)^{2n}$$

$$= x(0)(-j)^0 + x(1)(-j)^2 + x(2)(-j)^4 + x(3)(-j)^6$$

$$= 4 - 1 - 1 - 1$$

$$\underline{X(2) = 1}$$

$$X(3) = \sum_{n=0}^{3} x(n)(-j)^{3n}$$

$$= x(0)(-j)^0 + x(1)(-j)^3 + x(2)(-j)^6 + x(3)(-j)^9$$

$$= 4 + j + 1 - j$$

$$\underline{X(3) = 5}$$

ESE2014 Lab 3

1 b) $x(n) = \{2, 0, 0, 0, -1, 0, 0, 0\}$; $N = 8$

$$X(0) = \sum_{n=0}^{8} x(n)\left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)^{0 \cdot n}$$

$$e^{-j\left(\frac{2\pi}{N}\right)} = e^{j\left(\frac{\pi}{4}\right)}$$
$$= \cos\left(\frac{\pi}{4}\right) - j\sin\left(\frac{\pi}{4}\right)$$
$$= \frac{1}{\sqrt{2}} - j\left(\frac{1}{\sqrt{2}}\right)$$

$$= 2 + 0 + 0 - 1 + 0 + 0 + 0$$
$$\underline{X(0) = 1}$$

$$X(1) = 2 + 0 + 0 + 0 - 1\left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)^{1 \cdot 4} + 0 + 0 + 0$$
$$= 2 - \left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)^{2}\left(\frac{1}{\sqrt{2}} - j\left(\frac{1}{\sqrt{2}}\right)\right)^{2}$$
$$= 2 - (-j)(-j)$$
$$= 2 + 1$$
$$\underline{X(1) = 3}$$

$$\left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)^{2}$$
$$= \frac{1}{2} + \frac{j^2}{2} - 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{j}{\sqrt{2}}\right)$$
$$= \frac{1}{2} - \frac{1}{2} - j$$
$$= -j$$

$$X(2) = 2 - 1\left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)^{8}$$
$$= 2 - \left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)^{2}\left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)^{2}\left(\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}\right)^{2}\left(\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}\right)^{2}$$
$$= 2 - (-j)(-j)(-j)(-j)$$
$$= 2 - 1$$
$$\underline{X(2) = 1}$$

$$X(3) = 2 - 1\left(\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}\right)^{12}$$
$$= 2 - \left(\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}\right)^{4}\left(\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}\right)^{4} \cdot \left(\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}\right)^{4}$$
$$= 2 - (-j^2)(-j^2)(-j^2)$$
$$= 2 + 1$$
$$\underline{X(3) = 3}$$

$$X(4) = 2 - 1\left(\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}\right)^{16}$$
$$= 2 - 1$$
$$\underline{X(4) = 1}$$

$$X(5) = 2 - \left(\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}\right)^{20}$$
$$= 2 + 1$$
$$\underline{X(5) = 3}$$

$$X(6) = 2 - \left(\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}\right)^{24}$$
$$= 2 - 1$$
$$\underline{X(6) = 1}$$

$$X(7) = 2 - \left(\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}\right)^{28}$$
$$= 2 + 1$$
$$\underline{X(7) = 3}$$

We can verify these answers in Matlab using the FFT algorithm.
The result is as shown in the below image:

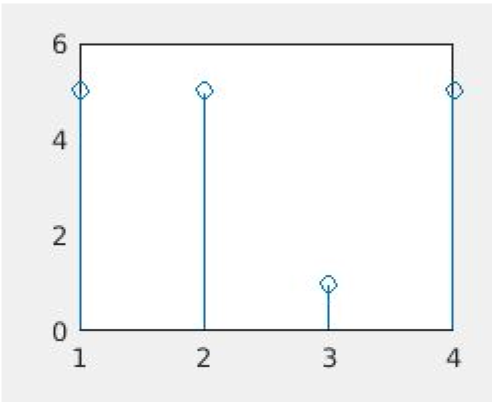


Fig: Result of 1a

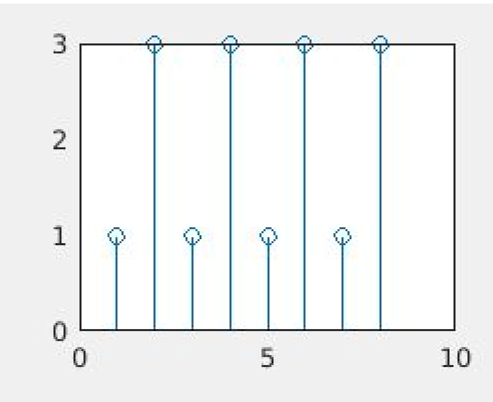


Fig: Result of 1b

DTFT is best explained by Steven W. Smith as:

"The Discrete-Time Fourier Transform (DTFT) is a member of the Fourier transform family that operates on aperiodic, discrete signals. The best way to understand the DTFT is how it relates to the DFT. To start, imagine that you acquire an N sample signal, and want to find its frequency spectrum. By using the DFT, the signal can be decomposed into sine and cosine waves, with frequencies equally spaced between zero and one-half of the sampling rate. (Padding the time domain signal with zeros makes the period of the time domain longer, as well as making the spacing between samples in the frequency domain narrower). As N approaches infinity, the time domain becomes aperiodic, and the frequency domain becomes a continuous signal. This is the DTFT, the Fourier transform that relates an aperiodic, discrete signal, with a periodic, continuous frequency spectrum."

DTFT is defined as:

$$X(\omega) = \sum_{k=-\infty}^{\infty} x[n] e^{-j\omega k}$$

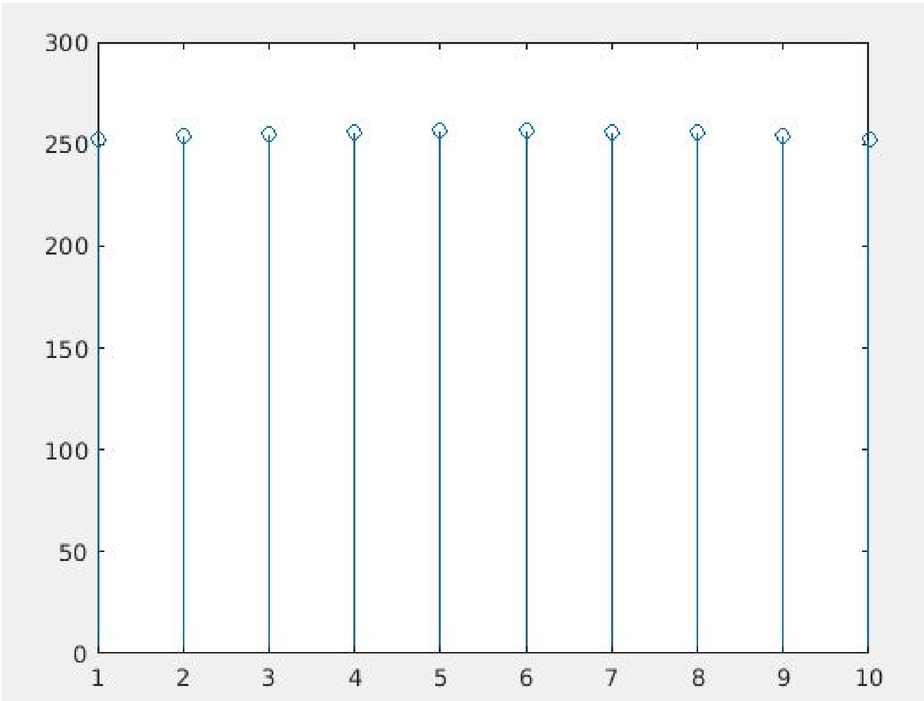
ω here is the frequency variable. Note that X(ω) is continuous in frequency even though x[n] is discrete in time-domain.

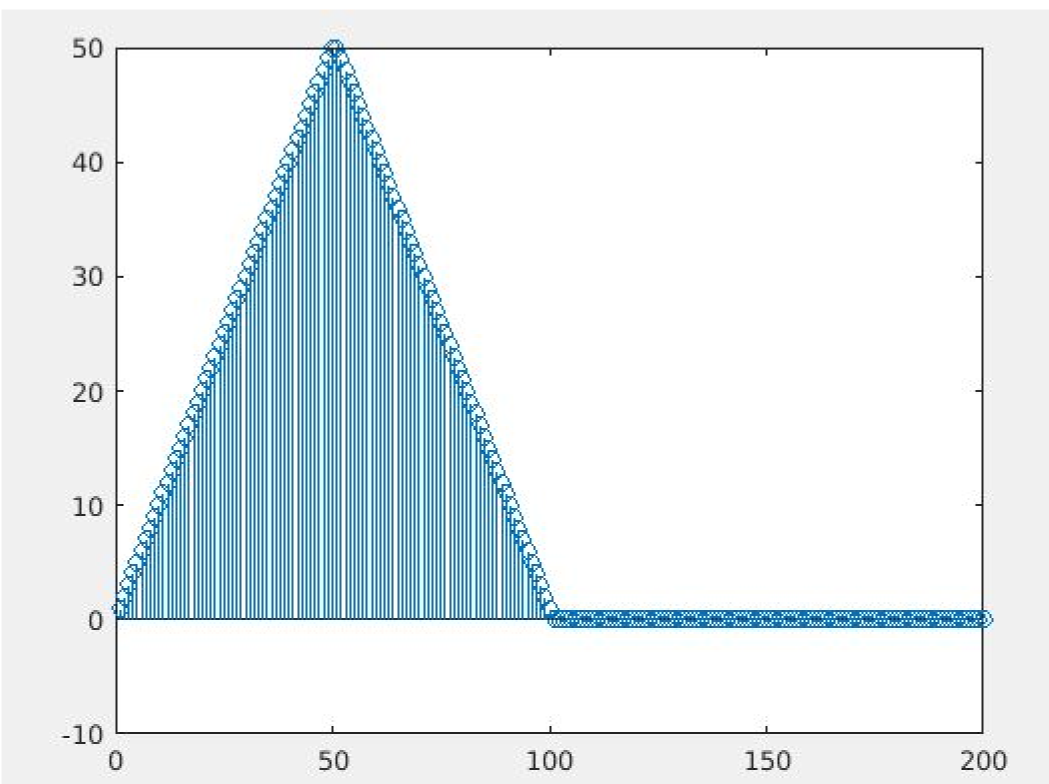
And IDTFT is defined as:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

ESE2014 Lab 3

2a)

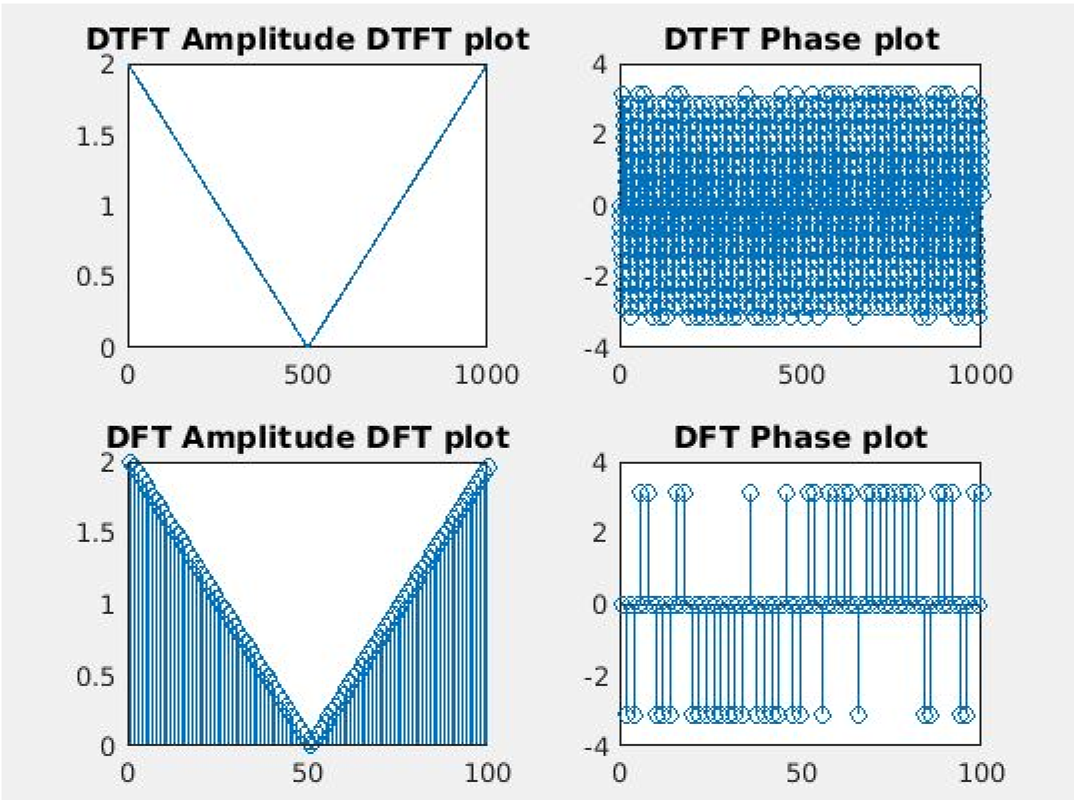




2c ans) As we can see from graph 2a and 2b increasing the resolution (number of points) gives a much better picture of how exactly the function is.

<u>ESE2014 Lab 3</u>

3)



e.

6. A 512-point DFT X(k) of a real-valued sequence x(n) has the DFT values:

$$X(0) = 20 + j\alpha; \quad X(5) = 20 + j30; \quad X(k_1) = -10 + j15; \quad X(152) = 17 + j23;$$

$$X(k_2) = 20 - j30; \quad X(k_3) = 17 - j23; \quad X(480) = -10 - j15; \quad X(256) = 30 + j\beta$$

And all other values are known to be zero.

a. Determine the real-valued coefficients α and β.
b. Determine the values of the integers k_1, k_2 and k_3.
c. Determine the energy of the signal x(n).
d. Express the sequence x(n) in a closed form.

ESE2014 Lab 3

(6) $X(0) = 20 + j\alpha$ $\quad X(k_1) = -10 + j15$

$X(k_2) = 20 - j30$ $\quad X(480) = -10 - j15$

$X(5) = 20 + j30$ $\quad X(152) = 17 + j23$ $\quad N = 512$ points

$X(k_3) = 17 - j23$ $\quad X(256) = 30 + j\beta$

a) w.k.t $\quad X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\left(\frac{2\pi}{N}\right)kn}$

$e^{-j\frac{2\pi}{N}} = \cos\left(\frac{2\pi}{512}\right) - j\sin\left(\frac{2\pi}{512}\right) = 0.987j \simeq j$

$X(0) = \sum_{n=0}^{N-1} x(n)\, j^{0n}$

$X(0) = \sum_{n=0}^{N-1} x(n)$

$\Rightarrow$ No imaginary component

$\alpha = 0$

b) $X(256) = \sum_{n=0}^{N-1} x(n)\, j^{256n}$

$X(256) = x(0) + x(1)(+1) + x(2)(-1) + x(3)(+1) + \ldots$ so on

$\Rightarrow$ No imaginary component

$\therefore \beta = 0$

b) $X(k) = X^*(N-k)$

$X(k_1) = -10 + 15j$

& $X(480) = -10 + 15j$

$X(k_1) = X^*(512 - 480)$

$\therefore \boxed{k_1 = 32}$

$X(k_2) = 20 - j30$

$X^*(5) = 20 + j30$

$X(k_2) = X^*(512 - 5)$

$\boxed{k_2 = 507}$

$X(k_3) = 17 - j23$

$X^*(152) = 17 + j23$

$X(k_3) = X^*(512 - 152)$

$\boxed{k_3 = 360}$

c) Energy of a signal $x(n)$ E can be expressed as

$$E = \frac{1}{N}\sum_{k=0}^{N-1} |F(k)|^2$$

i.e.

$$E = \frac{1}{512}\Big[|F(0)|^2 + |F(5)|^2 + |F(32|^2 + |F(152)|^2 + |F(256)|^2 + |F(360)|^2 + |F(480)|^2 + |F(507)|^2\Big]$$

$$= \frac{1}{512}\left[400 + 1300 + 325 + 818 + 900 + 818 + 325 + 1300\right]$$

$$\Rightarrow E = 12 \cdot 082$$

The sequence $x(n)$ can be expressed in closed form as

$$\tilde{x}(n) = \left(\frac{1}{N}\right)\sum_{k=0}^{N-1} X(k)\, e^{j\left(\frac{2\pi}{N}\right)kn}$$

where $n = 0, \pm 1, \pm 2, \ldots$

Conclusion

As we can clearly see, Matlab makes the calculations much more efficient and less time-consuming. Matlab can be used as a computational tool in signal analyzing and processing.

ESE2014 Lab 3

Appendix:

```
 %1a
clc; clear; close all;
 x1 = [4,1,-1,1]
 fft(x1);

%1b)
X2 =[2,0,0,0,-1,0,0,0]
fft(x2);

%2a
clear all;
clc;
 t= 0:1:100;
 x(1:50)= t(1:50)+1;
 x(51:100) = 100-t(51:100);
 jk=fft(x)
 x1=[jk(1),jk(10),jk(20),jk(30),jk(40),jk(50),jk(60),jk(70),jk(80),jk(90)]
 y=ifft(x1);
 stem(y);

%2b
clear all;
clc;
 t= 0:1:100;
 x(1:50)= t(1:50)+1;
 x(51:100) = 100-t(51:100);
 x200(1:100) = x
 x200(101:200) = 0
 jk=fft(x200)
 x1=jk(1:200);
 y=ifft(x1);
 stem(y);

%3
x=zeros(100,1);
for n = 0:1:100
x(n+1,:) = sinc((n-50)/2) * sinc((n-50)/2);
end

% stem(x);
```

```
dtftRes = fft(x,1000)%Assume continous in freq due to more points in freq dom
dftRes = fft(x,100)% Feels discrete due to less number of points in freq doma
subplot(2,2,1)
plot(abs(dtftRes)), title('DTFT Amplitude DTFT plot')
% Plot function further interpolates the gaps

subplot(2,2,3)
stem(abs(dftRes)), title('DFT Amplitude DFT plot')
subplot(2,2,2)
stem(angle(dtftRes)), title('DTFT Phase plot')
subplot(2,2,4)
stem(angle(dftRes)), title('DFT Phase plot')

function X = dft(x)
    le=length(x);
    xk=zeros(le,1);
    for k=0:le-1
        for n=0:le-1
            xk(k+1) = xk(k+1) + x(n+1)*exp(-1i*2*pi*k*n/le);
        end
    end
    X=xk;
end
```