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Laboratory Exercise Report – 5

Objective

The objective of this exercise is to understand the workings of Extended Kalman Filter (EKF) and implement EKF to a simple 2D navigation problem.

Methodology

Kalman filter working

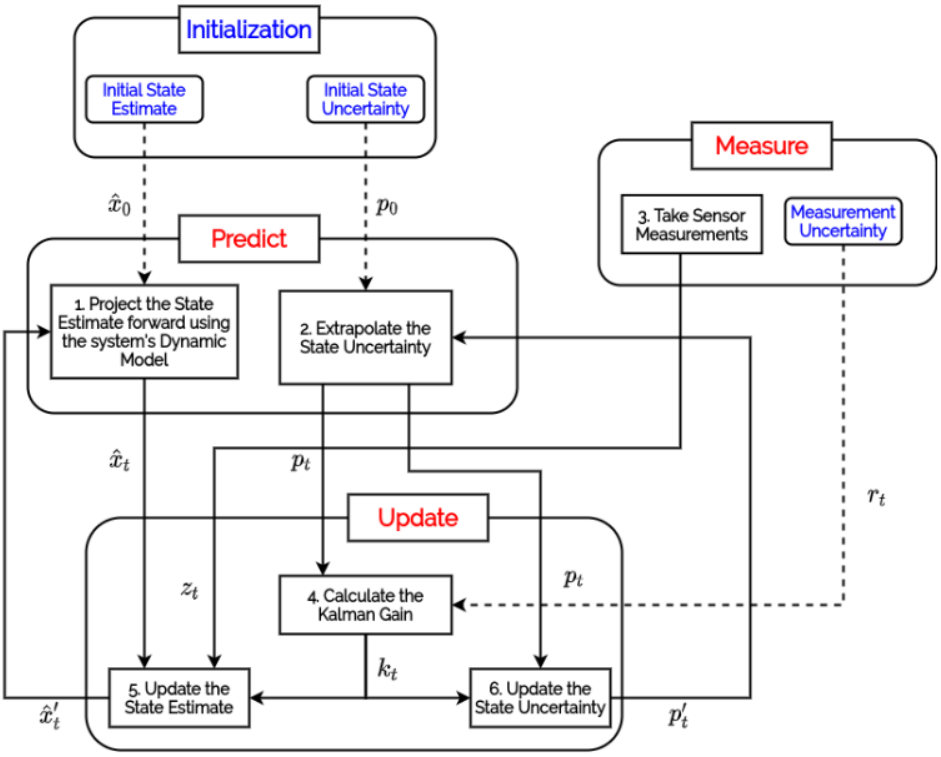

Figure 1 Flowchart for Kalman filter working (source:https://teyvonia.com/kalman-filter-1d-localization-python/))

Methodology to solve the problem

Problem Statement:

The problem involves implementing an Extended Kalman Filter (EKF) to estimate the trajectory of the moving car in a 2D plane using distance observations from four fixed control points. Initially, establish a system model describing the car's motion and a measurement model relating observed distances to the car's position. At each time step, predict the next state using the dynamic model, incorporating control inputs if available, and update the state estimate using the received distance measurements. This process iterates sequentially, continuously refining the estimated trajectory of the car by combining prediction and correction steps within the framework of the EKF.

1. All assumptions made

- Process noise is uncorrelated
- Noise is additive in nature
- Noise is Gaussian
- Noise is white noise, i.e. $E(\acute{\varepsilon}k\ \acute{\varepsilon}k\text{-}1^{T}) = 0$
- Velocities are changing slowly with time

2. State space model

$$Xk = \begin{bmatrix} x \\ y \\ vx \\ vy \end{bmatrix}$$

x: x-position

y: y-position

vx: velocity along x

vy: velocity along y

$$x_k = x_{k-1} + vx_{k-1}\Delta t + \varepsilon d$$

$$y_k = y_{k-1} + vy_{k-1}\Delta t + \varepsilon v$$

$$vx_k = vx_{k-1} + \varepsilon a$$

$$vy_k = vy_{k-1} + \varepsilon b$$

These are the state transition models

$$X_{k|k-1} = F_k \,.\, X_{k-1|k-1} + E_{X\ k-1}$$

Where,

$$F_k = \begin{bmatrix} 1 & 0 & dt & 0 \\ 0 & 1 & 0 & dt \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3. Measurement model

$$Zk = \begin{bmatrix} d1 \\ d2 \\ d3 \\ d4 \end{bmatrix}$$

Where ,

$$d1 = \sqrt{(x - x1)^2 + (y - y1)^2} + v1$$

$$d2 = \sqrt{(x - x2)^2 + (y - y2)^2} + v2$$

$$d3 = \sqrt{(x - x3)^2 + (y - y3)^2} + v3$$

$$d4 = \sqrt{(x - x4)^2 + (y - y4)^2} + v4$$

$$\overline{Zk} = H_k\, X\, k\text{-}1|k\text{-}1 + f(X0) - H_k X0$$

$$H_k == \frac{\partial f}{\partial x}|_{x=x0}$$

4. Linearization steps:

The measurement model mentioned is non-linear, it is linearized by Taylor series.

Linearized mathematical model: for the above distance equations

$$f\ (x_i\,,y_i\,) = \ f\ (x_i\,,y_i\,)_0 + [\frac{df}{dxi}]_0 \Delta x_i + [\frac{df}{dyi}]_0 \Delta y_i + [\frac{df}{dvxi}]_0 \Delta vx_i + [\frac{df}{dvyi}]_0 \Delta vy_i$$

5. Choosing Co-variances

$$P_{k-1|\,k-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Q_{\,k-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_{\,k} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

In case I, the covariance in predicted state, measurement and the process noise is assumed to be as identity.

6. Step by step process of state estimation using KF

Initialize: state vector Matrix

$$Xo = \begin{bmatrix} -9.764 \\ -9.764 \\ 0 \\ 0 \end{bmatrix}$$

The distance values in the state are initialize by solving first set of observation by least squares. The velocity is assumed to be slowly changing, and initially it is assumed to be zero.

Initialize: Weight Matrix

$$P_o = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Prediction: Perform prediction by using previously adjusted state vector xk −1 to predict the state vector and covariance matrix:

$$X_{k|k-1} = F_k . X_{k-1|k-1} + E_{X\,k-1}$$

$$P_{k|k-1} = F_k . P_{k-1|k-1} . F_k^T + Q_{k-1}$$

Kalman Gain:

$$K_k = P_{k|k-1} + K_k . (Z_k - H_k - X_{k|k-1} - E_{Z\,k-1})$$

Updating: The Sate vector and covariance matrix is updated on basis of the input observations in measurement model.

$$X_{k|k} = X_{k-1|k-1} + K_k . (\overline{Zk} - Z_k)$$

$$P_{k|k-1} = (I - K_k . H_k) . P_{k|k-1}$$

Results

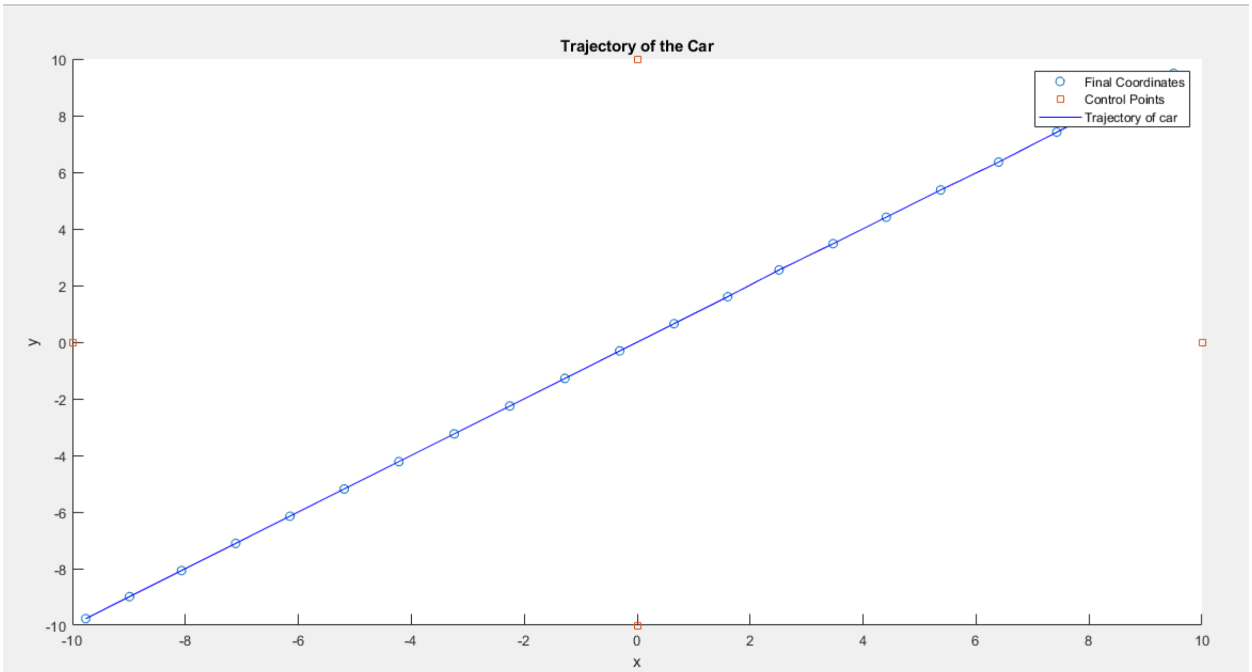

Figure 2 Trajectory of car

The trajectory of the car is represented by a series of points in a 2D plane. Each point corresponds to a specific position of the car at a given time. The coordinates of these points are provided in the table below:

Points	x	y	Points	x	y
1	-9.7646	-9.7646	12	0.6525	0.6525
2	-8.9894	-8.9894	13	1.6024	1.6024
3	-8.0686	-8.0686	14	2.5120	2.5120
4	-7.1089	-7.1089	15	3.4691	3.4691
5	-6.1490	-6.1490	16	4.4086	4.4086
6	-5.1867	-5.1867	17	5.3724	5.3724
7	-4.2196	-4.2196	18	6.3988	6.3988
8	-3.2410	-3.2410	19	7.4278	7.4278
9	-2.2569	-2.2569	20	8.4637	8.4637
10	-1.2808	-1.2808	21	9.4964	9.4964
11	-0.3109	-0.3109			

Discussion

1. How did you choose the initial state and covariance?

The initial state vector (Xk_1) serves as the starting point for estimating the system's state. In the provided instance, it was set to [-9.764; -9.764; 0; 0], assuming initial velocity is zero and the first set of distance is solved by Least squares.

The initial estimate for the trajectory are x0, y0 can be taken as any value but the value chosen will affect the number of iterations the loop will take to converge, in order to find the optimum initial estimates, solve the observation equation simultaneously.

$$9.72 = \sqrt{(x-(-10))^2 + (y-0)^2}$$

$$9.72 = \sqrt{(x-0)^2 + (y-(-10))^2}$$

Solving the above equations, the value of x and y is found to be: (-9.764, -9.764)

Regarding the initial covariance matrix (Pk_1), it reflects the uncertainty associated with each element of the state vector. The example employs the identity matrix, implying uniform uncertainty across all state variables.

2. How will you go about choosing the process noise covariance and measurement noise covariance?

Process Noise:

- Process noise represents the uncertainty in the system dynamics that are not accounted for by the state transition model (F).
- Q determines how much the actual state of the system can deviate from the predicted state due to unmodeled dynamics or disturbances.
- Choosing Q involves understanding the dynamics of the system and the sources of uncertainty.

- If the system dynamics are well-understood and modeled accurately, Q can be relatively small.
- If there are uncertainties or disturbances in the system dynamics, Q should be appropriately tuned to capture these uncertainties.
- Q can be estimated empirically from data or through system identification techniques.

  In this case Q is taken as diag(0.001,0.001,0.001,0.001)

Measurement noise Cavariance:

- Measurement noise represents the uncertainty in the sensor measurements or observations.
- R determines how much weight the Kalman filter assigns to the sensor measurements versus the predicted state.
- Choosing R involves understanding the characteristics of the sensors, such as accuracy, precision, and noise characteristics.
- If the sensors are highly accurate and reliable, R can be small.
- If the sensors are noisy or have varying levels of accuracy, R should be adjusted accordingly to account for this uncertainty.
- R can be estimated through sensor characterization experiments or by analyzing historical measurement data.

Here, it is considered as identity.

3. Generate the results for different values of process noise and measurement noise covariances. Do you observe any differences? If yes, try to explain the reasons for these differences.

Case I: Ideal case

$$Q_{k-1} = \begin{bmatrix} 0.001 & 0 & 0 & 0 \\ 0 & 0.001 & 0 & 0 \\ 0 & 0 & 0.001 & 0 \\ 0 & 0 & 0 & 0.001 \end{bmatrix} R_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

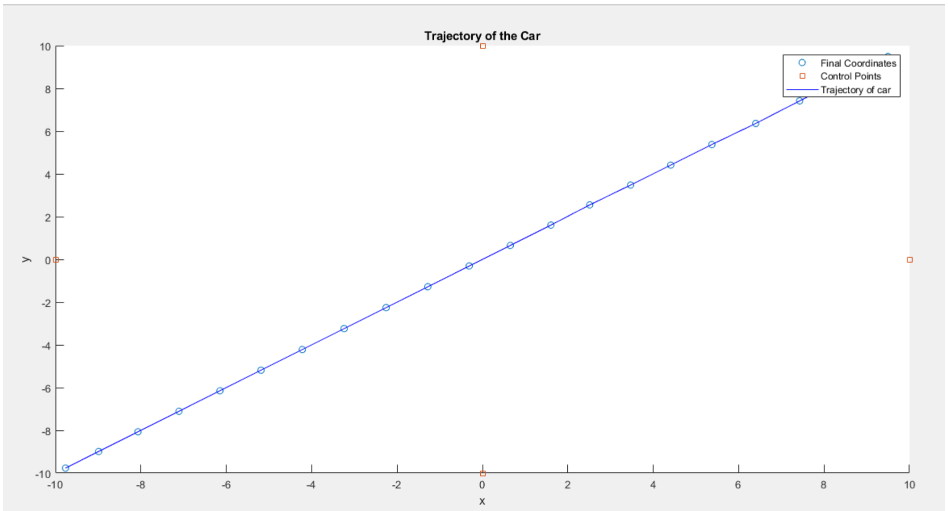

Figure 3 trajectory of car

Figure 4 Graphs plotted for accelerometer data, with Q, R as identity

$$Q_{k-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.01 & 0 \\ 0 & 0 & 0 & 0.01 \end{bmatrix} R_k = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

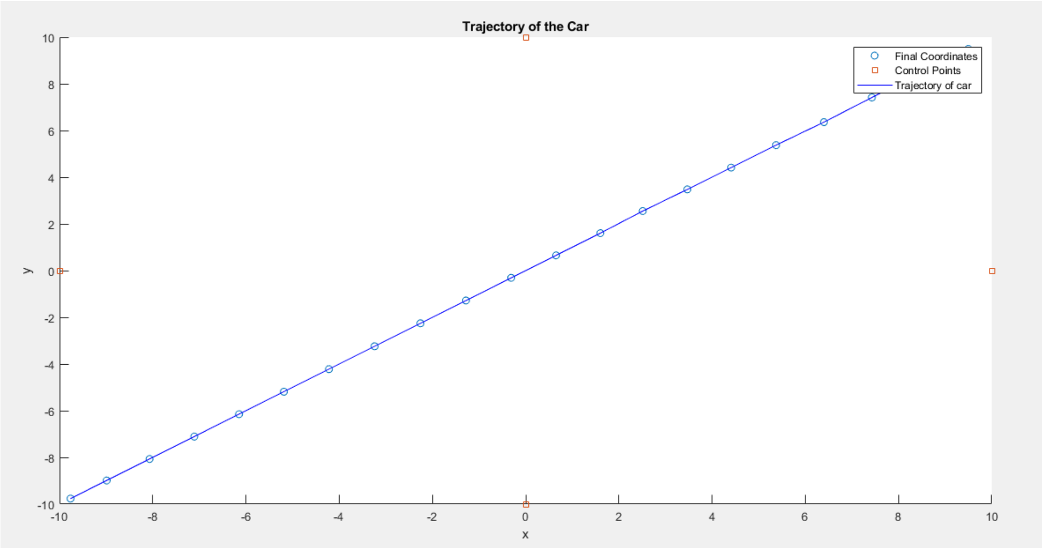

Figure 5 Trajectory of car

As the Q is increased for the values of velocity, that is the confidence on the values of the velocity is reduced, opposing the assumption that velocity changes slowly, the values obtained for the trajectory changes, moreover when R values are increased, the confidence on the measurement values if decreased, the filter will give more weight to predicted value than the measured value. The values of the trajectory obtained are different and thus the trajectory changes.

The coordinates obtained after applying the changes are:

Points	x	y	Points	x	y
1	-9.7646	-9.7646	12	0.4179	0.4179
2	-9.0926	-9.0926	13	1.3669	1.3669
3	-8.2380	-8.2380	14	2.3054	2.3054
4	-7.3204	-7.3204	15	3.2247	3.2247
5	-6.3910	-6.3910	16	4.1428	4.1428
6	-5.4545	-5.4545	17	5.0892	5.0892
7	-4.5048	-4.5048	18	6.0703	6.0703
8	-3.5286	-3.5286	19	7.1379	7.1379
9	-2.5305	-2.5305	20	8.2119	8.2119
10	-1.5342	-1.5342	21	9.2732	9.2732
11	-0.5507	-0.5507			

4. Plot a graph of trace of posterior estimation error covariance vs. time. What can you understand from this graph?

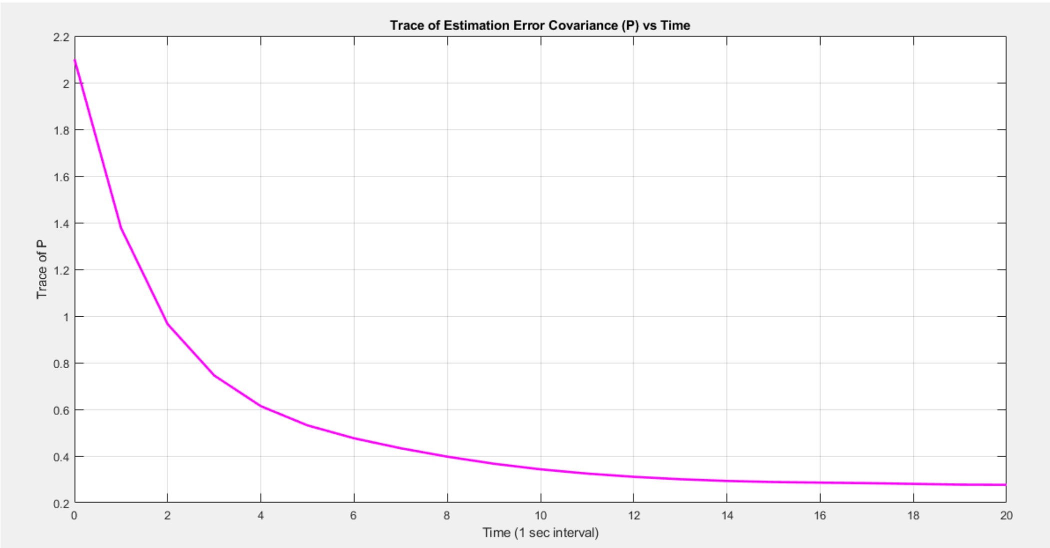

Figure 6 Trace of P vs Time

This trace value is indicative of the overall uncertainty in the estimated state of the system over time. In this case, the trace values decrease steadily from approximately 2.10 to 0.28 across iterations. A decreasing trace signifies that the Extended Kalman Filter is becoming progressively more confident in its estimation of the car's trajectory as more information is incorporated from measurements and model predictions. Lower trace values indicate reduced uncertainty in the estimated state, suggesting improved accuracy in tracking the car's movement over time.

5. Did you need to linearize either the state transition or measurement model? Why/why not?

Yes, in this scenario, the state transition model (represented by the matrix *Fk*) is linear, as it describes the evolution of the state variables over time in a predictable manner. However, the measurement model (represented by the matrix *Hk*) is nonlinear because it relates the state variables to the observed distances, which involve trigonometric functions or other nonlinear transformations due to the 2D geometry. Consequently, linearization is not required for the state transition model since it's already linear, but it is necessary for the measurement model to make it compatible with the Extended Kalman Filter framework. The Extended Kalman Filter can effectively handle such nonlinearities by linearizing the measurement model around the current state estimate, enabling estimation of the car's trajectory despite the nonlinear relationship between the state and measurements.

6. Learnings from the lab

This laboratory exercise offers valuable insights into the practical application of Extended Kalman Filters (EKF) for trajectory estimation in a 2D scenario. By varying process noise and measurement noise, the experiment shows the impact of uncertainty on the estimated trajectory. Through this lab, the importance of tuning filter parameters to strike a balance between incorporating noisy measurements and maintaining stability in trajectory estimation is understood. Additionally, understanding of how the EKF copes with nonlinear measurement models is understood. The lab also gives insight on the previous exercise done for sequential and least square adjustment on the same model. The results obtained by all the method are fairly identical but kalman filter is computationally less heavy as compared to previous two methods.

Conclusions

In this laboratory experiment, the effects of adjusting the process noise covariance matrix (Q) and measurement noise covariance matrix (R) in an Extended Kalman Filter (EKF) were explored. By reducing the values in Q and increasing the values in R, the filter's confidence in the accuracy of the state transition model was enhanced while diminishing its confidence in the accuracy of sensor measurements. Consequently, the Extended Kalman Filter tended to rely more on the predicted state from the model and less on the sensor measurements for state estimation. These adjustments influenced the filter's responsiveness to sudden changes and noise in the sensor measurements, potentially resulting in smoother but less sensitive state estimates. Overall, the experiment highlighted the critical role of Q and R in balancing the trade-off between modeling accuracy and robustness to uncertainties in Extended Kalman Filter applications.

References

[1] P. D. Groves, “Principles of GNSS, Inertial, and Multi-sensor Integrated Navigation Systems”.

[2] “W3 recommended draft for accelerometer”.