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LECTURE NOTES

MATHEMATICS-1 FOR MECHNICALENGINEERING STREAM (22MATM11)

MODULE- 2

DIFFERENTIAL CALCULUS- 2

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Series Expansion :-

Let $y = f(x)$ be a real valued function defined in the (a,b) and it can be expressed as a series given below.

1. $$f(b) = f(a) + \frac{(b-a)^1}{1!} f'(a) + \frac{(b-a)^2}{2!} f''(a) + \frac{(b-a)^3}{3!} f'''(a) + \cdots \quad (1)$$

Let, $b - a = h$

$\Rightarrow b = a + h$

$$\Rightarrow f(a+h) = f(a) + \frac{h^1}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \cdots \quad (2)$$

Let, $a + h = x$

$\Rightarrow x = h - a$

$\Rightarrow h = x - a$

$$\therefore \Rightarrow f(x) = f(a) + \frac{x-a}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots \quad (3)$$

$$y(x) = y(a) + \frac{(x-a)^1}{1!} y'(a) + \frac{(x-a)^2}{2!} y''(a) + \frac{(x-a)^3}{3!} y'''(a) + \cdots \quad (4)$$

Therefore, eqn (3) and (4) are called the tailors series expansion of $y = f(x)$ about $x = a$ (or) in the powers of $(x-a)$

2. When, $(a=0)$ the series eqn (3) and (4) can be expressed as,

(3) $\Rightarrow$ $$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

(OR)

(4) $\Rightarrow$ $$y(x) = y(0) + \frac{x}{1!} y'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + \cdots$$

Which is called the "Maclaurin's series of Expansion".

Problems:-

1. Find Tailor's series expansion of $\sin(x)$ in the powers of $(x - \pi/2)$

$\Rightarrow$ Let, $y = \sin x$ and $a = \pi/2$

$\therefore y(a) = y(\pi/2) \Rightarrow \sin\left(\frac{\pi}{2}\right) = 1$

$y'(a) = \cos x \Rightarrow y'(a) = y'(\pi/2) \Rightarrow \cos\left(\frac{\pi}{2}\right) = 0$

$y''(a) = -\sin x \Rightarrow y''(a) = y''(\pi/2) \Rightarrow -\sin(\pi/2) = -1$

$y'''(a) = -\cos x \Rightarrow y'''(a) = y'''(\pi/2) \Rightarrow -\cos\left(\frac{\pi}{2}\right) = 0$

$y^{iv}(a) = \sin x \Rightarrow y^{iv}(a) = y^{iv}(\pi/2) \Rightarrow \sin(\pi/2) = 1$

$\therefore$ WKT,

$$y(x) = y(a) + \frac{(x-a)}{1!} y'(a) + \frac{(x-a)^2}{2!} y''(a) + \frac{(x-a)^3}{3!} y'''(a) + \cdots$$

$$\Rightarrow y(x) = 1 + \frac{(x-\pi/2)}{1!}(0) + \frac{(x-\pi/2)^2}{2!}(-1) + \frac{(x-\pi/2)^3}{3!}(0) + \frac{(x-\pi/2)^4}{4!}(1) - -$$

$$\Rightarrow \sin x = 1 - \frac{(x-\pi/2)^2}{2!} + \frac{(x-\pi/2)^4}{4!} + - - -$$

2. $\log x$

$\Rightarrow \therefore$ Let, $y = \log x$, $a = 1$

$y(a) = y(1) = \log(1) = 0$

$y'(a) = 1/x \Rightarrow y'(a) = y'(1) = 1/1 = 1$

$y''(a) = -\frac{1}{x^2} \Rightarrow y''(a) = y''(1) = -\frac{1}{1^2} = -1$

$y'''(a) = \frac{2}{x^3} \Rightarrow y'''(a) = y'''(1) = \frac{2}{1^3} = 2$

$y^{IV}(a) = -\frac{6}{x^4} \Rightarrow y^{IV}(a) = y^{IV}(1) = -\frac{6}{1^4} = -6$

$\therefore$ WKT,

$$\Rightarrow y(x) = y(a) + \frac{(x-a)}{1!}y'(a) + \frac{(x-a)^2}{2!}y''(a) + \frac{(x-a)^3}{3!}y'''(a) + \frac{(x-a)^4}{4!}y^{IV}(a) + --$$

$$\Rightarrow y(x) = 0 + \frac{(x-a)^1}{1!}(1) + \frac{(x-a)^2}{2!}(-1) + \frac{(x-a)^3}{3!}(2) + \frac{(x-a)^4}{4!}(-6) + ----$$

$$\Rightarrow \log x = \frac{(x-1)^1}{1!} - \frac{(x-1)^2}{2!} + \frac{2(x-1)^3}{3!} - \frac{6(x-1)^4}{4!} + ----$$

$$\Rightarrow \log x = \frac{(x-1)^1}{1!} - \frac{(x-1)^2}{2!} + \frac{2(x-1)^3}{6} - \frac{6(x-1)^4}{24} + ---$$

$$\Rightarrow \log x = \frac{(x-1)^1}{1} - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + ---$$

3. Expand the function e^x using Maclaurin's series.

$\Rightarrow$ Let, $y = e^x$

$\therefore \; y(x) = e^x \Rightarrow y(0) = e^0 = 1$

$y'(x) = e^x \Rightarrow y'(0) = e^0 = 1$

$y''(x) = e^x \Rightarrow y''(0) = e^0 = 1$

$\therefore$ WKT,

$$y(x) = y(0) + \frac{x}{1!} y'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + \frac{x^4}{4!} y^{iv}(0) - \cdots$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

4. Expand the function $\sqrt{1+\sin 2x}$ using Maclurin's series;

$\Rightarrow$ Let, $y = \sqrt{1+\sin 2x}$

$\Rightarrow y = \sqrt{\sin^2 x + \cos^2 x + 2\sin x \cos x}$

$\Rightarrow y = \sqrt{(\sin x + \cos x)^2}$

$\Rightarrow y = \sin x + \cos x$

$\therefore \; y(0) = \sin 0 + \cos 0 \Rightarrow 0 + 1 = 1$

$y'(x) = \cos x - \sin x \Rightarrow y'(0) = \cos(0) - \sin(0) = 1 - 0 = 1$

$y''(x) = -\sin x - \cos x \Rightarrow y''(0) = -\sin(0) - \cos(0) = 0 - 1 = -1$

$y'''(x) = -\cos x + \sin x \Rightarrow y'''(0) = -\cos(0) + \sin(0) = -1 + 0 = -1$

WKT,

$$y(x) = y(0) + \frac{x}{1!} y'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + \frac{x^4}{4!} y^{iv}(0) + \cdots$$

$$\Rightarrow \sqrt{1+\sin 2x} = 1 + \frac{x}{1!}(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(-1) + \cdots$$

$$\Rightarrow \sqrt{1+\sin 2x} = 1+\frac{x}{1!}-\frac{x^2}{2!}-\frac{x^3}{3!}- \cdots$$

5. Expand the function $y = \log(1+\sin x)$ using Maclurin's series upto the term containing x^4.

$\Rightarrow$ Let, $y = \log(1+\sin x)$

$$\Rightarrow \therefore y(x) = \log(1+\sin x) = y(0) = \log(1+\sin 0) = \log 0 = 1$$

$$\Rightarrow y'(x) = \frac{\cos x}{1+\sin x} \Rightarrow y'(0) = \frac{\cos 0}{1+\sin 0} = \frac{1}{1+0} = 1$$

$$\Rightarrow y''(x) = \frac{(1+\sin x)(-\sin x) - \cos x(\cos x)}{(1+\sin x)^2}$$

$$= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2}$$

$$= \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1+\sin x)^2}$$

$$= \frac{-\sin x - 1}{(1+\sin x)^2}$$

$$= \frac{-(1+\sin x)}{(1+\sin x)^2}$$

$$= -\frac{1}{(1+\sin x)}$$

$$= \frac{-1}{1+\sin(0)} = \frac{-1}{1+0} = -1$$

$$\Rightarrow y'''(x) = -\left[\frac{1+\sin x(0) - 1(\cos x)}{(1+\sin x)^2}\right]$$

$$y'''(x) = \frac{\cos x}{(1+\sin x)^2} \Rightarrow y'''(0) = \frac{\cos 0}{(1+\sin 0)^2} = \frac{1}{(1+0)^2} = 1$$

WKT,

$$y(x) = y(0) + \frac{x}{1!} y'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + ----$$

$$\Rightarrow \log(1+\sin x) = 1 + \frac{x}{1!}(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(1) + \frac{x^4}{4!}(-1) + ---$$

$$\Rightarrow \log(1+\sin x) = 1 + \frac{x}{1!} - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + -----$$

6. Expand $y = \log(1+\cos x)$ using maclurin's expansion.

$\Rightarrow$ Let, $y = \log(1+\cos x)$

$\therefore\ y(x) = \log(1+\cos x) \Rightarrow y(0) = \log(1+\cos 0) = \log(1+1) = \log 2.$

$$\Rightarrow y'(x) = \frac{-\sin x}{1+\cos x} = \frac{-\sin 0}{1+\cos 0} = \frac{-0}{1+1} = 0$$

$$\Rightarrow y''(x) = \frac{(1+\cos x)(-\cos x) - \sin x(-\sin x)}{(1+\cos x)^2}$$

$$= \frac{-\cos x - \cos^2 x + \sin^2 x}{(1+\cos x)^2}$$

$$= \frac{-\cos x - (\sin^2 x + \cos^2 x)}{(1+\cos x)^2}$$

$$= \frac{-\cos x - 1}{(1+\cos x)^2}$$

$$= \frac{-\cancel{(1+\cos x)}}{(1+\cos x)^{\cancel{2}}}$$

$$= \frac{-1}{1+\cos x} = \frac{-1}{1+\cos 0} = \frac{-1}{1+1} = -\frac{1}{2}$$

$$\Rightarrow y'''(x) = -\left[\frac{(1+\cos x)(0) - 1(-\sin x)}{(1+\cos x)^2}\right]$$

$$= \frac{\sin x}{(1+\cos x)^2}$$

$$= \frac{\sin 0}{(1+\cos 0)^2} = \frac{0}{(1+1)^2} = \frac{0}{4} = 0$$

$\therefore$ WKT,

$$\Rightarrow y(x) = y(0) + \frac{x}{1!} y'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + \text{----}$$

$$\Rightarrow \log(1+\cos x) = \log 2 + \frac{x}{1!}(0) + \frac{x}{2!}\left(-\frac{1}{2}\right) + \frac{x}{3!}(0) + \text{---}$$

$$\Rightarrow \log(1+\cos x) = \log 2 - \frac{x}{4} \text{----}$$

7. Expand $y = \log(\sec x + \tan x)$ using maclurin's series;

$\Rightarrow$ let, $y = \log(\sec x + \tan x)$

$\Rightarrow$ $y = \log(\sec x + \tan x) = \log(\sec 0 + \tan 0) = \log 1 = 0$

$$\Rightarrow y'(x) = \frac{1}{\sec x + \tan x}[\sec x . \tan x + \sec^2 x]$$

$$= \frac{\sec x . \tan x + \sec^2 x}{\sec x + \tan x}$$

$$= \frac{\sec[\cancel{\sec x + \tan x}]}{\cancel{\sec x + \tan x}}$$

$$= \sec x$$

$$= \sec 0 = 1$$

$$\Rightarrow y''(x) = \sec x . \tan x$$

$$= \sec 0 . \tan 0 = 1+0 = 0$$

$\Rightarrow y'''(x) = \sec x \cdot \tan x \cdot \tan x + \sec x \cdot \sec^2 x$

$= \sec x \cdot \tan^2 x + \sec^3 x$

$= \sec x\,(\sec^2 x - 1) + \sec^3 x$

$= \sec^3 x - \sec x + \sec^3 x$

$= 2\sec^3 x - \sec x$

$= 2\sec^3(0) - \sec(0) = 2 - 1 = 1$

WKT,

$$\Rightarrow y(x) = y(0) + \frac{x}{1!}y'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \cdots$$

$$\Rightarrow \log(\sec x + \tan x) = 0 + \frac{x}{1!}(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(1) + \frac{x^4}{4!}(0) + \cdots$$

$$\Rightarrow \log(\sec x + \tan x) = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

8. Expand $y = \log(\sec x)$ using maclurin's expansion upto the term containing x^4.

$\Rightarrow$ let, $y = \log(\sec x)$

$f(y) = \log(\sec x) = \log(\sec 0) = \log 1 = 0$

$$\Rightarrow y'(x) = \frac{1}{\sec x}[\sec x \cdot \tan x]$$

$$= \frac{\cancel{\sec x} \cdot \tan x}{\cancel{\sec x}} = \underset{[y']}{\tan x} = \tan 0 = \underset{[y_0']}{0}$$

$\Rightarrow y''(x) = \sec^2 x$

$= 1 + \tan^2 x$

$= 1 + (y')^2$

$= 1 + {y_0'}^2 = 1 + 0 = 1$

$\Rightarrow \quad y'''(x) = 0 + 2y'.y'' = 2y'.y''$

$y'''(0) = 2y_0'.y_0'' = 2(0)(1) = 0$

$\Rightarrow \quad y^{iv}(x) = 2[y'y''' + y''.y'']$

$= 2[y'y''' + (y'')^2]$

$= 2[y_0' y_0''' + (y_0'')^2]$

$y^{iv}(0) = 2[0 + (1)^2] = 2$

WKT,

$\Rightarrow \quad y(x) = y(0) + \frac{x}{1!} y'(0) + \frac{x^2}{2!} y''(0) + \cdots$

$\Rightarrow \log(\sec x) = 0 + \frac{x}{1!}(0) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(2) + \cdots$

$\Rightarrow \log(\sec x) = \frac{x^2}{2!} + \frac{2x^4}{4!} + \cdots$

9. $y = \tan x$

$\Rightarrow$ Let, $y = \tan x$

$y = \tan x \Rightarrow f(0) = \tan(0) = 0$

$\Rightarrow \quad y'(x) = \sec^2 x. \Rightarrow \cancel{\sec^2(0)} =$

$= 1 + \tan^2 x$

$= 1 + y^2$

$y'(0) = 1 + y_0^2 = 1 + 0 = 1$

$\Rightarrow \quad y''(x) = 0 + 2y.y'$

$= 2y_0 y_0' = 2(0)(1) = 0$

$\Rightarrow \quad y'''(x) = 2[y.y'' + (y')^2]$

$= 2[y_0 y_0'' + (y_0')^2]$

$= 2[0 - 1^2] = 2$

WKT,

$$\Rightarrow y(x) = y(0) + \frac{x}{1!} y'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + \cdots$$

$$\Rightarrow \tan x = 0 + \frac{x}{1!}(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(2) + \cdots$$

$$\Rightarrow \tan x = \frac{x}{1!} + \frac{2x^3}{3!} + \cdots$$

INDETERMINATE FORMS:-

An indeterminate form involves two functions whose limit cannot be determined and which are of the form.

$$\frac{0}{0}, \frac{\infty}{\infty}, \infty + \infty, \infty - \infty, 0 \times \infty, 0^0, 1^\infty, \infty^0$$

To evalvate these type of expressions we can use "L-Hospital Rule" which is a general method of evaluating indeterminate forms such as $\frac{0}{0}$ (or) $\frac{\infty}{\infty}$.

To evalvate the limits of indeterminate forms for the derivates in calculus, LH rule can be applied more than once.

Let $f(x)$ & $g(x)$ be the 2 functions, then the indeterm-inate form of these will be in the form of:-

$$\Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0} \quad \text{——— ①}$$

As per LH rule differentiate both $f(x)$ and $g(x)$ separately and apply the limit value until to get the determinate value.

$$\therefore L = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{0}{0'} \quad (\text{if})$$

① $\Rightarrow$ $L = \lim_{x \to a} \dfrac{f''(x)}{g''(x)} = \dfrac{0}{0}$ (if) ——— ②

② $\Rightarrow$ $L = \lim_{x \to a} \dfrac{f'''(x)}{g'''(x)} = K$

Examples:-

1. Evaluate the following indeterminate forms:-

i) $\lim_{x \to 0} \dfrac{\sin x}{x}$

$\Rightarrow$ $L = \lim_{x \to 0} \dfrac{\sin x}{x} = \dfrac{0}{0}$ ——— ①

Apply LH rule

① $\Rightarrow$ $L = \lim_{x \to 0} \dfrac{\cos x}{1}$

$L = \dfrac{\cos 0}{1} = \dfrac{1}{1} = 1$

ii) $\lim_{x \to 0} \dfrac{\tan x}{x}$

$\Rightarrow$ $L = \lim_{x \to 0} \dfrac{\tan x}{x} = \dfrac{0}{0}$ ——— ①

Apply LH rule

① $\Rightarrow$ $L = \lim_{x \to 0} \dfrac{\sec^2 x}{1}$

$L = \dfrac{\sec^2(0)}{1} = \dfrac{1^2}{1} = 1$

iii) $\lim_{x \to 0} \dfrac{a^x - b^x}{x}$

$\Rightarrow$ $L = \lim_{x \to 0} \dfrac{a^x - b^x}{x} = \dfrac{1-1}{0} = \dfrac{0}{0}$ ——— ①

apply LH rule

$$\Rightarrow L = \lim_{x \to 0} \frac{a^x \log a - b^x \log b}{1}$$

$$= \frac{a^0 \log a - b^0 \log b}{1}$$

$$= \log a - \log b$$

$$\Rightarrow L = \log\left[\frac{a}{b}\right]$$

Problems:-

Evaluate the following indeterminate forms:-

1. $\lim\limits_{x \to 0} \left[\frac{a^x + b^x}{2}\right]^{1/x}$

$$\Rightarrow \text{let, } L = \lim_{x \to 0} \left[\frac{a^x + b^x}{2}\right]^{1/x} = 1^\infty$$

Taking log on b.s

$$① \Rightarrow \log L = \lim_{x \to 0} \log\left[\frac{a^x + b^x}{2}\right]^{1/x}$$

$$\Rightarrow \log L = \lim_{x \to 0} \frac{1}{x} \log\left[\frac{a^x + b^x}{2}\right]$$

$$\Rightarrow \log L = \lim_{x \to 0} \frac{1}{x} \left[\log(a^x + b^x) - \log 2\right]$$

$$\Rightarrow \log L = \lim_{x \to 0} \frac{\log(a^x + b^x) - \log 2}{x} = \frac{0}{0} \quad \text{—②}$$

apply LH rule

$$② \Rightarrow \log L = \lim_{x \to 0} \frac{\frac{1}{a^x + b^x}\left[a^x \log a - b^x \log b\right] - 0}{1}$$

$$\Rightarrow \log l = \lim_{x\to 0} \frac{a^x \log a - b^x \log b}{a^x + b^x}$$

$$\Rightarrow \log L = \frac{\log a - \log b}{2}$$

$$\Rightarrow \log L = \frac{1}{2} \log (ab)$$

$$\Rightarrow \log L = \log (ab)^{1/2}$$

$$\Rightarrow L = (ab)^{1/2}$$

$$\Rightarrow L = \sqrt{ab}$$

2. $\lim_{x\to 0} \left[\frac{a^x + b^x + c^x}{3}\right]^{1/x}$

$\Rightarrow$ Let, $L = \lim_{x\to 0} \left[\frac{a^x + b^x + c^x}{3}\right]^{1/x}$

Taking log on b.s

$$\Rightarrow \log L = \lim_{x\to 0} \log \left[\frac{a^x + b^x + c^x}{3}\right]^{1/x}$$

$$\Rightarrow \log L = \lim_{x\to 0} \frac{1}{x} \log \left[\frac{a^x + b^x + c^x}{3}\right]$$

$$\Rightarrow \log L = \lim_{x\to 0} \frac{1}{x} \log [a^x + b^x + c^x] - \log 3$$

apply LH rule

$$\Rightarrow \log L = \lim_{x\to 0} \frac{\frac{1}{a^x + b^x + c^x} [a^x \log a + b^x \log b + c^x \log c]}{x} = \frac{0}{0}$$

~~apply LH rule.~~

② $\Rightarrow \log l = \lim_{x\to 0} \frac{a^x \log a + b^x \log b + c^x \log c}{a^x + b^x + c^x}$

$$\Rightarrow \log L = \frac{\log a + \log b + \log c}{3}$$

$$\Rightarrow \log L = \frac{1}{3}\log(a.b.c)$$

$$\Rightarrow \log L = \log(abc)^{1/3}$$

$$\Rightarrow L = (abc)^{1/3}$$

3. $$\lim_{x\to 0}\left[\frac{a^x+b^x+c^x+d^x}{4}\right]^{1/x}$$

$$\Rightarrow \text{Let, } L = \lim_{x\to 0}\left[\frac{a^x+b^x+c^x+d^x}{4}\right]^{1/x}$$

taking log on b.s

$$\Rightarrow \log L = \lim_{x\to 0}\log\left[\frac{a^x+b^x+c^x+d^x}{4}\right]^{1/x}$$

$$\Rightarrow \log L = \lim_{x\to 0}\frac{1}{x}\log\left[\frac{a^x+b^x+c^x+d^x}{4}\right]$$

$$\Rightarrow \log L = \lim_{x\to 0}\frac{1}{x}\log\left[a^x+b^x+c^x+d^x\right] - \log 4$$

$$\Rightarrow \log L = \lim_{x\to 0}\frac{\log\left[a^x+b^x+c^x+d^x\right]-\log 4}{x}$$

apply LH - rule

$$\Rightarrow \log L = \lim_{x\to 0}\frac{\frac{1}{a^x+b^x+c^x+d^x}\left[a^x\log a+b^x\log b+c^x\log c+d^x\log d\right]-0}{1}$$

$$\Rightarrow \log L = \lim_{x\to 0}\frac{a^x\log a+b^x\log b+c^x\log c+d^x\log d}{a^x+b^x+c^x+d^x}$$

$$\Rightarrow \log L = \frac{\log a+\log b+\log c+\log d}{4}$$

$$\Rightarrow \log l = \frac{1}{4}\log(abcd)$$

$$\Rightarrow \log l = \log(abcd)^{1/4}$$

$$\Rightarrow L = (abcd)^{1/4}$$

4. $\lim_{x \to 0} (\cos x)^{1/x^2}$

$\Rightarrow$ Let, $l = \lim_{x \to 0} (\cos x)^{1/x^2} = 1^{\infty}$ —— ①

Taking log on b.s

$\Rightarrow \log l = \lim_{x \to 0} \log (\cos x)^{1/x^2}$

$\Rightarrow \log l = \lim_{x \to 0} \frac{1}{x^2} \log (\cos x)$

$\Rightarrow \log l = \lim_{x \to 0} \frac{\log(\cos x)}{x^2} = \frac{0}{0}$ —— ②

apply LH-rule

$\Rightarrow \log l = \lim_{x \to 0} \frac{\frac{1}{\cos x}(-\sin x)}{2x}$

$\Rightarrow \log l = \lim_{x \to 0} \frac{-\frac{\sin x}{\cos x}}{2x}$

$\Rightarrow \log l = \lim_{x \to 0} -\frac{\tan x}{2x}$

$\Rightarrow \log l = -\frac{1}{2} \lim_{x \to 0} \frac{\tan x}{x} = \frac{0}{0}$ —— ③

apply LH-rule

$\Rightarrow \log l = -\frac{1}{2} \lim_{x \to 0} \frac{\sec^2 x}{1}$

$\Rightarrow \log l = -\frac{1}{2} \left[\frac{1}{1}\right]$

$\Rightarrow \log_e l = -\frac{1}{2}$

$\Rightarrow l = e^{-1/2}$

$\Rightarrow l = \frac{1}{e^{1/2}}$

$\Rightarrow l = \frac{1}{\sqrt{e}}$

Evaluate :-

1. $\lim_{x\to 0} \left[\frac{1}{x}\right]^{2\sin x}$

$\Rightarrow$ let, $L = \lim_{x\to 0} \left[\frac{1}{x}\right]^{2\sin x}$

apply log on b.s

$\Rightarrow \log L = \lim_{x\to 0} \log \left(\frac{1}{x}\right)^{2\sin x}$

$\Rightarrow \log L = \lim_{x\to 0} 2\sin x \log\left[\frac{1}{x}\right]$

$\Rightarrow \log L = 2 \lim_{x\to 0} \frac{\log(1/x)}{1/\sin x}$

$\Rightarrow \log L = 2 \lim_{x\to 0} \frac{-\log x}{\operatorname{cosec} x}$

$\Rightarrow \log L = -2 \lim_{x\to 0} \frac{\log x}{\operatorname{cosec} x} = \frac{\infty}{\infty}$ —— 2

apply LH - rule

$\Rightarrow \log L = -2 \lim_{x\to 0} \frac{1/x}{-\cot x \cdot \operatorname{cosec} x}$

$\Rightarrow \log L = 2 \lim_{x\to 0} \frac{1}{\frac{x\cos x}{\sin x} \cdot \frac{1}{\sin x}}$

$\Rightarrow \log L = 2 \lim_{x\to 0} \frac{\sin x}{\cos x \left[\frac{x}{\sin x}\right]}$

$\Rightarrow \log L = 2 \lim_{x\to 0} \frac{\sin x}{\cos x (1)}$

$\Rightarrow \log L \Rightarrow 2 \lim_{x\to 0} \tan x$

$\Rightarrow \log L = 2(0) = 0$

$\Rightarrow \log_e L = 0 \quad \Rightarrow L = e^0 \quad \Rightarrow L = 1$

2. $\lim_{x\to 0} \left[\frac{\tan x}{x}\right]^{1/x}$

$\Rightarrow$ let, $l = \lim_{x\to 0} \left[\frac{\tan x}{x}\right]^{1/x} = \infty^{\infty}$ —— ①

log on b.s

$\Rightarrow \log L = \lim_{x\to 0} \log \left[\frac{\tan x}{x}\right]^{1/x}$

$\Rightarrow \log L = \lim_{x\to 0} \frac{1}{x} \log \left[\frac{\tan x}{x}\right]$

$\Rightarrow \log L = \lim_{x\to 0} \frac{\log(\tan x) - \log(x)}{x} = \frac{0}{0}$ —— ②

Apply LH rule

$\Rightarrow \log L = \lim_{x\to 0} \frac{\frac{1}{\tan x}(\sec^2 x) - \frac{1}{x}}{1}$

$\Rightarrow \log L = \lim_{x\to 0} \frac{\sec^2 x}{\tan x} - \frac{1}{x}$

$\Rightarrow \log L = \lim_{x\to 0} \frac{1/\cos^2 x}{\sin x/\cos x} - \frac{1}{x}$

$\Rightarrow \log L = \lim_{x\to 0} \frac{1}{\sin x.\cos x} - \frac{1}{x}$

$\Rightarrow \log L = \lim_{x\to 0} \left[\frac{2}{2\sin x.\cos x} - \frac{1}{x}\right]$

$\Rightarrow \log L = \lim_{x\to 0} \left[\frac{2}{\sin 2x} - \frac{1}{x}\right]$

$\Rightarrow \log L = \lim_{x\to 0} \left[\frac{2x - \sin 2x}{\sin 2x}\right]$

$\Rightarrow \log L = \lim_{x\to 0} \left[\frac{2x - \sin 2x}{2x^2 \cancel{\left[\frac{\sin 2x}{2x}\right]}^{1}}\right]$

$$\Rightarrow \log l = \frac{1}{2} \lim_{x\to 0} \frac{2x - \sin 2x}{x^2} = \frac{0}{0} \text{ --- } (3)$$

$$\Rightarrow \log l = \frac{1}{2} \lim_{x\to 0} \frac{2 - 2\cos 2x}{2x}$$

$$\Rightarrow \log l = \frac{1}{2} \lim_{x\to 0} \frac{1 - \cos 2x}{x} = \frac{0}{0}$$

$$\Rightarrow \log l = \frac{1}{2} \lim_{x\to 0} \frac{0 - (-2)\sin 2x}{1}$$

$$\Rightarrow \log l = \frac{1}{2} \lim_{x\to 0} 2\sin 2x$$

$$\Rightarrow \log l = \underset{x\to 0}{Lt} \sin 2x$$

$$\Rightarrow \log_e l = 0$$

$$\Rightarrow l = e^0$$

$$\Rightarrow l = 1$$

3. $\underset{x\to 0}{Lt} \left[\frac{\sin x}{x}\right]^{1/x^2}$

$\Rightarrow$ let, $L = \underset{x\to 0}{Lt} \left[\frac{\sin x}{x}\right]^{1/x^2} = \infty$

log on b.s

$$\Rightarrow \log l = \underset{x\to 0}{Lt} \log \left[\frac{\sin x}{x}\right]^{1/x^2}$$

$$\Rightarrow \log l = \underset{x\to 0}{Lt} \frac{1}{x^2} \log \left[\frac{\sin x}{x}\right]$$

$$\Rightarrow \log l = \underset{x\to 0}{Lt} \frac{\log(\sin x) - \log(x)}{x^2} = \frac{0}{0}$$

apply LH rule

$$\Rightarrow \log l = \underset{x\to 0}{Lt} \frac{\frac{1}{\sin x}(\cos x) - \frac{1}{x}}{2x}$$

$$\Rightarrow \log l = \underset{x\to 0}{\text{lt}} \frac{\frac{\cos x}{\sin x} - \frac{1}{x}}{2x}$$

$$\Rightarrow \log l = \underset{x\to 0}{\text{lt}} \frac{\frac{x\cos x - \sin x}{x\sin x}}{2x}$$

$$\Rightarrow \log l = \frac{1}{2} \lim_{x\to 0} \frac{x\cos x - \sin x}{x^2 \sin x}$$

$$\Rightarrow \log l = \frac{1}{2} \lim_{x\to 0} \frac{x\cos x - \sin x}{x^3 \left[\frac{\sin x}{x}\right]}$$

$$\Rightarrow \log l = \frac{1}{2} \lim_{x\to 0} \frac{x\cos x - \sin x}{x^3} \text{ ——— (2)} = \frac{0}{0}$$

apply LH- rule

$$\Rightarrow \log l = \frac{1}{2} \lim_{x\to 0} \frac{\cos x - x\sin x - \cos x}{3x^2}$$

$$\Rightarrow \log l = \frac{1}{6} \lim_{x\to 0} \frac{-x\sin x}{x^2}$$

$$\Rightarrow \log l = -\frac{1}{6} \lim_{x\to 0} \frac{\sin x}{x}$$

$$\Rightarrow \log l = -\frac{1}{6}(1)$$

$$\Rightarrow \log_e l = -1/6$$

$$\Rightarrow l = e^{-1/6}$$

Partial Differentiation

Let, $z = f(x,y)$ be a real number function defined on any other interval, then the first and second order partial derivates can be denoted as :-

→ First order Partial Differentiation :-

1. $$\frac{\partial z}{\partial x} = \frac{\partial t}{\partial x} = z_x = f_x = p$$

$$\frac{\partial z}{\partial y} = \frac{\partial t}{\partial y} = z_y = f_y = q$$

→ Second Order Partial Differentiation :-

2. $$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 t}{\partial x^2} = z_{xx} = f_{xx} = r$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 t}{\partial y^2} = z_{yy} = f_{yy} = t$$

$$\frac{\partial}{\partial y}\left[\frac{\partial z}{\partial x}\right] = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 t}{\partial y \partial x} = z_{yx} = f_{yx} = s$$

$$\frac{\partial}{\partial x}\left[\frac{\partial z}{\partial y}\right] = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 t}{\partial x \partial y} = z_{xy} = f_{xy} = s$$

$$\therefore \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$$

Problems :-

1. Find the first and second order, the partial derivatives of ;

i) $z = xy^2 + x^2y$

$\Rightarrow \quad z = xy^2 + x^2y$ ——— ①

diff partially w.r to 'x'

$$\Rightarrow \frac{\partial z}{\partial x} = y^2 \frac{\partial}{\partial x}(x) + y \frac{\partial}{\partial x}(x^2)$$

$$\Rightarrow \frac{\partial z}{\partial x} = y^2(1) + y(2x)$$

$$\Rightarrow \frac{\partial z}{\partial x} = y^2 + 2xy \text{ ——— ②}$$

Similarly,

$$\Rightarrow \frac{\partial z}{\partial y} = x \cdot \frac{\partial}{\partial y}(y^2) + x^2 \frac{\partial}{\partial y}(y)$$

$$\Rightarrow \frac{\partial z}{\partial y} = 2xy + x^2 \text{ ——— ③}$$

diff ② w.r to (x)

$$\therefore \frac{\partial}{\partial x} \cdot \frac{\partial z}{\partial x} = y^2 \frac{\partial}{\partial x}(1) + 2y \frac{\partial}{\partial x}(x)$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} = y^2(0) + 2y(1)$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} = 2y \text{ ——— ④}$$

diff ③ w.r to (y).

$$\therefore \frac{\partial}{\partial y} \cdot \frac{\partial z}{\partial y} = 2x \frac{\partial}{\partial y}(y) + x^2 \frac{\partial}{\partial y}(1)$$

$$\Rightarrow \frac{\partial^2 z}{\partial y^2} = 2x(1)+0$$

$$\Rightarrow \frac{\partial^2 z}{\partial y^2} = 2x \quad \text{——— (5)}$$

diff eqn (3) w.r to (x)

$$\therefore \frac{\partial}{\partial x} \cdot \frac{\partial z}{\partial y} = 2y \cdot \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial x}(x^2)$$

$$\Rightarrow \frac{\partial^2 z}{\partial x \partial y} = 2y + 2x$$

diff eqn (2) w.r to (y)

$$\therefore \frac{\partial}{\partial y} \cdot \frac{\partial z}{\partial x} = \frac{\partial}{\partial y}(y^2) + 2x \cdot \frac{\partial}{\partial y}(y)$$

$$\Rightarrow \frac{\partial^2 z}{\partial y \partial x} = 2y + 2x.$$

iii) $z = x^2 \sin y - y^2 \sin x$

$$\Rightarrow z = x^2 \sin y - y^2 \sin x \quad \text{——— (1)}$$

diff eqn (1) partially w.r to x

$$\therefore \frac{\partial z}{\partial x} = \sin y \cdot \frac{\partial}{\partial x}(x^2) - y^2 \frac{\partial}{\partial x}(\sin x)$$

$$\Rightarrow \frac{\partial z}{\partial x} = 2x \sin y - y^2 \cos x \quad \text{——— (2)}$$

diff eqn (1) w.r to y

$$\Rightarrow \frac{\partial z}{\partial y} = x^2 \frac{\partial}{\partial y}(\sin y) - \sin x \cdot \frac{\partial}{\partial y}(y^2)$$

$$\Rightarrow \frac{\partial z}{\partial y} = 2x \cos y - 2y \cos x \quad \text{——— (3)}$$

$$\Rightarrow \frac{\partial}{\partial x}\cdot\frac{\partial z}{\partial x} = \frac{\partial^2 z}{\partial x^2} = 2\sin y\cdot\frac{\partial}{\partial x}(x) - y^2\frac{\partial}{\partial x}(\cos x)$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} = 2\sin y + y^2\sin x \quad \text{———} (4)$$

diff (3) w.r to y

$$\Rightarrow \frac{\partial}{\partial y}\cdot\frac{\partial z}{\partial y} = \frac{\partial^2 z}{\partial y^2} = x^2\frac{\partial}{\partial y}(\cos y) - 2\sin x\cdot\frac{\partial}{\partial y}(y)$$

$$\Rightarrow \frac{\partial^2 z}{\partial y^2} = -x^2\sin y - 2\sin x$$

diff (3) w.r to x

$$\therefore \frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}\cdot\frac{\partial z}{\partial y} = \cos y\cdot\frac{\partial}{\partial x}(x^2) - 2y\frac{\partial x}{\partial y}(\sin x)$$

$$= 2x\cos y - 2y\cos x$$

2. If $u = x^3 - 3xy^2 + x + e^x\cos y + 1$
Show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$$\Rightarrow u = x^3 - 3xy^2 + x + e^x\cos y + 1 \quad \text{———} (1)$$

diff w.r to (x) partially

$$\Rightarrow \frac{\partial u}{\partial x} = 3x^2 - 3y^2(1) + (1) + e^x\cos y + 0$$

$$\Rightarrow \frac{\partial u}{\partial x} = 3x^2 - 3y^2 + e^x\cos y + 1 \quad \text{———} (2)$$

$$\therefore \frac{\partial}{\partial x}\cdot\frac{\partial u}{\partial x} = 6x - 0 + e^x\cos y + 0$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = 6x + e^x\cos y \quad \text{———} (3)$$

Similarly,

eqn (1) diff w.r to (y)

$$\Rightarrow \frac{\partial u}{\partial y} = 0 - 3x(2y) + 0 + e^x(-\sin y) + 0$$

$$\Rightarrow \frac{\partial u}{\partial y} = -6xy - e^x \sin y$$

$$\therefore \frac{\partial}{\partial y} \cdot \frac{\partial u}{\partial y} = -6x(1) - e^x \cos y$$

$$\Rightarrow \frac{\partial^2 u}{\partial y^2} = -6x - e^x \cos y \quad \text{---(4)}$$

$\therefore$ LHS,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow \cancel{6x} + e^x \cancel{\cos y} - \cancel{6x} - e^x \cancel{\cos y}$$

$$\Rightarrow 0$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

3. If $r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$

Show that, $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} = \frac{2}{r}$

$$\Rightarrow r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2 \quad \text{---(1)}$$

diff w.r to (x) partially

$$\Rightarrow 2r \cdot \frac{dr}{dx} = 2(x-a)$$

$$\Rightarrow r \cdot \frac{dr}{dx} = (x-a)$$

$$\Rightarrow \frac{dr}{dx} = \frac{(x-a)}{r}$$

$$\Rightarrow \frac{\partial^2 r}{\partial x^2} = \frac{r(1) - (x-a)\frac{\partial r}{\partial x}}{r^2}$$

$$\Rightarrow \frac{\partial^2 r}{\partial x^2} = \frac{r - (x-a)\left(\frac{x-a}{r}\right)}{r^2}$$

$$\Rightarrow \frac{\partial^2 r}{\partial x^2} = \frac{r - \frac{(x-a)^2}{r}}{r^2}$$

$$\Rightarrow \frac{\partial^2 r}{\partial x^2} = \frac{r}{r^2} - \frac{(x-a)^2}{r^3}$$

$$\Rightarrow \frac{\partial^2 r}{\partial x^2} = \frac{1}{r} \cdot \frac{(x-a)^2}{r^3} \text{ ——— (2)}$$

Similarly,

$$\Rightarrow \frac{\partial^2 r}{\partial y^2} = \frac{1}{r} - \frac{(y-a)^2}{r^3} \text{ ——— (3)}$$

$$\Rightarrow \frac{\partial^2 r}{\partial z^2} = \frac{1}{r} - \frac{(z-a)^2}{r^3} \text{ ——— (4)}$$

LHS,

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2}$$

$$= \frac{1}{r} - \frac{(x-a)^2}{r^3} + \frac{1}{r} - \frac{(y-a)^2}{r^3} + \frac{1}{r} - \frac{(z-a)^2}{r^3}$$

$$= \frac{3}{r} - \frac{1}{r^3}\left[\left[(x-a)^2 + (y-b)^2 + (z-c)^2\right]\right]$$

$$= \frac{3}{r} - \frac{1}{r^3}\left[r^2\right]$$

$$= \frac{3}{r} - \frac{1}{r}$$

$$= \frac{2}{r}, \quad \text{RHS}$$

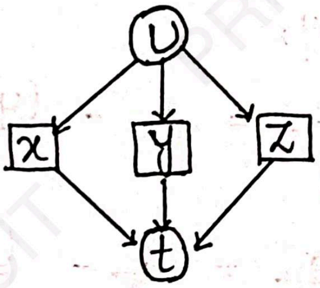
Total Derivatives :-

1. If $u = f(x, y, z)$ be a function in the independent variables of x, y, z then, the total derivaties of u with respect to x, y, z can be defined as,

$$\partial u = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$$

2. Let, $u = f(x, y, z)$, $x = x(t)$, $y = y(t)$, $z = z(t)$, then the total derivate of 'u' with respect to 't' can be defined as;

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial t}$$



3. Let $u = f(p, q, r)$ where, $p = p(x, y, z)$, $q = q(x, y, z)$ $r = r(x, y, z)$ then,

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial x} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial x} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial y} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial y} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial z} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial z} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial z}$$

1. Find the $\frac{du}{dt}$ for the function $u = xy^2 + x^2y$ where, $x = at$, $y = 2at$ and verify the same by the direct method.

$\Rightarrow$ $$u = xy^2 + x^2y \quad \text{---(1)}$$

where, $x = at$, $y = 2at$

$$\therefore \frac{\partial u}{\partial x} = y^2(1) + (2x)y$$

$$\Rightarrow \frac{\partial u}{\partial x} = y^2 + 2xy$$

$$\therefore \frac{\partial u}{\partial y} = x.(2y) + x^2(1)$$

$$\Rightarrow \frac{\partial u}{\partial y} = 2xy + x^2$$

$$\therefore \frac{dx}{dt} = a \quad , \quad \frac{dy}{dt} = 2a$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt}$$

$$\Rightarrow \frac{du}{dt} = [y^2 + 2xy][2a] + [2xy + x^2][2a]$$

$$\Rightarrow \frac{du}{dt} = a\left[(2at)^2 + 2(at)(2at)\right] + 2a\left[2(at)(2at) + (at)^2\right]$$

$$\Rightarrow \frac{du}{dt} = a\left[4a^2t^2 + 4a^2t^2\right] + 2a\left[4a^2t^2 + a^2t^2\right]$$

$$\Rightarrow \frac{du}{dt} = a\left[8a^2t^2\right] + 2a\left[5a^2t^2\right]$$

$$\Rightarrow \frac{du}{dt} = 8a^3t^2 + 10a^3t^2$$

$$\Rightarrow \frac{du}{dt} = 18a^3t^2$$

Direct method,

$\therefore (1) \Rightarrow u = (at)(2at)^2 + (at)^2(2at)$

$u = (at)(4a^2t^2) + (a^2t^2)(2at)$

$u = 4a^3t^3 + 2a^3t^3$

$u = 6a^3t^3$

$\therefore \dfrac{du}{dt} = 6a^3(3t^2)$

$\Rightarrow \dfrac{du}{dt} = 18a^3t^2$

2. Find $\dfrac{du}{dt}$ for the function $u = xy + yz + zx$ with $x = t\cos t$, $y = t\sin t$, $z = t$ at $t = \pi/4$

$\Rightarrow \quad u = xy + yz + zx \quad \text{——(1)}$

where, $x = t\cos t$, $y = t\sin t$, $z = t$

$\therefore \dfrac{\partial u}{\partial x} = y + 0 + z = y + z$

$\Rightarrow \dfrac{\partial u}{\partial x} = t\sin t + t$

$\therefore \dfrac{\partial u}{\partial y} = x + z + 0 = x + z$

$\Rightarrow \dfrac{\partial u}{\partial y} = t\cos t + t$

$\therefore \dfrac{\partial u}{\partial z} = 0 + y + x = y + x$

$\Rightarrow \dfrac{\partial u}{\partial z} = t\sin t + t\cos t$

$\therefore \dfrac{dx}{dt} = 1.\cos t + t(-\sin t) = \cos t - t\sin t$

$\dfrac{dy}{dt} = 1.\sin t + t\cos t = \sin t + t\cos t$

$$\frac{dz}{dt} = 1$$

$$\therefore \frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$

$$\Rightarrow \frac{du}{dt} = (t\sin t + t)(\cos t - t\sin t) + (t\cos t + t)(\sin t + t\cos t) + (\sin t + t\cos t)$$

at $t = \pi/4$

$$\Rightarrow \frac{du}{dt} = \left[\frac{\pi}{4\sqrt{2}} + \frac{\pi}{4}\right]\left[\frac{1}{\sqrt{2}} - \frac{\pi}{4\sqrt{2}}\right] + \left[\frac{\pi}{4\sqrt{2}} + \frac{\pi}{4}\right]\left[\frac{1}{\sqrt{2}} + \frac{\pi}{4\sqrt{2}}\right] + \left[\frac{\pi}{4\sqrt{2}} + \frac{\pi}{4\sqrt{2}}\right]$$

$$= \left[\frac{\pi}{4\sqrt{2}} + \frac{\pi}{4}\right]\left[\frac{1}{\sqrt{2}} - \cancel{\frac{\pi}{4\sqrt{2}}} + \frac{1}{\sqrt{2}} + \cancel{\frac{\pi}{4\sqrt{2}}}\right] + \frac{2\pi}{4\sqrt{2}}$$

$$= \left[\frac{\pi}{4\sqrt{2}} + \frac{\pi}{4}\right]\left[\frac{2}{\sqrt{2}}\right] + \frac{\pi}{2\sqrt{2}}$$

$$= \left[\frac{\pi}{4\sqrt{2}} + \frac{\pi}{4}\right]\sqrt{2} + \frac{\pi}{2\sqrt{2}}$$

$$= \frac{\pi}{4} + \frac{\pi}{4}\sqrt{2} + \frac{\pi}{2\sqrt{2}}$$

$$= \frac{\pi}{4} + \frac{\cancel{\sqrt{2}}\,\pi}{2\sqrt{2}\cancel{\sqrt{2}}} + \frac{\pi}{2\sqrt{2}}$$

$$= \frac{\pi}{4} + \frac{\cancel{2}\pi}{\cancel{2}\sqrt{2}}$$

$$= \frac{\pi}{4} + \frac{\pi}{\sqrt{2}}$$

3. Find $\frac{du}{dt}$ for the function $x = e^{x^2+y^2+z^2}$ where, $x = t^2+1$, $y = t\cos t$, $z = \sin t$ at $t=0$.

$$\Rightarrow \quad x = e^{x^2+y^2+z^2} \quad \text{———(1)}$$

where, $x = t^2+1$, $y = t\cos t$, $z = \sin t$

$$\Rightarrow \frac{\partial u}{\partial x} = 2x e^{x^2+y^2+z^2}$$

$$\Rightarrow \frac{\partial u}{\partial x} = 2(t^2+1)\, e^{(t^2+1)^2+(t^2\cos^2 t + \sin^2 t)}$$

$$\Rightarrow \frac{\partial u}{\partial y} = 2y \cdot e^{x^2+y^2+z^2}$$

$$\Rightarrow \frac{\partial u}{\partial y} = 2t\cos t\; e^{(t^2+1)^2+(t^2\cos^2 t + \sin^2 t)}$$

$$\Rightarrow \frac{\partial u}{\partial z} = 2z \cdot e^{x^2+y^2+z^2}$$

$$\Rightarrow \frac{\partial u}{\partial z} = 2\sin t \cdot e^{(t^2+1)^2+(t^2\cos^2 t + \sin^2 t)}$$

$$\therefore \frac{dx}{dt} = 2t$$

$$\frac{dy}{dt} = \cos t - t\sin t$$

$$\frac{dz}{dt} = \cos t$$

$$\therefore \frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$

$$\Rightarrow \frac{du}{dt} = \left[2(t^2+1)\, e^{(t^2+1)^2+(t^2\cos^2 t + \sin^2 t)}\right](2t) + \left[2t\cos t\; e^{(t^2+1)^2+(t^2\cos^2 t + \sin^2 t)}\right]\left[\cos t - t\sin t\right] + \left[2\sin t\; e^{(t^2+1)^2+(t^2\cos^2 t + \sin^2 t)}\right]\left[\cos t\right]$$

$$\Rightarrow \left[\frac{du}{dt}\right]_{t=0} = 0$$

4. If $u = f(y-z,\ z-x,\ x-y)$ show that,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

$\Rightarrow$ Let, $p = y-z$, $q = z-x$, $r = x-y$

then, $u = f(p, q, r)$ ——(2)

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial x} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial x} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}(0) + \frac{\partial u}{\partial q}(-1) + \frac{\partial u}{\partial r}(1)$$

$$\Rightarrow \frac{\partial u}{\partial x} = -\frac{\partial u}{\partial q} + \frac{\partial u}{\partial r} \quad \text{——(3)}$$

$$\therefore \frac{\partial u}{\partial y} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial y} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial y} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y}$$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}(1) + \frac{\partial u}{\partial q}(0) + \frac{\partial u}{\partial r}(-1)$$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r} \quad \text{——(4)}$$

$$\therefore \frac{\partial u}{\partial z} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial z} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial z} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial z}$$

$$\Rightarrow \frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}(-1) + \frac{\partial u}{\partial q}(1) + \frac{\partial u}{\partial r}(0)$$

$$\Rightarrow \frac{\partial u}{\partial z} = -\frac{\partial u}{\partial p} + \frac{\partial u}{\partial q} \quad \text{——(5)}$$

$(3)+(4)+(5) \Rightarrow \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z}$

$$\Rightarrow -\frac{\partial u}{\partial q} + \frac{\partial u}{\partial r} + \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r} - \frac{\partial u}{\partial p} + \frac{\partial u}{\partial q}$$

$\Rightarrow 0$

5. If $u = f(2x-3y,\ 3y-4z,\ 4z-2x)$ show that,

$$6u_x + 4u_y + 3u_z = 0 \quad \text{(or)}$$

$$\frac{1}{2}u_x + \frac{1}{3}u_y + \frac{1}{4}u_z = 0$$

$\Rightarrow$ Let, $p = 2x-3y$, $q = 3y-4z$; $r = 4z-2x$

$$\therefore \textcircled{1} \Rightarrow u = f(p,q,r) \quad \text{——(2)}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}(2) + \frac{\partial u}{\partial q}(0) + \frac{\partial u}{\partial r}(-2)$$

$$\Rightarrow \frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial p} - 2\frac{\partial u}{\partial r}$$

$$\Rightarrow \frac{1}{2}\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r} \quad \text{——(3)}$$

$$\therefore \frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y}$$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}(-3) + \frac{\partial u}{\partial q}(3) + \frac{\partial u}{\partial r}(0)$$

$$\Rightarrow \frac{1}{3}\frac{\partial u}{\partial y} = -\frac{\partial u}{\partial p} + \frac{\partial u}{\partial r} \quad \text{——(4)}$$

$$\therefore \frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z}$$

$$\Rightarrow \frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}(0) + \frac{\partial u}{\partial q}(-4) + \frac{\partial u}{\partial r}(4)$$

$$\Rightarrow \frac{1}{4}\frac{\partial u}{\partial z} = -\frac{\partial u}{\partial q} + \frac{\partial u}{\partial r} \quad \text{——(5)}$$

(3) + (4) + (5)

$$\Rightarrow \frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z}$$

$$\Rightarrow \cancel{\frac{\partial u}{\partial p}} - \cancel{\frac{\partial u}{\partial r}} - \cancel{\frac{\partial u}{\partial p}} + \cancel{\frac{\partial u}{\partial r}} - \cancel{\frac{\partial u}{\partial q}} + \cancel{\frac{\partial u}{\partial r}} = 0$$

$$\Rightarrow \frac{1}{2}U_x + \frac{1}{3}U_y + \frac{1}{4}U_z = 0 .$$

$$\Rightarrow \frac{6U_x + 4U_y + 3U_z}{12} = 0$$

$$\Rightarrow 6U_x + 4U_y + 3U_z = 0$$

6. If $u = f\left[\frac{y-x}{xy}, \frac{z-x}{zx}\right]$ show that,

$$x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0 .$$

$\Rightarrow$ Let, $p = \frac{y-x}{xy}$, $q = \frac{z-x}{zx}$

then, $u = f(p, q)$ —(1)

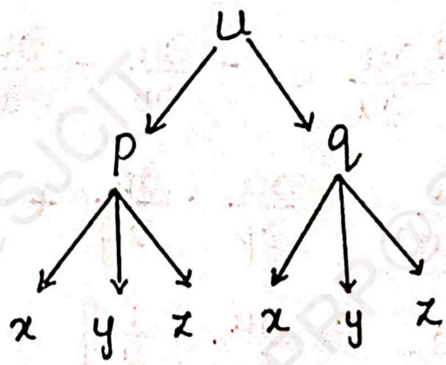
$$\therefore \frac{\partial p}{\partial x} = \frac{xy(-1) - (y-x)(y)}{(xy)^2}$$

$$\Rightarrow \frac{\partial p}{\partial x} = \frac{-xy - y^2 + xy}{(xy)^2}$$

$$\Rightarrow \frac{\partial p}{\partial x} = \frac{-y^2}{x^2y^2} = -\frac{1}{x^2}$$

$$\therefore \frac{\partial p}{\partial y} = \frac{xy(1) - (y-x)(x)}{(xy)^2}$$

$$\Rightarrow \frac{\partial p}{\partial y} = \frac{xy - xy + x^2}{x^2y^2}$$



$$\Rightarrow \frac{\partial p}{\partial y} = \frac{x^2}{x^2y^2}$$

$$\Rightarrow \frac{\partial p}{\partial y} = \frac{1}{y^2}$$

$$\therefore \frac{\partial p}{\partial z} = 0$$

$$\Rightarrow \frac{\partial q}{\partial x} = \frac{zx(-1)-(z-x)(z)}{(zx)^2}$$

$$\Rightarrow \frac{\partial q}{\partial x} = \frac{-\cancel{zx}-z^2+\cancel{zx}}{z^2x^2}$$

$$\Rightarrow \frac{\partial q}{\partial x} = \frac{-\cancel{z^2}}{\cancel{z^2}x^2}$$

$$\Rightarrow \frac{\partial q}{\partial x} = -\frac{1}{x^2}$$

$$\Rightarrow \frac{\partial q}{\partial x} = \text{`0'}$$

$$\therefore \frac{\partial q}{\partial z} = \frac{(zx)(1)-(z-x)(x)}{(zx)^2}$$

$$\Rightarrow \frac{\partial q}{\partial z} = \frac{\cancel{zx}-\cancel{zx}+x^2}{z^2x^2}$$

$$\Rightarrow \frac{\partial q}{\partial z} = \frac{\cancel{x^2}}{z^2\cancel{x^2}}$$

$$\Rightarrow \frac{\partial q}{\partial z} = \frac{1}{z^2}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\left[-\frac{1}{x^2}\right] + \frac{\partial u}{\partial q}\left[-\frac{1}{x^2}\right]$$

$$\Rightarrow x^2\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial p} - \frac{\partial u}{\partial q} \quad \text{——— (2)}$$

$$\therefore \frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial y}$$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\left[\frac{1}{y^2}\right] + \frac{\partial u}{\partial q}(0)$$

$$\Rightarrow y^2\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p} \quad \text{——— (3)}$$

$$\therefore \frac{\partial u}{\partial z} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial z} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial z}$$

$$\Rightarrow \frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}(0) + \frac{\partial u}{\partial q}\left[\frac{1}{z^2}\right]$$

$$\Rightarrow z^2 \frac{\partial u}{\partial z} = \frac{\partial u}{\partial q} \quad \text{———(4)}$$

$(2) + (3) + (4) =$

$$x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = -\cancel{\frac{\partial u}{\partial p}} - \cancel{\frac{\partial u}{\partial q}} + \cancel{\frac{\partial u}{\partial p}} + \cancel{\frac{\partial u}{\partial q}}$$

$$\Rightarrow x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = 0$$

7. If $u = f(x-y, y-z, z-x)$ then show that,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0 \quad \text{(or)} \quad U_x + U_y + U_z = 0$$

$$\Rightarrow \quad u = f(x-y, y-z, z-x) \quad \text{———(1)}$$

Let, $p = (x-y)$ $q = (y-z)$ $r = (z-x)$ then

$$u = f(p, q, r)$$

$$\therefore \frac{\partial p}{\partial x} = 1, \quad \frac{\partial p}{\partial y} = -1, \quad \frac{\partial p}{\partial z} = 0$$

$$\frac{\partial q}{\partial x} = 0, \quad \frac{\partial q}{\partial y} = 1, \quad \frac{\partial q}{\partial z} = -1$$

$$\frac{\partial r}{\partial x} = -1, \quad \frac{\partial r}{\partial y} = 0, \quad \frac{\partial r}{\partial z} = 1$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \frac{\partial p}{\partial x} + \frac{\partial u}{\partial q} \frac{\partial q}{\partial x} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}(1) + \frac{\partial u}{\partial q}(0) + \frac{\partial u}{\partial r}(-1)$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r} \quad \text{——(2)}$$

$$\therefore \frac{\partial u}{\partial y} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial y} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial y} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y}$$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}(-1) + \frac{\partial u}{\partial q}(1) + \frac{\partial u}{\partial r}(0)$$

$$\Rightarrow \frac{\partial u}{\partial y} = -\frac{\partial u}{\partial p} + \frac{\partial u}{\partial q} \quad \text{——(3)}$$

$$\therefore \frac{\partial u}{\partial z} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial z} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial z} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial z}$$

$$\Rightarrow \frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}(0) + \frac{\partial u}{\partial q}(-1) + \frac{\partial u}{\partial r}(1)$$

$$\Rightarrow \frac{\partial u}{\partial z} = -\frac{\partial u}{\partial q} + \frac{\partial u}{\partial r} \quad \text{——(4)}$$

(2) + (3) + (4) $\Rightarrow \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z}$

$$= \cancel{\frac{\partial u}{\partial p}} - \cancel{\frac{\partial u}{\partial r}} - \cancel{\frac{\partial u}{\partial p}} + \cancel{\frac{\partial u}{\partial q}} - \cancel{\frac{\partial u}{\partial q}} + \cancel{\frac{\partial u}{\partial r}}$$

$$= 0$$

8. If $u = f\left[\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right]$ then, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

$$\Rightarrow u = f\left[\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right]$$

Let, $p = \frac{x}{y}$, $q = \frac{y}{z}$, $r = \frac{z}{x}$

then, $u = f(p, q, r)$ ——(1)

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial x} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial x} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \cdot \frac{1}{y} + \frac{\partial u}{\partial x}(z) \text{ ——— } (2)$$

$$\therefore \frac{\partial u}{\partial y} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial y} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial y} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y}$$

$$\Rightarrow \frac{\partial u}{\partial y} = -\frac{1}{y^2} \frac{\partial u}{\partial p} + \frac{1}{z} \frac{\partial u}{\partial q} \text{ ——— } (3)$$

$$\therefore \frac{\partial u}{\partial z} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial z} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial z} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial z}$$

$$\Rightarrow \frac{\partial u}{\partial z} = -\frac{1}{z^2} \frac{\partial u}{\partial p} + \frac{1}{x} \frac{\partial u}{\partial r} \text{ ——— } (4)$$

(2) + (3) + (4)

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{1}{y} \frac{\partial u}{\partial p} + z \frac{\partial u}{\partial r} - \frac{1}{y^2} \frac{\partial u}{\partial p} + \frac{1}{z} \frac{\partial u}{\partial q} - \frac{1}{z^2} \frac{\partial u}{\partial q} + \frac{\partial u}{\partial x} \frac{1}{x}$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 0$$

Jacobians :-

1. Suppose ; $U = U(x, y)$, $V = v(x, y)$ be the 2 functions in the variables of x & y , then the jacobians of u, v with respect to x, y can be defined as;

$$J = J\left[\frac{u, v}{x, y}\right] = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

and its inverse can be defined as ;

$$J' = J'\left[\frac{x, y}{u, v}\right] = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

2. Suppose, $u = u(x,y,z)$ $v = v(x,y,z)$, $w = w(x,y,z)$ then, the jacobian of u, v, w with respect to x, y, z can be defined as;

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = J\left[\frac{u, v, w}{x, y, z}\right] = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

and its inverse can be defined as;

$$J' = \frac{\partial(x,y,z)}{\partial(u,v,w)} = J'\left[\frac{u, v, w}{x, y, z}\right] = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

where, $J \,\&\, J' = 1$

Questions :-

1. Find the jacobian of u,v with x,y where,
$u = e^x \cos y$, $v = e^x \sin y$

$\Rightarrow$ $\therefore \dfrac{\partial u}{\partial x} = e^x \cos y$, $\dfrac{\partial u}{\partial y} = -e^x \sin y$

$\dfrac{\partial v}{\partial x} = e^x \sin y$, $\dfrac{\partial v}{\partial y} = e^x \cos y$

$$\therefore J = J\left[\frac{u,v}{x,y}\right] = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{vmatrix}$$

$$= e^x . e^x \begin{vmatrix} \cos y & -\sin y \\ \sin y & \cos y \end{vmatrix}$$

$$= e^{2x} [\cos^2 y + \sin^2 y]$$

$$\frac{\partial(u,v)}{\partial(x,y)} = e^{2x}$$

2. If $u = x^2+y^2+z^2$, $v = xy+yz+zx$, $w = x+y+z$ then, find $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$

$\Rightarrow$ Given ;

$u = x^2+y^2+z^2$

$v = xy+yz+zx$

$w = x+y+z$

$\therefore \quad U_x = 2x \quad , \quad U_y = 2y \quad , \quad U_z = 2z$

$V_x = y+z \quad , \quad V_y = x+z \quad , \quad V_z = y+x.$

$W_x = 1 \quad , \quad W_y = 1 \quad , \quad W_z = 1.$

$\therefore$ WKT,

$$\Rightarrow \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} U_x & U_y & U_z \\ V_x & V_y & V_z \\ W_x & W_y & W_z \end{vmatrix}$$

$$= \begin{vmatrix} 2x & 2y & 2z \\ y+z & x+z & y+x \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} x & y & z \\ y+z & z+x & x+y \\ 1 & 1 & 1 \end{vmatrix}$$

$R_2 = R_2 + R_1$

$$= 2\begin{vmatrix} x & y & z \\ x+y+z & x+y+z & x+y+z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2(x,y,z)\begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2(x,y,z)(0)$$

$$\Rightarrow \frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$$

3. If $u = 3x+2y-z$, $v = x-2y+z$, $w = x^2+2xy-xz$ then, show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$

$\Rightarrow$ Given,

$$u = 3x+2y-z$$
$$v = x-2y+z$$
$$w = x^2+2xy-xz$$

$\therefore$ $u_x = 3$ $\quad$ $u_y = 2$ $\quad$ $u_z = -1$

$v_x = 1$ $\quad$ $v_y = -2$ $\quad$ $v_z = 1$

$w_x = 2x+2y-z$ $\quad$ $w_y = 2x$ $\quad$ $w_z = -x$

$\therefore$ WKT,

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 2 & -1 \\ 1 & -2 & 1 \\ 2x+2y-z & 2x & -x \end{vmatrix}$$

$$= 3[2x-2x] - 2[-x-[2x+2y-z]] - [2x+2(2x+2y-z)]$$

$$= 3(0) - 2[-3x-2y+z] - [6x+4y-2z]$$

$$= \cancel{6x} + \cancel{4y} - \cancel{2z} - \cancel{6x} - \cancel{4y} + \cancel{2z}$$

$$\Rightarrow \frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$$

4. If $u = x + 3y^2 - z^3$, $V = 4x^2yz$, $w = 2z^2 - xy$ evaluate $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ at the point $(1,-1,0)$

$\Rightarrow$ Given,

$u = x + 3y^2 - z^3$

$V = 4x^2yz$

$w = 2z^2 - xy$

$P = (1,-1,0)$

$\therefore$ $U_x = 1 \Rightarrow (U_x)_P = 1$

$U_y = 6y \Rightarrow (U_y)_P = -6$

$U_z = -3z^2 \Rightarrow (U_z)_P = 0$

$\therefore$ $V_x = 8xyz \Rightarrow (V_x)_P = 0$

$V_y = 4x^2z \Rightarrow (V_y)_P = 0$

$V_z = 4x^2y \Rightarrow (V_z)_P = -4$

$\therefore$ $w_x = -y \Rightarrow (w_x)_P = -(-1) = 1$

$w_y = -x \Rightarrow (w_y)_P = -1$

$w_z = 4z \Rightarrow (w_z)_P = 0$

WKT,

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} U_x & U_y & U_z \\ V_x & V_y & V_z \\ w_x & w_y & w_z \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -6 & 0 \\ 0 & 0 & -4 \\ 1 & -1 & 0 \end{vmatrix}$$

$$= 1[0-4] + 6[0+4]$$

$$= -4 + 24 = 20$$

5. If $u = x+y+z$, $v = y+z$, $w = z$, then find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

$\Rightarrow$ Given;

$$u = x+y+z$$
$$v = y+z$$
$$w = z$$

$\therefore$ $u_x = 1$ $u_y = 1$ $u_z = 1$

$v_x = 0$ $v_y = 1$ $v_z = 1$

$w_x = 0$ $w_y = 0$ $w_z = 1$

$\Rightarrow$ $\therefore$ $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 1[1-0] - 1[0-0] + 1[0-0]$$

$$= 1$$

6. If $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$, then, show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 4$

$\Rightarrow$ Given,

$$u = \frac{yz}{x}$$
$$v = \frac{zx}{y}$$
$$w = \frac{xy}{z}$$

$\therefore \quad U_x = -\frac{yz}{x^2} \qquad U_y = \frac{z}{x} \qquad U_z = \frac{y}{x}$

$V_x = \frac{z}{y} \qquad V_y = -\frac{zx}{y^2} \qquad V_z = \frac{x}{y}$

$W_x = \frac{y}{z} \qquad W_y = \frac{x}{z} \qquad W_z = -\frac{xy}{z^2}$

WKT,

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} U_x & U_y & U_z \\ V_x & V_y & V_z \\ W_x & W_y & W_z \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & -\frac{zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & -\frac{xy}{z^2} \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{yz}{x^2} & \frac{zx}{x^2} & \frac{yx}{x^2} \\ \frac{zy}{y^2} & -\frac{zx}{y^2} & \frac{yx}{y^2} \\ \frac{yz}{z^2} & \frac{xz}{z^2} & -\frac{xy}{z^2} \end{vmatrix}$$

$$= \frac{1}{x^2 y^2 z^2} \begin{vmatrix} -yz & zx & xy \\ yz & -zx & xy \\ yz & zx & -xy \end{vmatrix}$$

$$= \begin{matrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{matrix} = (-1)(1-1) - 1(-1-1) + 1(1+1)$$

$$= 0 - (-2) + (2)$$

$$= 2 + 2 = 4$$

7. If $x = r\sin\theta(\cos\phi)$; $y = r\sin\theta(\sin\phi)$, $z = r\cos\theta$, then, show that $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$

=> Soln:- $x = r\sin\theta(\cos\phi)$

$y = r\sin\theta(\sin\phi)$

$z = r\cos\theta$

$\frac{\partial x}{\partial r} = \sin\theta\cos\phi$, $\frac{\partial y}{\partial r} = \sin\theta\sin\phi$ $\frac{\partial z}{\partial r} = \cos\theta$

$\frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi$, $\frac{\partial y}{d\theta} = r\cos\theta\sin\phi$, $\frac{\partial z}{\partial\theta} = -r\sin\theta$

$\frac{\partial x}{\partial\phi} = -r\sin\theta\cos\phi$, $\frac{\partial y}{\partial\phi} = r\sin\theta\cos\phi$, $\frac{\partial z}{\partial\phi} = 0$

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta \begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta\,[\sin\theta\cos\phi\,[0+\sin\theta\cos\phi] - \cos\theta\cos\phi\,[0-\cos\theta\cos\phi] - \sin\phi\,[-\sin^2\theta\sin\phi - \cos^2\theta\sin\phi]]$$

$= r^2 \sin\theta \, [\cos^2\phi \, (\sin^2\theta + \cos^2\theta) + \sin^2\phi \, (1)]$

$= r^2 \sin\theta \, [\cos^2\phi + \sin^2\phi]$

$J = r^2 \sin\theta$

Maxima and Minima for two variables :-

1. Let the given function as $u = f(x, y)$

2. Find $\frac{\partial f}{\partial x}$. $\frac{\partial f}{\partial y}$ and make them equal to zero for maxima or minima that is, $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$

   By solving the above let's have (x_0, y_0) (x_1, y_1) (x_2, y_2) are called extreme or end points.

3. Find the second order partial derivatives of $u = f(x, y)$ say,

$$\frac{\partial^2 f}{\partial x^2} = A, \quad \frac{\partial^2 f}{\partial x \partial y} = B, \quad \frac{\partial^2 f}{\partial y^2} = C$$

and find the status of the end points as given below.

i. If $AC - B^2 > 0$ and $A < 0$, we say that the function may, have the maximum at $P(x_0, y_0)$ and the maximum value is $f(x_0, y_0)$

ii. If $AC - B^2 > 0$ and $A > 0$, we say that the function may have minimum value at $P(x_0, y_0)$ and the minimum value is $f(x_0, y_0)$

iii) If $AC - B^2 = 0$ (or) $AC - B^2 < 0$ we say that there is no maximum nor minimum at $P(x_0, y_0)$ and those points are said to be saddle points and follow the same verification at all the extreme points.

Problems:-

1. Find the extreme value of the function
$f(x,y) = x^3 + y^3 - 3x - 12y + 20$

$\Rightarrow$ Given,

$$f(x,y) = x^3 + y^3 - 3x - 12y + 20 \quad \text{——(1)}$$

diff (1) w.r to (x) partially

$$(1) \Rightarrow \frac{\partial f}{\partial x} = 3x^2 + 0 - 3 + 0 - 0$$

$$\Rightarrow \frac{\partial f}{\partial x} = 3x^2 - 3 \quad \text{——(2)}$$

Illy,

$$\Rightarrow \frac{\partial f}{\partial y} = 0 + 3y^2 - 0 - 12 + 0$$

$$\Rightarrow \frac{\partial f}{\partial y} = 3y^2 - 12 \quad \text{——(3)}$$

From max/min

$\frac{\partial f}{\partial x} = 0$ $\qquad$ $\frac{\partial f}{\partial y} = 0$

$\Rightarrow 3x^2 - 3 = 0$ $\qquad$ $3y^2 - 12 = 0$

$\Rightarrow x^2 - 1 = 0$ $\qquad$ $y^2 - 4 = 0$

$\Rightarrow x^2 = 1$ $\qquad$ $y^2 = 4$

$\Rightarrow x = \pm 1$ $\qquad$ $y = \pm 2$

$\therefore$ The extreme points are :-

$(-1,-2)$ $(-1,2)$ $(1,-2)$ $(1,2)$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 6x = A$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 0 = B$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = 6y = C$$

Point	A	B	C	$AC-B^2$	status
$(-1,-2)$	$-6<0$	0	-12	$72>0$	max
$(-1,2)$	$-6<0$	0	12	$-72<0$	saddle
$(1,-2)$	$6>0$	0	-12	$-72<0$	saddle
$(1,2)$	$6>0$	0	12	$72>0$	min

$$\max f = (-1)^3 + (-2)^3 - 3(-1) - 12(-2) + 20$$

$$\max f = -1 - 8 + 3 + 24 + 20 = 38$$

$$\min f = (1)^3 + (2)^3 - 3(1) - 12(2) + 20$$

$$\min f = 1 + 8 - 3 - 24 + 20 = 2$$

2. Find the extreme points of the function $f(x,y) = 2(x^2-y^2) - x^4 + y^4$

$\Rightarrow$ Soln:- $f(x,y) = 2(x^2-y^2) - x^4 + y^4$ ——(1)

diff (1) w.r to (x) & (y) partially

(1) $\Rightarrow$ $\frac{\partial f}{\partial x} = 4x - 4x^3$ ——(2)

$$(2) \Rightarrow \frac{\partial f}{\partial y} = -4y + 4y^3 \text{ ——— } (3)$$

$\therefore$ From max|min

$\Rightarrow 4x - 4x^3 = 0$ $\qquad$ $-4y + 4y^3 = 0$

$\Rightarrow x - x^3 = 0$ $\qquad$ $-y + y^3 = 0$

$\Rightarrow x(1 - x^2) = 0$ $\qquad$ $y(-1 + y^2) = 0$

$\Rightarrow x = 0, \; 1 - x^2 = 0$ $\qquad$ $y = 0, \; -1 + y^2 = 0$

$\Rightarrow x = 0, \; 1 - x^2 = 0$ $\qquad$ $y = 0, \; -1 + y^2 = 0$

$\Rightarrow x = 0, \; x = 1$ $\qquad$ $y = 0, \; y = 1$

$\therefore$ The extreme points are :-

$(0,0) \quad (0,1) \quad (1,0) \quad (1,1)$

$$\frac{\partial^2 f}{\partial x^2} = 4 - 12x^2 = A$$

$$\frac{\partial^2 f}{\partial x \partial y} = 0 = B$$

$$\frac{\partial^2 f}{\partial y^2} = -4 + 12y^2 = C$$

Points	A	B	C	$AC - B^2$	Status
$(0,0)$	$4>0$	0	-4	$-16<0$	saddle
$(0,1)$	$4>0$	0	8	$32>0$	min
$(1,0)$	$-8<0$	0	-4	$32>0$	max
$(1,1)$	$-8<0$	0	8	-64	saddle

$$\max f = 2(1-0) - 1 + 0 = 2 - 1 = 1$$

$$\min f = 2(0-1) - 0 + 1 = -2 + 1 = -2$$

3. Examine the function $f(x,y) = 2+2x+2y-x^2-y^2$

$\Rightarrow$ Soln :-

$$f(x,y) = 2+2x+2y-x^2-y^2 \quad \text{—(1)}$$

$$\frac{\partial f}{\partial x} = 2-2x \quad \text{—(2)}$$

$$\frac{\partial f}{\partial y} = 2-2y \quad \text{—(3)}$$

$\therefore$ for max|min

$$\frac{\partial f}{\partial x} = 0 \qquad \frac{\partial f}{\partial y} = 0$$

$\Rightarrow$ $2-2x=0$ $\qquad$ $2-2y=0$

$\Rightarrow$ $1-x=0$ $\qquad$ $1-y=0$

$\Rightarrow$ $x=1$ $\qquad$ $y=1$

$\therefore$ $P=(1,1)$

$$\therefore \frac{\partial f}{\partial x^2} = -2 = A$$

$$\frac{\partial^2 f}{\partial x \partial y} = 0 = B$$

$$\frac{\partial^2 f}{\partial y^2} = -2 = C$$

$\therefore$ $A + P(1,1)$

$AC - B^2 = 4 > 0$ & $A < 0$

$f(x,y)$, fmax

$$\max f = 2+2(1)+2(1)-1^2-1^2$$

$$\max f = 4$$

4. Examine the function $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

$\Rightarrow \quad f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2$ ——(1)

$$\frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 30x + 72 \text{ ——(2)}$$

$$\frac{\partial f}{\partial y} = 6xy - 30y \text{ ——(3)}$$

For max/min

$$\frac{\partial f}{\partial x} = 0 \quad \& \quad \frac{\partial f}{\partial y} = 0$$

and (3) $\Rightarrow 6xy - 30y = 0$

$\Rightarrow xy - 5y = 0$

$\Rightarrow y(x-5) = 0$

$\Rightarrow y = 0 \;,\; x - 5 = 0$

$\Rightarrow y = 0 \;,\; x = 5$

when $y = 0$

(4) $\Rightarrow x^2 - 10x + 24 = 0$

$\Rightarrow x^2 - 4x - 6x + 24 = 0$

$\Rightarrow x(x-4) - 6(x-4) = 0$

$\Rightarrow (x-4)(x-6) = 0$

$\Rightarrow x = 4, 6$

$\therefore$ Now points are $(4,0)\,(6,0)$

when, $x = 5$

(4) $\Rightarrow 25 + y^2 - 50 + 24 = 0$

$\Rightarrow y^2 - 1 = 0$

$\Rightarrow y^2 = 1$

$\Rightarrow y = 1$

the point are $(5,1)$

$$\frac{\partial^2 f}{\partial x^2} = 6x - 30 = A$$

$$\frac{\partial^2 f}{\partial x \partial y} = 6y = B$$

$$\frac{\partial^2 f}{\partial x \partial y} = 6x - 30 = C$$

Points	A	B	C	$AC - B^2$	Status
$(4,0)$	$-6<0$	0	-6	$36>0$	max
$(6,0)$	$+6>0$	0	6	$36>0$	min
$(5,1)$	0	0	0	$-36<0$	Saddle

5. Examine the function $f(x,y) = x^4 + y^4 - 2(x-y)^2$

$\Rightarrow$ Soln:- $f(x,y) = x^4 + y^4 - 2(x-y)^2$ —— (1)

diff (1) partially w.r to x & y

$$(1) \Rightarrow \frac{\partial f}{\partial x} = 4x^3 - 4(x-y)(1)$$

$$\Rightarrow \frac{\partial f}{\partial x} = 4x^3 - 4(x-y) \quad \text{—— (2)}$$

and

$$(1) \Rightarrow \frac{\partial f}{\partial y} = 0 + 4y^3 - 4(x-y)(-1)$$

$$\Rightarrow \frac{\partial f}{\partial y} = 4y^3 + 4(x-y) \quad \text{—— (3)}$$

for max | min

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0$$

$(3) \Rightarrow 4x^3 - 4(x-y) = 0$

$\Rightarrow x^3 - (x-y) = 0$

$\Rightarrow x^3 = x - y$ —— (4)

$(3) \Rightarrow 4y^3 + 4(x-y) = 0$

$\Rightarrow y^3 + (x-y) = 0$

$\Rightarrow y^3 = -(x-y)$ —— (5)

$\Rightarrow y^3 = -x^3$

$\Rightarrow x^3 + y^3 = 0$

$\Rightarrow (x+y)(x^2 - xy + y^2) = 0$

$\Rightarrow x+y = 0 \quad , \quad x^2 - xy + y^2 = 0$

$\Rightarrow x = -y$ —— (6)

$(8) \Rightarrow y^3 = -|-y-y|$

$\Rightarrow y^3 = 2y$

$\Rightarrow y^3 - 2y = 0$

$\Rightarrow y = 0 \quad , \quad y^2 - 2 = 0$

$\Rightarrow y = 0 \quad , \quad y^2 = 2$

$\Rightarrow y = 0 \quad , \quad y = \sqrt{2}$

$\Rightarrow x = 0 \quad , \quad x = \sqrt{2}$

The extreme points are $(0,0)$ $(-\sqrt{2}, \sqrt{2})$

$\therefore \dfrac{\partial^2 f}{\partial x^2} = 12x^2 - 4 = A$

$\dfrac{\partial^2 f}{\partial x \partial y} = 4 = B$

$\dfrac{\partial^2 f}{\partial y^2} = 12y^2 - 4 = C$

Points	A	B	C	$AC-B^2$	Status
$(0,0)$	$-4<0$	4	-4	0	Saddle
$(-\sqrt{2},\sqrt{2})$	$20>0$	4	20	$384>0$	min

6. Examine the function $f(x,y) = 2x^2 + 6xy^2 - 3y^2 - 150x$

$\Rightarrow$ Given,

$$f(x,y) = 2x^2 + 6xy^2 - 3y^2 - 150x \quad \text{——(1)}$$

$$(1) \Rightarrow \frac{\partial f}{\partial x} = 6x^2 + 6y^2 - 150 \quad \text{——(2)}$$

$$(1) \Rightarrow \frac{\partial f}{\partial y} = 12xy - 9y^2 \quad \text{——(3)}$$

for max|min

$$(2) \Rightarrow \frac{\partial f}{\partial x} = 0$$

$$\Rightarrow 6x^2 + 6y^2 - 150 = 0$$

$$\Rightarrow x^2 + y^2 - 25 = 0 \quad \text{——(4)}$$

$$(3) \Rightarrow \frac{\partial f}{\partial y} = 0$$

$$\Rightarrow 12xy - 4y^2 = 0$$

$$\Rightarrow 3y(4x - 3y) = 0$$

$$\Rightarrow y = 0 \quad , \quad 4x - 3y = 0$$

$$\Rightarrow y = 0 \quad , \quad 4x = 3y$$

$$\Rightarrow y = 0 \quad , \quad x = \frac{3y}{4}$$

$\therefore$ when $y = 0$

$$(4) \Rightarrow x^2 - 25 = 0$$

$$\Rightarrow x^2 = 25$$

$$\Rightarrow x = 5$$

$\therefore \quad P = (5,0)$

when, $x = \frac{34}{4}$

$$(4) \Rightarrow \left(\frac{34}{4}\right)^2 + y^2 - 25 = 0$$

$$\Rightarrow \frac{9y^2}{16} + y^2 - 25 = 0$$

$$\Rightarrow 25y^2 - 400 = 0$$

$$\Rightarrow y^2 - 16 = 0$$

$$\Rightarrow y^2 = 16$$

$$\Rightarrow y = 4$$

$\therefore Q = (3,4)$

$$\frac{\partial^2 f}{\partial x^2} = 12x = A$$

$$\frac{\partial^2 f}{\partial x \partial y} = 12y = B$$

$$\frac{\partial^2 f}{\partial y^2} = 12x - 18y = C$$

Points	A	B	C	$AC-B^2$	status
(5,0)	$60>0$	0	60	$3600>0$	min
(3,4)	$36>0$	48	-36	$-3600>0$	saddle

Additional Problems :-

1. If $u = x+y+z$, $uv = y+z$, $uvw = z$, evaluate $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$

$\Rightarrow$ Given,

$$u = x+y+z \quad \text{———} (1)$$

$$uv = y+z \quad \text{———} (2)$$

$$uvw = z \quad \text{———} (3)$$

$(3) \Rightarrow \quad z = uvw$

$(2) \Rightarrow \quad uv = y + uvw$

$\Rightarrow \quad y = uv - uvw$

$(1) \Rightarrow \quad u = x + uv$

$\Rightarrow \quad x = u - uv$

$$\therefore \frac{\partial x}{\partial u} = 1-v \qquad \frac{\partial x}{\partial v} = -u \qquad \frac{\partial x}{\partial w} = 0$$

$$\frac{\partial y}{\partial u} = v - vw \qquad \frac{\partial y}{\partial v} = u - uw \qquad \frac{\partial y}{\partial w} = -uv$$

$$\frac{\partial z}{\partial u} = vw \qquad \frac{\partial z}{\partial v} = uw \qquad \frac{\partial z}{\partial w} = uv$$

$\therefore$ WKT,

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= U.UV \begin{vmatrix} 1-V & -1 & 0 \\ V-VW & 1-W & -1 \\ VW & W & 1 \end{vmatrix}$$

$$R_2 = R_2 + R_3$$

$$= U^2.V \begin{vmatrix} 1-V & -1 & 0 \\ V & 1 & 0 \\ VW & W & 1 \end{vmatrix}$$

$$R_1 = R_1 + R_2$$

$$= U^2V \begin{vmatrix} 1 & 0 & 0 \\ V & 1 & 0 \\ VW & W & 1 \end{vmatrix}$$

$$= U^2V(1-0)$$

$$= U^2V$$

2. If $z = \dfrac{x^2+y^2}{x+y}$, show that $\left[\dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right]^2 = 4\left[1 - \dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right]$

$\Rightarrow$ Given:-

$$z = \frac{x^2+y^2}{x+y} \quad \text{———} (1)$$

$$\therefore \frac{\partial z}{\partial x} = \frac{(x+y)(2x) - (x^2+y^2)(1)}{(x+y)^2}$$

$$= \frac{2x^2 + 2xy - x^2 - y^2}{(x+y)^2}$$

$$= \frac{x^2 + 2xy - y^2}{(x+1)^2} \quad \text{———} (2)$$

Similarly,

$$\Rightarrow \frac{\partial z}{\partial y} = \frac{(x+y)(2y) - (x^2+y^2)(1)}{(x+y)^2}$$

$$= \frac{2xy + 2y^2 - x^2 - y^2}{(x+y)^2}$$

$$= \frac{-x^2 + 2xy + y^2}{(x+y)^2}$$

$$\therefore \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = \left[\frac{x^2+2xy-y^2}{(x+y)^2}\right] - \left[\frac{-x^2+2xy+y^2}{(x+y)^2}\right]$$

$$= \frac{1}{(x+y)^2}\left[x^2 + \cancel{2xy} - y^2 + x^2 - \cancel{2xy} - y^2\right]$$

$$= \frac{2x^2 - 2y^2}{(x+y)^2}$$

$$= \frac{2(x^2-y^2)}{(x+y)^2}$$

SDBS

$$\therefore \left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = \frac{4(x^2-y^2)^2}{(x+y)^4} \quad \text{——} (3)$$

Similarly,

$$4\left[1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right] = 4\left\{1 - \left[\frac{x^2-2x-y^2}{(x+y)^2}\right] - \left[\frac{-x^2+2xy+y^2}{(x+y)^2}\right]\right\}$$

$$= 4\left[\frac{(x+y)^2 - (x^2+2xy-y^2) - (-x^2+2xy+y^2)}{(x+y)^2}\right]$$

$$= 4\left[\frac{x^2 + \cancel{2xy} + y^2 - \cancel{x^2} - \cancel{2xy} + y^2 + \cancel{x^2} - 2xy - y^2}{(x+y)^2}\right]$$

$$= \frac{4(x^2 - 2xy + y^2)}{(x+y)^2}$$

$$= \frac{4(x-y)^2}{(x+y)^2}$$

$$= \frac{4(x-y)^2(x+y)^2}{(x+y)^2(x+y)^2}$$

$$= \frac{4(x^2-y^2)^2}{(x+y)^2} \quad \text{——— (4)}$$

$$\therefore \left(\frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y}\right)^2 = 4\left[1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right]$$

3. If $z = e^{ax+by} f(ax-by)$ show that

$$b\left[\frac{\partial z}{\partial x}\right] + a\left[\frac{\partial z}{\partial y}\right] = 2abz$$

$\Rightarrow \quad z = e^{ax+by} f(ax-by)$ ——— (1)

$$\therefore \frac{\partial z}{\partial x} = e^{ax+by}(a) f(ax-by) + e^{ax+by} f(ax-by)(a)$$

$$b\frac{\partial z}{\partial x} = ab\,e^{ax+by} f(ax-by) + ab\,e^{ax+by} f(ax-by) \quad \text{——— (2)}$$

Similarly,

$$\therefore \frac{\partial z}{\partial y} = e^{ax+by}\, b\, f(ax-by) + e^{ax+by} f(ax-by)(-b)$$

$$a\frac{\partial z}{\partial y} = ab\,e^{ax+by} f(ax-by) + e^{ax+by} f(ax-by)(-ab)$$

$$a\frac{\partial z}{\partial y} = ab.\,e^{ax+by} f(ax-by) - e^{ax+by} f(ax-by)(ab) \quad \text{——— (3)}$$

$(2) + (3) \Rightarrow$

$$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = ab\,e^{ax+by} f(ax-by) + ab\,e^{ax+by} f(ax-by) + ab\,e^{ax+by} f(ax-by) - ab\,e^{(ax+by)} f(ax-by)$$

$$= 2abe^{ax+by} f(ax-by) = 2abz$$

4. If $u = f\left[\frac{x}{z}, \frac{y}{z}\right]$ then prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

$\Rightarrow$ Let, $p = \frac{x}{z}$, $q = \frac{y}{z}$

then, $u = f[p, q]$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\cdot\frac{1}{z} + \frac{\partial u}{\partial q}\, 0$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{1}{z}\frac{\partial u}{\partial p}$$

$$\Rightarrow x\frac{\partial u}{\partial x} = \frac{x}{z}\frac{\partial u}{\partial p} \quad \text{———} (1)$$

$$\therefore \frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial y}$$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}(0) + \frac{\partial u}{\partial q}\left(\frac{1}{z}\right)$$

$$\Rightarrow y\frac{\partial u}{\partial y} = \frac{y}{z}\frac{\partial u}{\partial q} \quad \text{———} (2)$$

$$\therefore \frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial z}$$

$$\Rightarrow \frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\left(-\frac{x}{z^2}\right) + \frac{\partial u}{\partial q}\left[-\frac{y}{z^2}\right]$$

$$\Rightarrow z\frac{\partial u}{\partial z} = -\frac{x}{z^2}\frac{\partial u}{\partial p} - \frac{y}{z^2}\frac{\partial u}{\partial q} \quad \text{———} (3)$$

(1) + (2) + (3)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{x}{z}\cancel{\frac{\partial u}{\partial p}} + \frac{y}{z}\cancel{\frac{\partial u}{\partial q}} - \frac{x}{z}\cancel{\frac{\partial u}{\partial p}} - \frac{y}{z^2}\cancel{\frac{\partial u}{\partial p}}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

5. If $z = f(x+ay) + g(x-ay)$, prove that

$$\frac{\partial^2 z}{\partial y^2} = a^2\left[\frac{\partial^2 z}{\partial x^2}\right]$$

$\Rightarrow$ Given,

$$z = f(x+ay) + g(x-ay) \text{ ——— (1)}$$

diff partially w.r to x

$$\therefore \frac{\partial z}{\partial x} = f'(x+ay)(1) + g'(x-ay)(1)$$

$$\Rightarrow \frac{\partial}{\partial x}\left[\frac{\partial z}{\partial x}\right] = \frac{\partial^2 z}{\partial x^2} = f''(x+ay)(1) + g''(x-ay)(1)$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} = f''(x+ay) + g''(x-ay) \text{ ——— (2)}$$

$$\therefore \frac{\partial z}{\partial y} = f'(x+ay)a + g'(x-ay)(-a)$$

$$\Rightarrow \frac{\partial^2 z}{\partial y^2} = f''(x+ay)\,a.a + g''(x-ay)(-a)(-a)$$

$$\Rightarrow \frac{\partial^2 z}{\partial y^2} = a^2 f''(x+ay) + a^2 g''(x-ay)$$

$$\Rightarrow \frac{\partial^2 z}{\partial y^2} = a^2\left[f''(x+ay) + g''(x-ay)\right]$$

$$\Rightarrow \frac{\partial^2 z}{\partial y^2} = a^2\left[\frac{\partial^2 z}{\partial x^2}\right]$$

6. Find the extreme value of $f(x,y) = x^3+3x^2+4xy+y^2$

$\Rightarrow$ Given,

$$f(x,y) = x^3 + 3x^2 + 4xy + y^2 \text{ ——— (1)}$$

diff partially w.r to 'x'

$$\frac{\partial f}{\partial x} = 3x^2 + 6x + 4y \text{ ——— (2)}$$

diff partially w.r to 'y'

$$\frac{\partial f}{\partial y} = 0+0+4x+2y$$

$$\frac{\partial f}{\partial y} = 4x+2y \quad \text{——(3)}$$

For max | min

$$\frac{\partial f}{\partial x} = 0 \qquad\qquad \frac{\partial f}{\partial y} = 0$$

$(2) \Rightarrow \quad 3x^2+6x+4y$ ——(4)

$(3) \Rightarrow \quad 4x+2y=0$

$\Rightarrow \quad 2y = -4x$

$\Rightarrow \quad y = -2x$ ——(5)

$(4) \Rightarrow \quad 3x^2+6x+4(-2x)$

$\Rightarrow \quad 3x^2+6x-8x$

$\Rightarrow \quad 3x^2-2x$

$\Rightarrow \quad x(3x-2)=0$

$\Rightarrow \quad x=0 \quad , \quad 3x-2=0$

$\Rightarrow \quad x=0 \quad , \quad x = 2/3$

$\Rightarrow \quad y=0 \quad , \quad y = \frac{-4}{3}$

$\therefore \quad P(0,0) \qquad Q\left(\frac{2}{3}, -\frac{4}{3}\right)$

$$\therefore \quad \frac{\partial^2 f}{\partial x^2} = 6x+6 = A$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = 4 = B$$

$$\frac{\partial^2 f}{\partial y^2} = 2 = C$$

At $P(0,0)$

$A = 6(0)+6 = 6$

$B = 4$

$C = 2$

$\therefore AC - B^2 = 6(2) - 4^2$

$= 12 - 16$

$= -4 < 0$

$\therefore f(x,y)$ is neither max nor min at $P(0,0)$

At $Q\left[\frac{2}{3}, -\frac{4}{3}\right]$

$A = 6\left[\frac{2}{3}\right] + 6 = 10 > 0$

$B = 4$

$C = 2$

$AC - B^2 = (10)(2) - 4^2$

$= 20 - 16$

$= 4 > 0$

$\therefore f(x,y)$ is minima at $Q\left[\frac{2}{3}, -\frac{4}{3}\right]$

7. Examine the function $f(x,y) = xy(a-x-y)$ for extreme values:-

$\Rightarrow$ Given:-

$$f(x,y) = xy(a-x-y)$$

$$f(x,y) = axy - x^2y - xy^2 \quad \text{---(1)}$$

$$\therefore \frac{\partial f}{\partial x} = ay - 2xy - y^2 \quad \text{---(2)}$$

$$\frac{\partial f}{\partial y} = ax - x^2 - 2yx \quad \text{---(3)}$$

For max/min

$$\frac{\partial f}{\partial x} = 0 \quad , \quad \frac{\partial f}{\partial y} = 0$$

$$ay - 2xy - y^2 = 0$$

$$\Rightarrow y(a - 2x - y) = 0$$

$$\Rightarrow y = 0 \ , \ a - 2x - y = 0$$

$$\Rightarrow y = 0 \ , \ y = 2x - a$$

$$ax - x^2 - 2yx = 0$$

$$\Rightarrow x(a - x - 2y) = 0$$

$$\Rightarrow x = 0 \ , \ a - x - 2y = 0$$

$$\Rightarrow x = 0 \ , \ a - x - 2(2x - a) = 0$$

$$\Rightarrow x = 0 \ , \ a - x - 4x + 2a = 0$$

$$\Rightarrow x = 0 \ , \ -5x + 3a = 0$$

$$\Rightarrow x = 0 \ , \ x = \frac{3a}{5}$$

$$\therefore x = 0 \ , \ y = 0$$

$$P(0,0)$$

$$x = \frac{3a}{2} \qquad y = 2\left[\frac{3a}{5}\right] - a$$

$$= \frac{6a - 5a}{5} = \frac{a}{5}$$

$$Q\left[\frac{3a}{5}, \frac{a}{5}\right]$$

$$\frac{\partial^2 f}{\partial x^2} = -2y = A$$

$$\frac{\partial^2 f}{\partial x \partial y} = a - 2x - 2y = B$$

$$\frac{\partial^2 f}{\partial y^2} = -2x = C$$

$\therefore$ At $P(0,0)$

$A = 0$

$B = a$

$C = 0$

$$\therefore AC - B^2 = -a^2 < 0$$

$P(0,0)$ is saddle part.

At $Q\left(\frac{3a}{5}, \frac{a}{5}\right)$

$$A = -2\left[\frac{a}{5}\right] = \frac{-2a}{5} < 0$$

$$B = a - 2\left[\frac{3a}{5}\right] - \frac{2a}{5}$$

$$= a - \frac{6a}{5} - \frac{2a}{5}$$

$$= \frac{5a - 6a - 2a}{5}$$

$$= -\frac{3a}{5}$$

$$C = -2\left[\frac{3a}{5}\right]$$

$$= -\frac{6a}{5}$$

$$\therefore AC - B^2 = \left[\frac{-2a}{5}\right]\left[\frac{-6a}{5}\right] - \left[\frac{3a}{5}\right]^2$$

$$= \frac{12a^2}{25} - \frac{9a^2}{25}$$

$$= \frac{3a^2}{25} > 0$$

$\therefore$ $f(x,y)$ is maximum at $Q\left[\frac{3a}{5}, \frac{a}{5}\right]$

$$\therefore \max f = a\left[\frac{3a}{5}\right]\left[\frac{a}{5}\right] - \left[\frac{3a}{5}\right]^2\left[\frac{a}{5}\right] - \left[\frac{3a}{5}\right]\left[\frac{a}{5}\right]^2$$

$$= \cancel{\frac{3a^3}{25}} - \frac{9a^3}{25} - \cancel{\frac{3a^3}{25}}$$

$$\max f = \frac{-9a^3}{25}, \quad \forall \quad a > 0$$