

COMPREHENSIVE EXAM

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1. PERSONAL BACKGROUND

2. INTRODUCTION

2.1. Definition and Examples. Throughout we work over the field of complex numbers \mathbb{C} . A smooth projective variety X is called a *Fano Variety* if its anticanonical divisor $-K_X$ is ample. For any Cartier divisor D on a variety X , $\mathcal{O}_X(D)$ will denote the corresponding invertible sheaf, and, in particular, $\mathcal{O}_X(-K_X)$ is the anticanonical sheaf on X where $-K_X$ is the anticanonical divisor of X .

To give an example, for any positive integer n , \mathbb{P}^n is a n -dimensional Fano variety since its anticanonical sheaf is $\mathcal{O}(n+1)$ which is ample by II.7 in [5]. In fact \mathbb{P}^1 is the only 1-dimensional Fano variety up to isomorphism. Fano varieties of dimension 2 are called *del Pezzo surfaces* and their classification is given in [6]. Clearly, \mathbb{P}^3 is a 3-dimensional Fano variety. + **should I include 1 more example, say of hypersurfaces?**.

Fano varieties of dim 3 with Picard rank $\rho = 1$ are called *prime* Fano 3-folds and their classification was first completed by Iskovskikh using the birational method of double projection from a line in [1] and [2]. The classification was later reworked by S. Mukai in [7] using the biregular vector bundle method. Fano 3-folds with $\rho \geq 2$ were all classified by Mori and Mukai in **to cite**. A more recent Fatighenti et al. recent classification.

In this paper, we aim to understand the classification of Fano 3-folds.

2.2. Potential Research?

2.3. Plan. We begin by giving a Iskovskih's proof of the boundedness of the index r of smooth Fano 3-folds in the next subsection. In section 3, we describe the classification for Fano 3-fold with Picard rank 1. Iskovskih classified all smooth Fano 3-folds over \mathbb{C} with index $r \geq 2$ in [1]. We describe them in 3.1. In 3.2, we describe hyperelliptic Fano 3-folds following [1]. It turns out all Fano 3-folds except a few have very ample anticanonical divisor and $r = 1$. An outline of their classification is given in 3.3 following [2].

S. Mukai gave a more explicit description of Fano 3-folds with genus $g \geq 7$ and Gushel did it independently for $g = 6, 8$. We describe these in 3.4. The exposition here primarily follows [7] and [cite](#).

In section 4, we describe Fano 3-folds with Picard rank at least 2 following [Multiple citations](#).

2.4. Notations. In this section, we establish the basic results that are important for subsequent classification. Let X be a Fano 3-fold. Let D be a divisor(class) on X and let us write \mathcal{L} for the invertible sheaf(class) corresponding to D , that is, $\mathcal{L} = \mathcal{O}_X(D)$. We also write $H^0(X, D)$ for the finite-dimensional vector space over \mathbb{C} of global sections of X . The symbol $|\mathcal{L}|$ or $|D|$ will denote the complete linear system of effective divisors formed by the divisors of zeroes of sections in $H^0(X, \mathcal{L})$. Also we write $\dim |D|$ for $\dim_{\mathbb{C}}(H^0(X, \mathcal{L})) - 1$. For an arbitrary coherent sheaf \mathcal{F} on X , we will write $h^i(X, \mathcal{F})$ for $\dim H^i(X, \mathcal{F})$. We consider all vector spaces over \mathbb{C} . For a n -dimensional vector space V , $\text{Gr}(s, n)$ denotes the space of s -dimensional subspaces of V , called the Grassmannian.

For a Fano 3-fold X , we define the integer $g = g(X) = -K_X^3/2 + 1$ to be the *genus* of X .

Proposition 2.1. *If $F \in |-K_X|$ is a smooth surface, $C \in |\mathcal{O}_F(-K_X)|$ is a curve, and C has genus $g = g(C) = h^1(\mathcal{O}_C)$, then the following assertions are true:*

- (i) $-K_X^3 = 2g - 2$.
- (ii) *If $-K_X$ is very ample, then $\phi_{|-K_X|}(X) = X_{2g-2}$ is a smooth variety of degree $-K_X^3 = 2g - 2$ in \mathbb{P}^{g+1} , the hyperplane sections of which are K3 surfaces, and the curves sections of which are canonical curves $C_{2g-2} \subset \mathbb{P}^{g-1}$ of genus g .*

A sort of converse to Proposition 2.1 is given in [2], 1.2.

Note that since $h^1(\mathcal{O}_X) = 0$ by Kodaira Vanishing theorem, the first chern map $H^1(X, \mathcal{O}_X^*) \rightarrow H^2(X, \mathbb{Z})$ is injective. Thus the Picard group $\text{Pic } X$ coincides with the Néron-Severi group $NS(X)$ making $\text{Pic } X$ a finitely generated abelian group. The *Picard rank* of X is defined as the rank of $\text{Pic } X$, denoted by $\rho(X)$ or simply ρ .

The following result is due to Šokurov [3].

Theorem 2.2. *Let X be a Fano 3-fold. There exists a divisor $H \in \text{Pic}(X)$ and a natural number r such that $-K_X = rH$ and the linear system $|H|$ contains a smooth surface.*

The maximal such integer r is called the *index* of the Fano 3-fold X . By the Riemann-Roch theorem, Serre duality, the Kodaira vanishing theorem and the adjunction formula we have from [1], 1.9,

Proposition 2.3. *(fix notation- H should be S) If $r \geq 2$, then the canonical invertible sheaf of H is given by*

$$\mathcal{O}_H(K_H) \simeq \mathcal{O}_H \otimes \mathcal{O}_X(-(r-1)H).$$

proof does not seem too important, so not writing it. But I am not sure. maybe I should omit thispropotion here and include it in the proof of the following corollary. Think and Ask!

Corollary 2.4. *Let $S \in |H|$ be a smooth surface. Then S is a del Pezzo surface.*

Proof. From the [proposition](#) and $r \geq 2$, $-K_S = (r-1)H$ is ample. □

The following result establishes the boundedness of the index of Fano 3-folds.

Proposition 2.5. *Let X have index $r \geq 2$, and suppose that the linear system $|H|$ contains a smooth surface S . Then*

- (i) $r \leq 4$;
- (ii) if $r = 2$ then $1 \leq S^3 \leq 9$;
- (ii) if $r = 3$ then $S^3 = 2$;
- (iv) if $r = 4$ then $S^3 = 1$.

Proof. Let $S \in |H|$ be a del Pezzo surface. Then by [cite delpezzo?](#)

$$1 \leq K_S^2 \leq 9.$$

Plugging in the formula for K_S from 2.3, we get

$$1 \leq (r-1)^2 S^3 \geq 9.$$

Now S^3 is a positive integer as $-K_X$ is ample and $r \geq 2$, so considering possibilities for positive integer values of S^3 gives $r \leq 4$. This proves (i). For (ii) and (iv), using $r = 2$ and 4 respectively gives us possible values of S^3 . If $r = 3$, then from the last inequality $S^3 = 1$ or 2. For $S^3 = 1$, we get a contradiction

$$2g - 2 = -K_X^3 = (3H)^3 = 27,$$

so $S^3 = 2$. □

Definition 2.6. Set $d = d(X) = S^3$. If H is very ample, then $d(X)$ is the *degree* of $\phi_{|H|}(X)$ in $\mathbb{P}^{\dim |H|}$.

Proposition 2.7. (i) $\text{Pic } X \simeq H^2(X, \mathbb{Z})$.
 (ii) $\text{Pic } X$ is torsion-free.

Proof. The exponential exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \rightarrow 1$$

induces the long exact cohomological sequence

$$\cdots \rightarrow H^1(X, \mathcal{O}_X) \rightarrow H^1(X, \mathcal{O}_X^*) \xrightarrow{\delta} H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathcal{O}_X) \rightarrow \cdots$$

By Kodaira vanishing theorem, $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$, so δ is an isomorphism. This shows (i). Let $S \in |-K_X|$. It is a K3 surface by [1], 1.5 and [if?](#) $r \geq 2$, a del Pezzo surface by 2.4. In any case, $H^2(S, \mathbb{Z})$ is torsion-free. Further by Lefschetz hyperplane theorem, $H^2(X, \mathbb{Z}) \hookrightarrow H^2(S, \mathbb{Z})$. Thus by (i), $\text{Pic } X$ is also torsion-free. □

Given a Fano 3-fold X , we label it by a pair of numbers $\rho - N$ where ρ is the Picard rank of X and N is the number in the classification found in [6]. A most recent classification table for Fano 3-folds along with some of their associated invariants and information about their birational geometry, zero section description(due to [Fatighenti](#)) class can be found on [9].

3. FANO THREEFOLDS WITH $\rho = 1$

Let X be a smooth Fano 3-fold of index r .

3.1. Fano 3-folds with $r \geq 2$. The following theorem by Iskovskikh([1]) completely classifies Fano 3-folds with index $r \geq 2$.

Theorem 3.1. *Let X be a Fano 3-fold of index $r \geq 2$. Then the following assertions hold:*

- (i) 1-16, 1-17: *If $r \geq 3$, then $\phi_{|H|} : X \xrightarrow{\sim} \mathbb{P}^3$ is an isomorphism for $r = 4$, and $\phi_{|H|} : X \xrightarrow{\sim} X_2 \subset \mathbb{P}^4$ is an isomorphism of X with a smooth quadric of \mathbb{P}^4 for $r = 3$.*
- (ii) *If $r = 2$, then a variety X only exists for $1 \leq d \leq 7$; for $d \geq 3$, $\phi_{|H|} : X \xrightarrow{\sim} X_d \subset \mathbb{P}^{d+1}$ is an embedding of X as a subvariety X_d of degree d in \mathbb{P}^{d+1} , with X_d projectively normal; and if $d \geq 4$, then X_d is the intersection of the quadrics containing it. Conversely, for any $d \geq 3$, every smooth projectively normal 3-fold $X_d \subset \mathbb{P}^{d+1}$ not lying in any hyperplane is a Fano 3-fold, and has index $r = 2$, apart from the case $r = 4, d = 8$, when X_8 is the image of \mathbb{P}^3 in \mathbb{P}^9 under the Veronese embedding.*
- (iii) *If $r = 2$ and $3 \leq d \leq 7$, then; for $d = 7$, X_7 is the projection of the Veronese 3-fold $X_8 \subset \mathbb{P}^9$ from some point of X_8 ;
for $d = 6$, $X_6 \simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ in its Segre embedding;
1-15: for $d = 5$, $X_5 \subset \mathbb{P}^6$ is unique up to projective equivalence, and can be realized in either of the following two ways:
(a) as the birational image of a quadric $W \subset \mathbb{P}^4$ under the map defined by the linear system $|\mathcal{O}_W(2) - Y|$ of quadrics passing through a twisted cubic Y ;
(b) as the section of the Grassmannian $Gr(2, 5)$ of lines in \mathbb{P}^4 by 3 hyperplanes in general position;
1-14: for $d = 4$, X_4 is any smooth intersection of two quadrics in \mathbb{P}^5 ;
1-13: for $d = 3$, X_3 is any smooth cubic hypersurface of \mathbb{P}^4 .*
- (iv) *If $r = 2$ and $d = 1$ or 2 , then:
1-12: for $d = 2$, $\phi_{|H|} : X \rightarrow \mathbb{P}^3$ is a double covering with smooth ramification surface $D_4 \subset \mathbb{P}^3$ of degree 4, and any such variety is a Fano 3-fold with $r = 2$ and $d = 2$; and every Fano 3-fold with $r = 2$ and $d = 2$ can be realized as a smooth hypersurface of degree 4 in $\mathbb{P}(1, 1, 1, 1, 2)$;
1-11: for $d = 1$, $\phi_{|H|} : X \rightarrow \mathbb{P}^2$ is a rational map with a single point of indeterminacy, and with irreducible elliptic fibres; and X can be realized in either of the following two ways:
(a) $\phi_{|-K_X|} : X \rightarrow W_4$ is any double cover of the cone W_4 over the Veronese surface $F_4 \subset \mathbb{P}^5$, having smooth ramification divisor $D \subset W_4$ by a cubic hypersurface not passing through the vertex of the cone;
(b) any smooth hypersurface of degree 6 in $\mathbb{P}(1, 1, 1, 2, 3)$.*

Note that in the theorem, for $r = 2, d = 6$, Picard group is $\text{Pic } X \simeq \mathbb{Z}^{\oplus 3}$, so in this case X is a Fano 3-fold with $\rho(X) = 3(3-27)$. For $r = 2, d = 7$ is 2-35? **Probably not?**

3.2. Hyperelliptic Fano 3-folds with $r = 1$. For a Fano 3-fold of index r , Theorem 2.2 implies that the fundamental linear system $|H|$ is without fixed components and base points and so by [1], 2.2, it follows that $\deg \phi_{|H|} = 1$ or 2 . Here we study the case when $\deg \phi_{|H|} = 2$.

Definition 3.2. A Fano 3-fold X of index $r = 1$ is *hyperelliptic* if its anticanonical map $\phi_{|-K_X|}$ is a morphism and is of degree $\deg \phi_{|-K_X|} = 2$.

Theorem 3.3. *Let X be a hyperelliptic Fano variety, and let $\phi_{|-K_X|} : X \rightarrow Y \subset \mathbb{P}^{g+1}$ be the corresponding morphism of degree 2. Then Y is nonsingular and X is uniquely determined by the pair (Y, D) , where $D \subset Y$ is the ramification divisor of $\phi_{|-K_X|}$. For $\rho(X) = 1$, the pair (Y, D) belongs to one of the following families (and if D is a smooth divisor, then for each pair (Y, D) there exists a Fano 3-fold X):*

- (i) 1-1: $Y \simeq \mathbb{P}^3$, and D is a smooth hypersurface of degree 6; in this case X can be realized alternatively as a smooth hypersurface of degree 6 in the weighted projective space $\mathbb{P}(1, 1, 1, 1, 3)$.
- (ii) 1-2 b): $Y \simeq X_2$ is a smooth quadric in \mathbb{P}^4 and $D \in |\mathcal{O}_{X_2}(4)|$; that is, $D = X_2 \cap X_4$, where X_4 is a quartic in \mathbb{P}^4 . In this case X can also be realized as a smooth complete intersection in the weighted projective space $\mathbb{P}(1, 1, 1, 1, 1, 2)$: X is the intersection of a quadric cone and a hypersurface of degree 4:

$$\begin{aligned} F_2(x_0, \dots, x_4) &= 0 \\ F_4(x_0, \dots, x_5) &= 0. \end{aligned}$$

For the proof, see [1], 7.3-7.6.

3.3. Fano 3-folds with $r = 1$. Let X be a Fano 3-fold with index $r = 1$.

Definition 3.4. A smooth complete irreducible 3-fold X over \mathbb{C} is called a *Fano 3-fold of the principal series* if the anticanonical divisor $-K_X$ is very ample.

From [Isk77,78- cite the exact result or conclude](#), it follows that all Fano 3-folds are of the principal series with the exceptions of hyperelliptic Fano 3-folds, 1-11, and [cannot identify Fano 3-fold in 3.1-b](#)). [Also how are these not very ample??](#) From this point, we will consider our Fano 3-folds to be of principal series and write $X_{2g-2} \subset \mathbb{P}^{g+1}$ for a Fano 3-fold of the principal series in its anticanonical embedding.

Šokurov showed the existence of line on such Fano 3-folds with Picard rank 1 in [4]. Under this assumption and using the method of double projection from a line, Iskovskikh showed that there exist Fano 3-folds with $\rho = 1$ and $r = 1$ for genus $g \leq 10$ and $g = 12$ but not for $g = 11$.

For genus $g = 3, 4$, and 5 , Fano 3-folds with $r = 1$, we have

Proposition 3.5. *A Fano 3-fold $X_{2g-2} \subset \mathbb{P}^{g+1}$ is a complete intersection only for $g = 3, 4$ or 5 , and we have that*

1-2 a): $X_4 \subset \mathbb{P}^4$ is a quartic hypersurface,

1-3 : $X_6 = V_{2,3}$ is an intersection of a quadric and a cubic in \mathbb{P}^5 ,

1-4 : $X_8 = V_{2,2,2}$ is an intersection of 3 quadrics in \mathbb{P}^6 .

Conversely, each smooth complete intersection of the types indicated is a Fano 3-fold the principal series.

See [2], 1.3 for a proof. [To do some computations for genus g, index r and \$\rho\$ and that will complete \$r = \rho = 1\$ discussion for these genera, I believe. Necessary?](#)

Example 3.6. ($g = 6$) Let us denote by V a section of Plücker embedding $\text{Gr}(2, 5) \subset \mathbb{P}^9$ by two hyperplanes in general position and a quadric. By adjunction formula, this is a Fano variety of index 1. [genus, Picard rank computations important? I think I should maybe delete this example here because later in 3.4, Gushel's theorem gives a description of it anyway. my mainreason to include it here to show that Iskovskikh gave genus g=6 Fano 3-fold too..](#)

Iskovskikh asserts that it is possible to show that every Fano 3-fold $X_{10} \subset \mathbb{P}^7$ with Picard rank 1 is a section of Grassmannian $\text{Gr}(2, 5)$. See [2], 1.5.

The following result is the main classification theorem by Iskovskikh for Fano 3-folds with genus $g \geq 7$.

Theorem 3.7. *Let $X = X_{2g-2} \subset \mathbb{P}^{g+1}$ be a Fano 3-fold with $\rho = 1$ and $g \geq 7$. Let $\pi_{2Z} : X \rightarrow W \subset \mathbb{P}^{g-6}$ be the double projection from a sufficiently general line $Z \subset X$. Let E denote the hyperplane section of W . Then the following assertions hold:*

- (i) $g \leq 12$.
- (ii) If $g = 12$, then $W = W_5 \subset \mathbb{P}^6$ is a prime Fano 3-fold with index 2 and degree 5 (with possibly one singular point); the map $\rho_Y : W \rightarrow X$ inverse to π_{2Z} is given by the linear system $|3E - 2Y|$, with $Y \subset W$ a normal rational curve of degree 5 in \mathbb{P}^5 .
- (iii) There do not exist any prime Fano 3-folds with $g = 11$.
- (iv) If $g = 10$, then $W = W_2 \subset \mathbb{P}^4$ is a quadric and $\rho_Y : W \rightarrow X$ is given by the linear system $|5E - 2Y|$, where Y is a smooth curve of genus 2 and degree 7 in \mathbb{P}^4 .
- (v) If $g = 9$, then $W = \mathbb{P}^3$ and $\rho_Y : \mathbb{P}^3 \rightarrow X$ is given by the linear system $|7E - 2Y|$, where Y is a smooth curve of genus 3 and degree 7.
- (vi) If $g = 8$, then $\pi_{2Z} : X \rightarrow \mathbb{P}^2$ is a rational map with fibres (after resolving the determinacy) curves of genus 2, and such that the inverse images of lines of \mathbb{P}^2 are rational surfaces.
- (vii) If $g = 7$, then $\pi_{2Z} : X \rightarrow \mathbb{P}^1$ is a rational map whose general fiber (after resolving the indeterminacy) is a del Pezzo surface of degree 5 with 8 points blown up; X is a rational 3-fold, and the projection from a line maps it into a complete intersection of 3 quadrics of \mathbb{P}^6 containing a smooth rational ruled surface $R_3 \subset \mathbb{P}^4$.

The proof of (i) and the construction of Fano 3-folds with given genus in (ii)-(vii) uses the birational technique of projection and double projection from lines on X . See [2], §6 for details. For an alternative proof of boundedness of genus for Fano 3-folds using vector bundles, see [8].

3.4. Mukai and Gushel's description of Fano 3-folds with $g \geq 6$. The double projection method does not work... **To complete** To classify Fano n -folds with canonical curve section (very ample) of genus $g \geq 6$, Mukai first showed the existence of a good vector bundle \mathcal{E} on X . Then using the linear system $|\mathcal{E}|$, we can embed X into a Grassmannian variety and describe its image. We find that X is a linear section of some homogeneous space. We describe these in the following examples. For Fano n -folds, very ample $-K_X$ with $\rho = 1$ and genus $g \geq 6$, the dimension n cannot be arbitrarily large. In fact, the maximum dimension in these cases $n(g)$ is attained by a variety $X_{2g-2}^{n(g)}$ as below:

- Example 3.8.** (i) $g = 7$. Let V be a 9-dimensional vector space with F as a non degenerate symmetric bilinear form on V and S be the space of spinors of F . Here $n(g) = 10$. Denote by $X_{12}^{10} \subset \text{Gr}(4, 9)$ the set of all 4-dimensional subspaces W of V with $F(W, W) = 0$. Then X_{12}^{10} is a smooth 10-dimensional subvariety of Grassmannian $\text{Gr}(4, 9)$ and can be embedded in \mathbb{P}^{15} by the spinor coordinates.
- (ii) $g = 9$. Let V be a 6-dimensional vector space and F be a non degenerate skew-symmetric bilinear form on V . Here $n(g) = 6$. Let us denote by X_{16}^6 the set of all 3-dimensional subspaces W of V such that $F(W, W) = 0$. Then $X_{16}^6 \subset \text{Gr}(3, 6)$ is a homogeneous space isomorphic to $U(3)/O(3)$ as varieties, hence is a smooth 6-dimensional subvariety of degree 16 in $\text{Gr}(3, 6) \subset \mathbb{P}^{19}$.
- (iii) $g = 10$. Let V be a 7-dimensional vector space and F be a non degenerate skew-symmetric 4-linear form on V . Here $n(g) = 5$. Denote by X_{18}^5 the set of all 5-dimensional subspaces W of V such that $F(W, W, W, W) = 0$. Then $X_{18}^5 \subset \text{Gr}(5, 7)$ is a homogeneous space, hence a smooth 5-dimensional subvariety of degree 18 in $\text{Gr}(5, 7) \subset \mathbb{P}^{20}$.
- (iv) $g = 12$. Let V be a 7-dimensional vector space and F_1, F_2 , and F_3 be three linearly independent skew-symmetric bilinear forms on V . Denote by X the set of 3-dimensional subspaces W of V with $F_1(W, W) = F_2(W, W) = F_3(W, W) = 0$. If the subspace $F_1 \wedge V^\vee + F_2 \wedge V^\vee + F_3 \wedge V^\vee$ of $\wedge^3 V^\vee$ contains no vectors of the form $f_1 \wedge f_2 \wedge f_3 \neq 0$ for $f_1, f_2, f_3 \in V^\vee$ then X is a smooth 3-dimensional subvariety of degree 22, denoted by X_{22}^3 . Here $n(g) = 3$.

Theorem 3.9. Let X be a Fano n -manifold of dimension $n \geq 3$ with $\rho = 1$, genus $g \geq 6$ over $k \subset \mathbb{C}$. Then there exists a k -vector space V and a space M of multilinear forms on V such that X is isomorphic to a linear section of $\Sigma_g(V, M) \subset \mathbb{P}_k^{g+n(g)-2}$.

In particular, for $n = 3$ and $7 \leq g \leq 10$, a Fano 3-fold $X_{2g-2} \subset \mathbb{P}^{g+1}$ is obtained as a complete intersection of the homogeneous space $\Sigma_{2g-2}^{n(g)}$ and a linear subspace of codimension $n(g) - 3$ in $\mathbb{P}(V) = \mathbb{P}^{g+n(g)-2}$. And by Lefschetz theorem, it follows that such $X = X_{2g-2}$ has $\text{Pic}(X) \simeq \mathbb{Z}(-K_X)$. For Fano 3-folds of genus 6 and 8 over \mathbb{C} , the theorem was proved independently by Gushel. For $g = 6$, we have

Theorem 3.10. *Let $X = X_{10} \subset \mathbb{P}^7$ be an anticanonically embedded Fano threefold of index 1 and genus 6 with $\rho(X) = 1$. Then X is one of the following:*

- (i) *a section of the Grassmannian $\text{Gr}(2, 5)$ embedded by Plücker embedding into $\mathbb{P}(\wedge^2 \mathbb{C}^5)$ by a subspace of codimension 2 and a quadric,*
- (ii) *the section by a quadric of a cone $W = W_5 \subset \mathbb{P}^7$ over a nonsingular del Pezzo threefold $V = V_5 \subset \mathbb{P}^6$ of degree 5.*

Threefolds of type (i) and (ii) are not isomorphic.

See [10] for details on the proof. A similar method can be applied to study Fano threefolds of genus 8, see [11] and [12].

Theorem 3.11. *Let $X = X_{14} \subset \mathbb{P}^9$ be an anticanonically embedded Fano threefold of index 1 and genus 8 with $\rho(X) = 1$. Then X is a section of the Grassmannian $\text{Gr}(2, 6)$ embedded by Plücker embedding into $\mathbb{P}(\wedge^2 \mathbb{C}^6) = \mathbb{P}^{14}$ by a subspace of codimension 5.*

As a consequence, we see that the Fano 3-folds with Picard rank $\rho = 1$ appear as one of the following:

- (i) Sections of Grassmannians(1-5 a), 1-6, 1-7, 1-8, 1-9, 1-10, 1-15)
- (ii) Complete intersections in projective or weighted projective space (1-2 a), 1-3, 1-4, 1-11, 1-12, 1-13, 1-14, 1-16, 1-17, 1-11, 1-12)
- (iii) Hyperelliptic (1-1, 1-2 b))

to decide about 1-5b) after Gushel

4. FANO THREEFOLDS WITH $\rho \geq 2$

outline from mm(1981). Some details from others yet to add because what I have now is just not sufficient.

In this section, we consider Fano 3-folds with $B_2 \geq 2$. The main result is

Theorem 4.1. *There are exactly 88 types of Fano 3-folds with $B_2 \geq 2$ up to deformations.*

We begin with some definitions.

Definition 4.2. A Fano 3-fold is imprimitive if it is isomorphic to the blow-up of a Fano 3-fold along a smooth irreducible curve. A Fano 3-fold is primitive if it is not imprimitive.

Definition 4.3. A smooth variety over a smooth surface S is a conic bundle if every geometric fibre of $X \rightarrow S$ is isomorphic to a conic, i.e., a scheme of zeroes of a non zero homogeneous form of degree 2 on \mathbb{P}^2 .

The following theorem gives a complete classification for primitive Fano 3-folds.

Theorem 4.4. *Let X be a primitive Fano 3-fold. Then we have*

- (1) $B_2 \leq 3$,
- (2) if $B_2 = 2$, then X is a conic bundle over \mathbb{P}^2 , and
- (3) if $B_2 = 3$, then X is a conic bundle over $\mathbb{P}^1 \times \mathbb{P}^1$ and has either a divisor $D \simeq \mathbb{P}^1 \times \mathbb{P}^1$ such that $\mathcal{O}_D(D) \simeq \mathcal{O}(-1, -1)$ or another conic bundle structure over $\mathbb{P}^1 \times \mathbb{P}^1$.

The following proposition is important for classifying imprimitive Fano 3-folds.

Proposition 4.5. *On a Fano 3-fold X with $B_2 = 2$, there are two smooth rational curves C_1 and C_2 and two numerically effective divisors H_1 and H_2 such that $(C_i \cdot H_j) = \delta_{ij}$ for all $i, j = 1, 2$.*

It turns out that imprimitive Fano 3-folds can be obtained from successive curve-blow-ups of primitive Fano 3-folds by using their conic bundle structure or the existence of lines on Fano 3-folds with $B_2 = 2$. In the latter case, there can be several possibilities for each of the extremal rays and each possibility leads to an imprimitive Fano 3-fold with $B_2 = 2$. Combining all this information we get Table 2 in [refer tables](#).

Since the blowing-up of a Fano 3-fold raises B_2 by 1(refer?), we obtain $B_2 \geq 3$ imprimitive Fano 3-folds by the blowing-up of a Fano 3-fold Y along a smooth irreducible curve C . The following Propositions give strong necessary conditions on $C \subset Y$.

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