Algorithm Analysis and Design

Growth of Functions

Week 2
Lecture – 4,5, and 6

Overview

- A way to describe behaviour of functions *in the limit*. We're studying ***Asymptotic*** efficiency.
- Describe *growth* of functions.(i.e. The order of growth of the running time of an algorithm)
- Focus on what's important by abstracting away low-order terms and constant factors.
- How we indicate running times of algorithms.
- A way to compare "sizes" of functions through different notations ***(i.e. Asymptotic Notations):***

$O \approx \leq$

$\Omega \approx \geq$

$\Theta \approx =$

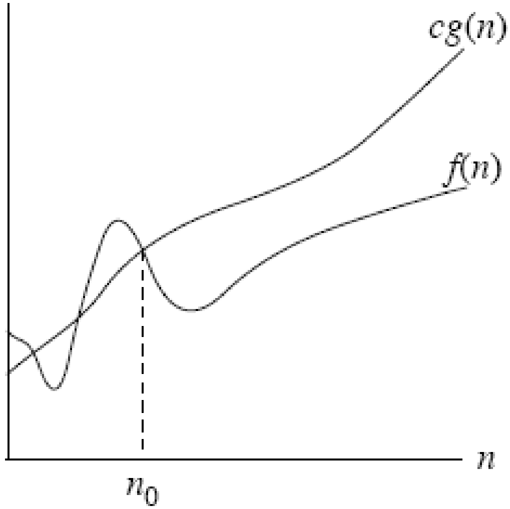
$o \approx <$

$\omega \approx >$

Asymptotic notation (Big Oh)

***O*-notation**

$O(g(n)) = \{ f(n) :$ there exist positive constants c and n_0 such that $0 \le f(n) \le cg(n)$ for all $n \ge n_0 \}$.



- Another view, probably easier to use i.e. $\lim_{n\to\infty} \frac{f(n)}{g(n)} \le c, where\ c > 0\ and\ n_0 \ge 1$

$g(n)$ is an ***asymptotic upper bound*** for $f(n)$.

If $f(n) \in O(g(n))$, we write $f(n) = O(g(n))$ (will precisely explain this soon).

Asymptotic notation (Big Oh)

Example: $2n^2 = O(n^3)$, with $c = 1$ and $n_0 = 2$.

Examples of functions in $O(n^2)$:

n^2
$n^2 + n$
$n^2 + 1000n$
$1000n^2 + 1000n$
Also,
n
$n/1000$
$n^{1.99999}$
$n^2/\lg\lg\lg n$

Asymptotic notation (Big Oh)

Example 1

$Prove\ that\quad f(n) = 2n + 3\ \in\ O(n)$

Asymptotic notation (Big Oh)

Example 1

$Prove\ that \quad f(n) = 2n + 3 \ \in \ O(n)$

$\Longrightarrow 2n + 3 \leq cg(n)$

$\Longrightarrow 2n + 3 \leq c(n)$

$\Longrightarrow 2n + 3 \leq 5n, \qquad n \geq 1$

$Hence f(n) = O(n)$

$For, f(n) = O(n^2)\ is\ also\ true$

$For, f(n) = O(2^n)\ is\ also\ true$

$But\ for, f(n) = O(\lg n)\ is\ not\ true$

Because

$$1 < \lg n < \sqrt{n} < n < n^2 < n^3 < \cdots < 2^n < 3^n < \cdots < n^n$$

Asymptotic notation (Big Oh)

Example 2

$Prove\ that\ \ f(n) = 2n^2 + 3n + 4\ \in\ O(n^2)$

Asymptotic notation (Big Oh)

Example 2

$Prove\ that\ \ f(n) = 2n^2 + 3n + 4\ \in\ O(n^2)$

$\Longrightarrow 2n^2 + 3n + 4 \leq 2n^2 + 3n^2 + 4n^2$

$\Longrightarrow 2n^2 + 3n + 4 \leq 11n^2\ \ , where\ c = 11\ and\ \ n \geq 1$

$Hence f(n) = O(n^2)$

Asymptotic notation (Big Oh)

Example 3

$If\ f(n) = 2^{n+1}\ and\ g(n) =\ 2^{n}\ the\ prove\ that\ f(n) \in\ \mathrm{O}(g(n))$

Asymptotic notation (Big Oh)

Example 3

$If\ f(n) = 2^{n+1}\ and\ g(n) =\ 2^n\ the\ prove\ that\ f(n) \in\ O(g(n))$

$\Longrightarrow 2^{n+1} = 2^n . 2$

So, as per the definition of Big Oh

$f(n) \leq cg(n)$

Hence

$\Longrightarrow 2^{n+1} \leq 2^n . 2$

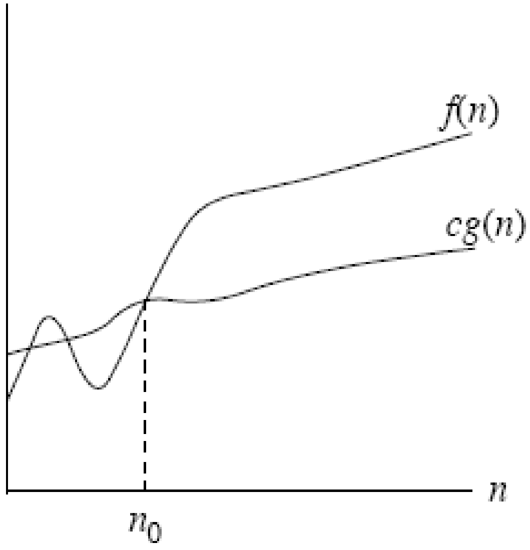
$\Longrightarrow 2^{n+1} \leq 2. 2^n\ \ for\ all\ n \geq 1\ and\ c > 0$

$Hence,\ \ f(n) \in\ O(g(n))$

Asymptotic notation (Big Omega)

Ω-notation

$\Omega(g(n)) = \{f(n) :$ there exist positive constants c and n_0 such that
$0 \le cg(n) \le f(n)$ for all $n \ge n_0\}$.



- Another view, probably easier to use i.e. $\lim_{n \to \infty} \frac{f(n)}{g(n)} \ge c, where\ c > 0\ and\ n_0 \ge 1$

$g(n)$ is an ***asymptotic lower bound*** for $f(n)$.

Asymptotic notation (Big Omega)

Example: $\sqrt{n} = \Omega(\lg n)$, with $c = 1$ and $n_0 = 16$.

Examples of functions in $\Omega(n^2)$:

n^2
$n^2 + n$
$n^2 - n$
$1000n^2 + 1000n$
$1000n^2 - 1000n$
Also,
n^3
$n^{2.00001}$
$n^2 \lg \lg \lg n$
2^{2^n}

Asymptotic notation (Big Omega)

Example 4

$Prove\ that\ \ f(n) = 2n^2 + 3n + 4\ \in\ \Omega(n^2)$

Asymptotic notation (Big Omega)

Example 4

$Prove\ that\ \ f(n) = 2n^2 + 3n + 4\ \in\ \Omega(n^2)$

$\Longrightarrow 2n^2 + 3n + 4 \geq 1 * n^2$

$Hence\ f(n) = \Omega(n^2)\ \ where\ c = 1\ and\ \ n \geq 1$

Asymptotic notation (Big Omega)

Example 5

$If\ f(n) = 3n + 2, \qquad g(n) = n^2\ \ show\ that\ f(n)\ \notin \Omega(g(n))$

Asymptotic notation (Big Omega)

Example 5

$If\ f(n) = 3n + 2, \qquad g(n) = n^2\ \ show\ that\ f(n)\ \notin \Omega(g(n))$

$$\Rightarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} > 0$$

$$\Rightarrow \lim_{n\to\infty} \frac{3n+2}{n^2} > 0$$

$$\Rightarrow \lim_{n\to\infty} \frac{n\left(3+\frac{2}{n}\right)}{n^2} > 0$$

$$\Rightarrow \lim_{n\to\infty} \frac{\left(3+\frac{2}{n}\right)}{n} > 0$$

$$\Rightarrow 0 > 0\ is\ false, Hence\ f(n)\ \notin \Omega(g(n))$$

Asymptotic notation (Big Omega)

Example 6

$If\ f(n) = 2^n + n^2\ \ and\ \ \ g(n) = 2^n\ \ show\ that\ f(n)\ \in \Omega(g(n))$

Asymptotic notation (Big Omega)

Example 6

$If\ f(n) = 2^n + n^2\ \ and\ \ \ g(n) = 2^n\ \ show\ that\ f(n)\ \in \Omega(g(n))$

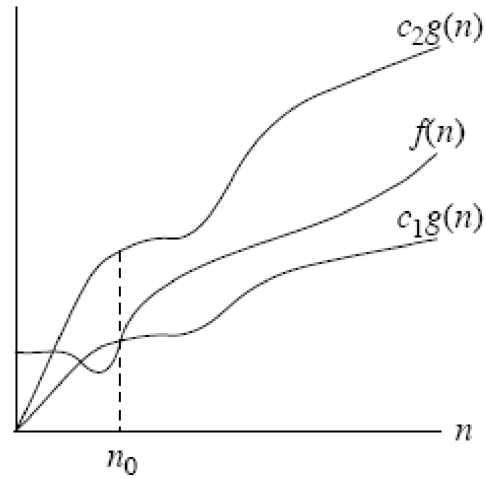
$\Longrightarrow\ 2^n + n^2 > \ 2^n\ for\ all\ n \geq 1\ and\ c = 1$

$Hence, f(n)\ \in \Omega\big(g(n)\big) is\ true$

Asymptotic notation (Theta)

Θ-notation

$\Theta(g(n)) = \{f(n) :$ there exist positive constants c_1, c_2, and n_0 such that $0 \le c_1 g(n) \le f(n) \le c_2 g(n)$ for all $n \ge n_0\}$.



- Another view, probably easier to use i.e. $\lim_{n\to\infty} \frac{f(n)}{g(n)} = c, where\ c > 0\ and\ n_0 \ge 1$

$g(n)$ is an ***asymptotically tight bound*** for $f(n)$.

Example: $n^2/2 - 2n = \Theta(n^2)$, with $c_1 = 1/4$, $c_2 = 1/2$, and $n_0 = 8$.

Asymptotic notation (Theta)

Example 7

$Show\ that\ f(n) = 10n^3 + 5n^2 + 17 \in \Theta(n^3)$

Asymptotic notation (Theta)

Example 7

$Show\ that\ f(n) = 10n^3 + 5n^2 + 17 \in \Theta(n^3)$

As per the definition of θ notation $C_1 g(n) \leq f(n) \leq C_2 g(n)$

$\Longrightarrow 10n^3 \leq 10n^3 + 5n^2 + 171 < 10n^3 + 5n^3 + 17n^3$

$\Longrightarrow 10n^3 \leq 10n^3 + 5n^2 + 171 < 32n^3$

$So,\qquad C_1 = 10 \text{ and } C_2 = 32 \text{ for all n} \geq 1$

$Hence, Proved$

Asymptotic notation (Theta)

Example 8

$Show\ that\ f(n) = (n + a)^b \in \Theta\left(n^b\right)$

Asymptotic notation (Theta)

Example 8

$Show\ that\ f(n) = (n+a)^b \in \Theta\left(n^b\right)$

As per the definition of θ notation $\Longrightarrow \lim\limits_{n\to\infty} \frac{f(n)}{g(n)} = c\ for\ all\ n \geq 1\ and\ c > 0$

$$\Longrightarrow \lim_{n\to\infty} \frac{(n+a)^b}{n^b}$$

$$\Longrightarrow \lim_{n\to\infty} \frac{n^b\left(1+\frac{a}{n}\right)^b}{n^b}$$

$$\Longrightarrow \lim_{n\to\infty} \left(1+\frac{a}{n}\right)^b \qquad \therefore \frac{a}{\infty} = 0$$

$\Longrightarrow 1\ which\ is\ a\ constant$

$Hence\ , f(n) = (n+a)^b \in \Theta\left(n^b\right)$ is true

Asymptotic notation (Little Oh)

o-notation

$$o(g(n)) = \{f(n) : \text{ for all constants } c > 0, \text{ there exists a constant } n_0 > 0 \text{ such that } 0 \le f(n) < cg(n) \text{ for all } n \ge n_0\}.$$

Another view, probably easier to use: $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0.$

$n^{1.9999} = o(n^2)$
$n^2/\lg n = o(n^2)$
$n^2 \ne o(n^2)$ (just like $2 \not< 2$)
$n^2/1000 \ne o(n^2)$

Asymptotic notation (Little Oh)

Example 9

$If\ f(n) = 2n, \qquad g(n) = \ n^2\ \ Prove\ that\ f(n) = o(g(n))$

Asymptotic notation (Little Oh)

Example 9

$If\ f(n) = 2n, \qquad g(n) = \ n^2\ \ Prove\ that\ f(n) = o(g(n))$

$$\Longrightarrow \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$$

$$\Longrightarrow \lim_{n \to \infty} \frac{2n}{n^2}$$

$$\Longrightarrow \lim_{n \to \infty} \frac{2}{n}$$

$$\Longrightarrow 0$$

$Which\ is\ True, Hence\ f(n) = o(g(n))$

Asymptotic notation (Little Oh)

Example 10

$If\ f(n) = 2n^2, \qquad g(n) = \ n^2\ \ Prove\ that\ f(n) \neq o(g(n))$

Asymptotic notation (Little Oh)

Example 10

$If\ f(n) = 2n^2, \quad g(n) = n^2\ \ Prove\ that\ f(n) \neq o(g(n))$

$$\Longrightarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

$$\Longrightarrow \lim_{n\to\infty} \frac{2n^2}{n^2}$$

$$\Longrightarrow \lim_{n\to\infty} 2$$

$$\Longrightarrow 2 \neq 0$$

$Which\ is\ True, Hence\ f(n) \neq o(g(n))$

Asymptotic notation (Little omega)

ω-notation

$$\omega(g(n)) = \{f(n) : \text{ for all constants } c > 0, \text{ there exists a constant } n_0 > 0 \text{ such that } 0 \le cg(n) < f(n) \text{ for all } n \ge n_0\}.$$

Another view, again, probably easier to use: $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$.

$n^{2.0001} = \omega(n^2)$
$n^2 \lg n = \omega(n^2)$
$n^2 \ne \omega(n^2)$

Asymptotic notation (Little omega)

Example 11

$If\ f(n) = 2n^2 + 16\ and\ g(n) = n^2\ show\ that\ f(n) \neq \omega\big(g(n)\big)$

Asymptotic notation (Little omega)

Example 11

$If\ f(n) = 2n^2 + 16\ and\ g(n) = n^2\ show\ that\ f(n) \neq \omega\big(g(n)\big)$

$$\Longrightarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$$

$$\Longrightarrow \lim_{n\to\infty} \frac{n^2\left(2 + \frac{16}{n^2}\right)}{n^2}$$

$$\Longrightarrow \lim_{n\to\infty} \left(2 + \frac{16}{n^2}\right)$$

$$\Longrightarrow \lim_{n\to\infty} (2 + 0)$$

$$\Longrightarrow \lim_{n\to\infty} 2$$

$So\ 2 \neq \infty\ is\ true\ , Hence\ f(n) \neq \omega\big(g(n)\big)$

Asymptotic notation (Little Oh omega)

Example 12

$If\ f(n) = n^2\ and\ g(n) = \log n\ show\ that\ f(n) \in \omega\big(g(n)\big)$

Asymptotic notation (Little Oh omega)

Example 12

$If\ f(n) = n^2\ and\ g(n) = \log n\ show\ that\ f(n) \in \omega\big(g(n)\big)$

$$\Longrightarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$$

$$\Longrightarrow \lim_{n\to\infty} \frac{n^2}{\log n}$$

Apply L Hospital Rule

$$\Longrightarrow \lim_{n\to\infty} \frac{2n}{1/n}$$

$$\Longrightarrow \lim_{n\to\infty} 2n^2$$

$$\Longrightarrow \lim_{n\to\infty} \infty$$

$Which\ is\ true\ as\ per\ \omega - notation, Hence\ f(n) \in \omega\big(g(n)\big)$

Comparisons of functions

Transitivity:

$f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n)) \Rightarrow f(n) = \Theta(h(n))$.
Same for O, Ω, o, and ω.

Reflexivity:

$f(n) = \Theta(f(n))$.
Same for O and Ω.

Symmetry:

$f(n) = \Theta(g(n))$ if and only if $g(n) = \Theta(f(n))$.

Transpose symmetry:

$f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$.
$f(n) = o(g(n))$ if and only if $g(n) = \omega(f(n))$.

Comparisons:

- $f(n)$ is ***asymptotically smaller*** than $g(n)$ if $f(n) = o(g(n))$.
- $f(n)$ is ***asymptotically larger*** than $g(n)$ if $f(n) = \omega(g(n))$.

No trichotomy. Although intuitively, we can liken O to $\leq$, Ω to $\geq$, etc., unlike real numbers, where $a < b$, $a = b$, or $a > b$, we might not be able to compare functions.

Example: $n^{1+\sin n}$ and n, since $1 + \sin n$ oscillates between 0 and 2.

Standard notations and common functions

Monotonicity

- $f(n)$ is ***monotonically increasing*** if $m \le n \Rightarrow f(m) \le f(n)$.
- $f(n)$ is ***monotonically decreasing*** if $m \ge n \Rightarrow f(m) \ge f(n)$.
- $f(n)$ is ***strictly increasing*** if $m < n \Rightarrow f(m) < f(n)$.
- $f(n)$ is ***strictly decreasing*** if $m > n \Rightarrow f(m) > f(n)$.

Exponentials

Useful identities:

$$\begin{aligned} a^{-1} &= 1/a , \\ (a^m)^n &= a^{mn} , \\ a^m a^n &= a^{m+n} . \end{aligned}$$

Can relate rates of growth of polynomials and exponentials: for all real constants a and b such that $a > 1$,

$$\lim_{n\to\infty} \frac{n^b}{a^n} = 0 ,$$

which implies that $n^b = o(a^n)$.

A suprisingly useful inequality: for all real x,

$$e^x \geq 1 + x .$$

As x gets closer to 0, e^x gets closer to $1 + x$.

Logarithms

Notations:

$$\begin{aligned}
\lg n &= \log_2 n & &\text{(binary logarithm)}\ , \\
\ln n &= \log_e n & &\text{(natural logarithm)}\ , \\
\lg^k n &= (\lg n)^k & &\text{(exponentiation)}\ , \\
\lg \lg n &= \lg(\lg n) & &\text{(composition)}\ .
\end{aligned}$$

Logarithm functions apply only to the next term in the formula, so that $\lg n + k$ means $(\lg n) + k$, and *not* $\lg(n + k)$.

In the expression $\log_b a$:

- If we hold b constant, then the expression is strictly increasing as a increases.
- If we hold a constant, then the expression is strictly decreasing as b increases.

Useful identities for all real $a > 0$, $b > 0$, $c > 0$, and n, and where logarithm bases are not 1:

$$\begin{aligned}
a &= b^{\log_b a}\ , \\
\log_c(ab) &= \log_c a + \log_c b\ , \\
\log_b a^n &= n \log_b a\ , \\
\log_b a &= \frac{\log_c a}{\log_c b}\ , \\
\log_b(1/a) &= -\log_b a\ , \\
\log_b a &= \frac{1}{\log_a b}\ , \\
a^{\log_b c} &= c^{\log_b a}\ .
\end{aligned}$$

Factorials

$n! = 1 \cdot 2 \cdot 3 \cdot n$. Special case: $0! = 1$.

Can use ***Stirling's approximation***,

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \Theta\left(\frac{1}{n}\right)\right) ,$$

to derive that $\lg(n!) = \Theta(n \lg n)$.

Thank u