Lab -4

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In this lab, we numerically analyze bifurcation in one-dimensional non-linear systems. Particularly, we concentrate on local bifurcations, which occur when stability of equilibrium point changes between stable and unstable.

I. INTRODUCTION

At times, we want to analyze the behaviour of the system when one of the parameters is altered by slight values. When we discover points where a slight change can change the system's phase space, that change is called Bifurcation. In formal terms, it is defined as the topological change of a system's phase space that occurs when some parameters are slightly altered with respect to the critical thresholds.

II. BIFURCATION

Bifurcations can also be implemented as real-life examples such as binary storage information in computer memory, excitations of neurons, transition of ecosystem states and many more.

Local bifurcations are defined by change in stability of equilibrium points. They can be easily analyzed by the concept of linear stability.
Global Bifurcations can't be explained just by local equilibrium points. Therefore, we'll not analyze this type of bifurcation in detail.

$$\frac{dx}{dt} = r - x^2$$

We know, equilibrium points of the above equation are:

$$r - x_{eq}^2 = 0 \Rightarrow x_{eq} = \pm\sqrt{r}$$

$$\frac{dF}{dx} = -2x, \text{ where } F(x) = r - x^2$$

Now,

$$\left(\frac{dF}{dx}\right)_{x=x_{eq}} = 0 \Rightarrow \pm 2\sqrt{r} = 0 \Rightarrow r = 0$$

Case-1: $x_{eq} = +\sqrt{r}$

$$\frac{dF}{dx} = -2x_{eq} < 0$$

Therefore, it is a stable equilibrium point.
Case-2: $x_{eq} = -\sqrt{r}$

$$\frac{dF}{dx} = -2x_{eq} > 0$$

Therefore, it is an unstable equilibrium point.
The above results are true only if $r > 0$. It is quite obvious that, for $r < 0$, equilibrium points does not exist.

III. MODEL

A. First Order Autonomous System in one variable

$$\text{Let } \frac{dx}{dt} = r - x^2$$

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$$\frac{dF}{dx} = -2x, \text{ where } F(x) = r - x^2$$

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$$\frac{dF}{dx} = -2x_{eq} > 0$$

Therefore, it is an unstable equilibrium point.
The above results are true only if $r > 0$. It is quite obvious that, for $r < 0$, equilibrium points do not exist.

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B. Pitchfork Bifurcation

Let's consider a case when an equilibrium point is splitted into three parts. If the original equilibrium point was stable, then two of the splitted points will be stable and one will be unstable. This is known as **Supercritical Pitchfork bifurcation**. On the contrary, if the original equilibrium point was unstable, then two of the splitted points will be unstable and one will be stable. This is known as **Subcritical Pitchfork bifurcation**.

Let's demonstrate Supercritical Pitchfork Bifurcation via an example:

$$\frac{dx}{dt} = rx - x^3$$

The equilibrium points are $0, \pm\sqrt{r}$. $x_{eq} = \pm\sqrt{r}$ are always stable, if they exist ($r >= 0$) and $x_{eq} = 0$ is unstable. If $r = 0$, all the equilibrium points coincide, and thus, $x = 0$ will be a stable equilibrium point.

Let's demonstrate Subcritical Pitchfork Bifurcation via an example:

$$\frac{dx}{dt} = rx + x^3$$

The equilibrium points are $0, \pm\sqrt{-r}$. $x_{eq} = \pm\sqrt{-r}$ are always unstable, if they exist ($r <= 0$) and $x_{eq} = 0$ is stable. If $r = 0$, all the equilibrium points coincide, and thus, $x = 0$ will be an unstable equilibrium point.

IV. RESULTS

We are given the problem of two companies competing against each other for maximum profit. Let x be the market share of A, y be the market share of B.
We know, there are no other competitors in this industry, therefore, we can say:

$$x + y = 1$$

$$\text{We are given with: } \frac{dx}{dt} = ax(1-x)(x-y)$$

By the above equations, we can convert it into a one-variable equation:

$$\frac{dx}{dt} = ax(1-x)(2x-1)$$

The equilibrium points are $0, 1/2, 1$.

Now we have to calculate the stability of the critical points that we have deduced. This is done by calculating the gradient of the function at points in the neighbourhood of the critical points. If the critical point acts as a sink, meaning if we plot the graph of $\frac{dx}{dt}$ vs x, for the x-intercepts where the points to the left are positive and those to the right are negative, we call them as stable points. In this case, we have to consider 2 cases:

1. When $a > 0$
2. When $a < 0$

To check for the stability we have to calculate the double differential equation of the given expression. If after substituting the value in the double differential equation we get a positive value then we say that that point is unstable and if it comes out to be negative then we say it is stable.
Double Differential:

$$-6x^2 + 6x - 1 - r = 0$$

Now we substitute each critical point value that we calculated earlier.
For $a > 0$:

1. When $x = 0$
$$\frac{d^2x}{dt^2} = a(-6(0) + 6(0) - 1)$$
$$\frac{d^2x}{dt^2} = (-1)a$$
This will always be negative for positive values of a, therefore we conclude that $x = 0$ is stable.

2. When $x = 0.5$
$$\frac{d^2x}{dt^2} = a(-6(0.5) + 6(0.5) - 1)$$
$$\frac{d^2x}{dt^2} = a(-1.5 + 3 - 1)$$
$$\frac{d^2x}{dt^2} = (0.5)a$$
This will always be positive for positive values of a, therefore we conclude that $x = 0.5$ is unstable.

3. When $x = 1$
$$\frac{d^2x}{dt^2} = a(-6(1) + 6(1) - 1)$$
$$\frac{d^2x}{dt^2} = (-1)a$$
This will always be negative for positive values of a, therefore we conclude that $x = 1$ is stable.

Similarly, For $a < 0$:

1. When $x = 0$
$$\frac{d^2x}{dt^2} = a(-6(0) + 6(0) - 1)$$
$$\frac{d^2x}{dt^2} = (-1)a$$
This will always be positive for negative values of a, therefore we conclude that $x = 0$ is unstable.

2. When $x = 0.5$
$$\frac{d^2x}{dt^2} = a(-6(0.5) + 6(0.5) - 1)$$
$$\frac{d^2x}{dt^2} = a(-1.5 + 3 - 1)$$
$$\frac{d^2x}{dt^2} = (0.5)a$$
This will always be negative for negative values of a, therefore we conclude that $x = 0.5$ is stable.

3. When $x = 1$
$$\frac{d^2x}{dt^2} = a(-6(1) + 6(1) - 1)$$
$$\frac{d^2x}{dt^2} = (-1)a$$
This will always be positive for negative values of a, therefore we conclude that $x = 1$ is unstable.

To conclude when $a > 0$ the stable points are x = 0 and x = 1, and when $a < 0$ the stable point is x = 0.5.

For $a = 1$, $p = 0.5$, we get the equation as:
$$\frac{dx}{dt} = -2x^3 + 3x^2 - (1 + r)x + \frac{r}{2}$$

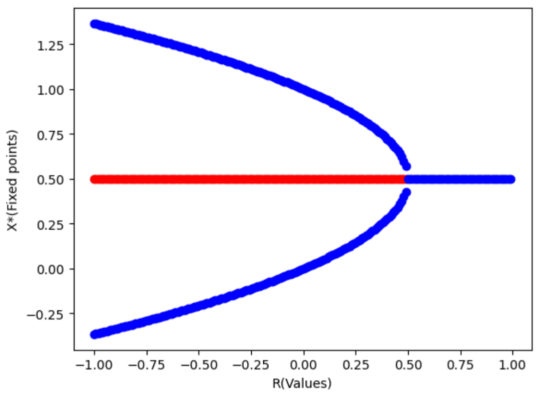

FIG. 1: In this model, we obtain a supercritical pitchfork bifurcation.

We now make an additional assumption that this local market is also influenced by a much larger global market. So, the updated growth equation of Market A can be written as:
$$\frac{dx}{dt} = ax(1 - x)(2x - 1) + r(p - x)$$
By simplifying the above equation, we get:
$$\frac{dx}{dt} = -2ax^3 + 3ax^2 - (a + r)x + rp$$
For equilibrium points (critical condition):
$$\frac{d}{dx}(-2ax^3 + 3ax^2 - (a + r)x + rp) = 0$$
$$-6ax^2 + 6ax - (a + r) = 0$$
For the equilibrium points to exist, the Determinant of the quadratic equation must be ≥ 0
$$(6a)^2 - 4(-6a)(-(a + r)) \geq 0$$
$$36a^2 - 24a(a + r) \geq 0$$
$$a(r - a/2) \leq 0$$
$$\boxed{0 \leq r \leq a/2}$$
Since it is independent of p. Therefore, $\boxed{p \in \mathbb{R}}$

Now, we analyze if we could get bifurcation of $f(x)$ by manipulating the values of r and p (assuming $a = 1$) The equation is:
$$y = x(1 - x)(2x - 1) + r(p - x)$$
We plotted nature of roots of $f(x)$ using different values of r and p, which is demonstrated as:

(1) For $r = 0.3, p = 0.6$, we get:

FIG. 2: By linear stability analysis, we can easily figure out that saddle equilibrium point is obtained very near $x = 0$, which shows bifurcation, as this result was not at all expected under normal conditions of r and p.

(2) For $r = 0.3, p = 0.4$, we get:

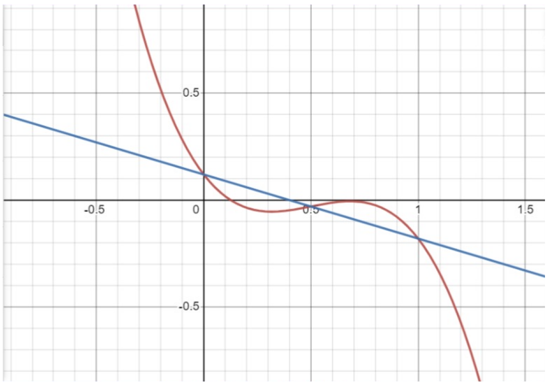

FIG. 3: Similarly, saddle equilibrium point is obtained very near $x = 1$, which also show bifurcation.

The above plots clearly show that the line $y = r(p-x)$ pushes the curve $y = x(1-x)(2x-1)$ to the equilibrium. Therefore, the nature of the function will depend on the values of the line $y = r(p-x)$, which depends on the varied possibilities of the values of r and p.

$y_{min} = r(p-x)$ will give us the equilibrium point of the given function.

$$6x^2 - 6x + 1 = 0$$

Required Root, $x = 0.317$, for which $y = -0.08$ We have to find such combinations for which $r(p - 0.317) = 0.08$ holds true.

The graph for the values of p vs values of r:

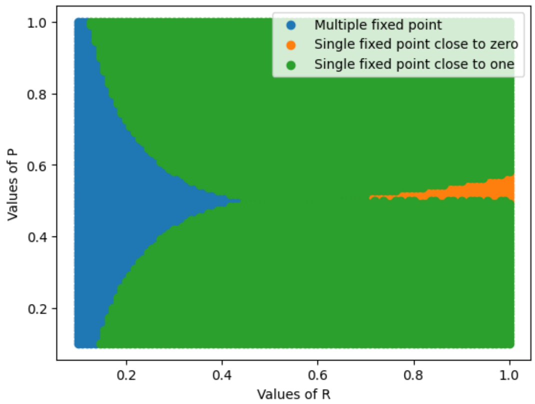

FIG. 4:

We can analyze from the Fig.1 graph, if the company's market share is above 50% and significantly influenced by the global market then we need to decrease the strength of influence from the global market by introducing offers, discounts, and reducing import exports from the global market. Making changes in the supply system to decrease the global connect will get us more money to focus on our growth and increase our market share.

If the company's market share is below 50%, then we need to introduce some form of marketing strategies like advertisements through newspapers and TV. We need to make our product available globally at better rates and provide greater benefits than the competing company in an effort to cross a threshold of 50%.
We can flip the market by varying values of r and p.

V. CONCLUSIONS

In this lab, we understood the use of 1-D bifurcations and their broad usage to analyze complicated differential equations that cannot be solved analytically. We deeply studied different types of bifurcations and how they can be implemented to analyze the behaviour of complex equations.

So by now, we have done the bifurcation analysis with respect to r and p and we have seen that we can obtain a stable fixed point that is either close to 0 or close to 1 by manipulating one of the variables if one is fixed. This gives us the condition to flip the market. For instance, if the current condition has $p = 1/2$ and we want a stable fixed point near 1 then we can calculate the value of r from $y = r(p-x)$ by substituting x and y as obtained by analyzing maxima point near one in the cubic equation.

If I were the company director, I would look at Fig 2 and obtain the r,p, and want to manipulate those parameters and take them to the green region. We will have to see if changing the values of r or p given any constraint on the other will lead us to the region we want. For instance, if our global market share(p) is 0.5 and our strength of influence in the global market(r) is 0.7(high)(Orange region) then we would want to work on our product and improve quality and customer services. So basically we will want to invest more in such strategies that increase our global strength while not focusing on profits or increase in global share. If our global market share is low even though our global market influence is low(blue region) then we need to focus on advertisement and marketing in the global as well as local market.

[1] A. Shiflet and G. Shiflet, *Introduction to Computational Science: Modeling an Simulation for the Sciences*, Princeton University Press.3, 276 (2006).

[2] A. Einstein and N. Rosen, Phys. Rev.**48**, 73 (1935).