Design and Analysis of Algorithm

Dynamic Programming (Matrix Chain Multiplication)



Lecture - 58

Overview

- Dynamic Programming is a general algorithm design technique for solving problems defined by recurrences with overlapping subproblems
- Invented by American mathematician "Richard Bellman) in the year 1950s to solve optimization problems and later assimilated by Computer Science.
- "Programming" here means "planning"

- "Method of solving complex problems by breaking them down into smaller sub-problems, solving each of those sub-problems just once, and storing their solutions."
- The problem solving approach looks like Divide and conquer approach.(which is not true)

Difference between Dynamic programming and Divide and Conquer approach.

Divide & Conquer	Dynamic Programming
Partitions a problem into independent smaller sub-problems	Partitions a problem into overlapping sub-problems
Doesn't store solutions of sub- problems. (Identical sub-problems may arise - results in the same computations are performed repeatedly.)	Stores solutions of sub- problems: thus avoids calculations of same quantity twice
3. Top down algorithms: which logically progresses from the initial instance down to the smallest sub-instances via intermediate sub-instances.	3. Bottom up algorithms: in which the smallest sub-problems are explicitly solved first and the results of these used to construct solutions to progressively larger sub-instances

Is a Four-step methods

- 1. Characterize the structure of an optimal solution.
- 2. Recursively define the value of an optimal solution.
- 3. Compute the value of an optimal solution, typically in a bottom-up fashion.
- 4. Construct an optimal solution from computed information.

Problems:

- 1. 0/1 Knapsack Problem
- 2. Floyd-Warshall Algorithm
- 3. Longest Common Sub-sequence
- 4. Matrix Chain Multiplication

Problem 4: Matrix Chain Multiplication

Problem:

"Given dimensions $p_0, p_1, p_2, \ldots, p_n$ corresponding to matrix sequence $\langle A_1, A_2, \ldots, A_n \rangle$ of n matrices, where for $i=1,2,\ldots,n$, matrix A_i has dimension $p_{i-1} \times p_i$, determine the "multiplication sequence" that minimizes the number of scalar multiplications in computing $\langle A_1, A_2, \ldots, A_n \rangle$."

Problem 4: Matrix Chain Multiplication

Problem:

That is determine ho to parenthesize the multiplication.

Example:

$$A_{1}, A_{2}, A_{3}, A_{4} = ((A_{1}A_{2})(A_{3}A_{4}))$$

$$(A_{1}(A_{2}(A_{3}A_{4})))$$

$$(A_{1}((A_{2}A_{3})A_{4}))$$

$$(((A_{1}A_{2})(A_{3}A_{4}))$$

$$(((A_{1}A_{2})A_{3})A_{4})$$

Problem 4: Matrix Chain Multiplication

Given a $p \times q$ matrix A, a $q \times r$ matrix B and a $r \times s$ matrix C, then ABC can be computed in two ways (AB)C and A(BC):

The number of multiplications needed are:

```
mult[(AB)C] = pqr + prs,

mult[A(BC)] = qrs + pqs.
```

When p = 5, q = 4, r = 6 and s = 2, then

```
mult[(AB)C] = 180,
mult[A(BC)] = 88.
```

Which is a big difference. Hence the implication is the the multiplication "sequence" (parenthesization) is very important.

Problem 4: Matrix Chain Multiplication

Step 1: Characterize the structure of an optimal solution

- Decompose thee problem into subproblems:
 - For each pair $1 \le i \le j \le n$, determine the multiplication sequence for $A_{i...j} = A_i, A_{i+1}, \dots, A_j$ that minimize the number of multiplications.
 - Clearly, $A_{i...i}$ is a $p_{i-1} \times p_i$ matrix.
- High-Level Parenthesization for $A_{i,j}$
 - For any optimal multiplication sequence, at the last step you are multiplying two matrices $A_{i...k}$ and $A_{k+1...j}$ for some k. That is,

$$A_{i...j} = (A_i ... A_k) (A_{k+1} ... A_j) = A_{i...k} A_{k+1...j}$$

Example

$$A_{3..6} = ((A_3(A_4A_5))(A_6)) = A_{3..5}A_{6..6}$$
. (Here $k = 5$.)

Problem 4: Matrix Chain Multiplication

Step 1: Characterize the structure of an optimal solution

- Thus the problem of determining the optimal sequence of multiplications is divided into 2 questions:
 - How do we decide where to split the chain (what is the value of k)?

(Search all possible values of k)

• How do we parenthesize the sub chains $A_{i..k}$ and $A_{k+1..j}$?

(Problem has optimal substructure property that $A_{i...k}$ and $A_{k+1...j}$ must be optimal so the same procedure can be applied recursively)

Problem 4: Matrix Chain Multiplication

Step 1: Characterize the structure of an optimal solution

- What is Optimal Substructure Property?
 - If final "optimal" solution of $A_{i...j}$ involves splitting into $A_{i...k}$ and $A_{k+1...j}$ at final step then parenthesization of $A_{i...k}$ and $A_{k+1...j}$ in final optimal solution must also be optimal for the subproblems "standing alone":
 - If parenthisization of $A_{i..k}$ was not optimal we could replace it by a better parenthesization and get a cheaper final solution, leading to a contradiction.
 - Similarly, if parenthisization of $A_{k+1...j}$ was not optimal we could replace it by a better parenthesization and get a cheaper final solution, also leading to a contradiction.

Problem 4: Matrix Chain Multiplication

Step 2: Recursively define the value of optimal solution.

• For $1 \le i \le j \le n$, let m[i,j] denote the minimum number of multiplications needed to compute $A_{i...j}$. The optimum cost can be described by the following recursive definition.

$$m[i,j] = \begin{cases} 0 & \text{if } i = j \\ \min_{i \le k < j} \{m[i,k] + m[k+1,j] + p_{i-1}p_k p_j\} & \text{if } i < j \end{cases}$$

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

```
MATRIX - CHAIN - ORDER(p)
 1 n \leftarrow length[p] - 1
2 let m[1...n, 1...n] and s[1...n - 1, 2...n] be new tables.
3 \ for \ i \leftarrow 1 \ to \ n
4 m[i,i] \leftarrow 0
5 \ for \ l \leftarrow 2 \ to \ n \triangleright l \ is \ the \ chain \ length.
    for i \leftarrow 1 to n - l + 1
j \leftarrow i + l - 1
8 m[i,j] \leftarrow \infty
9 for k \leftarrow i to j - 1
10
             q \leftarrow m[i,k] + m[k+1,j] + p_{i-1}p_kp_j
11
            if q < m[i,j]
12
                 then m[i,j] \leftarrow q
13
                       s[i,j] \leftarrow k
14 return m and s
```

Lets illustrate the example with the help of an example.

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of dimension is :

(30, 35, 15, 5, 10, 20, 25)

Solution:

Here

p_0	p_1	p_2	p_3	p_4	p_5	p_6
30	35	15	5	10	20	25

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of dimension is :

(30, 35, 15, 5, 10, 20, 25)

Solution:

Here

p_0	p_1	p_2	p_3	p_4	p_5	p_6
30	35	15	5	10	20	25

Matrix	A_1	A_2	A_3	A_4	A_5	A_6
Dimensions	30 x 35	35 x 15	15 x 5	5 x 10	10 x 20	20 x 25

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

		p_2				
30	35	15	5	10	20	25

Solution:

	1	2	3	4	5	6
1	0					
1 2 ⁷ / ₂ 3		0				
3	3		0			
4		Z3		0		
4 5 6			A		0	
6				3		0
		m	mati		7	_

	2	3	4	5	6
1					
2					
3					
4					
5					

s matrix

$$1 n \leftarrow length[p] - 1$$

2 let m[1..n, 1..n] and s[1..n - 1, 2..n] be new tables.

$$3 \ for \ i \leftarrow 1 \ to \ n$$

4
$$m[i,i] \leftarrow 0$$

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p_1	p_2	p_3	p_4	p_5	p_6
30	35	15	5	10	20	25

Solution:

11

13

_	1	2	3	4	5	6
1	0	8				
27)	0				
3	(3)		0			
4		33		0		
5			A		0	
6				3		0
		m	mati		7	

	2	3	4	5	6
1					
2					
3					
4					
5					

s matrix

then $m[i,j] \leftarrow q$

 $s[i, j] \leftarrow k$

if q < m[i,j]

$$l = 2$$
, $i = 1, j = 2, k = 1$
 $q =$

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p ₁	p_2	p_3	p_4	p_5	p_6
30	35	15	5	10	20	25

Solution:

	1	2	3	4	5	6
1	0	15750				
27		0				
3	(3		0			
4		73		0		
5			A		0	
6				3		0
		m	matı		76	

	2	3	4	5	6
1	1				
2					
3					
4					
5					

$$5 for l \leftarrow 2 to n$$

$$6 for i \leftarrow 1 to n - l + 1$$

$$7 j \leftarrow i + l - 1$$

$$8 m[i,j] \leftarrow \infty$$

$$9 for k \leftarrow i to j - 1$$

$$10 q \leftarrow m[i,k] + m[k + 1,j] + p_{i-1}p_kp_j$$

$$11 if q < m[i,j]$$

$$12 then m[i,j] \leftarrow q$$

$$13 s[i,j] \leftarrow k$$

$$l = 2$$
 , $i = 1, j = 2, k = 1$
 $q = 0 + 0 + 30 * 35 * 15 = 15750$

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p_1	p_2	p_3	p_4	p_5	p_6
30	35	15	5	10	20	25

Solution:

	1	2	3	4	5	6
1	0	15750				
2 ⁷ / ₂		0	8			
3	(3)		0			
4		(3)		0		
5			A		0	
6				75		0
		m	matı		70	

	2	3	4	5	6
1	1				
2					
3					
4					
5					

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p_1	p_2	p_3	p_4	p_5	p_6
30	35	15	5	10	20	25

Solution:

	1	2	3	4	5	6
1	0	15750				
2 ⁷ / ₂		0	2625			
3	(3))	0			
4		(3)		0		
5			A		0	
6				75		0
		m	mati		76	

	2	3	4	5	6
1	1				
2		2			
3					
4					
5					

$$5 for l \leftarrow 2 to n$$

$$6 for i \leftarrow 1 to n - l + 1$$

$$7 j \leftarrow i + l - 1$$

$$8 m[i,j] \leftarrow \infty$$

$$9 for k \leftarrow i to j - 1$$

$$10 q \leftarrow m[i,k] + m[k + 1,j] + p_{i-1}p_kp_j$$

$$11 if q < m[i,j]$$

$$12 then m[i,j] \leftarrow q$$

$$13 s[i,j] \leftarrow k$$

$$l = 2$$
, $i = 2, j = 3, k = 2$
 $q = 0 + 0 + 30 * 15 * 5 = 2625$

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p_1	p_2	p_3	p_4	p_5	p_6
30	35	15	5	10	20	25

Solution:

	1	2	3	4	5	6
1	0	15750				
2 ⁴ / ₂		0	2625			
3	3		0	8		
4		(3))	0		
5			(A		0	
6				7,		0
		m	mati		76	_

	2	3	4	5	6
1	1				
2		2			
3					
4					
5					

s matrix

l = 2, i = 3, i = 4, k = 3

q =

$$5 \ for \ l \leftarrow 2 \ to \ n$$
 $6 \ for \ i \leftarrow 1 \ to \ n - l + 1$
 $7 \ j \leftarrow i + l - 1$
 $8 \ m[i,j] \leftarrow \infty$
 $9 \ for \ k \leftarrow i \ to \ j - 1$
 $10 \ q \leftarrow m[i,k] + m[k + 1,j] + p_{i-1}p_kp_j$
 $11 \ if \ q < m[i,j]$
 $12 \ then \ m[i,j] \leftarrow q$
 $13 \ s[i,j] \leftarrow k$

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

1	00	p_1	p_2	p_3	p_4	p_5	p_6
	30	35	15	5	10	20	25

Solution:

	1	2	3	4	5	6
1	0	15750				
2 ⁴ / ₂	_	0	2625			
3	3		0	750		
4		(3))	0		
5			(A		0	
6				75		0
		m	matı		76	

	2	3	4	5	6
1	1				
2		2			
3			3		
4					
5					

$$l = 2$$
, $i = 3$, $j = 4$, $k = 3$
 $q = 0 + 0 + 15 * 5 * 10 = 750$

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p_1	p_2	p_3	p_4	p_5	p_6
30	35	15	5	10	20	25

Solution:

	1	2	3	4	5	6
1	0	15750				
1 2 ⁷ / ₂ 3	_	0	2625			
3	3		0	750		
4		3		0	8	
5			A)	0	
6				(35		0
		m	mati		76	

	2	3	4	5	6
1	1				
2		2			
3			3		
4					
5					

5 for
$$l \leftarrow 2$$
 to n
6 for $i \leftarrow 1$ to $n - l + 1$
7 $j \leftarrow i + l - 1$
8 $m[i,j] \leftarrow \infty$
9 for $k \leftarrow i$ to $j - 1$
10 $q \leftarrow m[i,k] + m[k + 1,j] + p_{i-1}p_kp_j$
11 if $q < m[i,j]$
12 then $m[i,j] \leftarrow q$
13 $s[i,j] \leftarrow k$

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p_1	p_2	p_3	p_4	p_5	p_6
30	35	15	5	10	20	25

Solution:

	1	2	3	4	5	6	
1	0	15750					
2 ⁷ / ₂	_	0	2625				
3	3		0	750			
4		33		0	1000		
5			(A) _	0		
6				(75		0	
m matrix							

	2	3	4	5	6
1	1				
2		2			
3			3		
4				4	
5					

s matrix

q = 0 + 0 + 5 * 10 * 20 = 1000

l = 2, i = 4, j = 5, k = 4

5 for
$$l \leftarrow 2$$
 to n
6 for $i \leftarrow 1$ to $n - l + 1$
7 $j \leftarrow i + l - 1$
8 $m[i,j] \leftarrow \infty$
9 for $k \leftarrow i$ to $j - 1$
10 $q \leftarrow m[i,k] + m[k + 1,j] + p_{i-1}p_kp_j$
11 if $q < m[i,j]$
12 then $m[i,j] \leftarrow q$
13 $s[i,j] \leftarrow k$

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p_1	p_2	p_3	p_4	p_5	p_6
30	35	15	5	10	20	25

Solution:

	1	2	3	4	5	6	
1	0	15750					
2 ⁴ / ₂	_	0	2625				
3	3		0	750			
4		3		0	1000		
5			A		0	8	
6				35		0	
	m matrix						

	2	3	4	5	6
1	1				
2		2			
3			3		
4				4	
5					

$$5 for l \leftarrow 2 to n$$

$$6 for i \leftarrow 1 to n - l + 1$$

$$7 j \leftarrow i + l - 1$$

$$8 m[i,j] \leftarrow \infty$$

$$9 for k \leftarrow i to j - 1$$

$$10 q \leftarrow m[i,k] + m[k + 1,j] + p_{i-1}p_kp_j$$

$$11 if q < m[i,j]$$

$$12 then m[i,j] \leftarrow q$$

$$13 s[i,j] \leftarrow k$$

$$l = 2$$
, $i = 4$, $j = 5$, $k = 4$
 $q =$

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p_1	p_2	p_3	p_4	p_5	p ₆
30	35	15	5	10	20	25

Solution:

	1	2	3	4	5	6	
1	0	15750					
2 ⁷ / ₂	_	0	2625				
3	3		0	750			
4		3		0	1000		
5		7 0					
6				3)	0	
	m matrix (%)						

	2	3	4	5	6
1	1				
2		2			
3			3		
4				4	
5					5

$$5 for l \leftarrow 2 to n$$

$$6 for i \leftarrow 1 to n - l + 1$$

$$7 j \leftarrow i + l - 1$$

$$8 m[i,j] \leftarrow \infty$$

$$9 for k \leftarrow i to j - 1$$

$$10 q \leftarrow m[i,k] + m[k + 1,j] + p_{i-1}p_kp_j$$

$$11 if q < m[i,j]$$

$$12 then m[i,j] \leftarrow q$$

$$13 s[i,j] \leftarrow k$$

$$l = 2$$
, $i = 4$, $j = 5$, $k = 4$
 $q = 0 + 0 + 10 * 20 * 25 = 5000$

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p ₁	p_2	p_3	p_4	p_5	p_6
30	35	15	5	10	20	25

Solution:

	1	2	3	4	5	6
1_	0	15750	8			
27	$)_{\overline{a}}$	0	2625			
3	13		0	750		
4		33)	0	1000	
5			A		0	5000
6				75		0
			_		~	

	2	3	4	5	6
1	1				
2		2			
3			3		
4				4	
5					5

m matrix

s matrix

q = 0 + 2625 + 30 * 35 * 5 = 7875

l = 3, i = 1, j = 3, k = 1

5 for
$$l \leftarrow 2$$
 to n
6 for $i \leftarrow 1$ to $n - l + 1$
7 $j \leftarrow i + l - 1$
8 $m[i,j] \leftarrow \infty$
9 for $k \leftarrow i$ to $j - 1$
10 $q \leftarrow m[i,k] + m[k + 1,j] + p_{i-1}p_kp_j$
11 if $q < m[i,j]$
12 then $m[i,j] \leftarrow q$
13 $s[i,j] \leftarrow k$

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p_1	p_2	p_3	p_4	p_5	p_6
30	35	15	5	10	20	25

Solution:

13

	1	2	3	4	5	6
1	0	15750	8			
33		0	2625			
3	43		0	750		
4		(3))	0	1000	
5			A		0	5000
6				75		0
					1	

	2	3	4	5	6
1	1				
2		2			
3			3		
4				4	
5					5

m matrix

 $s[i, j] \leftarrow k$

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p ₁	p_2	p_3	p_4	p_5	p_6
30	35	15	5	10	20	25

Solution:

13

	1	2	3	4	5	6
1	0	15750	7875			
53	$)_{\frown}$	0	2625			
3	13		0	750		
4		33)	0	1000	
5			A		0	5000
6				75		0
					1	

	2	3	4	5	6
1	1	1			
2		2			
3			3		
4				4	
5					5

m matrix

then $m[i,j] \leftarrow q$

 $s[i, j] \leftarrow k$

5 for
$$l \leftarrow 2$$
 to n
6 for $i \leftarrow 1$ to $n - l + 1$
7 $j \leftarrow i + l - 1$
8 $m[i,j] \leftarrow \infty$
9 for $k \leftarrow i$ to $j - 1$
10 $q \leftarrow m[i,k] + m[k + 1,j] + p_{i-1}p_kp_j$ $q = 15750 + 0 + 30 * 15 * 5 = 18000$
11 if $q < m[i,j]$

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p_1	p_2	p_3	p_4	p_5	p_6
30	35	15	5	10	20	25

Solution:

	1	2	3	4	5	6
1	0	15750	7875			
2 ⁷ / ₂		0	2625	∞		
3	(3))	0	750		
4		13		0	1000	
5			Z)	0	5000
6				75		0
		m	mati		76	

	2	3	4	5	6
1	1	1			
2		2			
3			3		
4				4	
5					5

s matrix

q = 0 + 750 + 35 * 15 * 10 = 6000

l = 3, i = 2, j = 4, k = 2

5 for
$$l \leftarrow 2$$
 to n
6 for $i \leftarrow 1$ to $n - l + 1$
7 $j \leftarrow i + l - 1$
8 $m[i,j] \leftarrow \infty$
9 for $k \leftarrow i$ to $j - 1$
10 $q \leftarrow m[i,k] + m[k + 1,j] + p_{i-1}p_kp_j$
11 if $q < m[i,j]$
12 then $m[i,j] \leftarrow q$
13 $s[i,j] \leftarrow k$

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p_1	p_2	p_3	p_4	p_5	p_6
30	35	15	5	10	20	25

Solution:

13

	1	2	3	4	5	6
1	0	15750	7875			
2 ⁷ / ₂		0	2625	8		
3	13		0	750		
4		Z)	0	1000	
5			(A		0	5000
6				75		0
		m	mati		76	

	2	3	4	5	6
1	1	1			
2		2	3		
3			3		
4				4	
5					5

 $s[i, j] \leftarrow k$

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p_1	p_2	p_3	p_4	p_5	p_6
30	35	15	5	10	20	25

Solution:

13

	1	2	3	4	5	6
1	0	15750	7875			
27		0	2625	4375		
2 ⁴ / ₂	(3)		0	750		
4		3)	0	1000	
5			(A		0	5000
6				75	<u> </u>	0
					1	

	2	3	4	5	6
1	1	1			
2		2	3		
3			3		
4				4	
5					5

m matrix

then $m[i, j] \leftarrow q$

 $s[i, j] \leftarrow k$

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p_1	p_2	p_3	p_4	p_5	p_6
30	35	15	5	10	20	25

Solution:

	1	2	3	4	5	6
1	0	15750	7875			
2 ⁷ / ₂	_	0	2625	4375		
3	3		0	750	∞	
4		(3))	0	1000	
5			A		0	5000
6				75		0
		m	mati	rix	76	

	2	3	4	5	6
1	1	1			
2		2	3		
3			3		
4				4	
5					5

$$5 \text{ for } l \leftarrow 2 \text{ to } n$$

 $6 \text{ for } i \leftarrow 1 \text{ to } n - l + 1$
 $7 \text{ } j \leftarrow i + l - 1$
 $l = 3, j = 5, k = 3$
 $q = 0 + 1000 + 15 * 5 * 20 = 2500$

$$\begin{array}{lll}
7 & j \leftarrow l + l - 1 \\
8 & m[i,j] \leftarrow \infty \\
9 & for k \leftarrow i to j - 1 \\
10 & q \leftarrow m[i,k] + m[k + 1,j] + p_{i-1}p_kp_j \\
11 & if q < m[i,j] \\
12 & then m[i,j] \leftarrow q \\
13 & s[i,j] \leftarrow k
\end{array}$$

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p_1	p_2	p_3	p_4	p_5	p_6
30	35	15	5	10	20	25

Solution:

13

	1	2	3	4	5	6
1	0	15750	7875			
2 ⁷ / ₂	_	0	2625	4375		
3	3		0	750	8	
4		(3)		0	1000	
5			A A		0	5000
6				(35		0
		m	mati		76	

	2	3	4	5	6
1	1	1			
2		2	3		
3			3		
4				4	
5					5

s matrix

5 for
$$l \leftarrow 2$$
 to n
6 for $i \leftarrow 1$ to $n - l + 1$
7 $j \leftarrow i + l - 1$
8 $m[i,j] \leftarrow \infty$
9 for $k \leftarrow i$ to $j - 1$
10 $q \leftarrow m[i,k] + m[k + 1,j] + p_{i-1}p_kp_j$ $q = 750 + 0 + 15 * 10 * 20 = 3750$
11 if $q < m[i,j]$

then $m[i, j] \leftarrow q$

 $s[i, j] \leftarrow k$

$$l = 3$$
, $i = 3$, $j = 5$, $k = 3$
 $q = 0 + 1000 + 15 * 5 * 20 = 2500$
 $l = 3$, $i = 3$, $j = 5$, $k = 4$
 $q = 750 + 0 + 15 * 10 * 20 = 2750$

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p_1	p_2	p_3	p_4	p_5	p_6
30	35	15	5	10	20	25

Solution:

13

	1	2	3	4	5	6
1	0	15750	7875			
2 ⁷ / ₂	_	0	2625	4375		
3	3		0	750	2500	
4		(3))	0	1000	
5			A		0	5000
6				75		0
m matrix						

	2	3	4	5	6
1	1	1			
2		2	3		
3			3	3	
4				4	
5					5

 $5 for l \leftarrow 2 to n$ $6 for i \leftarrow 1 to n - l + 1$ $7 j \leftarrow i + l - 1$ $8 m[i,j] \leftarrow \infty$ $9 for k \leftarrow i to j - 1$ $10 q \leftarrow m[i,k] + m[k+1,j] + p_{i-1}p_kp_j$ 11 if q < m[i,j] $12 then m[i,j] \leftarrow q$

 $s[i, j] \leftarrow k$

$$l = 3$$
, $i = 3$, $j = 5$, $k = 3$
 $q = 0 + 1000 + 15 * 5 * 20 = 2500$
 $l = 3$, $i = 3$, $j = 5$, $k = 4$

$$q \leftarrow m[i,k] + m[k+1,j] + p_{i-1}p_kp_j$$
 $q = 750 + 0 + 15 * 10 * 20 = 3750$

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p_1	p_2	p_3	p_4	p_5	p ₆
30	35	15	5	10	20	25

Solution:

_	1	2	3	4	5	6
1	0	15750	7875			
27	_	0	2625	4375		
3	3		0	750	2500	
4		3		0	1000	∞
5			A) _	0	5000
6				(3		0
		m	mati	rix	76	
$5 for l \leftarrow 2$	to n					

	2	3	4	5	6
1	1	1			
2		2	3		
3			3		
4				4	
5					5

s matrix

$$\begin{array}{l}
 l = 3, i = 4, j = 6, k = 4 \\
 q = 0 + 5000 + 5 * 10 * 25 = 6250
 \end{array}$$

6
$$for i \leftarrow 1 to n - l + 1$$

7 $j \leftarrow i + l - 1$
8 $m[i,j] \leftarrow \infty$
9 $for k \leftarrow i to j - 1$
10 $q \leftarrow m[i,k] + m[k + 1,j] + p_{i-1}p_kp_j$
11 $if \ q < m[i,j]$
12 $then \ m[i,j] \leftarrow q$
13 $s[i,j] \leftarrow k$

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p_1	p_2	p_3	p_4	p_5	p_6
30	35	15	5	10	20	25

Solution:

13

	1	2	3	4	5	6	
1	0	15750	7875				
2 ⁷ / ₂	_	0	2625	4375			
3	3		0	750	2500		
4		3		0	1000	∞	
5			A		0	5000	
6				Z)	0	
	m matrix						

	2	3	4	5	6
1	1	1			
2		2	3		
3			3		
4				4	
5					5

s matrix

5 for
$$l \leftarrow 2$$
 to n
6 for $i \leftarrow 1$ to $n - l + 1$
7 $j \leftarrow i + l - 1$
8 $m[i,j] \leftarrow \infty$
9 for $k \leftarrow i$ to $j - 1$
10 $q \leftarrow m[i,k] + m[k + 1,j] + p_{i-1}p_kp_j$
11 if $q < m[i,j]$

then $m[i, j] \leftarrow q$

 $s[i, j] \leftarrow k$

$$l = 3$$
, $i = 4$, $j = 6$, $k = 4$
 $q = 0 + 5000 + 5 * 10 * 25 = 6250$
 $l = 3$, $i = 4$, $j = 6$, $k = 5$
 $q = 1000 + 0 + 5 * 20 * 25 = 3500$

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p_1	p_2	p_3	p_4	p_5	p_6
30	35	15	5	10	20	25

Solution:

_	1	2	3	4	5	6
1 [0	15750	7875			
27	_	0	2625	4375		
3	3	_	0	750	2500	
4		3		0	1000	3500
5			A		0	5000
6				7,		0
		m	mati	rix	(36))
$5 for l \leftarrow 2$	to n					

	2	3	4	5	6
1	1	1			
2		2	3		
3			3	3	
4				4	5
5					5

s matrix

$$l = 3$$
, $i = 4$, $j = 6$, $k = 4$
 $q = 0 + 5000 + 5 * 10 * 25 = 6250$

$$l = 3$$
, $i = 4$, $j = 6$, $k = 5$
 $q = 1000 + 0 + 5 * 20 * 25 = 3500$

6
$$for i \leftarrow 1 to n - l + 1$$

7 $j \leftarrow i + l - 1$
8 $m[i,j] \leftarrow \infty$
9 $for k \leftarrow i to j - 1$
10 $q \leftarrow m[i,k] + m[k + 1,j] + p_{i-1}p_kp_j$
11 $if \ q < m[i,j]$
12 $then \ m[i,j] \leftarrow q$
13 $s[i,j] \leftarrow k$

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p ₁	p_2	p_3	p_4	p_5	p_6
30	35	15	5	10	20	25

Solution:

13

_	1	2	3	4	5	6
1	0	15750	7875	8		
53)	0	2625	4375		
3	12		0	750	2500	
4		3		0	1000	3500
5			A		0	5000
6						0
		100		ui.	7	

	2	3	4	5	6
1	1	1			
2		2	3		
3			3	3	
4				4	5
5					5

s matrix l = 3, i = 1, j = 4

m matrix 6 $5 for l \leftarrow 2 to n$ k = 1, q = 0 + 4375 + 30 * 35 * 10 = 14875 $for i \leftarrow 1 to n - l + 1$ $i \leftarrow i + l - 1$ $m[i,j] \leftarrow \infty$ for $k \leftarrow i$ to j-110 $q \leftarrow m[i,k] + m[k+1,j] + p_{i-1}p_kp_j$ 11 if q < m[i,j]then $m[i,j] \leftarrow q$

 $s[i, j] \leftarrow k$

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p_1	p_2	p_3	p_4	p_5	p_6
30	35	15	5	10	20	25

Solution:

13

	1	2	3	4	5	6
1	0	15750	7875	∞		
33		0	2625	4375		
3	T.		0	750	2500	
4		13		0	1000	3500
5			A		0	5000
6				75		0
			_		1	

	2	3	4	5	6
1	1	1			
2		2	3		
3			3	3	
4				4	5
5					5

m matrix

 $s[i, j] \leftarrow k$

m matrix
$$j \in S$$
 matrix $j \in S$ matrix $j \in S$ for $j \in S$ to $j \in S$ matrix $j \in S$ for $j \in S$ matrix $j \in S$ for $j \in S$ matrix $j \in S$ m

10
$$q \leftarrow m[i,k] + m[k+1,j] + p_{i-1}p_kp_j$$

11 $if \ q < m[i,j]$
12 $then \ m[i,j] \leftarrow q$

for $k \leftarrow i$ to j-1

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

 $5 for l \leftarrow 2 to n$

p_0	p_1	p_2	p_3	p_4	p_5	p_6
30	35	15	5	10	20	25

Solution:

	1	2	3	4	5	6
1	0	15750	7875	9375		
24		0	2625	4375		
3	13		0	750	2500	
4		3	\rangle	0	1000	3500
5			1 7 x		0	5000
6				Z,		0
					1	

	2	3	4	5	6
1	1	1	3		
2		2	3		
3			3	3	
4				4	5
5					5

m matrix

s matrix

$$l = 3$$
, $i = 1, j = 4$
 $k = 1, q = 0 + 4375 + 30 * 35 * 10 = 14875$
 $k = 2, q = 15750 + 750 + 30 * 15 * 10 = 18000$
 $k = 3, q = 7875 + 0 + 30 * 5 * 10 = 9375$

10
$$q \leftarrow m[i,k] + m[k+1,j] + p_{i-1}p_kp_j$$

11 $if \ q < m[i,j]$
12 $then \ m[i,j] \leftarrow q$
13 $s[i,j] \leftarrow k$

for $i \leftarrow 1$ to n - l + 1

for $k \leftarrow i$ to j-1

 $i \leftarrow i + l - 1$

 $m[i,j] \leftarrow \infty$

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p_1	p_2	p_3	p_4	p_5	p_6
30	35	15	5	10	20	25

Solution:

	1	2	3	4	5	6
1	0	15750	7875	9375		
27		0	2625	4375		
3	13		0	750	2500	
4		3	\rangle	0	1000	3500
5			A		0	5000
6				75		0
					1	

	2	3	4	5	6
1	1	1	3		
2		2	3		
3			3	3	
4				4	5
5					5

m matrix

m matrix
$$j \in \{for\ l \in 2\ to\ n\}$$
 $j \in \{i + l = 1, j = 4\}$ $j \in \{i + l = 1, j = 4\}$ $j \in \{i + l = 1, j = 4\}$ $j \in \{i + l = 1, j = 4\}$ $k = 1, q = 0 + 4375 + 30 * 35 * 10 = 14875$ $k = 2, q = 15750 + 750 + 30 * 15 * 10 = 18000$ min $k = 3, q = 7875 + 0 + 30 * 5 * 10 = 9375$ $k = 3, q = 7875 + 0 + 30 * 5 * 10 = 9375$

10
$$q \leftarrow m[i,k] + m[k+1,j] + p_{i-1}p_kp_j$$

11 $if \ q < m[i,j]$
12 $then \ m[i,j] \leftarrow q$
13 $s[i,j] \leftarrow k$

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p_1	p_2	p_3	p_4	p_5	p_6
30	35	15	5	10	20	25

Solution:

13

	1	2	3	4	5	6	
1	0	15750	7875	9375			
2 ⁴ / ₂		0	2625	4375	∞		
3	13		0	750	2500		
4		(3)		0	1000	3500	
5			A		0	5000	
6				À.)	0	
m matrix 7							

	2	3	4	5	6
1	1	1	3		
2		2	3		
3			3	3	
4				4	5
5					5

s matrix l = 3, i = 2, j = 5

	III IIIauix	0	j S Hiddix _
5 <i>f</i>	$or l \leftarrow 2 to n$		l = 3, i = 2, j = 5
•	$for i \leftarrow 1 to n - l + 1$		k = 2, q = 0 + 4375 + 35 * 15 * 20 = 10500
7	$j \leftarrow i + l - 1$		
8	$m[i,j] \leftarrow \infty$		
9	$for k \leftarrow i to j - 1$		
10	$q \leftarrow m[i,k] + m[k+1,j] +$	$\vdash p_{i-}$	$1p_kp_j$
11	if q < m[i,j]		
12	then $m[i,j] \leftarrow q$		

 $s[i,j] \leftarrow k$

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p_1	p_2	p_3	p_4	p_5	p_6
30	35	15	5	10	20	25

Solution:

	1	2	3	4	5	6
1	0	15750	7875	9375		
27		0	2625	4375	∞	
3	13		0	750	2500	
4		33		0	1000	3500
5			(A		0	5000
6				Z)	0
					7	

	2	3	4	5	6
1	1	1	3		
2		2	3		
3			3	3	
4				4	5
5					5

m matrix

m matrix
$$5 \text{ for } l \leftarrow 2 \text{ to } n$$
 $5 \text{ for } i \leftarrow 2 \text{ to } n - l + 1$ $5 \text{ for } i \leftarrow 1 \text{ to } n - l + 1$ $5 \text{ for } i \leftarrow 1 \text{ to } n - l + 1$ $5 \text{ for } i \leftarrow 1 \text{ to } n - l + 1$ $5 \text{ for } i \leftarrow 1 \text{ to } n - l + 1$ $5 \text{ for } i \leftarrow 1 \text{ to } n - l + 1$ $5 \text{ for } i \leftarrow 1 \text{ to } n - l + 1$ $5 \text{ for } i \leftarrow 1 \text{ to } n - l + 1$ $5 \text{ for } i \leftarrow 1 \text{ to } n - l + 1$ $6 \text{ for$

10
$$q \leftarrow m[i,k] + m[k+1,j] + p_{i-1}p_kp_j$$

11 $if \ q < m[i,j]$
12 $then \ m[i,j] \leftarrow q$
13 $s[i,j] \leftarrow k$

for $k \leftarrow i$ to j-1

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p_1	p_2	p_3	p_4	p_5	p_6
30	35	15	5	10	20	25

Solution:

	1	2	3	4	5	6
1	0	15750	7875	9375		
27		0	2625	4375	8	
3	The state of the s		0	750	2500	
4		3		0	1000	3500
5			Z		0	5000
6				(Z,		0
		m	mat	rix	76	

	2	3	4	5	6
1	1	1	3		
2		2	3		
3			3	3	
4				4	5
5					5

s matrix l = 3, i = 2, j = 5

 $5 for l \leftarrow 2 to n$ k = 2, q = 0 + 4375 + 35 * 15 * 20 = 10500for $i \leftarrow 1$ to n - l + 1k = 3, q = 2625 + 1000 + 35 * 5 * 20 = 7125 $i \leftarrow i + l - 1$ $m[i,j] \leftarrow \infty$ k = 4, q = 4375 + 0 + 35 * 10 * 20 = 11375for $k \leftarrow i$ to j-110 $q \leftarrow m[i,k] + m[k+1,j] + p_{i-1}p_kp_i$ 11 if q < m[i,j]then $m[i, j] \leftarrow q$ 13 $s[i, j] \leftarrow k$

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

 $5 for l \leftarrow 2 to n$

p_0	p_1	p_2	p_3	p_4	p_5	p_6
30	35	15	5	10	20	25

Solution:

	1	2	3	4	5	6		
1	0	15750	7875	9375				
27		0	2625	4375	7125			
3	13		0	750	2500			
4		3)	0	1000	3500		
5			Z		0	5000		
6				7,)	0		
	m matrix 7							

	2	3	4	5	6
1	1	1	3		
2		2	3	3	
3			3	3	
4				4	5
5					5

mii

s matrix

$$l = 3$$
, $i = 2$, $j = 5$
 $k = 2$, $q = 0 + 4375 + 35 * 15 * 20 = 10500$
 $k = 3$, $q = 2625 + 1000 + 35 * 5 * 20 = 7125$

k = 4, q = 4375 + 0 + 35 * 10 * 20 = 11375

10
$$q \leftarrow m[i,k] + m[k+1,j] + p_{i-1}p_kp_j$$

 $i \leftarrow i + l - 1$

 $m[i,j] \leftarrow \infty$

for $i \leftarrow 1$ to n - l + 1

for $k \leftarrow i$ to i - 1

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p_1	p_2	p_3	p_4	p_5	p ₆
30	35	15	5	10	20	25

Solution:

13

	1	2	3	4	5	6
1	0	15750	7875	9375		
27	_	0	2625	4375	7125	
3	3		0	750	2500	∞
4		3)_	0	1000	3500
5			A		0	5000
6				Z		0
		m	mat	rix	76)

 $s[i, j] \leftarrow k$

	2	3	4	5	6
1	1	1	3		
2		2	3	3	
3			3	3	
4				4	5
5					5

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p_1	p_2	p_3	p_4	p_5	p_6
30	35	15	5	10	20	25

Solution:

	1	2	3	4	5	6
1	0	15750	7875	9375		
27	_	0	2625	4375	7125	
3	3		0	750	2500	∞
4		133		0	1000	3500
5			A)	0	5000
6				13		0
		m	mat	rix	A.C.	

	2	3	4	5	6
1	1	1	3		
2		2	3	3	
3			3	3	
4				4	5
5					5

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p_1	p_2	p_3	p_4	p_5	p ₆
30	35	15	5	10	20	25

Solution:

	1	2	3	4	5	6
1	0	15750	7875	9375		
27	_	0	2625	4375	7125	
3	3		0	750	2500	8
4		(3)		0	1000	3500
5			Z		0	5000
6				Z		0
		m	mat	rix	180	

	2	3	4	5	6
1	1	1	3		
2		2	3	3	
3			3	3	
4				4	5
5					5

m matrix s = 3, i = 3, j = 6 $for i \leftarrow 1 to n - l + 1$ k = 3, q = 0 + 3500 + 15 * 5 * 25 = 5375 $j \leftarrow i + l - 1$ k = 4, q = 750 + 5000 + 15 * 10 * 25 = 9500 $mi[i,j] \leftarrow \infty$ $mi[i,j] \leftarrow \infty$ mi[i,j]

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p_1	p_2	p_3	p_4	p_5	p ₆
30	35	15	5	10	20	25

Solution:

	1	2	3	4	5	6
1	0	15750	7875	9375		
27	_	0	2625	4375	7125	
3	3		0	750	2500	5375
4		(3)		0	1000	3500
5			A		0	5000
6				Z		0
		m	mat	rix	A C)

	2	3	4	5	6
1	1	1	3		
2		2	3	3	
3			3	3	3
4				4	5
5					5

m matrix s = 3, i = 3, j = 6 $for i \leftarrow 1 to n - l + 1$ k = 3, q = 0 + 3500 + 15 * 5 * 25 = 5375 k = 4, q = 750 + 5000 + 15 * 10 * 25 = 9500 $mi[i,j] \leftarrow \infty$ $mi[i,j] \leftarrow \infty$ m

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p ₁	p_2	p_3	p_4	p_5	p_6
30	35	15	5	10	20	25

Solution:

_	1	2	3	4	5	6
1	0	15750	7875	9375	∞	
27		0	2625	4375	7125	
3	13		0	750	2500	5375
4		3		0	1000	3500
5			Z		0	5000
6				75		0
		m	mat	rix	7 de	

	2	3	4	5	6
1	1	1	3		
2		2	3	3	
3			3	3	3
4				4	5
5					5

m matrix $\begin{cases} 5 & \text{for } l \leftarrow 2 & \text{to } n \\ 6 & \text{for } i \leftarrow 1 & \text{to } n - l + 1 \\ 7 & \text{j} \leftarrow i + l - 1 \\ 8 & m[i,j] \leftarrow \infty \\ 9 & \text{for } k \leftarrow i & \text{to } j - 1 \\ 10 & q \leftarrow m[i,k] + m[k+1,j] + p_{i-1}p_kp_j \\ 11 & \text{if } q < m[i,j] \\ 12 & \text{then } m[i,j] \leftarrow q \\ 13 & s[i,j] \leftarrow k \end{cases}$

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p_1	p_2	p_3	p_4	p_5	p_6
30	35	15	5	10	20	25

Solution:

	1	2	3	4	5	6
1	0	15750	7875	9375	8	
27		0	2625	4375	7125	
3	The state of the s)	0	750	2500	5375
4		13		0	1000	3500
5			Z		0	5000
6				75		0
		m	mat	rix	76	

	2	3	4	5	6
1	1	1	3		
2		2	3	3	
3			3	3	3
4				4	5
5					5

m matrix $\begin{cases} 5 & \text{for } l \leftarrow 2 & \text{to } n \end{cases}$ $\begin{cases} 5 & \text{for } i \leftarrow 2 & \text{to } n \end{cases}$ $\begin{cases} 1 & \text{extremath} \\ 1 & \text{for } i \leftarrow 1 & \text{to } n - l + 1 \end{cases}$ $\begin{cases} 1 & \text{for } i \leftarrow 1 & \text{for } i \leftarrow 1 \end{cases}$ $\begin{cases} 1 & \text{for } i \leftarrow 1 & \text{for } i \leftarrow 1 \end{cases}$ $\begin{cases} 1 & \text{for } i \leftarrow 1 \\ \text{for } i \leftarrow 1 & \text{for } i \leftarrow 1 \end{cases}$ $\begin{cases} 1 & \text{for } i \leftarrow 1 \\ \text{for } i \leftarrow 1 \end{cases}$ $\begin{cases} 1 & \text{for } i \leftarrow 1 \\ \text{for } i \leftarrow 1 \end{cases}$ $\begin{cases} 1 & \text$

10
$$q \leftarrow m[i,k] + m[k+1,j] + p_{i-1}p_kp_j$$

11 $if \ q < m[i,j]$
12 $then \ m[i,j] \leftarrow q$
13 $s[i,j] \leftarrow k$

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p_1	p_2	p_3	p_4	p_5	p_6
30	35	15	5	10	20	25

Solution:

13

	1	2	3	4	5	6	
1_	0	15750	7875	9375	∞		
27		0	2625	4375	7125		
3	13		0	750	2500	5375	
4		33		0	1000	3500	
5			(A		0	5000	
6				75		0	
m matrix							

 $s[i,j] \leftarrow k$

	2	3	4	5	6
1	1	1	3		
2		2	3	3	
3			3	3	3
4				4	5
5					5

*15 * 20 = 27250

5 * 20 = 11875

	m matrix	s matrix
5 <i>f</i>	for $l \leftarrow 2$ to n	l=4 , $i=1,j=5$
6	$for i \leftarrow 1 to n - l + 1$	k = 1, q = 0 + 7125 + 30 * 35 * 20 = 28125
7	$j \leftarrow i + l - 1$	k = 2, q = 15750 + 2500 + 30 * 15 * 20 = 27
8	$m[i,j] \leftarrow \infty$	k = 3, q = 7875 + 1000 + 30 * 5 * 20 = 1187
9	$for k \leftarrow i to j - 1$, I
10	$q \leftarrow m[i,k] + m[k+1,j] + p$	$_{i-1}p_{k}p_{j}$
11	if q < m[i,j]	
12	then $m[i,i] \leftarrow a$	

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

then $m[i,j] \leftarrow q$

 $s[i, j] \leftarrow k$

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p_1	p_2	p_3	p_4	p_5	p_6
30	35	15	5	10	20	25

Solution:

13

	1	2	3	4	5	6
1_	0	15750	7875	9375	8	
24		0	2625	4375	7125	
3	3		0	750	2500	5375
4		Z		0	1000	3500
5			A A)	0	5000
6				75		0
		m	mati	rix	1 7	

	2	3	4	5	6
1	1	1	3		
2		2	3	3	
3			3	3	3
4				4	5
5					5

m matrix $\begin{cases} 5 & \text{for } l \leftarrow 2 & \text{to } n \end{cases}$ $\begin{cases} 1 & \text{for } l \leftarrow 2 & \text{to } n \end{cases}$ $\begin{cases} 1 & \text{for } l \leftarrow 2 & \text{to } n \end{cases}$ $\begin{cases} 1 & \text{for } l \leftarrow 1 & \text{for } l \leftarrow 1 \end{cases}$ $\begin{cases} 1 & \text{for } l \leftarrow 1 & \text{for } l \leftarrow 1 \end{cases}$ $\begin{cases} 1 & \text{for } l \leftarrow 1 \end{cases}$ \begin{cases}

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

then $m[i,j] \leftarrow q$

 $s[i, j] \leftarrow k$

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p_1	p_2	p_3	p_4	p_5	p_6
30	35	15	5	10	20	25

Solution:

13

	1	2	3	4	5	6				
1_	0	15750	7875	9375	11875					
27		0	2625	4375	7125					
3	3		0	750	2500	5375				
4		33	\sim	0	1000	3500				
5			Z Z		0	5000				
6				75		0				
	m matrix									

	2	3	4	5	6
1	1	1	3	3	
2		2	3	3	
3			3	3	3
4				4	5
5					5

m matrix $\begin{cases} 5 \text{ for } l \leftarrow 2 \text{ to } n \end{cases}$ $\begin{cases} 1 = 4, i = 1, j = 5 \end{cases}$ $\begin{cases} 6 \text{ for } i \leftarrow 1 \text{ to } n - l + 1 \end{cases}$ $\begin{cases} k = 1, q = 0 + 7125 + 30 * 35 * 20 = 28125 \end{cases}$ $\begin{cases} k = 2, q = 15750 + 2500 + 30 * 15 * 20 = 275 \end{cases}$ min $\begin{cases} m[i,j] \leftarrow \infty \end{cases}$ $\begin{cases} k = 3, q = 7875 + 1000 + 30 * 5 * 20 = 11875 \end{cases}$ $\begin{cases} k = 4, q = 9375 + 0 + 30 * 10 * 20 = 15375 \end{cases}$ 10 $\begin{cases} q \leftarrow m[i,k] + m[k+1,j] + p_{i-1}p_kp_j \end{cases}$ 11 $\begin{cases} if \ q < m[i,j] \end{cases}$

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p_1	p_2	p_3	p_4	p_5	p ₆
30	35	15	5	10	20	25

Solution:

13

	1	2	3	4	5	6			
1	0	15750	7875	9375	11875				
2 ⁴ / ₂		0	2625	4375	7125	8			
3	(3)		0	750	2500	5375			
4		13		0	1000	3500			
5			Z		0	5000			
6				75		0			
	m matrix								

 $s[i, j] \leftarrow k$

	2	3	4	5	6
1	1	1	3	3	
2		2	3	3	
3			3	3	3
4				4	5
5					5

m matrix
$$s$$
 matrix $l = 4$, $i = 2$, $j = 6$
6 for $i \leftarrow 1$ to $n - l + 1$ $k = 2$, $q = 0 + 5375 + 35 * 15 * 25 = 15375$
7 $j \leftarrow i + l - 1$
8 $m[i,j] \leftarrow \infty$
9 for $k \leftarrow i$ to $j - 1$
10 $q \leftarrow m[i,k] + m[k + 1,j] + p_{i-1}p_kp_j$
11 if $q < m[i,j]$
12 then $m[i,j] \leftarrow q$

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p_1	p_2	p_3	p_4	p_5	p_6
30	35	15	5	10	20	25

Solution:

	1	2	3	4	5	6		
1	0	15750	7875	9375	11875			
2 ⁷ / ₂		0	2625	4375	7125	∞		
3	13		0	750	2500	5375		
4		33		0	1000	3500		
5			A		0	5000		
6				75		0		
	m matrix							

	2	3	4	5	6
1	1	1	3	3	
2		2	3	3	
3			3	3	3
4				4	5
5					5

m matrix
$$s$$
 matrix $l = 4$, $i = 2$, $j = 6$
6 for $i \leftarrow 1$ to $n - l + 1$ $k = 2$, $q = 0 + 5375 + 35 * 15 * 25 = 15375$
7 $j \leftarrow i + l - 1$ $k = 3$, $q = 2625 + 3500 + 35 * 5 * 25 = 10500$
8 $m[i,j] \leftarrow \infty$
9 for $k \leftarrow i$ to $j - 1$

10
$$q \leftarrow m[i,k] + m[k+1,j] + p_{i-1}p_kp_j$$

11 $if \ q < m[i,j]$
12 $then \ m[i,j] \leftarrow q$
13 $s[i,j] \leftarrow k$

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p ₁	p_2	p_3	p_4	p_5	p_6
30	35	15	5	10	20	25

Solution:

13

	1	2	3	4	5	6		
1	0	15750	7875	9375	11875			
2 ⁷ / ₂		0	2625	4375	7125	8		
3	13		0	750	2500	5375		
4		3		0	1000	3500		
5			A		0	5000		
6				(Z		0		
	m matrix							

then $m[i, j] \leftarrow q$

 $s[i, j] \leftarrow k$

	2	3	4	5	6
1	1	1	3	3	
2		2	3	3	
3			3	3	3
4				4	5
5					5

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

then $m[i,j] \leftarrow q$

 $s[i, j] \leftarrow k$

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p ₁	p_2	p_3	p_4	p_5	p ₆
30	35	15	5	10	20	25

Solution:

	1	2	3	4	5	6	
1	0	15750	7875	9375	11875		
2 ⁷ / ₂		0	2625	4375	7125	∞	
3	(3)		0	750	2500	5375	
4		73		0	1000	3500	
5			A		0	5000	
6				75)	0	
	m matrix (%)						

	2	3	4	5	6
1	1	1	3	3	
2		2	3	3	
3			3	3	3
4				4	5
5					5

m matrix
$$s matrix$$
 $l = 4, i = 2, j = 6$
6 for $i \leftarrow 1$ to $n - l + 1$ $k = 2, q = 0 + 5375 + 35 * 15 * 25 = 15375$
7 $j \leftarrow i + l - 1$ $k = 3, q = 2625 + 3500 + 35 * 5 * 25 = 10500$
8 $m[i,j] \leftarrow \infty$ $k = 4, q = 4375 + 5000 + 35 * 10 * 25 = 18125$
9 for $k \leftarrow i$ to $j - 1$ $k = 5, q = 7125 + 0 + 35 * 20 * 25 = 24625$
10 $q \leftarrow m[i,k] + m[k+1,j] + p_{i-1}p_kp_j$
11 if $q < m[i,j]$

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p_1	p_2	p_3	p_4	p_5	p ₆
30	35	15	5	10	20	25

Solution:

	1	2	3	4	5	6		
1	0	15750	7875	9375	11875			
2 ⁷ / ₂		0	2625	4375	7125	10500		
3	(3)		0	750	2500	5375		
4		73)	0	1000	3500		
5			A		0	5000		
6				Z		0		
	m matrix							

	2	3	4	5	6
1	1	1	3	3	
2		2	3	3	3
3			3	3	3
4				4	5
5					5

m matrix s = 1 for $l \leftarrow 2$ to n l = 4, l = 2, l = 6 l = 6 l = 4, l = 2, l = 6 l = 4, l = 2, l = 6 l = 4, l = 2, l = 6 l = 4, l = 2, l = 6 l = 4, l = 2, l = 6 l = 4, l = 2, l = 6 l = 4, l = 2, l = 6 l = 4, l = 2, l = 6 l = 4, l = 2, l = 6 l = 4, l = 2, l = 6 l = 4, l = 2, l = 6 l = 4, l = 2, l = 6 l = 6 l = 6, l

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p_1	p_2	p_3	p_4	p_5	p ₆
30	35	15	5	10	20	25

Solution:

13

1 0 15750 7875 9375 11875 ∞ 0 2625 4375 7125 10500 3 0 750 2500 5375 4 0 1000 3500 5 0 5000 6 m matrix		1	2	3	4	5	6
3	1	0	15750	7875	9375	11875	8
4 3 0 1000 3500 5 0 5000 6 3 0	27		0	2625	4375	7125	10500
5 6 0 5000 0	3	13		9	750	2500	5375
6	4		3		9	1000	3500
3,	5			A		0	5000
m matrix	6				Z		0
1 24	_	_	m	mat	rix	76	

 $s[i, j] \leftarrow k$

	2	3	4	5	6
1	1	1	3	3	
2		2	3	3	3
3			3	3	3
4				4	5
5					5

m matrix
$$s$$
 matrix $l = 4$, $i = 1, j = 6$
 $for i \leftarrow 1 to n - l + 1$ $k = 1, q = 0 + 10500 + 30 * 35 * 25 = 36750$
 $for k \leftarrow i to j - 1$
 $for k \leftarrow m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j$
 $for k \leftarrow m[i, j] \leftarrow q$

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p_1	p_2	p_3	p_4	p_5	p_6
30	35	15	5	10	20	25

Solution:

	1	2	3	4	5	6
1	9	15750	7875	9375	11875	∞
53		0	2625	4375	7125	10500
3	45)	0	750	2500	5375
4		13		Q	1000	3500
5			Z		0	5000
6				Z		0
_	_	m	mat	rix	76)

	2	3	4	5	6
1	1	1	3	3	
2		2	3	3	3
3			3	3	3
4				4	5
5					5

m matrix
$$s$$
 matrix $l = 4$, $i = 1, j = 6$
 $for i \leftarrow 1 to n - l + 1$ $k = 1, q = 0 + 10500 + 30 * 35 * 25 = 36750$
 $k = 2, q = 15750 + 5375 + 30 * 15 * 25 = 32375$
 $m[i,j] \leftarrow \infty$
 $mi[i,j] \leftarrow \infty$

10
$$q \leftarrow m[i,k] + m[k+1,j] + p_{i-1}p_kp_j$$

11 $if \ q < m[i,j]$
12 $then \ m[i,j] \leftarrow q$
13 $s[i,j] \leftarrow k$

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

then $m[i, j] \leftarrow q$

 $s[i, j] \leftarrow k$

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p_1	p_2	p_3	p_4	p_5	p_6
30	35	15	5	10	20	25

Solution:

	1	2	3	4	5	6
1	0	15750	7875	9375	11875	∞
27		0	2625	4375	7125	10500
3	3		0	750	2500	5375
4		33		0	1000	3500
5			1 A		0	5000
6				Z		0
		m	mat	rix	76)

	2	3	4	5	6
1	1	1	3	3	
2		2	3	3	3
3			3	3	3
4				4	5
5					5

m matrix
$$\begin{cases} 5 & \text{for } l \leftarrow 2 & \text{to } n \\ 6 & \text{for } i \leftarrow 1 & \text{to } n - l + 1 \\ 7 & \text{j} \leftarrow i + l - 1 \\ 8 & m[i,j] \leftarrow \infty \\ 9 & \text{for } k \leftarrow i & \text{to } j - 1 \\ 10 & q \leftarrow m[i,k] + m[k+1,j] + p_{i-1}p_kp_j \\ 11 & \text{if } q < m[i,j] \end{cases}$$

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

then $m[i,j] \leftarrow q$

 $s[i, j] \leftarrow k$

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p_1	p_2	p_3	p_4	p_5	p_6
30	35	15	5	10	20	25

Solution:

	1	2	3	4	5	6
1	0	15750	7875	9375	11875	∞
27		0	2625	4375	7125	10500
3	3		0	750	2500	5375
4		73		0	1000	3500
5			A		0	5000
6				73		0
		m	mat	rix	76)
7	^ .					

	2	3	4	5	6
1	1	1	3	3	
2		2	3	3	3
3			3	3	3
4				4	5
5					5

m matrix
$$\begin{cases} 5 & \text{for } l \leftarrow 2 & \text{to } n \\ 6 & \text{for } i \leftarrow 1 & \text{to } n - l + 1 \\ 7 & \text{j} \leftarrow i + l - 1 \\ 8 & m[i,j] \leftarrow \infty \\ 9 & \text{for } k \leftarrow i & \text{to } j - 1 \\ 10 & q \leftarrow m[i,k] + m[k+1,j] + p_{i-1}p_kp_j \end{cases}$$

$$k = 1, q = 0 + 10500 + 30 * 35 * 25 = 36750$$

$$k = 1, q = 0 + 10500 + 30 * 35 * 25 = 36750$$

$$k = 2, q = 15750 + 5375 + 30 * 15 * 25 = 32375$$

$$k = 3, q = 7875 + 3500 + 30 * 5 * 25 = 15125$$

$$k = 4, q = 9375 + 5000 + 30 * 10 * 25 = 21875$$

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

then $m[i,j] \leftarrow q$

 $s[i,j] \leftarrow k$

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p_1	p_2	p_3	p_4	p_5	p_6
30	35	15	5	10	20	25

Solution:

12

	1	2	3	4	5	6
1	0	15750	7875	9375	11875	∞
27		0	2625	4375	7125	10500
3	13		0	750	2500	5375
4		3		0	1000	3500
5			A		0	5000
6				Z		0
		100	mat	riv.	79	

	2	3	4	5	6
1	1	1	3	3	
2		2	3	3	3
3			3	3	3
4				4	5
5					5

	$for l \leftarrow 2 to n$ $m matrix \qquad 6$ $l = 4, i = 1, j = 6$
	for $i \leftarrow 1$ to $n - l + 1$ $k = 1, q = 0 + 10500 + 30 * 35 * 25 = 36750$
	$j \leftarrow i + l - 1$ $k = 2, a = 15750 + 5375 + 30 * 15 * 25 = 32375$
6	$V = \{ A = A \} = A = A = A = A = A = A = A = A$
	$0 \qquad q \leftarrow m[i,k] + m[k+1,i] + n, n,n,k = 4, q = 9375 + 5000 + 30 * 10 * 25 = 21875$
1	k - 2, q - 13/30 + 33/3 + 30 * 13 * 23 - 323/3

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

then $m[i,j] \leftarrow q$

 $s[i,i] \leftarrow k$

Example 1: Find an optimal parenthesization of a matrix chain product whose sequence of

dimension is:

p_0	p_1	p_2	p_3	p_4	p_5	p_6
30	35	15	5	10	20	25

Solution:

12

13

	1	2	3	4	5	6
1	0	15750	7875	9375	11875	15125
27		0	2625	4375	7125	10500
3	3		0	750	2500	5375
4		33)	0	1000	3500
5			A		0	5000
6				Z		0
2 to n		m	mati	rix	75)

	2	3	4	5	6
1	1	1	3	3	3
2		2	3	3	3
3			3	3	3
4				4	5
5					5

Problem 4: Matrix Chain Multiplication

Step 3: Compute optimal cost.

```
MATRIX - CHAIN - ORDER(p)
 1 n \leftarrow length[p] - 1
2 let m[1...n, 1...n] and s[1...n - 1, 2...n] be new tables.
3 \ for \ i \leftarrow 1 \ to \ n
4 m[i,i] \leftarrow 0
5 for l \leftarrow 2 to n \triangleright l is the chain length.
    for i \leftarrow 1 to n - l + 1
     j \leftarrow i + l - 1
8 m[i,j] \leftarrow \infty
9 for k \leftarrow i to j - 1
10
             q \leftarrow m[i,k] + m[k+1,j] + p_{i-1}p_kp_j
11
            if q < m[i,j]
12
                 then m[i,j] \leftarrow q
13
                       s[i,i] \leftarrow k
14 return m and s
```

A simple inspection of the nested loop structure of MATRIX-CHAIN-ORDER yields a running time of $O(n^3)$ for the algorithm. The loops are nested three deep, and each loop index (l, i, and k) takes on at most n-1 values.

Problem 4: Matrix Chain Multiplication

Step 4: Construct / print the optimal solution.

Example 1: Find an optimal parenthesization of a matrix chain product whose

sequence of dimension is:

p_0						
30	35	15	5	10	20	25

```
PRINT - OPTIMAL - PARENS(s, i, j)

1 if i = j

2 then print "A<sub>i</sub>"

3 else print "("

4 PRINT - OPTIMAL - PARENS(s, i, s[i, j])

5 PRINT - OPTIMAL - PARENS(s, s[i, j] + 1, j)

6 print ")"
```

Lets see, how in the discussed example the call PRINT - OPTIMAL - PARENS(s, 1, 6) prints the parenthesization ((A1 (A2 A3)) ((A4 A5)A6)).

Problem 4: Matrix Chain Multiplication

Step 4: Construct / print the optimal solution.

Example 1: Find an optimal parenthesization of a matrix chain product whose

sequence of dimension is:

	_						_
PRINT - OPTIMAL - PARENS(s, i, j)	1 [1	1	3	3	3]
$\begin{array}{l} 1 \ if \ i = j \\ 2 \ the model \ mint \ "A \ " \end{array}$	2		2	3	3	3	
<pre>2 then print "A_i" 3 else print "("</pre>	3			3	3	3	
4 $PRINT - OPTIMAL - PARENS(s, i, s[i, j])$	4				4	5	
5 $PRINT - OPTIMAL - PARENS(s, s[i,j] + 1,j)$	5		s matrix			5	
6 print ")" S Matrix				IX		_	

Problem 4: Matrix Chain Multiplication

Step 4: Construct / print the optimal solution.

Example 1: Find an optimal parenthesization of a matrix chain product whose 6

sequence of dimension is:

```
PRINT - OPTIMAL - PARENS(s, i, j)
1 if i = j
   then print "A_i"
                                                                                                   3
   else print "("
        PRINT - OPTIMAL - PARENS(s, i, s[i, j])
5
         PRINT - OPTIMAL - PARENS(s, s[i, j] + 1, j)
                                                                                                   5
                                                                                  s matrix
         print")"
                                               POP(S,1,6)
```

Problem 4: Matrix Chain Multiplication

Step 4: Construct / print the optimal solution.

Example 1: Find an optimal parenthesization of a matrix chain product whose

Problem 4: Matrix Chain Multiplication

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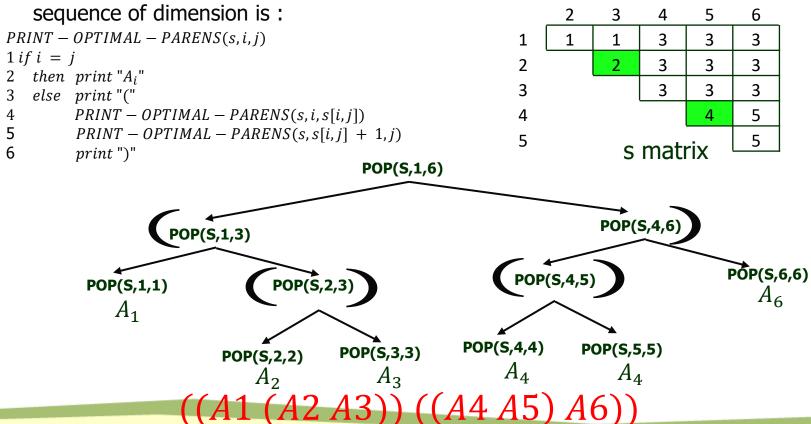
Example 1: Find an optimal parenthesization of a matrix chain product whose

sequence of dimension is: PRINT - OPTIMAL - PARENS(s, i, j)1 if i = jthen print " A_i " else print "(" PRINT - OPTIMAL - PARENS(s, i, s[i, j])5 PRINT - OPTIMAL - PARENS(s, s[i, j] + 1, j)5 s matrix print")" POP(S,1,6) POP(S,4,6) POP(S,1,3) POP(S,6,6) POP(S,4,5) POP(S,2,3) POP(S,1,1)

Problem 4: Matrix Chain Multiplication

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Problem 4: Matrix Chain Multiplication

Step 4: Construct / print the optimal solution.

Example 1: Find an optimal parenthesization of a matrix chain product whose

sequence of dimension is:

```
POP(s, 1, 6), s[1, 6] = 3, (A1A2A3)(A4A5A6)

POP(s, 1, 3), s[1, 3] = 1, ((A1)(A2A3))(A4A5A6)

POP(s, 4, 6), s[4, 6] = 5, ((A1)(A2A3))((A4A5)(A6))

POP(s, 2, 3), s[2, 3] = 2, ((A1)((A2)(A3)))((A4A5)(A6))

POP(s, 4, 5), s[4, 5] = 4, ((A1)((A2)(A3)))(((A4)(A5))(A6))

Hence the product is computed as follows
```

(A1(A2A3))((A4A5)A6).

Problem 4: Matrix Chain Multiplication

Example 2: Find an optimal parenthesization of a matrix-chain product whose sequence of dimensions is $\langle 5, 10, 3, 12, 5, 50, 6 \rangle$

Example 3: Find an optimal parenthesization of a matrix-chain product whose sequence of dimensions is $\langle 5, 4, 6, 2, 7 \rangle$

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Self practice

