MA 5102 - Algebraic Topology

Assignment

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Topological Properties of the Orthogonal Group

1 Notation

- $M_n(\mathbb{R}) \coloneqq$ set of $n \times n$ matrices with real entries
- $GL_n(\mathbb{R}) \coloneqq$ set of $n \times n$ invertible matrices with real entries
- $O(n) \coloneqq \{Q \in GL_n(\mathbb{R}) \mid QQ^t = \mathbb{I}_n\}$ is called the set of $n \times n$ real, orthogonal matrices
- $SO(n) \coloneqq \{Q \in O(n) \mid \det(Q) = 1\}$ is called the set of special orthogonal matrices

2 Preliminaries

2.1 The Orthogonal Matrices as a *Group*

Theorem 2.1. *$O(n)$ forms a group under the operation of matrix multiplication.*

Proof: Consider $Q, P \in O(n)$. Then,

$$QP(QP)^t = QPP^tQ^t = \mathbb{I}_n \quad \text{(closure under multiplication)},$$

$$Q\mathbb{I}_n = Q = \mathbb{I}_nQ \quad (\mathbb{I}_n \text{ is the identity element}),$$
$$QQ^t = \mathbb{I}_n \quad (Q^t \text{ forms the inverse}).$$

We notice that $O(n)$ forms a subgroup of $GL_n(\mathbb{R})$.
Remark. For all $Q \in O(n)$, we have that $\det(Q) = \pm 1$. *Proof*: $\det(\mathbb{I}_n) = 1 = \det(QQ^t) = \Big(\det(Q)\Big)^2 \implies \det(Q) = \pm 1$.

Theorem 2.2. *$SO(n)$ forms a normal subgroup of $O(n)$.*

Proof: Consider $Q \in O(n)$ and $P \in SO(n)$. Then, $QPQ^{-1} \in O(n)$ since $Q, P, Q^{-1} \in O(n)$ and $O(n)$ is closed under multiplication. Moreover,

$$\det(QPQ^{-1}) = \det(Q)\det(P)\det(Q^{-1}) = \det(P) = 1.$$

Thus, $QPQ^{-1} \in SO(n)\ \forall\ Q \in O(n) \implies SO(n) \lhd O(n) \preccurlyeq GL_n(\mathbb{R})$.

2.2 The Orthogonal Group with a *Topology*

We can use the Euclidean metric on $\mathbb{R}^{n^2}$ to assign a metric to $M_n(\mathbb{R})$.

Theorem 2.3. *Let $A, B \in M_n(\mathbb{R})$. Then,*

$$d : M_n(\mathbb{R}) \to M_n(\mathbb{R}); \quad d(A,B) \coloneqq \sqrt{\sum_{1 \le i,j \le n} \Big(A_{ij} - B_{ij}\Big)^2},$$

is a metric on $M_n(\mathbb{R})$.

Proof: We check the 3 properties a metric d must follow:
$d(A,B) = 0 \implies A_{ij} = B_{ij} \; \forall \; 1 \leq i,j \leq n \implies A = B$.
$d(A,B) = \sqrt{\sum_{1\leq i,j\leq n}\Big(A_{ij} - B_{ij}\Big)^2} = d(B,A)$.

$$d(A,B) = \sqrt{\sum_{1\leq i,j\leq n}\Big(A_{ij} - B_{ij}\Big)^2} = \sqrt{\sum_{1\leq i,j\leq n}\Big(A_{ij} - C_{ij} + C_{ij} B_{ij}\Big)^2}$$

$$\leq \sqrt{\sum_{1\leq i,j\leq n}\Big(A_{ij} - C_{ij}\Big)^2} + \sqrt{\sum_{1\leq i,j\leq n}\Big(C_{ij} - B_{ij}\Big)^2} = d(A,C) + d(C,B)$$

Hence d defined as above forms a metric on $M_n(\mathbb{R})$. It is important to note that we have induced the notion of a metric by viewing the entries of $M_n(\mathbb{R})$ in the Euclidean space $\mathbb{R}^{n^2}$ and using the standard Euclidean metric.

3 The Orthogonal Group as a Topological Group

3.1 Topological Group

Theorem 3.1. *The matrix group $O(n)$ is a topological group.*

Proof: To prove this, we must show that matrix multiplication and inverses are both continuous operations on $M_n(\mathbb{R})$. However, we note that matrix multiplication can be expressed as a polynomial in the entries. Since polynomials are continuous functions, we get that multiplication of matrices is element-wise continuous and thus continuous on the whole matrix. The inverse of a matrix $A \in M_n(\mathbb{R})$ is given by $A^{-1} := \frac{\text{adj}(A)}{\det(A)}$. Since the adjoint and determinant are both polynomials on the entries of A, they are continuous functions (and the determinant is non-zero for invertible matrices). Hence, we get that the inverse is also a continuous map.

3.2 Compactness of O(n)

Lemma 3.2. *$O(n)$ is a closed subset of $\mathbb{R}^{n^2}$.*

Proof: Consider $f : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ defined as $f(Q) = QQ^t$. Then $\Big(f(Q)\Big)_{ij} = \sum_{k=1}^{n} Q_{ik}Q_{jk}$ which is a polynomial in the entries of Q and polynomials are continuous functions. Since f is element wise continuous, we conclude f is a continuous map on $M_n(\mathbb{R})$. Now, $O(n) = f^{-1}(\{\mathbb{I}_n\})$ which is the inverse of a closed set. Hence, $O(n)$ is closed.

Lemma 3.3. *$O(n)$ is a closed subset of $\mathbb{R}^{n^2}$.*

Proof: $O(n) \subseteq \{A \in M_n(\mathbb{R}) \mid \sum_{i=1}^{n} A_{ij}^2 = 1\} \; \forall \; 1 \leq j \leq n$. Clearly this set is bounded by the open ball given by $d(A,0) = \sum_{1\leq i,j\leq n} A_{ij}^2 = n^2$ where 0 is the zero matrix with all entries 0.

Theorem 3.4. *$O(n)$ is compact.*

Proof: As an application of the Heine-Borel theorem, since $O(n)$ is a closed and bounded subset of Euclidean $\mathbb{R}^{n^2}$, we have that O(n) is compact.

3.3 Path-Connectedness

Theorem 3.5. *$O(n)$ is disconnected.*

Proof: Note that the function $\det : M_n(\mathbb{R}) \to \mathbb{R}$ is a continuous function. Now, $\det : O(n) \to \{-1, 1\} \subset \mathbb{R}$. Since continuous functions map connected sets to connected sets, we have that $O(n)$ is disconnected.

Theorem 3.6. *$SO(n)$ is path-connected.*

Proof: We begin by inspecting the simple case of $SO(2)$ and build our way upwards. Any generic matrix $P \in SO(2)$ is given by $P = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$. To show $SO(2)$ is path-connected, it is sufficient to show there exists a path between any arbitrary element in $SO(2)$ and the identity matrix. Consider the function $\gamma : [0, 1] \to SO(2)$ defined as

$$\gamma(t) = \begin{pmatrix} \cos(t\theta) & -\sin(t\theta) \\ \sin(t\theta) & \cos(t\theta) \end{pmatrix}.$$

Then observe that $\forall\, t \in [0, 1]$, $\gamma(t) \in SO(2)$ and $\gamma(0) = \mathbb{I}_2$ while $\gamma(1) = P$. Thus, we have shown $SO(2)$ is path-connected. Similarly, to show $SO(3)$ is path connected, consider $P \in SO(3)$. Since $SO(3) \lhd O(3) \implies \exists\, Q \in O(3)$ and $P'_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix} \in SO(3)$ such that,

$$P = QP'Q^{-1} = QP'_\theta Q^t.$$

Thus, consider the path $\gamma(t) = QP'_{t\theta}Q^t$. Then, $\gamma(0) = \mathbb{I}_3$ and $\gamma(1) = P$. Thus, we have shown $SO(3)$ is path connected. More generally, for any $P \in SO(n)$, there exists $Q \in O(n)$ and $P'_\theta \in SO(n)$ given by $P'_\theta = \begin{pmatrix} R_{\theta_1} & & & & \\ & R_{\theta_2} & & & \\ & & R_{\theta_3} & & \\ & & & \ddots & \\ & & & & R_{\theta_m} \end{pmatrix}$ such that $P = QP'Q^t$. Every R_{θ_k} is either a 2×2 rotation matrix or equal to $[1]$. Then the path connecting P to $\mathbb{I}_n$ is given by $\gamma(t) = QP'_{t\theta}Q^t$ where $\gamma(0) = \mathbb{I}_n$ and $\gamma(1) = P$. Thus we have shown that $SO(n)$ is path connected.

Theorem 3.7. *$O(n) \setminus SO(n)$ is path-connected.*

Proof: We will show there exists a path between any two elements $X, Y \in O(n) \setminus SO(n)$. Consider $Q \in O(n) \setminus SO(n)$. Then, $QX, QY \in SO(n)$ since $QX, QY \in O(n)$ $\big(O(n)$ is closed under multiplication$\big)$. Moreover $\det(QX) = \det(Q)\det(X) = -1 \times -1 = 1 = \det(QY) \implies QX, QY \in SO(n)$. Using the above result, we know $SO(n)$ is path-connected implying there exists a path connecting QX and QY called $\gamma(t)$ such that $\gamma(0) = QX$ and $\gamma(1) = QY$. Then the path $\phi(t) = Q^{-1}\gamma(t)$ is a well defined continuous function since matrix multiplication is continuous (according to Theorem 3.1). Clearly $\phi(0) = Q^{-1}\gamma(0) = Q^{-1}QX = X$ and similarly $\phi(1) = Y$. Thus we have shown there exists between any two elements $X, Y \in O(n) \setminus SO(n)$ which implies it is a connected set.
Remark: It follows that $O(n)$ has exactly two path components.

4 Gram-Schmidt as a Deformation Retract

Definition 4.1 (Gram-Schmidt orthogonalization)**.** The Gram-Schmidt orthogonalization (GS) of a matrix A of the form $A \coloneqq (v_1, v_2, \ldots, v_n)$ where $v_i \in \mathbb{R}^n$ are the column vectors of A is given by, $GS : GL_n(\mathbb{R}) \times [0, \frac{1}{2}] \to GL_n(\mathbb{R})$ is defined as,

$$GS(A,t) \coloneqq \Big((1-2t)v_1 + 2tu_1, \ldots, (1-2t)v_n + 2tu_n\Big),$$

where each u_i is defined as

$$\begin{aligned} u_1 &= v_1, \\ u_2 &= v_2 - \frac{\langle u_1, v_2\rangle}{\|u_1\|}u_1, \\ u_n &= v_n - \sum_{k=1}^{n-1} \frac{\langle u_k, v_n\rangle}{\|u_k\|}u_k. \end{aligned}$$

Since Gram-Schmidt orthogonalization generates an orthogonal set of n vectors from a linearly independent set of n vectors, the image of A is equal to image of $GS(A,t)$ implying $GS(A,t)$ attains full rank for all t. Hence for all $t \in [0, \frac{1}{2}]$, $GS(A,t) \in GL_n(\mathbb{R})$.

Definition 4.2 (Normalization map)**.** The normalization map for a matrix A as defined above, given by $N : GL_n(\mathbb{R}) \times [\frac{1}{2}, 1] \to GL_n(\mathbb{R})$ defined by:

$$N(A,t) \coloneqq \Big(v_1(\frac{2t-1}{\|v_1\|} + 2 - 2t), \ldots, v_n(\frac{2t-1}{\|v_n\|} + 2 - 2t)\Big).$$

The map N is designed such that $N(A, \frac{1}{2}) = A$ and $N(A,1) = \Big(\frac{v1}{\|v1\|}, \ldots, \frac{v_n}{\|v_n\|}\Big)$. We must check that for all $t \in [\frac{1}{2}, 1]$, $N(A,t) \in GL_n(\mathbb{R})$. Computing the determinant, given by

$$\det\Big(N(A,t)\Big) = \det(A)\prod_{i=1}^{n}(\frac{2t-1}{\|v_i\|} + 2 - 2t).$$

We must check that $\frac{2t-1}{\|v_i\|} + 2 - 2t \neq 0 \ \forall\ 1 \leq i \leq n$. Assume this is true for some i. Then,

$$\begin{aligned} \frac{2t-1}{\|v_i\|} + 2 - 2t = 0 \\ \implies 2t - 1 = 2t\|v_i\| - 2\|v_i\| \\ \implies t = \frac{1 - 2\|v_i\|}{2(1 - \|v_i\|)}. \end{aligned}$$

Plotting the function $y = \frac{1-2x}{2(1-x)}$on a graphing-calculator (Desmos) shows that it has no solutions for $y \in [\frac{1}{2}, 1]$. Hence, we have shown a contradiction which implies that $\det\Big(N(A,t)\Big) \neq 0 \implies N(A,t) \in GL_n(\mathbb{R})\ \forall\ t \in [\frac{1}{2}, 1]$.

Definition 4.3 (Gram-Schmidt orthonormalization)**.** Defining the Gram-Schmidt orthonormalization (GSO), given by $GSO : GL_n(\mathbb{R}) \times [0,1] \to GL_n(\mathbb{R})$ and defined by

$$GSO(A,t) = \begin{cases} GS(A,t), & \text{if } t \in [0, \frac{1}{2}) \\ N(A,t), & \text{if } t \in [\frac{1}{2}, 1] \end{cases}.$$

This is well-defined since $GS(A,t)$ and $N(A,t)$ agree at $t = \frac{1}{2}$.

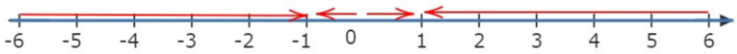

Figure 1: Deformation retraction of $GL_1(\mathbb{R})$ $\Big((-\infty,0)\cup(0,\infty)\Big)$ to $O(1)$ $\Big(\{-1,1\}\Big)$

Theorem 4.4. *The GSO as defined above is a strong deformation retraction of $GL_n(\mathbb{R})$ into $O(n)$.*

Proof: Observe that $GSO(A,0) = A \in GL_n(\mathbb{R})$. Next, $GSO(A,1)$ is a matrix with ortho-normal columns implying $GSO(A,1) \in O(n)$. Further, any element $A \in O(n)$ remains fixed since $\langle u_k, v_i\rangle = 0$ for all $1 \leq i \neq k \leq n$ and therefore, for all $t \in [0,\frac{1}{2}]$, we have $GS(A,t) = A$. For such a matrix,$N(A,t) = A$ for $t \in [\frac{1}{2},1]$ (As $\|v_i\| = 1$ for all $1 \leq i \leq n$). Thus, this is a strong deformation retraction. For the simple case of $n = 1$, this deformation retraction can be visualized as given in Figure 1.