Laplace Transform

Defn

Let $f(t)$ be a fx^n when $t > 0$, multiply it with e^{-at} and integrate $0 \to \infty$.

$$F(s) = \int_0^\infty e^{-at} f(t)\,dt$$ this is Laplace Transform. $L\{f(t)\}$ or $\bar{f}(s)$

Linear property

A transformation T is linear if $\forall\, f_1(t), f_2(t)$ and const c_1, c_2

$$T[c_1 f_1(t) + c_2 f_2(t)] = c_1 T[f_1(t)] + c_2 T[f_2(t)].$$

Th) Laplace is linear

Proof) $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$$L\{c_1 f_1(t) + c_2 f_2(t)\} = \int_0^\infty e^{-st}(c_1 f_1(t) + c_2 f_2(t))\,dt$$

$$= c_1 \int_0^\infty e^{-st} f_1(t) + c_2 \int_0^\infty e^{-st} f_2(t)$$

$$= c_1 L\{f_1(t)\} + c_2 L\{f_2(t)\}.$$

Th) Laplace transform exists in $Re(s) > a$ if

i) $\forall\, b > 0$, $[0, b]$ can be broken into finite subintervals in each of which $f(t)$ is continuous and $f(t)$ approaches definite limit.

ii) $|e^{-at} f(t)| \leq M$ or $|f(t)| \leq Me^{at}$ $\forall\, t > 0$ and $M > 0$

Proof) $L\{f(t)\} = \int_0^\infty e^{-at} f(t)\,dt$

$$\left|\int_0^\infty e^{-at} f(t)\,dt\right| \leq \int_0^\infty |e^{-st} f(t)|\,dt = \int_0^\infty |e^{-st}|\,|f(t)|\,dt$$

$$\leq \int_0^\infty e^{-(x+iy)t} M e^{at}\,dt$$

$$= M \int_0^\infty e^{-t(x-a)}\,dt$$

Laplace Transforms

① $\boxed{L(1) = \frac{1}{s}}$ $s > 0$

Proof $L(1) = \int_0^\infty e^{-st} \cdot 1\,dt$

$$= \int_0^\infty e^{-st}\,dt = \left[\frac{e^{-st}}{s}\right]_0^\infty = \left[\frac{1}{s} - \frac{0}{s}\right] = \frac{1}{s}$$

② $\boxed{L(t^n) = \Gamma(n+1)/s^{n+1}}$, $n > -1$, $s > 0$

Proof) $L(t^n) = \int_0^\infty e^{-st} t^n \, dt = \int_0^\infty e^{-st} t^{(n+1)-1} dt = \frac{\Gamma(n+1)}{s^{n+1}}$

③ $\boxed{L(e^{at}) = 1/s-a}$ $s > a$

Proof) $L(e^{at}) = \int_0^\infty e^{at} e^{-st} dt = \int_0^\infty e^{(a-s)t} dt = \lim_{B\to\infty} \int_0^B e^{-(s-a)t} dt$

$= \frac{1}{s-a} \lim_{B\to\infty} [1 - e^{-(s-a)B}] = \frac{1}{s-a}$

④ $\boxed{L(\sin at) = a/a^2+s^2}$ $s > 0$

Proof) $L(\sin at) = \int_0^\infty \sin at \, e^{-st} dt = \lim_{B\to\infty} \int_0^B e^{-st} \sin at \, dt$

$= \lim_{B\to\infty} \frac{-e^{-st}}{s^2+a^2}(-s\sin at - a\cos at)\Big|_0^B = \lim_{B\to\infty} \left[\frac{a}{a^2+s^2} - \frac{e^{-sB}}{s^2+a^2}(\cdots)\right]$

$= \frac{a^2}{s^2+a^2}$

⑤ $\boxed{L(\cos at) = s/s^2+a^2}$ $s > 0$

Proof) $L(\cos at) = \int_0^\infty e^{-st} \cos at \, dt = \lim_{B\to\infty} \int_0^B e^{-st} \cos at \, dt$

$= \lim_{B\to\infty} \frac{e^{-st}}{a^2+s^2}(-s\cos at + a\sin at)\Big|_0^B = \frac{s}{a^2+s^2}$

⑥ $\boxed{L(\sinh at) = a/s^2-a^2}$ $s > |a|$

Proof) $L(\sinh t) = \int_0^\infty e^{-st} \sinh at = \lim_{B\to\infty} \int_0^B e^{-st} \frac{(e^{at} - e^{-at})}{2} dt$

$= \lim_{B\to\infty} \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)} - \frac{e^{-(s+a)t}}{-(s+a)}\right]_0^B$

$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{2a}{2(s^2-a^2)} = \frac{a}{s^2-a^2}$

⑦ $\boxed{L(\cosh t) = s/s^2-a^2}$, $s > |a|$

Proof) $L(\cosh t) = \int_0^\infty e^{-st} \cosh at \, dt = \lim_{B\to\infty} \int_0^B \frac{e^{-st}(e^{at} + e^{-st})}{2} dt$

$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{2s}{2(s^2-a^2)} = \frac{s}{s^2-a^2}$

Th3 Shifting Property

$L\{f(t)\} = \bar{f}(s)$ then $L\{e^{at}f(t)\} = \bar{f}(s-a)$

Proof) $L(f(t)) = \int_0^{\infty} e^{-st} f(t)\,dt = \bar{f}(s)$

$$L(e^{at}f(t)) = \int_0^{\infty} e^{-st} f(t) e^{at}\,dt = \int_0^{\infty} e^{-(s-a)t} f(t)\,dt$$
$$= \bar{f}(s-a)$$

Hence

(8) $L(e^{at}t^n) = \dfrac{\Gamma(n+1)}{(s-a)^{n+1}}$ (9) $L(e^{at}\sin bt) = \dfrac{b}{(s-a)^2+b^2}$

(10) $L(e^{at}\cos bt) = \dfrac{(s-a)}{(s-a)^2+b^2}$ (11) $L(e^{at}\sinh bt) = \dfrac{b}{(s-a)^2-b^2}$

(12) $L(e^{at}\cosh bt) = \dfrac{s-a}{(s-a)^2-b^2}$ [conditions all same but $s \Rightarrow s-a$]

Imp Integrals

(1) $\int e^{ax}\cos bx\,dx = \dfrac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$

(2) $\int e^{ax}\sin bx\,dx = \dfrac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$

Th4 Second Shifting Property

if $L(f(t)) = \bar{f}(s)$, $g(t) = \begin{cases} f(t-a) & t>a \\ 0 & t<a \end{cases}$ then $L(g(t)) = e^{-at}\bar{f}(s)$

Proof) $L(g(t)) = \int_0^{\infty} e^{-st} g(t)\,dt = \int_0^{a} e^{-st}\cdot 0\,dt + \int_a^{s} e^{-st} f(t-a)\,dt$

$$= e^{-at}\int_0^{\infty} e^{-st} f(t)\,dt = e^{-at}\bar{f}(s)$$

Th5 Scaling Property

if $L(f(t)) = \bar{f}(s)$, then $L(f(at)) = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$

Proof) $L(f(at)) = \int_0^{\infty} e^{-st} f(at) = \lim_{B\to\infty} \int_0^{aB} e^{-s/a\,t} f(x)\,dx$

$$= \frac{1}{a}\int_0^{\infty} e^{-s/a\,t} f(t)\,dt = \frac{1}{a}\bar{f}(s/a)$$

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Th6 $L\{f(t)\} = \bar{f}(s)$, then $L\{t^n f(t)\} = (-1)^n \frac{d^n}{s^n} \bar{f}(s)$

Proof) $\underline{n=1}$

$$\bar{f}(s) = L(f(t)) = \int_0^\infty e^{-st} f(t)\,dt$$

$$\frac{d}{ds}\bar{f}(s) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty \frac{\partial}{\partial s}\left(e^{-st} f(t)\right)dt$$

$$= \int_0^\infty -t e^{-st} f(t)\,dt$$

$\underline{n=k \Rightarrow n=k+1}$

$$\Rightarrow L(t f(t)) = \frac{d}{ds}\bar{f}(s)$$

$$\int_0^\infty e^{-st}\left(t^m f(t)\right)dt = (-1)^m \frac{d^m}{ds^m}\bar{f}(s)$$

$$\frac{d}{ds}\int_0^\infty \quad '' \quad = (-1)^m \frac{d^{m+1}}{ds^{m+1}}\bar{f}(s)$$

$$\int_0^\infty \left(-t e^{-st}\right) t^m f(t)\,dt = \quad ''$$

$$\int_0^\infty e^{-st} t^{m+1} f(t)\,dt = (-1)^{m+1} \frac{d^{m+1}}{ds^{m+1}}\bar{f}(s)$$

Proved

Th7 if $L(f(t)) = \bar{f}(s)$, then $L\left(\frac{1}{t} f(t)\right) = \int_s^\infty \bar{f}(s)\,ds$

Proof) $\bar{f}(s) = \int_0^\infty e^{-st} f(t)\,dt$

$$\int_s^\infty \bar{f}(s)\,ds = \int_s^\infty \int_0^\infty e^{-st} f(t)\,dt\,ds$$

$$= \int_0^\infty \int_s^\infty e^{-st} f(t)\,ds\,dt = \int_0^\infty \underline{\int_s^\infty e^{-st}\,ds}\, f(t)\,dt$$

$$= \int_0^\infty \frac{e^{-st}}{t} f(t)\,dt = L\left(\frac{1}{t} f(t)\right)$$

Th8 a) if $L(f(t)) = \bar{f}(s)$, then $L(f'(t)) = s\bar{f}(s) - f(0)$

b) if $f(t)$ and first $n-1$ derivatives are continuous, then

$$L(f^n(t)) = -f^{n-1}(0) - sf^{n-2}(0) - \cdots - s^{n-1}f(0) + s^n\bar{f}(s)$$

Proof) a) $L(f'(t)) = \int_0^\infty e^{-st} f'(t)\,dt = \lim_{B\to\infty} e^{-st} f(t)\Big|_0^B + s\int_0^B e^{-st} f(t)\,dt$

$$L(f'(t)) = -f(0) + s\bar{f}(s)$$

b) $L(f''(t)) = L\left(\frac{d f'(t)}{dt}\right) = -f'(0) + sL(f'(t))$
$= -f'(0) - sf(0) + s^2 \bar{f}(s)$
by chain effect, we get the result.

Th9 If $L[f(t)] = \bar{f}(s)$, then $L\left(\int_0^t f(u)\,du\right) = \frac{1}{s}\bar{f}(s)$

Proof Let $g(t) = \int_0^t f(u)\,du \Rightarrow g'(t) = f(t)$ & $g(0) = 0$
$L(g'(t)) = -g(0) + sL(g(t))$
$\Rightarrow L(f(t)) = -0 + sL\left(\int_0^t f(u)\,du\right)$
$\Rightarrow L\left(\int_0^t f(u)\,du\right) = \frac{1}{s} L(f(t)) = \frac{1}{s}\bar{f}(s)$

Th10 if $f(t)$ is periodic with T then
$$L(f(t)) = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

Proof $L(f(t)) = \int_0^T e^{-st} f(t)\,dt + \int_T^{2T} e^{-st} f(t)\,dt + \cdots$
$= \left(1 + e^{-sT} + e^{-2sT} + \cdots\right) \int_0^T e^{-st} f(t)\,dt$
$= \frac{1}{1-e^{-sT}} \int_0^T e^{-sT} f(t)\,dt$

Th11 $L(u(t)) = 1/s$
Proof) $L(u(t)) = \int_0^\infty 1 \cdot e^{-st}\,dt = \left.\frac{e^{-st}}{-s}\right|_0^\infty = \frac{1}{s}$

Th12 $f(t) = \begin{cases} f_1(t) & t < a_1 \\ f_2(t) & a_1 \le t < a_2 \\ f_3(t) & t > a_2 \end{cases}$ is eq to; $f(t) = f_1(t) + \{f_2(t) - f_1(t)\} u(t-a_1) + \{f_3(t) - f_2(t)\} u(t-a_2)$

Also true till n partitions.

Th13 $L(f(t)) = L(f_1(t)) + e^{-a_1 t} L(f_2(t+a_1) - f_1(t+a_1)) + e^{-a_2 t}($
$f_3(t+a_2) - f_2(t+a_2))$
from above,

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Inverse Laplace Theorem

↳ ~~If $L(f(t)) = \bar{f}(s)$ then $f(t)$ is inv Laplace~~

$$f(t) = L^{-1}(\bar{f}(s))$$

Examples

↳ Just opposite of earlier.

Linear property

Th 14 Inverse Transform is linear

Proof) $L(f_1(t)) = \bar{f}_1(s) \quad L(f_2(t)) = \bar{f}_2(s)$

$$f_1(t) = L^{-1}(\bar{f}_1(s)),\ f_2(t) = L^{-1}(\bar{f}_2(s))$$

$$L(c_1 f_1(t) + c_2 f_2(t)) = c_1 \bar{f}_1(s) + c_2 \bar{f}_2(s)$$

$$L^{-1}(c_1 \bar{f}_1(s) + c_2 \bar{f}_2(s)) = c_1 f_1(t) + c_2 f_2(t)$$

$$= c_1 L^{-1}(\bar{f}_1(s)) + c_2 L^{-1}(f_2(s))$$

Shifting Property

Th 17 $L^{-1}(\bar{f}(s-a)) = e^{at} L^{-1}(\bar{f}(s))$

Proof Let $L^{-1}(\bar{f}(s)) = f(t) \Rightarrow L(e^{at} f(t)) = \bar{f}(s-a)$

$$\therefore L^{-1}(\bar{f}(s-a)) = e^{at} ft = e^{at} \bar{f}(s)$$

Second Shifting Property

Th 18 if $L^{-1}(\bar{f}(s)) = f(t)$, then $L^{-1}(e^{-at} \bar{f}(s)) = \begin{cases} f(t-a) & t > a \\ 0 & t < 0 \end{cases}$

$$= f(t-a)u(t-a)$$

Proof) ~~$g(t) = f(t-a)u(t-a)$~~

~~$L(g(t)) = e^{-at} \bar{f}(s)$~~

$$L^{-1}(e^{-at} \bar{f}(s)) = g(t) = f(t-a)\, u(t-a)$$

Scaling Property

Th 19 $L^{-1}(\bar{f}(as)) = \frac{1}{a} f\left(\frac{t}{a}\right)$

Proof) $L\left(\frac{1}{a} f\left(\frac{t}{a}\right)\right) = \frac{1}{a} L\left(f\left(\frac{t}{a}\right)\right)$

$$= \frac{1}{a} \frac{1}{1/a} \bar{f}\left(\frac{s}{1/a}\right)$$

$$= \bar{f}(as)$$

Th20 $L^{-1}(\bar{f}^n(s)) = (-1)^n t^n L^{-1}(\bar{f}(s))$

Proof $L(f(t)) = \bar{f}(s)$

$L(t^n f(t)) = (-1)^n \frac{d^n}{ds^n} \bar{f}(s) = (-1)^n \bar{f}^n(s)$

$(-1)^n L(t^n f(t)) = \bar{f}^n(s)$

$L^{-1}(\bar{f}^n(s)) = (-1)^n t^n f(t)$

$= (-1)^n t^n L^{-1}(\bar{f}(s))$

Th21 $L^{-1}(s^n \bar{f}(s)) = f^n(t)$

Proof $L(f^n(t)) = -f^{n-1}(0) - s f^{n-2}(0) - \cdots - s^{n-1} f(0) + s^n \bar{f}(s)$

$= s^n \bar{f}(s)$

$L^{-1}(s^n \bar{f}(s)) = f^n(t)$

Th22 $L^{-1}\left(\frac{\bar{f}(s)}{s}\right) = \int_0^t f(u)\,du$

Proof) $L\left(\int_0^t f(u)\,du\right) = \frac{1}{s}\bar{f}(s)$

$\Rightarrow L^{-1}\left(\frac{1}{s}\bar{f}(s)\right) = \int_0^t f(u)\,du$

Th22 $L^{-1}\left(\int_t^\infty f(u)\,du\right) = f(t)/t$

Proof) $L\left(\frac{1}{t} f(t)\right) = \int_t^\infty f(u)\,du$

$L^{-1}\left(\int_t^\infty f(u)\,du\right) = \frac{1}{t} f(t)$

Convulutions

$f * g = \int_0^t f(u)\, g(t-u)\,du$

Th24 $f * g = g * f$

Proof) $f * g = \int_0^t f(u)\, g(t-u)\,du = -\int_t^0 f(t-v)\, g(v)\,dv = \int_0^t g(v) f(t-v)\,dv$

$= g * f$

Th25 Convulution Thm

if $L^{-1}(\bar{f}(s)) = f(t)$ and $L^{-1}(\bar{g}(s)) = g(t)$,

then, $L^{-1}(\bar{f}(s) \cdot \bar{g}(s)) = \int_0^t f(u)\, g(t-u)\,du = f * g$.

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Solving ODE

Eg $(D^2+6D+9)y=0$

Ans Let $L(y)=\bar{y}$

$L(y'')+6L(y')+9L(y)=0$

$s^2\bar{y}-sy(0)-y'(0)+6s\bar{y}-6y(0)+9\bar{y}=0$

$(s^2+6s+9)\bar{y}=7+s$

$\bar{y}=\frac{s+7}{s^2+6s+9}=\frac{1}{(s+3)}+\frac{4}{(s+3)^2}$

$y=L^{-1}\left(\frac{1}{s+3}\right)+L^{-1}\left(\frac{4}{(s+3)^2}\right)$

$=e^{-3t}+4e^{-3t}L^{-1}\left(\frac{1}{s^2}\right)$

$=e^{-3t}+4e^{-3t}t$

$\boxed{y=(4t+1)e^{-3t}}$ ← soln