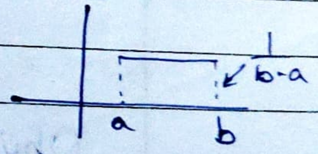
[Uniform Dist]

Defn: over $[a,b]$ p.d.f is

$$f(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{else} \end{cases}$$



Obvio, i) $f(x) \geq 0$

ii) $\int_{-\infty}^{\infty} f(x)\,dx = \int_a^b f(x)\,dx = \int_a^b \frac{dx}{b-a} = 1$

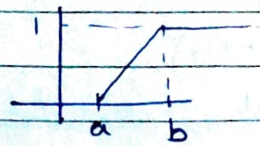
Distribution Func

$\hookrightarrow F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt$

① if $x < a$, $F(x) = 0$

② if $x \in [a,b]$, $f(x) = \int_a^x \frac{dx}{b-a} = \frac{x-a}{b-a}$

③ if $x > b$, $f(x) = \int_a^b \frac{dx}{b-a} = 1$



Mean

$E[X] = \frac{1}{2}(a+b)$

Proof) $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$

$= \int_{-\infty}^{a} 0 \cdot x\,dx + \int_a^b \frac{x}{b-a}\,dx + \int_b^{\infty} 0 \cdot x\,dx$

$= \frac{1}{b-a} \frac{x^2}{2}\Big|_a^b = \frac{b+a}{2}$

Variance

$Var(X) = (a-b)^2/12$

Proof) $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

$= \int_a^b x^2 \frac{1}{b-a}\,dx = \frac{1}{b-a} \frac{x^3}{3}\Big|_a^b = \frac{b^2+a^2+ba}{3}$

$Var(X) = E(X^2) - E(X)^2$

$= \frac{b^2+a^2+ba}{3} - \frac{a^2+b^2+2ab}{4} = \frac{a^2+b^2-2ab}{12} = \frac{(a-b)^2}{12}$

Exponential Distribution

Defn: parameter λ, pdf is

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

Obviously, i) $f(x) \ge 0$

ii) $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} \lambda e^{-\lambda x} = \lim_{B \to \infty} \int_0^B \lambda e^{-\lambda x}$

$= \lim_{B \to \infty} \frac{-\lambda}{\lambda} e^{-\lambda x}\Big|_0^B = (1 - e^{-\lambda(\infty)}) \Rightarrow 1$

it's a pdf!

Distribution func

if $x<0$, $F(x) = \int_{-\infty}^{0} f(u)\,du = 0$

if $x\geq 0$, $F(x) = \int_{0}^{x} \lambda e^{-\lambda t}\,dt = \lambda \frac{e^{-\lambda t}}{\lambda}\Big|_0^x$

$= 1-e^{-\lambda x}$

Mean

$E[X] = 1/\lambda$

Proof) $E[X] = \int_{-\infty}^{\infty} x f(u)\,du = \int_{0}^{\infty} x\lambda e^{-\lambda x}\,du$

$= \frac{1}{\lambda}\int_0^{\infty} ue^{-u}\,du = \frac{1}{\lambda}\frac{\Gamma(2)}{1} = \frac{1}{\lambda}$

Variance

$Var(x) = 1/\lambda^2$

Proof) $E(X^2) = \int_{-\infty}^{\infty} x^2 f(u)\,du = \int_0^{\infty} x^2 \lambda e^{-\lambda x}\,du$

$= \frac{1}{\lambda^2}\int_0^{\infty} u^2 e^{-u}\,du = \frac{1}{\lambda^2}\frac{\Gamma(3)}{1} = \frac{2}{\lambda^2}$

$Var(x) = E(X^2) - E(X)^2$

$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

Memoryless Property

Thm If X is an exp dist R.V then

$P(X\geq s+t/X\geq s) = P(X\geq t)$ $(\forall s,t>0)$

Proof $P(X\geq s) = \int_s^{\infty} f(x)\,dx = \lim_{B\to\infty}\int_s^{B} \lambda e^{-\lambda t}\,dt$

$= e^{-\lambda s}$

$\therefore P(X\geq s+t/X\geq s) = \frac{P((X\geq s+t)\cap(X\geq s))}{P(X\geq s)} = \frac{P(X\geq s+t)}{P(X\geq s)}$

$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = P(X\geq t)$

Geometric RV

↳ Do exp till you get success

pdf $= f(u) = q^{i-1}p$.

__/__/__

Mean

$E[X] = 1/p$

proof) $E[X] = \sum_{i=1}^{\infty} i q^{i-1} p = \sum_{i=1}^{\infty} (i-1) q^{i-1} p + \sum_{i=1}^{\infty} q^{i-1} p$

$= \sum_{j=0}^{\infty} j q^{j} p + 1$

$= q \sum_{j=1}^{\infty} j q^{j-1} p + 1$

$= q E[X] + 1$

$\Rightarrow p E[X] = 1$

$\Rightarrow E[X] = 1/p$

Variance

$Var(X) = 1-p/p^2$

Proof $E[X^2] = \sum_{j=0}^{\infty} j^2 q^j p + 2 \sum_{j=1}^{\infty} j q^j p + 1$

$= q E[X^2] + 2q E[X] + 1$

$\rightarrow p E[X^2] = \frac{2q+p}{p^2} = \frac{q+1}{p^2}$

$Var(X) = \frac{q+1}{p^2} - \frac{1}{p} = \frac{1-p}{p^2}$

Gamma Distribution

$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x} (\lambda x)^{\alpha-1}}{\Gamma(\alpha)} & x \geq 0 \\ 0 & x < 0 \end{cases}$

Mean $E[X] = \alpha/\lambda$

Variance $Var[X] = \frac{\alpha}{\lambda^2}$

Normal Distribution

↳ parameters μ & σ, pdf is

$$f(x) = \frac{1}{\sigma(\sqrt{2\pi})} e^{-(x-\mu)^2/2\sigma^2}$$

X is normal variate with μ & σ, denoted by $X \sim N(\mu, \sigma)$

Obviously, i) $f(x) \geq 0$

ii) $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\mu} f(x)dx + \int_{\mu}^{\infty} f(x)dx$

$$= \lim_{B_1 \to \infty} -\int_{\frac{B_1+\mu}{\sigma\sqrt{2}}}^{0} \frac{1}{\sqrt{\pi}} e^{-u^2} du + \lim_{B_2 \to \infty} \int_{0}^{\frac{B_2-\mu}{\sigma(\sqrt{2})}} \frac{1}{\sqrt{\pi}} e^{-v^2} dv$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-u^2} du = \frac{2}{\sqrt{\pi}} \lim_{B \to \infty} \int_0^{B} e^{-u^2} du$$

$$= \frac{2}{\sqrt{\pi}} \lim_{B \to \infty} \int_0^{B^2} \frac{1}{2} e^{-z} z^{-1/2} dz$$

It is a pdf!

$$= \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-z} z^{\frac{1}{2}-1} dz = \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right) = 1$$

Dist Func

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-(t-\mu)^2/2\sigma^2} dt$$

Mean

$E[X] = \mu$

Proof) $E[X] = \int_{-\infty}^{\infty} x f(x)dx = \int_{-\infty}^{\infty} (x-\mu) f(x)dx + \mu \int_{-\infty}^{\infty} f(x)dx$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu) e^{-(x-\mu)^2/2\sigma^2} dx + \mu$$

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu) e^{-(x-\mu)^2/2\sigma^2} = \lim_{\substack{B_1 \to \infty \\ B_2 \to \infty}} \frac{1}{2} \int_{(B_1+\mu)^2}^{(B_2+\mu)^2} e^{-z/2\sigma^2} dz$$

$$= \lim_{\substack{B_1 \to \infty \\ B_2 \to \infty}} \sigma^2 \left[e^{-(B_1+\mu)^2/2\sigma^2} - e^{-(B_2+\mu)^2/2\sigma^2} \right]$$

$= 0$

Hence, $E[X] = 0 + \mu = \mu$

Variance

$$E((X-\mu)^2)=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}(x-\mu)^2 e^{-(x-\mu)^2/2\sigma^2}\,dx$$

$$=\frac{1}{\sigma\sqrt{2\pi}}\lim_{B_1\to\infty}\int_{-B_1}^{\mu}(x-\mu)^2 e^{-(x-\mu)^2/2\sigma^2}\,\frac{dx}{\sigma\sqrt{2}}$$

$$+\frac{1}{\sqrt{\pi}}(\text{same})$$

$$=\frac{2\sigma^2}{\sqrt{\pi}}\lim_{B_1\to\infty}\int_{0}^{\frac{B_1+\mu}{\sigma\sqrt{2}}}u^2e^{-u^2}\,du+\frac{2\sigma^2}{\sqrt{\pi}}\lim_{B\to\infty}\int_{0}^{B}2u^2e^{-u^2}\,du$$

$$=\frac{2\sigma^2}{\sqrt{\pi}}\lim_{B\to\infty}\int_{0}^{B^2}e^{-z}z^{1/2}\,dz=\frac{2\sigma^2}{\sqrt{\pi}}\Gamma(3/2)=\frac{2\sigma^2}{\sqrt{\pi}}\cdot\frac{1}{2}\sqrt{\pi}$$

$$\boxed{Var(x)=\sigma^2}$$

Curve $\Rightarrow$ 

$\Rightarrow$ if $\mu=0$ and $\sigma=1$, it is called a standard normal distribution. $X\sim N(0,1)$

pdf $\quad \varphi(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$

df $\quad \Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-t^2/2}\,dt$

Th: if X has normal dist with μ,σ the $Z=\frac{X-\mu}{\sigma}$ has standard normal dist

Proof) $X\sim N(\mu,\sigma)$ & $Z=\frac{X-\mu}{\sigma}\Rightarrow X=\mu+\sigma Z$

$$F(z)=P\left(\frac{X-\mu}{\sigma}\le z\right)=P(X\le\mu+z\sigma)$$

$$=\left(\lim_{B\to\infty}\int_{-B}^{\mu+\sigma z}e^{-(x-\mu)^2/2\sigma^2}\,dx\right)\times\frac{1}{\sigma\sqrt{2\pi}}$$

$$=\frac{1}{2\sqrt{\pi}}\int_{-\infty}^{z}e^{-t^2/2}\,dt$$

$$f(z)=\frac{dF}{dz}=\frac{1}{\sqrt{2\pi}}e^{-z^2/2}\Rightarrow Z\sim N(0,1)$$

/ /

Normal approximation to Binomial

Binomial $P(X=i) = {}^nC_i\, p^i (1-p)^i$

if $n \to \infty$ & p is not small

i) $X \to N(np, \sqrt{(1-p)np})$

ii) $Z = \dfrac{X-np}{\sqrt{(1-p)np}} \to N(0,1)$

Joint Distribution Function

$F(a,b) = P\{x < a,\ y \le b\}$

$P(a_1 < x \le a_2,\ b_1 < x \le b_2) = F(a_2,b_2) + F(a_1,b_1) - F(a_1,b_2) - F(a_2,b_1)$

joint prob mass function

$\hookrightarrow p(x,y) = P(x=X, y=Y)$

$P_X(x) = P(X=x)$

$= P[\cup_j (X=x, Y=y_j)]$

$= \sum_j P(X=x, Y=y) = \sum_j p(x, y_j)$

$P_Y(y) = \sum_i p(x_i, y)$

Bivariate R.V if (x,y) are finite / coutably finite

Discrete | Continuous

Joint pmf | Conditional pmf

$P_X(x) / P_Y(y)$ | $P_{x|y}(x$

~~Bivariate Discrete R.v~~

$P_{ij} = P(X=x_i, Y=y_j)$ i) $P_{ij} \ge 0\ \forall i,j$ ii) $\sum_j \sum_i P_{ij} = 1$

Eg

	$y=1$	$y=2$	$y=3$	$y=4$
$x=1$	$1/4$	$1/16$	$1/16$	$1/8$
$x=2$	$1/16$	$1/8$	$1/4$	$1/16$

Ans i) $P_{ij} \ge 0$

ii) $\sum\sum P_{ij} = (\text{col}1) + (\text{col}2) + \dots + (\text{col}4) = 1$

its pdf!

Marginal pdf

$p(X=x_i) = \sum_j P_{ij} = P_{i1} + P_{i2} + \dots$

marginal pmf of y : $= P(y=y_j) = \sum_i P_{ij} = P_{1j} + P_{2j} + \dots$

Using last eg

X	$P_x(x)$
1	8/16
2	8/16

Y	$P_y(y)$
4	3/16
1	5/16
2	3/16
3	5/16

Independent R.V

$P_x(x)\, P_y(y) = P_{xy}(1,2)$

Conditional pdf

$P\left\{\frac{X=x_i}{Y=y_j}\right\} = \frac{P(X=x_i, Y=y_j)}{P(Y=y_j)}$

Using earlier eg,

$P_{\frac{x}{y}}\left(\frac{X}{Y=1}\right) = ?$

X	$P_{x/y}\left(\frac{X}{Y=1}\right)$
1	$\frac{1/4}{5/16} = 4/5$
2	$\frac{1/16}{5/16} = 1/5$

Eg joint pmf (x,y) is $p(x,y) = k(2x+3y)$ $x = 0-2$ $y = 1-3$

Ans i) find k

x\y	1	2	3	↓
0	3k	6k	9k	18k
1	5k	8k	11k	24k
2	7k	10k	13k	30k
⇒	15k	24k	33k	

$72k = 1$

$k = 1/72$

ii) Marginal pmf of x

X	$P_x(X)$
0	18/72
1	24/72
2	30/72

iii) Marginal pmf of Y

Y	$P_y(y)$
1	15/72
2	24/72
3	33/72

iv) Cond pmf of x when Y=1

$\boxed{X}$	0	1	2
$\boxed{P_{x/y}\left(\frac{X}{Y=1}\right)}$	3/15	5/15	7/15

v) Cond pmf of y when X=2

Y	1	2	3
$P_{y/x}\left(\frac{Y}{X=2}\right)$	7/30	10/30	13/30

vi) ~~pdf of (X+Y)~~

let $Z = X+Y$

X	0	0	1	0	1	2	1	2	2
Y	1	2	1	3	2	1	3	2	3
Z	1	2		3			4		5
P	3/72	11/72		24/72			21/72		13/72

(add)

Q Two balls are picked → 2R, 2G, 4W.

X → red / Y → green 2 balls drawn

Ans

x\y	0	1	2
0	${}^4C_2/{}^9C_2$	$2\cdot{}^4C_1/{}^9C_2$	–
1	–	–	–
2	–	–	–
3	–	–	–

i) ~~Joint~~
ii) ~~Marginal~~
iii) ~~conditional~~ of X s.t Y=1

PDF (joint)

$P\left(x-\frac{dx}{2} \le X \le x+\frac{dx}{2} \ \& \ y-\frac{dy}{2} \le Y \le y+\frac{dy}{2}\right) = f(x,y)\,dx\,dy$

it is joint pdf if

i) $f(x,y) \geq 0$

ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

Eg $f(x,y) = xy^2 + \frac{x^2}{8} \quad 0 \le x \le 2 \quad 0 \le y < 1$

Ans i) positive

ii) $\int_0^2 \int_0^1 \left(xy^2 + \frac{x^2}{8}\right) dy\,dx$

$= \int_0^2 \frac{x}{2} + \frac{x^2}{8}\,dx = \frac{1}{6} \times 4 + \frac{8}{24} = 1$ It is pdf.

Marginal pdf

$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \qquad f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$

if $f(x,y) = f_X(x)\,f_Y(y)$, independent

/ /

Conditional PDF

$$f\left(\frac{x}{y}\right) = \frac{f(x,y)}{f_Y(y)} \qquad f\left(\frac{y}{x}\right) = \frac{f(x,y)}{f_X(x)}$$

Eg $F(x,y) = k$ find k and marginal pdf and cond pdf

$0 \le x \le y \le 2$

Ans

$$\int_0^2 \int_x^2 k\,dy\,dx = 1$$

$$k \int_0^2 (2-x)\,dx = 1 \Rightarrow k = \frac{1}{2}$$

Marginal

$$f_X(x) = \int_x^2 \frac{1}{2}\,dy = \frac{1}{2}(2-x)$$

$$f_Y(y) = \int_0^y \frac{1}{2}\,dx = \frac{1}{2}y$$

Cond $f\left(\frac{x}{y}\right) = \frac{1/2}{1/2 \cdot y} = \frac{1}{y}$ $\qquad f\left(\frac{y}{x}\right) = \frac{\frac{1}{2}}{\frac{1}{2}(2-x)} = \frac{1}{2-x}$

Correlation Coefficient

$$r = \frac{\mathrm{cov}(x,y)}{\sigma_x \sigma_y} \qquad \mathrm{cov}(x,y) = E(xy) - E(x)E(y)$$

$$= \frac{\sum xy}{n} - \frac{\sum x}{n} - \frac{\sum y}{n}$$

Covarience

$$\mathrm{cov}(X,Y) = E((X-\bar{X})(Y-\bar{Y})) = E[XY - YE[X] - XE[Y] + E[X]E[Y]]$$

$$= E[XY] - E[X]E[Y]$$

① if independent, $\mathrm{Cov}(XY) = 0$

② $E[X]$, $\mathrm{Var}(x) - \sigma_x^2$

$$X^* = \frac{X - E[X]}{\sigma_x} \Rightarrow E(X^*) = \frac{1}{\sigma_x} - 0$$

$$\mathrm{Var}(X^*) = E[X^* - E[X^*]] = E[X^*]^2 = E\left[\frac{X - E[X]}{\sigma_x}\right]^2$$

$$= \frac{1}{\sigma_x^2} E(X - E[X])^2 = \frac{\sigma_x^2}{\sigma_x^2} = 1$$

$X \sim (E(X), \mathrm{Var}(X))$ or $X^* \sim N(0,1)$

_/_/_

③ $Var(aX+bY) = a^2 Var(x) + b^2 Var(y) + 2ab\, Cov(x,y)$

Proof) $Var(aX+bY) = E[aX+bY - E[aX+bY]]^2 = E[a(x-E(x)) + b(y-E(y))]^2$

$= a^2 E[x-E(x)]^2 + b^2 E(y-E(y))^2 + 2ab\, Cov(x,y)$

$= a^2 Var(x) + b^2 Var(y) + 2ab\, Cov(x,y)$

④ $Cov(X+a, Y+b) = cov(x,y)$

Proof $E((x+a - E(x+a))(y+b - E(y+b)))$

$= E((x-E(x))(y-E(y)))$

$= cov(x,y)$

⑤ $Cov(ax, by) - E[ax - E(ax)](by - E(by))$

Proof $= ab\, E((x-E(x))(y-E(x)))$

$= ab\, Cov(x,y)$

Correlation coeff

cont

$r = \frac{Cov(x,y)}{\sigma_x \sigma_y}$ → cont $\sigma_x = \sqrt{E(x-E(x))}$

disc calc using graph.

Moment Generating Func

$M_x(t) = E(e^{tx})$

$= \sum_r e^{tx_r} p_r$ or $\int_{-\infty}^{\infty} e^{tx} f(x)\,dx$

Result

① $M_x^{(k)}(t)\Big|_{t=0} = M_x^{(k)}(0) = E(x^k) = \alpha_k$

② $M_x(t) = \sum_{k=0}^{\infty} \left(\frac{M_x^{(k)}(0)}{k!}\right) t^k$

③ $M_{x-a}(t) = e^{-at} M_x(t)$

④ $M_{cx}(t) = M_x(ct)$

⑤ $M_{c+x}(t) = e^{ct} M_x(t)$

⑥ $M_{\Sigma x_i}(t) = \prod_{i=1}^{n} M_{x_i}(t)$

__/__/__

Tchebycheff's Inequality

For rv with $E(x) = \mu$ & $Var(x) = \sigma^2$ for any $k > 0$

① $P\{|(x-\mu)| \geq k\sigma\} \leq \frac{1}{k^2}$

or ② $P\{|(x-\mu)| < k\sigma\} \geq 1 - \frac{1}{k^2}$

Proof) Let $x \to$ c.r.v

$$\sigma^2 = E\{x - E[x]\}^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$$

$$= \left(\int_{-\infty}^{\mu-k\sigma} + \int_{\mu-k\sigma}^{\mu+k\sigma} + \int_{\mu+k\sigma}^{\infty}\right)(x-\mu)^2 f(x)dx$$

$$\sigma^2 \geq \left(\int_{-\infty}^{\mu-k\sigma} + \int_{\mu+k\sigma}^{\infty}\right)(x-\mu)^2 f(x)dx$$

$x \leq \mu - k\sigma$ $\qquad$ $x \geq \mu + \sigma k$

$\mu - x \geq k\sigma$ $\qquad$ $x - \mu \geq k\sigma$

$$\to \sigma^2 \geq k^2\sigma^2 \left[\left(\int_{-\infty}^{\mu-k\sigma} + \int_{\mu+k\sigma}^{\infty}\right) f(x)dx\right]$$

$$= k^2\sigma^2 \, P\{|x-\mu| \geq k\sigma\}$$

$$P\{|x-\mu| \geq k\sigma\} \leq \frac{1}{k^2}$$