NUMERICAL METHODS - 2

Numerical sol^n of ODE of 1st order and 1st degree:

Consider a differential eq^n of 1st order and 1st degree of the form $\frac{dy}{dx} = f(x, y)$ with initial cond^n $y(x_0) = y_0$. The problem of finding y is called IVP (initial value problem cond^n)

Numerical methods for solving an IVP:

(i) Taylor's series method:

Taylor series expansion of $y(x)$ about the point x_0 is given by $y(x) =$

$$y(x) = y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots$$

where y', y'', y''' etc denote derivatives of y at x_0

(1) Employ Taylor series method to find y at $x = 0.1$ correct to 4 decimal places for the initial value problem $\frac{dy}{dx} = x - y^2$, $y(0) = 1$

→ Taylor series formula.

$$y(x) = y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \cdots$$

given $y(0) = 1$

$\Rightarrow$ $y_0 = 1$ at $x_0 = 0$

$$y(x) = y(0) + x\, y'(0) + \frac{x^2}{2!} y''(0) + \cdots \quad \text{—— (1)}$$

Consider $y' = x - y^2$; $y'(0) = 0 - y(0)^2 = 0 - (1)^2 = \boxed{-1}$

$y'' = 1 - 2yy'$; $y''(0) = 1 - 2y(0)\,y'(0) = 1 - 2(1)(-1) = \boxed{3}$

$y''' = 0 - 2[yy'' + y'y']$; $y'''(0) = -2y(0)y''(0) - 2y'(0)^2$

$= -2yy'' - 2y'^2$ $\quad = -2(1)(3) - 2(-1)^2$

$= \boxed{-8}$

$y^{iv} = -2[yy''' + y''y' + y'y'' + y''y']$

$= ~~-2[yy~~$

$= -2[(1)(-8) + 3(-1) + (-1)(3) + 3(-1)]$

$= -2[-8-3-3-3] = \boxed{34}$

substitute in ①,

$$y(x) = 1 - x + \frac{3x^2}{2} - \frac{8x^3}{6} + \frac{34x^4}{24}$$

$$\boxed{y(0.1) = 0.9138}$$

② use Taylor series method to find y at $x = 0.1$ given $\frac{dy}{dx} = x^2y - 1$, $y(0) = 1$ considering upto 4^{th} degree.

$\rightarrow$ $y(x) = y_0 + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \ldots$

Given $y(0) = 1 \Rightarrow y_0 = 1$ at $x_0 = 0$

$y(0.1) = 1 + x.y'(0) + \frac{x^2}{2!}y''(0) + \ldots$ —— ①

Consider $y' = x^2y - 1$; $y'(0) = 0 - 1 = \boxed{-1}$

$y'' = x^2y' + 2xy$; $y''(0) = 0 + 0 = \boxed{0}$

$y''' = x^2y'' + 2xy' + 2xy' + 2y$; $y'''(0) = 0+0+0+2 = \boxed{2}$

$y^{iv} = x^2y''' + 2xy'' + 2[xy'' + y'] + 2[xy'' + y'] + 2y'$

$= x^2y''' + 2xy'' + 4xy'' + 4y' + 2y'$

$y^{iv}(0) = 0 + 0 + 0 + (-4) + (-2) = \boxed{-6}$

Sub in ①,

$$y(x) = 1 + x(-1) + \frac{x^2}{2}(0) + \frac{x^3}{6}(2) + \frac{x^4}{24}\times(-6)$$

$$y(x) = 1 - x + \frac{x^3}{3} - \frac{x^4}{4}$$

$$\boxed{y(0.1) = 0.9003}$$

③ Use Taylor series method to find $y(0.1)$ correct to ~~4~~ 5 decimal places given $\frac{dy}{dx} = \sqrt{x^2+y}$ with $y(0) = 0.8$

$$y(x) = y_0 + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \dots$$

given $y(0) = 0.8$ at $x_0 = 0$

$$y(x) = 0.8 + x\,y'(0) + \frac{x^2}{2!}y''(0) + \dots \quad ①$$

$y' = \sqrt{x^2+y}$; $y'(0) = \sqrt{0+0.8} = \boxed{0.89443}$

y' square on both sides,

$y'^2 = x^2 + y$

$2y'y'' = 2x + y'$; $2y'(0)\,y''(0) = 2(0) + y'(0)$

$$y''(0) = \frac{2(0)+y'(0)}{2y'(0)}$$

$$= \frac{0.89443}{2\times 0.89443} = \boxed{0.5}$$

$2y'y''' + 2y''y'' = 2 + y''$; $2\times 0.89443\,y'''(0) + 2(0.5)^2 = 2 + 0.5$

$2\times 0.89443\,y'''(0) = 2.5 - 0.5$

$y'''(0) = 1.11803$

$2[y'y^{iv} + y'''y''] + 2[y''y''' + y''y'''] = \cancel{0} + y'''$

~~$2[0.8(1.11803) + (0.5)(0.894$~~

$2(0.89443)(y^{iv}) + (1.11803)(0.5)] + 2[(0.5)(1.11803) + (1.11803)(0.5)] = 1.11803$

$= -1.24999$

$y(x) = 0.8 + 0.89443x - 0.25x^2 + 0.18634x^3 - 0.05208x^4$

$= -0.05208x^4 + 0.18634x^3 + 0.25x^2 + 0.089443x + 0.8$

$\boxed{y(0.1) = 0.89212}$

(4) Find y(0.1) and y(0.2) using taylor series method upto 4th degree terms given

$\frac{dy}{dx} = 2y + 3e^x, \quad y(0) = 0$

→ $y(x) = y_0 + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \cdots$

given $y(0) = 0$ at $x_0 = 0$

$y(x) = 0 + xy'(0) + \frac{x^2}{2!}y''(0) + \cdots$ —— (1)

$y' = 2y + 3e^x \;; \; y'(0) = 2y(0) + 3e^{x_0} = \boxed{3}$

$y'' = 2y' + 3e^x \;; \; y''(0) = 2(3) + 3e^0 = \boxed{9}$

$y''' = 2y'' + 3e^x \;; \; y'''(0) = 2(9) + 3e^0 = \boxed{21}$

$y^{IV} = 2y''' + 3e^x \;; \; y^{IV}(0) = 2(21) + 3e^0 = \boxed{45}$

$y(x) = 0 + x \times 3 + \frac{x^2}{2} \times 9 + \frac{x^3}{6} \times 21 + \frac{x^4}{24} \times 45$

$y(0.1) = 0.3487$

$y(0.2) = 0.8110$

(5) Solve $y' = x + y^2$, $y(0) = 1$ using Taylor series method at $x = 0.1, 0.2$ considering upto 4th degree term

$\rightarrow$ $y(x) = y_0 + (x - x_0)\, y_0'(x_0) + \frac{(x-x_0)}{2!}\, y''(x_0) + \ldots$

given $y(0) = 1$ at $x_0 = 0$

$y(x) = \cancel{y}\, 1 + x y'(0) + \frac{x^2}{2!}\, y''(0) + \ldots$ —— ①

$y' = x + y^2$; $y'(0) = 0 + 1^2 = \boxed{1}$

$y'' = 1 + 2yy'$; $y''(0) = 1 + 2(1)(1) = \boxed{3}$

$y''' = 0 + 2[yy'' + y'y']$; $\cancel{y'''(0) = 0 + 2(1) = 2}$

$y'''(0) = 2[1(3) + 1^2] = \boxed{8}$

$y^{IV} = 0 + 2[yy''' + y''y' + y'y'' + y''y']$

$y^{IV}(0) = 2[1(8) + 3(1) + 1(3) + 3(1)] = \boxed{34}$

$y(x) = 1 + x \times 1 + \frac{x^2}{2} \times 3 + \frac{x^3}{6} \times 8 + \frac{x^4}{24} \times 34$

$y(0.1) = 1.1165$

$y(0.2) = 1.2729$

② Modified Euler's method:

Consider IVP $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$. To find

y at x_1 i.e to find $y(x_1) = y_1$ where $x_1 = x_0 + h$

h is the step size.

step (i) Apply Euler's formula given by

$y_1(0) = y_0 + h f(x_0, y_0)$ regarded as initial approxiamation for y_1

step (ii) Apply modified Euler's formula successively to obtain better approximations for y_1

Formulae: $y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right] \text{ etc}$$

① Using modified Euler's method, find y at $x = 0.1$ given $\frac{dy}{dx} = 3x + \frac{y}{2}$, $y(0) = 1$, $h = 0.1$

perform 2 iterations

→ given $\frac{dy}{dx} = f(x,y) = 3x + \frac{y}{2}$

$y(0) = 1 \Rightarrow y_0 = 1$ at $x_0 = 0$

$\therefore f(x_0, y_0) = 3x_0 + \frac{y_0}{2} = 3(0) + \frac{1}{2} = \boxed{0.5}$

To find $y(0.1)$:

Here $x_1 = x_0 + h = 0 + 0.1 = 0.1$

($\therefore$ It is a one stage problem)

Euler's formula,

$y_1(0) = y_0 + h f(x_0, y_0)$

$= 1 + 0.1(0.5) \Rightarrow \boxed{y_1^{(0)} = 1.05}$

Modified Euler's formulae,

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)}) \right]$$

I $$y_1^{(1)} = 1 + \frac{0.1}{2}\left[0.5 + 3x_1 + \frac{y_1^{(0)}}{2} \right]$$

$$y_1^{(1)} = 1 + 0.05\left[0.5 + 3(0.1) + \frac{1.05}{2} \right]$$

$$\boxed{y_1^{(1)} = 1.0663}$$

II $$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)}) \right]$$

$$y_1^{(2)} = 1 + \frac{0.1}{2}\left[0.5 + 3x_1 + \frac{y_1^{(1)}}{2} \right]$$

$$y_1^{(2)} = 1 + 0.05\left[0.5 + 3(0.1) + \frac{1.0663}{2} \right]$$

$$\boxed{y_1^{(2)} = 1.0667}$$

Thus $y(0.1) = 1.0667$ after 2 iterations.

2. Use modified Euler's method to solve $\frac{dy}{dx} = \log_{10}(x+y)$, $y(0) = 2$, to find $y(0.2)$

→ $h = 0.2$ since $x_0 = 0$ and $x_1 = 0.2$

$\frac{dy}{dx} = f(x,y) = \log_{10}(x+y)$

$y(0) = 2 \Rightarrow y_0 = 2$ at $x_0 = 0$

$\therefore f(x_0, y_0) = \log_{10}(x_0 + y_0)$

$= \log_{10}(0+2) = \boxed{0.3010}$

To find $y(0.2)$:

Euler's formula.

$y_1(0) = y_0 + hf(x_0, y_0)$

$y_1^{(0)} = 2 + 0.2(0.3010) = \boxed{2.0602}$

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Modified Euler's formulae,

I $y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0,y_0) + f(x_1, y_1^{(0)})]$

$y_1^{(1)} = 2 + \frac{0.2}{2}[0.3010 + \log_{10}(x_1 + y_1^{(0)})]$

$y_1^{(1)} = 2 + 0.1[0.3010 + \log_{10}(0.2 + 2.0602)]$

$\boxed{y_1^{(1)} = 2.0655}$

II $y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0,y_0) + f(x_1, y_1^{(1)})]$

$y_1^{(2)} = 2 + \frac{0.2}{2}[0.3010 + \log_{10}(x_1 + y_1^{(0)})]$

$= 2 + 0.1[0.3010 + \log_{10}(0.2 + 2.0655)]$

$\boxed{y_1^{(2)} = 2.0656}$

III $y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0,y_0) + f(x_1, y_1^{(2)})]$

$= 2 + \frac{0.2}{2}[0.3010 + \log_{10}(0.2 + 2.0656)]$

$\boxed{y_1^{(3)} = 2.0656}$

Thus $y(0.2) = 2.0656$ after 3 iterations.

③ Use modified Euler's formula to solve $\frac{dy}{dx} = x + |\sqrt{y}|$ at $x = 0.2$ by taking $h = 0.2$ given $y(0) = 1$

$\frac{dy}{dx} = x + |\sqrt{y}|$ means that ~~modulu~~

Modulus sign indicates only +ve values of $\sqrt{y}$ to be taken

i.e $f(x,y) = x + \sqrt{y}$

given $y(0) = 1 \Rightarrow y_0 = 1$ at $x_0 = 0$

$\boxed{h = 0.2}$

$\therefore f(x_0, y_0) = x_0 + \sqrt{y_0} = 0 + \sqrt{1} = \boxed{1}$

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Euler's formula,

$y_1^{(0)} = y_0 + hf(x_0, y_0)$

$= 1 + 0.2(1) \Rightarrow \boxed{y_1^{(0)} = 1.2}$

Modified Euler's formula,

I $y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})]$

$= 1 + \frac{0.2}{2}[1 + x_1 + \sqrt{y_1^{(0)}}]$

$= 1 + 0.1[1 + 0.2 + \sqrt{1.2}] \Rightarrow \boxed{y_1^{(1)} = 1.2295}$

II $y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$

$= 1 + 0.1[1 + 0.2 + \sqrt{1.2295}] \Rightarrow \boxed{y_1^{(2)} = 1.2309}$

III $y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(2)})]$

$= 1 + 0.1[1 + 0.2 + \sqrt{1.2309}] \Rightarrow \boxed{y_1^{(3)} = 1.2309}$

Thus, $y = 1.2309$ at $x = 0.2$ //

4 Using modified Euler's method find $y(0.1)$ given $\frac{dy}{dx} = x^2 + y$, $y(0) = 1$ taking $h = 0.05$ and perform two modifications in each stage.

→ 1st stage: $\frac{dy}{dx} = f(x, y) = x^2 + y$

$y(0) = 1 \Rightarrow y_0 = 1$ at $x_0 = 0$

$\therefore f(x_0, y_0) = x_0^2 + y_0 = 0 + 1 = \boxed{1}$

given $\boxed{h = 0.05}$

Euler's formula,

$y_1^{(0)} = y_0 + hf(x_0, y_0)$

$= 1 + 0.05(1) \Rightarrow \boxed{y_1^{(0)} = 1.05}$

Modified Euler's formula,

I. $y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})]$

$= 1 + \frac{0.05}{2}[1 + x_1^2 + y_1^{(0)}]$ $\quad x_1 = x_0 + h = 0 + 0.05 = 0.05$

$= 1 + 0.0250\ [1 + (0.05)^2 + 1.05]$

$\boxed{y_1^{(1)} = 1.0513}$

II $y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$

$= 1 + 0.0250\ [1 + (0.05)^2 + 1.0513]$

$\boxed{y_1^{(2)} = 1.0513}$

Thus $y(0.05) = 1.0513$.

$x_1 = x_0 + x_0 = 0.05 + 0.05$
$\boxed{x_1 = 0.1}$
To find y at $x = 0.1$ i.e to find $y(0.1)$

2nd stage : Let $x_0 = 0.05$, $y_0 = 1.0513$

$f(x_0, y_0) = x_0^2 + y_0 = (0.05)^2 + 1.0513 = 1.0538$

given $\boxed{h = 0.05}$

Euler's formula,

$y_1^{(0)} = y_0 + h\, f(x_0, y_0)$

$= 1.0513 + 0.05(1.0538) = \boxed{y_1^{(0)} = 1.1040}$

Modified Euler's formula,

I $y_1^{(1)} = 1.0513 + 0.0250\ [1.0538 + (0.1)^2 + 1.1040]$

$\boxed{y_1^{(1)} = 1.1055}$

II $y_1^{(2)} = 1.0513 + 0.0250\ [1.0538 + (0.1)^2 + 1.1055]$

$\boxed{y_1^{(2)} = 1.1055}$

(5) given $\frac{dy}{dx} = 1 + \frac{y}{x}$, $y(1) = 2$ compute $y(1.2)$ by modified Euler's method, taking $h = 0.1$

given $\frac{dy}{dx} = 1 + \frac{y}{x}$

$y(1) = 2 \Rightarrow y_0 = 2$ at $x_0 = 1$

$f(x_0, y_0) = 1 + \frac{y}{x} = 1 + \frac{2}{1} = \boxed{3}$

Given $\boxed{h = 0.1}$

Euler's formula,

$y_1^{(0)} = y_0 + h f(x_0, y_0) = 2 + 0.1(3) = \boxed{y_1^{(0)} = 2.3}$

Modified Euler's formula,

$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)}) \right]$

$x_1 = x_0 + h$
$= 1 + 0.1$
$= 1.1$

$= 2 + \frac{0.1}{2}\left[3 + 1 + \frac{2.3}{1.1} \right]$

$= 2.3045$

$y_1^{(2)} = 2 + \frac{0.1}{2}\left[3 + 1 + \frac{2.3045}{1.1} \right] \Rightarrow 2.3048$

$y_1^{(3)} = 2 + \frac{0.1}{2}\left[3 + 1 + \frac{2.3048}{1.1} \right] \Rightarrow 2.3048$ //

2nd stage: Let $x_0 = 1.1$, $y_0 = 2.3048$

$x_1 = 1.1 + 0.1 = 1.2$

$f(x_0, y_0) = 1 + \frac{2.3048}{1.1} = 3.0953$

Euler's: $y_1^{(0)} = 2.3048 + 0.1(3.0953) \Rightarrow 2.6143$

Modified euler's: $y_1^{(1)} = 2.3048 + \frac{0.1}{2}\left[3.0953 + 1 + \frac{2.6143}{1.2} \right] = 2.6185$

$y_1^{(2)} = 2.3048 + \frac{0.1}{2}\left[3.0953 + 1 + \frac{2.6185}{1.2} \right] = 2.6187$

$y_1^{(3)} = 2.3048 + \frac{0.1}{2}\left[3.0953 + 1 + \frac{2.6187}{1.2} \right] = 2.6187$

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Runge Kutta method of 4th order:

consider IVP, $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$ to find $y(x_1) = y_1$,
RK method of 4th order is given by
$y(x_1) = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$ where

$K_1 = h f(x_0, y_0)$
$K_2 = h f(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2})$
$K_3 = h f(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2})$
$K_4 = h f(x_0 + h, y_0 + K_3)$

(1) Use RK method of 4th order to find y at $x = 0.5$ for an IVP $(x+y)\frac{dy}{dx} = 1$, $y(0.4) = 1$

→ given $\frac{dy}{dx} = f(x,y) = \frac{1}{x+y}$

given $y(0.4) = 1 \Rightarrow y_0 = 1$ at $x_0 = 0.4$

<u>To find $y(0.5)$:</u>

$h = 0.1$

RK method of 4th order:

$y(x_1) = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$ —— (1)

where $K_1 = h f(x_0, y_0)$
$= 0.1 f(0.4, 1)$
$= 0.1 \times \frac{1}{0.4+1} \Rightarrow \boxed{K_1 = 0.0714}$

$K_2 = h f(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2})$
$= 0.1 f\left(0.4 + \frac{0.1}{2}, 1 + \frac{0.0714}{2}\right)$

$= 0.1 f(0.45, 1.0357)$
$= 0.1 \times \frac{1}{0.45 + 1.0357} \Rightarrow \boxed{K_2 = 0.0673}$

$K_3 = h\, f\left(x_0 + \frac{h}{2},\ y_0 + \frac{K_2}{2}\right)$

$= 0.1\, f\left(0.4 + \frac{0.1}{2},\ 1 + \frac{0.0673}{2}\right)$

$= 0.1\, f(0.45,\ 1.0337)$

$= 0.1 \times \frac{1}{0.45 + 1.0337} \Rightarrow \boxed{K_3 = 0.0674}$

$K_4 = h\, f(x_0 + h,\ y_0 + K_3)$

$= 0.1\, f(0.4 + 0.1,\ 1 + 0.0674)$

$= 0.1\, f(0.5,\ 1.0674)$

$= 0.1 \times \frac{1}{0.5 + 1.0674} \Rightarrow \boxed{K_4 = 0.0638}$

substituting in eq^n 1,

$y(0.5) = 1 + \frac{1}{6}\left[0.0714 + 2 \times 0.0673 + 2 \times 0.0674 + 0.0638\right]$

$\boxed{y(0.5) = 1.0674}$

2. Use RK method of 4th order to find $y(0.1)$ with $h = 0.1$ for the eq^n $\frac{dy}{dx} + y + xy^2 = 0$, $y(0) = 1$

→ given $\frac{dy}{dx} = f(x, y) = -y - xy^2$

given $y(0) = 1 \Rightarrow y_0 = 1$ at $x_0 = 0$

To find $y(0.1)$:

$h = 0.1$

$y(x_1) = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$ —— 1

where, $K_1 = h\, f(x_0, y_0)$

$= 0.1\left[-(1)^2 - (0)(1)^2\right] = \boxed{K_1 = -0.1}$

$K_2 = h f\left(x_0 + \frac{h}{2},\ y_0 + \frac{K_1}{2}\right)$

$= 0.1\, f\left[0 + \frac{0.1}{2},\ 1 + \frac{(-0.1)}{2}\right]$

$= 0.1\, f(0.05,\ 0.95)$

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$= 0.1 \times [-0.95 - (0.05)(0.95)^2] \Rightarrow \boxed{K_2 = -0.0995}$

$K_3 = h f(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2})$

$= 0.1 f(0 + \frac{0.1}{2}, 1 + \frac{(-0.0995)}{2})$

$= 0.1 f(0.05, 0.9503)$

$= 0.1 \times [-0.9503 - (0.05)(0.9503)^2] \Rightarrow \boxed{K_3 = -0.0995}$

$K_4 = h f(x_0 + h, y_0 + K_3)$

$= 0.1 f(0 + 0.1, 1 + (-0.0995))$

$= 0.1 f(0.1, 0.9005)$

$= 0.1 \times [-0.9005 - (0.1)(0.9005)^2] \Rightarrow \boxed{K_4 = -0.0982}$

sub in ①,

$y(0.1) = 1 + \frac{1}{6}(-0.1 + 2(-0.0995) + 2(-0.0995) - 0.0982)$

$\boxed{y(0.1) = 0.9006}$

③ Apply RK method of 4th order to find $y(0.2)$ for an IVP $\frac{dy}{dx} = \frac{y-x}{y+x}$, $y(0) = 1$ taking $h = 0.2$

given $\frac{dy}{dx} = f(x,y) = \frac{y-x}{y+x}$

given $y(0) = 1 \Rightarrow y_0 = 1$ at $x_0 = 0$

To find $y(0.2)$:

$h = 0.2$

$y(x_1) = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$ —— ①

where, $K_1 = h f(x_0, y_0)$

$= 0.2 \left[\frac{1-0}{1+0}\right] \Rightarrow \boxed{K_1 = 0.2}$

$K_2 = h f(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2})$

$= 0.2 f(0 + \frac{0.2}{2}, 1 + \frac{0.2}{2})$

$= 0.2 f(0.1, 1.1)$

$= 0.2 \times \left[\frac{1.1 - 0.1}{1.1 + 0.1}\right] \Rightarrow \boxed{K_2 = 0.1667}$

$K_3 = hf\left(x_0+\frac{h}{2}, y_0+\frac{K_2}{2}\right)$

$= 0.2\, f\left(0+\frac{0.2}{2}, 1+\frac{0.1667}{2}\right)$

$= 0.2\, f(0.1, 1.0834)$

$= 0.2\left[\frac{1.0834-0.1}{1.0834+0.1}\right] \Rightarrow \boxed{K_3 = 0.1662}$

$K_4 = hf(x_0+h, y_0+K_3)$

$= 0.2\, f(0+0.2, 1+0.1662)$

$= 0.2\, f(0.2, 1.1662)$

$= 0.2\left[\frac{1.1662-0.2}{1.1662+0.2}\right] \Rightarrow \boxed{K_4 = 0.1414}$

Sub in ①,

$y(0.2) = 1+\frac{1}{6}\left[0.2+2\times0.1667+2\times0.1662+0.1414\right]$

$\boxed{y(0.2) = 1.1679}$

④ Apply RK method of 4th order to find $y(0.2)$ for an IVP $\frac{dy}{dx} = \frac{y^2-x^2}{y^2+x^2}$, $y(0)=1$ taking $h=0.2$

given $\frac{dy}{dx} = \frac{y^2-x^2}{y^2+x^2}$

given $y(0)=1 \Rightarrow y_0=1$ at $x_0=0$

To find $y(0.2)$:

$h=0.2$

$y(x_1) = y_0+\frac{1}{6}\left[K_1+2K_2+2K_3+K_4\right]$ —— ①

Where, $K_1 = hf(x_0, y_0)$

$= 0.2\left[\frac{1^2-0^2}{1^2+0^2}\right] \Rightarrow \boxed{K_1 = 0.2}$

$K_2 = hf\left(x_0+\frac{h}{2}, y_0+\frac{K_1}{2}\right)$

$= 0.2\, f\left(0+\frac{0.2}{2}, 1+\frac{0.2}{2}\right)$

$= 0.2\left[\frac{1.1^2-0.1^2}{1.1^2+0.1^2}\right] \Rightarrow \boxed{K_2 = 0.1967}$

$$K_3 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{K_2}{2}\right)$$

$$= 0.2\, f\left(0 + \frac{0.2}{2},\ 1 + \frac{0.1967}{2}\right)$$

$$= 0.2\, f(0.1,\ 1.0984)$$

$$= 0.2\left[\frac{(1.0984)^2 - (0.1)^2}{(1.0984)^2 + (0.1)^2}\right] \Rightarrow \boxed{K_3 = 0.1967}$$

$$K_4 = hf(x_0 + h,\ y_0 + K_3)$$

$$= 0.2\, f(0 + 0.2,\ 1 + 0.1967)$$

$$= 0.2 f(0.2,\ 1.1967)$$

$$= 0.2\left[\frac{(1.1967)^2 - (0.2)^2}{(1.1967)^2 + (0.2)^2}\right] \Rightarrow \boxed{K_4 = 0.1891}$$

sub in (1),

$$y(0.2) = 1 + \frac{1}{6}[0.2 + 2\times 0.1967 + 2\times 0.1967 + 0.1891]$$

$$\boxed{y(0.2) = 1.1960}$$

(5) Using RK method of 4th order find y at $x = 0.1$ for an IVP $\frac{dy}{dx} = 2y + 3e^x$, $y(0) = 0$

given $\frac{dy}{dx} = 2y + 3e^x$

given $y(0) = 0 \Rightarrow y_0 = 0$ at $x_0 = 0$

To find $y(0.1)$:

$\boxed{h = 0.1}$

$$y(x_1) = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4] \quad \text{—— (1)}$$

Where, $K_1 = hf(x_0, y_0)$

$$= 0.1[2(0) + 3e^0] \Rightarrow \boxed{K_1 = 0.3}$$

$$K_2 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{K_1}{2}\right)$$

$0.1 f(0.05, 0.15)$ $= 0.1\, f\left(0 + \frac{0.1}{2},\ 0 + \frac{0.3}{2}\right)$

$$= 0.1\left[2(0.15) + 3e^{0.05}\right] \Rightarrow \boxed{K_2 = 0.3454}$$

$$K_3 = hf\left(x_0+\frac{h}{2},\ y_0+\frac{K_2}{2}\right)$$
$$= 0.1\, f\left[0+\frac{0.1}{2},\ 0+\frac{0.3454}{2}\right]$$
$$= 0.1\, f(0.05,\ 0.1727)$$
$$= 0.1\, f\left[2(0.1727)+3e^{0.05}\right] \Rightarrow \boxed{K_3 = 0.3499}$$

$$K_4 = hf(x_0+h,\ y_0+K_3)$$
$$= 0.1 f(0+0.1,\ 0+0.3499)$$
$$= 0.1 f(0.1,\ 0.3499)$$
$$= 0.1\left[2(0.3499)+3e^{0.1}\right] \Rightarrow \boxed{K_4 = 0.4015}$$

sub in ①,

$$y(0.1) = 0+\frac{1}{6}\left[0.3+2\times 0.3454+2\times 0.3499+0.4015\right]$$
$$\boxed{y(0.1) = 0.3487}$$

Predictor-corrector method:

Consider differential eqn $\frac{dy}{dx} = f(x,y)$ with the set of 4 pre determined values of y : $y(x_0)=y_0$, $y(x_1)=y_1$, $y(x_2)=y_2$, $y(x_3)=y_3$.
Where x_0, x_1, x_2, x_3 are equally spaced values of x with width 'h' to find $y(x_4)=y_4$.

Milne's predictor-corrector method (3 iterations)

predictor formula: $y_4^{(p)} = y_0 + \frac{4h}{3}\left(2y_1' - y_2' + 2y_3'\right)$

Corrector formula: $y_4^{(c)} = y_2 + \frac{h}{3}\left(y_2' + 4y_3' + y_4'\right)$

① given $\frac{dy}{dx} = 2e^x - y$, $y(0)=2$, $y(0.1)=2.010$, $y(0.2)=2.040$, $y(0.3)=2.090$. compute y at $x=0.4$ by Milne's method. Apply corrector formula twice.

x	y	$y' = 2e^x - y$
$x_0 = 0$	$y_0 = 2$	$y_0' = 0$
$x_1 = 0.1$	$y_1 = 2.010$	$y_1' = 0.2003$
$x_2 = 0.2$	$y_2 = 2.040$	$y_2' = 0.4028$
$x_3 = 0.3$	$y_3 = 2.090$	$y_3' = 0.6097$
$x_4 = 0.4$	$y_4 = ?$ 2.1623	$y_4' = ?$ 0.8213

Milne's predictor formula:

$$y_4^{(P)} = y_0 + \frac{4h}{3}\left(2y_1' - y_2' + 2y_3'\right)$$

$$y_4^{(P)} = 2 + \frac{4(0.1)}{3}\left[2\times 0.2003 - 0.4028 + 2\times 0.6097\right]$$

$$\boxed{y_4^{(P)} = 2.1623}$$

$$\therefore\ y_4' = 2e^{x_4} - y_4 = 2e^{0.4} - 2.1623 \Rightarrow \boxed{y_4' = 0.8213}$$

Milne's Corrector formula:

$$y_4^{(C)} = y_2 + \frac{h}{3}\left(y_2' + 4y_3' + y_4'\right)$$

$$y_4^{(C)} = 2.040 + \frac{0.1}{3}\left[0.4028 + 4\times 0.6097 + 0.8213\right]$$

$$\boxed{y_4^{(C)} = 2.1621}$$

$$y_4' = 2e^{x_4} - y_4 = 2e^{0.4} - 2.1621 \Rightarrow \boxed{y_4' = 0.8215}$$

$$y_4^{(C)} = 2.040 + \frac{0.1}{3}\left[0.4028 + 4\times 0.6097 + 0.8215\right]$$

$$\boxed{y_4^{(C)} = 2.1621}$$

Thus, $y(0.4) = 2.1621$ //

② given $\frac{dy}{dx} = x - y^2$, $y(0) = 0$, $y(0.2) = 0.02$, $y(0.4) = 0.0795$, $y(0.6) = 0.1762$. Compute y at $x = 0.8$ by milne's method.

→

x	y	$y' = x - y^2$
$x_0 = 0$	$y_0 = 0$	$y_0' = 0$
$x_1 = 0.2$	$y_1 = 0.02$	$y_2' = 0.1996$
$x_2 = 0.4$	$y_2 = 0.0795$	$y_2' = 0.3937$
$x_3 = 0.6$	$y_3 = 0.1762$	$y_3' = 0.5690$
$x_4 = 0.8$	$y_4 = ?$	$y_4' = ?$

Milne's predictor formula:

$$y_4^{(P)} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right]$$

$$= 0 + \frac{4(0.2)}{3}\left[2 \times 0.1996 - 0.3937 + 2 \times 0.5690\right]$$

$$\boxed{y_4^{(P)} = 0.3049}$$

$$y_4' = x_4 - y_4^2 = 0.8 - (0.3049)^2 = \underline{0.7070}$$

milne's corrector formula:

$$y_4^{(C)} = y_2 + \frac{h}{3}\left(y_2' + 4y_3' + y_4'\right)$$

$$= 0.0795 + \frac{0.2}{3}\left[0.3937 + 4 \times 0.5690 + 0.7070\right]$$

$$\boxed{y_4^{(C)} = 0.3046}$$

$$y_4' = 0.8 - (0.3046)^2 = \underline{0.7072}$$

again milne's corrector formula,

$$y_4^{(C)} = 0.0795 + \frac{0.2}{3}\left[0.3937 + 4 \times 0.5690 + 0.7072\right]$$

$$\boxed{y_4^{(C)} = 0.3046}$$

Thus $y(0.8) = 0.3046$ //

(3) Apply milne's predictor-corrector method to find $y(1.4)$ from $\frac{dy}{dx} = x^2 + \frac{y}{2}$ given that $y(1) = 2$, $y(1.1) = 2.2156$, $y(1.2) = 2.4549$, $y(1.3) = 2.7514$

x	y	$y' = x^2 + y/2$
$x_0 = 1$	$y_0 = 2$	$y'_{(0)} = 2$
$x_1 = 1.1$	$y_1 = 2.2156$	$y'_{(1)} = 2.3178$
$x_2 = 1.2$	$y_2 = 2.4549$	$y'_2 = 2.6675$
$x_3 = 1.3$	$y_3 = 2.7514$	$y'_3 = 3.0657$
$x_4 = 1.4$	$y_4 = ?$	$y'_4 = ?$

Milne's predictor formula:

$$y_4^{(P)} = y_0 + \frac{4h}{3}[2y'_1 - y'_2 + 2y'_3]$$

$$= 2 + \frac{4(0.1)}{3}[2\times 2.3178 - 2.6675 + 2\times 3.0657]$$

$$\boxed{y_4^{(P)} = 3.0799}$$

$$y'_4 = x_4^2 + \frac{y_4}{2} = (1.4)^2 + \frac{3.0799}{2} = \underline{3.5}$$

Milne's corrector formula:

$$y_4^{(C)} = y_2 + \frac{h}{3}[y'_2 + 4y'_3 + y'_4)$$

$$= 2.4549 + \frac{0.1}{3}[2.6675 + 4\times 3.0657 + 3.\overset{5}{\cancel{0799}}]$$

$$\boxed{y_4^{(C)} = 3.0692}$$

$$y'_4 = (1.4)^2 + \frac{3.0692}{2} = \underline{3.4946}$$

again milne's corrector formula,

$$y_4^{(C)} = 2.4549 + \frac{0.1}{3}[2.6675 + 4\times 3.0657 + 3.4946]$$

$$\boxed{y_4^{(C)} = 3.0691}$$

$$y'_4 = \underline{3.4946}$$

$$y_4^{(C)} = 2.4549 + \frac{0.1}{3}[2.6675 + 4\times 3.0657 + 3.4946]$$

$$\boxed{y_4^{(C)} = 3.0691}$$

Thus $y(1.4) = 3.0691$ //

④ The following table gives solⁿ of $5xy' + y^2 - 2 = 0$. Find the value of y at $x = 4.4$ using Milne's predictor - corrector method.

x	4	4.1	4.2	4.3	4.4
y	1	1.0049	1.0097	1.0143	1.0187 ?

x	y	$y' = \frac{-y^2+2}{5x}$
$x_0 = 4$	$y_0 = 1$	$y_0' = 0.05$
$x_1 = 4.1$	$y_1 = 1.0049$	$y_1' = 0.0483$
$x_2 = 4.2$	$y_2 = 1.0097$	$y_2' = 0.0467$
$x_3 = 4.3$	$y_3 = 1.0143$	$y_3' = 0.0452$
$x_4 = 4.4$	$y_4 = 1.0187$?	$y_4' = ?$
~~$x_5 = 4.5$~~	~~y_5~~ $= ?$	

Milne's predictor formula,

$$y_4^{(P)} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right]$$

$$= 1 + \frac{4(0.1)}{3}\left[2\times 0.0483 - 0.0467 + 2\times 0.0452\right]$$

$$\boxed{y_4^{(P)} = 1.0187}$$

$$y_4' = \frac{-(1.0187)^2 + 2}{5\times 4.4} = \underline{0.0437}$$

Milne's corrector formula,

$$y_4^{(C)} = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right]$$

$$= 1.0097 + \frac{0.1}{3}\left[0.0467 + 4\times 0.0452 + 0.0437\right]$$

$$\boxed{y_4^{(C)} = \cancel{1.00}\ 1.0187}$$

Apply Milne's predictor - corrector formula to compute $y(0.4)$ given $\frac{dy}{dx} = 2e^x y$ and following table

x	0	0.1	0.2	0.3
y	2.4	2.473	3.129	4.059

x	y	$y' = 2e^x y$
$x_0 = 0$	$y_0 = 2.4$	$y_0' = 4.8$
$x_1 = 0.1$	$y_1 = 2.473$	$y_1' = 5.4662$
$x_2 = 0.2$	$y_2 = 3.129$	$y_2' = 7.6435$
$x_3 = 0.3$	$y_3 = 4.059$	$y_3' = 10.9582$
$x_4 = 0.4$	$y_4 = ?$	$y_4' = ?$

Milne's predictor formula,

$$y_4^{(P)} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right]$$

$$y_4^{(P)} = 2.4 + \frac{4(0.1)}{3}\left[2\times 5.4662 - 7.6435 + 2\times 10.9582\right]$$

$$y_4^{(P)} = 5.7607$$

$$y_4' = 2e^{0.4}\times 5.7607 = 17.1879$$

Milne's Corrector formula.

$$y_4^{(C)} = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right]$$

$$y_4^{(C)} = 3.129 + \frac{0.1}{3}\left[7.6435 + 4\times 10.9582 + 17.1879\right]$$

$$\boxed{y_4^{(C)} = 5.4178}$$

$$y_4' = 2e^{0.4}\times 5.4178 = \underline{16.1648}$$

$$y_4^{(C)} = 3.129 + \frac{0.1}{3}\left[7.6435 + 4\times 10.9582 + 16.1648\right]$$

$$\boxed{y_4^{(C)} = 5.3837}$$

$$y_4' = 2e^{0.4}\times 5.3837 = \underline{16.0631}$$

$$y_4^{(C)} = 3.129 + \frac{0.1}{3}\left[7.6435 + 4\times 10.9582 + 16.0631\right]$$

$$\boxed{y_4^{(C)} = 5.3803}$$