COMPREHENSIVE EXAM

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Contents

1. Personal Background

Basically Copy.

2. Introduction

Notations, definition(index, ρ, degree, etc., extremal rays and stuff?), basic examples, goal(s), other preliminaries about invariants, definitions, etc., potential research?

2.1. Definition and Examples.

Throughout we work over the field of complex numbers $\mathbb{C}$. A nonsingular projective variety X is called a *Fano Variety* if the anticanonical divisor $-K_X$ is ample. For any Cartier divisor D on a variety X, $\mathcal{O}_X(D)$ will denote the corresponding invertible sheaf, and, in particular, $\mathcal{O}_X(-K_X)$ is the canonical sheaf on X where $-K_X$ is the canonical divisor of X.
Q.2 Complete linear system, $h^i(X)$, Proposition 1.3, build upto Prop 1.6 & as a consequence genus, and the following para- NS(X) =Pic(X) has no torsion, $\rho = b_2$, fundamental divisor H, index. Then Def 1.13 for degree.
For any positive integer n, $\mathbb{P}^n$ is a n-dimensional Fano variety since its anticanonical sheaf is $\mathcal{O}(n+1)$ which is ample by Hart. In fact $\mathbb{P}^1$ is the only 1-dimensional Fano variety. Fano varities of dimension 2 are called *del Pezzo surfaces* and their classification is given in AG V. Clearly, $\mathbb{P}^3$ is a 3-dimensional Fano variety. + 1 more example of hypersurfaces.
Fano varieites of dim 3 with $\rho = 1$ are called *prime* Fano 3-folds and they are completely classified by Iskovskikh in Isko-1,2. Fano 3-folds with $\rho \geq 2$ were all classified by mori and mukai in refer with timeline?.
In this paper(?), we aim to understand the classification of Fano 3-folds.

3. Fano Threefolds with $\rho = 1$

V_d notation, etc that will be used in next section
For the classification here, we follow Isko-1,2.

4. Fano Threefolds with $\rho \geq 2$

outline from mm(1981). Some details from others yet to add because what I have now is just not sufficient.
In this section, we consider Fano 3-folds with $B_2 \geq 3$. The main result is

Theorem 4.1. *There are exactly* 88 *types of Fano* 3*-folds with* $B_2 \geq 2$ *up to deformations.*

We begin with some definitions.

Definition 4.2. A Fano 3-fold is imprimitive if it is isomorphic to the blow-up of a Fano 3-fold along a smooth irreducible curve. A Fano 3-fold is primitive if it is not imprimitive.

Definition 4.3. A smooth variety over a smooth surface S is a conic bundle if every geometric fibre of $X \to S$ is isomorphic to a conic, i.e., a scheme of zeroes of a non zero homogeneous form of degree 2 on $\mathbb{P}^2$.

The following theorem gives a complete classification for primitive Fano 3-folds.

Theorem 4.4. *Let X be a primitive Fano* 3*-fold. Then we have*

(1) $B_2 \leq 3$,
(2) *if* $B_2 = 2$, *then X is a conic bundle over* $\mathbb{P}^2$, *and*
(3) *if* $B_2 = 3$, *then X is a conic bundle over* $\mathbb{P}^1 \times \mathbb{P}^1$ *and has either a divisor* $D \simeq \mathbb{P}^1 \times \mathbb{P}^1$ *such that* $\mathcal{O}_D(D) \simeq \mathcal{O}(-1,-1)$ *or another conic bundle structure over* $\mathbb{P}^1 \times \mathbb{P}^1$.

The following proposition gives a classification for Fano 3-folds with $B_2 = 2$.

Proposition 4.5. *On a Fano* 3*-fold X with* $B_2 = 2$, *there are two smooth rational curves* C_1 *and* C_2 *and two numerically effective divisors* H_1 *and* H_2 *such that* $(C_i \cdot H_j) = \delta_{ij}$ *for all* $i, j = 1, 2$.

Theorem and Proposition give a complete classification for Fano 3-folds which are either primitive or have $B_2 = 2$, including the imprimitive ones. This leaves us to consider the imprimitive Fano 3-folds with $B_2 \geq 3$. It turns out they can be obtained from successive curve-blow-ups of primitive Fano 3-folds by using their conic bundle structure or the existence of lines on Fano 3-folds with $B_2 = 2$. In the latter case, there can be several possibilities for each of the extremal rays and each possibility leads to an imprimitive Fano 3-fold with $B_2 = 2$. Combining all this information we get Table 2 in refer tables.
Since the blowing-up of a Fano 3-fold raises B_2 by 1(refer?), we obtain $B_2 \geq 3$ imprimitive Fano 3-folds by the blowing-up of a Fano 3-fold Y along a smooth irreducible curve C. The following Propositions give strong necessary conditions on $C \subset Y$.

5. References