Model specification

We will repress the location notation just to simply the further expressions. In other words, let $\boldsymbol{x}_k(\boldsymbol{s}) = \boldsymbol{x}_k$, $k \in \{m, g, p\}$ where m relates to the data from the SDM, g from genetic data and p from pollen data.

Level 1 - Data collection specific models

$$\begin{aligned} x_m | x^{(m)} &= \beta_0 + \beta_1 x^{(m)} + \epsilon_m \quad & \epsilon_m \sim N(0, \sigma_m^2) \\ x_g | x^{(g)} &= \alpha_0 + \alpha_1 x^{(g)} + \epsilon_g \quad & \epsilon_g \sim N(0, \sigma_g^2) \\ x_p | x^{(p)} &= \gamma_0 + \gamma_1 x^{(p)} + \epsilon_p \quad & \epsilon_p \sim N(0, \sigma_p^2) \end{aligned}$$

where $x^{(k)}$ indicates the values of $\boldsymbol{x}$, the latent spatial process, that relate to model k, $k \in \{m, g, p\}$. Each of these models has a sample size n_k, $k \in \{m, g, p\}$. The sample sizes might be different and they might not add up to n, the length of $\boldsymbol{x}$, in the case where one location appears in more than one data collection model.
Since the $\boldsymbol{x}^{(k)}$ are essentially subsets of $\boldsymbol{x}$ we can write them in terms of $\boldsymbol{x}$ and a matrix that chooses the values of $\boldsymbol{x}$ that appear in $\boldsymbol{x}^{(k)}$. Let these be the following:

$$\boldsymbol{x}^{(m)} = \boldsymbol{M}\boldsymbol{x} \qquad \boldsymbol{x}^{(g)} = \boldsymbol{G}\boldsymbol{x} \qquad \boldsymbol{x}^{(p)} = \boldsymbol{P}\boldsymbol{x}$$

Note that $\boldsymbol{K}\boldsymbol{K}^T = \boldsymbol{I}_{n_k}$ but $\boldsymbol{K}^T\boldsymbol{K} \neq \boldsymbol{I}_n$, for $\boldsymbol{K} \in \{\boldsymbol{M}, \boldsymbol{G}, \boldsymbol{P}\}$ and its corresponding $k \in \{m, g, p\}$.

Level 2 - Latent spatial process

$$\boldsymbol{x} \sim GP(\mu, \boldsymbol{\Sigma})$$

where μ is a common mean across al locations and $\boldsymbol{\Sigma}_{i,j} = \sigma^2 e^{-\frac{h_{i,j}}{\phi}}$ is the exponential covariance function where $h_{i,j} = ||\boldsymbol{s}_i - \boldsymbol{s}_j||$ is the distance between the i-th and j-th sites. Note that we can also write the covariance matrix as $\boldsymbol{\Sigma} = \sigma^2 \boldsymbol{V}$, where $\boldsymbol{V}$ is a correlation matrix.

Level 3 - Priors

$$\beta_0, \alpha_0, \gamma_0 \sim N(0, \tau)$$

$$\beta_1, \alpha_1, \gamma_1 \sim N(1, \tau)$$

$$\sigma_m^2, \sigma_g^2, \sigma_p^2, \sigma^2 \sim IG(a, b)$$

$$\mu \sim N(0, \omega)$$

$$\phi \sim Unif(A, B)$$

Posterior distributions

Conjugate

The following are the known conditional posterior distributions so that we might use a Gibbs sampler. For notation, *e.e.* will stand for "everything else". Also, $\bar{x}_k = \frac{1}{n_k}\mathbf{1}_{n_k}^T \boldsymbol{x}_k$ stands for the average of $\boldsymbol{x}_k$, whereas $\bar{x}^{(k)} = \frac{1}{n_k}\mathbf{1}_{n_k}^T \boldsymbol{x}^{(k)}$ stand for the average of the latent process $\boldsymbol{x}$ associated to model k, $k \in \{m, g, p\}$.

$$\beta_0 | e.e. \sim N\left(\frac{n_m}{\sigma_m^2 \rho_m}(\bar{x}_m - \beta_1 \bar{x}^{(m)}), \frac{1}{\rho_m}\right)$$

$$\alpha_0 | e.e. \sim N\left(\frac{n_g}{\sigma_g^2 \rho_g}(\bar{x}_g - \alpha_1 \bar{x}^{(g)}), \frac{1}{\rho_g}\right)$$

$$\gamma_0 | e.e. \sim N\left(\frac{n_p}{\sigma_p^2 \rho_p}(\bar{x}_p - \gamma_1 \bar{x}^{(p)}), \frac{1}{\rho_p}\right)$$

where $\rho_k = \frac{n_k}{\sigma_k^2} + \frac{1}{\tau}$, $k \in \{m, g, p\}$.

$$\beta_1 | e.e. \sim N\left(\frac{\frac{n_m}{\sigma_m^2}(c_m - \beta_0 \bar{x}^{(m)}) + \frac{1}{\tau}}{\eta_m}, \frac{1}{\eta_m}\right)$$

$$\alpha_1 | e.e. \sim N\left(\frac{\frac{n_g}{\sigma_g^2}(c_g - \alpha_0 \bar{x}^{(g)}) + \frac{1}{\tau}}{\eta_g}, \frac{1}{\eta_g}\right)$$

$$\gamma_1 | e.e. \sim N\left(\frac{\frac{n_p}{\sigma_p^2}(c_p - \gamma_0 \bar{x}^{(p)}) + \frac{1}{\tau}}{\eta_p}, \frac{1}{\eta_p}\right)$$

where $c_k = \frac{1}{n_k}\boldsymbol{x}_k^T \boldsymbol{x}^{(k)}$ is the average of the cross-products and $\eta_k = \frac{n_k s_k}{\sigma_k^2} + \frac{1}{\tau}$, where $s_k = \frac{1}{n_k}\sum_{i=1}^{n_k}\left[x_i^{(k)}\right]^2 = \frac{1}{n_k}\boldsymbol{x}^T \boldsymbol{K}^T \boldsymbol{K}\boldsymbol{x}$ is the mean squares of the latent process associated to model k, $\boldsymbol{K} \in \{\boldsymbol{M}, \boldsymbol{G}, \boldsymbol{P}\}$ and its corresponding $k \in \{m, g, p\}$.

$$\sigma_m^2 | e.e. \sim IG\left(\frac{n_m}{2} + a, b + \frac{1}{2}\sum_{i=1}^{n_m}(x_{m,i} - \beta_0 - \beta_1 x_i^{(m)})^2\right)$$

$$\sigma_g^2 | e.e. \sim IG\left(\frac{n_g}{2} + a, b + \frac{1}{2}\sum_{i=1}^{n_g}(x_{g,i} - \alpha_0 - \alpha_1 x_i^{(g)})^2\right)$$

$$\sigma_p^2 | e.e. \sim IG\left(\frac{n_p}{2} + a, b + \frac{1}{2}\sum_{i=1}^{n_p}(x_{p,i} - \gamma_0 - \gamma_1 x_i^{(p)})^2\right)$$

Finally, for the latent process:

$$\mu | e.e. \sim N\left(\frac{\mathbf{1}_n^T \boldsymbol{\Sigma}^{-1}\boldsymbol{x}}{\mathbf{1}_n^T \boldsymbol{\Sigma}^{-1}\mathbf{1}_n + \frac{1}{\omega}}, \frac{1}{\mathbf{1}_n^T \boldsymbol{\Sigma}^{-1}\mathbf{1}_n + \frac{1}{\omega}}\right)$$

$$\sigma^2 | e.e. \sim IG\left(\frac{n}{2} + a, \frac{1}{2}(\boldsymbol{x} - \mu\mathbf{1}_n)^T \boldsymbol{V}^{-1}(\boldsymbol{x} - \mu\mathbf{1}_n) + b\right)$$

$$\boldsymbol{x}|e.e. \sim N_n\left(\tilde{\boldsymbol{\Sigma}}\tilde{\boldsymbol{\mu}}, \tilde{\boldsymbol{\Sigma}}\right)$$

where,

$$\tilde{\boldsymbol{\mu}} = \mu\boldsymbol{\Sigma}^{-1}\mathbf{1}_n + \frac{\beta_1}{\sigma_m^2}\boldsymbol{M}^T(\boldsymbol{x}_m - \beta_0\mathbf{1}_{n_m}) + \frac{\alpha_1}{\sigma_g^2}\boldsymbol{G}^T(\boldsymbol{x}_g - \alpha_0\mathbf{1}_{n_g}) + \frac{\gamma_1}{\sigma_p^2}\boldsymbol{P}^T(\boldsymbol{x}_p - \gamma_0\mathbf{1}_{n_p})$$

$$\tilde{\boldsymbol{\Sigma}} = \left[\boldsymbol{\Sigma}^{-1} + \frac{\beta_1^2}{\sigma_m^2}\boldsymbol{M}^T\boldsymbol{M} + \frac{\alpha_1^2}{\sigma_g^2}\boldsymbol{G}^T\boldsymbol{G} + \frac{\gamma_1^2}{\sigma_p^2}\boldsymbol{P}^T\boldsymbol{P}\right]^{-1}$$

Non-Conjugate

ϕ has no know posterior distribution, however we can state what the density is proportional to so that we can use Metropolis-Hastings. Note that $\boldsymbol{V}$ is a function of ϕ so for this last part we will use the notation $\boldsymbol{V}(\phi)$ to indicate this.

$$\pi(\phi|e.e.) \propto |\boldsymbol{V}(\phi)|^{-1/2} \cdot \exp\left\{\frac{1}{2\sigma^2}(\boldsymbol{x} - \mu\mathbf{1}_n)^T\boldsymbol{V}(\phi)^{-1}(\boldsymbol{x} - \mu\mathbf{1}_n)\right\} \cdot 1_{\{A<\phi<B\}}$$

where $|\boldsymbol{V}(\phi)|$ is the determinant of the matrix.