14 Spatial Analysis and Inference

LEARNING OBJECTIVES

The second of two chapters on spatial analysis focuses on five areas: analyses that address concepts of area and centrality, analyses of surfaces, analyses that are oriented to design, and statistical inference. The chapter begins with area-based techniques, including measures of area and shape. Techniques based on the concept of centrality seek to find central or representative points for geographic distributions. Surface techniques include measures of slope, aspect, and visibility along with the delineation of watersheds and channels. Design is concerned with choosing locations on the Earth's surface for various kinds of activities and with the positioning of points, lines, and areas and associated attributes to achieve defined objectives. Statistical inference concerns the extent to which it is possible to reason from limited samples or study areas to conclusions about larger populations or areas, but is often complicated by the nature of geographic data.

After studying this chapter you will understand:

- Methods for measuring properties of areas.
- Measures that can be used to capture the centrality of geographic phenomena.
- Techniques for analyzing surfaces and for determining their hydrologic properties.
- Techniques for the support of spatial decisions and the design of landscapes according to specific objectives.
- Methods for generalizing from samples, and the problems of applying methods of statistical inference to geographic data.

14.1 The Purpose of Area-Based Analyses

One of the ways in which humans simplify geography and address the Earth's infinite complexity is by ascribing characteristics to entire areas rather than to individual points. *Regional* geography relies heavily on this process to make parsimonious descriptions of the geographic world, a practice discussed earlier in Chapter 3 in the context of representation, and in Chapter 2 as one aspect of the nature of geographic data. At a technical level, this results in the creation not of points but of polygons (Section 7.2, and note that the term *polygon* is often used in geographic information science and systems [GISS] to refer to any area, whether or not its edges are straight.) In Chapter 5 we discussed some of the issues of uncertainty that arise as a result of this process, when areas are not truly homogeneous or when their boundaries are not precisely known.

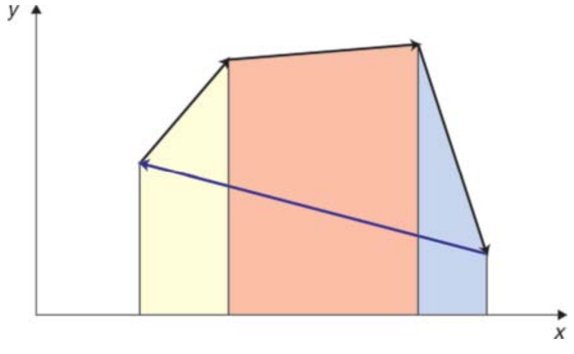
14.1.1 Measurement of Area

Many types of interrogation ask for measurements—we might want to know the total area of a parcel of land, or the distance between two points, or the length of a stretch of road—and in principle all these measurements are obtainable by simple calculations inside a geographic information (GI) system. Comparable measurements by hand from maps can be very tedious and error-prone. In fact, it was the ability of the computer to make accurate evaluations of area quickly that led the Canadian government to fund the development of the world's first GI system, the Canada Geographic Information System, in the mid-1960s (see the brief history of GI systems in Table 1.4), despite the primitive state and high costs of computing at that time. Evaluation of area by hand is a messy and soul-destroying business. The *dot-counting* method uses transparent sheets on which randomly located dots have been printed—an area on the map is estimated by counting the number of dots falling within it. In the *planimeter* method a mechanical device is used to trace the area's boundary, and the required measure accumulates on a dial on the machine.

Humans have never devised good manual tools for making measurements from maps, especially measurements of area.

By comparison, measurement of the area of a digitally represented polygon is trivial and totally reliable. The common procedure or algorithm calculates and sums the areas of a series of trapezia, formed by dropping perpendiculars to the *x*-axis, as shown in Figure 14.1. By making a simple change to the algorithm, it is also possible to use it to compute a polygon's centroid (see Section 14.2.1 for a discussion of centroids; note that the term *centroid* is often used in GISS whenever a polygon is collapsed to a point, whether or not the point is technically the mathematical centroid of the polygon). It is advisable to be cautious, however, to ensure that the coordinate system is appropriate. For example, if a polygon's vertices are coded in latitude and longitude, the result of computing area may be a measurement in "square degrees." But except at the equator, the length of a degree of longitude varies with latitude and is always less than the length of a degree of latitude. On the other hand, measurement of area using projected coordinates, such as Universal Transverse Mercator (UTM) or U.S. State Plane (Sections 4.8.2 and 4.8.3), will give a result in square meters or square feet, respectively. Note, however, that neither of these projected coordinate systems has equal-area properties, so the result will be somewhat distorted. Studies that require accurate measures of area should use only equal-area projections.

Figure 14.1 The algorithm for calculation of the area of a polygon given the coordinates of the polygon's vertices. The polygon consists of the three black arrows, plus the blue arrow forming the fourth side. Trapezia are dropped from each edge to the *x*-axis, and their areas are calculated as (difference in *x*) times (average of *y*). The trapezia for the first three edges, shown in yellow, orange, and blue, are summed. When the fourth trapezium is formed from the blue arrow, its area is negative because its start point has a larger *x* than its endpoint. When this area is subtracted from the total, the result is the correct area of the polygon.



14.1.2 Measurement of Shape

GI systems are also used to characterize the *shapes* of areas. In many countries the system of political representation is based on the concept of districts or constituencies, which are used to define who will vote for each place in the legislature (Box 14.1). In the United States and the UK, and in many other countries that derived their system of representation from the UK, one place is reserved in the legislature for each district. It is expected that districts will be compact in shape, and the manipulation of a district's shape to achieve certain overt or covert objectives is termed gerrymandering, after an early governor of Massachusetts, Elbridge Gerry (the shape of one of the state's districts was thought to resemble a salamander, with the implication that it had been manipulated to achieve a certain outcome in the voting). The construction of voting districts is an example of the principles of aggregation and zone design discussed in Section 5.2.1.

Anomalous shape is the primary means of detecting gerrymanders of political districts.

Geometric shape was the aspect that alerted Gerry's political opponents to the manipulation of districts, and today shape is measured whenever GI systems are used to aid in the drawing of political district boundaries, as must occur by law in the United States after every decennial census. An easy way to define shape is by comparing the perimeter length of an area to its area measure. Normally the square root of area is used to ensure that the numerator and denominator are both measured in the same units. A common measure of shape or compactness is

$$P/2\sqrt{\pi A}$$

where P is the perimeter length and A is the area. The factor of twice the square root of π (3.54) ensures that the most compact shape, a circle, returns a shape of 1.0, and the most distended and contorted shapes return much higher values. Box 14.1 explores these ideas for a Congressional District in California.

Application Box 14.1

Shape and Congressional Districts

One way to bias the outcome of an election in the United States is to draw (or *gerrymander*) district boundaries to include a majority of people likely to vote for a particular party. In Southern California the 23rd Congressional District had long been drawn to include the liberal-leaning coastal communities such as Santa Barbara and to exclude the more conservative agricultural areas of the interior. Its long, thin shape following the coastline had a shape index of 4.45. Representative Lois Capps, a Democrat, had held the seat through seven biennial elections.

Redistricting is required by law following each U.S. decennial census. To try to reduce the creation of "safe" seats by gerrymandering, a new process of districting was put in place in California that required districts to be drawn by a committee of citizens, rather than politicians. The result, after the 2010 census, was a new electoral map of California. Figure 14.2 shows how the old 23rd District was merged into a new 24th District that included both liberal-leaning and conservative-leaning areas; its much more compact shape has an index of 2.28. After a stiff fight, Lois Capps held the seat in the 2012 election with a majority of 53.8%.

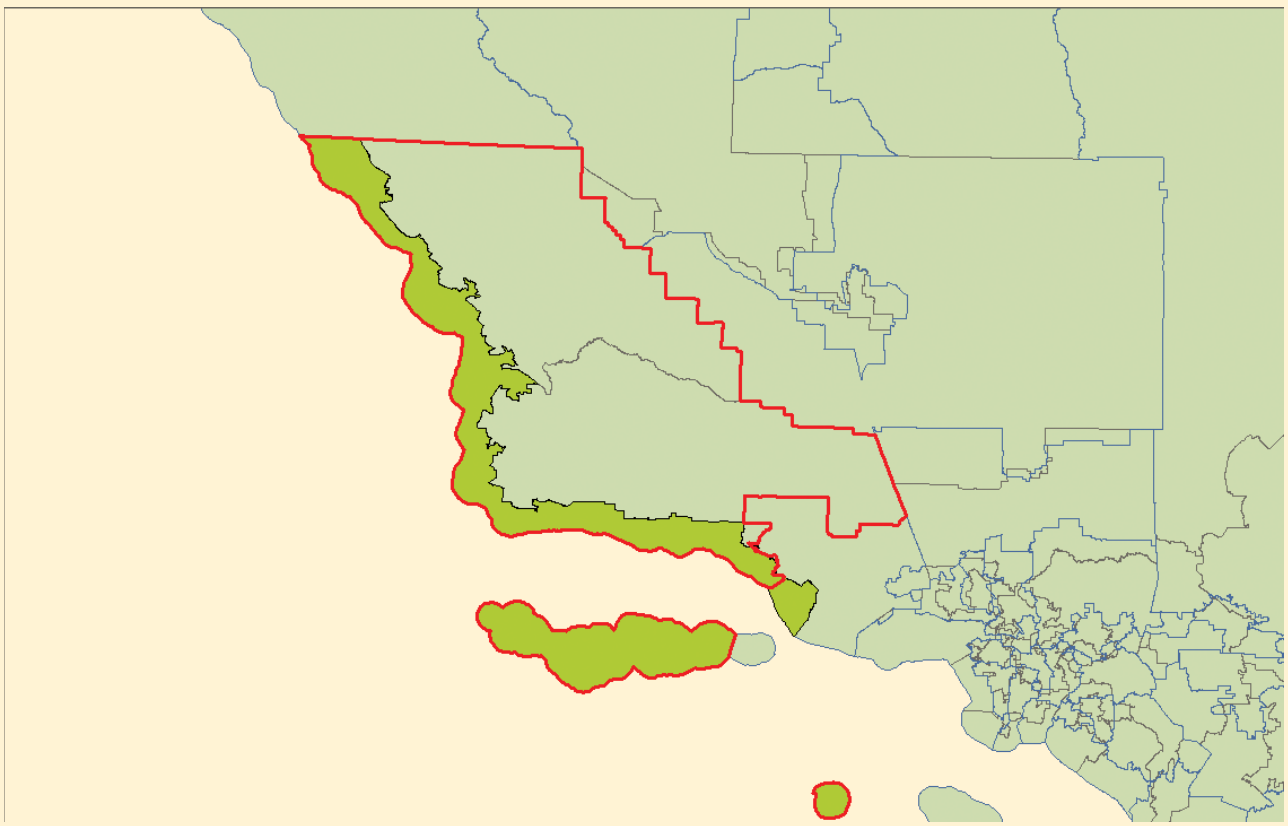


Figure 14.2 Redistricting following the 2010 U.S. census, coupled with a new opposition to gerrymandering in California, transformed the old 23rd Congressional District (yellow) into a new, more compact 24th District (outlined in red).

14.2 Centrality

The topics of generalization and abstraction were discussed in Section 3.8 as ways of reducing the complexity of data. This section reviews a related topic, that of numerical summaries. If we want to describe the nature of summer weather in an area, we cite the *average* or *mean*, knowing that there is substantial variation around this value, but that it nevertheless gives a reasonable *expectation* about what the weather will be like on any given day. The mean (the more formal term) is one of a number of measures of *central tendency*, all of which attempt to create a summary description of a series of numbers in the form of a single number. Another is the *median*, the value such that one-half of the numbers are larger and one-half are smaller. Although the mean can be computed only for numbers measured on interval or ratio scales, the median can be computed for ordinal data. For nominal data the appropriate measure of central tendency is the *mode*, or the most common value. For definitions of nominal, ordinal, interval, and ratio see Box 2.1. Special methods must be used to measure central tendency for cyclic data; they are discussed in texts on directional data, for example, by Mardia and Jupp.

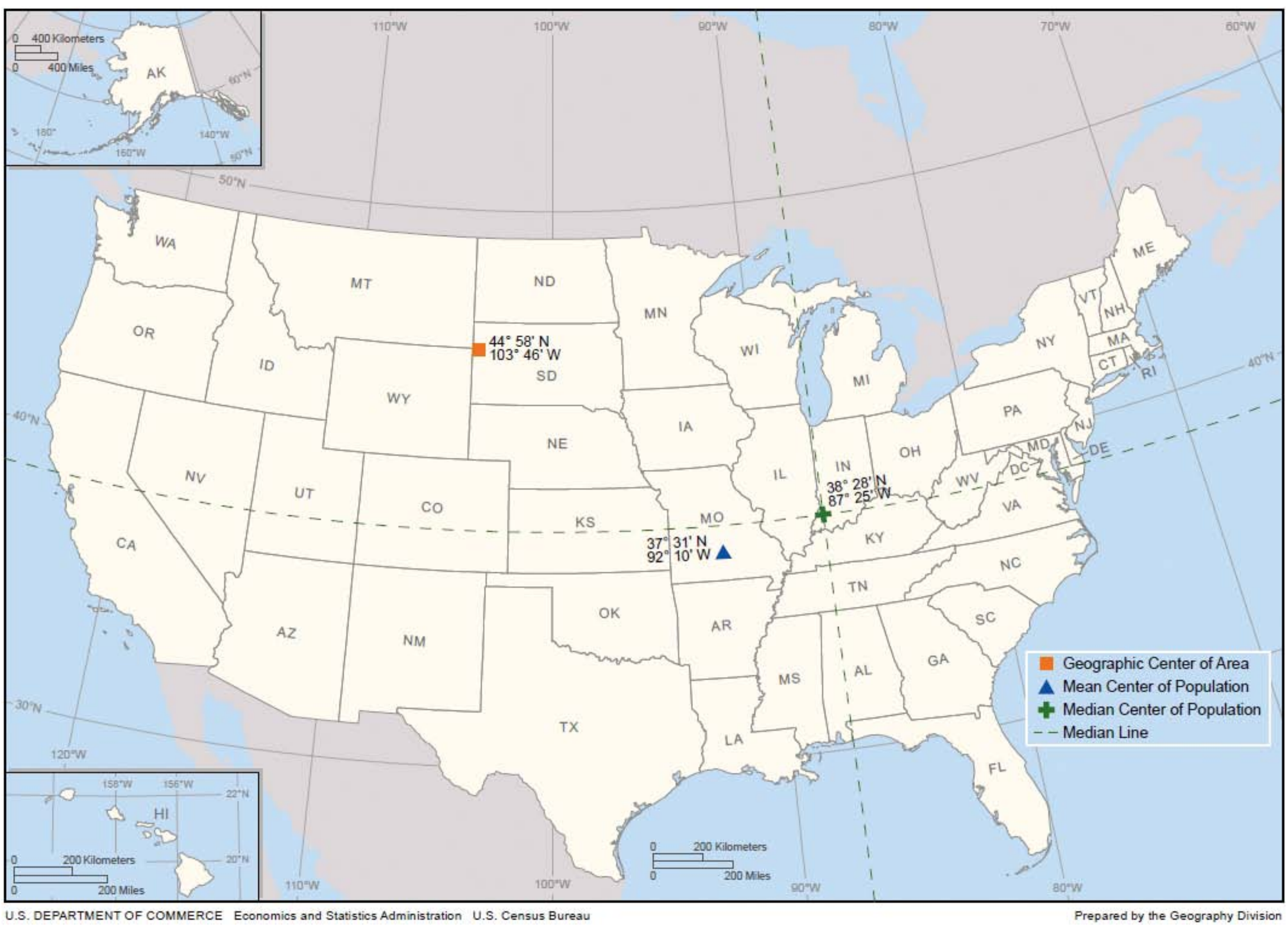
14.2.1 Centers

The spatial equivalent of the mean would be some kind of center, calculated to summarize the positions of a number of points. Early in U.S. history the Bureau of the Census adopted a practice of calculating a representative center for the U.S. population. As agricultural settlement advanced across the West in the Nineteenth Century, the repositioning of the center every 10 years captured the popular imagination. Today, the movement west has slowed and shifted more to the south (Figure 14.3) and by the next census may even have reversed.

Centers are the two-dimensional equivalent of the mean.

The mean of a set of numbers has several properties. First, it is calculated by summing the numbers and dividing by the number of numbers. Second, if we take any value *d* and sum the squares of the differences between the numbers and *d*, then when *d* is set equal to the mean this sum is minimized (Figure 14.4). Third, the mean is the point about which the set of numbers would balance if we made a physical model such as the one shown in Figure 14.5 and suspended it.

Figure 14.3 The mean center of the U.S. population, determined from the results of the 2010 census. Also shown is the median center, such that half of the population is to the north and half to the south, half to the east and half to the west.



Source: U.S. Bureau of the Census, public domain

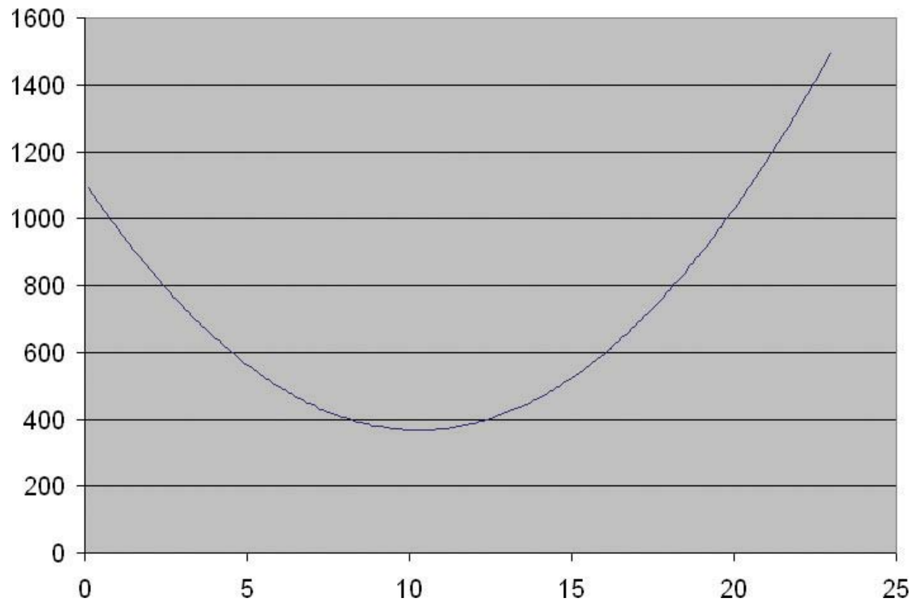


Figure 14.4 Seven points are distributed along a line, at coordinates 1, 3, 5, 11, 12, 18, and 22. The curve shows the sum of distances squared from these points and how it is minimized at the mean [(1+3+5+11+12+18+22)/7 = 10.3].

These properties extend easily into two dimensions. Figure 14.6 shows a set of points on a flat plane, each one located at a point (x_i,y_i) and with weight w_i. The centroid or mean center is found by taking the weighted average of the *x* and *y* coordinates:

$$\bar{x} = \sum_i w_i x_i \Big/ \sum_i w_i$$

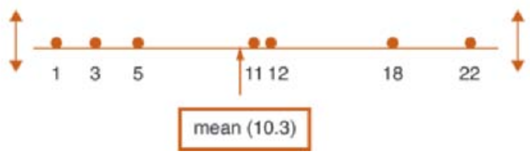
$$\bar{y} = \sum_i w_i y_i \Big/ \sum_i w_i$$

It also is the point that minimizes the sum of squared distances, and it is the balance point.

Just like the mean, the centroid is a useful summary of a distribution of points. Although any single centroid may not be very interesting, a comparison of centroids for different sets of points or for different times can provide useful insights.

The centroid is the most convenient way of summarizing the locations of a set of points.

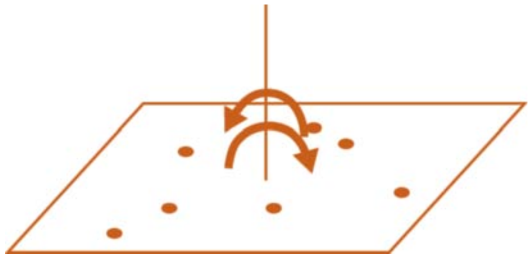
Figure 14.5 The mean is also the balance point, the point about which the distribution would balance if it were modeled as a set of equal weights on a weightless, rigid rod.



The property of minimizing functions of distance (the square of distance in the case of the centroid) makes centers useful for different reasons. Of particular interest is the location that minimizes the sum of distances, rather than the sum of squared distances because this could be the most effective location for any service that is intended to serve a dispersed population. The point that minimizes total straight-line distance is known as the point of minimum aggregate travel, or MAT, and is discussed in detail in Section 14.4, which is devoted to methods of design.

All the methods described in this and the following section are based on plane geometry and simple *x*,*y* coordinate systems. If the curvature of the Earth's surface is taken into account, the centroid of a set of points must be calculated in three dimensions and will always lie under the surface. More useful perhaps are versions of the MAT and centroid that minimize

Figure 14.6 The centroid or mean center replicates the balance-point property in two dimensions—the point about which the two-dimensional pattern would balance if it were transferred to a weightless, rigid plane and suspended.



distance over the curved surface (the minimum total distance and the minimum total of squared distances, respectively, using great circle distances: see Sections 4.7 and 13.3.1).

14.2.2 Dispersion

Central tendency is the obvious choice if a set of measurements must be summarized in a single value, but what if there is the opportunity for a second summary value? Here the measure of choice for measurements with interval or ratio properties is the *standard deviation*, or the square root of the mean squared difference from the mean:

$$s = \sqrt{\sum_i (x_i - \bar{x})^2 / n}$$

where n is the number of observations, s is the standard deviation, x_i refers to the ith observation, and $\bar{x}$ is the mean of the observations. In weighted form the equation becomes

$$s = \sqrt{\sum_i w_i (x_i - \bar{x})^2 / \sum_i w_i}$$

where w_i is the weight given to the ith observation. The *variance*, or the square of the standard deviation (the mean squared difference from the mean), is often encountered, but it is not as convenient a measure for descriptive purposes. Standard deviation and variance are considered more appropriate measures of dispersion than the *range* (the difference between the highest and lowest numbers) because as averages they are less sensitive to the specific values of the extremes.

The standard deviation has also been encountered in Section 5.3.2.2 in a different guise, as the root mean squared error (RMSE), a measure of dispersion of observations about a true value. Just as in that instance, the Gaussian distribution provides a basis for generalizing about the contents of a sample of numbers, using the mean and standard deviation as the parameters of a simple bell curve. If data follow a Gaussian distribution, then approximately 68% of values lie within one standard deviation of the mean and approximately 5% of values lie outside two standard deviations.

These ideas convert very easily to the two-dimensional case. A simple measure of dispersion in two dimensions is the *mean distance from the centroid*. In some applications it may be desirable to give greater weight to more distant points. For example, if a school is being located, then students living at distant locations are comparatively disadvantaged. They can be given greater weight if each distance is squared, such that a student twice as far away receives four times the weight. This property is minimized by locating the school at the centroid.

Mean distance from the centroid is a useful summary of dispersion.

Measures of dispersion can be found in many areas of GI systems. The breadth of the kernel function of density estimation (Section 13.3.5) can be thought of as a measure of how broadly a pile of sand associated with each point is dispersed. RMSE is a measure of the dispersion inherent in positional errors (Section 5.3.2.2).

14.3 Analysis of Surfaces

Continuous fields of elevation provide the basis for many types of analysis in GI systems. More generally, any field formed by measurements of an interval or ratio variable, such as air temperature, rainfall, or soil pH, can also be conceptualized as a surface and analyzed using the same set of tools, though the results may make little sense in some cases. This section is devoted first to simple measurement of surface slope and aspect. It then introduces techniques for determining paths over surfaces, watersheds and channels, and finally intervisibility.

The various ways of representing continuous fields were first discussed in Section 3.5.2 and later in greater technical detail in Section 7.2. The techniques discussed in this section begin with a raster representation, termed a digital elevation model (DEM) in the case of terrain representation. In most cases the value recorded will be the elevation at the center of each raster cell, though in some cases it may be the mean elevation over the cell; it is always important to check a dataset's documentation on this issue before using the dataset.

The digital elevation model is the most useful representation of terrain in a GI system.

14.3.1 Slope and Aspect

Knowing the exact elevation of a point above sea level is important for some applications, including prediction of the effects of global warming and rising sea levels on coastal cities. For many applications, however, the value of a DEM lies in its ability to produce derivative measures through transformation, specifically measures of slope and aspect, both of which are also conceptualized as fields. Imagine taking a large sheet of plywood and laying it on the Earth's surface so that it touches at the point of interest. The magnitude of steepest tilt of the sheet defines the *slope* at that point, and the direction of steepest tilt defines the *aspect* (Box 14.2).

Technical Box 14.2

Calculation of Slope Based on the Elevations of a Point and Its Eight Neighbors

There are many ways of estimating the slope in each cell of a DEM, depending on the equations that are used. Described here is the most common, though by no means the only, approach (refer to Figure 14.7 for point numbering):

$$b = (z_3 + 2z_6 + z_9 - z_1 - 2z_4 - z_7)/8D$$

$$c = (z_1 + 2z_2 + z_3 - z_7 - 2z_8 - z_9)/8D$$

where b and c are tan(slope) in the x and y directions, respectively, D is the grid point spacing, and z_i denotes elevation at the ith point, as shown in Figure 14.7. These equations give the four diagonal neighbors of Point 5 only half the weight of the other four neighbors in determining slope at Point 5.

Now we can calculate slope and aspect, as follows:

$$\tan(slope) = \sqrt{b^2 + c^2}$$

where *slope* is the angle of slope in the steepest direction.

$$\tan(aspect) = b/c$$

where *aspect* is the angle between the y-axis and the direction of steepest slope, measured clockwise.

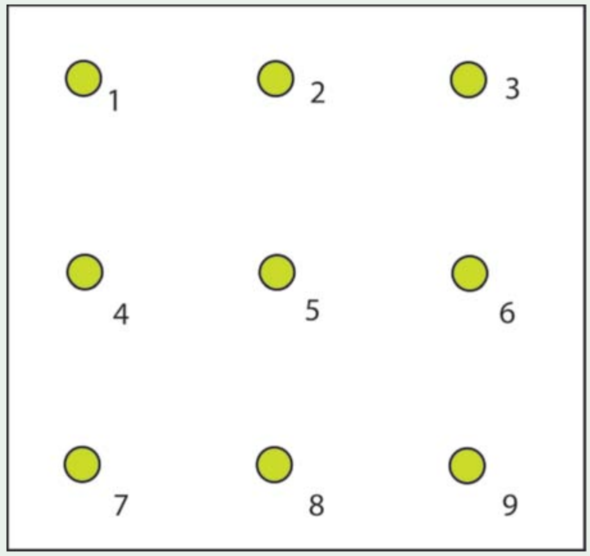


Figure 14.7 Calculation of the slope at Point 5 based on the elevation of it and its eight neighbors.

Because *aspect* varies from 0 to 360, an additional test is necessary that adds 180 to *aspect* if c is positive.

This sounds straightforward, but it is complicated by a number of issues. First, what if the plywood fails to sit firmly on the surface, but instead pivots because the point of interest happens to be a peak or a ridge? In mathematical terms, we say that the surface at this point *lacks a well-defined tangent*, or that the surface at this point is *not differentiable*, meaning that it fails to obey the normal rules of continuous mathematical functions and differential calculus. The surface of the Earth has numerous instances of sharp breaks of slope, rocky outcrops, cliffs, canyons, and deep gullies that defy this simple mathematical approach to slope, and this is one of the issues that led Benoît Mandelbrot to develop his theory of fractals (see Section 2.8).

A simple and satisfactory alternative is to take the view that slope must be measured at a particular resolution. To measure slope at a 30-m resolution, for example, we evaluate elevation at points 30 m apart and compute slope by comparing them (equivalent in concept to using a plywood sheet 30 m across). The value this gives is specific to the 30-m spacing, and a different spacing (or different-sized sheet of plywood) would have given a different result. In other words, *slope is a function of resolution* or scale, and it makes no sense to talk about slope without at the same time talking about a specific resolution or level of detail. This is convenient because slope is easily computed in this way from a DEM with the appropriate resolution.

The spatial resolution used to calculate slope and aspect should always be specified.

A second issue is the existence of several alternative *measures* of slope, and it is important to know which one is used in a particular software package and application. Slope can be measured as an *angle*, varying from 0 to 90 degrees as the surface ranges from horizontal to vertical. But it can also be measured as a percentage or ratio, defined as *rise over run*, and unfortunately there are two different ways of defining run. Figure 14.8 shows the two options, depending on whether run means the horizontal distance covered between two points, or the diagonal distance (the *adjacent* or the *hypotenuse* of the right-angled

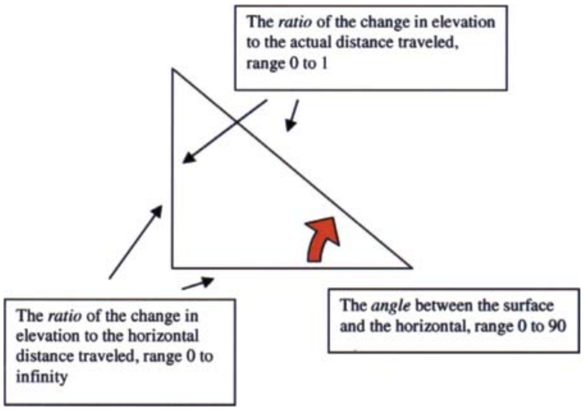


Figure 14.8 Three alternative definitions of slope. To avoid ambiguity we use the angle, which varies between 0 and 90 degrees.

triangle, respectively). In the first case (opposite over adjacent), slope as a ratio is equal to the tangent of the angle of slope and ranges from zero (horizontal) through 1 (45 degrees) to infinity (vertical). In the second case (opposite over hypotenuse), slope as a ratio is equal to the sine of the angle of slope and ranges from zero (horizontal) through 0.707 (45 degrees) to 1 (vertical). To avoid confusion we will use the term *slope* only to refer to the measurement in degrees and will call the other options tan(slope) and sin(slope), respectively.

When a GI system calculates slope and aspect from a DEM, it does so by estimating slope at each of the data points of the DEM, by comparing the elevation at that point to the elevations of surrounding points. But the number of surrounding points used in the calculation varies, as do the weights given to each of the surrounding points in the calculation. Box 14.2 shows this idea in practice, using one of the most common methods, which employs eight surrounding points and gives them different weights depending on how far away they are.

14.3.2 Modeling Travel on a Surface

One way to interpret a buffer drawn around a point (Figure 13.17) is that it represents the distance that could be traveled from the point in a given time, assuming constant travel speed. But if travel speed were not uniform and instead were represented by a continuous field of *friction*, then the buffer would be modified to expand more in some directions than in others. This function is often termed a *spread*, and like many techniques discussed in this section, it is an example of a function that is far easier to execute on a raster representation. The total friction associated with a route is calculated by summing the friction values of the cells along the route, and because there are many possible routes from an origin to a

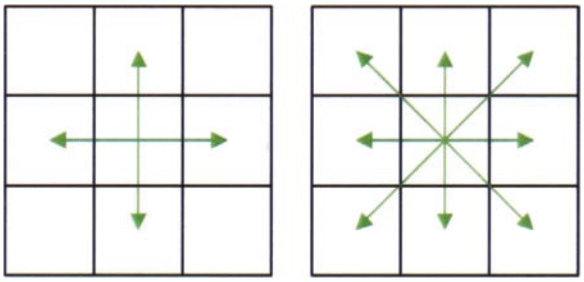
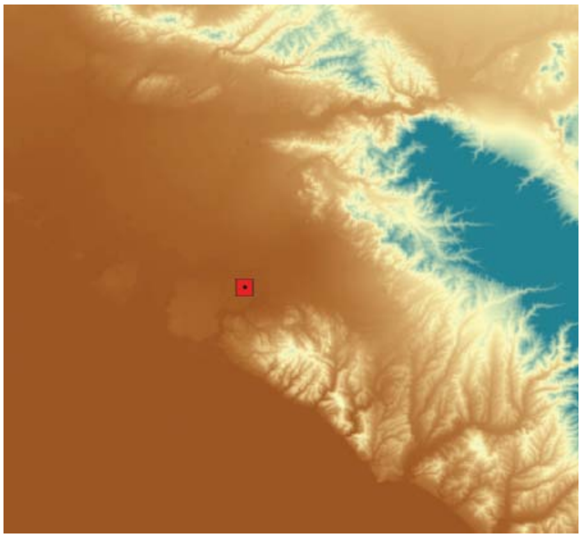


Figure 14.9 The rook's-case (left) and queen's-case (right) move sets, defining the possible moves from one cell to another in solving the problem of optimum routing across a friction surface.

destination, it is necessary to select the route that minimizes total friction. But first one must select a *move set*, the set of possible moves that can be made between cells along the route. Figure 14.9 shows the two most commonly used move sets. When a diagonal move is made, it is necessary to multiply the associated friction by 1.414 ($\sqrt{2}$) to account for the move's greater length.

Figure 14.10 shows an example DEM, in this case with 30-m point spacing and covering an area roughly corresponding to Orange County, California. Archaeologists are often interested in the impact of terrain on travel because it may help to explain ancient settlement patterns and communication paths. If we assume that travelers will avoid steep slopes and high terrain, then a possible measure of friction would combine slope s and elevation e in an expression such as $s + e/100$ where s is measured in degrees and e in feet. Applying this to the Orange County DEM,

Figure 14.10 A digital elevation model of an area of Southern California, including most of Orange County. High ground is shown in blue. The Pacific Ocean covers the lower left, and the red symbol is located at Santa Ana (John Wayne) Airport.



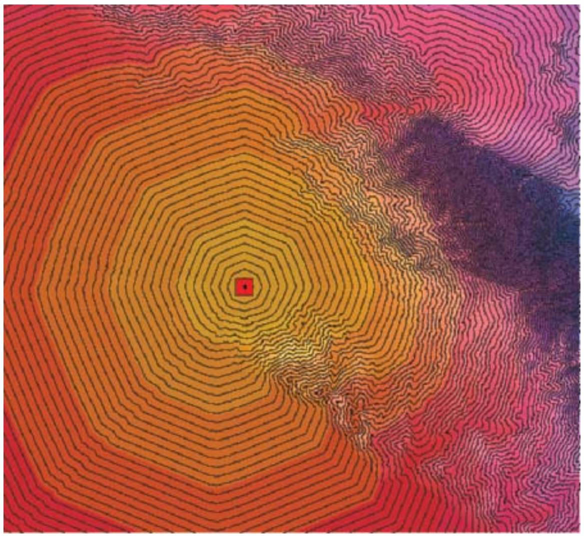


Figure 14.11 Contours of equal travel cost from an origin. Friction or travel "cost" is determined by a combination of ground elevation and ground slope, reflecting the conditions faced by pre-Columbian populations in this part of California. Note the effect of the Santa Ana River gorge in providing easy access to the Inland Empire (upper right).

Figure 14.11 shows the effects of traveling away from a location near the current Santa Ana airport. Contours show points of equal total friction ("cost") from the start point. Notice how the gorge formed by the Santa Ana River provides an easy route across the high, steep ground and into what is now the Inland Empire.

14.3.3 Computing Watersheds and Channels

A DEM provides an easy basis for predicting how water will flow over a surface, and therefore of many useful hydrologic properties. Consider the DEM shown in Figure 14.12. We assume that water can flow from a cell to any of its eight neighboring cells, down the direction of steepest slope. Because only eight such directions are possible in this raster representation, we assume that water will flow to the lowest of the eight neighbors, provided at least one neighbor is lower. If no neighbor is lower, water is instead assumed to pond, perhaps forming a lake, until it rises high enough to overflow. Figure 14.12A shows the outflow directions predicted for each cell and outflow accumulation in the one cell that has no lower neighbor. Because this cell is at the edge of the area, we assume that it will spill over the boundary, forming the area's outflow.

A *watershed* is defined as the area upstream of a point—in other words, the area that drains through that point. Once we know the flow directions, it is

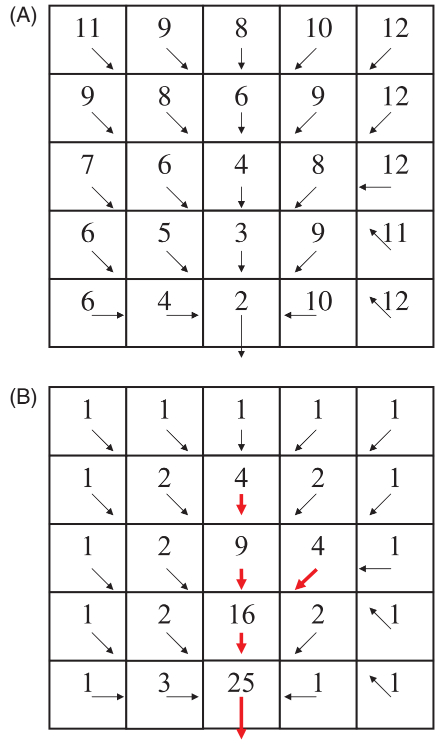


Figure 14.12 Hydrologic analysis of a sample DEM: (A) the DEM and inferred flow directions using the queen's case move set; and (B) accumulated flows in each cell and eroded channels based on a threshold flow of 4 units.

easy to identify a point and the associated upstream area that forms that point's watershed. Note that any point on the map has an associated watershed and that watersheds therefore may overlap.

This solution represents what a hydrologist would term *overland* flow. When flow accumulates sufficiently, it begins to erode its bed and form a channel. If we could establish an appropriate threshold, expressed in terms of the number of upstream cells that drain through a given cell, then we could map a network of channels. Figure 14.12B shows an example using a threshold of four cells.

In reality some landscapes have closed depressions that fill with water to form lakes. Other landscapes, particularly those developed on soluble rocks such as limestone or gypsum, contain closed depressions that drain underground. But a DEM generated by any of the conventional means will also likely contain elevation errors, some of which will appear as artificial closed depressions. So a GI system will commonly include a routine to "fill" any such "closed depressions," allowing them to overflow and add to the general surface runoff. This filling step will need to be conducted before any useful hydrologic analysis of a landscape can be made.

Figure 14.13 shows the result of applying a filling step and then computing the drainage network for

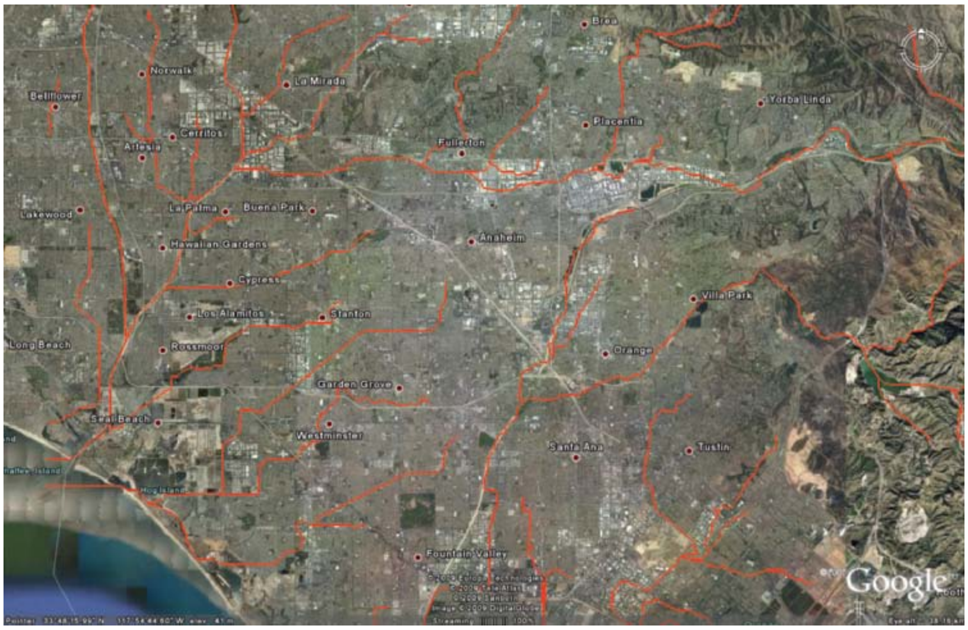


Figure 14.13 Analysis of the Orange County DEM predicts the river channels shown in red in this Google Earth mashup. The Santa Ana River appears to flow out of the gorge shown in the upper right, and then far to the west before emptying into the Pacific near Seal Beach. In reality, it turns south and empties near Newport Beach in the bottom center. See text for explanation.

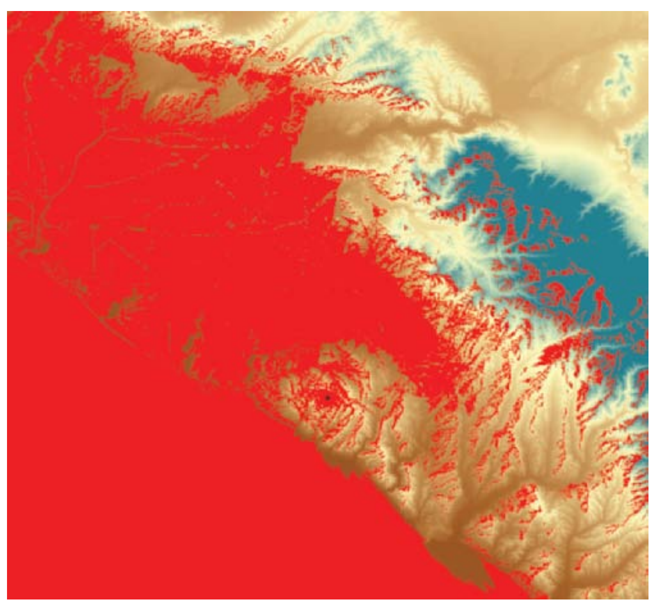
the Orange County DEM, using a channel-forming threshold of 10,000 cells or 9 sq km. The analysis correctly predicts a large river flowing into the top right of the area shown, through the Santa Ana River gorge. However, rather than flowing south to reach the Pacific below the middle center of the figure as the real river does, the predicted channel flows several kilometers further west before turning south, entering the Pacific near Seal Beach. Interestingly, this is the historic course of the river, before several severe floods in the Nineteenth Century. The current concrete channel, clearly visible in the figure, is delimited by levées, which are not large enough to affect the DEM given its 30-m point spacing.

14.3.4 Computing Visibility

One of the most powerful forms of surface analysis centers on the ability to compute intervisibility: can Point A on the surface be seen from Point B? This often takes the form of determining a point's *viewshed*—the area of surface that can be seen by an observer located at the point whose eye is elevated some specified height above the surface. Viewsheds have been used to plan the locations of observation points, transmitters, and mobile-phone towers. They have been used to analyze the locations adopted for prehistoric burial mounds, to test the hypothesis that people sought to be buried at conspicuous points where their remains could dominate the landscape. An interesting analysis by GI scientist Anne Knowles showed that a significant factor in the outcome of the Battle of Gettysburg, a turning point in the American Civil War, was the visibility of the battlefield achieved by the two commanders from their respective headquarters.

Figure 14.14 shows the viewshed of an observation point located on the low hills near Newport

Figure 14.14 The area visible from a 200-ft tower located on the hills near Newport Beach, California.



Beach, again using the Orange County DEM. In this case the eye has been elevated 200 ft above the surface, enough to command a view of the flatter areas in the northwest, but still not enough to view areas to the north and east of the higher hills in the area.

14.4 Design

In this section we look at the analysis of spatial data not for the purpose of discovering anomalies, or testing hypotheses about process, as in previous sections, but with the objective of creating improved designs—in short, changing the world. These objectives might include designs that minimize travel distance, maximize someone's profit, or minimize the costs and environmental impacts of construction of some new development. Recently the term *geodesign* has become popular as a description of this kind of application; geodesign can be defined as "geography by design," or the application of GI systems to improving the world at geographic scales. Box 14.3 describes the work of Carl Steinitz, one of the leaders of the geodesign movement.

The three principles of retailing are often said to be *location*, *location*, and *location*, and over the years many GIS applications have been directed at applications that involve, in one way or another, the search for optimum designs. The methods for finding centers described in Section 14.2.1 were shown to have useful design-oriented properties, and the modeling of travel across a surface discussed in Section 14.3.2 can also be interpreted as a design problem, for routing power lines, highways, or tanks. This section includes discussion of a wider selection of these so-called *normative* methods, or methods developed for application to the solution of practical problems of design.

Normative methods apply well-defined objectives to the design of systems.

Design methods are often implemented as components of systems built to support decision making—so-called *spatial-decision support systems*, or SDSSs. Complex decisions are often contentious, with many *stakeholders* interested in the outcome and in arguing for one position or another. SDSSs are specially adapted GI systems that can be used during the decision-making process to provide instant feedback on the implications of various proposals and the

Biographical Box 14.3

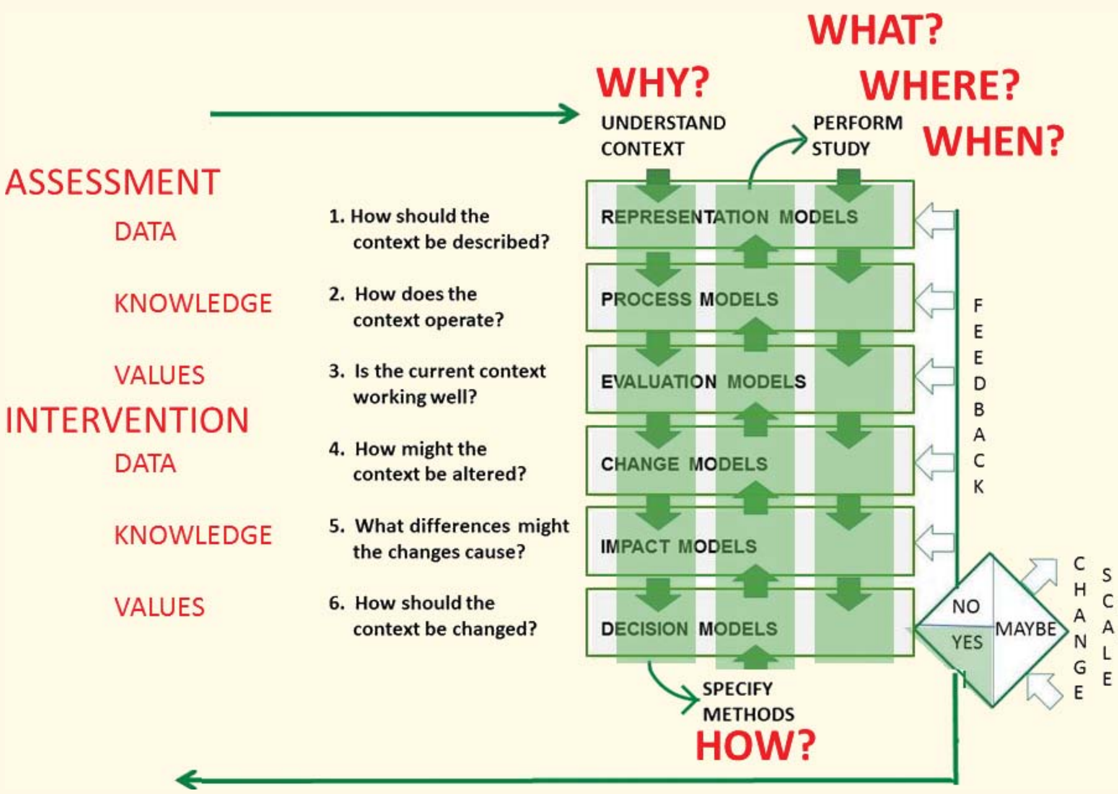
Carl Steinitz, Geodesigner

Carl Steinitz (Figure 14.15) is the Alexander and Victoria Wiley Professor of Landscape Architecture and Planning, Emeritus, at Harvard University Graduate School of Design. In 1966, Steinitz received his PhD degree in City and Regional Planning, with a major in urban design, from the Massachusetts Institute of Technology (MIT). He also holds a Master of Architecture degree from MIT and a Bachelor of Architecture degree from Cornell University. In 1965 he began his affiliation with the Harvard Graduate School of Design as an initial research associate in the Laboratory for Computer Graphics and Spatial Analysis. He has been Professor of Landscape Architecture and Planning at the Graduate School of Design since 1973.


Courtesy: Carl Steinitz

Figure 14.15 Carl Steinitz, Geodesigner.

Professor Steinitz has devoted much of his academic and professional career to improving methods to analyze large land areas and make design decisions about conservation and development. His applied research and teaching focus on highly valued landscapes that are undergoing substantial pressures for change. Professor Steinitz has directed studies in as wide ranging locales as the Gunnison region of Colorado; the Monadnock region of New Hampshire; the region of Camp Pendleton, California; the Gartenreich Worlitz in Germany; the West Lake in Hangzhou, China; Coiba National Park in Panama; Cagliari, Italy; and the regions of Castilla La Mancha and Valencia in Spain. Figure 14.16 summarizes the process that has evolved through these projects.



Source: Carl Steinitz

Figure 14.16 A schematic model of the geodesign process, developed by Carl Steinitz to capture the various stages used in his practical applications of geodesign principles.

evaluation of what-if scenarios. SDSSs typically have special user interfaces that present only those functions relevant to the application.

The methods discussed in this section fall into several categories. The next section discusses methods for the optimum location of points and extends the method introduced earlier for the MAT. The second section discusses routing on a network and its manifestation in the *traveling-salesperson problem* (TSP). The methods also divide between those that are designed to locate points and routes on networks and those designed to locate points and routes in continuous space without respect to the existence of roads or other transportation links.

14.4.1 Point Location

The MAT problem is an instance of location in continuous space and finds the location that minimizes total distance with respect to a number of points. The analogous problem on a network would involve finding that location on the network that minimizes total distance to a number of points, also located on the network, using routes that are constrained to the network. Figure 14.17 shows the contrast between continuous and network views, and Chapter 7 discusses data models for networks.

A very useful theorem first proved by Louis Hakimi reduces the complexity of many location problems on networks. Figure 14.17B shows a typical basis for network location. The links of the network come together in *nodes*. The weighted points are also located on the network and also form nodes. For example, the task might be to find the location that minimizes total distance to a distribution of customers, with all customers aggregated into these weighted points. The weights in this case would be counts of customers. The Hakimi theorem proves that for this problem of minimizing distance the only locations that have to be considered are the nodes; it is impossible for the optimum location to be anywhere else. It is easy to see why this should be so. Think of

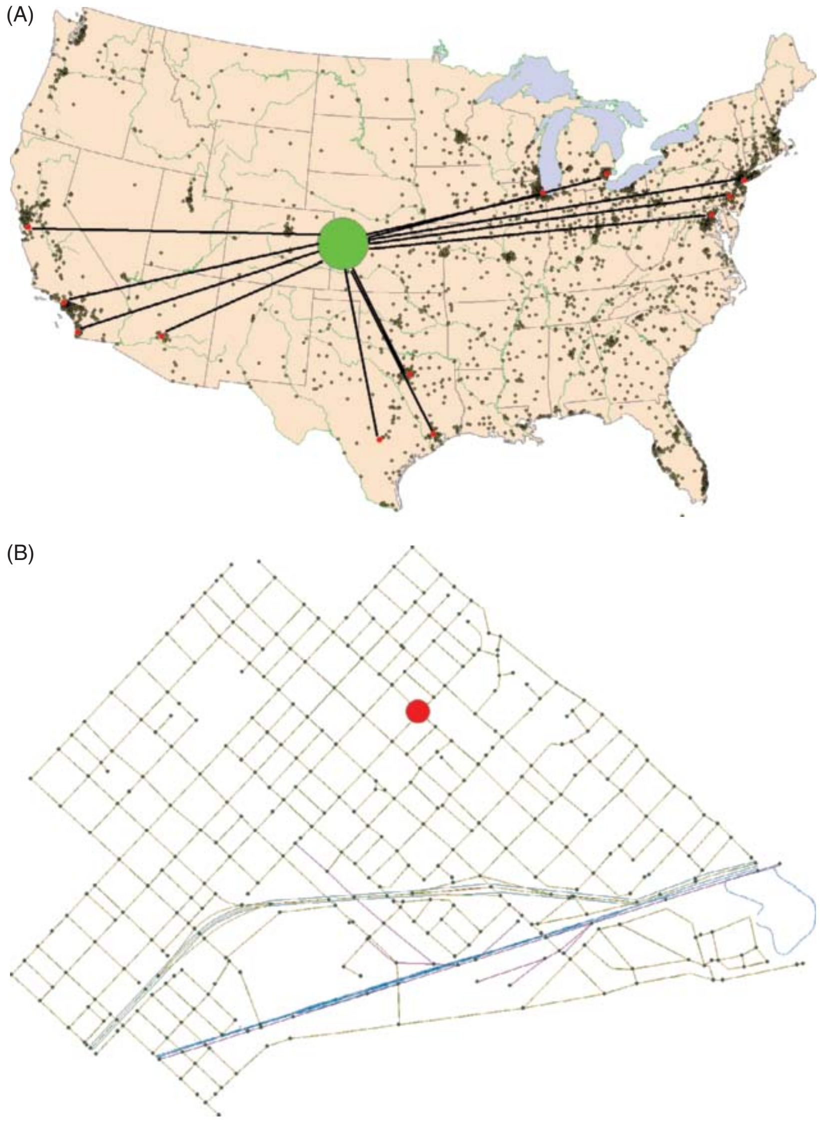


Figure 14.17 Search for the best locations for a central facility to serve dispersed customers. In (A) the problem is solved in continuous space, with straight-line travel, for a warehouse to serve the 12 largest U.S. cities. In continuous space there is an infinite number of possible locations for the site. In (B) a similar problem is solved at the scale of a city neighborhood on a network, where Hakimi's theorem states that only junctions (nodes) in the network and places where there is weight need to be considered, making the problem much simpler, but where travel must follow the street network.

a trial point located in the middle of a link, away from any node. Think of moving it slightly in one direction along the link. This moves it toward some weights and away from others, but every unit of movement results in the same increase or decrease in total weighted distance. In other words, the total distance traveled to the location is a linear function of the location along the link. Because the function is linear, it cannot have a minimum mid-link, so the minimum must occur at a node.

Optimum location problems can be solved in either discrete or continuous space, depending largely on scale.

The MAT problem on a network is known as the *1-median* problem, and the *p-median* problem seeks optimum locations for any number *p* of central facilities such that the sum of the distances between each weight and the *nearest* facility is minimized. A typical practical application of this problem is in the location of central public facilities, such as libraries, schools, or agency offices, when the objective is to locate for maximum total accessibility.

Many problems of this nature have been defined for different applications and implemented in GI systems. Whereas the median problems seek to minimize total distance, the *coverage* problems seek to minimize the *furthest* distance traveled, on the grounds that dealing with the worst case of accessibility is often more attractive than dealing with average accessibility. For example, it may make more sense to a city fire department to locate so that a response is possible to *every* property in less than five minutes, than to worry about minimizing the *average* response time. Coverage problems find applications in the location of emergency facilities, such as fire stations (Figure 14.18), where it is desirable that every possible emergency be covered within a fixed number of minutes of response time, or when the objective is to minimize the worst-case response time, to the furthest possible point.

All these problems are referred to as *location-allocation* problems because they involve two types of decisions: where to *locate* and how to *allocate* demand for service to the central facilities. A typical location-allocation problem might involve the selection of sites for supermarkets. In some cases the allocation of demand to sites is controlled by the designer, as it is in the case of school districts when students have no choice of schools. In other cases allocation is a matter of choice, and good designs depend on the ability to predict how consumers will choose among the available options. Models that make such predictions are known as *spatial interaction models*, and their use is an important application of GI systems in market research.

Location-allocation involves two types of decisions: where to locate and how to allocate demand for a service.

Figure 14.18 GI systems can be used to find locations for fire stations that result in better response times to emergencies.

14.4.2 Routing Problems

Point-location problems are concerned with the design of fixed locations. Another area of optimization is in routing and scheduling, or decisions about the optimum tracks followed by vehicles. A commonly encountered example is in the routing of delivery vehicles (Box 14.4). These examples show a base location, a depot that serves as the origin and final destination of delivery vehicles; and a series of stops that need to be made. There may be restrictions on the times at which stops must be made. For example, a vehicle delivering home appliances may be required to visit certain houses at certain times, when the residents are home. Vehicle routing and scheduling solutions are used by parcel delivery companies, school buses, on-demand public transport vehicles, and many other applications.

Underlying all routing problems is the concept of the *shortest path*—the path through the network between a defined origin and destination that minimizes distance or some other measure based on distance, such as travel time. Attributes associated with the network's links, such as length, travel speed, restrictions on travel direction, and level of congestion, are often taken into account. Many people are now familiar with the routine solution of the shortest path problem by Web sites such as Google Maps or MapQuest.com (Section 1.5.1), which solve many millions of such problems per day for travelers, and by similar methods to those used by in-vehicle navigation systems. They use standard algorithms developed decades ago, long before the advent of GI systems. The path that is strictly shortest is often not suitable because it involves too many turns or uses too many narrow streets, and algorithms will often be programmed to find longer routes that use faster highways, particularly freeways. Routes in Los Angeles, for example, can often be caricatured as (1) shortest route from origin to nearest freeway, (2) follow freeway network, and (3) shortest route from nearest freeway to destination, even though this route may be far from the shortest. The latest generation of route finders is also beginning to accommodate time of day, time

Applications Box 14.4

Routing Service Technicians for Sears

Sears manages one of the largest home-appliance repair businesses in the world, with six distinct geographic regions that include 50 independent districts. More than 10,000 technicians throughout the United States complete approximately 11 million in-home service orders each year. Decisions on dividing the daily orders among the company's technician teams, and on routing each team optimally, used to be made by dispatchers who relied on their own intuition and personal knowledge. But human intuition can be misleading and is lost when dispatchers are sick or retire.

Several years ago Sears teamed with Esri (Redlands, CA) to build the Computer-Aided Routing System (CARS) and the Capacity Area Management System (CAMS). CAMS manages the planned capacity of available service technicians assigned to geographic work areas, and CARS provides daily street-level geocoding and optimized routing for the mobile service technicians (Figure 14.19). The mobile Sears Smart Toolbox application provides service technicians with repair information for products, such as schematic diagrams. It also contains a GI system module for mobile mapping and routing, which gives in-vehicle navigation capabilities to assist in finding service locations and minimizing travel time. TeleAtlas ('s-Hertogenbosch, the Netherlands, now part of TomTom) provides the accurate street data that is critical for supporting geocoding and routing.

The system has had a major impact on Sears' business, saving the business tens of millions of dollars annually.

Source: Esri

Figure 14.19 Screenshot of the system used by drivers for Sears to schedule and navigate a day's workload.

of the week, and real-time information on traffic congestion.

The simplest routing problem with multiple destinations is the so-called traveling-salesperson problem, or TSP. In this problem there are a number of places that must be visited in a tour from the depot, and the distances between pairs of places are known. The problem is to select the best tour from all possible orderings, in order to minimize the total distance traveled. In other words, the optimum is to

Table 14.1 The number of possible tours in a traveling-salesperson problem.

Number of places to visit	Number of possible tours
3	1
4	3
5	12
6	60
7	360
8	2520
9	20160
10	181440

be selected out of the available tours. If there are *n* places to be visited, including the depot, then there are (*n*-1)! possible tours (the symbol ! indicates the product of the integers from 1 up to and including the number, known as the number's *factorial*). Because it is irrelevant, however, whether any given tour is conducted in a forward or backward direction, the effective number of options is (*n*-1)!/2. Unfortunately, this number grows very rapidly with the number of places to be visited, as Table 14.1 shows.

A GI system can be very effective at solving routing problems because it is able to examine vast numbers of possible solutions quickly.

The TSP is an instance of a problem that becomes quickly unsolvable for large *n*. Instead, designers adopt procedures known as *heuristics*, which are algorithms designed to work quickly and to come close to providing the best answer, although not guaranteeing that the best answer will be found. One not-very-good heuristic for the TSP is to proceed always to the closest unvisited destination (a so-called *greedy* approach) and finally to return to the start. Many spatial optimization problems, including location-allocation and routing problems, are solved today by the use of sophisticated heuristics. Because of the complexity of road networks, for example, route finders will often use a variant of the A* algorithm, a heuristic, rather than exhaustively examine every possible route.

14.5 Hypothesis Testing

This last section reviews a major area of statistics—the testing of hypotheses and the drawing of inferences—and its relationship to GI science and spatial analysis. Much work in statistics is *inferential*; that is, it uses information obtained from samples to make general conclusions about a larger population, on the assumption that the sample came from that population. The concept of inference was introduced in Section 2.4 as a way of reasoning about the properties of a larger group from the properties of a sample. At that point several problems associated with inference from geographic data were raised. This section revisits and elaborates on that topic and discusses the particularly thorny issue of spatial hypothesis testing.

For example, suppose we were to take a random and independent sample of 1,000 people and ask them how they might vote in the next U.S. Presidential election. By *random and independent*, we mean that every person of voting age in the general population has an equal chance of being chosen and that the choice of one person does not make the choice of any others—parents, neighbors—more or less likely. Suppose that 45% of the sample said they would support Hillary Clinton. Statistical theory then allows us to give 45% as the best estimate of the proportion who would vote for Clinton among the *general* population, and it also allows us to state a *margin of error*, or an estimate of how much the true proportion among the population will differ from the proportion among the sample. A suitable expression of margin of error is given by the 95% confidence limits, or the range within which the true value is expected to lie 19 times out of 20. In other words, if we took 20 different samples, all of size 1000, there would be a scatter of outcomes, and 19 out of 20 of them would lie within these 95% confidence limits. In this case a simple analysis using the *binomial distribution* shows that the 95% confidence limits are 3%; in other words, 19 times out of 20 the true proportion lies between 42 and 48%.

This example illustrates the *confidence limits* approach to inference, in which the effects of sampling are expressed in the form of uncertainty about the properties of the population. An alternative that is commonly used in scientific reasoning is the *hypothesis-testing* approach. In this case our objective is to test some general statement about the population— for example, that 50% will support Clinton in the next election (and 50% will support the other candidate—in other words, there is no real preference in the electorate). We take a sample, and we then ask whether the evidence from the sample supports the general statement. Because there is uncertainty associated with any sample, unless it includes the entire population, the answer is never absolutely certain. In this example and using our confidence limits approach, we know that if 45% were found to support Clinton in the sample, and if the margin of error was 3%, it is highly unlikely that the true proportion in the population is as high as 50%. Alternatively, we could state the 50% proportion in the population as a *null hypothesis* (we use the term *null* to reflect the absence of something, in this case a clear choice) and determine how frequently a sample of 1,000 from such a population would yield a proportion as low as 45%. Again the answer is very small; in fact, the probability is 0.0008. But it is not zero, and its value represents the chance of making an error of inference—of rejecting the hypothesis when in fact it is true.

Methods of inference reason from information about a sample to more general information about a larger population.

These two concepts—confidence limits and inferential tests—are the basis for statistical testing and form the core of introductory statistics texts. There is no point in reproducing those introductions here, and the reader is simply referred to them for discussions of the standard tests—*F*, *t*, χ^2, and so on. The focus here is on the problems associated with using these approaches with geographic data in a GI science context. The next section reviews the inferential tests associated with one popular descriptive statistic for spatial data, the Moran index of spatial dependence that was discussed in Section 2.7. The following section discusses the general issues and points to ways of resolving them.

14.5.1 Hypothesis Tests on Geographic Data

Although inferential tests are standard practice in much of science, they are very problematic for geographic data. The reasons have to do with fundamental properties of geographic data, many of which were introduced in Chapter 2, and others have been encountered at various stages in this book.

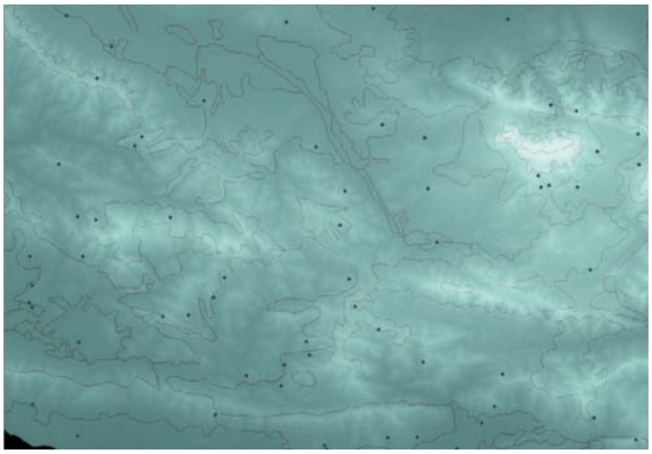
Many inferential tests propose the existence of a population, from which the sample has been obtained by some well-defined process. We saw in Section 2.4 how difficult it is to think of a geographic dataset as a sample of all of the datasets that might have been. It is equally difficult to think of a dataset as a sample of some larger area of the Earth's surface, for two major reasons.

First, the samples in standard statistical inference are obtained independently (Section 2.4). But a geographic dataset is often *all there is* in a given area—it *is* the population. Perhaps we could regard a dataset as a sample of a larger area. But in this case the sample would not have been obtained randomly; instead, it would have been obtained by systematically selecting all cases within the area of interest. Moreover, the samples would not have been independent. Because of spatial dependence, which we have understood to be a pervasive property of geographic data (Chapter 2), it is very likely that there will be similarities between neighboring observations.

A GIS project often analyzes all the data there is about a given area, rather than a sample.

Figure 14.20 shows a typical instance of geographic sampling. In this case the objective is to explore the relationship between topographic elevation and vegetation cover class, based on a DEM and a map of vegetation, in the area north of Santa Barbara. We might suspect, for example, that certain types of vegetation are encountered only at higher altitudes. One way to do this would be by randomly sampling at a set of points, recording the elevation and vegetation cover class at each, and then examining the results in a routine statistical analysis. In Figure 14.20, roughly 50 sample points are displayed. But why 50? If we increased the number to 500, our statistical test would have more data, and consequently more power to detect and evaluate any relationship. But why stop at 500? Here as in Box 13.4 the geographic case appears to create the potential for endless proliferation of data. Tobler's First Law (Sections 2.2 and 13.3.4) tells us, however, that after a while additional data values will not be truly informative because they could have been predicted from previously sampled values. We cannot have it both ways—if we believe in spatial interpolation (Section 13.3.6), we cannot at the same time believe in independence of geographic samples, despite the fact that this is a basic assumption of statistical tests. In effect, a geographic area can only yield a limited number of truly independent samples, from points spaced sufficiently far apart. Beyond that number, the new samples are not truly independent and in reality add nothing to the power of a test.

Figure 14.20 A randomly placed sample of points used to examine the relationship between vegetation cover class (delimited by the boundaries shown) and elevation (whiter areas are higher), in an area north of Santa Barbara.



Finally, the issue of spatial *heterogeneity* (Section 2.2) also gets in the way of inferential testing. The Earth's surface is highly variable, and there is no such thing as an average place on it. The characteristics observed on one map sheet are likely to be substantially different from those on other map sheets, even when the map sheets are neighbors. So the census tracts of a city are certainly not acceptable as a random and independent sample of all census tracts, even the census tracts of an entire nation. They are not independent, and they are not random. Consequently, it is very risky to try to infer the properties of *all* census tracts from the properties of all the tracts in any *one* area. The concept of sampling, which is the basis for statistical inference, does not transfer easily to the spatial context.

The Earth's surface is very heterogeneous, making it difficult to take samples that are truly representative of any large region.

Before using inferential tests on geographic data, therefore, it is advisable to ask two fundamental questions:

- Can I conceive of a larger *population* that I want to make inferences about?
- Are my data acceptable as a *random* and an *independent* sample of that population?

If the answer to either of these questions is *no*, then inferential tests are not appropriate.

Given these arguments, what options are available? One strategy that is sometimes used is to discard data until the proposition of independence becomes acceptable—until the remaining data points are so far apart that they can be regarded as essentially independent. But no scientist is happy throwing away data.

Another approach is to abandon inference entirely. In this case the results obtained from the data are descriptive of the study area, and no attempt is made to generalize. This approach, which uses local statistics to observe the *differences* in the results of analysis over space, represents an interesting compromise between the nomothetic and idiographic positions outlined in Section 1.3. Generalization is very tempting, but the heterogeneous nature of the Earth's surface makes it very difficult. If generalization is required, then it can be accomplished by appropriate experimental design—by replicating the study in a sufficient number of distinct areas to warrant confidence in a generalization.

Another, more successful approach exploits the special nature of spatial analysis and its concern with detecting pattern. Consider the example in Figure 13.21, where the Moran index was computed at +0.4011, an indication that high values tend to be surrounded by high values and low values by low values—a positive spatial autocorrelation. It is reasonable to ask whether such a value of the Moran index could have arisen by chance because even a random arrangement of a limited number of values typically will not give the theoretical value of 0 corresponding to no spatial dependence (actually the theoretical value is very slightly negative). In this case 51 values are involved, arranged over the 51 features on the map (the 50 states plus the District of Columbia). If the values were arranged randomly, how far would the resulting values of the Moran index differ from 0? Would they differ by as much as 0.4011? Intuition is not good at providing answers.

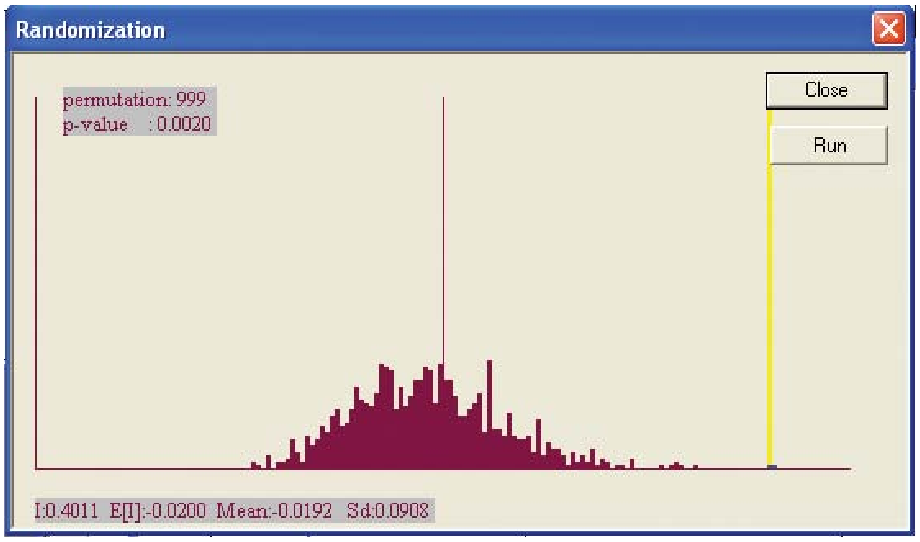
In such cases a simple test can be run by simulating random arrangements. The software used to prepare this illustration includes the ability to make such simulations, and Figure 14.21 shows the results of simulating 999 random rearrangements. It is clear that the actual value is well outside the range of what is possible for random arrangements, leading to the conclusion that the apparent spatial dependence is real.

How does this *randomization* test fit within the normal framework of statistics? The null hypothesis being evaluated is that the distribution of values over the 51 features is random, each feature receiving a value that is independent of neighboring values. The population is the set of all possible arrangements, of which the data represent a sample of one. The test then involves comparing the test statistic—the Moran Index—for the actual pattern against the distribution of values produced by the null hypothesis. So the test is a perfect example of standard hypothesis testing, adapted to the special nature of spatial data and the common objective of discovering pattern.

Randomization tests are uniquely adapted to testing hypotheses about spatial pattern.

Finally, a large amount of research has been devoted to devising versions of inferential tests that cope effectively with spatial dependence and spatial heterogeneity. Software that implements these tests is now widely available, and interested readers are urged to consult the appropriate sources. GeoDa is an excellent comprehensive software environment for such tests (available via geodacenter.asu.edu), and many of these methods are available as extensions of standard GI systems.

Figure 14.21 Randomization test of the Moran Index computed in Figure 13.20. The histogram shows the results of computing the index for 999 rearrangements of the 51 values on the map. The yellow line on the right shows the actual value, which is very unlikely to occur in a random arrangement, reinforcing the conclusion that there is positive spatial dependence in the data.



14.6 Conclusion

This chapter has covered the conceptual basis of many of the more sophisticated techniques of spatial analysis that are available in GI systems. Box 14.5 is devoted to Doug Richardson, Executive Director of the Association of American Geographers, who has done much to promote spatial analysis and spatial technologies at the U.S. National Institutes of Health and has himself developed and exploited many new

Biographical Box 14.5

Doug Richardson, A Leader in Academic Geography

Doug Richardson (Figure 14.22) is the Executive Director of the Association of American Geographers (AAG). During the past ten years, he has led a highly successful organizational renewal of the AAG and has built strong academic, research, publishing, and financial foundations for the organization's future.

Prior to joining the AAG, Doug founded and for 18 years was the president of GeoResearch, Inc., a private-sector scientific research company specializing in geographic science and technology, including GISS, spatial modeling, and GPS. GeoResearch developed and patented the world's first real-time interactive GPS/GIS technologies, leading to far-reaching changes in the ways in which geographic information is collected, mapped, integrated, and used within geography, as well as in society at large. The technologies and methods pioneered by GeoResearch are now at the heart of a wide array of real-time interactive mapping, navigation, location-based business, geographic research, mobile computing, military operations, and large-scale operations management applications of most major industries and governments. Doug sold his company and its core patents in 1998.

Courtesy: Doug Richardson

Figure 14.22 Doug Richardson.

Doug continues to conduct research at the AAG and to publish across multiple dimensions of geography, ranging from GI science to the GeoHumanities, and from international health research to interactions between science, innovation, and human rights. As lead author of a 2013 *Science* article entitled "Spatial Turn in Health Research," for example, he explores how geographic science and technologies are opening profound new possibilities for understanding the prevalence, diffusion, and causes of disease, and its treatment. He holds a Bachelor's degree from the University of Michigan and a PhD in Geography from Michigan State University. He currently is a member of the U.S. National Geospatial Advisory Committee.

technological developments in the areas of GI systems and GPS.

The last section of the chapter raised some fundamental issues associated with applying methods and theories that were developed for nonspatial data to the spatial case. Spatial analysis is clearly not a simple and straightforward extension of nonspatial analysis, but instead raises many distinct problems, as well as some exciting opportunities. The two chapters on spatial analysis have only scratched the surface of this large and rapidly expanding field.

Questions for Further Study

1. Parks and other conservation areas have geometric shapes that can be measured by comparing park perimeter length to park area, using the methods reviewed in this chapter. Discuss the implications of shape for park management, in the context of (a) wildlife ecology and (b) neighborhood security.
2. What exactly are *multicriteria* methods? Examine one or more of the methods in the Eastman chapter referenced in Further Reading, summarizing the issues associated with (a) measuring variables to support multiple criteria, (b) mixing variables that have been measured on different scales (e.g., dollars and distances), and (c) finding solutions to problems involving multiple criteria.
3. Besides being the basis for useful summary measures, fractals also provide interesting ways of simulating geographic phenomena and patterns. Browse the Web for sites that offer fractal simulation software, or investigate one of many commercially available packages. What other uses of fractals in GI science can you imagine?
4. Every point on the Earth's surface has an antipodal point—the point that would be reached by drilling an imaginary hole straight through the Earth's center. Britain, for example, is approximately antipodal to New Zealand. If one-third of the Earth's surface is land, you might expect that one-third of all of the land area would be antipodal to points that are also on land, but a quick look at an atlas will show that the proportion is actually far less than that. In fact, the only substantial areas of land that have antipodal land are in South America (and their antipodal points in China). How is spatial dependence relevant here, and why does it suggest that the Earth is not so surprising after all?

Further Reading

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