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Basic Probability Theory

Set: collection of distinct & well defined objects
$A \subseteq B \Rightarrow A$ is subset of B

Random Exp: Event (E) is random if
i) all possible outcomes of E are known
ii) it is impossible to predict which outcome will come at particular performance of E
iii) E can be repeated under identical conditions infinite times

Eventspace: Set of all possible outcomes $\Rightarrow$ called S.
Event: An circumstance $\rightarrow \in S$.
Impossible Event: $P(E)=0$
Certain Event: $P(E)=1$
Compliment Event: Not of any event.
Mutually exclusive: $A \cap B = \phi$
Exhaustive: consider
Equally Likely: $P(E_1)=P(E_2)$

Probability
↳ Let E be a random exp such that its space S contains a finite number, n of event points, which are equally likely. if A is connected to $m(A)$ of such points, then probability of A is

$$P(A) = \frac{m(A)}{n}$$

a) $0 \leq P(A) \leq 1$ b) $P(S)=1$ c) $P(\varphi)=0$ d) $P(A')=1-P(A)$

Axiomatic Def^n

$A_1)$ $P(A) \geq 0 \ \forall A \in S$

$A_2)$ $P(S) = 1$

$A_3)$ if $A_1, \ldots, A_n, \ldots$ be finite or infinite no of pairwise mutually exclusive events ie $A_i \cap A_j = \phi$ for $i \neq j$ then.

$$P(A_1 \cup A_2 \ldots \cup A_n) = P(A_1) + P(A_2) - \ldots + P(A_n) + \ldots$$

Th1 Probability of impossible event is zero

Proof
$$S + \varphi = S$$
$$S\varphi = S$$
$$P(S+\varphi) = P(S)$$
$$P(S) + P(\varphi) = P(S)$$
$$P(\varphi) = 0$$

Th2 $P(A') = 1 - P(A)$ and $0 \leq P(A) \leq 1$

Ans
$$A + \bar{A} = S$$
$$P(A + \bar{A}) = P(S)$$
$$P(A) + P(\bar{A}) = 1$$
$$P(\bar{A}) = 1 - P(A)$$

Since $P(A) \geq 0 \Rightarrow 0 \leq P(A) \leq 1$

Th3 $P(A'B) = P(B) - P(AB)$

Proof
$$AB + \bar{A}B = B$$
$$P(AB + \bar{A}B) = B$$
$$P(AB) + P(\bar{A}B) = P(B)$$
$$P(\bar{A}B) = P(B) - P(AB)$$

Th4 $P(A+B) = P(A) + P(B) - P(AB)$

Proof
$$P(A+B) = P(A + \bar{A}B)$$
$$= P(A) + P(\bar{A}B)$$
$$= P(A) + P(B) - P(AB)$$

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Th5 $P(A+B+C) = \Sigma P(A) - \Sigma P(AB) + P(ABC)$

Proof: by Th4

Derivation of Classical Defn

$U_1 + U_2 + \dots + U_n = S$

$P(U_1 + \dots + U_n) = P(S)$

$\Sigma P(U_i) = 1$

Let them be equal prob

$\Rightarrow P(U_1) = P(U_2) = \dots = P(U_n) = \frac{1}{n}$

$A = U_1 + U_2 + \dots + U_m$

$P(A) = \frac{1}{n} + \dots + \frac{1}{n}$

$= \frac{m(A)}{n}$

Th6 $P(AB) \le P(A) \le P(A+B) \le P(A) + P(B)$

Proof

$AB + AB' = A$

$P(AB + AB') = P(A)$

$P(AB) + P(AB') = P(A)$

$P(AB') \geq 0$ so, $P(AB) \le P(A)$

$P(AB) \le P(A)$

$P(A+B) = P(A) + P(B) - P(AB)$

$= P(A) + P(A'B)$

$P(A) \le P(A+B) < P(A) + P(B)$

Th7 $P(A_1 + A_2 + \dots + A_n) \le P(A_1) + P(A_2) + \dots + P(A_n)$ [Boole's Identity]

Proof $P(A_1 + A_2) \le P(A_1) + P(A_2)$

~~True for 1, 2.~~

for $n = k, \Rightarrow n = k+1$

$P(A_1 + \dots + A_k) \le P(A_1) + P(A_2) + \dots + P(A_k)$

$P(A_1 + \dots + A_k + A_{k+1}) = P(A_1 + A_2 + \dots + A_k + A_{k+1})$

$\le P(A_1 + \dots + A_k) + P(A_{k+1})$

$\le P(A_1) + P(A_2) + \dots + P(A_{k+1})$

By induction, proved.

Thm i) $P(A_1 A_2 \ldots A_n) \geq 1 - \sum_{i=1}^{n} P(\bar{A}_i)$ [Bonferroni's Inequality]

ii) $P(A_1 \ldots A_n) \geq \sum_{j=1}^{n} P(A_j) - (n-1)$

Proof i) $P(\bar{A}_1 + \bar{A}_2 + \ldots + \bar{A}_n) \leq P(\bar{A}_1) + P(\bar{A}_2) + \ldots + P(\bar{A}_n)$

$$1 - P(A_1 A_2 \ldots A_n) \leq \sum_{i=1}^{n} P(\bar{A}_i)$$

$$P(A_1 A_2 \ldots A_n) \geq 1 - \sum_{i=1}^{n} P(\bar{A}_i)$$

ii) $P(\bar{A}_i) = 1 - P(A_i)$

$$1 - \sum_{1}^{n} P(\bar{A}_i) = 1 - \left[\sum_{1}^{n} (1 - P(A_i))\right]$$

$$= \sum_{1}^{n} P(A_i) - (n-1)$$

$$P(A_1 A_2 \ldots A_n) \geq \sum_{i=1}^{n} P(A_i) - (n-1)$$

Th: If $A_1, A_2, \ldots, A_n$ are pairwise mutually exclusive events,

$$P(B) = \sum_{i=1}^{n} P(A_i)P(B/A_i)$$

Proof: $A_i A_j = \phi$ if $i \neq j$

$A_1 + A_2 + \ldots + A_n = S$

$B = SB = (A_1 + A_2 + \ldots + A_n)B = A_1B + \ldots + A_nB$

$(A_iB)(A_jB) = (A_i A_j)B = \phi B = \phi$

$P(B) = P(A_1B + A_2B + \ldots + A_nB)$

$= P(A_1B) + P(A_2B) + \ldots + P(A_nB)$

$= \sum P(A_iB)$

$= \sum P(A_i) P(B/A_i)$

Now, $P(A_i/B) = \dfrac{P(A_iB)}{P(B)}$

$= \dfrac{P(A_i)P(B/A_i)}{P(B)}$

$P(A_i/B) = \dfrac{P(A_i)\,P(B/A_i)}{\sum P(A_i)\,P(B/A_i)}$ Bays Thm

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Independent Events

↳ if ~~B~~ $P(A/B) = A$ & $P(B/A) = B$, $P(A) \neq P(B) \neq 0$.

if $P(A)$ & $P(B)$ independent

$$P(AB) = P(A).P(B)$$

Defn1 if $P(AB) = P(A).P(B)$, ~~B~~ A, B are independent

Defn2 A, B, C are pairwise independent if.

$$P(AB) = P(A)P(B)$$
$$P(BC) = P(B)P(C)$$
$$P(CA) = P(C)P(A)$$

Defn3 A, B, C are mutually independent if.

~~P(ABC)=P(A)~~

$P(ABC) = P(A)P(B)P(C)$ & pairwise ind

Random Var

↳ a variable which takes numerical values determined by outcoms of a random experiment

$(X=x) = \{w \in S \mid X(w)=x\}$

$(X \leq a) = \{w \in S \mid X(w) \leq a\}$

$(X > b) = \{w \in S \mid X(w) > b\}$

Th1 if X_1, X_2 are random vars. of S, even $X_1 + X_2$ is.

Proof Let X_1, X_2 be random vars. of S in Event E

Let $w \in S$ be outcome of E,

So, $X_1(w)$ & $X_2(w) \in \mathbb{R}$

$\Rightarrow X_1(w) + X_2(w) \in \mathbb{R}$

$\Rightarrow (X_1+X_2)(w) \in \mathbb{R}$

Also for $X_1 - X_2$

proved.

Th2 if X_1, X_2 are random vars of S, C_1, C_2 are consts,

CX_1, $C_1X_1 + C_2X_2$, X_1X_2 are all random vars.

Proof same as Th1

Th3 even max (X_1, X_2) is a random var

Proof Same as Th1

Distribution func

Let X be a random var on S associated with a exp.

The cumulative dist func (cdf) on X is

$F_X(x) = P(X \leq x)$, $P(X \leq x) = \{w \in S \mid X(w) \leq x\}$

Properties

1 i) $P(a \leq x \leq b) = F(b) - F(a)$

Proof $X < a$ & $a \leq X \leq b$ are mutually excl.

$(X \leq b) = (X \leq a) \cup (a < X \leq b)$

$P(X \leq b) = P(X \leq a) + P(a < x \leq b)$

$P(a \leq x \leq b) = P(x \leq b) - P(x \leq a)$

$= F(b) - F(a)$

ii) $P(a \le x \le b) = P(x=a) + F(b) - F(a)$

Proof: $P(a \le x \le b) = P(x=a \cup (a < x \le b))$

$= P(x=a) + P(a < x \le b)$

$= P(x=a) + F(b) - F(a)$

iii) $P(a < x < b) = F(b) - F(a) - P(x=b)$

Proof $P(a < x \le b) = P(a < x < b \cup x=b)$

$= P(a < x < b) + P(x=b)$

$P(a < x < b) = P(a < x \le b) - P(x=b)$

$= F(b) - F(a) - P(x=b)$

iv) $P(a \le x < b) = F(b) - F(a) - P(x=b) + P(x=a)$

Proof $P(a \le x < b) = P(x=a \cup a < x < b)$

$= P(x=a) + P(a < x < b)$

$= P(x=a) + F(b) - F(a) - P(x=b)$

2 i) $0 \le F(x) \le 1$

Proof $F(x) = P(X \le x)$

$\Rightarrow 0 \le P(X \le x) \le 1$

$\Rightarrow 0 \le F(x) \le 1$

ii) $x < y \Rightarrow F(x) \le F(y)$

Proof $x < y \Rightarrow F(y) - F(x) = P(x < X \le y)$

$\Rightarrow F(y) - F(x) \ge 0$

$\Rightarrow F(y) \ge F(x)$

3 $F(-\infty) = \lim_{x \to -\infty} F(x) = 0 \quad F(\infty) = \lim_{x \to \infty} F(x) = 1$

Proof $F(\infty) = \lim_{x \to \infty} F(x) = \lim_{x \to \infty} P(X \le x) = P(S) = 1$

$F(-\infty) = \lim_{x \to -\infty} F(x) = \lim_{x \to -\infty} P(X \le x) = P(\phi) = 0$

④ $F(x)$ is continuous on right.

Proof $P(a < X \le a+h) = P(X=a) + F(a+h) - F(a)$

$\lim_{h\to 0^+} P(a < X \le a+h) = P(X=a) + \lim_{h\to 0^+} F(a+h) - F(a)$

$P(X=a) = P(X=a) + \lim_{h\to 0^+} F(a+h) - F(a)$

$\lim_{h\to 0^+} F(a+h) = F(a)$

⑤ $P(X=a) = F(a) - \lim_{h\to 0^+} F(a-h) = F(a) - \lim_{x\to a^-} F(x)$

Proof $P(a-h < X \le a) = F(a) - F(a-h)$

$P(X=a) = \lim_{h\to 0^+} P(a-h < X \le a) = \lim_{h\to 0^+} [F(a) - F(a-h)]$

$= F(a) - \lim_{h\to 0^+} F(a-h)$

$= F(a) - \lim_{x\to a^-} F(x) = F(a) - F(a-0)$

Discrete Random Var.

↳ if a random var takes finite/infinite of distinct vals

range of Discrete is finite or countable infinite

Prob mass func

↳ for discrete X,

$P(X = x_i) = p(x_i) = P_i$

i) $P_i \ge 0$ ii) $\sum_{i=0}^{\infty} P_i = 1$

discrete dist func

$F(x) = P(X = x_0) + P(X = x_1) + \dots + (X = x_i)$

$= \sum_{k=0}^{i} P_k \quad x_i \le x < x_{i+1}$

Continuous random var

↳ X is a cont random var s.t.

$$P\left(x - \frac{dx}{2} \le X \le x + \frac{dx}{2}\right) = f(x)\,dx$$

i) $f(x) \geq 0 \ \forall x \in \mathbb{R}$

ii) $\int_{\mathbb{R}} f\,dx = 1$

iii) $\int_E f(x)\,dx = P(E)$

$\Rightarrow P(a \le X \le b) = \int_a^b f(x)\,dx$

$\Rightarrow P(X = a) = P(a \le X \le a) = \int_a^a f(x)\,dx = 0$

$\Rightarrow P(a < x < b) = P(a \le x < b) = P(a < x \le b) = P(a \le x \le b) = F(b) - F(a)$

$$f(x)\,dx = dF(x)$$

$$f(x) = F'(x)$$

Cont dist func

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt$$

$\Rightarrow F(-\infty) = 0$

$\Rightarrow F(\infty) = 1$

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Mean / Expectation

$$E(x) = \sum_{i=1}^{\infty} p_i x_i \quad s.t \sum_{j=1}^{\infty} p_i = 1$$

if cont, $E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$. $\quad s.t \int_{-\infty}^{\infty} f(x)\,dx = 1$

Properties

1. i) if $X = a$; then $E(x) = a$
   ii) if $a \leq X \leq b$, then $a \leq E(x) \leq b$
2. if $Y = aX$, $E(Y) = a\,E(X)$
   if $Y = a \pm bX$, $E(Y) = a \pm b\,E(X)$
3. $E(XY) = E(X)\,E(Y)$, if indepedent
4. if $Z = aX \pm bY$; $E(Z) = a\,E(X) \pm b\,E(Y)$
5. if $g(x)$ is a function,

$$E\{g(x)\} = \sum p_i\, g(x_i) \quad \text{(disc.)}$$

$$= \int_{-\infty}^{\infty} g(x) f(x)\,dx \quad \text{(cont)}$$

Variance

i) $Var(X) = E[(X-m)^2] = E[\{x - E(x)\}^2] = \sum_{i=1}^{\infty} (x_i - m)^2 p_i$

ii) $= \int_{-\infty}^{\infty} (x-m)^2 f(x)\,dx$

~~Standard Deviation~~

$\hookrightarrow SD = \sqrt{Var(X)} = \sigma$

Properties

i) $Var(x) = E(x^2) - E(x)^2 = E(x^2) - m^2$

ii) $Var(aX+b) = a^2 Var(X)$

iii) $Var(a) = 0$

iv) $Var(X) = E[X(X-1)] - m(m-1)$

Proof

i) $Var(x) = E[(x-m)^2]$

$= E[x^2 - 2mx + m^2]$

$= E[x^2] - 2m E[x] + m^2$

$= E[x^2] - m^2$

ii) $Var(aX+b) = E[(aX+b-am-b)^2]$

$= E[a^2(X-m)^2]$

$= a^2 E[(X-m)^2]$

$= a^2 Var(X)$

iii) $Var(a) = E[[a - E(a)]^2]$

$= E[(a-a)^2] = 0$

iv) $Var(x) = E[\{x-M\}^2]$

$= E[x^2 - x + x - 2mx + m^2]$

$= E[x^2 - x] + E(x) - 2mE(x) + u^2$

$= E[x(x-1)] - m(m-1)$

Coefficient of variance

$$CV = \frac{SD}{mean} \times 100$$

3) $x \le |x|$

$x - |x| \le 0$

$E(x - |x|) \le 0$

$E(x) - E(|x|) \le 0$

$\Rightarrow E(x) \le E[|x|]$

$-E(x) \le E(|x|)$

$|E(x)| \le E(|x|)$

Binomial

r successes in n trials

$$P(X=r) = {}^{n}C_{r}\, p^{r} q^{n-r} = {}^{n}C_{r}\, p^{r} (1-p)^{n-r}$$

$$p+q=1$$

$$F(x) = P(X \le x) = \sum {}^{n}C_{i}\, p^{i} (1-p)^{n-i}$$

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$$P(r) = \frac{n!}{(n-r)!\,r!} q^{n-r} p^r$$

$$P(x=r+1) = \frac{n!}{(r+1)!\,(n-r-1)!} p^{r+1} q^{n-r-1}$$

$$\frac{P(r+1)}{P(r)} = \frac{n-r}{r+1}\frac{p}{q}$$

$$P(r+1) = \frac{n-r}{r+1}\frac{p}{q} P(r)$$

$\Rightarrow$ Mean $= np$ Var $= npq$

Proof: $B(n,p)$

$$E[x] = \sum r P(r) = \sum r\, {}^nC_r\, p^r q^{n-r} = \sum \frac{n!}{r!\,(n-r)!} p^r q^{n-r}$$

$$= np \sum \frac{(n-1)!}{(r-1)!\,(n-r)!} p^{r-1} q^{n-r} = np \sum_{1}^{n} {}^{n-1}C_{r-1} p^{r-1} q^{n-r}$$

$$= np \sum_{0}^{n-1} {}^{n-1}C_r\, p^r q^{n-r-1}$$

$$= np\,(p+q)^{n-1}$$

$$= np$$

$$E[x(x-1)] = \sum r(r-1) P(r)$$

$$= \sum r(r-1) \frac{n!}{r!\,(n-r)!} p^r q^{n-r}$$

$$= n(n-1)p^2 \sum \frac{(n-2)!}{(r-2)!\,(n-r)!} p^{r-2} q^{n-r}$$

$$= (n-1)np^2 \sum {}^{n-2}C_r\, p^r q^{n-r-2} = n(n-1)p^2 (p+q)^{n-2}$$

$$= n(n-1)p^2$$

$$Var(x) = E[x(x-1)] - m(m-1)$$

$$= n(n-1)p^2 - np(p-1) = np(1-p) = npq$$

$$SD = \sqrt{npq}$$

Poisson

$$P(r) = \begin{cases} \frac{e^{-\lambda}\lambda^r}{r!} & r=1,2 \\ 0 & \text{elsr} \end{cases}$$

i) $P(r) > 0$

ii) $\sum P(r) = e^{-\lambda} \sum \frac{\lambda^r}{r!} = 1$

Mean $= \lambda$ $\quad Var(x) = \lambda$

Proof $\quad X \sim P(\lambda)$

$$P(r) = \begin{cases} \frac{e^{-\lambda}\lambda^r}{r!} & r=1,2 \ldots \\ 0 & \text{else} \end{cases}$$

$$E(x) = \sum r P(r) = \sum_{1}^{\infty} r \frac{e^{-\lambda}\lambda^r}{r!}$$

$$= \lambda e^{-\lambda} \sum_{1}^{\infty} \frac{\lambda^{r-1}}{(r-1)!}$$

$$= \lambda e^{-\lambda} \underbrace{\sum_{0}^{\infty} \frac{\lambda^r}{r!}}_{\to e^{\lambda}}$$

$$= \lambda$$

$$E(x(x-1)) = \sum r(r-1) P(r) = \sum r(r-1) \frac{e^{-\lambda}\lambda^r}{r!}$$

$$= \lambda^2 e^{-\lambda} \sum_{2}^{\infty} \frac{\lambda^{r-2}}{(r-2)!} = \lambda^2 e^{-\lambda} \sum_{0}^{\infty} \frac{\lambda^r}{r!}$$

$= \lambda^2$

$Var(x) = \lambda^2 + (\lambda)(\lambda - 1)$

$= \lambda$

$S.D = \sqrt{\lambda}$

<u>Thm</u> i) no of trials $(n \to \infty)$

ii) prob of success $(p \to 0)$

iii) $\lambda = np \to$ finite const

$$P(r) = {}^nC_r p^r (1-p)^{n-r} \to \frac{e^{-\lambda}\lambda^r}{r!} \text{ as } n \to \infty \text{ \& } np = \lambda$$

<u>Proof:</u>

$$P(r) = {}^nC_r p^r (1-p)^r$$

$$= \frac{n(n-1)\cdots(n-r+1)}{r!}\left(\frac{\lambda}{n}\right)^r\left(1-\frac{\lambda}{n}\right)^{n-r}$$

$$= \frac{\lambda^r}{r!}\frac{(n)(n-1)\cdots(n-r+1)}{n^r}\frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^r}$$

$$= \frac{\lambda^r}{r!}\left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\cdots\left(\frac{n-r+1}{n}\right)\left(\frac{\left(\left(1-\frac{\lambda}{n}\right)^{-\frac{n}{\lambda}}\right)^{-\lambda}}{\left(1-\frac{\lambda}{n}\right)^r}\right)$$

$$\lim_{n\to\infty} P(r) = \frac{\lambda^r}{r!}e^{-\lambda}$$

<u>$r=0$</u> $\quad P(0) = {}^nC_0 p^0 (1-p)^n = \left(1-\frac{\lambda}{n}\right)^n$

$= e^{-\lambda}$