Algorithm Analysis and Design

Recurrence Equation

(Solving Recurrence using Iteration Methods)

Lecture – 7 and 8

Overview

- A ***recurrence*** is a function is defined in terms of
  - one or more base cases, and
  - itself, with smaller arguments.

Examples:

- $$T(n) = \begin{cases} 1 & \text{if } n = 1\,, \\ T(n-1)+1 & \text{if } n > 1\,. \end{cases}$$
  Solution: $T(n) = n$.

- $$T(n) = \begin{cases} 1 & \text{if } n = 1\,, \\ 2T(n/2)+n & \text{if } n \geq 1\,. \end{cases}$$
  Solution: $T(n) = n \lg n + n$.

- $$T(n) = \begin{cases} 0 & \text{if } n = 2\,, \\ T(\sqrt{n})+1 & \text{if } n > 2\,. \end{cases}$$
  Solution: $T(n) = \lg \lg n$.

- $$T(n) = \begin{cases} 1 & \text{if } n = 1\,, \\ T(n/3)+T(2n/3)+n & \text{if } n > 1\,. \end{cases}$$
  Solution: $T(n) = \Theta(n \lg n)$.

Overview

- Many technical issues:
  - Floors and ceilings

    [Floors and ceilings can easily be removed and don't affect the solution to the recurrence. They are better left to a discrete math course.]
  - Exact vs. asymptotic functions
  - Boundary conditions

Overview

In algorithm analysis, the recurrence and it's solution are expressed by the help of asymptotic notation.

- Example: $T(n) = 2T(n/2) + \Theta(n)$, with solution $T(n) = \Theta(n \lg n)$.
  - The boundary conditions are usually expressed as $T(n) = O(1)$ for sufficiently small n..
  - But when there is a desire of an exact, rather than an asymptotic, solution, the need is to deal with boundary conditions.
  - In practice, just use asymptotics most of the time, and ignore boundary conditions.

Recursive Function

- Example

$A(n)$

$\{$

$If(n > 1)$

$Return(A(n-1))$

$\}$

The relation is called recurrence relation

The Recurrence relation of given function is written as follows.

$$T(n) = T(n-1) + 1$$

Recursive Function

- To solve the Recurrence relation the following methods are used:

  1. **Iteration method**
  2. Recursion-Tree method
  3. Master Method
  4. Substitution Method

Iteration Method(Example 1)

- In Iteration method the basic idea is to expand the recurrence and express it as a summation of terms dependent only on 'n' (i.e. the number of input) and the initial conditions.

Example 1:

Solve the following recurrence relation by using Iteration method.

$$T(n) = \begin{cases} T(n-1) + 1 & if\ n > 1 \\ 1 & if\ n = 1 \end{cases}$$

Iteration Method (Example 1)

It means $T(n) = T(n-1) + 1 \ \ if\ n > 1\ \ and\ \ T(n) = 1\ when\ n = 1 ---(1)$

$Put\ n = n - 1\ in\ equation\ 1, we\ get$

$T(n-1) = T(n-2) + 1$

$Put\ the\ value\ of\ T(n-1)\ in\ equation\ 1, we\ get$

$T(n) = T(n-2) + 2 --------------------(2)$

$Put\ n = n - 2\ in\ equation\ 1, we\ get$

$T(n-2) = T(n-3) + 1$

$Put\ the\ value\ of\ T(n-2)\ in\ equation\ 2, we\ get$

$T(n) = T(n-3) + 3 --------------------(3)$

……….

$T(n) = T(n-k) + k --------------------(k)$

Iteration Method (Example 1)

$Let\ T(n-k) = T(1) = 1 \quad (As\ per\ the\ base\ condition\ of\ recurrence)$

$So\ n-k = 1$

$\Rightarrow k = n-1$

$Now\ put\ the\ value\ of\ k\ in\ equation\ k$

$T(n) = T\big(n-(n-1)\big) + n - 1$

$T(n) = T(1) + n - 1$

$T(n) = 1 + n - 1$

$T(n) = n$

$\therefore\ T(n) = \Theta(n)$

Iteration Method (Example 2)

Example 1:

Solve the following recurrence relation by using Iteration method.

$$T(n) = \begin{cases} 2T\left(\frac{n}{2}\right) + 3n^2 & if\ n > 1 \\ 11 & if\ n = 1 \end{cases}$$

Iteration Method (Example 2)

$$It\ means\ T(n) = 2T\left(\frac{n}{2}\right) + 3n^2 \quad if\ n > 1\ \ and\ \ T(n) = 11\ when\ n = 1 - -(1)$$

$$Put\ n = \frac{n}{2}\ in\ equation\ 1, we\ get$$

$$T\left(\frac{n}{2}\right) = 2T\left(\frac{n}{4}\right) + 3\ \left(\frac{n}{2}\right)^2$$

$$T\left(\frac{n}{2}\right) = 2T\left(\frac{n}{2^2}\right) + 3\ \left(\frac{n}{2}\right)^2$$

$$Put\ the\ value\ of\ T\left(\frac{n}{2}\right)\ in\ equation\ 1, we\ get$$

$$T(n) = 2\left[2T\left(\frac{n}{2^2}\right) + 3\ \left(\frac{n}{2}\right)^2\right] + 3n^2$$

$$T(n) = 2^2T\left(\frac{n}{2^2}\right) + 2.3\frac{n^2}{4} + 3n^2$$

$$T(n) = 2^2T\left(\frac{n}{2^2}\right) + 3\frac{n^2}{2} + 3n^2\ \ ----------------(2)$$

Iteration Method (Example 2)

$$Put\ n = \frac{n}{4}\ in\ equation\ 1, we\ get$$

$$T\left(\frac{n}{4}\right) = 2T\left(\frac{n}{8}\right) + 3\ \left(\frac{n}{4}\right)^2$$

$$T\left(\frac{n}{4}\right) = 2T\left(\frac{n}{2^3}\right) + 3\ \left(\frac{n}{4}\right)^2$$

$$Put\ the\ value\ of\ T\left(\frac{n}{4}\right)\ in\ equation\ 2, we\ get$$

$$T(n) = 2^2\left[2T\left(\frac{n}{8}\right) + 3\frac{n^2}{16}\right] + 3\frac{n^2}{2} + 3n^2$$

$$T(n) = 2^2\left[2T\left(\frac{n}{2^3}\right) + 3\ \left(\frac{n}{4}\right)^2\right] + 3\frac{n^2}{2} + 3n^2$$

$$T(n) = 2^3T\left(\frac{n}{2^3}\right) + 4.3\frac{n^2}{16} + 3\frac{n^2}{2} + 3n^2$$

Iteration Method (Example 2)

$$T(n) = 2^3 T\left(\frac{n}{2^3}\right) + 3\frac{n^2}{2^2} + 3\frac{n^2}{2} + 3n^2 ----------------(3)$$

$\ldots \ldots .$

$$T(n) = 2^i T\left(\frac{n}{2^i}\right) + \cdots + \cdots + \cdots + 3\frac{n^2}{2^2} + 3\frac{n^2}{2} + 3n^2 ---------(i^{th}\ term)$$

$\mathit{and\ the\ series\ terminate\ when}\ \dfrac{n}{2^i} = 1$

$\Rightarrow n = \ 2^i$

$\mathit{Taking}\ \log\ \mathit{both\ side}$

$\Rightarrow \log_2 n = i \log_2 2$

$\Rightarrow i = \ \log_2 n \quad (\mathit{because}\ \log_2 2 = 1)$

Iteration Method (Example 2)

$Hence\ we\ can\ write\ the\ i^{th}\ term\ as\ follows$

$$\Rightarrow T(n) = 3n^2 + 3\frac{n^2}{2} + 3\frac{n^2}{2^2} + \cdots + \cdots + \cdots + \ 2^i T\left(\frac{n}{2^i}\right)$$

$$\Rightarrow T(n) = 3n^2 + 3\frac{n^2}{2} + 3\frac{n^2}{2^2} + \cdots + \cdots + \cdots + \ 2^{\log_2 n} T(1)$$

$$\Rightarrow T(n) = 3n^2 + 3\frac{n^2}{2} + 3\frac{n^2}{2^2} + \cdots + \cdots + \cdots + \ 2^{\log_2 n}.11$$

$$\Rightarrow T(n) = 3n^2 + 3\frac{n^2}{2} + 3\frac{n^2}{2^2} + \cdots + \cdots + \cdots + \ n^{\log_2 2}.11 \qquad [\,As\ \log_2 2 = 1]$$

$$\Rightarrow T(n) = 3n^2 + 3\frac{n^2}{2} + 3\frac{n^2}{2^2} + \cdots + \cdots + \cdots + \ n.11$$

$$\Rightarrow T(n) = \left[3n^2 + 3\frac{n^2}{2} + 3\frac{n^2}{2^2} + \cdots + \cdots + \cdots\right] + 11.n$$

$$\Rightarrow T(n) = 3n^2\left[1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \cdots + \cdots\right] + 11.n$$

Iteration Method (Example 2)

$As\ we\ know\ that\ \ Sum\ of\ infinite\ Geometric\ series\ is$

$$= a\ +\ ar\ +\ ar^2\ + \ldots + ar^{(n-1)} = \sum_{i=0}^{\infty} ar^i = a\left(\frac{1}{1-r}\right) = \frac{a}{1-r}$$

$$\Rightarrow T(n) \leq 3n^2\left[\frac{1}{1-\frac{1}{2}}\right] + 11\,n$$

$$\Rightarrow T(n) \leq 3n^2.2 + 11\,n$$

$$\Rightarrow T(n) \leq 6n^2 + 11\,n$$

$Hence\ T(n) = O(n^2)$

Iteration Method (Example 3)

Example 3:

Solve the following recurrence relation by using Iteration method.

$$T(n) = \begin{cases} 8T\left(\frac{n}{2}\right) + n^2 & if\ n > 1 \\ 1 & if\ n = 1 \end{cases}$$

Iteration Method (Example 3)

$$It\ means\ T(n) = 8T\left(\frac{n}{2}\right) + n^2\ \ if\ n > 1\ \ and\ \ T(n) = 1\ when\ n = 1 ---- (1)$$

$$Put\ n = \frac{n}{2}\ in\ equation\ 1, we\ get$$

$$T\left(\frac{n}{2}\right) = 8T\left(\frac{n}{4}\right) + \left(\frac{n}{2}\right)^2$$

$$Put\ the\ value\ of\ T\left(\frac{n}{2}\right)\ in\ equation\ 1, we\ get$$

$$T(n) = 8\left[8T\left(\frac{n}{4}\right) + \left(\frac{n}{2}\right)^2\right] + n^2$$

$$T(n) = 8^2T\left(\frac{n}{4}\right) + 8\frac{n^2}{4} + n^2 ---------------- (2)$$

$$Put\ n = \frac{n}{4}\ in\ equation\ 1, we\ get$$

$$T\left(\frac{n}{4}\right) = 8T\left(\frac{n}{8}\right) + \left(\frac{n}{4}\right)^2$$

Iteration Method (Example 3)

$$Put\ the\ value\ of\ T\left(\frac{n}{4}\right)\ in\ equation\ 2, we\ get$$

$$T(n) = 8^2\left[8T\left(\frac{n}{8}\right) + \left(\frac{n}{4}\right)^2\right] + 8\frac{n^2}{4} + n^2$$

$$T(n) = 8^3T\left(\frac{n}{8}\right) + 8^2\frac{n^2}{4^2} + 8\frac{n^2}{4} + n^2 ---------------(3)$$

$$\dots\dots$$

$$T(n)$$

$$= 8^kT\left(\frac{n}{2^k}\right) + 8^{k-1}\frac{n^2}{4^{k-1}} + \dots + \dots + \dots + 8^2\frac{n^2}{4^2} + 8\frac{n^2}{4} + n^2 ----(k^{th}\ term)$$

$$T(n) = 8^kT\left(\frac{n}{2^k}\right) + n^2\left[\frac{8^{k-1}}{4^{k-1}} + \frac{8^{k-2}}{4^{k-2}} \dots + \dots + \dots + \frac{8^2}{4^2} + \frac{8}{4} + 1\right]$$

$$T(n) = 8^kT\left(\frac{n}{2^k}\right) + n^2[2^{k-1} + 2^{k-2} \dots + \dots + \dots + 2^2 + 2 + 1] -----(4)$$

Iteration Method (Example 3)

$$and\ the\ series\ terminate\ when\ \frac{n}{2^k} = 1$$

$$\Rightarrow n = \ \ 2^k$$

$$Taking \log both\ side$$

$$\Rightarrow \log_2 n = k \log_2 2$$

$$\Rightarrow k = \ \ \log_2 n \qquad (because\ \ \log_2 2 = 1)$$

$$Now, apply\ the\ value\ of\ \ k = \log_2 n\ \ and\ \frac{n}{2^k} = 1\ \ in\ equation\ 4$$

$$T(n) = 8^{\log_2 n} T(1) + \ n^2 \left[2^{\log_2 n - 1} + 2^{\log_2 n - 2} + \cdots + 2^2 + 2 + 1\right] - -(5)$$

Iteration Method (Example 3)

$As\ we\ know\ that\ \ Sum\ of\ finite\ Geometric\ series\ is$

$$= a\ +\ ar\ +ar^2+\ldots+ar^n = \sum_{i=0}^{n} ar^i = a\left(\frac{r^{n+1}-1}{r-1}\right)$$

$Here, n(Total\ number\ of\ terms) = \log_2 n\,, a = 1\ and\ r = 2.$

$Hence\ equation\ 5\ can\ be\ written\ as\ follows$

$$T(n) = 8^{\log_2 n}T(1) +\ n^2\left[2^{\log_2 n-1} + 2^{\log_2 n-2}\ldots + \cdots + \cdots + 2^2 + 2 + 1\right]$$

$$T(n) = 8^{\log_2 n} +\ n^2\left(\frac{2^{\log_2 n}-1}{2-1}\right)$$

Iteration Method (Example 3)

$$T(n) = n^{\log_2 8} + \ n^2\left(\frac{n^{\log_2 2} - 1}{1}\right)$$

$$T(n) = n^{\log_2 8} + \ n^2\left(n^{\log_2 2} - 1\right)$$

$$T(n) = n^{\log_2 8} + \ n^2\left(n^{\log_2 2} - 1\right)$$

As $\log_2 8 = 3$ *and* $\log_2 2 = 1$

$$T(n) = n^3 + \ n^2(n^1 - 1)$$

$$T(n) = n^3 + \ n^2(n - 1)$$

$$T(n) = n^3 + n^3 + n^2$$

$$T(n) = 2n^3 + n^2$$

$$Hence\ T(n) = O(n^3)$$

Iteration Method (Example 4)

Example 4:

Solve the following recurrence relation by using Iteration method.

$$T(n) = \begin{cases} 7T\left(\frac{n}{2}\right) + n^2 & if\ n > 1 \\ 1 & if\ n = 1 \end{cases}$$

$(i.e. Strassion\ Algorithm)$

Iteration Method (Example 4)

$It\ means\ T(n) = 7T\left(\frac{n}{2}\right) + n^2 \quad if\ n > 1\ \ and\ \ T(n) = 1\ when\ n = 1 ---- (1)$

$Put\ n = \frac{n}{2}\ in\ equation\ 1, we\ get$

$$T\left(\frac{n}{2}\right) = 7T\left(\frac{n}{4}\right) + \left(\frac{n}{2}\right)^2$$

$Put\ the\ value\ of\ T\left(\frac{n}{2}\right)\ in\ equation\ 1, we\ get$

$$T(n) = 7\left[7T\left(\frac{n}{4}\right) + \left(\frac{n}{2}\right)^2\right] + n^2$$

$$T(n) = 7^2T\left(\frac{n}{4}\right) + 7\frac{n^2}{4} + n^2 ---------------- (2)$$

$Put\ n = \frac{n}{4}\ in\ equation\ 1, we\ get$

$$T\left(\frac{n}{4}\right) = 7T\left(\frac{n}{8}\right) + \left(\frac{n}{4}\right)^2$$

Iteration Method (Example 4)

Put the value of $T\left(\frac{n}{4}\right)$ *in equation* 2, *we get*

$$T(n) = 7^2\left[7T\left(\frac{n}{8}\right) + \left(\frac{n}{4}\right)^2\right] + 7\frac{n^2}{4} + n^2$$

$$T(n) = 7^3T\left(\frac{n}{8}\right) + 7^2\frac{n^2}{4^2} + 7\frac{n^2}{4} + n^2 ------------------(3)$$

… …

$$T(n) = 7^kT\left(\frac{n}{2^k}\right) + 7^{k-1}\frac{n^2}{4^{k-1}} + \cdots + \cdots + \cdots + 7^2\frac{n^2}{4^2} + 7\frac{n^2}{4} + n^2 ---(k^{th}\ term)$$

$$T(n) = 7^kT\left(\frac{n}{2^k}\right) + n^2\left[\frac{7^{k-1}}{4^{k-1}} + \frac{7^{k-2}}{4^{k-2}} \ldots + \cdots + \cdots + \frac{7^2}{4^2} + \frac{7}{4} + 1\right]$$

$$T(n) = 7^kT\left(\frac{n}{2^k}\right) + n^2\left[\sum_{i=0}^{k-1}\left(\frac{7}{4}\right)^i\right] -----(4)$$

Iteration Method (Example 4)

$Put\ the\ value\ of\ T\left(\frac{n}{4}\right)\ in\ equation\ 2, we\ get$

$$T(n) = 7^2\left[7T\left(\frac{n}{8}\right) + \left(\frac{n}{4}\right)^2\right] + 7\frac{n^2}{4} + n^2$$

$$T(n) = 7^3T\left(\frac{n}{8}\right) + 7^2\frac{n^2}{4^2} + 7\frac{n^2}{4} + n^2 ----------------(3)$$

$\dots\dots$

$$T(n) = 7^kT\left(\frac{n}{2^k}\right) + 7^{k-1}\frac{n^2}{4^{k-1}} + \cdots + \cdots + \cdots + 7^2\frac{n^2}{4^2} + 7\frac{n^2}{4} + n^2 ---(k^{th}\ term)$$

$$T(n) = 7^kT\left(\frac{n}{2^k}\right) + n^2\left[\frac{7^{k-1}}{4^{k-1}} + \frac{7^{k-2}}{4^{k-2}} \dots + \cdots + \cdots + \frac{7^2}{4^2} + \frac{7}{4} + 1\right]$$

$$T(n) = 7^kT\left(\frac{n}{2^k}\right) + n^2\left[\sum_{i=0}^{k-1}\left(\frac{7}{4}\right)^i\right] -----(4)$$

Iteration Method (Example 4)

$As\ we\ know\ that\ \ Sum\ of\ \ finite\ Geometric\ series\ is$

$$= a\ +\ ar\ +\ ar^2\ + \ldots +\ ar^n = \sum_{i=0}^{n} ar^i = a\left(\frac{r^{n+1} - 1}{r - 1}\right)$$

$Here, n(Total\ number\ of\ \ terms) = \log_2 n\ , a = 1\ and\ r = 2.$

$Hence\ equation\ 4\ can\ be\ written\ as\ follows$

$$T(n) = 7^{\log_2 n} +\ n^2\left(\frac{\left(\frac{7}{4}\right)^{\log_2 n} - 1}{0.75}\right)$$

Iteration Method (Example 4)

$$T(n) = n^{\log_2 7} + \; n^2 \left(\frac{n^{\log_2\left(\frac{7}{4}\right)} - 1}{0.75} \right)$$

$$T(n) = n^{\log_2 7} + \; n^2 \left(\frac{n^{\log_2 7 - \log_2 4} - 1}{0.75} \right)$$

$$T(n) = n^{2,8} + \; n^2 \left(\frac{n^{2.80-2} - 1}{0.75} \right)$$

As $\log_2 7 = 2.80 \; and \; \log_2 4 = 2$

$$T(n) = n^{2.8} + \; n^2 \left(\frac{n^{0.8} - 1}{0.75} \right)$$

$$Hence \; T(n) = O(n^{2.8})$$

Iteration Method

Iteration Method

Iteration Method

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Thank u