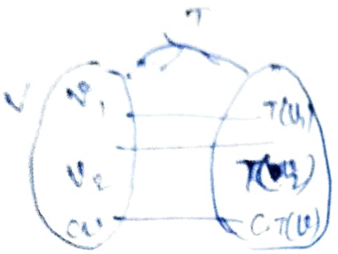
Linear Transformation:

let V & W are two vector spaces over a same field F.

A function $T: V \to W$ is called a linear transformation if it satisfies the following:-

(i) $T(v_1 + v_2) = T(v_1) + T(v_2) \quad \forall\, v_1, v_2 \in V$

(ii) $T(c.v) = c.T(v) \quad \forall\, c \in F,\ v \in V$



If $c \in F$
$x \in V$
$\Rightarrow c.v \in V$

Example

(1) $T: \mathbb{R}^2 \to \mathbb{R}^2$, s.t. $T(x,y) = (x,0)$

(2) $T: \mathbb{R}^2 \to \mathbb{R}^2$ s.t $T(x,y) = (x+y, 8x-y)$.

let $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$

To prove

(i) $T[(x_1,y_1) + (x_2,y_2)] = T(x_1,y_1) + T(x_2,y_2)$

(ii) $T[c(x,y)] = c.T(x,y)$

By definition $T(x_1,y_1) = (x_1, 0)$

$T(x_2,y_2) = (x_2, 0)$

LHS $T[(x_1,y_1) + (x_2,y_2)] = T[(x_1+x_2, y_1+y_2)]$

$= T(x_1+x_2, y_1+y_2)$

$= (x_1+x_2, 0)$

RHS $T(x_1,y_1) + T(x_2,y_2) = (x_1, 0) + (x_2, 0) = (x_1+x_2, 0)$

LHS = RHS (i) ✓

$T[c(x,y)] : T(cx, cy) = (cx, 0)$ LHS

$c \cdot T(x,y) = c \cdot (x, 0) = (cx, 0)$ RHS

LHS = RHS
$\hookrightarrow$ (ii) ✓

(2) $T : \mathbb{R}^2 \to \mathbb{R}^2$ s. t $T(x,y) = (x+y, x-y)$.

Let $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$

$T(x_1, y_1) = (x_1+y_1, x_1-y_1)$
$T(x_2, y_2) = (x_2+y_2, x_2-y_2)$

$T[(x_1, y_1) + (x_2, y_2)] = T[x_1+x_2, y_1+y_2]$

$= [x_1+x_2+y_1+y_2, x_1+x_2-y_1-y_2]$

$T(x_1, y_1) + T(x_2, y_2) = (x_1+y_1, x_1-y_1) + (x_2+y_2, x_2-y_2)$

$= (x_1+x_2+y_1+y_2, x_1+x_2-y_1-y_2)$

LHS = RHS $\to$ (i) ✓

$T[c(x,y)] = T(cx, cy) = (cx+cy, cx-cy)$

$c \cdot T(x,y) = c(x+y, x-y) = (cx+cy, cx-cy)$

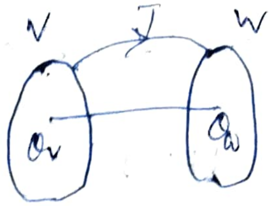
LHS = RHS
$\hookrightarrow$ (ii) ✓

$\therefore$ $T : V \to W$ is a linear transformation.

If $T: V \to W$ be a linear transformation, then show that

(i) $T(\theta_V) = \theta_W$

(ii) $T(-v) = -T(v)$



$T(cv) = c \cdot T(v) \; \forall \; c \in F, \, v \in V$

If $c = -1$ then $T(cv) = T(-v)$

$= c \cdot T(v)$

$= (-1) \cdot T(v)$

$= -T(v)$

$T(v_1 + v_2) = T(v_1) + T(v_2)$

In particular if $v_1 = v_2 = \theta_V$.

$\therefore T(\theta_V + \theta_V) = T(\theta_V) + T(\theta_V)$

$\theta_V \to$ zero vector

$\Rightarrow T(\theta_V) = T(\theta_V) + T(\theta_V)$

$\Rightarrow T(\theta_V) - T(\theta_V) = T(\theta_V)$

$\Rightarrow T(\theta_V) = \theta_W$.

If $T: \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $T(x, y) = (0, y) \; \forall \; (x, y) \in \mathbb{R}^2$ then find the set $= \{ v : T(v) = \theta_W \}$

$\Rightarrow$ S is non empty because $\theta_V \in S$

$T(x, y) = \theta_W = (0, 0)$

$\Rightarrow (0, y) = (0, 0)$

$y = 0$.

$S = \{ (x, 0) : x \in \mathbb{R} \} = x$-axis.

For any linear transformation $T: V \to W$ the set $\{ v \in V \mid T(v) = \theta_W \}$ is called the kernel of the transformation i.e. $\ker(T) = \{v \in V \mid T(v) = 0_W\}$

P.T.

\# $\ker(T)$ ~~forms a~~ forms a subspace of V.

$\Rightarrow$ let $v_1, v_2 \in \ker(T)$ and $c \in F$.

~~$v_1 + v_2$~~

we have to show that $v_1 + v_2 \in \ker(T)$ &

$c \cdot v \in \ker(T)$.

$T(v_1) = 0_W$

$T(v_2) = 0_W$

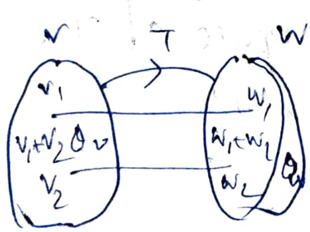
$$
\begin{aligned}
T(v_1 + v_2) &= T(v_1) + T(v_2) \\
&= 0_W + 0_W \\
&= 0_W
\end{aligned}
$$

$$
\begin{aligned}
\& \quad T(c \cdot v) &= c \cdot T(v) \\
&= c \cdot 0_W \\
&= 0_W
\end{aligned}
$$

$v_1 + v_2 \in \ker(T)$ & $c v \in \ker(T)$

$\Rightarrow \ker(T)$ is a subspace of $\mathbb{R}^2$.

Range of a L.T.



$T(V)$ or $R(T)$, the range of T forms a subspace of W.

$T(0_W) = 0_W \in R(T)$

Now let $w_1, w_2 \in R(T)$

$\Rightarrow w_1 = T(v_1)$ for some $v_1 \in V$

$w_2 = T(v_2)$ for some $v_2 \in V$

Now $w_1 + w_2 = T(v_1) + T(v_2) = T(v_1 + v_2)$

$\Rightarrow w_1 + w_2 \in R(T)$ ——— (1)

Let $c \in F$ & $w \in R(T)$

$\therefore w = T(v)$ for some $v \in V$.

$\therefore cw = c.T(v)$

$= T(cv)$

$cw \in R(T)$. ——(2)

By 1 & 2 it is proved that $R(T)$ is a subspace of W.

subspace = subset + vector space.

~~S is R(T~~

Since $R(T)$ is a subspace

$\Rightarrow R(T)$ is a vector space

$\Rightarrow R(T)$ has basis

$\Rightarrow R(T)$ has Dimension

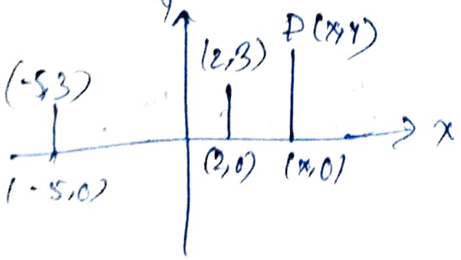
$\therefore$ The dimension of $R(T)$ = Rank of the transformation

<u>Example:</u> Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ s.t

$T(x,y) = (x,0)$ → projection of x-axis.

Find the range and null space of T.

What is Rank & Nullity of T?



Null space

↓

$x : T(x) = 0$

$\therefore T(x,y) = (0,0)$

$\Rightarrow (x,0) = (0,0)$

or $x = 0$.

Null space $= \{(0,y) : y \in \mathbb{R}\}$

$=$ y axis

$\therefore$ Range $= \{T(x,y)\}$

$= \{(x,0) : x \in \mathbb{R}\}$

A basis of the null space $= \{(0,1)\}$

$\therefore$ Dimension $= 1 =$ Nullity.

A basis of the range $= \{(1,0)\}$

$\therefore$ Dimension $= 1 =$ Rank

Example :- (1) $T: \mathbb{R}^3 \to \mathbb{R}^2$ s.t

$T(x,y,z) = (x-y, 2z)$

Find the range & Null space of T.

What is rank and ~~what is~~ Nullity?

$\Rightarrow$

Null space

$T(x,y,z) = (0,0)$

or $(x-y, 2z) = (0,0)$

or $x-y=0 \qquad 2z=0$

$x=y \qquad z=0.$

Null space $= \{(x,x,0) : x \in \mathbb{R}\}$

A basis of null space $= \{(1,1,0)\}$

Nullity $= 1$.

$\therefore$ Range $= \{T(x,y,z)\}$

$= \{(x-y, 2z), \; x,y,z \in \mathbb{R}\}$.

Basis of range $= \{(1,0),(0,1)\} = \mathbb{R}^2$

Rank $= 2 =$ dimension of range of T

$= \dim(\mathbb{R}^2)$

$= 2$

Rank(T) + Nullity(T) $= 1+2 = 3 =$ Dimension of Domain.

* Rank - Nullity Theorem:

If $T: V \to W$ be a linear transformation and $\dim(V) =$ finite then.

Rank (T) + Nullity $(T) = \mathrm{Dim}(V)$

Let $\mathrm{Dim}(V) = \infty$

$$\left.\begin{array}{l} \text{Rank} + \text{Nullity}(T) = \infty \\ 1 + \infty = \infty \\ 2 + \infty = \infty \\ \vdots \end{array}\right\} \text{Not defined.}$$

Dim(V) is always finite.

set of all real-polynomials over $\mathbb{R}$.

$\downarrow$

Infinite Dimensional vector space.

$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n + a_{n+1} x^{n+1} \cdots$

Basis $= \{1, x, x^2, \ldots\}$

Dimension $= \infty$.

Check Rank - Nullity theorem for the given examples:-

(i) $T: \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $T(x,y) = (x-y, 0)$

$\Rightarrow$ Null space $= \{ T(x,y) = (0,0)$

or $(x-y, 0) = (0,0)$

or $x = y$.

$\therefore$ Null space $= (x, x) : x, \in \mathbb{R}$

Nullity $= 1$ Basis $= (1,1)$.

Range $= T(x,y) = (x+y, 0)$, $x, y \in \mathbb{R}$

Basis $= (1,0)$

Rank $= 1$

Rank (T) + Nullity (T) = $1 + 1 = 2 = \mathrm{Dim}(V)$

(ii) $T: \mathbb{R}^3 \to \mathbb{R}^2$ s.t $T(x,y,z) = (x+y, z)$

$\Rightarrow$ Null space $= T(x,y,z) = (0,0)$

$\Rightarrow (x+y, z) = (0,0)$

$\Rightarrow x+y = 0 \quad | \quad z = 0$

$\Rightarrow x = -y \quad | \quad$ or

Null space $= (x, -x, 0), \; x \in \mathbb{R}$

Basis $= (1,-1,0)$ Nullity $= 1$

Range $= T(x,y,z) = (x+y, z), \; x,y,z \in \mathbb{R}$

Basis $= \{(1,0), (0,1)\}$ Rank $= 2$

Nullity (T) + Rank $(T) = 1+2 = 3 = \operatorname{Dim}(V)$

(iii) $T: P_3(\mathbb{R}) \to P_2(\mathbb{R})$ s.t $T(a_0 + a_1x + a_2x^2 + a_3x^3) = a_1 + 2a_2x + 3a_3x^2$.

$\Rightarrow$ Null space $= T: P_3(\mathbb{R}) = (0)$

$a_1 + 2a_2x + 3a_3x^2 = 0$

or $a_1 + 2a_2x + 3a_3x^2 = 0 + 0.x + 0.x^2$

or $a_1 = 0, \; a_2 = 0 \; a_3 = 0$

Null space $= (a_0)$

Basis $= (1)$ Nullity $= 1$.

Range $= a_1 + 2a_2x + 2a_3x^2$

Basis $= (1, x, x^2)$, Rank $= 3$

Nullity + Rank $= 1+3 = \operatorname{Dim}(T)$

(i) Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a L.T. (linear transformation) such that,

$T(1,0) = (1,4)$

$T(1,1) = (2,5)$

What is $T(2,3)$?

(ii) Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a L.T s.t $T(1,1) = (1,0,2)$, $T(2,3) = (1,-1,4)$

What is $T(8,11)$?

$\Rightarrow$

$T: V \to W$

$T(v_1 + v_2) = T(v_1) + T(v_2)$

$T(\alpha v) = \alpha \cdot T(v)$

Let $(2,3) = \alpha(1,0) + \beta(1,1)$

$= (\alpha, 0) + (\beta, \beta)$

$= (\alpha+\beta, \beta)$

$\alpha+\beta = 2 \quad \beta = 3$

or $\alpha = -1$

$(2,3) = (-1)(1,0) + 3(1,1)$

$\therefore T(2,3) = T(-1(1,0) + 3(1,1))$

$= T(-1(1,0)) + T(3(1,1))$

$= -1T(1,0) + 3T(1,1)$

$= -1(1,4) + 3(2,5)$

$= (5,11)$

(11) Let $(8,11) = \alpha(1,1) + \beta(2,3)$

$= (\alpha+2\beta, \alpha+3\beta)$

$\alpha+2\beta = 8 \quad \alpha+3\beta = 11$

or $\alpha = 8-6$ $\quad$ or $\beta = 3$

or $\alpha = 2$

$(8,11) = 2(1,1) + 3(2,3)$

or $T(8,11) = T(2(1,1)) + T(3(2,3))$

$= 2T(1,1) + 3T(2,3)$

$= 2(1,0,2) + 3(1,-1,4)$

$= (5,-3,16)$

12. Is there a L.T. $T: \mathbb{R}^3 \to \mathbb{R}^2$ s.t $T(1,0,3) = (1,1)$ and $T(-2,0,-6) = (2,1)$

$(-2,0,-6) = -2(1,0,3)$

$T(-2,0,-6) = T(-2(1,0,3))$

$= -2\,T(1,0,3)$

$= -2(1,1)$

$= (-2,-2) \neq (2,1)$

So, there is no linear transformation.

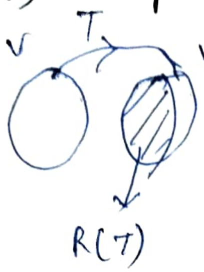
So, $\nexists$ such linear transformation.

Theorem

Let $T: V \to W$ be a L.T.

suppose, $\beta = \{v_1, v_2, \dots, v_n\}$ be a basis of V.

Then $R(T) = \text{span}\{T(v_1), T(v_2), \dots, T(v_n)\}$.



$\Rightarrow$ $(1, x, x^2, x^3)$

$T: P_3(\mathbb{R}) \to P_2(\mathbb{R})$

$\{1, x, x^2\}$

$T(f(x)) = f'(x) = \frac{d}{dx}(f(x))$

$T(1) = \frac{d}{dx}(1) = 0$

$T(x) = 1$

$T(x^2) = 2x$

$T(x^3) = 3x^2$

$\{0, 1, 2x, 3x^2\}$

$a_0 + a_1 x + a_2 x^2 \in P_2(\mathbb{R})$

$= 1 + \frac{1}{2}(2x) + \frac{1}{3}(3x^2)$

$\alpha_1 \cdot 1 + \alpha_2 \cdot 2x + \alpha_3 \cdot 3x^2 = 0 = 0 + 0.x + 0.x^2$

$\Rightarrow \alpha_1 = 0$ | $2\alpha_2 = 0$, or, $\alpha_2 = 0$ | $3\alpha_3 = 0$ or $\alpha_3 = 0$

9. Let $T: P_2(\mathbb{R}) \to M_2(\mathbb{R})$ be a L.T s.t —

$$T(f(x)) = \begin{pmatrix} f(1) - f(2) & 0 \\ 0 & f(0) \end{pmatrix}$$

$B = \{1, x, x^2\}$

$T(1) = \begin{pmatrix} 1-1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$T(x) = \begin{pmatrix} 1-2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \longrightarrow$ $f(x) = x$
$f(1) = 1$
$f(2) = 2$
$f(0) = 0$

$T(x^2) = \begin{pmatrix} 1-4 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix}$

$\hookrightarrow f(x) = x^2$
$f(1) = 1^2 = 1$
$f(2) = 2^2 = 4$
$f(0) = 0^2 = 0$

Range of T i.e.

$= R(T) = \text{span}\{T(1), T(x), T(x^2)\}$

$= \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix} \right\}$

$M_3 = 3(M_1)$

or $M_1 = \frac{1}{3} M_3$

Image of T i.e. $R(T)$ is actually spanned by –

$\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \right\}$ or $\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix} \right\}$

and they are linearly independent.

$\dim(R(T)) = 2$

They are basis of $R(T)$.

~~Rank of~~ $\text{Rank}(T) = 2$.

$T: V \rightarrow W$ be linear transformation & $\dim(V) =$ finite

then $\text{Rank}(T) + \text{Nullity}(T) = \text{Dim}(V)$

By Rank - Nullity theorem

$2 + \text{Nullity}(T) = 3$

$\Rightarrow \text{Nullity}(T) = 1$

$T : P_2(\mathbb{R}) \to M_2(\mathbb{F})$

$\downarrow$

Dim = 3.

Q. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a Linear transformation such that

$T(x, y) = (x - y, y)$

Find Rank (T) & Nullity (T) . .

$\Rightarrow$ Null space of $T = \{ v \in V : T(v) = 0 \}$

or $T(x, y) = (0, 0)$

or $(x - y, y) = (0, 0)$

$y = 0 \quad x - y = 0$

or $x = y = 0$

$\Rightarrow$ Null space of $T = \{(0, 0)\}$

Nullity $(T) = 0$

$V = \{0\}$

Basis $= \phi$

Dimension $(V) = 0$.

By Rank - Nullity Theorem

$\text{Rank}(T) + 0 = \dim(\mathbb{R}^2) = 2$

or $\text{Rank}(T) = 2$.

Theorem

Let $T : V \to W$ be a Linear Transformation.

T is 1-1 iff $N(T) = \{0\}$

$f : A \to B$

$O(A) = m$

$O(B) = n$

f is 1-1 when —

(i) $m \geq n$ (ii) $m = n$ (iii) $m \leq n$.

f is onto when —

$m \geq n$

f is both 1-1 and onto when —

$m = n$.

$m \geq n \to$ onto

$m \leq n \to$ 1-1

$m = n \to$ both 1-1 & onto.

↳ f is bijective.

Theorem (*) $T: V \to W$ be a L.T

Then T is bijective / Invertible

· if T is both 1-1 & onto.

Theorem $T: V \to W$ be a L.T then —

T is 1-1 iff $N(T) = \{0\}$.

Theorem If $T: V \to W$ be a L.T. s.t. $\dim V = \dim W$ and they are finite, then the following are equivalent :—

(1) T is 1-1 $\Downarrow$

$\Uparrow$ (2) T is onto $\Downarrow$

$\Uparrow$ (3) Rank $(T) = \operatorname{Dim} V$

Let T is 1-1 $\Rightarrow N(T) = \{0\}$

$\Rightarrow$ Nullity $(T) = 0$

~~By Rank (T) + Nullity (T) = n~~

By rank - Nullity theorem

$\text{Rank}(T) + \underbrace{\text{Nullity}(T)}_{0} = n$ (suppose)

$\text{Rank}(T) = n$

$\Rightarrow T$ is onto.

H.W

Let $T: P_2(\mathbb{R}) \to P_3(\mathbb{R})$ s.t.

$T(f(x)) = f'(x) + \int_0^x 3 \cdot f(t)\, dt$

Find Rank & Nullity of T.

Matrix Representation

Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ s.t. $T(a_1, a_2) = (a_1 + 3a_2, 0, 2a_1 - 4a_2)$

$\beta_1' = \{(1,0), (0,1)\}$ $\beta_1'' = \{(1,0,0), (0,1,0), (0,0,1)\}$

$T(1,0) = (1,0,2) = \alpha_1(1,0,0) + \alpha_2(0,1,0) + \alpha_3(0,0,1)$

$T(0,1) = (3,0,-4)$

$= \beta_1(1,0,0) + \beta_2(0,1,0) + \beta_3(0,0,1)$

$\alpha_1 = 1,\ \alpha_2 = 0,\ \alpha_3 = 2$ | $\beta_1 = 3,\ \beta_2 = 0,\ \beta_3 = -$

∴ The matrix associated to the linear transformation is

$$[T]_{\beta_1'}^{\beta_1''} = \begin{pmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{pmatrix}_{3\times 2}$$

$\Rightarrow T: V \to W$ s.t $\dim(V) = m$ & $\dim(W) = n$

∴ matrix ~~with~~ will be of order $n \times m$

$$[T]_{\beta_1'}^{\beta_1''} = [T]_{n\times m}.$$

If $\beta_1' = \{(2,0), (0,1)\}$, $\beta_1'' = \{(1,0,0), (0,-1,0), (0,$

$T(2,0) = (2,0,4) = \alpha_1(1,0,0) + \alpha_2(0,-1,0) + \alpha_3(0,0$

$T(0,1) = (3,0,-4) = \beta_1(1,0,0) + \beta_2(0,-1,0) + \beta_3(0,0$

$\alpha_1 = 2\ \ \alpha_2 = 0\ \ \alpha_3 = 2$

$\beta_1 = 3,\ \beta_2 = 0\ \ \beta_3 = -2$

$\therefore$ ~~The~~ $[T]_{B_1}^{B_2''} = \begin{pmatrix} 2 & 3 \\ 0 & 0 \\ 2 & -2 \end{pmatrix}_{3\times 2}$

Q. $T: P_3(\mathbb{R}) \to P_2(\mathbb{R})$ be a L.T s.t $T(f(x)) = f'(x)$
find the matrix wrt the standard basis of $P_2(\mathbb{R})$ and $P_3(\mathbb{R})$

$B_1' = \{1, x, x^2, x^3\}$ $\quad B_1'' = \{1, x, x^2\}$

$T(1) = \frac{d}{dx}(1) = 0 = 0.1 + 0.x + 0.x^2$

$T(x) = \frac{d}{dx}(x) = 1 = 1.1 + 0.x + 0.x^2$

$T(x^2) = 2x = 0.1 + 2.x + 0.x^2$

$T(x^3) = 3x^2 = 0.1 + 0.x + 3.x^2$

$[T]_{B_1'}^{B_1''} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}_{3\times 4}$ $\Rightarrow$ Rank $[T]_{B_1'}^{B_1''} = 3$

$\Rightarrow$ Rank of $T = 3$.

Rank of T + Nullity $= 4$ (Rank of $P_3(\mathbb{R})$)

as 3 + nullity $= 4$

or Nullity $= 1$.

Range of $T = R(T) = \text{span}\{T(1), T(x), T(x^2), T(x^3)\}$

Another method $\quad N(T) = \{p(x) \in P_3(\mathbb{R}) \mid T(p(x)) = 0\} = \{a_0 : a_0 \in \mathbb{R}\}$

$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$T(p(x)) = a_1 + 2a_2 x + 3a_3 x^2 = 0 = 0 + 0.x + 0.x^2$

$a_1 = 0 \quad a_2 = 0 \quad a_3 = 0.$

$N(T) = \{a_0 : a_0 \in \mathbb{R}\}$ $\quad N(T) = \{1\}$

Nullity $= 1$, $\therefore$ Rank $= 3$

Rank + Nullity $= \dim(P_3(\mathbb{R})) = 4$

* Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear trans for ...
$T(a_1, a_2, a_3) = (2a_2 + a_3, -a_1 + 4a_2 + 5a_3, a_1 + a_3)$
Find $[T]_\beta^\beta$ where β is the standard ordered basis of $\mathbb{R}^3$.

$\Rightarrow$ $\beta = \{(1,0,0), (0,1,0), (0,0,1)\}$

$T(1,0,0) = (0,-1,1) = \alpha_1(1,0,0) + \alpha_2(0,1,0) + \alpha_3(0,0,1)$
$T(0,1,0) = (2,4,0) = \beta_1(1,0,0) + \beta_2(0,1,0) + \beta_3(0,0,1)$
$T(0,0,1) = (1,5,1) = \gamma_1(1,0,0) + \gamma_2(0,1,0) + \gamma_3(0,0,1)$

$\alpha_1 = 0$, $\beta_1 = 2$, $\gamma_1 = 1$
$\alpha_2 = -1$, $\beta_2 = 4$, $\gamma_2 = 5$
$\alpha_3 = 1$, $\beta_3 = 0$, $\gamma_3 = 1$

$$[T]_\beta^\beta = \begin{pmatrix} 0 & 2 & 1 \\ -1 & 4 & 5 \\ 1 & 0 & 1 \end{pmatrix}_{3\times 3}$$

* Let $T: M_2(\mathbb{R}) \to P_2(\mathbb{R})$ s.t
$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+b) + (2d)x + bx^2$.
Compute $[T]_\beta^\gamma$, where β and γ are the standard bases of $M_2(\mathbb{R})$ and $P_3(\mathbb{R})$ ~~repres~~ respectively.

$\Rightarrow$ $\beta = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$

$\gamma = \{1, x, x^2\}$

$T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1 + 0.x + 0.x^2$

$$T\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = 1 + 0.x + 1.x^2$$

$$T\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} = 0 + 0.x + 0.x^2$$

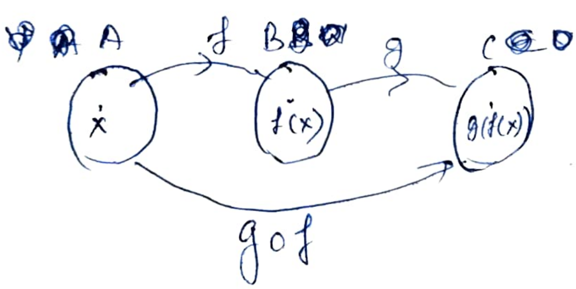
$$T\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} = 0 + 2x + 0.x^2$$

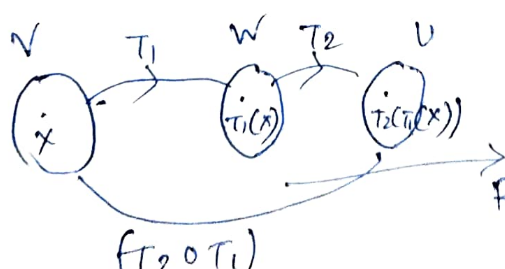
$$\therefore [T]_\beta^\gamma = \begin{pmatrix}1 & 1 & 0 & 0\\ 0 & 0 & 0 & 2\\ 0 & 1 & 0 & 0\end{pmatrix}_{3\times 4}$$

$\text{Rank}(T) = 3$

$$\therefore \text{Nullity} = \text{Dim}(M_2(\mathbb{R})) - 3$$
$$= 4 - 3 = 1.$$

Composition of Linear mapping / Transformation





Criteria

$\text{Range}(T_1) \subseteq \text{Domain}(T_2)$

$$(f\circ g)(x) = f(g(x))$$

$$(T_2\circ T_1)(v) = T_2(T_1(v))$$