COMPREHENSIVE EXAM

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1. Personal Background

2. Introduction

2.1. **Definition and Examples.** Throughout we work over the field of complex numbers $\mathbb{C}$. A smooth projective variety X is called a *Fano Variety* if its anticanonical divisor $-K_X$ is ample. For any Cartier divisor D on a variety X, $\mathcal{O}_X(D)$ will denote the corresponding invertible sheaf, and, in particular, $\mathcal{O}_X(-K_X)$ is the anticanonical sheaf on X where $-K_X$ is the anticanonical divisor of X.

To give an example, for any positive integer n, $\mathbb{P}^n$ is a n-dimensional Fano variety since its anticanonical sheaf is $\mathcal{O}_{\mathbb{P}^n}(n+1)$ which is ample by II.7 in [5]. In fact $\mathbb{P}^1$ is the only 1-dimensional Fano variety up to isomorphism. Fano varieties of dimension 2 are called *del Pezzo surfaces* and their classification is given in cite. Clearly, $\mathbb{P}^3$ is a 3-dimensional Fano variety. To see another example, consider a smooth hypersurface V of degree d in $\mathbb{P}^n$, $n \geq 2$. Then by the adjunction formula, [5], II.8

$$K_V = (d - n - 1)H$$

where H is the generator of Pic $\mathbb{P}^n$. Then if $d - n - 1 < 0$, $-K_V$ is ample and so V is a Fano variety.

Fano varieties of dim 3 with Picard rank $\rho = 1$ are called *prime* Fano 3-folds and their classification was first completed by Iskovskikh using the birational method of double projection from a line in [1] and [2]. The classification was later reworked by S. Mukai in [7] using the biregular vector bundle method. Fano 3-folds with $\rho \geq 2$ were all classified by Mori and Mukai in to cite. More recently, De Biase, Fatighenti, and Tanturri have obtained a description of a general member of each deformation family of Fano 3-folds as the zero locus of a homogeneous vector bundle in a product of Grassmannians and weighted projective spaces, cite.

The subject of this paper/article/survey/document is Fano 3-folds, we study their classical classification given by Iskovskikh, Mukai, and Mori.

2.2. **Potential Research.**

2.3. **Plan.** We begin by giving a Iskovskih's proof of the boundedness of the index r of smooth Fano 3-folds in the next subsection. In section 3, we describe he classification for Fano 3-fold with Picard rank 1. Iskovskih classified all smooth Fano 3-folds over $\mathbb{C}$ with index $r \geq 2$ in [1]. We describe them in 3.1. In 3.2, we describe hyperelliptic Fano 3-folds following [1]. It turns out all Fano 3-folds except a few have very ample anticanonical divisor and $r = 1$. An outline of their classification is given in 3.3 following [2].
S. Mukai gave a more explicit description of Fano 3-folds with genus $g \geq 7$ and Gushel did it independently for $g = 6, 8$. We describe these in 3.4. The exposition here primarily follows [7] and cite.
In section 4, we describe Fano 3-folds with Picard rank at least 2 following Multiple citations.

2.4. **Notations.** In this section, we establish the basic results that are important for subsequent classification. Let X be a Fano 3-fold. Let D be a divisor(class) on X and let us write $\mathcal{L}$ for the invertible sheaf(class) corresponding to D, that is, $\mathcal{L} = \mathcal{O}_X(D)$. We also write $H^0(X, D)$ for the finite-dimensional vector space over $\mathbb{C}$ of global sections of X. The symbol $|\mathcal{L}|$ or $|D|$ will denote the complete linear system of effective divisors formed by the divisors of zeroes of sections in $H^0(X, \mathcal{L})$. Also we write dim $|D|$ for $\dim_{\mathbb{C}}(H^0(X, \mathcal{L})) - 1$. For an arbitrary coherent sheaf $\mathcal{F}$ on X, we will write $h^i(X, \mathcal{F})$ for dim $H^i(X, \mathcal{F})$. We consider all vector spaces over $\mathbb{C}$. For a n-dimensional vector space V, $\mathrm{Gr}(s, n)$ denotes the space of s-dimensional subspaces of V, called the Grassmannian.

For a Fano 3-fold X, we define the integer $g = g(X) = -K_X^3/2 + 1$ to be the *genus* of X.

Proposition 2.1. *If $F \in |-K_X|$ is a smooth surface, $C \in |\mathcal{O}_F(-K_X)|$ is a curve, and C has genus $g = g(C) = h^1(\mathcal{O}_C)$, then the following assertions are true:*

(i) $-K_X^3 = 2g - 2$.

(ii) If $-K_X$ is very ample, then $\phi_{|-K_X|}(X) = X_{2g-2}$ is a smooth variety of degree $-K_X^3 = 2g-2$ in $\mathbb{P}^{g+1}$, the hyperplane sections of which are $K3$ surfaces, and the curves sections of which are canonical curves $C_{2g-2} \subset \mathbb{P}^{g-1}$ of genus g.

A sort of converse to Proposition 2.1 is given in [2], 1.2.
Note that since $h^1(\mathcal{O}_X) = 0$ by Kodaira Vanishing theorem, the first chern map $H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z})$ is injective. Thus the Picard group Pic X coincides with the Néron-Severi group $NS(X)$ making Pic X a finitely generated abelian group. The *Picard rank* of X is defined as the rank of Pic X, denoted by $\rho(X)$ or simply ρ.
The following result is due to Šokurov [3].

Theorem 2.2. *Let X be a Fano 3-fold. There exists a divisor $H \in Pic(X)$ and a natural number r such that $-K_X = rH$ and the linear system $|H|$ contains a smooth surface.*

The maximal such integer r is called the *index* of the Fano 3-fold X. By the Riemann-Roch theorem, Serre duality, the Kodaira vanishing theorem and the adjunction formula we have from [1], 1.9,

Proposition 2.3. *(fix notation- H should be S)If $r \geq 2$, then the canonical invertible sheaf of H is given by*

$$\mathcal{O}_H(K_H) \simeq \mathcal{O}_H \otimes \mathcal{O}_X(-(r-1)H).$$

proof does not seem too important, so not writing it. But I am not sure. maybe I should omit thispropsotion here and include it in the proof of the following corollary. Think and Ask!

Corollary 2.4. *Let $S \in |H|$ be a smooth surface. Then S is a del Pezzo surface.*

Proof. From the proposition and $r \geq 2$, $-K_S = (r-1)H$ is ample. □

The following result establishes the boundedness of the index of Fano 3-folds.

Proposition 2.5. *Let X have index $r \geq 2$, and suppose that the linear system $|H|$ contains a smooth surface S. Then*

(i) $r \leq 4$;
(ii) if $r = 2$ *then* $1 \leq S^3 \leq 9$;
(ii) if $r = 3$ *then* $S^3 = 2$;
(iv) if $r = 4$ *then* $S^3 = 1$.

Proof. Let $S \in |H|$ be a del Pezzo surface. Then by cite delpezzo?

$$1 \leq K_S^2 \leq 9.$$

Plugging in the formula for K_S from 2.3, we get

$$1 \leq (r-1)^2 S^3 \geq 9.$$

Now S^3 is a positive integer as $-K_X$ is ample and $r \geq 2$, so considering possibilities for positive integer values of S^3 gives $r \leq 4$. This proves (i). For (ii) and (iv), using $r = 2$ and 4 respectively gives us possible values of S^3. If $r = 3$, then from the last inequality $S^3 = 1$ or 2. For $S^3 = 1$, we get a contradiction

$$2g - 2 = -K_X^3 = (3H)^3 = 27,$$

so $S^3 = 2$.

□

Definition 2.6. Set $d = d(X) = S^3$. If H is very ample, then $d(X)$ is the *degree* of $\phi_{|H|}(X)$ in $\mathbb{P}^{\dim|H|}$.

Proposition 2.7. *(i) Pic* $X \simeq H^2(X, \mathbb{Z})$.
(ii) Pic X is torsion-free.

Proof. The exponential exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \xrightarrow{exp} \mathcal{O}_X^* \to 1$$

induces the long exact cohomological sequence

$$\cdots \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \xrightarrow{\delta} H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X) \to \dots$$

By Kodaira vanishing theorem, $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$, so δ is an isomorphism. This shows (i). Let $S \in |-K_X|$. It is a $K3$ surface by [1], 1.5 and if? $r \geq 2$, a del Pezzo surface by 2.4. In any case, $H^2(S, \mathbb{Z})$ is torsion-free. Further by Lefschetz hyperplane theorem, $H^2(X, \mathbb{Z}) \hookrightarrow H^2(S, \mathbb{Z})$. Thus by (i), Pic X is also torsion-free. □

Given a Fano 3-fold X, we label it by a pair of numbers $\rho - N$ where ρ is the Picard rank of X and N is the number in the classification found in [6]. A most recent classification table for Fano 3-folds along with some of their associated invariants and information about their birational geometry, zero section description(due to Fatighenti) class can be found on [9].

3. Fano Threefolds with $\rho = 1$

Let X be a smooth Fano 3-fold of index r.

3.1. **Fano 3-folds with $r \geq 2$.** The following theorem by Iskovskikh([1]) completely classifies Fano 3-folds with index $r \geq 2$.

Theorem 3.1. *Let X be a Fano 3-fold of index $r \geq 2$. Then the following assertions hold:*

(i) 1-16, 1-17: If $r \geq 3$, then $\phi_{|H|} : X \xrightarrow{\sim} \mathbb{P}^3$ is an isomorphism for $r = 4$, and $\phi_{|H|} : X \xrightarrow{\sim} X_2 \subset \mathbb{P}^4$ is an isomorphism of X with a smooth quadric of $\mathbb{P}^4$ for $r = 3$.

(ii) If $r = 2$, then a variety X only exists for $1 \leq d \leq 7$; for $d \geq 3$, $\phi_{|H|} : X \xrightarrow{\sim} X_d \subset \mathbb{P}^{d+1}$ is an embedding of X as a subvariety X_d of degree d in $\mathbb{P}^{d+1}$, with X_d projectively normal; and if $d \geq 4$, then X_d is the intersection of the quadrics containing it.
Conversely, for any $d \geq 3$, every smooth projectively normal 3-fold $X_d \subset \mathbb{P}^{d+1}$ not lying in any hyperplane is a Fano 3-fold, and has index $r = 2$, apart from the case $r = 4, d = 8$, when X_8 is the image of $\mathbb{P}^3$ in $\mathbb{P}^9$ under the Veronese embedding.

(iii) If $r = 2$ and $3 \leq d \leq 7$, then; for $d = 7$, X_7 is the projection of the Veronese 3-fold $X_8 \subset \mathbb{P}^9$ from some point of X_8;
for $d = 6$, $X_6 \simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ in its Segre embedding;
1-15: for $d = 5$, $X_5 \subset \mathbb{P}^6$ is unique up to projective equivalence, and can be realized in either of the following two ways:
  (a) as the birational image of a quadric $W \subset \mathbb{P}^4$ under the map defined by the linear system $|\mathcal{O}_W(2) - Y|$ of quadrics passing through a twisted cubic Y;
  (b) as the section of the Grassmannian $Gr(2, 5)$ of lines in $\mathbb{P}^4$ by 3 hyperplanes in general position;

1-14: for $d = 4$, X_4 is any smooth intersection of two quadrics in $\mathbb{P}^5$;
1-13: for $d = 3$, X_3 is any smooth cubic hypersurface of $\mathbb{P}^4$.

(iv) If $r = 2$ and $d = 1$ or 2, then:
1-12: for $d = 2$, $\phi_{|H|} : X \to \mathbb{P}^3$ is a double covering with smooth ramification surface $D_4 \subset \mathbb{P}^3$ of degree 4, and any such variety is a Fano 3-fold with $r = 2$ and $d = 2$; and every Fano 3-fold with $r = 2$ and $d = 2$ can be realized as a smooth hypersurface of degree 4 in $\mathbb{P}(1, 1, 1, 1, 2)$;
1-11: for $d = 1$, $\phi_{|H|} : X \to \mathbb{P}^2$ is a rational map with a single point of indeterminacy, and with irreducible elliptic fibres; and X can be realized in either of the following two ways:
  (a) $\phi_{|-K_X|} : X \to W_4$ is any double cover of the cone W_4 over the Veronese surface $F_4 \subset \mathbb{P}^5$, having smooth ramification divisor $D \subset W_4$ by a cubic hypersurface not passing through the vertex of the cone;
  (b) any smooth hypersurface of degree 6 in $\mathbb{P}(1, 1, 1, 2, 3)$.

Note that in the theorem, for $r = 2$, $d = 6$, Picard group is Pic $X \simeq \mathbb{Z}^{\oplus 3}$, so in this case X is a Fano 3-fold with $\rho(X) = 3$(3-27). For $r = 2, d = 7$ is 2-35?Probably not?

3.2. **Hyperelliptic Fano 3-folds with $r = 1$.** For a Fano 3-fold of index r, Theorem 2.2 implies that the fundamental linear system $|H|$ is without fixed components and base points and so by [1], 2.2, it follows that deg $\phi_{|H|} = 1$ or 2. Here we study the case when deg $\phi_{|H|} = 2$.

Definition 3.2. A Fano 3-fold X of index $r = 1$ is *hyperelliptic* if its anticanonical map $\phi_{|-K_X|}$ is a morphism and is of degree deg $\phi_{|-K_X|} = 2$.

Theorem 3.3. *Let X be a hyperelliptic Fano variety, and let $\phi_{|-K_X|} : X \to Y \subset \mathbb{P}^{g+1}$ be the corresponding morphism of degree 2. Then Y is nonsingular and X is uniquely determined by the pair (Y, D), where $D \subset Y$ is the ramification divisor of $\phi_{|-K_X|}$. For $\rho(X) = 1$, the pair (Y, D) belongs to one of the following families(and if D is a smooth divisor, then for each pair (Y, D) there exists a Fano 3-fold X):*

(i) 1-1: $Y \simeq \mathbb{P}^3$, and D is a smooth hypersurface of degree 6; in this case X can be realized alternatively as a smooth hypersurface of degree 6 in the weighted projective space $\mathbb{P}(1,1,1,1,3)$.

(ii) 1-2 b): $Y \simeq X_2$ is a smooth quadric in $\mathbb{P}^4$ and $D \in |\mathcal{O}_{X_2}(4)|$; that is, $D = X_2 \cap X_4$, where X_4 is a quartic in $\mathbb{P}^4$. In this case X can also be realized as a smooth complete intersection in the weighted projective space $\mathbb{P}(1,1,1,1,1,2)$: X is the intersection of a quadric cone and a hypersurface of degree 4:

$$F_2(x_0, \dots, x_4) = 0$$
$$F_4(x_0, \dots, x_5) = 0.$$

For the proof, see [1], 7.3-7.6.

3.3. **Fano 3-folds with $r = 1$.** Let X be a Fano 3-fold with index $r = 1$.

Definition 3.4. A smooth complete irreducible 3-fold X over $\mathbb{C}$ is called a *Fano 3-fold of the principal series* if the anticanonical divisor $-K_X$ is very ample.

From Isk77,78- cite the exact result or conclude, it follows that all Fano 3-folds are of the principal series with the exceptions of hyperelliptic Fano 3-folds, 1-11, and cannot identify Fano 3-fold in 3.1-b). Also how are these not very ample?? From this point, we will consider our Fano 3-folds to be of principal series and write $X_{2g-2} \subset \mathbb{P}^{g+1}$ for a Fano 3-fold of the principal series in its anticanonical embedding.
Šokurov showed the existence of line on such Fano 3-folds with Picard rank 1 in [4]. Under this assumption and using the method of double projection from a line, Iskovskikh showed that there exist Fano 3-folds with $\rho = 1$ and $r = 1$ for genus $g \leq 10$ and $g = 12$ but not for $g = 11$.
For genus $g = 3, 4$, and 5, Fano 3-folds with $r = 1$, we have

Proposition 3.5. *A Fano 3-fold $X_{2g-2} \subset \mathbb{P}^{g+1}$ is a complete intersection only for $g = 3, 4$ or 5, and we have that*

1-2 a): $X_4 \subset \mathbb{P}^4$ is a quartic hypersurface,
1-3 : $X_6 = V_{2\cdot 3}$ is an intersection of a quadric and a cubic in $\mathbb{P}^5$,
1-4 : $X_8 = V_{2\cdot 2\cdot 2}$ is an intersection of 3 quadrics in $\mathbb{P}^6$.

Conversely, each smooth complete intersection of the types indicated is a Fano 3-fold the principal series.

See [2], 1.3 for a proof. To do some computations for genus g, index r and ρ and that will complete $r = \rho = 1$ discussion for these genera, I believe. Necessary?

Example 3.6. ($g = 6$) Let us denote by V a section of Plücker embedding $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$ by two hyperplanes in general position and a quadric. By adjunction formula, this is a Fano variety of index 1. genus, Picard rank computations important? I think I should maybe delete this example here because later in 3.4, Gushel's theorem gives a description of it anyway. my mainreason to include it here to show that Iskovskikh gave genus g=6 Fano 3-fold too..

Iskovskikh asserts that it is possible to show that every Fano 3-fold $X_{10} \subset \mathbb{P}^7$ with Picard rank 1 is a section of Grassmannian $\mathrm{Gr}(2,5)$. See [2], 1.5.

The following result is the main classification theorem by Iskovskikh for Fano 3-folds with genus $g \geq 7$.

Theorem 3.7. *Let $X = X_{2g-2} \subset \mathbb{P}^{g+1}$ be a Fano 3-fold with $\rho = 1$ and $g \geq 7$. Let $\pi_{2Z} : X \to W \subset \mathbb{P}^{g-6}$ be the double projection from a sufficiently general line $Z \subset X$. Let E denote the hyperplane section of W. Then the following assertions hold:*

(i) $g \leq 12$.
(ii) *If* $g = 12$, *then* $W = W_5 \subset \mathbb{P}^6$ *is a prime Fano* 3*-fold with index* 2 *and degree* 5 *(with possibly one singular point); the map* $\rho_Y : W \to X$ *inverse to* π_{2Z} *is given by the linear system* $|3E - 2Y|$, *with* $Y \subset W$ *a normal rational curve of degree* 5 *in* $\mathbb{P}^5$.
(iii) *There do not exist any prime Fano* 3*-folds with* $g = 11$.
(iv) *If* $g = 10$, *then* $W = W_2 \subset \mathbb{P}^4$ *is a quadric and* $\rho_Y : W \to X$ *is given by the linear system* $|5E - 2Y|$, *where* Y *is a smooth curve of genus* 2 *and degree* 7 *in* $\mathbb{P}^4$.
(v) *If* $g = 9$, *then* $W = \mathbb{P}^3$ *and* $\rho_Y : \mathbb{P}^3 \to X$ *is given by the linear system* $|7E - 2Y|$, *where* Y *is a smooth curve of genus* 3 *and degree* 7.
(vi) *If* $g = 8$, *then* $\pi_{2Z} : X \to \mathbb{P}^2$ *is a rational map with fibres(after resolving the determinacy) curves of genus* 2, *and such that the inverse images of lines of* $\mathbb{P}^2$ *are rational surfaces.*
(vii) *If* $g = 7$, *then* $\pi_{2Z} : X \to \mathbb{P}^1$ *is a rational map whose general fiber(after resolving the indeterminacy) is a del Pezzo surface of degree* 5 *with* 8 *points blown up;* X *is a rational* 3*-fold, and the projection from a line maps it into a complete intersection of* 3 *quadrics of* $\mathbb{P}^6$ *containing a smooth rational ruled surface* $R_3 \subset \mathbb{P}^4$.

The proof of (i) and the construction of Fano 3-folds with given genus in (ii)-(vii) uses the birational technique of projection and double projection from lines on X. See [2], §6 for details. For an alternative proof of boundedness of genus for Fano 3-folds using vector bundles, see [8].

3.4. **Mukai and Gushel's description of Fano 3-folds with $g \geq 6$.** While Iskovskikh's method gives the existence of prime Fano 3-folds with genus $g \geq 6$, Mukai gave a more explicit description of such projective varieties of dimension $n \geq 3$. To classify Fano n-folds of the principal series of genus $g \geq 6$, Mukai first showed the existence of a good vector bundle $\mathcal{E}$ on X. Then using the linear system $|\mathcal{E}|$, we can embed X into a Grassmannian variety and describe its image. We find that X is a linear section of some homogeneous space. We describe these homogeneous spaces in the following examples.

Example 3.8. For Fano n-folds, very ample $-K_X$ with $\rho = 1$ and genus $g \geq 6$, the dimension n cannot be arbitraily large. In fact, the maximum dimension in these cases $n(g) = 24 - 2g$ is attained by a variety $\Sigma_{2g-2}^{n(g)}$ as below:

(i) $g = 7$. Let V be a 9-dimensional vector space with F as a non degenerate symmetric bilinear form on V and S be the space of spinors of F. Here $n(g) = 10$. Denote by $\Sigma_{12}^{10} \subset \mathrm{Gr}(4,9)$ the set of all 4-dimensional subspaces W of V with $F(W,W) = 0$. Then Σ_{12}^{10} is a smooth 10-dimensional subvariety of Grassmannian $\mathrm{Gr}(4,9)$ and can be embedded in $\mathbb{P}^{15}$ by the spinor coordinates.
Here $\Sigma_{12}^{10} = SO_{10}(\mathbb{C})/P$ is a homogeneous space with P a maximal parabolic subgroup of $SO_{10}(\mathbb{C})$ and is unique up to isomorphism.
(ii) $g = 8$. The Grassmannian $\Sigma_{14}^8 := \mathrm{Gr}(2,6) \subset \mathbb{P}(\wedge^2\mathbb{C}^6) = \mathbb{P}^{14}$ is a smooth Fano variety of dimension 8 and index 8. Here $\Sigma_{14}^8 = SL_6(\mathbb{C})/P$ is a homogeneous space with P a maximal parabolic subgroup of $SL_6(\mathbb{C})$.
(iii) $g = 9$. Let V be a 6-dimensional vector space and F be a non degenrate skew-symmetric bilinear form on V. Here $n(g) = 6$. Let us denote by Σ_{16}^6 the set of all 3-dimensional subspaces W of V such that $F(W,W) = 0$. Then $\Sigma_{16}^6 \subset \mathrm{Gr}(3,6)$ is a homogeneous space isomorphic to $U(3)/O(3)$ as varieties, hence is a smooth 6-dimensional subvariety of degree 16 in $\mathrm{Gr}(3,6) \subset \mathbb{P}^{19}$.
(iv) $g = 10$. Let V be a 7-dimensional vector space and F be a non degenerate skew-symmetric 4-linear form on V. Here $n(g) = 5$. Denote by Σ_{18}^5 the set of all 5-dimensional subspaces W of V such that $F(W,W,W,W) = 0$. Then $\Sigma_{18}^5 \subset \mathrm{Gr}(5,7)$ is a homogeneous space, hence a smooth 5-dimensional subvariety of degree 18 in $\mathrm{Gr}(5,7) \subset \mathbb{P}^{20}$. Here Σ_{18}^5 is a homogeneous

space under the action of the exceptional Lie group of type G_2 and is isomorphic to G_2/P where P is a maximal parabolic subgroup of G_2.

(v) $g = 12$. Let V be a 7-dimensional vector space and F_1, F_2, and F_3 be three linearly independent skew-symmetric bilinear forms on V. Denote by X the set of 3-dimensional subspaces W of V with $F_1(W,W) = F_2(W,W) = F_3(W,W) = 0$. If the subspace $F_1 \wedge V^\vee + F_2 \wedge V^\vee + F_3 \wedge V^\vee$ of $\wedge^3 V^\vee$ contains no vectors of the form $f_1 \wedge f_2 \wedge f_3 \neq 0$ for $f_1, f_2, f_3 \in V^\vee$ then X is a smooth 3-dimensional subvariety of degree 22, denoted by Σ^3_{22}. Here $n(g) = 3$.

Index computation using mukai92b(uses a result based on root systems), degree of each $\Sigma, 6 \leq g \leq 10$ is $2g-2$, also follows from the same result. As a result we get genus of Fano X is g. Picard rank follows from the Lefschetz hyperplane theorem.

Theorem 3.9. *Let X be a prime Fano n-fold ($n \geq 3$) of index $n-2$ and genus g, $6 \leq g \leq 10$ over $k \subset \mathbb{C}$. Then there exists a k-vector space V and a space M of multilinear forms on V such that X is isomorphic to a linear section of $\Sigma_g(V, M) \subset \mathbb{P}_k^{g+n(g)-2}$. For $g = 12$, AG V statement from page 112 or muk89 statement as above? Cannot find that n can only be 3 in original papers as far as I have understood and read them.*

In particular, for $n = 3$ and $7 \leq g \leq 10$, a Fano 3-fold $X_{2g-2} \subset \mathbb{P}^{g+1}$ is obtained as a complete intersection of the homogeneous space $\Sigma^{n(g)}_{2g-2}$ and a linear subspace of codimension $n(g) - 3$ in $\mathbb{P}(V) = \mathbb{P}^{g+n(g)-2}$. And by Lefschetz theorem, it follows that such $X = X_{2g-2}$ has Pic $(X) \simeq \mathbb{Z}(-K_X)$. For Fano 3-folds of genus 6 and 8 over $\mathbb{C}$, the theorem was proved independently by Gushel. For $g = 6$, we have

Theorem 3.10. *Let $X = X_{10} \subset \mathbb{P}^7$ be an anticanonically embedded Fano threefold of index 1 and genus 6 with $\rho(X) = 1$. Then X is one of the following:*

(i) a section of the Grassmannian $Gr(2, 5)$ embedded by Plücker embedding into $\mathbb{P}(\wedge^2\mathbb{C}^5)$ by a subspace of codimension 2 and a quadric,

(ii) the section by a quadric of a cone $W = W_5 \subset \mathbb{P}^7$ over a nonsingular del Pezzo threefold $V = V_5 \subset \mathbb{P}^6$ of degree 5.

Threefolds of type (i) and (ii) are not isomorphic.

See [10] for details on the proof. A similar method can be applied to study Fano threefolds of genus 8, see [11] and [12].

Theorem 3.11. *Let $X = X_{14} \subset \mathbb{P}^9$ be an anticanonically embedded Fano threefold of index 1 and genus 8 with $\rho(X) = 1$. Then X is a section of the Grassmannian $Gr(2, 6)$ embedded by Plücker embedding into $\mathbb{P}(\wedge^2\mathbb{C}^6) = \mathbb{P}^{14}$ by a subspace of codimension 5.*

As a consequence, we see that the Fano 3-folds with Picard rank $\rho = 1$ appear as one of the following:

(i) Sections of Grassmannians(1-5 a), 1-6, 1-7, 1-8, 1-9, 1-10, 1-15)
(ii) Complete intersections in projective or weighted projective space (1-2 a), 1-3, 1-4, 1-11, 1-12, 1-13, 1-14, 1-16, 1-17, 1-11, 1-12)
(iii) Hyperelliptic (1-1, 1-2 b))

to decide about 1-5b) after Gushel

4. Fano Threefolds with $\rho \geq 2$

outline from mm(1981). Some details from others yet to add because what I have now is just not sufficient.

In this section, we consider Fano 3-folds with $B_2 \geq 2$. The main result is

Theorem 4.1. *There are exactly 88 types of Fano 3-folds with $B_2 \geq 2$ up to deformations.*

We begin with some definitions.

Definition 4.2. A Fano 3-fold is imprimitive if it is isomorphic to the blow-up of a Fano 3-fold along a smooth irreducible curve. A Fano 3-fold is primitive if it is not imprimitive.

Definition 4.3. A smooth variety over a smooth surface S is a conic bundle if every geometic fibre of $X \to S$ is isomorphic to a conic, i.e., a scheme of zeroes of a non zero homogeneous form of degree 2 on $\mathbb{P}^2$.

The following theorem gives a complete classification for primitive Fano 3-folds.

Theorem 4.4. *Let X be a primitive Fano 3-fold. Then we have*

(1) $B_2 \leq 3$,
(2) if $B_2 = 2$, then X is a conic bundle over $\mathbb{P}^2$, and
(3) if $B_2 = 3$, then X is a conic bundle over $\mathbb{P}^1 \times \mathbb{P}^1$ and has either a divisor $D \simeq \mathbb{P}^1 \times \mathbb{P}^1$ such that $\mathcal{O}_D(D) \simeq \mathcal{O}(-1,-1)$ or another conic bundle structure over $\mathbb{P}^1 \times \mathbb{P}^1$.

The following proposition is important for classifying imprimitive Fano 3-folds.

Proposition 4.5. *On a Fano 3-fold X with $B_2 = 2$, there are two smooth rational curves C_1 and C_2 and two numerically effective divisors H_1 and H_2 such that $(C_i \cdot H_j) = \delta_{ij}$ for all $i, j = 1, 2$.*

It turns out that imprimitive Fano 3-folds can be obtained from successive curve-blow-ups of primitive Fano 3-folds by using their conic bundle structure or the existence of lines on Fano 3-folds with $B_2 = 2$. In the latter case, there can be several possibilities for each of the extremal rays and each possibility leads to an imprimitive Fano 3-fold with $B_2 = 2$. Combining all this information we get Table 2 in refer tables.
Since the blowing-up of a Fano 3-fold raises B_2 by 1(refer?), we obtain $B_2 \geq 3$ imprimitive Fano 3-folds by the blowing-up of a Fano 3-fold Y along a smooth irreducible curve C. The following Propositions give strong necessary conditions on $C \subset Y$.

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