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LECTURE NOTES

MATHEMATICS-1 FOR MECHANICAL ENGINEERING STREAM (22MATM11)

MODULE-3

DIFFERENTIAL EQUATIONS-1

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Differential Equation of First Order and First Degree

An equation which contains one dependent variable and its derivatives with respect to one or more variables is called differential equations.

There are 2 types :-

1. Ordinary Differential Equations [ODE]
2. Partial Differential Equations [PDE]

1. Ordinary Differential Equations [ODE]

An equation which contains one dependent variables and its derivatives with respect to one independent variables is called Ordinary Differential Equations [ODE].

Ex:-

$$\Rightarrow x^2 \frac{d^2y}{dx^2} - 2 \left[\frac{dy}{dx} \right]^2 + y = 0$$

$$\Rightarrow x^2 \frac{d^2y}{dx^2} - 2xy \frac{dy}{dx} + 3y = 0$$

2. Partial Differential Equations [PDE]

An equation which contains one dependent variable and its derivatives with respect to two or more independent variables is called PDE.

Ex:-

$$\Rightarrow x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = z$$

$$\Rightarrow x^2 \frac{\partial^2 z}{\partial x^2} - 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial x} - 3 \frac{\partial z}{\partial y} + z = 0$$

Solution of ODE of 1st Order and 1st Degree

Step 1 :- Write the given differential equation to the standard form.

$$M(x, y) dx + N(x, y) dy = 0 \quad (1)$$

and identify M and N.

Step 2 :- Find $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$, if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$,

then eqn (1) is called an exact differential equation and follows the solution as,

$$\int M(x, y) dx + \int (\text{The terms which do not contain } x \text{ in } N) dy = C$$

Step 03 :- If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the given D.E can be called as non-exact D.E

Step 04 :- Non-exact D.E may have a solution by reading reducing to exact form as given below,

- * Find $f(x)$ by $\frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right)$ and find the integrating factor $IF = e^{\int f(x) dx}$.
- * Find $g(y)$ through $\frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) = g(y)$ and find the integrating factor $IF = e^{\int g(y) dy}$
- * Multiply the integrating factor to the given D.E on both sides and verify the exactness.

Problems :-

1. Solve $\frac{dy}{dx} + \frac{8x+3y-1}{3x+4y-2} = 0$

Sol:-
$$\begin{aligned} (3x+4y-2)dy + (8x+3y-1)dx &= 0 \\ (3x+4y-2)dx & \end{aligned}$$

$$\Rightarrow (3x+4y-2)dy + (8x+3y-1)dx = 0$$

$$\Rightarrow (8x+3y-1)dx + (3x+4y-2)dy = 0 \quad \text{--- (1)}$$

$$Md x + Nd y = 0$$

$$\therefore M = 8x+3y-1, \quad N = 3x+4y-2$$

$$\frac{\partial M}{\partial y} = 3, \quad \frac{\partial N}{\partial x} = 3$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Eqn (1) is an exact D.E

∴ The solution is;

$$\int(2x+3y-1) dx + \int(4y-2) dy = C$$

$$\Rightarrow 2 \int x dx + 3y \int 1 dx - \int 1 dx + 4 \int y dy - 2 \int 1 dy = C$$

$$\Rightarrow \frac{2x^2}{2} + 3yx - x + \frac{4y^2}{2} - 2y = C$$

$$\Rightarrow x^2 + 3yx - x + 2y^2 - 2y = C$$

2. Solve $(2x+y+1)dx + (x+2y+1)dy = 0$

$$\Rightarrow (2x+y+1) dx + (x+2y+1) dy = 0 \quad \text{--- (1)}$$

$$M dx + N dy = 0$$

$$M = 2x + y + 1, \quad N = x + 2y + 1$$

$$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = 1$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Eqn (1) is an exact D.E

∴ The solution is;

$$\int(2x+y+1) dx + \int(x+2y+1) dy = 0$$

$$\Rightarrow 2 \int x dx + y \int 1 dx + \int 1 dx + 2 \int y dy + \int 1 dy = 0$$

$$\Rightarrow \frac{2x^2}{2} + yx + x + \frac{2y^2}{2} + y = C$$

$$\Rightarrow x^2 + yx + y + x = C$$

3. Solve $(5x^4 + 3x^2y^2 - 2xy^3)dx + (2x^3y - 3x^2y^2 - 5y^4)dy = 0$

$$\Rightarrow (5x^4 + 3x^2y^2 - 2xy^3)dx + (2x^3y - 3x^2y^2 - 5y^4)dy = 0$$

$$Mdx + Ndy = 0$$

$$M = 5x^4 + 3x^2y^2 - 2xy^3, \quad N = 2x^3y - 3x^2y^2 - 5y^4$$

$$\frac{\partial M}{\partial y} = 6x^3y - 6xy^2 \quad \frac{\partial N}{\partial x} = 6x^2 - 6xy^2$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Eqn (1) is an EDE

∴ The soln is;

$$\int (5x^4 + 3x^2y^2 - 2xy^3)dx + \int (-5y^4)dy = 0$$

$$\Rightarrow 5 \int x^4 dx + 3y^2 \int x^2 dx - 2y^3 \int x dx - 5 \int y^4 dy = C$$

$$\Rightarrow 5 \frac{x^5}{5} + 3y^2 \frac{x^3}{3} - 2y^3 \frac{x^2}{2} - 5 \frac{y^5}{5} = C$$

$$\Rightarrow x^5 + x^3y^2 - x^2y^3 - y^5 = C$$

4. Solve, $\frac{dy}{dx} + \frac{y \cos x + \sin y + y}{\sin x + x \cos y + x} = 0$

$$\Rightarrow (\sin x + x \cos y + x)dy + (y \cos x + \sin y + y)dx = 0$$

$$\Rightarrow (y \cos x + \sin y + y)dx + (\sin x + x \cos y + x)dy = 0 \quad (1)$$

$$Mdx + Ndy = 0$$

$$M = y \cos x + \sin y + y, \quad N = \sin x + x \cos y + x$$

$$\frac{\partial M}{\partial y} = \cos x + \cos y + 1 \quad \frac{\partial N}{\partial x} = \cos x + \cos y + 1$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Eqn (1) is an EDE

∴ The solution is :-

$$\int(y\cos x + \sin y + y)dx + \int 0 \cdot dy = C$$

$$\Rightarrow y\sin x + \sin y \int 1 dx + y \int 1 dx = C$$

$$\Rightarrow y\sin x + \sin y x + xy = C$$

5. Solve $(1 + e^{x/y})dx + e^{x/y}(1 - x/y)dy = 0$

$$\Rightarrow (1 + e^{x/y})dx + e^{x/y}(1 - x/y)dy = 0 \quad \text{--- (1)}$$

$$M \cdot dx + N \cdot dy = 0$$

$$M = 1 + e^{x/y}$$

$$N = e^{x/y}(1 - x/y)$$

$$\frac{\partial M}{\partial y} = e^{x/y} - \frac{x}{y^2}$$

$$= -\frac{x}{y^2} e^{x/y}$$

$$\frac{\partial N}{\partial x} = e^{x/y} \left[-\frac{1}{y} \right] + \left[1 - \frac{x}{y} \right] e^{x/y} \cdot \frac{1}{y}$$

$$= -\frac{e^{x/y}}{y} + \frac{e^{x/y}}{y} - \frac{x}{y^2} e^{x/y}$$

$$= -\frac{x}{y^2} e^{x/y}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Eqn (1) is an exact d.e

$$\therefore \int(1 + e^{x/y})dx + \int 0 \cdot dy = C$$

$$\Rightarrow x + \frac{e^{x/y}}{1/y} = C$$

$$\Rightarrow x + y e^{x/y} = C$$

$$6. \text{ Solve } [4x^3y^2 + y\cos(xy)]dx + [2x^4y + x\cos(xy)]dy$$

$$\Rightarrow [4x^3y^2 + y\cos(xy)]dx + [2x^4y + x\cos(xy)]dy = 0 \quad (1)$$

$$M \cdot dx + N \cdot dy = 0$$

$$M = 4x^3y^2 + y\cos(xy)$$

$$\Rightarrow \frac{\partial M}{\partial y} = 8x^3y + y(-\sin xy)x + \cos(xy) \quad (1)$$

$$= 8x^3y + xy(\sin xy) + \cos xy$$

$$= 8x^3y - xy\sin xy + \cos xy$$

$$N = 2x^4y + x\cos(xy)$$

$$\Rightarrow \frac{\partial N}{\partial x} = 8x^3y + x(-\sin xy)y + \cos(xy) \quad (1)$$

$$= 8x^3y - xy\sin xy + \cos xy$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Eqn (1) is an EDE

∴ The solution is,

$$\int [4x^3y^2 + y\cos(xy)]dx + \int 0 dy = C$$

$$\Rightarrow \frac{4x^4y^3}{4} + \frac{y\sin(xy)}{y} = C$$

$$\Rightarrow x^4y^3 + \sin(xy) = C$$

$$7. \text{ Solve } [y(1 + \frac{1}{x}) + \cos y]dx + [x + \log x - x\sin y]dy = 0$$

$$\Rightarrow [y(1 + \frac{1}{x}) + \cos y]dx + [x + \log x - x\sin y]dy = 0$$

$$M dx + N dy = 0$$

$$M = y\left(1 + \frac{1}{x}\right) + \cos y$$

$$N = x + \log x - x \sin y$$

$$M = y + \frac{y}{x} + \cos y$$

$$\frac{\partial N}{\partial x} = 1 + \frac{1}{x} - \sin y$$

$$\frac{\partial M}{\partial y} = 1 + \frac{1}{x} - \sin y$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Eqn (1) is an exact d.e

∴ The soln is,

$$\int \left(y\left(1 + \frac{1}{x}\right) + \cos y\right) dx + \int 0 dy = C$$

$$\Rightarrow y \int 1 \cdot dx + y \int \frac{1}{x} dx + \cos y \int 1 \cdot dx = C$$

$$\Rightarrow xy + y \log x + x \cos y = C$$

8. Solve $(x^2 + y^2 + x)dx + xy dy = 0$

$$\Rightarrow (x^2 + y^2 + x)dx + xy dy = 0$$

$$M dx + N dy$$

$$M = x^2 + y^2 + x$$

$$N = xy$$

$$\frac{\partial M}{\partial y} = 2y$$

$$\frac{\partial N}{\partial x} = y$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Let,

$$\frac{1}{N} \left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right] = \frac{1}{xy} (2y - y)$$

$$= \frac{y}{xy} = \frac{1}{x} = f(x)$$

$$\begin{aligned}\therefore \text{IF} &= e^{\int f(x) dx} \\ &= e^{\int \frac{1}{x} dx} \\ &= e^{\log x} \\ &= x\end{aligned}$$

$$\begin{aligned} \text{If } \Rightarrow \text{Eqn(1)} \times \text{IF} &\Rightarrow x(x^2 + y^2 + x) dx + x \cdot xy dy = 0 \\ &\Rightarrow (x^3 + xy^2 + x^2) dx + x^2 y dy = 0 \quad \text{--- (2)} \\ &\Rightarrow M' dx + N' dy = 0 \end{aligned}$$

$$\begin{aligned}\therefore \frac{\partial M'}{\partial y} &= 2xy, \quad \frac{\partial N'}{\partial x} = 2xy \\ \therefore \frac{\partial M'}{\partial y} &= \frac{\partial N'}{\partial x}\end{aligned}$$

eqn(2) is an EDE

\therefore The solution is,

$$\begin{aligned} \int [x^3 + xy^2 + x^2] dx + \int 0 dy &= C \\ \Rightarrow \int x^3 dx + y^2 \int x dx + \int x^2 dx &= C \\ \Rightarrow \frac{x^4}{4} + y^2 \frac{x^2}{2} + \frac{x^3}{3} &= C \\ \Rightarrow 3x^4 + 6x^2y^2 + 4x^3 &= 12C \end{aligned}$$

$$\begin{aligned} \text{q. } (4xy + 3y^2 - x) dx + x(x+2y) dy &= 0 \\ \Rightarrow (4xy + 3y^2 - x) dx + x(x+2y) dy &= 0 \quad \text{--- (1)} \end{aligned}$$

$$M dx + N dy = 0$$

$$M = 4xy + 3y^2 - x$$

$$N = x(x+2y) = x^2 + 2xy$$

$$\frac{\partial M}{\partial y} = 4x + 6y$$

$$\frac{\partial N}{\partial x} = 2x + 2y$$

Let,

$$\frac{1}{N} \left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right] = \frac{1}{x(x+2y)} [2x+4y]$$
$$= \frac{2(x+2y)}{x(x+2y)}$$
$$= \frac{2}{x} = f(x)$$

$$\text{I.F.} = e^{\int f(x) dx}$$
$$= e^{\int \frac{2}{x} dx}$$
$$= e^{2 \int \frac{1}{x} dx}$$
$$= e^{2 \log_e x}$$
$$= e^{\log_e x^2}$$
$$= x^2$$

$$\textcircled{1} \times \text{IF} \Rightarrow x^2(4xy + 3y^2 - x) dx + x^2 \cdot x(x+2y) dy = 0$$

$$\Rightarrow (4x^3y + 3x^2y^2 - x^3) dx + (x^4 + 2x^3y) dy = 0$$

$$\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}$$

eqn (2) is EDE

∴ The solution is,

$$\int (4x^3y + 3y^2x^2 - x^3) dx - \int 0 dy = C$$

$$\Rightarrow 4y \int x^3 dx + 3y^2 \int x^2 dx - \int x^3 dx = C$$

$$\Rightarrow \frac{4yx^4}{4} + \frac{3y^2x^3}{3} - \frac{x^4}{4} = C$$

$$\Rightarrow yx^4 + y^2x^3 - \frac{x^4}{4} = C$$

$$10. \text{ Solve } (x^3 + y^3 + 6x) dx + (xy)^2 dy = 0$$

$$\Rightarrow (x^3 + y^3 + 6x) dx + (xy)^2 dy = 0 \quad (1)$$

$$M dx + N dy = 0$$

$$M = x^3 + y^3 + 6x$$

$$N = xy^2$$

$$\frac{\partial M}{\partial y} = 3y^2 \quad \frac{\partial N}{\partial x} = y^2$$

$$\text{let, } \frac{1}{N} \left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right] = \frac{1}{xy^2} [3y^2 - y^2]$$

$$= \frac{1}{xy^2} [2y^2]$$

$$= \frac{2}{x} = f(x)$$

$$\text{I.F} = e^{\int f(x) dx}$$

$$= e^{\int \frac{2}{x} dx}$$

$$= e^{2 \int \frac{1}{x} dx}$$

$$= e^{2 \log x}$$

$$= e^{\log x^2}$$

$$= x^2$$

$$(1) \times \text{IF} = x^2 [x^3 + y^3 + 6x] dx + x^2 [xy^2] dy = 0$$

$$= x^5 + x^3 y^3 + 6x^3 dx + x^3 y^2 dy = 0 \quad (2)$$

$$\therefore \frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}$$

eqn (2) is an EDE

The solution is,

$$\int x^5 + x^3 y^3 + 6x^3 dx + \int 0 dy = C$$

$$\Rightarrow \int x^5 dx + y^3 \int x^3 dx + 6 \int x^3 dx = C$$

$$\Rightarrow \frac{x^6}{6} + y^3 \frac{x^3}{3} + 6 \frac{x^4}{4} = C$$

$$\Rightarrow 2x^6 + 4x^3y^3 + 18x^4 = 12C$$

11. Solve $(y \log y)dx + (x - \log y)dy = 0$

$$\Rightarrow (y \log y)dx + (x - \log y)dy = 0 \quad \text{--- (1)}$$

$$Mdx + Ndy = 0$$

$$M = y \log y$$

$$N = x - \log y$$

$$\begin{aligned}\frac{\partial M}{\partial y} &= 1 \cdot \log y + y \cdot \frac{1}{y} \\ &= \log y + 1\end{aligned}$$

$$\frac{\partial N}{\partial x} = 1$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$\text{let, } \frac{1}{M} \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] = \frac{1}{y \log y} \left[1 - (\log y + 1) \right]$$

$$\begin{aligned}&= \frac{1}{y \log y} [x - \log y - 1] \\&= -\frac{\log y}{y \log y} \\&= -\frac{1}{y} = g(y)\end{aligned}$$

$$\begin{aligned}\therefore \text{IF} &= e^{\int g(y)dy} \\&= e^{\int -\frac{1}{y} dy} \\&= e^{-\log y} \\&= e^{\log(\bar{y}^{-1})} \\&= e^{\log(\frac{1}{y})} \\&= \frac{1}{y}\end{aligned}$$

$$① \times I.F \Rightarrow \frac{1}{y} [y \log y] dx + \frac{1}{y} [x - \log y] dy = 0$$

$$\Rightarrow \log y dx + \left[\frac{x}{y} - \frac{\log y}{y} \right] dy = 0$$

$$\Rightarrow \log y dx + \left[\frac{x}{y} - \frac{\log y}{y} \right] dy = 0$$

$$\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}$$

The solution is,

$$\int \log y dx + \int \left[-\frac{\log y}{y} \right] dy = C$$

$$\Rightarrow \log y \int 1 dx + \int \frac{1}{y} \log y dy = C \quad (3)$$

$$\text{Let, } I = \int \frac{1}{y} \log y$$

$$\text{formula: } \int u v dx = u \int v dx - \int (u' \int v dx) dx$$

$$\Rightarrow I = \log y \cdot \int \frac{1}{y} dx - \int \left(\frac{1}{y} \int \frac{1}{y} dy \right) dx$$

$$\Rightarrow I = \log y \cdot \log y - \int \frac{1}{y} \log y dy$$

$$\Rightarrow I = (\log y)^2 - I$$

$$\Rightarrow I + I = (\log y)^2$$

$$\Rightarrow 2I = (\log y)^2$$

$$\Rightarrow I = \frac{1}{2} (\log y)^2$$

from eqn (3)

$$\Rightarrow x \log y - \frac{1}{2} (\log y)^2 = C$$

12. Solve $y(2xy+1)dx - xdy = 0$

$$\Rightarrow y(2xy+1)dx - xdy = 0 \quad \text{--- (1)}$$

$$Mdx + Ndy = 0$$

$$M = y(2xy+1)$$

$$N = -x$$

$$M = 2xy^2 + y$$

$$\frac{\partial N}{\partial x} = -1$$

$$\frac{\partial M}{\partial y} = 4xy + 1$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$\text{let, } \frac{1}{M} \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] = \frac{1}{y(2xy+1)} [-1 - 4xy - 1]$$

$$= \frac{-2 - 4xy}{y(2xy+1)}$$

$$= \frac{-2(1 + 2xy)}{y(1 + 2xy)}$$

$$= -\frac{2}{y} = g(y)$$

$$\text{I.F.} = e^{\int g(y) dy}$$

$$= e^{-2 \int \frac{1}{y} dy}$$

$$= e^{-2 \log y}$$

$$= e^{\log_e(\frac{1}{y^2})}$$

$$= \frac{1}{y^2}$$

$$\textcircled{1} \times \text{I.F.} \Rightarrow \frac{1}{y^2} y(1 + 2xy)dx - \frac{x}{y^2} dy = 0$$

$$\Rightarrow \frac{1}{y} (1 + 2xy)dx - \frac{x}{y^2} dy = 0$$

$$\Rightarrow \left[\frac{1}{y} + ax \right] dx - \frac{x}{y^2} dy = 0$$

$$\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}$$

\therefore The solution is,

$$\Rightarrow \int \left(\frac{1}{y} + ax \right) dx + \int 0 dy = C$$

$$\Rightarrow \frac{1}{y} \int 1 dx + a \int x dx = C$$

$$\Rightarrow \frac{x}{y} + x^2 = C$$

Solution of Linear Differential Equation of First Order and First Degree.

Step 01 :- Write the given differential equation to the standard form.

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Step 02 :- Identify P and Q and find the integrating factor

$$IF = e^{\int P(x) dx}$$

Step 03 :- Write the soln as given below

$$y \times IF = \int Q(x) \times IF dx + C$$

Similarly, for the D.E

$$\frac{dx}{dy} + P(y)x = Q(y)$$

we get the soln

$$x \times IF = \int Q(y) \times IF dy + C$$

$$\text{when, } IF = e^{\int P(y) dy}$$

Bernoulli's Equation

Step 01 :- Define the Bernoulli's equation

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \quad (1)$$

Step 02 :- Divide y^n on b.s and reduce the equation
as,

$$\text{eqn (1)} \Rightarrow \frac{1}{y^n} \frac{dy}{dx} + P(x) \cdot \frac{1}{y^{n-1}} = Q(x) \quad (2)$$

Step 03 :- Assume

$$\frac{1}{y^{n-1}} = u \text{ and differentiate the eqn w.r.t 'x'}$$
$$\Rightarrow \frac{1}{y^n} \frac{dy}{dx} = \frac{1}{1-n} \frac{du}{dx}$$

Step 04 :- Reduce the eqn (2) to the linear form and
follow the same procedure.

Step 05 :- The similar process can be applicable for the
Bernoulli's equation.

$$\frac{dx}{dy} + P(y)x = Q(y)x^n$$

1. Solve $\frac{dy}{dx} - \frac{y}{x} = 2x^2$

$$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = 2x^2$$

$$\Rightarrow \frac{dy}{dx} + \left[-\frac{1}{x}\right]y = 2x^2$$

$$\text{where, } P = -\frac{1}{x} \quad Q = 2x^2$$

$$\begin{aligned}
 \therefore I.F. &= e^{\int P(x) dx} \\
 &= e^{\int -\frac{1}{x} dx} \\
 &= e^{-\log x} \\
 &= e^{\log_e(1/x)} \\
 &= \frac{1}{x}
 \end{aligned}$$

The solution is,

$$\begin{aligned}
 y \times I.F. &= \int Q \times I.F. dx + C \\
 \Rightarrow y \left[\frac{1}{x} \right] &= \int 2x^2 \cdot \frac{1}{x} dx + C \\
 \Rightarrow \frac{y}{x} &= 2 \int x dx + C \\
 \Rightarrow \frac{y}{x} &= 2 \frac{x^2}{2} + C \\
 \Rightarrow \frac{y}{x} &= x^2 + C
 \end{aligned}$$

Q. Solve, $\frac{dy}{dx} + \frac{y}{x} = y^2 x$

$$\begin{aligned}
 \Rightarrow \frac{dy}{dx} + \frac{y}{x} &= y^2 x \quad \text{--- (1)} \\
 \Rightarrow \frac{dy}{dx} + \frac{1}{x}(y) &= y^2 x
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow \frac{1}{y^2} \frac{dy}{dx} + \frac{1}{x} \frac{1}{y} &= \frac{xy^2}{y^2} \\
 \Rightarrow \frac{1}{y^2} \frac{dy}{dx} + \frac{1}{x} \cdot \frac{1}{y} &= x \quad \text{--- (2)}
 \end{aligned}$$

$$\text{Let, } \frac{1}{y^2} = u \quad \text{--- (3)}$$

diff (3) w.r.t to x

$$(3) \Rightarrow \frac{1}{y^2} \frac{dy}{dx} = \frac{du}{dx}$$

$$\Rightarrow \frac{1}{y^2} \frac{dy}{dx} = - \frac{du}{dx}$$

$$(2) \Rightarrow - \frac{du}{dx} + \frac{1}{x} u = x$$

$$\Rightarrow \frac{du}{dx} - \frac{1}{x} u = -x$$

$$\Rightarrow \frac{du}{dx} + \left[-\frac{1}{x} \right] u = -x$$

$$\therefore P = -\frac{1}{x}, Q = -x$$

$$I.F = e^{-\int \frac{1}{x} dx}$$

$$= e^{-\log x}$$

$$= e^{\log(\frac{1}{x})}$$

$$= \frac{1}{x}$$

The solution is,

$$u \times I.F = \int Q(x) \times I.F dx + C$$

$$\Rightarrow u\left(\frac{1}{x}\right) = - \int x \cdot \frac{1}{x} dx + C$$

$$\Rightarrow \frac{u}{x} = - \int 1 dx + C$$

$$\Rightarrow \frac{u}{x} = -x + C$$

$$\Rightarrow \frac{1}{xy} = -x + C$$

$$\Rightarrow \frac{1}{xy} + x = C$$

$$3. \text{ Solve, } \frac{dy}{dx} + xy = xy^3$$

$$\Rightarrow \frac{dy}{dx} + xy = xy^3 \quad \dots (1)$$

$$\Rightarrow \frac{1}{y^3} \frac{dy}{dx} + \frac{xy}{y^3} = \frac{xy^3}{y^3}$$

$$\Rightarrow \frac{1}{y^3} \frac{dy}{dx} + \frac{x}{y^2} = x \quad \dots (2)$$

$$\text{Let, } \frac{1}{y^2} = u \quad \dots (3)$$

diff w.r.t 'x'

$$(3) \Rightarrow -\frac{2}{y^3} \frac{dy}{dx} = \frac{du}{dx}$$

$$\Rightarrow \frac{1}{y^3} \frac{dy}{dx} = -\frac{1}{2} \frac{du}{dx}$$

$$(2) \Rightarrow -\frac{1}{2} \frac{du}{dx} + xu = x \quad (\div(2))$$

$$\Rightarrow \frac{du}{dx} - 2xu = -2x$$

$$\Rightarrow \frac{du}{dx} + (-2x)u = -2x$$

$$P = -2x, Q = -2x$$

$$\begin{aligned} \therefore I.F &= e^{\int P(x) dx} \\ &= e^{-2 \int x dx} \\ &= e^{-2 \frac{x^2}{2}} \\ &= e^{-x^2} \end{aligned}$$

\therefore The solution is,

$$ux \cdot I.F = \int Q(x) \times I.F dx + C$$

$$\Rightarrow ue^{-x^2} = \int (-2x)e^{-x^2} dx + C$$

$$\Rightarrow \frac{e^{-x^2}}{y^2} = \int e^{-x^2} (-2x) dx + C \quad \text{--- (4)}$$

$$\text{Let, } -x^2 = t$$

$$\Rightarrow -2x dx = dt$$

$$\therefore (4) \Rightarrow \frac{e^{-x^2}}{y^2} = \int e^t dt + C$$

$$\Rightarrow \frac{e^{-x^2}}{y^2} = e^t + C$$

$$\Rightarrow \frac{e^{-x^2}}{y^2} = e^{x^2} + C$$

$$\Rightarrow e^{-x^2} = (e^{x^2} + C) y^2$$

4. Solve $x \left[\frac{dy}{dx} \right] + y = x^3 y^6$

$$\Rightarrow x \left[\frac{dy}{dx} \right] + y = x^3 y^6$$

$$\Rightarrow \frac{dy}{dx} + \frac{1}{x} y = x^2 y^6 \quad \text{--- (1)}$$

$$\Rightarrow \frac{1}{y^6} \frac{dy}{dx} + \left[\frac{1}{x} \right] \left[\frac{1}{y^5} \right] = x^2 \quad \text{--- (2)}$$

$$\text{let, } \frac{1}{y^5} = u$$

$$\Rightarrow -\frac{5}{y^6} \cdot \frac{dy}{dx} = \frac{du}{dx}$$

$$\Rightarrow \frac{1}{y^6} \frac{dy}{dx} = -\frac{1}{5} \frac{du}{dx}$$

$$(2) \Rightarrow -\frac{1}{5} \frac{du}{dx} + \frac{1}{x} u = x^2$$

$$\Rightarrow \frac{du}{dx} + \left[-\frac{5}{x} \right] u = -5x^2 \quad \text{--- (3)}$$

$$\therefore P = -\frac{5}{x}, \quad Q = -5x^2$$

$$I.F = e^{\int P(x) dx}$$

$$= e^{-5 \int \frac{1}{x} dx}$$

$$= e^{-5 \log x}$$

$$= e^{\log_e(1/x^5)}$$

$$= \frac{1}{x^5}$$

\therefore The solution is,

$$u \times I.F = \int Q \times I.F dx + C$$

$$\Rightarrow u\left(\frac{1}{x^5}\right) = - \int 5x^2 \cdot \frac{1}{x^5} dx + C$$

$$\Rightarrow \frac{u}{x^5} = -5 \int \frac{1}{x^3} dx + C$$

$$\Rightarrow \frac{u}{x^5} = -5 \frac{x^{3+1}}{-3+1} + C$$

$$\Rightarrow \frac{u}{x^5} = \frac{-5}{-2} x^2 + C$$

$$\Rightarrow \frac{1}{x^5 y^5} = \frac{5}{2} x^2 + C$$

$$5. \text{ Solve, } \frac{dy}{dx} - y \tan x = y^2 \sec x$$

$$\Rightarrow \frac{dy}{dx} - y \tan x = y^2 \sec x$$

$$\Rightarrow \frac{1}{y^2} \frac{dy}{dx} - (\tan x) \left(\frac{1}{y} \right) = \sec x \quad (1)$$

$$\text{let, } \frac{1}{y} = u$$

$$\Rightarrow -\frac{1}{y^2} \frac{dy}{dx} = \frac{du}{dx}$$

$$\Rightarrow \frac{1}{y^2} \frac{dy}{dx} = -\frac{du}{dx}$$

$$(1) \Rightarrow -\frac{du}{dx} - (\tan x) u = \sec x$$

$$\Rightarrow \frac{du}{dx} + \tan x \cdot u = -\sec x \quad (2)$$

$$P = \tan x, Q = -\sec x$$

$$\begin{aligned}\therefore I.F &= e^{\int \tan x dx} \\ &= e^{\log |\sec x|} \\ &= \sec x\end{aligned}$$

∴ The solution is,

$$u \sec x = - \int \sec x \cdot \sec x dx + C$$

$$\Rightarrow \frac{1}{y} \sec x = - \int \sec^2 x dx + C$$

$$\Rightarrow \frac{1}{y} \sec x = - \tan x + C$$

$$\Rightarrow \frac{1}{y} \sec x + \tan x = C$$

$$6. \text{ Solve, } \frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$$

$$\Rightarrow \frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$$

$$\Rightarrow \frac{1}{\cos^2 y} \frac{dy}{dx} + 2x \frac{\sin y \cos y}{\cos^2 y} = x^3$$

$$\Rightarrow \sec^2 y \frac{dy}{dx} + 2x \frac{\sin y}{\cos y} = x^3$$

$$\Rightarrow \sec^2 y \frac{dy}{dx} + 2x \tan y = x^3 \quad (1)$$

$$\text{let, } \tan y = u$$

$$\sec^2 y \cdot \frac{du}{dx} = \frac{du}{dx}$$

$$\Rightarrow \frac{du}{dx} + 2x u = x^3$$

$$\therefore P = 2x, Q = x^3$$

$$\begin{aligned} \therefore I.F. &= e^{\int 2x dx} \\ &= e^{x^2} \end{aligned}$$

The solution is;

$$ue^{x^2} = \int x^3 e^{x^2} dx + C$$

$$\Rightarrow e^{x^2} \tan y = \int x^3 \cdot e^{x^2} \cdot x dx + C \quad (2)$$

$$\text{let, } x^2 = t$$

$$2x dx = dt$$

$$x dx = \frac{1}{2} dt$$

$$(2) \Rightarrow e^{x^2} \tan y = \frac{1}{2} \int t \cdot e^t dt + C$$

$$\Rightarrow e^{x^2} \tan y = \frac{1}{2} e^t (t-1) + c$$

$$\Rightarrow e^{x^2} \tan y = \frac{1}{2} e^{x^2} (x^2 - 1) + c$$

$$\text{Solve } xy(1+xy^2) \frac{dy}{dx} = 1$$

\Rightarrow Given,

$$\Rightarrow xy(1+xy^2) \frac{dy}{dx} = 1$$

$$\Rightarrow (xy + x^3y^3) \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{xy + x^3y^3}$$

$$\Rightarrow \frac{dx}{dy} = xy + x^3y^3$$

$$\Rightarrow \frac{dx}{dy} - xy = x^3y^3$$

$$\text{let, } \frac{1}{x} = u$$

$$\Rightarrow -\frac{1}{x^2} \frac{dx}{dy} = \frac{du}{dy}$$

$$\Rightarrow \frac{1}{x^2} \frac{dx}{dy} = -\frac{du}{dy}$$

$$\therefore (1) \Rightarrow -\frac{du}{dy} - yu = y^3$$

$$\Rightarrow \frac{du}{dy} + yu = -y^3$$

$$\therefore p(y) = y, \quad q(y) = -y^3$$

$$\begin{aligned} I.F &= e^{\int p(y) dy} \\ &= e^{\int y dy} \\ &= e^{y^2/2} \end{aligned}$$

\therefore The solution is,

$$\begin{aligned} ue^{y^2/2} &= - \int y^3 \cdot e^{y^2/2} dy + c \\ \Rightarrow \frac{1}{2} e^{y^2/2} &= - \int y^2 \cdot e^{y^2/2} \cdot y dy + c \quad \text{--- (2)} \end{aligned}$$

$$\text{Let, } \frac{y^2}{2} = t$$

$$\Rightarrow y^2 = 2t$$

$$\Rightarrow 2y dy = 2dt$$

$$\Rightarrow y dy = dt$$

$$\therefore (2) \Rightarrow \frac{1}{2} e^{y^2/2} = - \int y^2 \cdot e^t dt + c$$

$$\Rightarrow \frac{1}{2} e^{y^2/2} = - \frac{y^3}{3} e^t + c$$

Orthogonal Trajectories :-

If the given two curves can intersect perpendicularly then those curves are said to be orthogonally to each other.

Orthogonal Trajectory In Cartesian Form:-

Step 1 :- Let the given curve be $F(x, y, c) = 0$ --- (1)

Step 2 :- Diff eqn (1) w.r.t 'x' and get the differential equation as

$$f\left[x, y, \frac{dy}{dx}\right] = 0 \quad \text{--- (2)}$$

Step 3 :- To get orthogonally trajectory, replace
 $\frac{dy}{dx} = -\frac{dx}{dy}$ in eqn (2), we get

$$(2) \Rightarrow f\left[x, y, -\frac{dx}{dy}\right] = 0 \quad \text{--- (3)}$$

Step 4 :- Solve eqn (3) and get the solution, say
 $G(x, y, c') = 0$, which is called the
orthogonal trajectory of the given curve.

Orthogonal Trajectory In Polar Form :-

Step 1 :- Let the given polar curve be
 $F(r, \theta, c) = 0 \quad \text{--- (1)}$

Step 2 :- Diff eqn (1) w.r.t 'θ' and get the
 $F(r, \theta, \frac{dr}{d\theta}) = 0 \quad \text{--- (2)}$

Step 3 :- To get orthogonally trajectory, replace
 $\frac{dr}{d\theta} = -r^2 \frac{d\theta}{dr}$, we get

$$(2) \Rightarrow f\left(r, \theta, -r^2 \frac{d\theta}{dr}\right) = 0 \quad \text{--- (3)}$$

Step 4 :- Solve eqn (3) and get the solution
 $G(r, \theta, c') = 0$ which is called the
orthogonal trajectory of the polar curves.

Problems :-

1. Find the orthogonally trajectory of the parabola $y^2 = 4ax$, where 'a' is the parameter.

\Rightarrow Given,

$$y^2 = 4ax \quad \text{--- (1)}$$

diff (1) w.r.t to 'x'

$$\therefore (1) \Rightarrow 2y \frac{dy}{dx} = 4a$$

$$\Rightarrow y^2 = 2y \frac{dy}{dx} \cdot x$$

$$\Rightarrow y = 2x \frac{dy}{dx} \quad \text{--- (2)}$$

replace , $\frac{dy}{dx} = -\frac{dx}{dy}$

$$\therefore (2) \Rightarrow y = 2x \left[-\frac{dx}{dy} \right]$$

$$\Rightarrow y = -2x \frac{dx}{dy}$$

$$\Rightarrow y dy = -2x \cdot dx$$

$$\Rightarrow \int y dy = -2 \int x dx$$

$$\Rightarrow \frac{y^2}{2} = -2 \frac{x^2}{2} + C$$

$$\Rightarrow x^2 + \frac{y^2}{2} = C$$

Q. Show that the parabola $y^2 = 4a(x+a)$ is self-orthogonal.

$$\Rightarrow y^2 = 4a(x+a) \quad \text{---(1)}$$

diff (1) w.r.t 'x'

$$(1) \Rightarrow 2yy_1 = 4a(1+0)$$

$$\Rightarrow yy_1 = 2a$$

$$\Rightarrow a = \frac{yy_1}{2}$$

$$\therefore (1) \Rightarrow y^2 = 4 \left[\frac{yy_1}{2} \right] \left[x + \frac{yy_1}{2} \right]$$

$$\Rightarrow y^2 = 4 \frac{yy_1}{2} \left[\frac{2x + yy_1}{2} \right]$$

$$\Rightarrow y^2 = yy_1(2x + yy_1)$$

$$\Rightarrow y = y_1(2x + yy_1) \quad \text{---(2)} \quad \text{Replacing, } y_1 = -\frac{1}{y},$$

$$(2) \Rightarrow y = 2x \left[-\frac{1}{y_1} \right] + y \left[-\frac{1}{y_1} \right]^2$$

$$\Rightarrow y = -\frac{2x}{y_1} + \frac{y}{y_1^2}$$

$$\Rightarrow y = -\frac{2xy_1 + y}{y_1^2}$$

$$\Rightarrow yy_1^2 = -2xy_1 + y$$

$$\Rightarrow y = 2xy_1 + yy_1^2 \quad \text{---(3)}$$

Eqn (2) & (3) are equal

\therefore Eqn is self orthogonally.

3. Find the orthogonal trajectory of $\frac{x^2}{a^2} + \frac{y^2}{a^2+\lambda} = 1$

where ' λ ' is the parameter.

$$\Rightarrow \text{Given : } \frac{x^2}{a^2} + \frac{y^2}{a^2+\lambda} = 1 \quad \text{--- (1)}$$

diff (1) w.r.t to 'x'

$$\therefore (1) \Rightarrow \frac{\partial x}{\partial x} + \frac{\partial y y_1}{\partial x + \lambda} = 0$$

$$\Rightarrow \frac{x}{a^2} + \frac{y y_1}{a^2 + \lambda} = 0$$

$$\Rightarrow \frac{y y_1}{a^2 + \lambda} = -\frac{x}{a^2}$$

$$\Rightarrow \frac{1}{a^2 + \lambda} = \frac{-x}{a^2 y y_1}$$

$$\Rightarrow a^2 + \lambda = \frac{a^2 y y_1}{-x}$$

$$\therefore (1) \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{\frac{-a^2 y y_1}{x}} = 1$$

$$\Rightarrow x^2 - \frac{y}{y_1/x} = a^2$$

$$\Rightarrow x^2 - \frac{xy}{y_1} = a^2 \quad \text{--- (2)}$$

$$\text{let, } y_1 = -\frac{1}{y_1}$$

$$\therefore (2) \Rightarrow x^2 - \frac{xy}{(-1/y_1)} = a^2$$

$$\Rightarrow x^2 - xyy_1 = a^2$$

$$\Rightarrow yy_1 = \frac{a^2 - x^2}{x}$$

$$\Rightarrow y \frac{dy}{dx} = \frac{a^2}{x} - \frac{x^2}{x}$$

$$\Rightarrow y dy = \left[\frac{a^2}{x} - x \right] dx$$

$$\Rightarrow \int y \cdot dy = a^2 \int \frac{1}{x} dx - \int x \cdot dx$$

$$\Rightarrow \frac{y^2}{2} = a^2 \log x - \frac{x^2}{2} + C$$

$$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} = a^2 \log x + C$$

4. Show that $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$ is self orthogonal
where ' λ ' is the parameter.

$$\Rightarrow \text{Given : } \frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1 \quad (1)$$

diff (1) w. r. to 'x'

$$\therefore (1) \Rightarrow \frac{\partial x}{a^2+\lambda} + \frac{\partial y y_1}{b^2+\lambda} = 0$$

$$\Rightarrow \frac{x}{a^2+\lambda} + \frac{y y_1}{b^2+\lambda} = 0$$

$$\Rightarrow (b^2+\lambda)x + (a^2+\lambda)y y_1 = 0$$

$$\Rightarrow b^2x + \lambda x + a^2y y_1 + \lambda y y_1 = 0$$

$$\Rightarrow (x + y y_1)\lambda = -(b^2x + a^2y y_1)$$

$$\Rightarrow \lambda = -\frac{(b^2x + a^2y y_1)}{x + y y_1}$$

$$\begin{aligned}
 \therefore a^2 + \lambda &= a^2 - \frac{(b^2x + a^2yy_1)}{x+yy_1} \\
 &= \frac{a^2(x+yy_1) - (b^2x + a^2yy_1)}{x+yy_1} \\
 &= \frac{a^2x + a^2yy_1 - b^2x - a^2yy_1}{x+yy_1} \\
 &= \frac{(a^2 - b^2)x}{x+yy_1}
 \end{aligned}$$

$$\begin{aligned}
 \therefore b^2 + \lambda &= b^2 - \frac{(b^2x + a^2yy_1)}{x+yy_1} \\
 &= \frac{b^2x + b^2yy_1 - b^2x - a^2yy_1}{x+yy_1} \\
 &= \frac{- (a^2 - b^2)yy_1}{x+yy_1}
 \end{aligned}$$

$$\therefore (1) \frac{x^2}{(a^2 - b^2)x} + \frac{y^2}{-(a^2 - b^2)yy_1} = 1$$

$$\Rightarrow \frac{x(x+yy_1)}{1} - \frac{y(x+yy_1)}{y_1} = a^2 - b^2$$

$$\Rightarrow (x+yy_1) \left(x - \frac{y}{y_1} \right) = a^2 - b^2 \quad \text{--- (2)}$$

$$\text{Let, } y_1 = -\frac{1}{y_1}$$

$$\therefore (2) \Rightarrow \left[x + y \left[-\frac{1}{y_1} \right] \right] \left[x - \frac{y}{(-y_1)} \right] = a^2 - b^2$$

$$\Rightarrow \left[x - \frac{y}{y_1} \right] [x + yy_1] = a^2 - b^2 \quad (3)$$

\therefore Eqn (2) & (3) are equal

\therefore The eqn is self orthogonal.

5. Find the orthogonal trajectory of the given polar curve.

$$i) r = a(1 - \cos\theta)$$

$$\Rightarrow \text{Given: } r = a(1 - \cos\theta) \quad (1)$$

diff. w.r.t. θ

$$\therefore (1) \Rightarrow \frac{dr}{d\theta} = a(0 + \sin\theta)$$

$$\Rightarrow \frac{dr}{d\theta} = a\sin\theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a\sin\theta}{r}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a\sin\theta}{a(1 - \cos\theta)}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \cot(\theta/2) \quad (2)$$

$$\text{but, } \frac{dr}{d\theta} = -r^2 \frac{d\theta}{dr}$$

$$\therefore (2) \Rightarrow \frac{1}{r} \left[-r^2 \frac{d\theta}{dr} \right] = \cot(\theta/2)$$

$$\Rightarrow -r \frac{d\theta}{dr} = \cot\left(\frac{\theta}{2}\right)$$

$$\Rightarrow -\frac{1}{r} dr = -\frac{1}{\cot(\theta/2)} d\theta$$

$$\Rightarrow \int \frac{1}{r} dr = - \int \tan(\theta/2) d\theta$$

$$\Rightarrow \log r = -\frac{\log |\sec(\theta/2)|}{1/2} + \log k$$

$$\Rightarrow \log r = -2 \log |\sec(\theta/2)| + \log k$$

$$\Rightarrow \log r = \log |\cos^2(\theta/2)| + \log k$$

$$\Rightarrow \log r = \log |k \cos^2(\theta/2)|$$

$$\Rightarrow r = k \cos^2\left[\frac{\theta}{2}\right]$$

$$\Rightarrow r = k \left[\frac{1 + \cos\theta}{2} \right]$$

$$\Rightarrow r = \frac{k}{2} (1 + \cos\theta)$$

$$\Rightarrow r = b (1 + \cos\theta)$$

iib $r = a (1 + \cos\theta)$

$$\Rightarrow r = a (1 + \cos\theta) \quad \text{---(1)}$$

diff (1) w.r.t to ' θ '

$$\Rightarrow \frac{dr}{d\theta} = a(0 - \sin\theta)$$

$$\Rightarrow \frac{dr}{d\theta} = -a \sin \theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\frac{a \sin \theta}{r}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\frac{a \sin \theta}{a(1 + \cos \theta)}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\frac{a \sin(\theta/2) \cos(\theta/2)}{a \cos^2(\theta/2)}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\tan(\theta/2) \quad \text{--- (2)}$$

$$\text{but, } \frac{dr}{d\theta} = -r^2 \frac{d\theta}{dr}$$

$$\therefore (2) \Rightarrow \frac{1}{r} \left[-r^2 \frac{d\theta}{dr} \right] = -\tan(\theta/2)$$

$$\Rightarrow -r \frac{d\theta}{dr} = -\tan(\theta/2)$$

$$\Rightarrow -\frac{1}{r} dr = -\tan(\theta/2) d\theta$$

$$\Rightarrow \int -\frac{1}{r} dr = \int -\tan(\theta/2) d\theta$$

$$\Rightarrow \log r = \frac{\log |\sec(\theta/2)|}{1/2} + \log K$$

$$\Rightarrow \log r = 2 \log |\sec(\theta/2)| + \log K$$

$$\Rightarrow \log r = \log \sec^2(\theta/2) + \log K$$

$$\Rightarrow \log r = \log K \sec^2(\theta/2)$$

$$\Rightarrow r = K \sec^2(\theta/2)$$

$$\text{iii} \quad r^n = a^n \sin n\theta$$

$$\Rightarrow r^n = a^n \sin n\theta \quad (1)$$

diff (1) w.r.t to θ

$$\therefore (1) \Rightarrow nr^{n-1} \cdot \frac{dr}{d\theta} = a^n \cos n\theta \cdot n$$

$$\Rightarrow \frac{r^n}{r} \cdot \frac{dr}{d\theta} = a^n \cos n\theta$$

$$\Rightarrow \frac{1}{r} \cdot \frac{dr}{d\theta} = \frac{a^n \cos n\theta}{r^n}$$

$$\Rightarrow \frac{1}{r} \cdot \frac{dr}{d\theta} = \frac{a^n \cos n\theta}{a^n \sin n\theta}$$

$$\Rightarrow \frac{1}{r} \cdot \frac{dr}{d\theta} = \cot n\theta \quad (2)$$

$$\therefore \text{but } \frac{dr}{d\theta} = -r^2 \frac{d\theta}{dr}$$

$$\therefore (2) \Rightarrow \frac{1}{r} \left[-r^2 \frac{d\theta}{dr} \right] = \cot n\theta$$

$$\Rightarrow -r \frac{d\theta}{dr} = \cot n\theta$$

$$\Rightarrow \int \frac{1}{r} dr = - \int \frac{1}{\cot n\theta} d\theta$$

$$\Rightarrow \log r = - \int \tan n\theta d\theta$$

$$\Rightarrow \log r = - \frac{\log |\sec n\theta|}{n} + \log b$$

$$\Rightarrow n \log r = - \log |\sec n\theta| + n \log b$$

$$\Rightarrow \log r^n = \log |\cos n\theta| + \log b^n$$

$$\Rightarrow \log (r^n) = \log (b^n \cos n\theta)$$

$$\Rightarrow r^n = b^n \cos n\theta$$

$$iv) \frac{2a}{r} = 1 - \cos \theta$$

$$\Rightarrow 2a = r(1 - \cos \theta) \quad \text{--- (1)}$$

diff w.r.t θ

$$\Rightarrow 0 = r(a + \sin \theta) + (1 - \cos \theta) \frac{dr}{d\theta}$$

$$\Rightarrow 0 = r \sin \theta + (1 - \cos \theta) \frac{dr}{d\theta}$$

$$\Rightarrow -\frac{r \sin \theta}{1 - \cos \theta} = \frac{dr}{d\theta}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\frac{\sin \theta}{1 - \cos \theta}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\frac{2 \sin(\theta/2) \cos(\theta/2)}{2 \sin^2(\theta/2)}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\cot(\theta/2) \quad \text{--- (2)}$$

$$\text{But, } \frac{dr}{d\theta} = -r^2 \frac{d\theta}{dr}$$

$$\therefore (2) \Rightarrow \frac{1}{r} \left[-r^2 \frac{d\theta}{dr} \right] = -\cot(\theta/2)$$

$$\Rightarrow -r \frac{d\theta}{dr} = -\cot(\theta/2)$$

$$\Rightarrow \int \frac{1}{r} \frac{d\theta}{dr} = \int \frac{1}{\cot(\theta/2)} d\theta$$

$$\Rightarrow \log r = \int \tan(\theta/2) d\theta$$

$$\Rightarrow \log r = 2 \log |\sec(\theta/2)| + \log K$$

$$\Rightarrow \log r = \log \sec^2(\theta/2) + \log K$$

$$\Rightarrow \log r = \log |\sec^2(\theta/2) \cdot K|$$

$$\Rightarrow r = K \sec^2(\theta/2)$$

6.

$$\Rightarrow \text{Given : } r = 2a(\cos\theta + \sin\theta) \quad \text{--- (1)}$$

$$\Rightarrow \frac{dr}{d\theta} = 2a(-\sin\theta + \cos\theta) \quad \text{--- (2)}$$

$$(2) \div (1) \Rightarrow \frac{dr/d\theta}{r} = \frac{2a(\cos\theta - \sin\theta)}{2a(\cos\theta + \sin\theta)}$$

$$\Rightarrow \frac{1}{r} \cdot \frac{dr}{d\theta} = \frac{1 - \tan\theta}{1 + \tan\theta}$$

$$\Rightarrow \frac{1}{r} \cdot \frac{dr}{d\theta} = \tan\left[\frac{\pi}{4} - \theta\right]$$

$$\text{but } \frac{dr}{d\theta} = -r^2 \frac{d\theta}{dr}$$

$$\therefore (2) \Rightarrow \frac{1}{r} \left[-r^2 \cdot \frac{d\theta}{dr} \right] = \tan\left[\frac{\pi}{4} - \theta\right]$$

$$\Rightarrow -r \cdot \frac{d\theta}{dr} = \tan\left[\frac{\pi}{4} - \theta\right]$$

$$\Rightarrow \frac{1}{r} \cdot \frac{d\theta}{dr} = \frac{1}{\tan\left[\frac{\pi}{4} - \theta\right]} \cdot d\theta$$

$$\Rightarrow \int \frac{1}{r} \cdot \frac{d\theta}{dr} = \int \cot\left(\frac{\pi}{4} - \theta\right) \cdot d\theta$$

$$\Rightarrow \log r = -\frac{\log |\sec(\pi/4 - \theta)|}{-1} + \log b$$

$$\Rightarrow \log r = \log |\sin(\pi/4 - \theta)| + \log b$$

$$\Rightarrow \log r = \log |b \sin(\pi/4 - \theta)|$$

$$\Rightarrow r = b \sin(\pi/4 - \theta)$$

$$\Rightarrow r = b \left[\sin \frac{\pi}{4} \cos\theta - \cos \frac{\pi}{4} \sin\theta \right]$$

$$\Rightarrow \tau = \frac{b}{\sqrt{2}} (\cos \theta - \sin \theta)$$

Applications of Differential Equations

Newton's Law of cooling :-

Let 'T' be the temperature of the body and ' T_0 ' be the surrounding medium / room temperature at any time 't', then the Newton's law can state that "The rate of change of body temperature is directly proportional to the difference of the temperature of the body and its surrounding medium."

$$\therefore \frac{dT}{dt} \propto (T - T_0)$$

$$\Rightarrow \frac{dT}{dt} = -k(T - T_0)$$

$$\Rightarrow \frac{1}{T - T_0} dT = -k dt$$

$$\Rightarrow \int \frac{1}{T - T_0} dT = -kt + C$$

$$\Rightarrow \log_e [T - T_0] = -kt + C$$

$$\Rightarrow T - T_0 = e^{-kt + C}$$

$$\Rightarrow T - T_0 = e^{-kt} \cdot e^C$$

$$\Rightarrow T = T_0 + e^C \cdot e^{-kt}$$

$$\Rightarrow \boxed{T = T_0 + \lambda e^{-kt}}$$

1. A body originally 80°C cools down to 60°C in 20 min the temp of the air being 40°C , what will be the temperature of the body after 40 min from the original.

\Rightarrow Given,

$$\text{Air temperature } [T_0] = 40^{\circ}\text{C}$$

$$\text{WKT, } T = T_0 + \lambda e^{-kt} \quad \dots (1)$$

$$\Rightarrow T = 40 + \lambda e^{-kt} \quad \dots (2)$$

Also given,

The body given temp $[T] = 80^{\circ}\text{C}$ at $t=0$

$$\therefore (2) \Rightarrow 80 = 40 + \lambda e^{-kt}$$

$$\Rightarrow 80 - 40 = \lambda$$

$$\Rightarrow \lambda = 40$$

$$\therefore (2) \Rightarrow T = 40 + 40 e^{-kt} \quad \dots (3)$$

Also given, The body temp is reduced to 60°C after 20 min.

$$\therefore T = 60^{\circ}\text{C}$$

$$(3) \Rightarrow 60 = 40 + 40 e^{-20K}$$

$$\Rightarrow 40 e^{-20K} = 20$$

$$\Rightarrow e^{-20K} = \frac{20}{40}$$

$$\Rightarrow e^{-20K} = 0.5$$

$$\Rightarrow -20K = \log_e(0.5)$$

$$\Rightarrow -20K = -0.69314$$

$$\Rightarrow K = \frac{0.69314}{20} = 0.03465$$

$$\therefore (2) \Rightarrow T = 40 + 40 e^{-(0.03465)t}$$

$$\therefore \text{At, } t=40$$

$$\Rightarrow T = 40 + 40 e^{-(0.03465)(40)}$$

$$\Rightarrow T = 40 + 40 e^{-1.386}$$

$$\Rightarrow T = 40 + 40(0.25)$$

$$\Rightarrow T = 40 + 10$$

$$\Rightarrow \boxed{T = 50^\circ\text{C}}$$

2. If the temperature of air is 30°C and a metal ball cools from 100°C to 70°C in 15 min. Find how long will it take for the metal ball to reach the temperature of 40°C ?

\Rightarrow Given,

$$\text{Air temperature } [T_0] = 30^\circ\text{C}$$

$$\text{WKT, } T = T_0 + \lambda e^{-kt} \quad (1)$$

$$\Rightarrow T = 30 + \lambda e^{-kt} \quad (2)$$

Given,

$$\text{Temperature of metal ball } [T] = 100^\circ\text{C at } t=0$$

$$\therefore \text{eqn (2)} : \Rightarrow 100 = 30 + \lambda e^0$$

$$\Rightarrow 100 - 30 = \lambda$$

$$\Rightarrow \lambda = 70$$

$$(2) \Rightarrow T = 30 + \lambda e^{-kt}$$

$$T = 30 + 70 e^{-kt} \quad (3)$$

Also given, the metal ball temp reduced to 70°C after 15 min.

$$(3) \Rightarrow 70 = 30 + 70 e^{-15k}$$

$$\Rightarrow 70 e^{-15K} = 40$$

$$\Rightarrow e^{-15K} = \frac{40}{70}$$

$$\Rightarrow e^{-15K} = 0.5714$$

$$\Rightarrow -15K = \log_e(0.5714)$$

$$\Rightarrow -15K = -0.5596$$

$$\Rightarrow K = \frac{0.5596}{15}$$

$$\Rightarrow K = 0.0373$$

$$\therefore (3) \Rightarrow T = 30 + 70 e^{-(0.0373)t} \quad (4)$$

At, $T = 40^\circ\text{C}$

$$(4) \Rightarrow 40 = 30 + 70 e^{-(0.0373)t}$$

$$\Rightarrow 70 e^{-(0.0373)t} = 10$$

$$\Rightarrow e^{-(0.0373)t} = \frac{10}{70}$$

$$\Rightarrow e^{-(0.0373)t} = 0.1428$$

$$\Rightarrow -(0.0373)t = \log_e(0.1428)$$

$$\Rightarrow -(0.0373)t = -1.9463$$

$$\Rightarrow t = \frac{1.9463}{0.0373}$$

$$\Rightarrow t = 52.18 \text{ min}$$

3. Water at temperature 10°C takes 5 min to warm upto 20°C at room temperature of 40°C . Find the temperature of water after 20 min.

\Rightarrow Given,

$$\text{room temperature } [T_0] = 40^{\circ}\text{C}$$

WKT, The Newton's law of warming.

$$T = T_0 + \lambda e^{kt}$$

$$\Rightarrow T = 40 + \lambda e^{kt} \quad \dots(1)$$

Given, that the water is at 10°C at $t=0$

$$(1) \Rightarrow T = 40 + \lambda e^0$$

$$\Rightarrow 10 = 40 + \lambda$$

$$\Rightarrow \lambda = -30$$

$$\therefore (1) \Rightarrow T = 40 - 30 e^{-kt} \quad \dots(2)$$

Also given, the water was heated to 20°C after

5 min

$$\therefore (2) \Rightarrow 20 = 40 - 30 e^{-5k}$$

$$\Rightarrow -30 e^{-5k} = -20$$

$$\Rightarrow e^{-5k} = \frac{20}{30}$$

$$\Rightarrow e^{-5k} = 0.6666$$

$$\Rightarrow 5k = \log_e(0.6666)$$

$$\Rightarrow 5k = -0.4055$$

$$\Rightarrow k = -0.0811$$

$$(2) \Rightarrow T = 40 - 30 e^{-(0.0811)t} \quad \dots(3)$$

\therefore At $t=20$ min

$$(3) \Rightarrow T = 40 - 30 e^{-(0.0811)(20)}$$

$$\Rightarrow T = 40 - 30 e^{-1.622}$$

$$\Rightarrow T = 40 - 30 (0.1975)$$

$$\Rightarrow T = 40 - 5.9250$$

$$\Rightarrow \boxed{T = 34.07^\circ\text{C}}$$

4. A body is heated at 110°C and placed in the air at 10°C after an hour its temperature becomes 60°C . How much additional time is required to cool 30°C at $t=0$.

\Rightarrow Given,

$$T_0 = 10^\circ\text{C}$$

WKT,

$$T = T_0 + \lambda e^{-kt}$$

$$\Rightarrow T = 10 + \lambda e^{-kt} \quad (1)$$

given the water is cooled to 60°C at $t=1 \text{ hour} = 60 \text{ min}$

$$\Rightarrow 60 = 10 + \lambda e^{-60k}$$

also given :-

$$\therefore T = 110^\circ\text{C} \text{ at } t = 0 \text{ min}$$

$$\therefore (2) \Rightarrow 110 = 10 + \lambda e^0$$

$$\lambda = 110 - 10$$

$$\lambda = 100$$

$$\therefore (2) \Rightarrow T = 10 + 100 e^{-kt} \quad (3)$$

also given, $T = 60^\circ\text{C}$ at $t = 60 \text{ min}$

$$(3) \Rightarrow 60 = 10 + 100 e^{-60k}$$

$$\Rightarrow \frac{50}{100} = e^{-60k}$$

$$\Rightarrow e^{-60K} = 0.5$$

$$\Rightarrow -60K = \log_e(0.5)$$

$$\Rightarrow K = \frac{0.69}{60}$$

$$\Rightarrow K = 0.0115$$

$$\therefore (3) \Rightarrow T = 10 + 100e^{-0.0115t}$$

At , $T = 30$

$$\Rightarrow 30 = 10 + 100e^{-(0.0115)t}$$

$$\Rightarrow 30 - 10 = 100e^{-(0.0115)t}$$

$$\Rightarrow e^{-(0.0115)t} = \frac{20}{100}$$

$$\Rightarrow -(0.0115)t = -0.2$$

$$\Rightarrow t = \frac{0.2}{0.0115}$$

$$\Rightarrow t = 17.39 \text{ min}$$

5. A copper ball at 80°C at cools down to 60°C in 20 min . If the temperature of the room is being 40°C , What will be the temperature of the ball after 40 min from the original.

\Rightarrow Given,

The room temperature, $T_0 = 40^\circ\text{C}$

WKT,

$$T = T_0 + \lambda e^{-kt} \quad (1)$$

$$\Rightarrow T = 40 + \lambda e^{-kt} \quad (2)$$

also given, $T = 80^\circ\text{C}$ at $t = 0$

$$(2) \Rightarrow 80 = 40 \lambda e^{\lambda t}$$

$$\Rightarrow \lambda = 40$$

$$\therefore (2) \Rightarrow T = 40 + 40 e^{-kt} \quad (3)$$

and given, $T = 60^\circ\text{C}$ at $t = 20\text{ min}$

$$\therefore (3) \Rightarrow 60 = 40 + 40 e^{-20k}$$

$$\Rightarrow 40 e^{-20k} = 20$$

$$\Rightarrow e^{-20k} = \frac{20}{40}$$

$$\Rightarrow e^{-20k} = 0.5$$

$$\Rightarrow -20k = \log_e(0.5)$$

$$\Rightarrow -20k = -0.6931$$

$$\Rightarrow k = \frac{0.6931}{20}$$

$$\Rightarrow k = 0.03465$$

$$\therefore (3) \Rightarrow T = 40 + 40 e^{-0.03465t} \quad (4)$$

\therefore At $t = 40\text{ min}$

$$\therefore T = 40 + 40 e^{(-0.03465)(40)}$$

$$\Rightarrow T = 40 + 40 e^{-1.386}$$

$$\Rightarrow T = 40 + 40 (0.250)$$

$$\Rightarrow T = 40 + 10$$

$$\Rightarrow \boxed{T = 50^\circ\text{C}}$$

6. A body is at 25°C , whose from 100°C to 75°C in one min. Find the temperature of the body at 3min.

Given,

$$T_0 = 25^\circ\text{C}$$

WKT,

$$T = T_0 + \lambda e^{-kt} \quad \text{--- (1)}$$

$$\Rightarrow T = 25 + \lambda e^{-kt} \quad \text{--- (2)}$$

$$\text{at } T = 100, t = 0$$

$$\therefore (2) \Rightarrow 100 = 25 + \lambda e^0$$

$$\Rightarrow 75 = \lambda$$

$$(2) \Rightarrow T = 25 + 75 e^{-kt} \quad \text{--- (3)}$$

also given, at $T = 75^\circ$, $t = 1\text{ min}$

$$(3) \Rightarrow 75 = 25 + 75 e^{-k(1)}$$

$$\Rightarrow 75 - 25 = 75 e^{-k}$$

$$\Rightarrow 50 = 75 e^{-k}$$

$$\Rightarrow e^{-k} = \frac{50}{75}$$

$$\Rightarrow e^{-k} = 0.6666$$

$$\Rightarrow -k = \log_e (0.6666)$$

$$\Rightarrow -k = -0.4055$$

$$\Rightarrow k = 0.4055$$

$$\therefore (3) \Rightarrow T = 25 + 75 e^{(-0.4055)t} \quad \text{--- (4)}$$

$$\text{at } t = 3\text{ min}$$

$$\therefore (4) \Rightarrow T = 25 + 75 e^{(-0.4055)(3)}$$

$$= 25 + 75 e^{-1.2165}$$

$$= 25 + 75 (0.2962)$$

$$= 25 + 22.215$$

$$= 47.215$$

Flow of Electricity

Non - Linear differential Equations :-

Solvable for p method

1. Let $y = f(x)$ be a function and let $p = \frac{dy}{dx}$, then the general non-linear differential equation can be defined as

$$A_0 p^n + A_1 p^{n-1} + A_2 p^{n-2} + \dots + A_n = 0 \quad (1)$$

where $A_0, A_1, A_2, \dots, A_n$ are the functions of the variables x & y and also eqn(1) is a polynomial of degree n , it follows as

$$2. \quad (1) \Rightarrow [p - f_1(x, y)] [p - f_2(x, y)] [p - f_3(x, y)] \dots [p - f_n(x, y)] = 0$$

$$\Rightarrow p - f_1(x, y) = 0, \quad p - f_2(x, y) = 0, \quad p - f_3(x, y) = 0 \dots \\ p - f_n(x, y) = 0$$

$$\Rightarrow p = f_1(x, y), \quad p = f_2(x, y), \quad p = f_3(x, y) \dots \quad p = f_n(x, y)$$

3. Solve the above equations and get the solutions

$$f_1(x, y, c_1) = 0, \quad f_2(x, y, c_2) = 0, \quad f_3(x, y, c_3) = 0 \dots \\ f_n(x, y, c_n) = 0$$

∴ The complete solution of given non-linear differential equation can be written as,

$$f_1(x, y, c_1) \cdot f_2(x, y, c_2) \cdot f_3(x, y, c_3) \dots f_n(x, y, c_n) = 0$$

1. Solve $xyp^2 - (x^2 + y^2)p + xy = 0$

$$\Rightarrow \text{Given : } xyp^2 - (x^2 + y^2)p + xy = 0 \quad \dots (1)$$

$$\Rightarrow xyp^2 - x^2p - y^2p + xy = 0$$

$$\Rightarrow xp(yp - x) - y(yp - x) = 0$$

$$\Rightarrow (px - y)(py - x) = 0$$

$$\Rightarrow px - y = 0, \quad py - x = 0$$

$$\Rightarrow px = y, \quad py = x$$

$$\Rightarrow p = \frac{y}{x}, \quad p = \frac{x}{y}$$

case (1) :

$$p = \frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x}$$

$$\Rightarrow \int \frac{1}{y} \cdot dy = \int \frac{1}{x} \cdot dx$$

$$\Rightarrow \log y = \log x + \log C_1$$

$$\Rightarrow \log y = \log (C_1 x)$$

$$\Rightarrow y = C_1 x$$

$$\Rightarrow y - C_1 x = 0$$

case (2) :

$$p = x/y$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{y}$$

$$\Rightarrow \int y \cdot dy = \int x \cdot dx$$

$$\Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + C_2$$

$$\Rightarrow y^2 = x^2 + 2C_2$$

$$\Rightarrow y^2 - x^2 - 2C_2 = 0$$

∴ The complete solution of equation (1) is

$$(y - C_1 x)(y^2 - x^2 - 2C_2) = 0$$

2. Solve $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$

⇒ Given, $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x} \quad \text{---(1)}$

but $\frac{dy}{dx} = P$

$$\therefore (1) \Rightarrow P - \frac{1}{P} = \frac{x^2 - y^2}{xy}$$

$$\Rightarrow \frac{P^2 - 1}{P} = \frac{x^2 - y^2}{xy}$$

$$\Rightarrow (P^2 - 1)xy = (x^2 - y^2)P$$

$$\Rightarrow xyp^2 - (x^2 - y^2)p - xy = 0$$

$$\Rightarrow xyp^2 - x^2p + y^2p - xy = 0$$

$$\Rightarrow xp(yp - x) + y(yp - x) = 0$$

$$\Rightarrow (yp - x)(xp + y) = 0$$

$$\Rightarrow yp - x = 0, \quad xp + y = 0$$

$$\Rightarrow yp = x, \quad xp = -y$$

$$\Rightarrow p = \frac{x}{y}, \quad p = -\frac{y}{x}$$

case (1) :-

$$P = \frac{x}{y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{y}$$

$$\Rightarrow \int y \cdot dy = \int x \cdot dx$$

$$\Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + C_1$$

$$\Rightarrow y^2 = x^2 + 2C_1$$

$$\Rightarrow y^2 - x^2 - 2C_1 = 0$$

case (2) :

$$P = -\frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y}{x}$$

$$\Rightarrow \int \frac{1}{y} dy = - \int \frac{1}{x} dx$$

$$\Rightarrow \log y = -\log x + \log C_2$$

$$\Rightarrow \log y = \log \left[\frac{C_2}{x} \right]$$

$$\Rightarrow y = \frac{C_2}{x}$$

$$\Rightarrow xy = C_2$$

$$\Rightarrow xy - C_2 = 0$$

∴ The solution is $(y^2 - x^2 - C_1)(xy - C_2) = 0$

3. Solve, $P^2 + 2Py \cot x - y^2 = 0$

⇒ Given, $P^2 + 2Py \cot x - y^2 = 0$

$$\therefore a = 1, b = 2y \cot x, c = -y^2$$

$$\therefore p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$p = \frac{-2y \cot x + \sqrt{(2y \cot x)^2 - 4(1)(-y^2)}}{2(1)}$$

$$p = \frac{-2y \cot x \pm \sqrt{(4y^2 \cot x)^2 + 4y^2}}{2}$$

$$p = \frac{-2y \cot x \pm 2y \sqrt{\cot^2 x + 1}}{2}$$

$$p = -y \cot x \pm y \cosec x$$

case (1) :-

$$p = -y \cot x + y \cosec x$$

$$\Rightarrow \frac{dy}{dx} = y [\cosec x - \cot x]$$

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = [-\cot x + \cosec x] dx$$

$$\Rightarrow \frac{1}{y} \cdot dy = \left[\frac{1}{\sin x} - \frac{\cos x}{\sin x} \right] dx$$

$$\Rightarrow \frac{1}{y} \cdot dy = \left[\frac{1 - \cos x}{\sin x} \right] dx$$

$$\Rightarrow \frac{1}{y} \cdot dy = \frac{2 \sin^2 x}{2 \sin(x/2) \cos(x/2)} dx$$

$$\Rightarrow \int \frac{1}{y} \cdot dy = \int \tan(x/2) dx$$

$$\Rightarrow \log y = \frac{\log |\sec(x/2)|}{1/2} + \log C_1$$

$$\Rightarrow \log y = 2 \log |\sec(\frac{x}{2})| + \log C_1$$

$$\Rightarrow \log y = \log |C_1 \sec^2(\frac{x}{2})|$$

$$\Rightarrow y = C_1 \sec^2(\frac{x}{2})$$

$$\Rightarrow y = \frac{C_1}{\cos^2(\frac{x}{2})}$$

$$\Rightarrow y = \frac{C_1}{(1 + \cos x)}$$

$$\Rightarrow y = \frac{2C_1}{1 + \cos x}$$

$$\Rightarrow y(1 + \cos x) = 2C_1$$

$$\Rightarrow y(1 + \cos x) - 2C_1 = 0$$

case (2) :-

$$\frac{1}{y} dy = -(\cot x + \operatorname{cosec} x)$$

$$\Rightarrow \frac{1}{y} dy = -\left[\frac{\cos x}{\sin x} + \frac{1}{\sin x}\right] dx$$

$$\Rightarrow \frac{1}{y} dy = -\frac{(1 + \cos x)}{\sin x} dx$$

$$\Rightarrow \frac{1}{y} dy = -\frac{2\cos^2 x}{2\sin(\frac{x}{2})\cos(\frac{x}{2})} dx$$

$$\Rightarrow \int \frac{1}{y} dy = - \int \cot(\frac{x}{2}) dx$$

$$\Rightarrow \log y = -\frac{\log |\sin(\frac{x}{2})|}{\frac{1}{2}} + \log C_2$$

$$\Rightarrow \log y = -2 \log |\sin(\alpha/2)| + \log C_2$$

$$\Rightarrow \log y = -\log |\sin^2(\alpha/2)| + \log C_2$$

$$\Rightarrow \log y = \log \left[\frac{C_2}{\sin^2(\alpha/2)} \right]$$

$$\Rightarrow y = \frac{C_2}{\sin^2(\alpha/2)}$$

$$\Rightarrow y = \frac{C_2}{\frac{(1-\cos x)}{2}}$$

$$\Rightarrow y = \frac{2C_2}{1-\cos x}$$

$$\Rightarrow y(1-\cos x) = 2C_2$$

$$\Rightarrow y(1-\cos x) - 2C_2 = 0$$

4. Solve $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$

$$\Rightarrow p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$$

$$\Rightarrow p[p^2 + 2xp - y^2p - 2xy^2] = 0$$

$$\Rightarrow p=0, p^2 + 2xp - y^2p - 2xy^2 = 0$$

$$\Rightarrow p=0, p[p+2x] - y^2[p+2x] = 0$$

$$\Rightarrow p=0, (p-y^2)(p+2x) = 0$$

$$\Rightarrow p=0, p-y^2=0, p+2x=0$$

$$\Rightarrow p=0, p=y^2, p=-2x$$

case (1) :- $P=0$

$$\Rightarrow \frac{dy}{dx} = 0$$

$$\Rightarrow \int dy = \int 0 dx$$

$$\Rightarrow y = C_1$$

$$\Rightarrow y - C_1 = 0$$

case (2) :-

$$P=y^2$$

$$\Rightarrow \frac{dy}{dx} = y^2$$

$$\Rightarrow \int \frac{1}{y^2} dy = \int dx$$

$$\Rightarrow -\frac{1}{y} = x + C_2$$

$$\Rightarrow x + \frac{1}{y} + C_2 = 0$$

case (3) :-

$$P = -2x$$

$$\Rightarrow \frac{dy}{dx} = -2x$$

$$\Rightarrow \int dy = -2 \int x dx$$

$$\Rightarrow y = -2 \frac{x^2}{2} + C_3$$

$$\Rightarrow y = -x^2 + C_3$$

$$\Rightarrow y + x^2 - C_3 = 0$$

\therefore The solution is, $(y - C_1)(x + \frac{1}{y} + C_2)(y + x^2 - C_3) = 0$

$$5. \text{ Solve, } y \left[\frac{dy}{dx} \right]^2 + (x-y) \frac{dy}{dx} - x = 0$$

$$\Rightarrow y \left[\frac{dy}{dx} \right]^2 + (x-y) \frac{dy}{dx} - x = 0$$

$$\Rightarrow y p^2 + (x-y)p - x = 0$$

$$\Rightarrow y p^2 - x p - y p - x = 0$$

$$\Rightarrow p(y p + x) - 1(y p + x) = 0$$

$$\Rightarrow (p-1)(y p + x) = 0$$

$$\Rightarrow p-1=0, \quad y p + x = 0$$

$$\Rightarrow p=1, \quad y p = -x$$

$$\Rightarrow p=1, \quad p = -\frac{x}{y}$$

$$\text{Case (1)} :- \quad p=1$$

$$\Rightarrow \frac{dy}{dx} = 1$$

$$\Rightarrow \int dy = \int dx$$

$$\Rightarrow y = x + C_1$$

$$\Rightarrow y - x - C_1 = 0$$

$$\text{Case (2)} :- \quad p = -\frac{x}{y}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

$$\Rightarrow y dy = -x dx$$

$$\Rightarrow \int y dy = - \int x dx$$

$$\Rightarrow \frac{y^2}{2} = - \frac{x^2}{2} + C_2$$

$$\Rightarrow y^2 + x^2 - 2C_2 = 0$$

Claurit's Equation:-

1. Write the given non-linear differential eqn to be in the form of $y = px + f(p)$, it is called the claurit's equation.
2. To get the general solution for claurit's equation, substitute $p=c$ in the claurit's equation, we get

$$y = cx + f(c) \quad (1)$$

3. To get similar solution, diff (1) partially w.r.t to 'c' and hence substitute the 'c' value in the general solution.

1. Solve the claurit's equation $y = px + \frac{a}{p}$

$$\Rightarrow \text{Given: } y = px + \frac{a}{p} \quad (1)$$

\therefore eqn(1) is in claurin's form

The solution is

$$y = cx + \frac{a}{c} \quad (2)$$

diff (2) partially w.r.t to 'c'

$$\therefore (2) \Rightarrow 0 = x - \frac{a}{c^2}$$

$$\Rightarrow \frac{a}{c^2} = x$$

$$\Rightarrow c^2 = \frac{a}{x}$$

$$\Rightarrow c = \sqrt{\frac{a}{x}}$$

$$\therefore (2) \Rightarrow y = \sqrt{\frac{a}{x}} \cdot x + \frac{a}{\sqrt{\frac{a}{x}}}$$

$$\Rightarrow y = \frac{\sqrt{a}}{\sqrt{x}} \sqrt{x} \sqrt{x} + \frac{\sqrt{a} \sqrt{a}}{\sqrt{a}/\sqrt{x}}$$

$$\Rightarrow y = \sqrt{a} \sqrt{x} + \sqrt{a} \sqrt{x}$$

$$\Rightarrow y = 2\sqrt{ax}$$

2. Show that equation $xp^2 + px - py - y + 1 = 0$ is a Clairaut's equation and hence find its general and singular solution.

\Rightarrow Given,

$$xp^2 + px - py + 1 = 0$$

$$\Rightarrow xp^2 + px - py - y = -1$$

$$\Rightarrow xp(p+1) - y(p+1) = -1$$

$$\Rightarrow (xp - y)(p+1) = -1$$

$$\Rightarrow (xp - y) = \frac{-1}{p+1}$$

$$\Rightarrow -y = -px - \frac{1}{p+1}$$

$$\Rightarrow y = px + \left[\frac{1}{p+1} \right] \quad (1)$$

\therefore eqn (1) is in Clairaut's form

\therefore The general solution is

$$y = cx + \frac{1}{c+1} \quad (2)$$

diff (2) w.r to 'c' partially

$$\therefore (2) \Rightarrow 0 = x - \frac{1}{(c+1)^2}$$

$$\Rightarrow \frac{1}{(c+1)^2} = x$$

$$\Rightarrow (c+1)^2 = \frac{1}{x}$$

$$\Rightarrow c+1 = \frac{1}{\sqrt{x}}$$

$$\Rightarrow c = \frac{1}{\sqrt{x}} - 1$$

\therefore The singular solution is

$$y = \left(\frac{1}{\sqrt{x}} - 1\right)x + \frac{1}{1/\sqrt{x}}$$

$$\Rightarrow y = \frac{x}{\sqrt{x}} - x + \sqrt{x}$$

$$\Rightarrow y = \sqrt{x} - x + \sqrt{x}$$

$$\Rightarrow y = 2\sqrt{x} - x$$

3. Show that the equation $xp^3 - yp^2 + 1 = 0$ is the Clairaut's eqn and hence find its solution.

$$\Rightarrow \text{Given, } xp^3 - yp^2 + 1 = 0$$

$$\Rightarrow xp^3 - yp^2 = -1$$

$$\Rightarrow p^2(px - y) = -1$$

$$\Rightarrow px - y = -\frac{1}{p^2}$$

$$\Rightarrow y = px + \frac{1}{p^2} \quad (1)$$

\therefore Eqn(1) is in clairaut's form

\therefore The general solution

$$(1) \Rightarrow y = cx + \frac{1}{c^2} \quad \text{--- (2)}$$

diff (2) w.r to 'c' partially

$$(2) \Rightarrow 0 = x - \frac{1}{c}$$

$$\Rightarrow \frac{1}{c} = x$$

$$\Rightarrow c = \frac{1}{x}$$

\therefore The singular soln is

$$y = \left(\frac{1}{x}\right)x + x^2$$

$$\Rightarrow y = 1 + x^2$$

$$\Rightarrow x^2 - y + 1 = 0$$

4. Reduce the equation $(px-y)(py-x) = 2p$ to the clairaut's form, taking the substitution $x=x^2$ and $y=y^2$ and hence find its solution.

\Rightarrow Given:-

$$(px-y)(py-x) = 2p \quad \text{--- (1)}$$

$$x = x^2 \quad \text{--- (2)}$$

$$y = y^2 \quad \text{--- (3)}$$

$$\therefore \frac{dx}{dx} = 2x, \quad \frac{dy}{dy} = 2y$$

$$\Rightarrow dx = 2x dx, \quad dy = 2y dy$$

$$\Rightarrow \frac{dy}{dx} = \frac{2y}{2x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} \left[\frac{dy}{dx} \right]$$

$$\Rightarrow P = \frac{\sqrt{y}}{\sqrt{x}} b$$

$$\Rightarrow b = \frac{\sqrt{x}}{\sqrt{y}} P$$

$$\therefore (1) \Rightarrow \left[\frac{\sqrt{x}}{\sqrt{y}} P \sqrt{x} - \sqrt{y} \right] \left[\frac{\sqrt{x}}{\sqrt{y}} P \cdot \sqrt{y} + \sqrt{x} \right] = 2 \frac{\sqrt{x}}{\sqrt{y}} P$$

$$\Rightarrow \left[\frac{Px}{\sqrt{y}} - \sqrt{y} \right] \sqrt{x} (P+1) = 2 \frac{\sqrt{x}}{\sqrt{y}} P$$

$$\Rightarrow \left[\frac{Px - y}{\sqrt{y}} \right] \sqrt{x} (P+1) = 2 \frac{\sqrt{x}}{\sqrt{y}} P$$

$$\Rightarrow (Px - y)(P+1) = 2P$$

$$\Rightarrow Px - y = \frac{2P}{P+1}$$

$$\Rightarrow -y = -Px + \frac{2P}{P+1}$$

$$\Rightarrow y = Px + \frac{2P}{P+1}$$

\therefore The solution is,

$$y = cx - \frac{2c}{c+1}$$

$$\Rightarrow y^2 = cx^2 - \frac{2c}{c+1}$$

5. Reduce the equation to the Clairaut's form
 $x^2(y - \beta x) = \beta^2 y$ taking the substitutions $x = x^2$,
 $y = y^2$ and hence find the general and singular
solution?

$$\Rightarrow x^2(y - \beta x) = \beta^2 y \quad \text{--- (1)}$$

$$x = x^2 \quad \text{--- (2)}$$

$$y = y^2 \quad \text{--- (3)}$$

$$\therefore \frac{dx}{dx} = 2x \quad \frac{dy}{dy} = 2y$$

$$\Rightarrow dx = 2x dx, \quad dy = 2y dy$$

$$\therefore \frac{dy}{dx} = \frac{y}{x} \left(\frac{dy}{dx} \right)$$

$$\Rightarrow P = \frac{\sqrt{y}}{\sqrt{x}} \beta$$

$$\Rightarrow \beta = \frac{\sqrt{x}}{\sqrt{y}} P$$

$$(1) \Rightarrow x \left[\sqrt{y} - \frac{\sqrt{x}}{\sqrt{y}} P \cdot \sqrt{x} \right] = \left(\frac{\sqrt{x}}{\sqrt{y}} P \right)^2 \sqrt{y}$$

$$\Rightarrow x \left[\frac{y - Px}{\sqrt{y}} \right] = \frac{x}{y} P^2 \sqrt{y}$$

$$\Rightarrow y - Px = \frac{P^2 x}{y}$$

$$\Rightarrow y - Px = P^2$$

$$\Rightarrow y = Px + P^2$$

\therefore The soln is

$$y = cx + c^2 \quad \text{--- (4)}$$

$$\Rightarrow y^2 = cx^2 + c^2 \quad (5)$$

diff w.r to 'c' partially

$$(5) \Rightarrow 0 = x^2 + 2c$$

$$\Rightarrow 2c = -x^2$$

$$\Rightarrow c = -\frac{x^2}{2}$$

$$(5) \Rightarrow y^2 = \left[-\frac{x^2}{2}\right] [x^2] + \left[-\frac{x^2}{2}\right]^2$$

$$\Rightarrow y^2 = -\frac{x^4}{2} + \frac{x^4}{4}$$

$$\Rightarrow y^2 = -\frac{x^4}{4}$$

$$\Rightarrow x^4 + 4y^2 = 0$$

6. Find the general soln of the equation

$(\beta x - y)(\beta y + x) = \alpha^2 \beta$ by reducing into Clairut's form by taking the substitution $x = x^2$, $y = y^2$

$$\Rightarrow (\beta x - y)(\beta y + x) = \alpha^2 \beta \quad (1)$$

$$x = x^2 \quad (2)$$

$$y = y^2 \quad (3)$$

$$\therefore \frac{dx}{dx} = 2x, \quad \frac{dy}{dy} = 2y$$

$$\Rightarrow dx = 2x dx, \quad dy = 2y dy$$

$$\therefore \frac{dy}{dx} = \frac{y}{x} \left[\frac{dy}{dx} \right]$$

$$\Rightarrow P = \frac{\sqrt{y}}{\sqrt{x}} \beta \quad \Rightarrow \beta = \frac{\sqrt{x}}{\sqrt{y}} P$$

$$(1) \Rightarrow \left[\frac{\sqrt{x}}{\sqrt{y}} P\sqrt{x} - \sqrt{y} \right] \left[\frac{\sqrt{x}}{\sqrt{y}} P\sqrt{y} + \sqrt{x} \right] = a^2 \frac{\sqrt{x}}{\sqrt{y}} P$$

$$\Rightarrow \left[\frac{Px}{\sqrt{y}} - \sqrt{y} \right] \left[\sqrt{x} [P+1] \right] = a^2 \frac{\sqrt{x}}{\sqrt{y}} P$$

$$\Rightarrow \left[\frac{Px - y}{\sqrt{y}} \right] \sqrt{x} (P+1) = a^2 \frac{\sqrt{x}}{\sqrt{y}} P$$

$$\Rightarrow (Px - y)(P+1) = a^2 P$$

$$\Rightarrow Px - y = \frac{a^2 P}{P+1}$$

$$\Rightarrow -y = -Px + \frac{a^2 P}{P+1}$$

$$\Rightarrow y = Px + \frac{a^2 P}{P+1}$$

\therefore The solution is;

$$y = \frac{Cx - a^2 c}{c+1}$$

$$\Rightarrow y^2 = Cx^2 - \frac{a^2 c}{c+1}$$