VECTOR SPACES AND LINEAR TRANSFORMATION

Let F be a field. A vector space over F, is a non-empty set V together with two operations such that for each $u, v \in V$ there is a unique element $u+v \in V$ and for each $\alpha \in F$ and $u \in V$ there is a unique element $\alpha u \in V$ and it satisfies the following conditions:

Abelian group

I. (i) $u+(v+w) = (u+v)+w$, for all $u, v, w \in V$ (Associativity)

(ii) $u+v = v+u$, for all $u, v \in V$ (commutativity)

(iii) There exists an element $0 \in V$ such that $u+0 = u = 0+u$, for all $u \in V$ (Existance of identity element)

(iv) For each $u \in V$, there exists a unique element $-u \in V$ such that $u+(-u) = 0 = (-u)+u$ (Existence of additive inverse)

II. There is an external composition in V over F called scalar multiplication.
i.e $\forall \alpha \in F$ and $u \in V \Rightarrow \alpha . u \notin V$
In otherwords V is closed with respect to scalar multiplication.

III. The two compositions i.e. vector addition and scalar multiplication satisfy the following postulates

$\forall a, b \in F$ and $v, w \in V$

(i) $(a+b)v = a.v + b.v$

(ii) $a.(b.v) = (ab).v$

(iii) $a.(v+w) = a.v + a.w$

(iv) $1.(v) = v$

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vectors in R^n :

The set of all ordered triples (a, b, c) of real numbers is called Euclidean 3-space and is denoted by R^3 and the set of all n-triples of real no's denoted by R^n, is called Euclidean n-space.

eg: The ordered pair $(2, -3)$ belongs to R^2; it is a 2-tuple of dimension two
The ordered triple $(7, 3, 6)$ belongs to R^3; it is a 3-tuple dimension three

problems on basis of vector spaces

① prove that the set of all vectors in a plane over the field of real numbers is vector space wrt vector addition and scalar multiplication.

Set of all vectors and R be the field of real numbers.
The elements of V are the ordered pair (x, y) where x and y belongs to R, $x, y \in R$
$V = \{(x, y) \mid x, y \in R\}$

I. To show that $(V, +)$ is an abelian group.

(i) Associativity: we have for all, $u, v, w \in V$
$u + (v + w) = (u + v) + w$.

(ii) commutativity: $\forall u, v \in V$, $u + v = v + u$

(iii) Existence of additive identity: For every vector $u \in V$, there exists a zero vector $0 \in V$, such that $u + 0 = 0 + u = u$

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(iv) Existence of additive inverse: For every vector $u \in V$, there exists a vector $-u \in V$ such that $u+(-u) = -u+u = 0$.

Thus, V is an abelian group wrt to vector addition.

II. Scalar multiplication in V

(i) For $u, v \in V$ and $\alpha \in R$, we have
$\alpha(u+v) = \alpha.u + \alpha.v$

(ii) For $u \in V$ and $a, b \in R$, we have
$(a+b)u = au + bu$

(iii) For $u \in V$ and $a, b \in R$, we have
$a(bu) = (ab)u$

(iv) For $u \in V$ and $1 \in R$, we have
$1.u = u$

$\therefore$ Set of all real no's scalar multiplication in V
Thus, V satisfies all the properties of vector addition and scalar multiplication and hence V is a vector space.

(2) Prove that the set C of all complex no's (i.e. set of all ordered pairs of real no's) is a vector space over the field 'R' of all real no's where vector addition is defined by
$(x_1, x_2) + (y_1, y_2) = (x_1+y_1, x_2+y_2)$, $\forall$ $(x_1, x_2)(y_1, y_2) \in C$ and scalar multiplication

is defined by $\alpha(x_1, x_2) = (\alpha x_1, \alpha x_2) \; \forall \alpha \in R$

→ I. Show that $(C, +)$ is an abelian group:

(i) Associativity: For all $(x_1, x_2)(y_1, y_2)(z_1, z_2) \in C$, we have,

$$(x_1, x_2) + [(y_1, y_2) + (z_1, z_2)] = (x_1, x_2) + [y_1 + z_1, y_2 + z_2]$$
$$= (x_1 + y_1 + z_1, \; x_2 + y_2 + z_2)$$
$$= (x_1 + y_1, \; x_2 + y_2) + (z_1, z_2)$$
$$= [(x_1, x_2) + (y_1, y_2)] + (z_1, z_2)$$

(ii) Commutativity:

For all $(x_1, x_2), (y_1, y_2) \in C$, we have

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, \; x_2 + y_2)$$
$$= (y_1 + x_1, \; y_2 + x_2)$$
$$= (y_1, y_2) + (x_1, x_2)$$

(iii) Existence of identity:

for $(x_1, x_2) \in C$, we have

$$(x_1, x_2) + (0, 0) = (x_1 + 0, \; x_2 + 0)$$
$$= (x_1, x_2)$$

Also $(0, 0) + (x_1, x_2) = (0 + x_1, \; 0 + x_2)$
$$= (x_1, x_2)$$

$\therefore (x_1, x_2) + (0, 0) = (x_1, x_2) = (0, 0) + (x_1, x_2)$

(iv) Existence of inverse:

For any $(x_1, x_2) \in C$, $\exists \, (-x_1, -x_2) \in C$ such that

$$(x_1, x_2) + (-x_1, -x_2) = (x_1 - x_1, \; x_2 - x_2)$$
$$= (0, 0)$$

Thus, $(C, +)$ is an abelian group wrt vector addition.

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Scalar multiplication in C:

(i) $\alpha(x_1 x_2 + y_1 y_2) = \alpha[x_1+y_1, x_2+y_2]$
$= \alpha(x_1+y_1), \alpha(x_2+y_2)$
$= (\alpha x_1 + \alpha y_1, \alpha x_2 + \alpha y_2)$
$= (\alpha x_1, \alpha x_2) + (\alpha y_1, \alpha y_2)$
$= \alpha(x_1, x_2) + \alpha(y_1, y_2)$

(ii) $(a+b)(x_1, x_2) = [(a+b)x_1, (a+b)x_2]$
$= [ax_1 + bx_1, ax_2 + bx_2]$
$= (ax_1, ax_2) + (bx_1, bx_2)$
$= a(x_1, x_2) + b(x_1, x_2)$

(iii) $a[b(x_1, x_2)] = a[bx_1, bx_2]$
$= (abx_1, abx_2)$
$= ab(x_1, x_2)$

(iv) $1.(x_1, x_2) = (x_1, x_2)$

Thus, C satisfies the properties of scalar multiplication.

$\therefore$ $\{C, +\}$ is a vector space over R.

③ Show that the set V of all real valued continuous functions of x defined on interval [0, 1] is a vector space over the field R of real no's wrt to vector addition and scalar multiplication defined by

$(f_1 + f_2)x = f_1(x) + f_2(x)$, for all $f_1, f_2 \in V$
$(\alpha f_1)x = \alpha f_1(x)$, for all $\alpha \in R$, $f_1 \in V$

→ I. $(V, +)$ is an abelian group

(i) Associativity:

Let $f_1, f_2, f_3 \in V$ be arbitarary

$$[(f_1+f_2)+f_3](x) = (f_1+f_2)(x) + f_3(x)$$
$$= [f_1(x) + f_2(x)] + f_3(x)$$
$$= f_1(x) + [f_2(x) + f_3(x)]$$
$$= f_1(x) + (f_2+f_3)(x)$$
$$= [f_1 + (f_2+f_3)](x)$$
$$(f_1+f_2)+f_3 = f_1 + (f_2+f_3)$$

(ii) commutativity:

Let $f_1, f_2 \in V$ be orbitary

$$(f_1+f_2)(x) = f_1(x) + f_2(x)$$
$$= f_2(x) + f_1(x)$$
$$= (f_2+f_1)(x), \text{ for all } x$$
$$f_1+f_2 = f_2+f_1$$

(iii) Existence of identity:

Define a function 0 such that $0(x) = 0$ (real no), for all $x \in [0,1]$

Also, 0 is a continous function and belongs to V

$$(0+f_1)(x) = 0(x) + f_1(x) = 0 + f_1(x) = f_1(x)$$
$$(f_1+0)(x) = f_1(x) + 0(x) = f_1(x)$$
$$0+f_1 = f_1+0 = f_1$$

Thus, the function 0 defined above is an additive identity element to V

(iv) Existence of Inverse:

For a function f_1, the function $-f_1$ defined by $(-f_1)(x) = -f_1(x)$ is called additive inverse, as

$$[f_1 + (-f_1)]x = f_1(x) + (-f_1)(x) = f_1(x) - f_1(x) = 0 = 0(x)$$

$|||^{ly}$ $[-f_1 + f_1](x) = 0(x)$

Thus the set V is an abelian grp under addition

<u>properties of Scalar multiplication in V:</u>

for $f_1, f_2 \in V$ and $\alpha \in R$, we have

$$[\alpha(f_1+f_2)]x = \alpha[(f_1+f_2)(x)]$$
$$= \alpha[f_1(x)+f_2(x)]$$
$$= \alpha f_1(x) + \alpha f_2(x)$$
$$= (\alpha f_1)x + (\alpha f_2)x$$
$$= (\alpha f_1 + \alpha f_2)x$$
$$\alpha(f_1+f_2) = \alpha f_1 + \alpha f_2$$

(ii) for $f_1 \in V$ and $a, b \in R$

$$[(a+b)f_1](x) = (a+b)f_1(x)$$
$$= af_1(x) + bf_1(x)$$
$$= (af_1)x + (bf_1)x$$
$$= (af_1 + bf_1)x$$
$$(a+b)f_1 = af_1 + bf_1$$

(iii) for $f_1 \in V$ and $a, b \in R$

$$[a(bf_1)]x = a[(bf_1)x]$$
$$= a[bf_1(x)]$$
$$= (ab)f_1(x)$$
$$= [(ab)f_1]x$$
$$a(bf_1) = (ab)f_1$$

(iv) For $f_1 \in V$ and $1 \in R$, we have

$$(1.f_1)x = 1.f_1(x) = f_1(x)$$
$$1f_1 = f_1$$

$\therefore$ Thus V satisfies all the properties of a vector space and hence V is a vector space

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Subspaces:

A non-empty subset W of a vector space V(F) is said to form a subspace of V if W is also a vector space over F with the same addition and scalar multiplication as for V

eg: Let $W_1 = \{(a,0,0) : a \in R\}$

$W_2 = \{(a,b,0) : a,b \in R\}$

Here W_1 is a subspace of W_2. Also W_1 and W_2 are subspace of R^3.

Necessary and sufficient condⁿ for a subspace:

Theorem 1: W is a subspace of V(F) iff

(i) W is non-empty

(ii) W is closed under vector addition i.e, $\forall w_1, w_2 \in W \Rightarrow w_1 + w_2 \in W$

(iii) W is closed under scalar multiplication i.e $\forall a \in F$ and $w \in W \Rightarrow a.w \in W$

Theorem 2: W is a subspace of V(F) iff

(i) W is non-empty

(ii) $\forall a, b \in F$ and $v, w \in W \Rightarrow a.v + b.w \in W$.

problems on subspaces:

① Show that W is a subspace of V(R) where $W: \{f \,/\, f(2) = f(1)\}$

→ $0 \in W$ since $0(2) = 0(1)$

$\Rightarrow 0 = 0$.

$\therefore$ W is a non-empty set

Let $f, g \in W$, then $f(2) = f(1)$ and $g(2) = g(1)$

$\forall a, b \in R$,

$(a.f + b.g)(2) = a.f(2) + b.g(2)$

$= a.f(1) + b.g(1)$

$= (a.f + b.g)(1)$

f(2) F(1)

$\therefore a.f + b.g \in W$

By Theorem 2, W is a subspace of $V(R)$

(2) Let $V = R^3$ be the Euclidean 3-space.

Let $W = \{(x, y, z) \mid ax + by + cz = 0 ; x, y, z \in R\}$

a, b, c being real no's. Show that W is a subspace of V. (OR) Show that any plane passing through the origin in a subspace of R^3

$\longrightarrow$ Let $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2)$ ~~be~~ be any two vectors of W where $x_1, x_2, y_1, y_2, z_1, z_2 \in R$.

Such that
$$\left.\begin{aligned} ax_1 + by_1 + cz_1 &= 0 \\ ax_2 + by_2 + cz_2 &= 0 \end{aligned}\right\} \text{—— (1)}$$

For $\alpha, \beta \in R$, $\alpha u + \beta v = \alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2)$

$$\alpha u + \beta v = (\alpha x_1, \alpha y_1, \alpha z_1) + (\beta x_2, \beta y_2, \beta z_2)$$
$$= (\alpha x_1 + \beta x_2, \alpha y_2 + \beta y_2, \alpha z_2 + \beta z_2)$$

Consider,

$$a(\alpha x_1 + \beta x_2) + b(\alpha y_1 + \beta y_2) + c(\alpha z_1 + \beta z_2)$$
$$= (a\alpha x_1 + a\beta x_2) + (b\alpha y_1 + b\beta y_2) + (c\alpha z_1 + c\beta z_2)$$
$$= (a\alpha x_1 + b\alpha y_1 + c\alpha z_1) + (a\beta x_2 + b\beta y_2 + c\beta z_2)$$
$$= \alpha(ax_1 + by_1 + cz_1) + \beta(ax_2 + by_2 + cz_2)$$
$$= \alpha(0) + \beta(0) \quad [\text{using } (1)]$$
$$= 0$$

$\therefore$ W is a subspace of V.

(3) Prove that the subset $W = \{(x, y, z) \mid x - 3y + 4z = 0\}$ of the vector space R^3 is a subspace of R^3.

→ Let $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2)$ be any two vectors of W, where $x_1, x_2, y_1, y_2, z_1, z_2 \in R$.

Such that
$$\left.\begin{array}{l} x_1 - 3y_1 + 4z_1 = 0 \\ x_2 - 3y_2 + 4z_2 = 0 \end{array}\right\} \text{ — } (1)$$

Consider, $u+v = (x_1, y_1, z_1) + (x_2, y_2, z_2)$
$= (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

Consider,

$x - 3y + 4z$
$= (x_1 + x_2) - 3(y_1 + y_2) + 4(z_1 + z_2)$
$= (x_1 + x_2) - (3y_1 + 3y_2) + (4z_1 + 4z_2)$
$= (x_1 - 3y_1 + 4z_1) + (x_2 - 3y_2 + 4z_2)$
$= 0 + 0$ (from 1)
$= 0$.

∴ W is a subspace of $V(R^3)$.

(4) Let $V = R^3$ be a vectorspace and consider W of V consisting of vectors of the form (a, a^2, b), where the second component is the square of the first. Is W a subspace of V?

Let $u = (a_1, a_1^2, b_1)$ and $v = (a_2, a_2^2, b_2)$ be two vectors in W.

Consider $u + v = (a_1, a_1^2, b_1) + (a_2, a_2^2 + b_2)$
$= [(a_1 + a_2), (a_1^2 + a_2^2), (b_1 + b_2)]$

The second component $a_1^2 + a_2^2$ need not be always equal to $(a_1 + a_2)^2$

Thus, W is not closed under addition.

∴ W is not a subspace of V.

(5) Show that intersection of 2 subspaces of a vectorspace V is also a subspace of V.

→ Let S and T be any 2 subspaces of vectorspace V, over the field F.

$\therefore S \cap T = \{\alpha \mid \alpha \in S \text{ and } \alpha \in T\}$

Since S and T are subspaces, $0 \in S$ and $0 \in T$

$\therefore 0 \in S \cap T$ and hence $S \cap T \neq \phi$

Consider any two scalars $c_1, c_2 \in F$,

$\therefore$ for every $\alpha, \beta \in S \cap T$ we have,

$\alpha, \beta \in S$ and $\alpha, \beta \in T$.

$\Rightarrow c_1\alpha + c_2\beta \in S$ and $c_1\alpha + c_2\beta \in T$

$\Rightarrow c_1\alpha + c_2\beta \in S \cap T$

$\Rightarrow S \cap T$ is a subspace of V.

<u>Note:</u> Union of two subspaces of V need not be a subspace of V.

This can be seen from following example:

$\Rightarrow$ Consider 2 subspaces of $V_2(R)$ as:

$S = \{(a,0) \mid a \in R\}$ and $T = \{(0,b) \mid b \in R\}$

$\therefore W = S \cup T = \{\alpha \mid \alpha \in S \text{ or } \alpha \in T\}$

Now, say $(1,0), (0,1) \in W$, but

$(1,0) + (0,1) = (1,1) \notin \underline{W}$. $S \cup T$

$\therefore$ W is not closed under vector addition

$\Rightarrow \overset{= S \cup T}{W}$ is not a vectorspace of $V_2(R)$.

(6) Let V be the vectorspace of all square matrices over R. Determine which of following are subspaces of V

(i) $W = \left\{ \begin{bmatrix} x & y \\ z & 0 \end{bmatrix} : x, y, z \in R \right\}$ (ii) $W = \left\{ \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} : x, y \in R \right\}$

(iii) $W = \{A : A \in V \text{ and } A \text{ is singular}\}$ (iv) $W = \{A : A \in V, A^2 = A\}$

(i) Let $A = \begin{bmatrix} x_1 & y_1 \\ z_1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} x_2 & y_2 \\ z_2 & 0 \end{bmatrix}$ be any two elements of W

If $a, b \in R$ then

$$aA + bB = a\begin{bmatrix} x_1 & y_1 \\ z_1 & 0 \end{bmatrix} + b\begin{bmatrix} x_2 & y_2 \\ z_2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} ax_1 & ay_1 \\ az_1 & 0 \end{bmatrix} + \begin{bmatrix} bx_2 & by_2 \\ bz_2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} ax_1 + bx_2 & ay_1 + by_2 \\ az_1 + bz_2 & 0 \end{bmatrix}$$

Which is a matrix of type $\begin{bmatrix} x & y \\ z & 0 \end{bmatrix}$ and $ax_1 + bx_2$, $ay_1 + by_2$, $az_1 + bz_2 \in R$.

$$\therefore aA + bB \in W$$

Thus, W is a subspace of V

(ii) Let $A = \begin{bmatrix} x_1 & 0 \\ 0 & y_1 \end{bmatrix}$ $B = \begin{bmatrix} x_2 & 0 \\ 0 & y_2 \end{bmatrix}$ be any 2 elements of W

If $a, b \in R$ then

$$aA + bB = a\begin{bmatrix} x_1 & 0 \\ 0 & y_1 \end{bmatrix} + b\begin{bmatrix} x_2 & 0 \\ 0 & y_2 \end{bmatrix}$$

$$= \begin{bmatrix} ax_1 + bx_2 & 0 \\ 0 & ax_2 + by_2 \end{bmatrix}$$

Which is a matrix of the type $\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$ and

$ax_1 + bx_2$, $ay_1 + by_2 \in R$.

$$\therefore aA + bB \in W$$

Thus, W is a subspace of V.

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Linear combination

Let V be a vector space over a field F and let $v_1, v_2 \dots v_n \in V$

Any vector of form $a_1v_1 + a_2v_2 + \dots + a_nv_n$ in V where $a_i \in F$ is called linear combination of $v_1, v_2 \dots v_n$

Linear dependence:

Let V be a vector space over the field F. The vectors v_1, v_2, v_n are said to be linearly dependent over F, if $\exists$ scalars $a_1, a_2 \dots a_n \in F$ not all zero but linear combination is zero.

i.e. $a_1v_1 + a_2v_2 + \dots + a_nv_n = 0$ but all $a_i \neq 0$, where $i \in N$.

Linear independence:

Let V be a vector space over the field F. The vectors $v_1, v_2 \dots v_n$ are said to be linearly independent over F, if $\exists$ scalars $a_1, a_2 \dots a_n \in F$ such that $a_1v_1 + a_2v_2 + \dots + a_nv_n = 0 \Rightarrow$ all $a_i = 0$ where $i \in N$

Linear span:

Let S be a subset of vector space V over the field F. The set of all linear combinations of vectors in S is called a linear span of S and denoted by $\alpha(S)$.

If $S = \phi$, then $\alpha(S) = 0$.

Basis or Base of vector space V:

Let V be a vector space over field F. The set of vectors $\{v_1, v_2 \dots v_n\}$ is called basis of V, if

(i) $v_1, v_2, \dots v_n$ are linearly independent

$v_1, v_2 \dots v_n$ span V i.e each vector of V can be uniquely expressed as linear comb^n of $v_1, v_2 \dots v_n$

No of elements in a basis of vector space V is called dimension of V and is denoted by dim V. If V contains a basis with n elements then $\dim V = n$.

Note: (i) The vector space {0} is defined to have dim 0 since empty set ϕ is independent and generates {0}

$\therefore \dim\{0\} =$ No of elements in ϕ

$= 0$ [since no element is in ϕ]

(ii) When a vector space is not of finite dimension, it is said to be of infinite dimension

Note: Two vectors v_1 and v_2 are linearly dependent iff one of them is a multiple of other

eg: Determine whether or not the vectors v_1, v_2 are linearly dependent

(i) $v_1 = (1,3,9)$ $v_2 = (2,4,1)$

→ No vector is multiple of other hence v_1 and v_2 are not linearly dependent

(ii) $v_1 = (3,4)$ $v_2 = (6,8)$

$v_2 = 2v_1$ hence v_1 and v_2 are linearly dependent vectors.

Problems on linear combination and linear span

1. Express the vector $(3,5,2)$ as a linear comb^n of the vectors $(1,1,0)$ $(2,3,0)$ $(0,0,1)$ ~~of~~ $V_3(R)$

→ Let $v = (3,5,2)$

Let $u_1 = (1,1,0)$, $u_2 = (2,3,0)$, $u_3 = (0,0,1)$

Consider $v = xu_1 + yu_2 + zu_3$ —— (1)

$\therefore (3,5,2) = x(1,1,0) + y(2,3,0) + z(0,0,1)$ —— (1)

$(3,5,2) = (x,x,0) + (2y,3y,0) + (0,0,z)$

$(3,5,2) = (x+2y,\ x+3y\ \cancel{+}\ z)$

we have,

$x+2y=3$; $x+3y=5$; $\boxed{z=2}$

Solving, $\boxed{x=-1}$ and $\boxed{y=2}$

Sub in eq^n ①,

$(3,5,2) = -1(1,1,0) + 2(2,3,0) + 2(0,0,1)$ //

② Write the vector $v=(1,3,9)$ as a linear combination of the vectors $u_1=(2,1,3)$, $u_2=(+1,-1,1)$, $u_3=(3,1,5)$

→ Let $v=(1,3,9)$

Let $u_1=(2,1,3)$, $u_2=(1,-1,1)$, $u_3=(3,1,5)$

consider $v = xu_1 + yu_2 + zu_3$

$\Rightarrow (1,3,9) = x(2,1,3) + y(1,-1,1) + z(3,1,5)$ — ①

$(1,3,9) = (2x, x, 3x) + (y,-y,y) + (3z, z, 5z)$

$(1,3,9) = (2x+y+3z,\ x-y+z,\ 3x+y+5z)$

we have,

$2x+y+3z=1$; $x-y+z=3$; $3x+y+5z=9$

solving,

$\boxed{x=-12}$ $\boxed{y=-5}$ $\boxed{z=10}$

Sub in ①,

$(1,3,9) = -12(2,1,3) - 5(1,-1,1) + 10(3,1,5)$ //

③ Write the vector $v=(4,2,1)$ as a linear comb^n of vectors $u_1=(1,-3,1)$, $u_2=(0,1,2)$, $u_3=(5,1,37)$

→ Let $v=(4,2,1)$

Let $u_1=(1,-3,1)$, $u_2=(0,1,2)$, $u_3=(5,1,37)$

consider $v = xu_1 + yu_2 + zu_3$

$(4,2,1) = x(1,-3,1) + y(0,1,2) + z(5,1,37)$

$= (x,-3x,x) + (0,y,2y) + (5z, z, 37z)$

$(x+5z,\ -3x+y+z,\ x+2y+37z)$

we have,

$x + 0y + 5z = 4$; $-3x + y + z = 2$; $x + 2y + 37z = 1$

is of form $AX = B$

where $A = \begin{bmatrix} 1 & 0 & 5 \\ -3 & 1 & 1 \\ 1 & 2 & 37 \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$

consider $[A:B] = \left[\begin{array}{ccc:c} 1 & 0 & 5 & 4 \\ -3 & 1 & 1 & 2 \\ 1 & 2 & 37 & 1 \end{array}\right]$

$R_2 \to R_2 + 3R_1$; $R_3 \to R_3 - R_1$

$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 0 & 5 & 4 \\ 0 & 1 & 16 & 14 \\ 0 & 2 & 32 & -3 \end{array}\right]$

$R_3 \longrightarrow R_3 - 2R_2$

$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 0 & 5 & 4 \\ 0 & 1 & 16 & 14 \\ 0 & 0 & 0 & -31 \end{array}\right]$

we observe that $\rho(A) = 2$, $\rho[A:B] = 3$

$\rho(A) \neq \rho[A:B] \Rightarrow$ given system of linear eq^n is inconsistent i.e, sol^n does not exist.

$\therefore$ v cannot be expressed as linear comb^n of given vectors u_1, u_2, u_3.

(4) Determine whether the matrix $\begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix}$ is a linear comb^n of $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ in the vector space M_{22} of 2x2 matrices.

$\Rightarrow$ $A = \begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ $C = \begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}$ $D = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$

Consider,

$$A = xB + yC + zD$$

$$\begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix} = x\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} + y\begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix} + z\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \quad \text{—} ①$$

$$\begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix} = \begin{bmatrix} x & 0 \\ 2x & x \end{bmatrix} + \begin{bmatrix} 2y & -3y \\ 0 & 2y \end{bmatrix} + \begin{bmatrix} 0 & z \\ 2z & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix} = \begin{bmatrix} x+2y & -3y+z \\ 2x+2z & x+2y \end{bmatrix}$$

$\Rightarrow$

$x + 2y = -1$

$-3y + z = 7$

$2x + 2z = 8$

$x + 2y = -1$ (same as eq^n ①)

Solving,

$x + 2y + 0z = -1$

$0x - 3y + 1z = 7$

$2x + 0y + 2z = 8$

$x = 3, \quad y = -2, \quad z = 1$

Sub in ①,

$$\begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix} = 3\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} - 2\begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix} + 1\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \quad //$$

⑤ Let $f(x) = 2x^2 - 5$, $g(x) = x + 1$, ST the fun^n $h(x) = 4x^2 + 3x - 7$ lies in the subspace span $\{f, g\}$ of P_2.

Note: If $h(x)$ can be expressed as a linear comb^n of $f(x)$ and $g(x)$, then the function $h(x)$ lies in the linear span.

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→ Let $h(x) = c_1 f(x) + c_2 g(x)$, (to find c_1, c_2)

$4x^2 + 3x - 7 = c_1(2x^2 - 5) + c_2(x+1)$ — ①

$4x^2 + 3x - 7 = (2c_1 x^2 - 5c_1) + (c_2 x + c_2)$

$4x^2 + 3x - 7 = 2c_1 x^2 + c_2 x + (c_2 - 5c_1)$

Equating coefficient of like terms on both sides,

$2c_1 = 4$ $\boxed{c_1 = 2}$	$\boxed{c_2 = 3}$	$c_2 - 5c_1 = -7$ $3 - 10 = -7$ $-7 = -7$ (True)

$\therefore c_1 = 2, \quad c_2 = 3$

Sub in ①,

$\therefore 4x^2 + 3x - 7 = 2(2x^2 - 5) + 3(x+1)$ //

$\therefore$ $h(x)$ is expressed as linear combⁿ of $f(x)$, ~~and~~ $g(x)$ and hence $h(x)$ lies in the linear span $\{f, g\}$ of P_2.

⑥ S.T the set $S = \{(1,2,4), (1,0,0), (0,1,0), (0,0,1)\}$ is linearly dependent

Consider $xv_1 + yv_2 + zv_3 + uv_4 = (0,0,0)$

$x(1,2,4) + y(1,0,0) + z(0,1,0) + u(0,0,1) = (0,0,0)$

$(x, 2x, 4x) + (y,0,0) + (0,z,0) + (0,0,u) = (0,0,0)$

$x+y, \; 2x+z, \; 4x+u = 0,0,0$

$x+y = 0 \Rightarrow x + y + 0z + 0u = 0$

$2x + z = 0 \Rightarrow 2x + 0y + z + 0u = 0$

$4x + u = 0 \Rightarrow 4x + 0y + 0z + u = 0$ is of form $AX = 0$

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 4 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1 \; ; \; R_3 \to R_3 - 4R_1$

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -4 & 0 & 1 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 2R_2$

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix}$$

$\rho(A) = 3 < 4$ (no of unknowns)

$\therefore$ given vectors are linearly dependent

(7) Prove that in $V_3(R)$, $\{(1,2,1)(2,1,0)(1,-1,2)\}$ are linearly dependent.

Consider $xv_1 + yv_2 + zv_3 = 0,0,0$

$x(1,2,1) + y(2,1,0) + z(1,-1,2) = (0,0,0)$

$x+2y+z, \quad 2x+y-z, \quad x+0y+2z = (0,0,0)$

$x+2y+z = 0$

$2x+y-z = 0$

$x+0y+2z = 0$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1, \quad R_3 \rightarrow R_3 - R_1$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -3 \\ 0 & -2 & 1 \end{bmatrix}$$

$R_3 \rightarrow 3R_3 - 2R_2$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -3 \\ 0 & 0 & 9 \end{bmatrix}$$

$\rho(A) = 3 =$ no. of unknowns this implies homogenous system of linear equation posses unique solution $x=0, y=0, z=0$

$\therefore$ The given vectors are linearly dependent.

(8) Define linearly independent set of vectors and linearly dependent set of vectors. Are the vectors $V_1 = (2,5,3)$, $V_2 = (1,1,1)$ and $V_3 = (4,-2,0)$ are linearly independent? Justify your answer.

→ Def^n:

$xV_1 + yV_2 + zV_3 = (0,0,0)$

$x(2,5,3) + y(1,1,1) + z(4,-2,0) = (0,0,0)$

$2x + y + 4z = 0$

$5x + y - 2z = 0$

$3x + y + 0z = 0$

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 5 & 1 & -2 \\ 3 & 1 & 0 \end{bmatrix}$$

$R_2 \longrightarrow 2R_2 - 5R_1$ $R_3 \longrightarrow 2R_3 - 3R_1$

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & -3 & -16 \\ 0 & -1 & -12 \end{bmatrix}$$

$R_3 \longrightarrow 3R_3 - R_2$

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & -3 & -16 \\ 0 & 0 & -20 \end{bmatrix}$$

$\rho(A) = 3$

$\therefore$ Given vectors are linearly independent

(9) Define a basis for vector space. Determine whether or not the vectors (1, 1, 2) (1, 2, 5) (5, 3, 4) from a basis of R^3

→ Def^n

given 3 vectors in R_3 form a basis if and only if they are linearly independent

Let $xV_1 + yV_2 + zV_3 = (0,0,0)$

$x(1,1,2) + y(1,2,5) + z(5,3,4) = (0,0,0)$

$x + y + 5z = 0$

$x + 2y + 3z = 0$

$2x + 5y + 4z = 0$ is of the form $AX = 0$

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 3 \\ 2 & 5 & 4 \end{bmatrix}$$

$R_2 \to R_2 - R_1$; $R_3 \to R_3 - 2R_1$

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & -2 \\ 0 & 3 & -6 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2 <$ no of unknowns

∴ The vectors are linearly dependent

∴ It does not form a basis of R_3

(10) Find the basis and dimension of the subspace spanned by the vectors $\{(2,4,2)\ (1,-1,0)\ (1,2,1)\ (0,3,1)\}$ in $V_3(R)$

(10) Let $A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & -1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & 1 \end{bmatrix}$

$R_2 \to 2R_2 - R_1 \quad R_3 \to 2R_3 - R_1$

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 0 & -6 & -2 \\ 0 & 0 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

$R_4 \to 2R_4 + R_2$

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 0 & -6 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Basis of the subspace spanned by the vectors $= \{(2,4,2)\ (0,-6,-2)\}$. Since non zero rows of echelon matrix form a basis of the subspace spanned by the rows of A.

$\therefore$ Dimension of the subspace $= 2$.

(11) Find the basis and dimension of subspace spanned by the subset $S = \left\{ \begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix}, \begin{bmatrix} 2 & -4 \\ -5 & 7 \end{bmatrix}, \begin{bmatrix} 1 & -7 \\ -5 & 1 \end{bmatrix} \right\}$

of the vector space of all 2×2 matrices over R.

Let A, B, C, D are matrices of S.

Then the coordinates A, B, C, D wrt to standard basis are $(1,-5,-4,2)$, $(1,1,-1,5)$, $(2,-4,-5,7)$ and $(1,-7,-5,1)$

Consider matrix A as:

$$A = \begin{bmatrix} 1 & -5 & -4 & 2 \\ 1 & 1 & -1 & 5 \\ 2 & -4 & -5 & 7 \\ 1 & -7 & -5 & 1 \end{bmatrix}$$

$R_2 \to R_2 - R_1$; $R_3 \to R_3 - 2R_1$, $R_4 \to R_4 - R_1$

$$\begin{bmatrix} 1 & -5 & -4 & 2 \\ 0 & 6 & 3 & 3 \\ 0 & 6 & 3 & 3 \\ 0 & -2 & -1 & -1 \end{bmatrix}$$

$R_4 \to 3R_4 + R_3$ $\quad R_3 \to R_3 - R_2$

$$A \sim \begin{bmatrix} 1 & -5 & -4 & 2 \\ 0 & 6 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basics of the subspace spanned by the vectors

Basics : $\{(1, -5, -4, 2), (0, 6, 3, 3)\}$

$\therefore$ Dimension of the subspace = 2

(12) Let W be the subspace of R^5, spanned by $x_1 = (1, 2, -1, 3, 4)$ $x_2 = (2, 4, -2, 6, 8)$ $x_3 = (1, 3, 2, 2, 6)$ $x_4 = (1, 4, 5, 1, 8)$ $x_5 = (2, 7, 3, 3, 9)$. Find a subset of vectors which forms basis of W.

$\rightarrow$

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 2 & 4 & -2 & 6 & 8 \\ 1 & 3 & 2 & 2 & 6 \\ 1 & 4 & 5 & 1 & 8 \\ 2 & 7 & 3 & 3 & 9 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$; $R_3 \to R_3 - R_1$; $R_4 \to R_4 - R_1$; $R_5 \to R_5 - 2R_1$

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 2 \\ 0 & 2 & 6 & -2 & 4 \\ 0 & 3 & 5 & -3 & 1 \end{bmatrix}$$

$R_4 \to R_4 - 2R_3$; $R_5 \to R_5 - 3R_3$

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & -5 \end{bmatrix}$$

$\therefore$ Basis of the subspace spanned by the vectors
$= \{(1,2,-1,3,4), (1,3,-1,2), (0,0,-4,0,-5)\}$
$\therefore$ dim = 3

(13) V is a vector space of polynomials over R. Find a basis and dimension of supspace W of V, spanned by the polynomials

$x_1 = t^3 - 2t^2 + 4t + 1$, $x_2 = 2t^3 - 3t^2 + 9t - 1$
$x_3 = t^3 + 6t - 5$, $x_4 = 2t^3 - 5t^2 + 7t + 5$

$$A = \begin{bmatrix} 1 & -2 & 4 & 1 \\ 2 & -3 & 9 & -1 \\ 1 & 0 & 6 & -5 \\ 2 & -5 & 7 & 5 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$; $R_3 \to R_3 - R_1$; $R_4 \to R_4 - 2R_1$

$$\begin{bmatrix} 1 & -2 & 4 & 1 \\ 0 & 1 & 1 & -3 \\ 0 & 2 & 2 & -6 \\ 0 & -1 & -1 & 3 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$; $R_4 \to R_4 + R_2$

$$\begin{bmatrix} 1 & -2 & 4 & 1 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis of the subspace spanned by the vectors
$= \{(1,-2,4,1), (0,1,1,-3)\}$
$\therefore$ dim = 2

Linear Transformation:

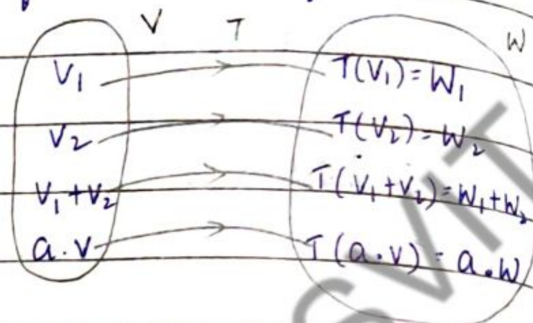
Let V and W be any two supspaces over the field F. A mapping T from V to W is called a linear transformation if,

(i) $v_1, v_2 \in V$

$T(v_1+v_2) = T(v_1) + T(v_2)$

(ii) $\forall a \in F$ and $\forall v \in V$

$T(a.v) = a.T(v)$.



problems on Linear Transformation

① Prove that $T: R^3 \to R^3$ defined by $T(a,b,c) = (3a, a-b, 2a+b+c)$ is a L.T

→ Let $u = (a_1, b_1, c_1)$ and $v = (a_2, b_2, c_2) \in R^3$ be orbitary

$\therefore T(u) = T(a_1, b_1, c_1) = (3a_1, a_1-b_1, 2a_1+b_1+c_1)$

$T(v) = T(a_2, b_2, c_2) = (3a_2, a_2-b_2, 2a_2+b_2+c_2)$

(i) $T(u+v) = T(\underset{A}{a_1+a_2}, \underset{B}{b_1+b_2}, \underset{C}{c_1+c_2})$

$= (3a_1+3a_2) a_1+a_2-b_1-b_2, 2a_1+2a_2+b_1+b_2+c_1+c_2)$

$= (3a_1, a_1-b_1, 2a_1+b_1+c_1) + (3a_2, a_2-b_2, 2a_2+b_2+c_2)$

$= T(u) + T(v)$

(ii) For any scalar $\alpha \in R$,

$T(\alpha u) = T(\underset{A}{\alpha a_1}, \underset{B}{\alpha b_1}, \underset{C}{\alpha c_1})$

$= (3\alpha a_1, \alpha a_1 - \alpha b_1, 2\alpha a_1 + \alpha b_1 + \alpha c_1)$

$= \alpha(3a_1, a_1-b_1, 2a_1+b_1+c_1)$

$= \alpha. T(u)$

$\therefore$ T is a linear transformation

(i) Prove that $T: R^2 \rightarrow R^2$ defined by $T(x,y) = (3x, x+y)$ is linear. Find the images of the vector $(1,3)$ and $(-1,2)$ under this transformation.

Let $u = (x_1, y_1)$ and $v = (x_2, y_2) \in R^2$ be orbitary

$\therefore T(u) = T(x_1, y_1) = (3x_1, x_1+y_1)$

$T(v) = T(x_2, y_2) = (3x_2, x_2+y_2)$

(i) $T(u+v) = (x_1+x_2, y_1+y_2)$

$= (3x_1+3x_2, x_1+x_2+y_1+y_2)$

$= (3x_1, x_1+y_1) + (3x_2, x_2+y_2)$

$= T(u) + T(v)$.

(ii) For any scalar $\alpha \in R$,

$T(\alpha u) = T(\alpha x_1, \alpha y_1)$

$= (3\alpha x_1, \alpha x_1 + \alpha y_1)$

$= \alpha (3x_1, x_1+y_1)$

$= \alpha . T(u)$

$\therefore T$ is a linear transformation.

To find the images of the vectors $(1,3)$ and $(-1,2)$ under a transformation, $T(x,y) = (3x, x+y)$, apply the transformation to each vector.

i.e. Image of $(1,3) = T(1,3)$

$= (3\times 1, 1+3)$

$= (3,4)$

Image of $(-1,2) = T(-1,2)$

$= (3\times -1, -1+2)$

$= (-3, 1)$

(8) Let P_n be the vector space of real polynomial functions of degree $\leq n$. Show that the transformation $T: P_2 \to P_1$ defined by $T(ax^2 + bx + c) = (a+b)x + c$ is linear.

(i) To prove that for any two polynomials $f(x)$ and $g(x)$ in P_n,

$T\{f(x) + g(x)\} = T\{f(x)\} + T\{g(x)\}$

Let $f(x)$ and $g(x)$ in P_n be two orbitary polynomials then,

$f(x) = a_1x^2 + b_1x + c_1$ and $g(x) = a_2x^2 + b_2x + c_2$

Consider LHS $= T\{f(x) + g(x)\}$

$$= T\{(a_1x^2 + b_1x + c_1) + (a_2x^2 + b_2x + c_2)\}$$

$$= T\{\underbrace{(a_1+a_2)}_{A}x^2 + \underbrace{(b_1+b_2)}_{B}x + \underbrace{(c_1+c_2)}_{C}\}$$

$$= (a_1 + a_2 + b_1 + b_2)x + (c_1 + c_2) \quad \text{--- (1)}$$

Consider RHS $= T\{f(x)\} + T\{g(x)\}$

$$= T\{a_1x^2 + b_1x + c_1\} + T\{a_2x^2 + b_2x + c_2\}$$

$$= \{(a_1+b_1)x + c_1\} + \{(a_2+b_2)x + c_2\}$$

$$= (a_1 + a_2 + b_1 + b_2)x + (c_1 + c_2) \quad \text{--- (2)}$$

from (1) and (2),

$T\{f(x) + g(x)\} = T\{f(x)\} + T\{g(x)\}$.

(ii) To prove that for any scalar α and any polynomial $T\{\alpha f(x)\} = \alpha T\{f(x)\}$

consider $f(x)$ in P_n be any orbitary polynomial then,

$f(x) = ax^2 + bx + c$.

Consider LHS $= T\{\alpha f(x)\}$

$= T\{\alpha a x^2 + \alpha b x + \alpha c\}$

$= (\alpha a + \alpha b)x + \alpha c$

$= \alpha\{(a+b)x + c\}$ —— ③

Consider RHS $= \alpha T\{f(x)\}$

$= \alpha T\{ax^2 + bx + c\}$

$= \alpha\{(a+b)x + c\}$ —— ④

from ③ and ④,

$T\{\alpha f(x)\} = \alpha T\{f(x)\}$

$\therefore$ T is a linear transformation.

④ verify that $T: V_2(R) \rightarrow V_2(R)$ defined by $T(x, y) = (x+6, y+2)$ is linear or not.

→ Let $u = (x_1, y_1)$ and $v = (x_2; y_2)$ be two orbitary vectors.

then $T(u) = T(x_1, y_1) = (x_1+6, y_1+2)$

$T(v) = T(x_2, y_2) = (x_2+6, y_2+2)$

(i) $T(u+v) = T(x_1+x_2, y_1+y_2)$

$= (x_1+x_2+6, y_1+y_2+2)$ —— ①

$T(u)+T(v) = T(x_1, y_1) + T(x_2, y_2)$

$= (x_1+6, y_1+2) + (x_2+6, y_2+2)$

$= (x_1+x_2+12, y_1+y_2+4)$ —— ②

from ① and ②,

$T(u+v) \neq T(u)+T(v)$

$\therefore$ T is not linear.

Matrix of a linear transformation:

Let $T: U \to V$ be the linear transformation, where U and V are vector space over field F

Let $B = \{u_1, u_2 \cdots u_n\}$ and $B' = \{v_1, v_2 \ldots, v_m\}$ be ordered basis for the finite dimensional vector spaces U and V respectively.

Since $T(u_1), T(u_2) \ldots T(u_n) \in V$ and $\{v_1, v_2 \ldots v_m\}$ spans V, each $T(u_i)$ can be expressed as a linear combination of vectors $v_1, v_2 \ldots v_m$

Let $T(u_1) = a_{11}v_1 + a_{21}v_2 + \ldots + a_{m1}v_m$

$T(u_2) = a_{12}v_1 + a_{22}v_2 + \ldots + a_{m2}v_m$

$- \quad - \quad - \quad - \quad - \quad -$

$T(u_n) = a_{1n}v_1 + a_{2n}v_2 + \ldots + a_{mn}v_m$ where $a_{ij} \in F$

The coefficient matrix of ~~g~~ this system of eq^n is

$$\begin{bmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{m1} \\ a_{12} & a_{22} & a_{32} & \cdots & a_{m2} \\ - & - & - & & - \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{mn} \end{bmatrix}$$

The transpose of this matrix is a matrix representation of T, called matrix of T with respect to ordered basis B and B' [or matrix associated with T wrt B and B'] It is denoted by $[T: B, B']$ and is given by.

$$[T: B, B'] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

(1) Find the matrix $T: R^3 \rightarrow R^3$ defined by
$T(x,y,z) = (x-y+z,\ 2x-z,\ x+y-2z)$

Let $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ $e_3 = (0,0,1) \in R^3$ be the standard bases.

Given $T(x,y,z) = (x-y+z,\ 2x-z,\ x+y-2z)$

$\therefore T(e_1) = T(1,0,0) = (1,2,1)$
$T(e_2) = T(0,1,0) = (-1,0,1)$
$T(e_3) = T(0,0,1) = (1,-1,-2)$

$\therefore$ Matrix of the LT is $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & -1 \\ 1 & 1 & -2 \end{bmatrix}$

(2) Find the matrix of the transformation
$T: V_3(R) \rightarrow V_2(R)$ defined by $T(x,y,z) = (y-x,\ y-z)$.

Given $e_1(1,0,0)$ $e_2 = (0,1,0)$, $e_3 = (0,0,1) \in V_3(R)$
Given $T(x,y,z) = (y-x,\ y-z)$
$T(e_1) = T(1,0,0) = (-1,0)$
$T(e_2) = T(0,1,0) = (1,1)$
$T(e_3) = T(0,0,1) = (0,-1)$

Matrix of L.T is $\begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$

(3) If $T: R^2 \rightarrow R^3$, $T(x_1,x_2) = (3x_1 - x_2,\ 2x_1 + 4x_2,\ 5x_1 - 6x_2)$
Find the LT relative to the standard bases.

Let $e_1(1,0)$ $e_2 = (0,1)$, $\in R^2$

Given $T(x_1,x_2) = (3x_1 - x_2,\ 2x_1 + 4x_2,\ 5x_1 - 6x_2)$
$T(e_1) = T(1,0) = (3,2,5)$
$T(e_2) = T(0,1) = (-1,4,-6)$
$\therefore$ Matrix of LT is $\begin{bmatrix} 3 & -1 \\ 2 & 4 \\ 5 & -6 \end{bmatrix}$

(4) QP Find the matrix of the linear transformation $T : V_2(R) \to V_3(R)$ such that $T(-1,1) = (-1,0,2)$ and $T(2,1) = (1,2,1)$

→ Let $(-1,1) = -1.e_1 + 1.e_2$; $(2,1) = 2e_1 + 1e_2$

$T(-1,1) = T(-1e_1 + 1e_2)$; $T(2,1) = T(2e_1 + 1e_2)$

we have,

$$-T(e_1) + T(e_2) = (-1,0,2) \quad \text{--- (1)}$$

$$2T(e_1) + T(e_2) = (1,2,1) \quad \text{--- (2)}$$

solving simultaneously,

eq^n (1) $-$ eq^n (2)

$$-3T(e_1) = (-2,-2,1)$$

$$\Rightarrow 3T(e_1) = (2,2,-1)$$

$$\Rightarrow T(e_1) = \left(\frac{2}{3}, \frac{2}{3}, \frac{-1}{3}\right)$$

sub $T(e_1)$ in eq^n (1),

$$T(e_2) = (-1,0,2) + T(e_1)$$

$$T(e_2) = (-1,0,2) + \left(\frac{2}{3}, \frac{2}{3}, \frac{-1}{3}\right)$$

$$T(e_2) = \left(\frac{-1}{3}, \frac{2}{3}, \frac{5}{3}\right)$$

$\therefore$ Matrix of L.T is $\begin{bmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ -1/3 & 5/3 \end{bmatrix}$

(5) $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 0 \end{bmatrix}$, determine the L.T $T: V_3(R) \to V_2(R)$ relative to the bases B_1 and B_2 is given by

(i) B_1 and B_2 are the standard bases of $V_3(R)$ and $V_2(R)$ respectively

(ii) $B_1 = \{(1,1,1), (1,2,3), (1,0,0)\}$

$B_2 = \{(1,1)\ (1,-1)\}$

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(i) Since B_1 and B_2 are standard bases we hence define

$T: V_3(R) \to V_2(R)$ by

$T(1,0,0) = 1(1,0) + 3(0,1) = (1,3)$

$T(0,1,0) = -1(1,0) + 1(0,1) = (-1,1)$

$T(0,0,1) = 2(1,0) + 0(0,1) = (2,0)$

Now, $(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$

$T(x,y,z) = xT(1,0,0) + yT(0,1,0) + zT(0,0,1)$

$T(x,y,z) = x(1,3) + y(-1,1) + z(2,0)$

$T(x,y,z) = (x-y+2z,\ 3x+y)$

(ii) Given $B_1 = \{(1,1,1)\ (1,2,3)\ (1,0,0)\}$

$B_2 = \{(1,1)\ (1,-1)\}$

Define $T: V_3(R) \longrightarrow V_2(R)$ by

$T(1,1,1) = 1(1,1) + 3(1,-1) = (4,-2)$

$T(1,2,3) = -1(1,1) + 1(1,-1) = (0,-2)$

$T(1,0,0) = 2(1,1) + 0(1,-1) = (2,2)$

Now, $(x,y,z) = c_1(1,1,1) + c_2(1,2,3) + c_3(1,0,0)$ — ①

$\cancel{T(x,y,z) = c_1T(1,1,1) + c_2T(1,2,3) + c_3T(1,0,0)}$

$\cancel{T(x,y,z) = c_1(4,-2) + c_2(0,-2) + c_3(2,2)}$

$(x,y,z) = (c_1+c_2+c_3,\ c_1+2c_2,\ c_1+3c_2)$

we have, $c_1+c_2+c_3 = x$ — ②

$c_1+2c_2 = y$ — ③

$c_1+3c_2 = z$ — ④

eq^n ③ $-$ eq^n ④

$-c_2 = y-z$

$\Rightarrow \boxed{c_2 = z-y}$

Sub c_2 in eq^n ③,

$c_1 = y-2c_2$

$c_1 = y-2(z-y)$

$c_1 = y-2z+2y \Rightarrow \boxed{c_1 = 3y-2z}$

Sub c_1 and c_2 in eq^n 2,

$c_3 = x - c_1 - c_2 = x - (3y - 2z) - (z - y)$

$c_3 = x - 3y + 2z - z + y$

$\boxed{c_3 = x - 2y + z}$

Sub c_1, c_2, c_3 in 1,

$(x,y,z) = (3y-2z)(1,1,1) + (z-y)(1,2,3) + (x-2y+z)(1,0,0)$

$T(x,y,z) = (3y-2z)T(1,1,1) + (z-y)T(1,2,3) + (x-2y+z)T(1,0,0)$

$= (3y-2z)(4,-2) + (z-y)(0,-2) + (x-2y+z)(2,2)$

$= (12y - 8z, -6y + 4z) + (0, -2z + 2y) + (2x - 4y + 2z, 2x - 4y + 2z)$

first terms addition and second terms addition

$T(x,y,z) = (2x + 8y - 6z, 2x - 8y + 4z)$

Range and Kernel of a LT:

__Range of T:__ Let $T: V \to W$ be a LT, then the range of T is the set of all images of the elements of V under T, denoted by $R(T)$

i.e, $R(T) = \{T(\alpha) \mid \alpha \in V\}$

$R(T)$ is also called the Range space. Clearly $R(T) \leq W$

__Kernel of T / Null space of T:__

Let $T: V \to W$ be a LT, then the Kernel of T is the set of all elements of V denoted by $N(T)$

i.e, $N(T) = \{\alpha \in V \mid T(\alpha) = 0\}$ where 0 is the zero vector of W.

clearly $N(T) \subseteq V$

Rank of LT: Let $T: V \to W$ be a LT. The dimension of the range space $R(T)$ is called the rank of the LT denoted by $r(T)$

i.e, $r(T) = \dim\{R(T)\}$

Nullity of LT: Let $T: V \longrightarrow W$ be a LT The dimension of the null space $N(T)$ is called the nullity of LT, denoted by $n(T)$. i.e, $n(T) = \dim(N(T))$

Rank-Nullity theorem: Let $T: V \to W$ be a LT and V be a finite dimensional vector space, then

$$r(T) + n(T) = \dim(V)$$

i.e, $\dim(R(T)) + \dim(N(T)) = \dim(V)$

(or) rank + nullity = dimension of the domain

problems:

① Let $T: V \to W$ be a LT defined by $T(x,y,z) = (x+y, x-y, 2x+z)$ Find the range, nullspace, rank, nullity and hence verify rank-nullity theorem.

→ Let $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$ be the standard bases.

Given $T(x,y,z) = (x+y, x-y, 2x+z)$

$T(e_1) = T(1,0,0) = (1,1,2)$

$T(e_2) = T(0,1,0) = (1,-1,0)$

$T(e_3) = T(0,0,1) = (0,0,1)$

(we should consider transpose of the matrix of linear transformation)

Consider $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

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$R_2 \to R_2 - R_1$

$$A \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$ is reduced to echelon form (having 3 non-zero rows)

$\therefore R(T)$ is the subspace generated by ~~the and~~ $(1,1,2)$, $(0,-2,-2)$ and $(0,0,1)$

i.e. $R(T) = \{x(1,1,2) + y(0,-2,2) + z(0,0,1)\}$

$R(T) = \{x, x-2y, 2x-2y+z\}$ is the range space

$$\boxed{\text{Rank of LT} = r(T) = \dim(R(T)) = 3}$$

<u>To find $N(T)$:</u>

Consider $T(x,y,z) = (0,0,0)$

$(x+y, x-y, 2x+z) = (0,0,0)$

$$\Rightarrow \left.\begin{aligned} x+y &= 0 \\ x-y &= 0 \\ 2x+z &= 0 \end{aligned}\right\} \begin{aligned} 2x = 0 &\Rightarrow \boxed{x=0} \\ &\Rightarrow \boxed{y=0} \\ &\Rightarrow \boxed{z=0} \end{aligned}$$

$\therefore N(T) = \{(0,0,0)\}$ is the null space

$\therefore \boxed{\dim(N(T)) = n(T) = 0}$

We have $r(T) + n(T) = \dim(\text{domain})$

$\Rightarrow 3+0 = 3$ (true)

Hence rank-nullity theorem is verified //

(2) Verify the rank nullity theorem for the linear transformation $T: V_3(R) \to V_2(R)$ defined by $T(x,y,z) = (y-x, y-z)$

Let $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$ be the standard basis

Given $T(x,y,z) = (y-x, y-z)$

$T(e_1) = T(1,0,0) = (-1,0)$

$T(e_2) = T(0,1,0) = (1,1)$

$T(e_3) = T(0,0,1) = (0,-1)$

Consider $A = \begin{bmatrix} -1 & 0 \\ 1 & 1 \\ 0 & -1 \end{bmatrix}$

$R_2 \longrightarrow R_2 + R_1$

$A \sim \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}$

$R_3 \longrightarrow R_3 + R_2$

$A \sim \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ reduced to echelon form (having 2 non-zero rows)

$\therefore$ $\boxed{\text{Rank of LT} = r(T) = 2}$

To find $N(T)$:

Consider $T(x,y,z) = (0,0)$

$(y-x, y-z) = (0,0)$

$\Rightarrow$ $\left.\begin{array}{l} y-x=0 \\ y-z=0 \end{array}\right\}$ $\begin{array}{l} \Rightarrow y=x \\ \Rightarrow y=z \end{array}$

$\therefore$ $x=y=z$

suppose $z=a$, $a \neq 0$ then $x=0$, $y=0$

$\therefore$ $N(T) = \{(a,a,a)\}$ is the null space

$\therefore$ $\boxed{\dim(N(T)) = n(T) = 1}$

We have $r(T) + n(T) = \dim(\text{domain})$

$\Rightarrow$ $2+1 = 3$ (true)

Hence rank-nullity theorem is verified

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(3) Find the kernel and range of the linear operator $T(x,y,z) = (x+y, z)$ of $R^3 \to R^2$.

$\to$ Let $e_1 = (1,0,0)$ $e_2 = (0,1,0)$ $e_3 = (0,0,1)$ be the standard bases.

Given $T(x,y,z) = (x+y, z)$

$T(e_1) = T(1,0,0) = (1,0)$

$T(e_2) = T(0,1,0) = (1,0)$

$T(e_3) = T(0,0,1) = (0,1)$

Consider $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$R_2 \to R_2 - R_1$

$A \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$

$R_2 \leftrightarrow R_3$

$A \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ is reduced to echelon form (having 2 non-zero rows)

$\therefore R(T)$ is the subspace generated by $(1,0)$ and $(0,1)$

i.e, $R(T) = \{x(1,0) + y(0,1)\}$

$= \{(x, y)$ is the range space

<u>To find N(T):</u>

consider $T(x,y,z) = (0,0)$

$\Rightarrow (x+y, z) = (0,0)$

$x+y = 0$; $\boxed{z = 0}$

$\Rightarrow x = -y$

$\Rightarrow \boxed{y = -x}$ and $\boxed{x = x}$ itself.

$\therefore N(T) = \{(x, -x, 0)\}$ is the null space.

(4) State rank-nullity theorem and verify the theorem for L.T $T: R^3 \to R^3$ defined by $T(x,y,z) = (x+2y-z, y+z, x+y-2z)$

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statement of rank-nullity theorem: (from theory)

Let $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$ be the standard bases.

Given $T(x,y,z) = (x+2y-z,\ y+z,\ x+y-2z)$

$T(e_1) = T(1,0,0) = (1,0,1)$

$T(e_2) = T(0,1,0) = (2,1,1)$

$T(e_3) = T(0,0,1) = (-1,1,-2)$

Consider $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & -2 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$; $R_3 \to R_3 + R_1$

$A \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$

$R_3 \to R_3 - R_2$

$A \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ reduced to echelon form (having 2 non-zero rows)

$\therefore$ $\boxed{\text{Rank of LT} = r(T) = 2}$

To find N(T):

consider $T(x,y,z) = (0,0,0)$

$(x+2y-z,\ y+z,\ x+y-2z) = (0,0,0)$

$$\left.\begin{aligned} x+2y-z &= 0 \\ y+z &= 0 \\ x+y-2z &= 0 \end{aligned}\right\}$$

on solving,

$\boxed{y=-z}$, ~~and~~ $\boxed{x=3z}$ and $\boxed{z=z}$ itself

$\therefore N(T) = \{(3z, -z, z)\}$ is the null space

$\therefore$ $\boxed{\dim(N(T)) = n(T) = 1}$

We have $r(T) + n(T) = \dim(\text{domain})$

$\Rightarrow 2+1 = 3$ (true)

Hence rank-nullity theorem is verified.

Inner product spaces and orthogonality.

<u>Inner product space</u>: Let V be a real vector space. Suppose to each pair of vectors $u, v \in V$ there is assigned a real no, denoted by $\langle u, v\rangle$. This function is called a (real) inner product of V if it satisfies the following axioms.

(i) Linear property: $\langle au_1 + bu_2, v\rangle = a\langle u_1, v\rangle + b\langle u_2, v\rangle$

(ii) symmetric property: $\langle u, v\rangle = \langle v, u\rangle$

(iii) positive definite property: $\langle u, u\rangle \geq 0$; and $\langle u, u\rangle = 0$ if and only if $u = 0$

(iv) Angle b/w two vectors u, v in inner product space V is given by $\cos\theta = \dfrac{\langle u, v\rangle}{\|u\|\|v\|}$

<u>Norm of a vector</u>: An inner product, $\langle u, u\rangle$ is non-negative for any vector u.

$$\|u\| = \sqrt{\langle u, u\rangle} \text{ or } \|u\|^2 = \langle u, u\rangle$$

This non-negative number is called the norm or length of u.

- Every non-zero vector v in V can be multiplied the reciprocal of its length to obtain the unit vector

$$\hat{v} = \frac{1}{\|v\|} \cdot v$$ which is a positive multiple of v.

This process is called normalizing v.

<u>Orthogonality</u>: Let V be an inner product space. The vectors $u, v \in V$ are said to be orthogonal and u is said to be orthogonal to v if

$$\langle u, v\rangle = 0.$$

Problems on inner product space and orthogonality.

① Consider the vectors $u = (1,2,4)$, $v = (2,-3,5)$ and $w = (4,2,-3)$ in R^3. Find

(i) $\langle u, v\rangle$ (ii) $\langle u, w\rangle$ (iii) $\langle v, w\rangle$ (iv) $\langle u+v, w\rangle$

(v) $\|u\|$ (vi) $\|v\|$ (vii) $\hat{u}$ (normalise u) (viii) $\hat{v}$

(ix) Angle b/w u and v.

(i) $\langle u, v\rangle = (1\times 2) + (2\times -3) + (4\times 5)$
$= 2 - 6 + 20 = 16$

(ii) $\langle u, w\rangle = (1\times 4) + (2\times 2) + (4\times -3)$
$= 4 + 4 - 12 = -4$

(iii) $\langle v, w\rangle = (2\times 4) + (-3\times 2) + (5\times -3)$
$= 8 - 6 - 15 = -13$

(iv) $\langle u+v, w\rangle = (3,-1,9)\cdot(4,2,-3)$
$= 12 - 2 - 27 = -17$

(v) $\|u\|^2 = \langle u, u\rangle$
$= (1,2,4)\cdot(1,2,4)$
$= 1 + 4 + 16 = 21$
$\Rightarrow \|u\| = \sqrt{21}$

(vi) $\|v\|^2 = \langle v, v\rangle$
$= (2,-3,5)\cdot(2,-3,5)$
$= 4 + 9 + 25 = 38$
$\Rightarrow \|v\| = \sqrt{38}$

(vii) $\hat{u} = \dfrac{u}{\|u\|} = \dfrac{(1,2,4)}{\sqrt{21}} = \left(\dfrac{1}{\sqrt{21}}, \dfrac{2}{\sqrt{21}}, \dfrac{4}{\sqrt{21}}\right)$

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(viii) $\hat{v} = \frac{v}{\|v\|} = \frac{(2,-3,5)}{\sqrt{38}} = \left(\frac{2}{\sqrt{38}}, \frac{-3}{\sqrt{38}}, \frac{5}{\sqrt{38}}\right)$

(ix) Angle b/w u and v is :

$$\cos\theta = \frac{\langle u, v\rangle}{\|u\|\,\|v\|}$$

$$\cos\theta = \frac{16}{\sqrt{21}\sqrt{38}}$$

$$\theta = \cos^{-1}\left(\frac{16}{\sqrt{21}\sqrt{38}}\right)$$

(2) Show that the function $f(x) = 3x-2$ and $g(x) = x$ are orthogonal in P_n with inner product $\langle f, g\rangle = \int_0^1 f(x)\, g(x)\, dx$

$$\langle f, g\rangle = \int_0^1 f(x)\, g(x)\, dx$$

$$= \int_0^1 (3x-2)(x)\, dx$$

$$= \int_0^1 (3x^2-2x)\, dx$$

$$\left[\frac{\cancel{3}x^3}{\cancel{3}} - \cancel{2}\frac{x^2}{\cancel{2}}\right]_0^1$$

$$= 1^3 - 1^2 = 0 //$$

$f(x), g(x)$ are orthogonal in P_n

(2) Define an inner product space. consider $f(t) = 3t-5$, $g(t) = t^2$, $\langle f, g\rangle = \int_0^1 f(t)\, g(t)\, dt$

Find $\langle f, g\rangle$

$\rightarrow$ $\langle f, g \rangle = \int_0^1 f(t)\, g(t)\, dt$

$= \int_0^1 (3t-5) \cdot t^2\, dt$

$= \int_0^1 (3t^3 - 5t^2)\, dt$

$= \left[3 \cdot \frac{t^4}{4} - 5 \cdot \frac{t^3}{3}\right]_0^1$

$= \left[\frac{3}{4} - \frac{5}{3}\right] \Rightarrow \frac{-11}{12}$ //

③ Consider the following polynomials in $P(t)$ and inner product
$f(t) = t+2$, $g(t) = 3t-2$, $h(t) = t^2 - 2t - 3$ and
$\langle f, g \rangle = \int_0^1 f(t)\, g(t)\, dt$.

(a) find $\langle f, g \rangle$ and $\langle f, h \rangle$

(b) find $\|f\|$ and $\|g\|$

(c) normalize f and g

$\rightarrow$ (a) $\langle f, g \rangle = \int_0^1 (t+2)(3t-2)\, dt = \int_0^1 (3t^2 + 4t - 4)\, dt$

$\langle f, g \rangle = \left[t^3 + 2t^2 - 4t\right]_0^1 \Rightarrow -1$ //

$\langle f, h \rangle = \int_0^1 (t+2)(t^2 - 2t - 3)\, dt = \left[\frac{t^4}{4} - \frac{7t^2}{2} - 6t\right]_0^1 = \frac{-37}{4}$

(b) $\|f\|^2 = \langle f, f \rangle = \int_0^1 (t+2)(t+2)\, dt \Rightarrow \frac{19}{3}$

$\|f\| = \frac{\sqrt{19}}{\sqrt{3}}$

$\|g\|^2 = \langle g, g \rangle = \int_0^1 (3t-2)(3t-2)\, dt \Rightarrow 1$

$\|g\| = \sqrt{1} = 1$

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(c) $\hat{f} = \frac{f}{\|f\|} = \frac{\sqrt{3}}{\sqrt{19}}(t+2)$

$\hat{g} = \frac{g}{\|g\|} = 3t-2$

(4) Let $M = M_{2,3}$ with inner product $\langle A, B\rangle = tr(B^T A)$ and let $A = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ $C = \begin{bmatrix} 3 & -5 & 2 \\ 1 & 0 & -4 \end{bmatrix}$

Find (a) $\langle A, B\rangle$, $\langle A, C\rangle$, $\langle B, C\rangle$

(b) $\langle 2A+3B, 4C\rangle$

(c) $\|A\|$ and $\|B\|$

→ (a) $\langle A, B\rangle = 9+16+21+24+25+24 = 119$

$\langle A, C\rangle = 27-40+14+6+0-16 = -9$

$\langle B, C\rangle = 3-10+6+4+0-24 = -21$

(b) $2A+3B = \begin{bmatrix} 21 & 22 & 23 \\ 24 & 25 & 26 \end{bmatrix}$ $4C = \begin{bmatrix} 12 & -20 & 8 \\ 4 & 0 & -16 \end{bmatrix}$

$\langle 2A+BC, 4C\rangle = 252-440+96+0-416 = -324$

(c) $\|A\|^2 = \langle A, A\rangle = 9^2+8^2+7^2+6^2+5^2+4^2 = 271 \Rightarrow \|A\| = \sqrt{271}$

$\|B\|^2 = \langle B, B\rangle = 1^2+2^2+3^2+4^2+5^2+6^2 = 91 \Rightarrow \|B\| = \sqrt{91}$

(5) verify the vectors $u = (1,1,1)$ $v = (1,2,-3)$ and $w = (1,-4,3)$ in R^3 are orthogonal or not

→ $\langle u, v\rangle = 1+2-3 = 0$

The vectors u and v are orthogonal

$\langle u, w\rangle = 1-4+3 = 0$

The vectors u and w are orthogonal

$\langle v, w\rangle = 1-8-9 = -16$

The vectors v and w are not orthogonal