ECE 657 Assignment 4
Group 47

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Problem 1. Suppose that the state of "fast speed" of a machine is denoted by the fuzzy set F with membership function $\mu_F(v)$. Then the state of "very fast speed", where the *linguistic hedge* "very" has been incorporated, may be represented by $\mu_F(v\text{-}v_o)$ with $v_o > 0$. Also, the state "presumably fast speed", where the linguistic hedge "presumably" has been incorporated, may be represented by $\mu_F^2(v)$.

(a) Discuss the appropriateness of the use of these membership functions to represent the respective linguistic hedges.

(b) In particular, if

$$F = \{\frac{0.1}{10}, \frac{0.3}{20}, \frac{0.6}{30}, \frac{0.8}{40}, \frac{1.0}{50}, \frac{0.7}{60}, \frac{0.5}{70}, \frac{0.3}{80}, \frac{0.1}{90}\}$$

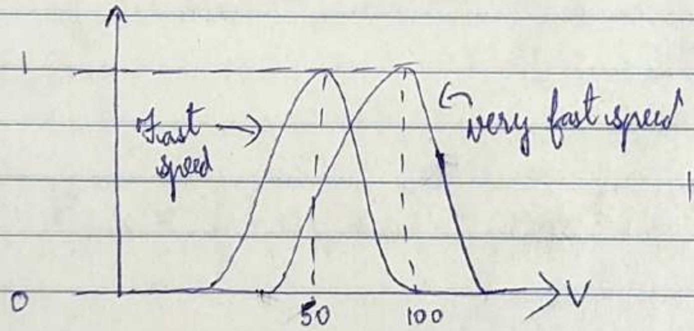
in the discrete universe $V = \{0, 10, 20, \ldots, 190, 200\}$ rev/s and $v_o = 50$ rev/s, determine the membership functions of "very fast speed" and "presumably fast speed". Display both membership functions over the discrete Universe V.

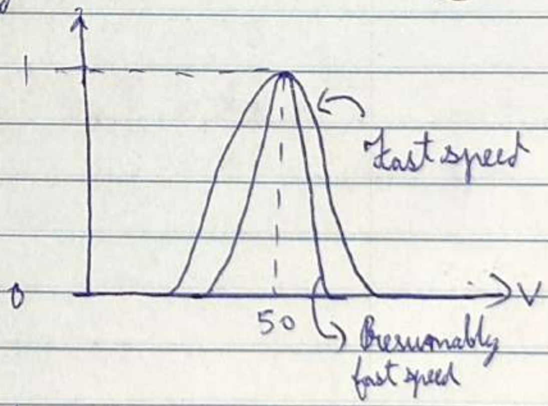
Assignment 4

Problem 1 Solution

a) i) "Very Fast speed"

~~The linguistic hedge "very" amplifies the speed state of the original membership function represented by fuzzy state "Fast speed".~~
By subtracting V_0 from V, we are shifting the membership function to the right, which means that the "very fast" state starts later than the original "fast" state. ~~So if we assume~~ When a membership function is moved to the right it justifies ~~as~~ on the linguistic hedge "very" as it includes membership of higher V values.





ii) "Presumably Fast speed"

The linguistic hedge "Presumably" amplifies the degree of the original membership function represented by state "Fast speed". By squaring we are contracting the membership function. As "presumably" means more probably in linguistic term so by contracting the funtion we ~~amplify the probab~~ are more sure about the memberships & more ~~an~~ sure / probable cases are taken care b)

$V_0 = 50$ revels

V	$V - V_0$ $(V-50)$	$\mu_F(V)$	$\mu_F(V-V_0)$	$\mu_F^2(V)$
0	~~-40~~ -50	0	0	0
10	-40	0.1	0	0.01
20	-30	0.3	0	0.09
30	-20	0.6	0	0.36
40	-10	0.8	0	0.64
50	0	1.0	0	1.0
60	10	0.7	0.1	0.49
70	20	0.5	0.3	0.25
80	30	0.3	0.6	0.09
90	40	0.1	0.8	0.01
100	50	0	1.0	0
110	60	0	0.7	0
120	70	0	0.5	0
130	80	0	0.3	0
140	90	0	0.1	0
150	100	0	0	0
160	110	0	0	0
170	120	0	0	0
180	130	0	0	0
190	140	0	0	0
200	150	0	0	0

-50, -40, -30, -20, -10 speeds are out of universe V so their membership function is 0.

"Very Fast speed" $= \mu_{VF} = \mu_F(v - v_0) = \left\{ \frac{0.1}{60}, \frac{0.3}{70}, \frac{0.6}{80}, \frac{0.8}{90}, \frac{1.0}{100}, \frac{0.7}{110}, \frac{0.5}{120}, \frac{0.3}{130}, \frac{0.1}{140} \right\}$

"Presumably Fast speed" $= \mu_{PF} = \mu_F^2(v) = \left\{ \frac{0.01}{10}, \frac{0.09}{20}, \frac{0.036}{30}, \frac{0.64}{40}, \frac{1.0}{50}, \frac{0.49}{60}, \frac{0.25}{70}, \frac{0.09}{80}, \frac{0.}{9} \right.$

Problem 2. Sketch the membership function $\mu_A(x) = e^{-\lambda(x-a)^n}$ for $\lambda = 2$, $n = 2$, and $a = 3$ for the support set $S = [0,6]$. On this sketch separately show the shaded areas that represent the following fuzziness measures given by M_1, M_2 and M_3:

(a) $M_1 = \int_S f(x)dx$ where $\begin{aligned} f(x) &= \mu_A(x) && \text{for } \mu_A(x) \leq 0.5 \\ &= 1 - \mu_A(x) && \text{for } \mu_A(x) > 0.5 \end{aligned}$

(b) $M_2 = \int_S \left| \mu_A(x) - \mu_{A_{1/2}}(x) \right| dx$

where $\mu_{A_{1/2}}$ is the α-cut of $\mu_A(x)$ for $\alpha = 1/2$

(c) $M_3 = \int_S \left| \mu_A(x) - \mu_{\overline{A}}(x) \right| dx$

where $\overline{A}$ is the complement of the fuzzy set A. Evaluate the values of M_1, M_2 and M_3 for the given membership function.

i. Establish relationships between M_1, M_2 and M_3.

ii. Indicate how these measures can be used to represent the degree of fuzziness of a membership function.

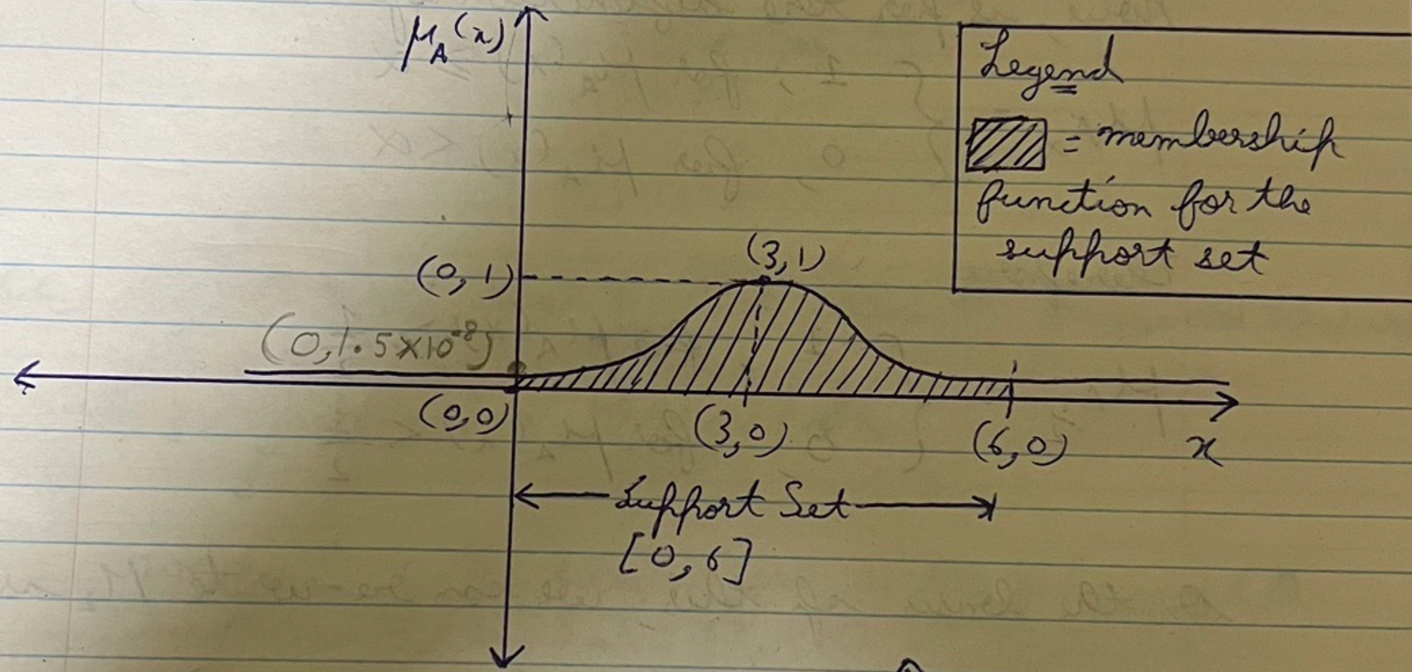
iii. Compare your results with the case $\lambda = 1$, $a = 3$, and $n = 2$ for the same support set, by showing the corresponding fuzziness measures on a sketch of the new membership function.

PROBLEM 2

* Membership function for $\mu_A(x) = e^{-\lambda(x-a)^n}$

where, $\lambda = 2$, $n = 2$ and $a = 3$

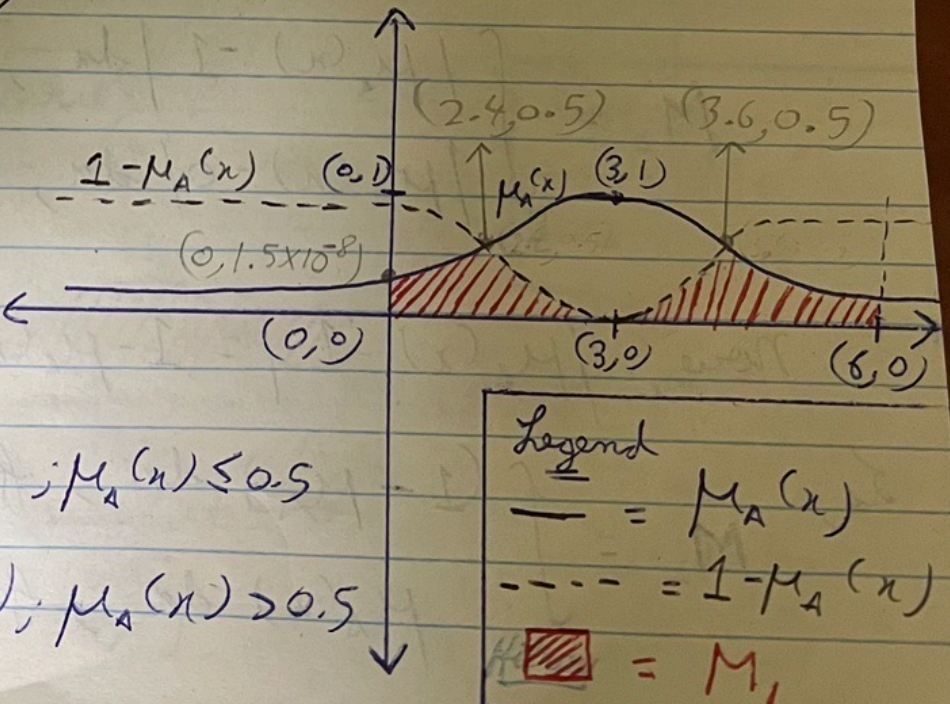
In other words, $\mu_A(x) = e^{-2(x-3)^2}$



(a) On this sketch, let's show the fuzziness measure

$M_1 = \int_S f(x)\,dx$

where $f(x) = \begin{cases} \mu_A(x) & ; \mu_A(x) \le 0.5 \\ 1-\mu_A(x) & ; \mu_A(x) > 0.5 \end{cases}$



(b) On this sketch; let's show the fuzziness measure

$$M_2 = \int_S \left| \mu_A(x) - \mu_{A_{1/2}}(x) \right| dx$$

where $\mu_{A_{\frac{1}{2}}}$ is the α - cut of $\mu_A(x)$ for $\alpha = \frac{1}{2}$

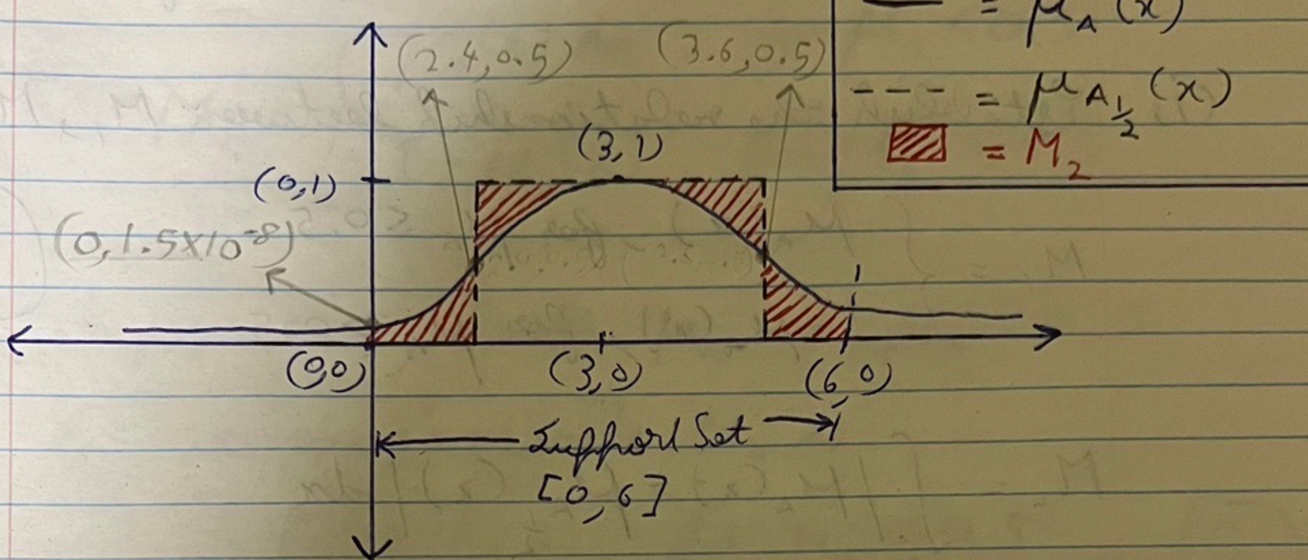
now; as per the definition of α-cut;

$$\mu_{A_\alpha} = \begin{cases} 1; & \text{for } \mu_A(x) \geq \alpha \\ 0; & \text{for } \mu_A(x) < \alpha \end{cases}$$

Therefore;

$$\mu_{A_{\frac{1}{2}}} = \begin{cases} 1; & \text{for } \mu_A(x) \geq \frac{1}{2} \\ 0; & \text{for } \mu_A(x) < \frac{1}{2} \end{cases}$$

Let's sketch M_2 now;



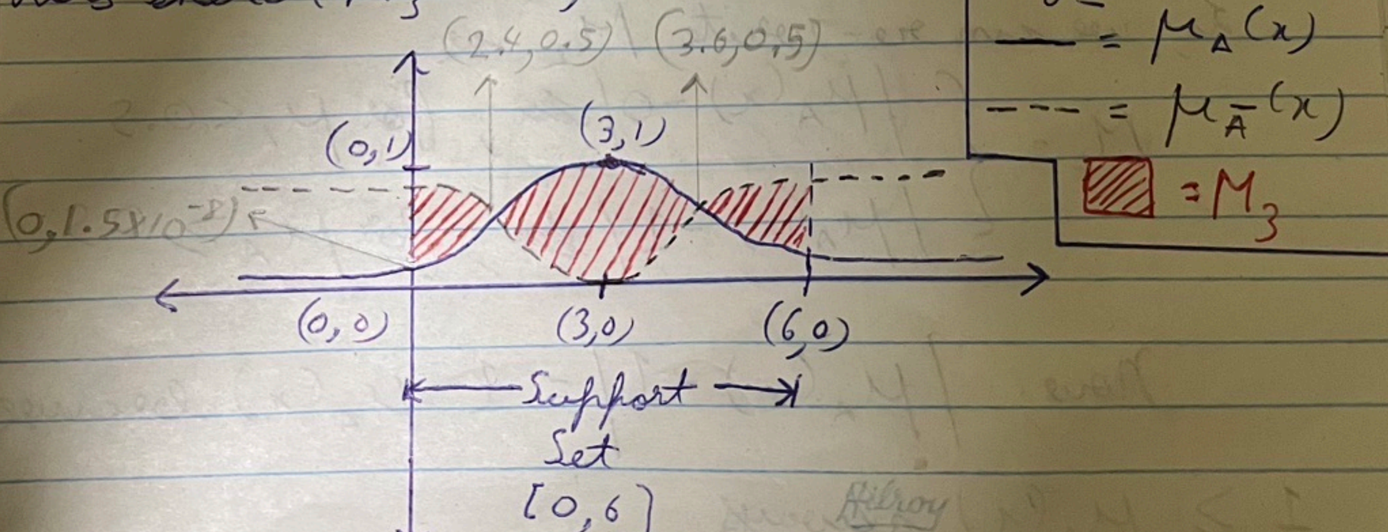
(c) On the next sketch; let's show the fuzziness measure

$$M_3 = \int_S \left| \mu_A(x) - \mu_{\bar{A}}(x) \right| dx$$

where $\bar{A}$ is the complement of the fuzzy set A

Therefore; $\mu_{\bar{A}}(x) = 1 - \mu_A(x)$

Let's sketch M_3 now;



Now, let's move on to the next parts of the question

(i) $\boxed{\text{Establish the relationship between } M_1, M_2 \text{ and } M_3}$

$$M_1 = \begin{cases} \mu_A(x) \ ; \text{ for } \mu_A \leq 0.5 \\ 1-\mu_A(x); \text{ for } \mu_A > 0.5 \end{cases} \quad \text{—— (1)}$$

$$M_2 = \int_S |\mu_A(x) - \mu_{A_{\frac{1}{2}}}(x)|\, dx$$

Now, as per the definition of α-cut,

$$\mu_{A_\alpha} = \begin{cases} 0 \ ; \text{ for } \mu_A < \alpha \\ 1 \ ; \text{ for } \mu_A \geq \alpha \end{cases}$$

Therefore,

$$\mu_{A_{\frac{1}{2}}} = \begin{cases} 0 \ ; \text{ for } \mu_A < \frac{1}{2} \ (\text{or } \mu_A < 0.5) \\ 1 \ ; \text{ for } \mu_A \geq \frac{1}{2} \ (\text{or } \mu_A \geq 0.5) \end{cases}$$

So, we can re-write M_2 as

$$M_2 = \begin{cases} |\mu_A(x) - 0| \ ; \text{ for } \mu_A < 0.5 \\ |\mu_A(x) - 1| \ ; \text{ for } \mu_A \geq 0.5 \end{cases}$$

Now, $|\mu_A(x) - 1| = 1 - \mu_A(x)$ because

$1 \geq \mu_A(x)$ always.

and; $\left| \mu_A(x) - 0 \right| = \mu_A(x)$

So:

$$M_2 = \begin{cases} \mu_A(x) & ; \text{ for } \mu_A < 0.5 \\ 1 - \mu_A(x) & ; \text{ for } \mu_A \geq 0.5 \end{cases}$$

At $\mu_A = 0.5$; both $\mu_A = 1 - \mu_A = 0.5$

So; we can re-write M_2 as

$$M_2 = \begin{cases} \mu_A(x) & ; \text{ for } \mu_A \leq 0.5 \\ 1 - \mu_A(x) & ; \text{ for } \mu_A > 0.5 \end{cases} \quad \text{— } 2$$

After comparing 1 and 2

$$M_1 = M_2 = \begin{cases} \mu_A(x) & ; \text{ for } \mu_A \leq 0.5 \\ 1 - \mu_A(x) & ; \text{ for } \mu_A > 0.5 \end{cases}$$

OR

$$\boxed{M_1 = M_2} \quad \text{— } 3$$

Now; $M_3 = \int_S |\mu_A(x) - \mu_{\bar{A}}(x)|\, dx$

$\bar{A}$ is the complement of the fuzzy set A

Therefore; $\mu_{\bar{A}}(x) = 1 - \mu_A(x)$

Let's re-write M_3

$$M_3 = \int_S |\mu_A(x) - (1 - \mu_A(x))|\, dx$$

$$M_3 = \int_S |\mu_A(x) - 1 + \mu_A(x)|\, dx$$

$$M_3 = \int_S |2\mu_A(x) - 1|\, dx$$

or;

$$M_3 = 2\int_S |\mu_A(x) - \tfrac{1}{2}|\, dx = 2\int_S |\mu_A(x) - 0.5|\, dx$$

now; $|\mu_A(x) - 0.5| = \mu_A(x) - 0.5$ if; $\mu_A(x) - 0.5 \geq 0$

or $\mu_A(x) \geq 0.5$

and; $|\mu_A(x) - 0.5| = 0.5 - \mu_A(x)$ if; $\mu_A(x) - 0.5 \leq 0$

or; if $\mu_A(x) \leq 0.5$

So, we can re-write M_3 as,

$$M_3 = \begin{cases} 2(0.5 - \mu_A(x)) & ; \text{ for } \mu_A \leq 0.5 \\ 2(\mu_A(x) - 0.5) & ; \text{ for } \mu_A > 0.5 \end{cases}$$

$$M_3 = \begin{cases} 1 - 2\mu_A(x) & ; \text{ for } \mu_A \leq 0.5 \\ 2\mu_A(x) - 1 & ; \text{ for } \mu_A > 0.5 \end{cases}$$

$$M_3 = \begin{cases} 1 - 2\mu_A(x) & ; \text{ for } \mu_A \leq 0.5 \\ -1 + 2\mu_A(x) & ; \text{ for } \mu_A > 0.5 \end{cases}$$

Now, $-1 + 2\mu_A(x)$ can be written as $(1-2) + 2\mu_A(x)$

or, $1 - 2 + 2\mu_A(x)$; or, $1 - 2(1 - \mu_A(x))$

So, $$M_3 = \begin{cases} 1 - 2(\mu_A(x)) & ; \text{ for } \mu_A \leq 0.5 \\ 1 - 2(1 - \mu_A(x)) & ; \text{ for } \mu_A > 0.5 \end{cases}$$

Now, as per 1 ; $M_1 = \mu_A(x)$ for $\mu_A \leq 0.5$
and, $M_1 = (1 - \mu_A(x))$ for $\mu_A > 0.5$

So, $$M_3 = \begin{cases} 1 - 2M_1 & ; \text{ for } \mu_A \leq 0.5 \\ 1 - 2M_1 & ; \text{ for } \mu_A > 0.5 \end{cases}$$

<u>OR</u> $M_3 = 1 - 2M_1$

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$$2M_1 = 1 - M_3$$

$$\boxed{M_1 = \frac{1}{2}\left[1 - M_3\right]} \quad \text{— } 4$$

Therefore; on the basis of 3 and 4 the relationship between M_1, M_2 and M_3 is

$$\boxed{M_1 = M_2 = \frac{1}{2}\left[1 - M_3\right]}$$

for the support set S

Indicate how these measures can be used to represent the degree of fuzziness of a membership function

(ii) All these fuzziness measures M_1, M_2 and M_3 are various forms of representing the magnitude of fuzziness of a membership function.

- $M_1 \Rightarrow$ Closeness to grade 0.5
- $M_2 \Rightarrow$ Distance from $\frac{1}{2}$-cut
- $M_3 \Rightarrow$ Inverse of distance from the complement

All these measures help to express the uncertainty & vagueness of the membership function.

The higher the uncertainity / vagueness, the higher the degree of fuzziness, and vice versa.

That's how all these measures help demonstrate the degrees of fuzziness.

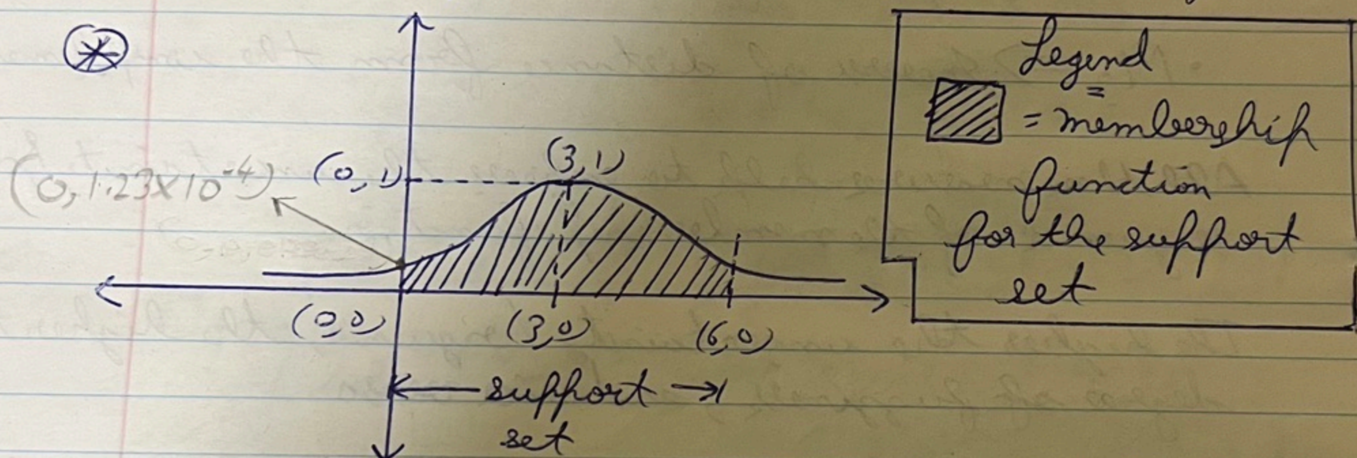
Compare your results with $\lambda = 1$, $a = 3$, $n = 2$ for the same support set by showing the corresponding measures on a sketch of the new membership function

(iii) With new values $\lambda = 1$ $a = 3$ and $n = 2$, our membership function becomes;

$$\mu_A(x) = e^{-(x-3)^2}$$

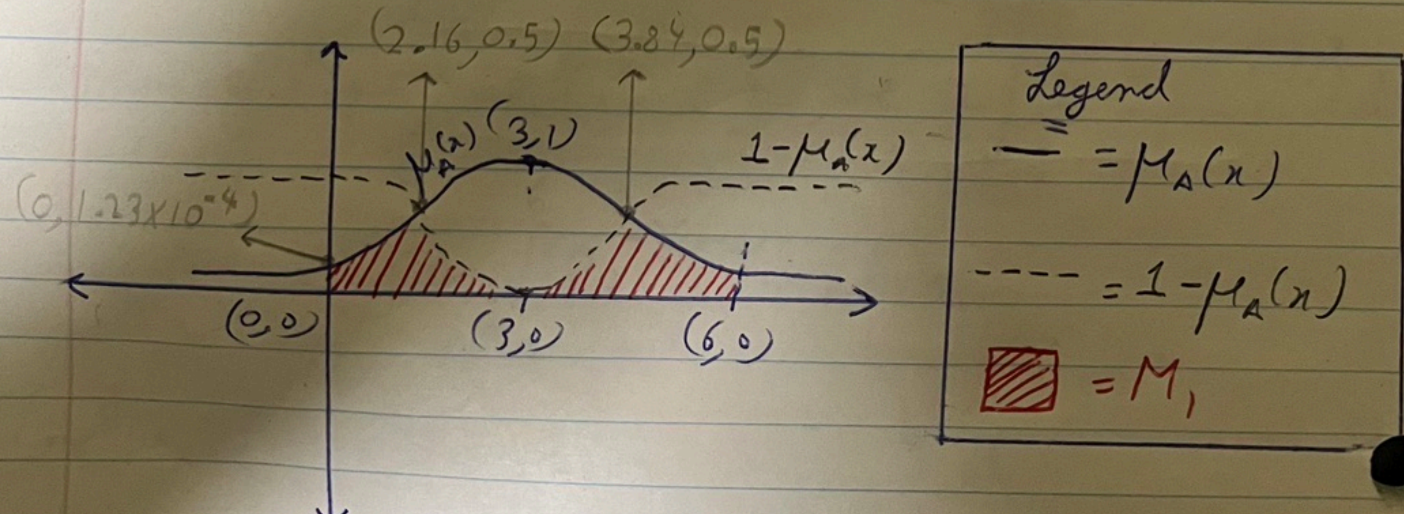
If we plot this function and M_1, M_2 and M_3, there is not going to be a big visual difference because the nature of this function is same. except, a few co-ordinates would change (pencil)

(*)

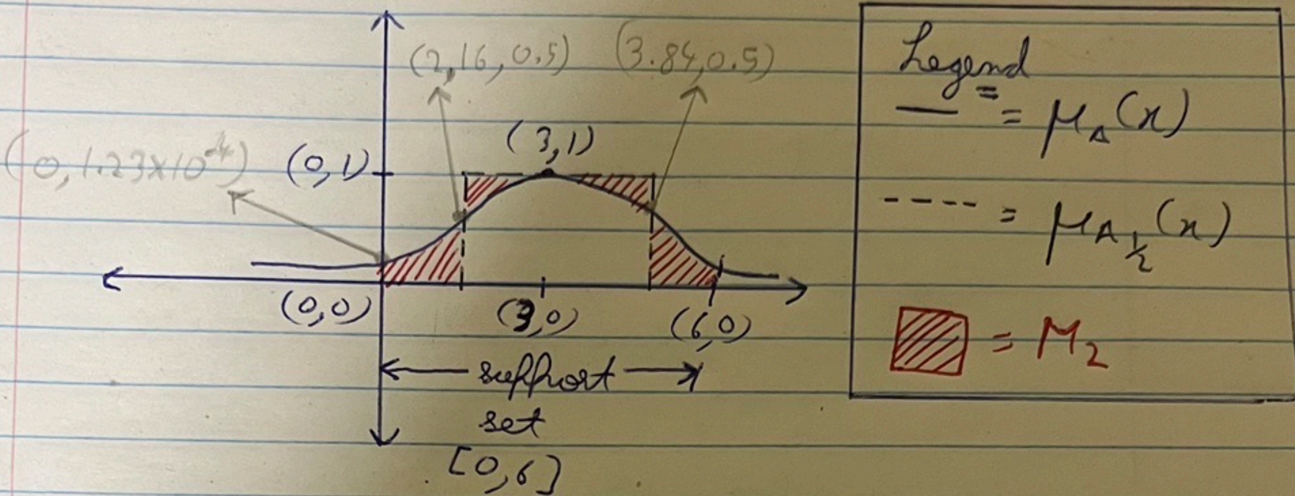


(*)

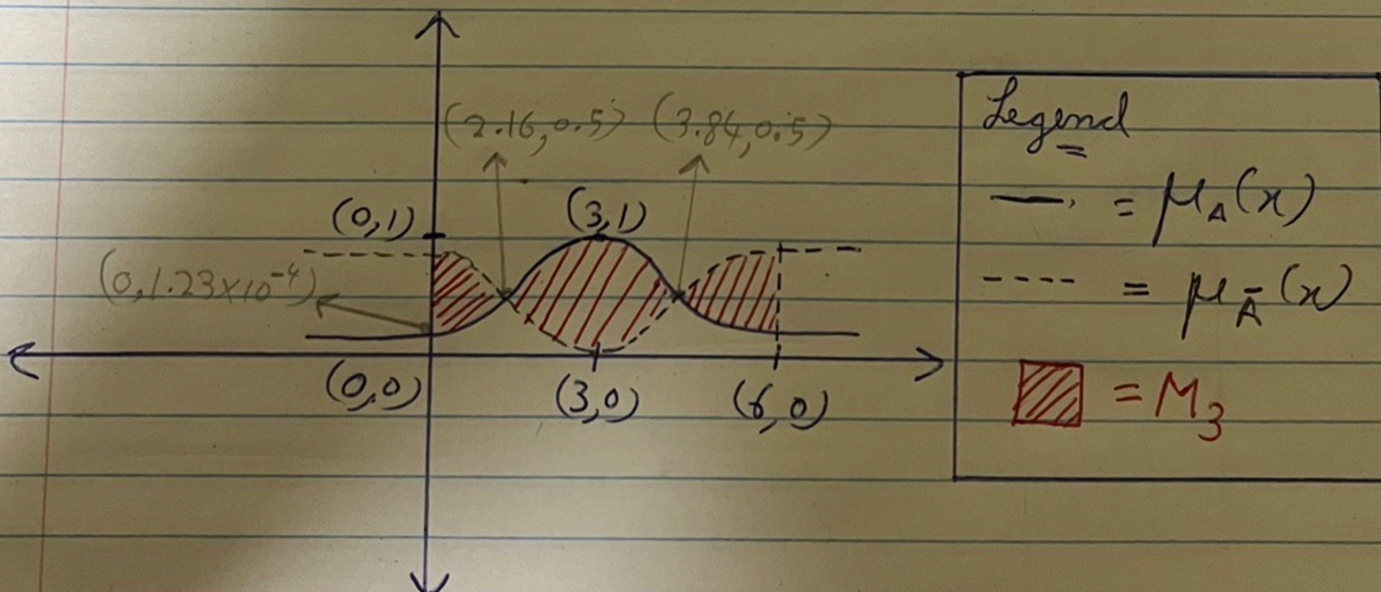
$$M_1 = \int_S f(x)\,dx \text{ where; } f(x) = \begin{cases} \mu_A(x); \text{ for } \mu_A \leq 0.5 \\ 1-\mu_A(x); \text{ for } \mu_A > 0.5 \end{cases}$$



$$\circledast \quad M_2 = \int_S \left| \mu_A(x) - \mu_{A_{\frac{1}{2}}}(x) \right| dx$$



$$\circledast \quad M_3 = \int_S \left| \mu_A(x) - \mu_{\bar{A}}(x) \right| dx$$



As we can see that the sketches are very similar to the first set of sketches except certain co-ordinates denoted by pencil.

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Question 3

Problem 3. The characteristic function χ_A of a crisp set A is analogous to the membership function of a fuzzy set, and is defined as follows:

$$\chi_A(x) = 1 \text{ if } x \in A$$
$$= 0 \text{ otherwise}$$

Using isomorphism of crisp sets and binary logic, show that

$$\chi_{A'} = 1 - \chi_A$$ proof $A' = 1 - A$

$$\chi_{A\cup B} = \max(\chi_A, \chi_B)$$

$$\chi_{A\cap B} = \min(\chi_A, \chi_B)$$

$$\chi_{A\to B}(x,y) = \min\left[1, \{1-\chi_A(x)+\chi_B(y)\}\right]$$ → different universe

where A and B are defined in the same universe X, except in the last case (implication) where A and B may be defined in two different universes X and Y.

What are the implications of these results?

Characteristic Function $\mathcal{X}_A$

A = Crisp set

Crisp	Fuzzy
Charateristic Function	Membership Function

$$\mathcal{X}_A(x) = \left\{ \begin{array}{ll} 1, & x \in A \\ 0, & \text{otherwise} \end{array} \right\}$$

Proof

$\mathcal{X}_{A'}(x)$ = the negation of the characteristic function of $\mathcal{X}_A(x)$.

ⓐ $\mathcal{X}_{A'} = 1 - \mathcal{X}_A$

Characteristic function of $A' = \mathcal{X}_{A'}(x)$

$$\mathcal{X}_{A'}(x) = \left\{ \begin{array}{ll} 1, & x \notin A \\ 0 & x \in A \end{array} \right\}$$

$\boxed{\text{Case 1:}}$

for an element x as a member of A

$\mathcal{X}_A(x) = 1$ by definition of $\mathcal{X}_A$

$\mathcal{X}_{A'}(x) = 0$ for same element x

$\therefore$ Since $\mathcal{X}_A(x) = 1$ and $\mathcal{X}_{A'}(x) = 0$

$$1 - \mathcal{X}_A(x) = 1 - 1 = 0$$

$\therefore \mathcal{X}_{A'}(x) = 1 - \mathcal{X}_A(x)$ as shown above.

$\boxed{\text{Case 2}}$

$\mathcal{X}_A(x) = 0$ for $\notin A$

$\mathcal{X}_{A'}(x) = 1$ (the complement)

Similarly

$$1 - \mathcal{X}_A(x) = 1 - 0 = 1 = \mathcal{X}_{A'}(x)$$

$$\therefore \mathcal{X}_{A'}(x) = 1 - \mathcal{X}_A(x)$$

The above shows that $\mathcal{X}_{A'}(x) = 1 - \mathcal{X}_A(x)$ and holds true for all x values in Universe X.

ⓑ $A \cup B$ - Contains all elements in A, elements in B and elements present in both set A and B

$\boxed{\text{proof} \rightarrow \mathcal{X}_{A\cup B} = \max(\mathcal{X}_A, \mathcal{X}_B)}$

Characteristic functions

$$\mathcal{X}_{A\cup B}(x) = \left\{ \begin{array}{ll} 1 & \text{for all } x \in A\cup B \\ 0 & \text{otherwise} \end{array} \right\}$$

$$\mathcal{X}_A(x) = \left\{ \begin{array}{ll} 1 & \text{for all } x \in A \\ 0 & \text{otherwise} \end{array} \right\}$$

$$\mathcal{X}_B(x) = \left\{ \begin{array}{ll} 1 & \text{for all } x \in B \\ 0 & \text{otherwise} \end{array} \right\}$$

$\boxed{\text{Case 1}}$ x is in A

$\mathcal{X}_{A\cup B}(x) = 1$ by definition of CF

$\mathcal{X}_A(x) = 1$ as $x \in A$

$\mathcal{X}_B(x) = 0$ as $x \notin B$

$\max[\chi_A(x), \chi_B(x)] =$

$\max[1, 0] = 1$ which is same as $\chi_{A\cup B}$

$\therefore \chi_{A\cup B} = \max[\chi_A(x), \chi_B(x)]$

holds true for Case 1 where x is only an element in set A

Case 2: x is in B and not in A

$\chi_{A\cup B}(x) = 1$

$\chi_A(x) = 0_{\; x\notin A}$; $\chi_B(x) = 1_{\; x\in B}$

$\max[\chi_A(x), \chi_B(x)] = \max[1, 1]$
$= 1$

$\chi_{A\cup B}(x)$ is also $= 1$

therefore $\chi_{A\cup B}(x) = \max[\chi_A(x), \chi_B(x)]$

holds for case 2

Case 3: element x is neither in set A or B

$\chi_{A\cup B}(x) = 0$

$\chi_A(x) = 0$ x not in A

$\chi_B(x) = 0$ x not in B

$\max[\chi_A(x), \chi_B(x)] = \max[0, 0]$
$= 0$

Which is equal to the value of $\chi_{A\cup B}(x)$ as defined in the characteristic eq.

$\therefore$

$\chi_{A\cup B}(x) = \max[0, 0] = \max[\chi_A(x), \chi_B(x)]$

holds true for case 3 where x is not an element of A or B.

Case 4 x is a member of set A and set B

$\chi_{A\cup B}(x) = 1$

$\chi_A(x) = 1_{\; x\in A}$; $\chi_B(x) = 1_{\; x\in B}$

$\max[\chi_A(x), \chi_B(x)] = \max[1, 1]$

$\max = 1$

$\therefore \chi_{A\cup B}(x) = \max[\chi_A(x), \chi_B(x)]$

and holds true for Case 4 as well.

All above cases show that for all possible values of x in Universe X, $\chi_{A\cup B} = \max[\chi_A, \chi_B]$.

(c) $A \cap B$ - Contains all elements in both set A and set B.

$\boxed{\text{Proof} \rightarrow \chi_{A\cap B} = \min(\chi_A, \chi_B)}$

$$\chi_{A\cap B}(x) = \begin{cases} 1 & x \in A \text{ and } B \\ 0 & \text{otherwise} \end{cases}$$ characteristic function of $A \cap B$

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & \text{otherwise} \end{cases}$$ characteristic function of set A

$$\chi_B(x) = \begin{cases} 1 & x \in B \\ 0 & \text{otherwise} \end{cases}$$ characteristic function of set B

Case 1 : where x is a member of both A and B

By definition of the characteristic equations above for this case,

$$\chi_{A\cap B}(x) = 1$$

$\chi_A(x) = 1$ as x is in A $[x \in A]$

$\chi_B(x) = 1$ as x is also in B $[x \in B]$

$\min[\chi_A(x), \chi_B(x)] =$

$\min[1, 1] = 1$ and $\chi_{A\cap B} = 1$

$\therefore$ $\chi_{A\cap B} = \min[\chi_A, \chi_B]$

is valid for this case

Case II : element x is in A and not in B

$\therefore$ $\chi_{A\cap B}(x) = 0$ per characteristic eq

$\chi_A(x) = 1$ x is in A

$\chi_B(x) = 0$ as $x \notin B$

$\min[\chi_A(x), \chi_B(x)] =$

$\min[1, 0] = 0$

$\therefore$ $\chi_{A\cap B} = \min[\chi_A(x), \chi_B(x)]$

is also valid and holds for this case.

Case 3: Element x is not in A and is present in B

$\chi_{A\cap B}(x) = 0$ (By definition)

$\chi_A(x) = 0$ $x \notin A$

$\chi_B(x) = 1$ $x \in B$

$\min[\chi_A(x), \chi_B(x)] =$

$\min[0, 1] = 0$

$0 = \chi_{A\cap B}(x)$ as defined by the characteristic func

It holds true for case 3 that

$$\chi_{A\cap B} = \min[\chi_A(x), \chi_B(x)]$$

Case 4: x is not in A and not in B

$\chi_{A\cap B}(x) = 0$

$\chi_A(x) = 0$; $x \notin A$

$\chi_B(x) = 0$; $x \notin B$

$\min[\chi_A(x), \chi_B(x)] = \min[0, 0] = 0$

$\therefore$ $\chi_{A\cap B}(x) = \min[\chi_A(x), \chi_B(x)]$

Therefore

$$\chi_{A \cap B} = \min [\chi_A, \chi_B]$$

holds true for all cases of element x in Universe X.

(4) Implication Operation of $\chi_{A \to B}$ between sets A & B in Universes X & Y

$$\boxed{\text{Proof} \to \chi_{A \to B}(x, y) = \min [1, \{1 - \chi_A(x) + \chi_B(y)]}$$

$$\chi_{A \to B}(x, y) = \begin{cases} 1 & \text{if } x \notin A, y \in B \\ 0 & \text{otherwise} \end{cases}$$

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases} \qquad \chi_B(y) = \begin{cases} 1 & \text{if } y \in A \\ 0 & \text{otherwise} \end{cases}$$

$\boxed{\text{Case 1:}}$ x is in A and y in B

$$\chi_{A \to B}(x, y) = 1 \qquad y \in B$$

$$\chi_A(x) = 1 \qquad \chi_B(y) = 1$$

$$\min [1, \{1 - \chi_A(x) + \chi_B(y)\}] =$$

$$\min [1, \{1 - 1 + 1\}] = \min [1, 1]$$

$$\min = \underline{1} \text{ and } \chi_{A \to B}(x, y) = 1$$

$$\therefore \chi_{A \to B}(x, y) = \min [1, \{1 - \chi_A(x) + \chi_B(y)]$$

$\boxed{\text{Case 2:}}$ x is in A and y not in B

$$\chi_{A \to B}(x, y) = 0 \qquad x \in A, y \notin B$$

$$\chi_A(x) = 1 \qquad \chi_B(y) = 0$$

$$\min [1, \{1 - \chi_A(x) + \chi_B(y)\}] =$$

$$\min [1, \{1 - 1 + 0\} = \min [1, 0]$$

$$\min = 0$$

Which is equal to $\chi_{A \to B}(x, y)$

$\boxed{\text{Case 3:}}$ x not in A and y in B

$$\chi_A(x) = 0 \; ; \; x \notin A \qquad \chi_{A \to B}(x, y) = 1 \text{ satisfies both conditions}$$

$$\chi_B(y) = 1 \; ; \; y \in B$$

$$\min [1, \{1 - \chi_A(x) + \chi_B(y)]$$

$$\min [1, \{1 - 0 + 1\}] = \min [1, 2]$$

$$\min = \underline{1}$$

In this case, $\chi_{A \to B} = \min [1, \{1 - \chi_A(x) + \chi_B(y)]$

$\boxed{\text{Case 4:}}$ x not in A & y not in B

By definition of the characteristic eq.

$$\chi_A(x) = 0 \qquad x \notin A$$

$$\chi_B(y) = 0 \qquad y \notin B \qquad \chi_{A \to B} = 1 \quad x \notin A$$

$$\therefore \min [1, \{1 - \chi_A(x) + \chi_B(y)]$$

$$\min [1, \{1 - 0 + 0\}] = \min [1, 1]$$

$$\min = \underline{1}$$

$$\chi_{A \to B} = \min [1, \{1 - \chi_A(x) + \chi_B(y)]$$

For all cases of elements x and y in set A and B for Universe X and Y respectively

$$\chi_{A \to B}(x, y) = \min [1, \{\chi_A(x) + \chi_B(x)]$$

holds true.

Implication of Results.

→ The results shows a mathematical relationship between Characteristic function and set operations.

→ It provides a connection between binary logic, Crisp sets and fuzzy logic where binary logic operations can be used to represent and manipulate sets.

→ It highlights the duality in set theory where set operations like Union and Intersection Correspond to the Maximum and minimum Operation respectively.

→ Lastly, it demonstrates that set operations and fuzzy set operations can be used for Convenient decision making and data manipulation in the real world.

Problem 4. Show that $max[0, x+ y - 1]$ is a t-norm. Also, determine the corresponding t-conorm (i.e., s-norm). *Hint:* Show that the non-decreasing, commutative, and associative properties and the boundary conditions are satisfied.

Show that $\underbrace{max[0, x+y-1]}$ is a t-norm

To proof t-norm

The properties of :

Commutativity ✓ Associativity

Non-decreasing ✓ Boundary conditions

should be satisfied

2 Commutativity → Order of elements should not matter

For all x and y in range $[0,1]$

$$max[0, x+y-1] = max[0, y+x-1]$$

let $x=0$ $y=1$

$$max[0, 0+1-1] = max[0, 1+0-1]$$

$$max[0, 0] = max[0, 0]$$

LHS = RHS

$\therefore$ $Max[0, x+y-1]$ is commutative

Non-decreasing →

For any x_1, x_2 in $[0,1]$

where $x_1 \leq x_2$

$$\underbrace{Max[0, x_1+y-1]}_{LHS} \leq \underbrace{max[0, x_2+y-1]}_{RHS}$$

In LHS

If $x_1+y-1 \geq 0$

$\therefore$ $max[0, x_1+y-1] = x_1+y-1$ ... 1

If $x_1+y-1 < 0$

$\therefore$ $max[0, x_1+y-1] = 0$ 2

2nd element in LHS = $max[0, x_1+y-1]$

let $K = max[0, x_1+y-1]$

Rewriting LHS

$$max[0, max[0, x_1+y-1]]$$

$$= max[0, K]$$

Evaluating RHS

If $x_1 \leq x_2$

$\therefore$ $x_1+y-1 \leq x_2+y-1$

Applying logic 1 & 2 to 2nd element in RHS

If $x_2+y-1 \geq 0$;

$max[0, x_2+y-1] = x_2+y-1$ and

If $x_2+y-1 < 0$;

$max[0, x_2+y-1] = 0$

$\therefore$ let s be $\max[0, x_2+y-1]$

Rewriting RHS

$\max[0, \max[0, x_2+y-1]]$

$RHS = \max[0, s]$

LHS RHS

$\max[0, k] \leq \max[0, s]$

from initial condition since

$x_1 \leq x_2$

$\therefore x_1 + \underbrace{y-1}_{\text{same}} \leq x_2 + \underbrace{y-1}_{\text{same}}$ as

other parts of the equation are same

$\therefore \max[0, x_1+y-1] \leq \max[0, x_2+y-1]$

The above shows that $\max[0, x+y-1]$ is non-decreasing when $x_1 \leq x_2$ and both are in $[0,1]$ range.

Similar for $y_1 \leq y_2$ in $[0,1]$

$\underbrace{x_1+y_1-1}_{\max[0,\ \uparrow\ ]} \leq \underbrace{x_2+y_1-1}_{\max[0,\ \uparrow\ ]}$ & $\underbrace{x_1+y_1-1}_{\max[0,\ \uparrow\ ]} \leq \underbrace{x_1+y_2-1}_{\max[0,\ \uparrow\ ]}$

$\max[0, \ ] \leq \max[0, \ ]$ & $\max[0, \ ] \leq \max[0, \ ]$

Boundary Conditions

When $x = 0$

$\max[0, x+y-1] = 0$

When $y = 0$

$\max[0, x+y-1] = 0$

For $x = 0$

$\max[0, 0+y-1]$

$\max[0, 0+y-1] = \max[0, y-1]$ ..①

If $y-1 \geq 0$ $\therefore \max[0, y-1] = y-1$

If $y-1 < 0$ $\therefore \max[0, y-1] = 0$

As y is in the range $[0,1]$

$y-1$ will always be < 1

$\therefore \max[0, y-1] = 0$

meaning $\max[0, 0+y-1] = 0$

as in ① above since they are equal

for $y = 0$

$\max[0, x+0-1] = \max[0, x-1]$

Applying the same logic as above for x with y.

If $x-1 \geq 0$ $\therefore$

$\max[0, x-1] = x-1$

If $x-1 < 0$ $\therefore \max = 0$

x is also in $[0,1]$ range

$\therefore x-1 < 1$ (always)

This shows that the

$\max[0, x-1] = 0$

and

$\max[0, x+0-1] = 0$

Both boundary conditions are satisfied for $\max[0, x+y-1]$

Associativity

can move brackets and get the same result

For all x, y, and z in $[0,1]$ range

$$\max[0, x + \max[0, \underbrace{y+z-1}_{w}] - 1] \quad (LHS)$$

$$=$$

$$\max[0, \max[0, \underbrace{x+y-1}_{m}] + z - 1] \quad (RHS)$$

$$\max[0, m]$$

If $m \geq 0 \therefore \max[0, m] = m$

Rewriting

$$\max[0, \max[0, m] + z - 1]$$

$$\max[0, m + z - 1]$$

$$m = x + y - 1$$

$$\max[0, x + y - 1 + z - 1]$$

$$\max[0, x + y + z - 2]$$

$$\max[0, w] \quad (LHS)$$

If $w \geq 0 \therefore \max[0, w] = w$

If $w < 0 \therefore \max[0, w] = 0$

Rewriting

$$\max[0, x + \max[0, w] - 1]$$

$$\max[0, x + \max[0, y + z - 1] - 1]$$

when $y + z - 1$ is the max of inner max

$$\max[0, x + y + z - 1 - 1]$$

$$\max[0, x + y + z - 2]$$

$\therefore$ LHS = RHS having the same expression proving that $\max[0, x+y-1]$ is associative.

Corresponding Snorm

Snorm = Complementary Tnorm

$$S(\cdot) = T'$$

for $\max[0, x + y - 1]$

Using De Morgan's law and direct substitution of min for Tnorm

$$1 - \min[1, 1 - x - y]$$