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Department of Mathematics

LECTURE NOTES

MATHEMATICS-1 FOR MECHANICAL ENGINEERING STREAM (22MATM11)

MODULE-1

DIFFERENTIAL CALCULUS-1

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CALCULUS AND LINEAR ALGEBRA

Module - 1

POLAR CURVES AND EVALUATES

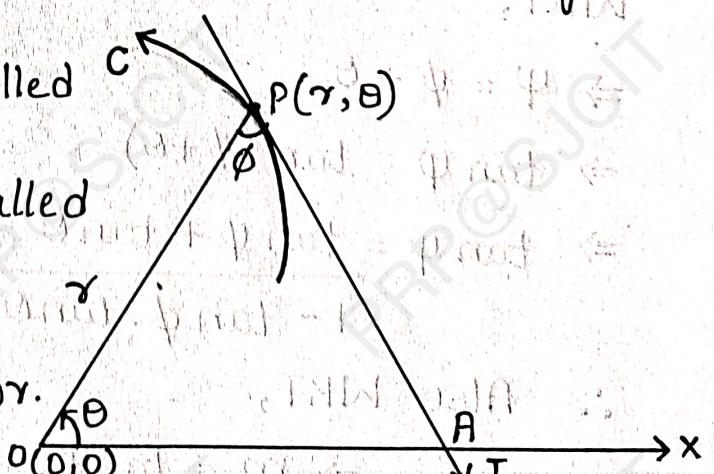
Introduction :-

Let 'P' be any point on the plane $O(0,0)$ be an origin, which extends the \overrightarrow{Ox} and $OP = r$ and 'C' be a curve at the point 'P' with the tangent 'T'.

Here, the point $O(0,0)$ is called pole,

the point $P(r, \theta)$ is called polar co-ordinates,

the distance $OP = r$ is called radius of vector.



' θ ' be the angle between radius of vector and initial line,

' ϕ ' is called the angle between the radius of vector and tangent to the polar curve ' C ' and the polar curve which is of the form either $r = f(\theta)$ or $f(r, \theta) = C$.

The entire above system is called "Polar System".

ANGLE BETWEEN THE RADIUS OF VECTOR AND TANGENT TO THE POLAR CURVE :-

Given,

$P(r, \theta)$ be any point on the plane,

Let, $OP = r$ be the radius of a vector, which makes the angle to its polar curve ' ϕ ' and ' ψ ' be the angle made by the tangent to the initial line.

Finally, ' θ ' be the angle between radius of vector and initial line \overrightarrow{OX} .

WKT,

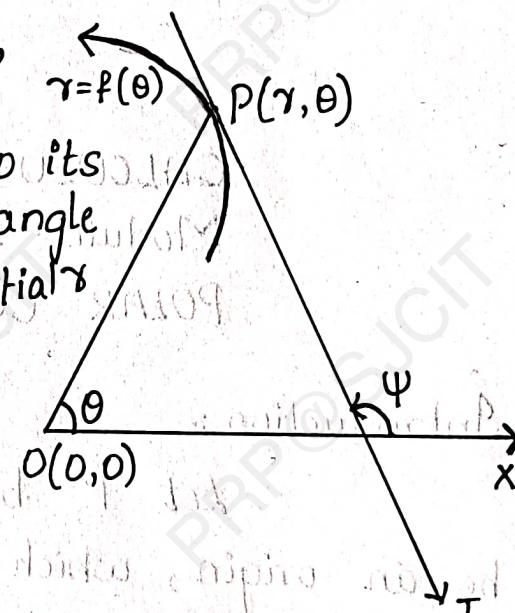
$$\Rightarrow \psi = \phi + \theta$$

$$\Rightarrow \tan \psi = \tan(\phi + \theta)$$

$$\Rightarrow \tan \psi = \frac{\tan \phi + \tan \theta}{1 - \tan \phi \cdot \tan \theta} \quad \text{①}$$

\therefore Also WKT,

$$\Rightarrow m = \tan \psi = \frac{dy}{dx}$$



$$\Rightarrow \tan \psi = \frac{dy}{dx}$$

$$\Rightarrow r \cos \theta = x \quad \text{--- (3)}$$

diff w.r.t 'θ'

$$(3) \Rightarrow \cos \theta \frac{dr}{d\theta} + r(-\sin \theta) = \frac{dx}{d\theta}$$

$$\Rightarrow \cos \theta \frac{dr}{d\theta} - r \sin \theta = \frac{dx}{d\theta}$$

$$(4) \Rightarrow \frac{dy}{d\theta} = \sin \theta \frac{dy}{d\theta} + r \cos \theta$$

From equation (2)

$$\Rightarrow \tan \psi = \frac{dy/d\theta}{dx/d\theta}$$

$$\Rightarrow \tan \psi = \frac{\sin \theta \cdot \frac{dy}{d\theta} + r \cos \theta}{\cos \theta \cdot \frac{dr}{d\theta} - r \sin \theta}$$

$$\frac{\sin \theta \cdot \frac{dy}{d\theta}}{\cos \theta \cdot \frac{dr}{d\theta}} + \frac{r \cos \theta}{\cos \theta \cdot \frac{dr}{d\theta}}$$

$$\Rightarrow \tan \psi = \frac{\frac{\cos \theta \cdot \frac{dy}{d\theta}}{\cos \theta \cdot \frac{dr}{d\theta}} - \frac{r \sin \theta}{\cos \theta \cdot \frac{dr}{d\theta}}}{\tan \theta + r \left[\frac{d\theta}{dr} \right]}$$

$$\Rightarrow \tan \psi = \frac{\tan \theta + r \left[\frac{d\theta}{dr} \right]}{1 - (\tan \theta) \left[r \frac{d\theta}{dr} \right]}$$

$$\Rightarrow \tan \theta + \tan \phi = \tan \theta + r \frac{d\theta}{dr} = 1 - \tan \theta \cdot \tan \phi$$

$$\Rightarrow \tan \phi = r \frac{d\theta}{dr}$$

$$\Rightarrow \cot \phi = \frac{1}{r} \frac{d\theta}{d\theta}$$

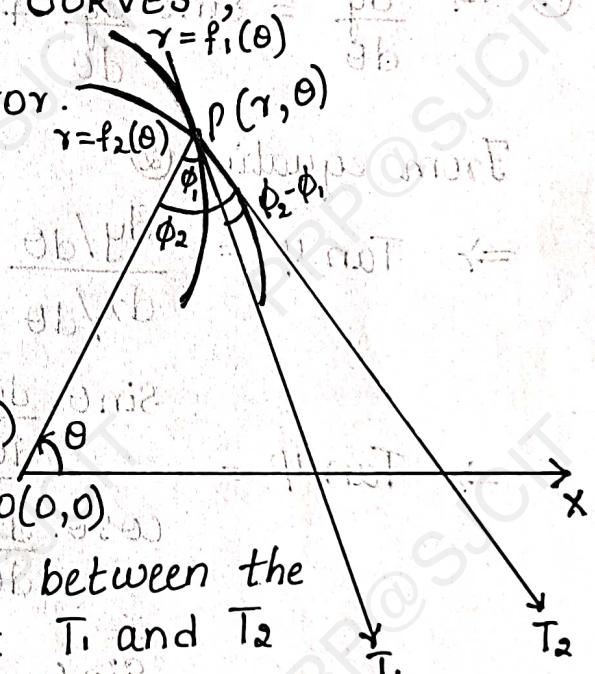
ANGLE BETWEEN Two POLAR CURVES

Let, $OP=r$ be the radius of vector.

$\gamma = f_1(\theta)$, $\gamma = f_2(\theta)$ be the two polar curves intersecting at 'P'.

Let, T_1 and T_2 be the two tangents to the curves, $\gamma = f_1(\theta)$ and $\gamma = f_2(\theta)$ at 'P'

Let, ϕ_1 and ϕ_2 be the angles between the radius of vector and tangent T_1 and T_2 respectively.



∴ The angle between the given two polar curves can be evaluated as ϕ_1 and ϕ_2 .
 If $|\phi_2 - \phi_1| = \frac{\pi}{2}$, then we can say that the given curves are intersecting perpendicular (or) Orthogonally.

PROVE THAT:

$$\frac{1}{P^2} = \frac{1}{r^2} + \frac{1}{r^4} \left[\frac{dr}{d\theta} \right]^2$$

using usual notations.

Let $OP = r$, be the radius of vector,

let $r = f(\theta)$ be a polar curve,

let 'T' be the tangent to $r = f(\theta)$
the polar curve at 'P',

let ' ϕ ' be the angle between radius
of vector and tangent to the polar
curve,

Let 'M' be the foot of perpendicular,

Let $OM = P$,

∴ From, ΔOPM :

$$\Rightarrow \sin \phi = \frac{OM}{OP}$$

$$\Rightarrow \sin \phi = \frac{P}{r}$$

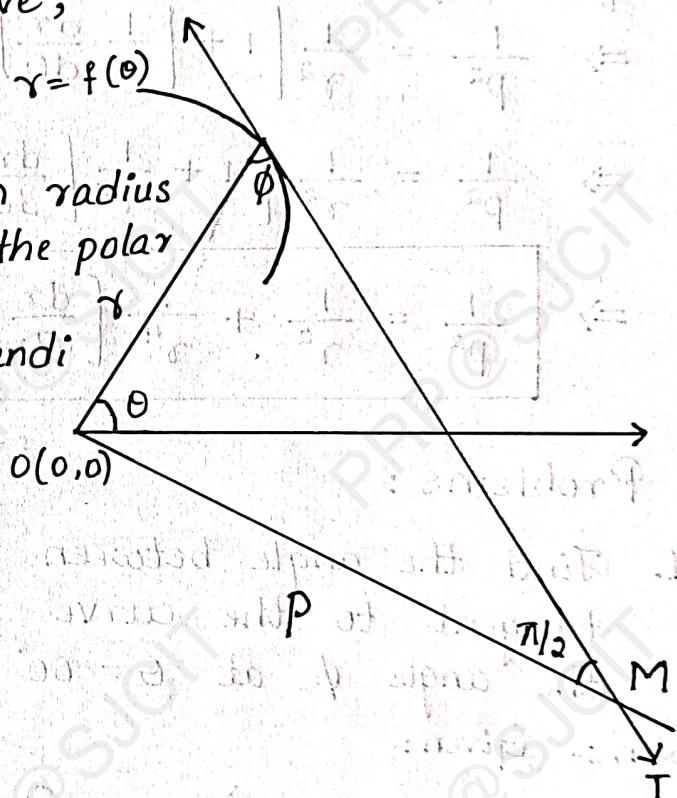
$$\Rightarrow P = r \sin \phi \quad \boxed{1} \text{ [SOBS]}$$

$$\Rightarrow P^2 = r^2 \sin^2 \phi \quad [\text{reciprocal}]$$

$$\Rightarrow \frac{1}{P^2} = \frac{1}{r^2 \sin^2 \phi}$$

$$\Rightarrow \frac{1}{P^2} = \frac{1}{r^2} \csc^2 \phi$$

$$\Rightarrow \boxed{\frac{1}{P^2} = \frac{1}{r^2} (1 + \cot^2 \phi)} \quad \boxed{2}$$



WKT,

$$\Rightarrow \tan \phi = \gamma \frac{d\theta}{dr}$$

$$\Rightarrow \cot \phi = \frac{1}{\gamma} \frac{dr}{d\theta}$$

from equation ②

$$\Rightarrow \frac{1}{p^2} = \frac{1}{\gamma^2} \left[1 + \left(\frac{1}{\gamma} \frac{dr}{d\theta} \right)^2 \right]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{\gamma^2} \left[1 + \frac{1}{\gamma^2} \left(\frac{dr}{d\theta} \right)^2 \right]$$

$$\Rightarrow \boxed{\frac{1}{p^2} = \frac{1}{\gamma^2} + \frac{1}{\gamma^4} \left(\frac{dr}{d\theta} \right)^2}$$

Hence proved.

Problems :

1. Find the angle between the radius of vector and tangent to the curve $r = a(1 - \cos\theta)$, also find the angle ϕ at $\theta = 60^\circ$.

Soln :- Given :-

$$\Rightarrow r = a(1 - \cos\theta) \quad \text{--- ①}$$

diff w.r.t θ

$$\Rightarrow \frac{dr}{d\theta} = a(0 - (-\sin\theta))$$

$$\Rightarrow \frac{dr}{d\theta} = a\sin\theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a\sin\theta}{r}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a\sin\theta}{a(1 - \cos\theta)}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a\sin\theta/2 \cdot \cos\theta/2}{a\sin^2\theta/2}$$

$$\Rightarrow \cot \phi = \cot \left[\frac{\theta}{2} \right]$$

$$\Rightarrow \phi = \frac{\theta}{2}$$

$$\therefore \text{where, } \theta = \frac{\pi}{3}$$

$$\Rightarrow \theta = \frac{\pi/3}{2} \Rightarrow \phi = \frac{\pi}{6}$$

2. Find the angle b/w the radius of vector and tangent to the curve $\gamma = a(1 + \cos \theta)$, also find the angle ϕ at $\theta = \pi/6$.

Soln:-

$$\Rightarrow \gamma = a(1 + \cos \theta) \quad \text{--- (1)}$$

diff w.r.t θ

$$\Rightarrow \frac{d\gamma}{d\theta} = -a \sin \theta$$

$$\Rightarrow \frac{1}{\gamma} \frac{d\gamma}{d\theta} = -\frac{a \sin \theta}{\gamma}$$

$$\Rightarrow \frac{1}{\gamma} \frac{d\gamma}{d\theta} = -\frac{a \sin \theta}{a(1 + \cos \theta)}$$

$$\Rightarrow \frac{1}{\gamma} \frac{d\gamma}{d\theta} = -\frac{a \sin \theta / 2 \cos \theta / 2}{a \cos^2 \theta / 2}$$

$$\Rightarrow \cot \phi = -\tan \left[\frac{\theta}{2} \right]$$

$$\Rightarrow \cot \phi = \tan \left[\frac{\pi}{2} + \frac{\theta}{2} \right]$$

$$\Rightarrow \cot \phi = \cot \left[\frac{\pi}{2} + \frac{\theta}{2} \right]$$

$$\Rightarrow \phi = \frac{\pi}{2} + \frac{\theta}{2}$$

$$\text{Where, } \theta = \frac{\pi}{6} \Rightarrow \phi = \frac{\pi}{2} + \frac{\pi/6}{\alpha} \\ \Rightarrow \phi = \frac{\pi}{2} + \frac{\pi}{12} \\ \Rightarrow \phi = \frac{6\pi + \pi}{12} \Rightarrow \boxed{\phi = \frac{7\pi}{12}}$$

3. Show that the given pairs of polar curves can intersect orthogonally.

$$\text{if } r = a(1 - \cos\theta), \quad r = b(1 + \cos\theta)$$

$$\Rightarrow r = a(1 - \cos\theta) \text{ --- ①} \quad \Rightarrow r = b(1 + \cos\theta) \text{ --- ②}$$

$$\Rightarrow \text{diff w.r.t } \theta \quad \Rightarrow \text{diff w.r.t } \theta$$

$$\Rightarrow \frac{dr}{d\theta} = a(0 - (-\sin\theta)) \quad \Rightarrow \frac{dr}{d\theta} = b(0 - \sin\theta)$$

$$\Rightarrow \frac{dr}{d\theta} = a\sin\theta \quad \Rightarrow \frac{dr}{d\theta} = -b\sin\theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a\sin\theta}{r} \quad \Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\frac{b\sin\theta}{r}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a\sin\theta}{a(1 - \cos\theta)} \quad \Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\frac{b\sin\theta}{b(1 + \cos\theta)}$$

$$\Rightarrow \cot\phi_1 = \frac{a\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{a\sin^2\frac{\theta}{2}} \quad \Rightarrow \cot\phi_2 = \frac{-a\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{a\cos^2\frac{\theta}{2}}$$

$$\Rightarrow \cot\phi_1 = \cot\frac{\theta}{2} \quad \Rightarrow \cot\phi_2 = -\tan\frac{\theta}{2}$$

$$\Rightarrow \phi_1 = \frac{\theta}{2} \quad \Rightarrow \cot\phi_2 = \cot\left(\frac{\theta}{2} + \frac{\pi}{2}\right)$$

$$\Rightarrow |\phi_2 - \phi_1| = \frac{\pi}{2} + \frac{\theta}{2} - \frac{\theta}{2}$$

$$\Rightarrow |\phi_2 - \phi_1| = \frac{\pi}{2}$$

$$\text{iii) } r = a(1 + \sin \theta) \quad | \quad r = b(1 - \sin \theta)$$

$$\Rightarrow r = a(1 + \sin \theta) \quad \text{--- ①}$$

diff w.r.t θ

$$\Rightarrow \frac{dr}{d\theta} = a \cos \theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a \cos \theta}{r}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a \cos \theta}{a(1 + \sin \theta)}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{\cos^2 \frac{\theta}{2} \sin^2 \frac{\theta}{2}}{\cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2} + 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{\left[\cos \frac{\theta}{2} + \sin \frac{\theta}{2} \right] \left[\cos \frac{\theta}{2} - \sin \frac{\theta}{2} \right]}{\left[\cos \frac{\theta}{2} + \sin \frac{\theta}{2} \right]^2}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{\cos \frac{\theta}{2} - \sin \frac{\theta}{2}}{\cos \frac{\theta}{2} + \sin \frac{\theta}{2}}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{\frac{\cos \frac{\theta}{2}}{\cos \frac{\theta}{2}} - \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}}}{\frac{\cos \frac{\theta}{2}}{\cos \frac{\theta}{2}} + \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}}}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{1 - \tan \frac{\theta}{2}}{1 + \tan \frac{\theta}{2}}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{\tan(\pi/4) - \tan(\theta/2)}{1 + \tan(\pi/4) \tan(\theta/2)}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \tan \left[\frac{\pi}{4} - \frac{\theta}{2} \right]$$

$$\Rightarrow \cot \phi_1 = \cot \left[\frac{\pi}{2} - \left[\frac{\pi}{4} - \frac{\theta}{2} \right] \right]$$

$$\Rightarrow \phi_1 = \frac{\pi}{2} - \frac{\pi}{4} + \frac{\theta}{2}$$

$$\Rightarrow \phi_1 = \frac{\pi}{4} + \frac{\theta}{2}$$

$$\gamma = b(1 - \sin \theta) \quad \text{--- ②}$$

$$\Rightarrow \frac{d\gamma}{d\theta} = -b \cos \theta$$

$$\Rightarrow \frac{1}{\gamma} \frac{d\gamma}{d\theta} = -\frac{b \cos \theta}{\gamma}$$

$$\Rightarrow \frac{1}{\gamma} \frac{d\gamma}{d\theta} = -\frac{b \cos \theta}{b(1 - \sin \theta)}$$

$$\Rightarrow \frac{1}{\gamma} \frac{d\gamma}{d\theta} = \frac{-\cos^2 \left[\frac{\theta}{2} \right] \sin^2 \left[\frac{\theta}{2} \right]}{\cos^2 \left[\frac{\theta}{2} \right] + \sin^2 \left[\frac{\theta}{2} \right] - 2 \sin \left[\frac{\theta}{2} \right] \cos \left[\frac{\theta}{2} \right]}$$

$$\Rightarrow \frac{1}{\gamma} \frac{d\gamma}{d\theta} = \frac{-[\cos \frac{\theta}{2} + \sin \frac{\theta}{2}] [\cos \frac{\theta}{2} - \sin \frac{\theta}{2}]}{[\cos \frac{\theta}{2} - \sin \frac{\theta}{2}]^2}$$

$$\Rightarrow \frac{1}{\gamma} \frac{d\gamma}{d\theta} = \frac{-\cos \frac{\theta}{2} + \sin \frac{\theta}{2}}{\cos \frac{\theta}{2} - \sin \frac{\theta}{2}}$$

$$\Rightarrow \cot \phi_2 = - \left| \frac{1 + \tan \frac{\theta}{2}}{1 - \tan \frac{\theta}{2}} \right|$$

$$\Rightarrow \cot \phi_2 = -\tan \left[\frac{\pi}{4} + \frac{\theta}{2} \right]$$

$$\Rightarrow \cot \phi_2 = \cot \left[\frac{\pi}{2} + \left[\frac{\pi}{4} + \frac{\theta}{2} \right] \right]$$

$$\Rightarrow \phi_2 = \frac{\pi}{2} + \frac{\pi}{4} + \frac{\theta}{2}$$

$$\Rightarrow |\phi_2 - \phi_1| = \left| \frac{\pi}{2} + \frac{\pi}{4} + \frac{\theta}{2} - \frac{\pi}{4} - \frac{\theta}{2} \right|$$

$$\Rightarrow |\phi_2 - \phi_1| = \frac{\pi}{2}$$

iii) $\gamma = \frac{a}{1 + \cos \theta}$

$$\Rightarrow \gamma(1 + \cos \theta) = a \quad \text{--- ①}$$

\Rightarrow diff w.r.t θ

$$\Rightarrow (1 + \cos \theta) \frac{d\gamma}{d\theta} + \gamma(-\sin \theta) = 0$$

$$\Rightarrow (1 + \cos \theta) \frac{d\gamma}{d\theta} = \gamma \sin \theta$$

$$\Rightarrow \frac{1}{\gamma} \frac{d\gamma}{d\theta} = \frac{\sin \theta}{1 + \cos \theta}$$

$$\Rightarrow \frac{1}{\gamma} \frac{d\gamma}{d\theta} = \frac{\sin \theta/2 \cos \theta/2}{\cos^2 \theta/2}$$

$$\Rightarrow \frac{1}{\gamma} \frac{d\gamma}{d\theta} = \frac{\sin \theta/2}{\cos \theta/2}$$

$$\Rightarrow \cot \phi_1 = \tan \frac{\theta}{2}$$

$$\Rightarrow \cot \phi_1 = \cot \left[\frac{\pi}{2} - \frac{\theta}{2} \right]$$

$$\Rightarrow \phi_1 = \frac{\pi}{2} - \frac{\theta}{2}$$

$$\Rightarrow |\phi_2 - \phi_1| = -\frac{\theta}{2} - \left[\frac{\pi}{2} - \frac{\theta}{2} \right]$$

$$\Rightarrow |\phi_2 - \phi_1| = -\frac{\theta}{2} - \frac{\pi}{2} + \frac{\theta}{2}$$

$$\Rightarrow |\phi_2 - \phi_1| = \frac{\pi}{2}$$

$$\gamma = \frac{b}{1 - \cos \theta}$$

$$\Rightarrow \gamma(1 - \cos \theta) = b \quad \text{--- ②}$$

\Rightarrow diff w.r.t θ

$$\Rightarrow (1 - \cos \theta) \frac{d\gamma}{d\theta} + \gamma(\sin \theta) = 0$$

$$\Rightarrow (1 - \cos \theta) \frac{d\gamma}{d\theta} = -\gamma \sin \theta$$

$$\Rightarrow \frac{1}{\gamma} \frac{d\gamma}{d\theta} = -\frac{\sin \theta}{1 - \cos \theta}$$

$$\Rightarrow \frac{1}{\gamma} \frac{d\gamma}{d\theta} = -\frac{\sin \theta/2 \cos \theta/2}{\cos^2 \theta/2}$$

$$\Rightarrow \frac{1}{\gamma} \frac{d\gamma}{d\theta} = -\cot \frac{\theta}{2}$$

$$\Rightarrow \cot \phi_2 = \cot \left[-\frac{\theta}{2} \right]$$

$$\Rightarrow \phi_2 = -\frac{\theta}{2}$$

$$\text{iv} \quad r^n = a^n \cos n\theta$$

$$\Rightarrow r^n = a^n \cos n\theta \quad \text{--- ①}$$

diff w.r to θ

$$\Rightarrow \cancel{r^{n-1}} \frac{dr}{d\theta} = a^n (-\sin n\theta)(\cancel{r})$$

$$\Rightarrow \frac{r^n}{r} \frac{dr}{d\theta} = -a^n \sin n\theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\frac{a^n \sin n\theta}{r^n}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\frac{a^n \sin n\theta}{a^n \cos n\theta}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\tan n\theta$$

$$\Rightarrow \cot \phi_1 = \cot \left[\frac{\pi}{2} + n\theta \right]$$

$$\Rightarrow \phi_1 = \frac{\pi}{2} + n\theta$$

$$\Rightarrow |\phi_2 - \phi_1| = n\theta - \frac{\pi}{2} - n\theta$$

$$\Rightarrow |\phi_2 - \phi_1| = \frac{\pi}{2}$$

$$\text{v} \quad r^n \cos n\theta = a^n$$

$$\Rightarrow r^n = \frac{a^n}{\cos n\theta} \quad \text{--- ②}$$

diff w.r to θ

$$\Rightarrow \cancel{r^{n-1}} \frac{dr}{d\theta} = 0 \frac{(\cos n\theta) - a^n (-\sin n\theta)(\cancel{r})}{(\cos n\theta)^2}$$

$$\Rightarrow \frac{r^n}{r} \frac{dr}{d\theta} = \frac{a^n \sin n\theta}{(\cos n\theta)^2}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a^n \sin n\theta}{(\cos n\theta)^2 r^n}$$

$$r^n = b^n \sin n\theta$$

$$\Rightarrow r^n = b^n \sin n\theta \quad \text{--- ③}$$

diff. w.r to θ

$$\Rightarrow \cancel{r^{n-1}} \frac{dr}{d\theta} = b^n (\cos n\theta)(\cancel{r})$$

$$\Rightarrow \frac{r^n}{r} \frac{dr}{d\theta} = b^n \cos n\theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{b^n \cos n\theta}{r^n}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{b^n \cos n\theta}{b^n \sin n\theta}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \cot n\theta$$

$$\Rightarrow \cot \phi_2 = \cot n\theta$$

$$\Rightarrow \phi_2 = n\theta$$

$$r^n \sin n\theta = b^n$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a^n \sin n\theta}{(\cos n\theta)^n} \left[\frac{a^n}{\cos n\theta} \right]$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{\sin n\theta}{\cos n\theta}$$

$$\Rightarrow \cot \phi_1 = \tan n\theta$$

$$\Rightarrow \cot \phi_1 = \cot \left[\frac{\pi}{2} - n\theta \right]$$

$$\Rightarrow \phi_1 = \frac{\pi}{2} - n\theta$$

$$r^n \sin n\theta = b^n \quad \text{--- ②}$$

$$\Rightarrow r^n = \frac{b^n}{\sin n\theta}$$

diff w.r.t θ

$$\Rightarrow rr^{n-1} \frac{dr}{d\theta} = \frac{o(\sin n\theta) - b^n (\cos n\theta)(n)}{(\sin n\theta)^2}$$

$$\Rightarrow \frac{r^n}{r} \frac{dr}{d\theta} = - \frac{b^n \cos n\theta}{(\sin n\theta)^2}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = - \frac{b^n \cos n\theta}{(\sin n\theta)^2 r^n}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = - \frac{b^n \cos n\theta}{(\sin n\theta)^2 \left(\frac{b^n}{\sin n\theta} \right)}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = - \frac{\cos n\theta}{\sin n\theta}$$

$$\Rightarrow \cot \phi_2 = -\cot n\theta$$

$$\Rightarrow \cot \phi_2 = \cot (-n\theta)$$

$$\Rightarrow \phi_2 = -n\theta$$

$$\Rightarrow |\phi_2 - \phi_1| = -n\theta - \frac{\pi}{2} + n\theta$$

$$\Rightarrow |\phi_2 - \phi_1| = \frac{\pi}{2}$$

$$\text{v/r } \gamma = 4 \sec^2\left(\frac{\theta}{2}\right)$$

$$\Rightarrow \gamma = \frac{4}{\cos^2\left(\frac{\theta}{2}\right)}$$

$$\Rightarrow \gamma \cos^2\left(\frac{\theta}{2}\right) = 4$$

$$\Rightarrow \frac{\gamma(1+\cos\theta)}{2} = 4$$

$$\Rightarrow \gamma(1+\cos\theta) = 8 \quad \text{--- (1)}$$

diff w.r.t θ

$$\Rightarrow (1+\cos\theta) \frac{d\gamma}{d\theta} + \gamma(-\sin\theta) = 0$$

$$\Rightarrow (1+\cos\theta) \frac{d\gamma}{d\theta} = \gamma \sin\theta$$

$$\Rightarrow \frac{1}{\gamma} \frac{d\gamma}{d\theta} = \frac{\sin\theta}{1+\cos\theta}$$

$$\Rightarrow \frac{1}{\gamma} \frac{d\gamma}{d\theta} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}$$

$$\Rightarrow \cot\phi_1 = \tan\left(\frac{\theta}{2}\right)$$

$$\Rightarrow \cot\phi_1 = \cot\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$$

$$\Rightarrow \phi_1 = \frac{\pi}{2} - \frac{\theta}{2}$$

$$\Rightarrow |\phi_2 - \phi_1| = -\frac{\theta}{2} - \frac{\pi}{2} + \frac{\theta}{2}$$

$$\Rightarrow |\phi_2 - \phi_1| = \frac{\pi}{2}$$

$$\gamma = q \cosec^2\left(\frac{\theta}{2}\right)$$

$$\Rightarrow \gamma = \frac{q}{\sin^2\left(\frac{\theta}{2}\right)}$$

$$\Rightarrow \gamma \sin^2\left(\frac{\theta}{2}\right) = q$$

$$\Rightarrow \frac{\gamma(1-\cos\theta)}{2} = q$$

$$\Rightarrow \gamma(1-\cos\theta) = 18 \quad \text{--- (2)}$$

diff w.r.t θ

$$\Rightarrow (1-\cos\theta) \frac{d\gamma}{d\theta} + \gamma \sin\theta = 0$$

$$\Rightarrow (1-\cos\theta) \frac{d\gamma}{d\theta} = -\gamma \sin\theta$$

$$\Rightarrow \frac{1}{\gamma} \frac{d\gamma}{d\theta} = -\frac{\sin\theta}{1-\cos\theta}$$

$$\Rightarrow \frac{1}{\gamma} \frac{d\gamma}{d\theta} = -2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$\Rightarrow \cot\phi_2 = -\cot\left(\frac{\theta}{2}\right)$$

$$\Rightarrow \cot\phi_2 = \cot\left[-\frac{\theta}{2}\right]$$

$$\Rightarrow \phi_2 = -\frac{\theta}{2}$$

4. Find the angle between the given polar curves;

$$\text{i} \text{f } r = 2 \sin \theta$$

$$r = 2 \cos \theta$$

$$\Rightarrow r = 2 \sin \theta \quad \text{--- ①}$$

$$\Rightarrow r = 2 \cos \theta \quad \text{--- ②}$$

diff w.r to θ

diff w.r to θ

$$\Rightarrow \frac{dr}{d\theta} = 2 \cos \theta$$

$$\Rightarrow \frac{dr}{d\theta} = -2 \sin \theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{2 \cos \theta}{r}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\frac{2 \sin \theta}{r}$$

$$\Rightarrow \cot \phi_1 = \frac{2 \cos \theta}{2 \sin \theta}$$

$$\Rightarrow \cot \phi_2 = -\frac{2 \sin \theta}{2 \cos \theta}$$

$$\Rightarrow \cot \phi_1 = \cot \theta$$

$$\Rightarrow \cot \phi_2 = -\tan \theta$$

$$\Rightarrow \phi_1 = \theta$$

$$\Rightarrow \phi_2 = \frac{\pi}{2} + \theta$$

$$\Rightarrow |\phi_2 - \phi_1| = \frac{\pi}{2} + \theta - \theta$$

$$\Rightarrow |\phi_2 - \phi_1| = \frac{\pi}{2}$$

$$\text{ii} \text{f } r = a \log \theta$$

$$r = \frac{a}{\log \theta}$$

$$\Rightarrow r = a \log \theta \quad \text{--- ①}$$

$$\Rightarrow r \log \theta = a \quad \text{--- ②}$$

diff w.r to θ

diff w.r to θ

$$\Rightarrow \frac{dr}{d\theta} = a \cdot \frac{1}{\theta}$$

$$\Rightarrow (\log \theta) \frac{dr}{d\theta} + r \left[\frac{1}{\theta} \right] = 0$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a}{r\theta}$$

$$\Rightarrow \log \theta \frac{dr}{d\theta} = -\frac{r}{\theta}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a}{a \log \theta(\theta)}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{1}{-\theta \log \theta}$$

$$\Rightarrow r \frac{d\theta}{dr} = \theta \log \theta$$

$$\Rightarrow r \frac{d\theta}{dr} = -\theta \log \theta$$

$$\Rightarrow \tan \phi_1 = \theta \log \theta \quad \text{--- ③}$$

$$\Rightarrow \tan \phi_2 = -\theta \log \theta \quad \text{--- ④}$$

WKT,

$$\Rightarrow \tan(\phi_1 - \phi_2) = \frac{\tan \phi_1 - \tan \phi_2}{1 + \tan \phi_1 \cdot \tan \phi_2}$$

$$\Rightarrow \tan(\phi_1 - \phi_2) = \frac{\theta \log \theta - (-\theta \log \theta)}{1 + \theta \log \theta (-\theta \log \theta)}$$

$$\Rightarrow \tan(\phi_1 - \phi_2) = \frac{\theta \log \theta + \theta \log \theta}{1 + \theta^2 (\log \theta)^2}$$

$$\Rightarrow \tan(\phi_1 - \phi_2) = \frac{2\theta \log \theta}{1 - \theta^2 (\log \theta)^2} \quad \textcircled{4}$$

From equation ① & ②

$$\Rightarrow \alpha \log \theta = \frac{\alpha}{\log \theta}$$

$$\Rightarrow \log \theta = \frac{1}{\log \theta}$$

$$\Rightarrow (\log \theta)^2 = 1$$

$$\Rightarrow \log_e \theta = 1$$

$$\Rightarrow \theta = e^1$$

$$\Rightarrow \theta = e$$

from equation ④

$$\Rightarrow \tan(\phi_1 - \phi_2) = \frac{2e \cdot 1}{1 - e^2 (1)^2}$$

$$\Rightarrow \tan(\phi_1 - \phi_2) = \frac{2e}{1 - e^2}$$

$$\Rightarrow |\phi_1 - \phi_2| = \tan^{-1} \left[\frac{2e}{1 - e^2} \right] = 2 \tan^{-1} e.$$

$$\text{iii} \quad r^2 \sin 2\theta = 4$$

$$\Rightarrow r^2 \sin 2\theta = 4 \quad \text{--- (1)}$$

diff wrt to θ

$$\Rightarrow 2r \frac{dr}{d\theta} \sin 2\theta + r^2 \cos 2\theta (2) = 0 \quad \Rightarrow 2r \frac{dr}{d\theta} = 16 \cos 2\theta (2)$$

$$\Rightarrow r \frac{dr}{d\theta} \sin 2\theta = -r^2 \cos 2\theta \quad \Rightarrow r \frac{dr}{d\theta} = 16 \cos 2\theta$$

$$\Rightarrow \frac{r}{r^2} \frac{dr}{d\theta} = -\frac{\cos 2\theta}{\sin 2\theta}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\cot 2\theta$$

$$\Rightarrow \cot \phi_1 = \cot (-2\theta)$$

$$\Rightarrow \phi_1 = -2\theta$$

$$\Rightarrow |\phi_2 - \phi_1| = |2\theta - (-2\theta)|$$

$$\Rightarrow |\phi_2 - \phi_1| = 4\theta \quad \text{--- (3)}$$

From (1) & (2)

$$\Rightarrow \frac{4}{\sin 2\theta} = \frac{16}{r^2} \sin 2\theta$$

$$\Rightarrow 1 = 4 \sin^2 2\theta$$

$$\Rightarrow \sin^2 2\theta = \frac{1}{4} \quad (\div 2)$$

$$\Rightarrow \sin 2\theta = \frac{1}{2}$$

$$\Rightarrow 2\theta = \sin^{-1} \left[\frac{1}{2} \right]$$

$$\Rightarrow 2\theta = \pi/6$$

$$\Rightarrow \theta = \pi/12$$

$$\therefore |\phi_2 - \phi_1| = 4 \left[\frac{\pi}{12} \right] = \frac{\pi}{3}$$

$$r^2 = 16 \sin 2\theta$$

$$\Rightarrow r^2 = 16 \sin 2\theta \quad \text{--- (2)}$$

diff wrt to θ

$$\Rightarrow 2r \frac{dr}{d\theta} \sin 2\theta + r^2 \cos 2\theta (2) = 0 \quad \Rightarrow 2r \frac{dr}{d\theta} = 16 \cos 2\theta$$

$$\Rightarrow r \frac{dr}{d\theta} = 16 \cos 2\theta$$

$$\Rightarrow \frac{r}{r^2} \frac{dr}{d\theta} = \frac{16 \cos 2\theta}{r^2}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{16 \cos 2\theta}{16 \sin 2\theta}$$

$$\Rightarrow \cot \phi_2 = \cot 2\theta$$

$$\Rightarrow \phi_2 = 2\theta$$

$$\text{iv} \quad r = a(1 - \cos \theta)$$

$$\Rightarrow r = a(1 - \cos \theta) \quad \text{--- ①}$$

diff w.r.t to θ

$$\Rightarrow \frac{dr}{d\theta} = a \sin \theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a \sin \theta}{r}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a \sin \theta}{a(1 - \cos \theta)}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a \sin \theta / 2 \cos \theta / 2}{2 \sin^2 \theta / 2}$$

$$\Rightarrow \cot \phi_1 = \cot \theta / 2$$

$$\Rightarrow \phi_1 = \frac{\theta}{2}$$

$$\Rightarrow |\phi_2 - \phi_1| = \frac{\pi}{2} + \theta - \frac{\theta}{2}$$

$$\Rightarrow |\phi_2 - \phi_1| = \frac{\pi}{2} + \frac{\theta}{2} \quad \text{--- ③}$$

From, ① & ②

$$\Rightarrow a(1 - \cos \theta) = 2a \cos \theta$$

$$\Rightarrow 1 - \cos \theta = 2 \cos \theta$$

$$\Rightarrow 1 = 2 \cos \theta + \cos \theta$$

$$\Rightarrow 1 = 3 \cos \theta$$

$$\Rightarrow \frac{1}{3} = \cos \theta$$

$$\Rightarrow \theta = \cos^{-1} \left[\frac{1}{3} \right]$$

$$\therefore |\phi_2 - \phi_1| = \frac{\pi}{2} + \frac{\cos^{-1} \left[\frac{1}{3} \right]}{2}$$

$$r = 2a \cos \theta$$

$$\Rightarrow r = 2a \cos \theta \quad \text{--- ②}$$

diff w.r.t to θ

$$\Rightarrow \frac{dr}{d\theta} = -2a \sin \theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\frac{2a \sin \theta}{r}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\frac{2a \sin \theta}{2a \cos \theta}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\frac{\sin \theta}{\cos \theta}$$

$$\Rightarrow \cot \phi_2 = -\tan \theta$$

$$\Rightarrow \cot \phi_2 = \cot \left[\frac{\pi}{2} + \theta \right]$$

$$\Rightarrow \phi_2 = \frac{\pi}{2} + \theta$$

PEDAL EQUATION Of POLAR CURVES :-

Let ' γ ' = $f(\theta)$ be a polar curve,

Let ' ϕ ' be the angle between radius of vector and tangent to the curve $f(\theta)$ and ' p ' be the perpendicular distance from pole to the tangent ' T ', then

$$p = \gamma \sin \phi \quad (\text{or}) \quad \frac{1}{p^2} = \frac{1}{\gamma^2} (1 + \cot^2 \phi) \quad (\text{or})$$

$$\frac{1}{p^2} = \frac{1}{\gamma^2} + \frac{1}{\gamma^4} \left[\frac{d\gamma}{d\theta} \right]^2$$

is called the pedal equation of a polar curve $\gamma = f(\theta)$, it is called $p - \gamma$ equation excluding ' θ '.

Problems :-

1. Find the pedal equation of a polar curve.

$$\text{if } \gamma = a(1 + \cos \theta)$$

$$\Rightarrow \gamma = a(1 + \cos \theta) \quad \text{--- ①}$$

diff w.r.t θ

$$\Rightarrow \frac{d\gamma}{d\theta} = -a\sin \theta$$

$$\Rightarrow \frac{1}{\gamma} \frac{d\gamma}{d\theta} = -\frac{a\sin \theta}{\gamma}$$

$$\Rightarrow \frac{1}{\gamma} \frac{d\gamma}{d\theta} = -\frac{a\sin \theta}{a(1 + \cos \theta)}$$

$$\Rightarrow \cot \phi = -\frac{a \sin \theta/2 \cancel{\cos \theta/2}}{a \cos^2 \theta/2}$$

$$\Rightarrow \cot \phi = -\tan \theta/2$$

WKT,

$$\Rightarrow \frac{1}{P^2} = \frac{1}{\gamma^2} [1 + \cot^2 \phi]$$

$$\Rightarrow \frac{1}{P^2} = \frac{1}{\gamma^2} [1 + \tan^2 \frac{\theta}{2}]$$

$$\Rightarrow \frac{1}{P^2} = \frac{1}{\gamma^2} \sec^2 \frac{\theta}{2}$$

$$\Rightarrow \frac{1}{P^2} = \frac{1}{\gamma^2 \cos^2 \frac{\theta}{2}}$$

$$\Rightarrow P^2 = \gamma^2 \cos^2 \frac{\theta}{2}$$

$$\Rightarrow P^2 = \gamma^2 \left[1 + \frac{\cos \theta}{2} \right]$$

$$\Rightarrow P^2 = \frac{\gamma^2}{2} [1 + \cos \theta]$$

$$\Rightarrow P^2 = \frac{\gamma^2}{2} \left[\frac{r}{a} \right]$$

$$\Rightarrow P^2 = \frac{\gamma^3}{2a}$$

$$\Rightarrow 2ap^2 = \gamma^3$$

i.e. $r = a(1 - \cos \theta) \quad \text{--- (2)}$

diff w.r.t θ

$$\Rightarrow \frac{dr}{d\theta} = a \sin \theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a \sin \theta}{r}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a \sin \theta}{a(1 - \cos \theta)}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a \sin \theta / 2 \cos \theta / 2}{a \sin^2 \theta / 2}$$

$$\Rightarrow \cot \phi = \cot \frac{\theta}{2}$$

WKT,

$$\Rightarrow \frac{1}{p^2} = \frac{1}{\gamma^2} [1 + \cot^2 \phi]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{\gamma^2} [1 + \cot^2 \theta/2]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{\gamma^2} [\cosec^2 \theta/2]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{\gamma^2 \sin^2 \theta/2}$$

$$\Rightarrow p^2 = \gamma^2 \sin^2 \theta/2$$

$$\Rightarrow p^2 = \gamma^2 \left[\frac{1 - \cos \theta}{2} \right]$$

$$\Rightarrow p^2 = \frac{\gamma^2}{2} [1 - \cos \theta]$$

$$\Rightarrow p^2 = \frac{\gamma^2}{2} \left[\frac{\gamma}{a} \right]$$

$$\Rightarrow p^2 = \frac{\gamma^3}{2a}$$

$$\Rightarrow 2ap^2 = \gamma^3$$

iii) $\gamma^m = a^m [\cos m\theta + \sin m\theta]$

\Rightarrow diff w.r.t θ

$$\Rightarrow m\gamma^{m-1} \frac{d\gamma}{d\theta} = a^m [-m\sin m\theta + m\cos m\theta]$$

$$\Rightarrow m \frac{\gamma^m}{\gamma} \frac{d\gamma}{d\theta} = m [\cos m\theta - \sin m\theta] a^m$$

$$\Rightarrow \frac{1}{\gamma} \frac{d\gamma}{d\theta} = \frac{(\cos m\theta - \sin m\theta) a^m}{\gamma^m}$$

$$\Rightarrow \cot \phi = \frac{(\cos m\theta - \sin m\theta) a^m}{a^m [\cos m\theta + \sin m\theta]}$$

$$\Rightarrow 1 + \cot^2 \phi = 1 + \frac{(\cos m\theta - \sin m\theta)^2}{(\cos m\theta + \sin m\theta)^2}$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{(\cos m\theta + \sin m\theta)^2 + (\cos m\theta - \sin m\theta)^2}{(\cos m\theta + \sin m\theta)^2}$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{2 (\cos^2 m\theta + \sin^2 m\theta)}{\left[\frac{\gamma^m}{a^m}\right]^2}$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{2}{\frac{\gamma^{2m}}{a^{2m}}}$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{2a^{2m}}{\gamma^{2m}}$$

∴ WKT,

$$\Rightarrow \frac{1}{P^2} = \frac{1}{\gamma^2} [1 + \cot^2 \phi]$$

$$\Rightarrow \frac{1}{P^2} = \frac{1}{\gamma^2} \frac{2a^{2m}}{\gamma^{2m}}$$

$$\Rightarrow \frac{1}{P^2} = \frac{2a^{2m}}{\gamma^{2m+2}}$$

$$\Rightarrow 2a^{2m} P^2 = \gamma^{2(m+1)}$$

$$\text{iv} \quad \gamma = a(e^{\theta \cot \alpha})$$

$$\Rightarrow \gamma = a e^{\theta \cot \alpha} \quad \text{--- ①}$$

diff w.r.t θ

$$\Rightarrow \frac{d\gamma}{d\theta} = a \cdot e^{\theta \cot \alpha} \cdot \cot \alpha$$

$$\Rightarrow \frac{1}{\gamma} \frac{d\gamma}{d\theta} = \frac{(a \cot \alpha) e^{\theta \cot \alpha}}{\gamma}$$

$$\Rightarrow \frac{1}{\gamma} \frac{d\gamma}{d\theta} = \frac{(\alpha \cot \alpha) e^{\cot \alpha}}{\alpha e^{\cot \alpha}}$$

$$\Rightarrow \cot \phi = \cot \alpha$$

$$\Rightarrow \phi = \alpha$$

WKT,

$$p = \gamma \sin \phi$$

$$\boxed{p = \gamma \sin \alpha}$$

$$\text{vr } \frac{l}{\gamma} = 1 + e \cos \theta$$

$$\Rightarrow \gamma(1 + e \cos \theta) = l \quad \text{--- ②}$$

diff w.r.t. 'θ'

$$\Rightarrow (1 + e \cos \theta) \frac{d\gamma}{d\theta} + \gamma(0 - e \sin \theta) = 0$$

$$\Rightarrow (1 + e \cos \theta) \frac{d\gamma}{d\theta} = \gamma e (\sin \theta)$$

$$\Rightarrow \frac{1}{\gamma} \frac{d\gamma}{d\theta} = \frac{e \sin \theta}{1 + e \cos \theta}$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{1 + (e \sin \theta)^2}{(1 + e \cos \theta)^2}$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{(1 + e \cos \theta)^2 + (e^2 \sin^2 \theta)}{(1 + e \cos \theta)^2}$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{1 + 2e \cos \theta + e^2 \cos^2 \theta + e^2 \sin^2 \theta}{(1 + e \cos \theta)^2}$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{1 + 2e \cos \theta + e^2 (\cos^2 \theta + \sin^2 \theta)}{(1 + e \cos \theta)^2}$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{1 + 2e \cos \theta + e^2}{(1 + e \cos \theta)^2}$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{1 + e^2 + 2 \left[\frac{l}{\gamma} - 1 \right]}{(l/\gamma)^2}$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{1 + e^2 + 2 \frac{l}{\gamma} - 2}{l^2/\gamma^2}$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{\gamma^2}{l^2} \left[e^2 + \frac{2l}{\gamma} - 1 \right]$$

∴ WKT

$$\Rightarrow \frac{1}{P^2} = \frac{1}{\gamma^2} (1 + \cot^2 \phi)$$

$$\Rightarrow \frac{1}{P^2} = \frac{1}{\gamma^2} \frac{\gamma^2}{l^2} \left[e^2 + \frac{2l}{\gamma} - 1 \right]$$

$$\Rightarrow \frac{1}{P^2} = \frac{1}{l^2} \left[e^2 + \frac{2l}{\gamma} - 1 \right]$$

$$\text{vif } \gamma^m = a^m \cos m\theta + b^m \sin m\theta$$

⇒ diff w.r.t to 'θ'

$$\Rightarrow m\gamma^{m-1} \frac{d\gamma}{d\theta} = -ma^m \sin m\theta + mb^m \cos m\theta$$

$$\Rightarrow m \frac{\gamma^m}{\gamma} \frac{d\gamma}{d\theta} = a^m (b^m \cos m\theta - a^m \sin m\theta)$$

$$\Rightarrow \frac{\gamma^m}{\gamma} \frac{d\gamma}{d\theta} = \frac{b^m \cos m\theta - a^m \sin m\theta}{\gamma^m}$$

$$\Rightarrow \frac{1}{\gamma} \frac{d\gamma}{d\theta} = \frac{b^m \cos m\theta - a^m \sin m\theta}{a^m \cos m\theta + b^m \sin m\theta}$$

$$\Rightarrow \cot^2 \phi = \frac{(b^m \cos m\theta - a^m \sin m\theta)^2}{(a^m \cos m\theta + b^m \sin m\theta)^2}$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{1 + (b^m \cos m\theta - a^m \sin m\theta)^2}{(a^m \cos m\theta + b^m \sin m\theta)^2}$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{(a^m \cos m\theta + b^m \sin m\theta)^2 + (b^m \cos m\theta - a^m \sin m\theta)^2}{(a^m \cos m\theta + b^m \sin m\theta)^2}$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{a^{2m} \cos^2 m\theta + b^{2m} \sin^2 m\theta + 2a^m b^m \cos m\theta \sin m\theta + b^{2m} \cos^2 m\theta + a^{2m} \sin^2 m\theta - 2a^m b^m \cos m\theta \sin m\theta}{(\gamma^m)^2}$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{(a^{2m} + b^{2m}) \cos^2 m\theta + (a^{2m} + b^{2m}) \sin^2 m\theta}{\gamma^{2m}}$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{(a^{2m} + b^{2m}) (\cos^2 m\theta + \sin^2 m\theta)}{\gamma^{2m}}$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{a^{2m} + b^{2m}}{\gamma^{2m}}$$

WKT

$$\Rightarrow \frac{1}{P^2} = \frac{1}{\gamma^2} [1 + \cot^2 \phi]$$

$$\Rightarrow \frac{1}{P^2} = \frac{1}{\gamma^2} \left[\frac{a^{2m} + b^{2m}}{\gamma^{2m}} \right]$$

ARC LENGTH Of DIFFERENT CURVES :-

- (i) Let $f(x, y) = C$ be a cartesian curve, then the arc length can be evaluated by the given formula.

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx} \right)^2} = \sqrt{1 + y'^2} = \sqrt{1 + y_i^2}$$

$$\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy} \right)^2} = \sqrt{1 + x'^2} = \sqrt{1 + x_i^2}$$

(2) Let $x = x(t)$, $y = y(t)$ be the parametric equation for x and y , then

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$\frac{ds}{dt} = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$

(3) Let, $r(\theta) = C$ be a polar curve, then

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left[\frac{dr}{d\theta}\right]^2}$$

$$\frac{ds}{d\theta} = \sqrt{1 + r^2 \left[\frac{d\theta}{dr}\right]^2}$$

CURVATURE AND RADIUS OF CURVATURE :-

The rate of change of direction of a curve with respect to distance along the curve is called curvature.

It can be denoted by ' k '.

The reciprocal of the curvature is called the radius of curvature.

It can be denoted by ' ρ '.

$$\therefore \text{The radius of curvature } \rho = \frac{1}{k} = \left| \frac{ds}{d\psi} \right|$$

Derivation of Radius of Curvature in cartesian form;

Let $f(x, y) = C$, curve in cartesian form,

Let 'T' be a tangent to the curve $f(x, y) = C$, having the slope $m = \tan \psi$.

$$\Rightarrow f(x, y) = C$$

$$\Rightarrow m = \tan \psi = y_1$$

$$\Rightarrow \psi = \tan^{-1}(y_1) \quad \text{--- } ①$$

diff. w.r.t. θ ,

$$\Rightarrow \frac{d\psi}{d\theta} = \frac{1}{1+y_1^2} \frac{dy_1}{dx}$$

$$\Rightarrow \frac{d\psi}{d\theta} = \frac{y_2}{1+y_1^2}$$

WKT

$$\frac{ds}{dx} = \sqrt{1+y_1^2}$$

also, WKT

$$\Rightarrow \rho = \frac{ds}{d\psi}$$

$$\Rightarrow \rho = \frac{\frac{ds}{dx}}{\frac{d\psi}{dx}}$$

$$\Rightarrow \rho = \frac{\sqrt{1+y_1^2}}{y_2/1+y_1^2}$$

$$\Rightarrow \rho = \frac{(1+y_1^2)^{1/2}(1+y_1^2)}{y_2}$$

$$\Rightarrow \rho = \frac{(1+y_1^2)^{3/2}}{y_2}, \quad y_2 \neq 0$$

Derivative of Radius of Curvature in Polar form :-

Let, $r = f(\theta)$ be a polar curves

at $P(r, \theta)$ with $OP = r$

radius of a curve having the tangent 'T'

Let ' ϕ ' be the angle between radius of a curve and tangent, let ' θ ' be the angle between radius of a curve and initial line.

' ψ ' be the angle made by the initial line.

WKT,

$$\Rightarrow \psi = \phi + \theta \quad \text{--- (1)}$$

diff w.r.t 's'

$$\Rightarrow \frac{d\psi}{ds} = \frac{d\phi}{ds} + \frac{d\theta}{ds}$$

$$\Rightarrow \frac{d\psi}{ds} = \frac{d\phi/d\theta}{ds/d\theta} + \frac{d\theta}{ds}$$

$$\Rightarrow \frac{d\psi}{ds} = \frac{d\phi}{d\theta} \cdot \frac{d\theta}{ds} + \frac{d\theta}{ds}$$

$$\Rightarrow \frac{d\psi}{ds} = \frac{d\theta}{ds} \left[\frac{d\phi}{d\theta} + 1 \right] \quad \text{--- (2)}$$

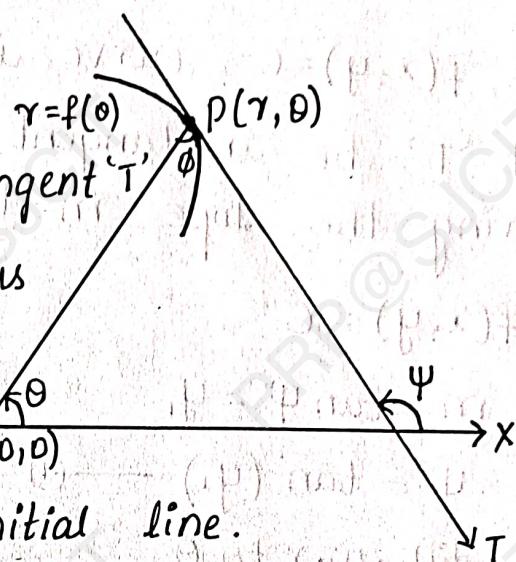
Also, WKT

$$\Rightarrow \tan \phi = r \frac{d\theta}{dr}$$

$$\Rightarrow \tan \phi = \frac{r}{(dr/d\theta)}$$

$$\Rightarrow \tan \phi = \frac{r}{r_1} \quad \text{where, } r_1 = \frac{dr}{d\theta}$$

$$\Rightarrow \phi = \tan^{-1} \left[\frac{r}{r_1} \right] \quad \text{--- (3)}$$



using diff ③ w.r.t θ

$$\Rightarrow \frac{d\phi}{d\theta} = \frac{1}{1 + \left(\frac{\gamma}{\gamma_1}\right)^2} \frac{d}{d\theta} \left[\frac{\gamma}{\gamma_1} \right]$$

$$\Rightarrow \frac{d\phi}{d\theta} = \left[\frac{1}{1 + \frac{\gamma^2}{\gamma_1^2}} \right] \gamma_1 \frac{d}{d\theta} (\gamma) - \gamma \frac{d}{d\theta} (\gamma_1)$$

$$\Rightarrow \frac{d\phi}{d\theta} = \frac{\gamma_1^2}{\gamma_1^2 + \gamma^2} \left[\frac{\gamma_1 \gamma_1 - \gamma \gamma_2}{\gamma_1^2 + \gamma^2} \right]$$

$$\Rightarrow \frac{d\phi}{d\theta} = \frac{\gamma_1^2 - \gamma \gamma_2}{\gamma_1^2 + \gamma^2}$$

$$\therefore ② \Rightarrow \frac{d\Psi}{ds} = \frac{d\theta}{ds} \left[\frac{\gamma_1^2 - \gamma \gamma_2}{\gamma_1^2 + \gamma^2} + 1 \right]$$

$$\Rightarrow \frac{1}{\frac{ds}{d\Psi}} = \frac{1}{\frac{d\theta}{ds}} \left[\frac{\gamma_1^2 - \gamma \gamma_2 + \gamma_1^2 + \gamma^2}{\gamma_1^2 + \gamma^2} \right]$$

$$\Rightarrow \frac{1}{\rho} = \frac{1}{\sqrt{\gamma^2 + \gamma_1^2}} \left[\frac{\gamma^2 + 2\gamma_1^2 - \gamma \gamma_2}{\gamma^2 + \gamma_1^2} \right]$$

$$\Rightarrow \frac{1}{\rho} = \frac{\gamma^2 + 2\gamma_1^2 - \gamma \gamma_2}{(\gamma^2 + \gamma_1^2)^{3/2}}$$

$$\Rightarrow \rho = \frac{(\gamma^2 + \gamma_1^2)^{3/2}}{\gamma^2 + 2\gamma_1^2 - \gamma \gamma_2}$$

RADIUS OF CURVATURE IN PARAMETRIC FORM:-

Let $x = x(t)$, $y = y(t)$ with two parametric equations, then the radius of curvature can be evaluated from the given formula.

$$\Rightarrow \rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - x''y'}$$

where,

$$x' = \frac{dx}{dt}, \quad y' = \frac{dy}{dt}$$

$$x'' = \frac{d^2x}{dt^2}, \quad y'' = \frac{d^2y}{dt^2}$$

The Radius of Curvature of a polar curve in pedal form :

$$\rho = r \frac{dr}{dp}$$

Problems:-

1. Find the radius of curvature of :-

$$y = a \log(\sec x/a)$$

\Rightarrow diff w.r.t x'

$$\Rightarrow \frac{dy}{dx} = \frac{a}{\sec \frac{x}{a}} \cdot \tan \left[\frac{x}{a} \right] \sec \left[\frac{x}{a} \right] \left(\frac{1}{a} \right)$$

$$\Rightarrow \frac{dy}{dx} = \tan \left[\frac{x}{a} \right]$$

diff w.r.t x'

$$\Rightarrow \frac{d^2y}{dx^2} = \sec^2 \left(\frac{x}{a} \right) \left(\frac{1}{a} \right)$$

$$\Rightarrow y_2 = \frac{1}{a} \sec^2 \left[\frac{x}{a} \right]$$

WKT

$$\Rightarrow \rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$\Rightarrow \rho = \frac{\left[1 + \tan^2\left(\frac{x}{a}\right)\right]^{3/2}}{\frac{1}{a} \sec^2\left(\frac{x}{a}\right)}$$

$$\Rightarrow \rho = \frac{a \left[\sec^2\left(\frac{x}{a}\right)\right]^{3/2}}{\sec^2\left(\frac{x}{a}\right)}$$

$$\Rightarrow \rho = \frac{a \sec^3\left(\frac{x}{a}\right)}{\sec^2\left(\frac{x}{a}\right)}$$

$$\Rightarrow \rho = a \sec\left(\frac{x}{a}\right)$$

2. Find the radius of curvature at any point of the parabola $y^2 = 4ax$

$$\Rightarrow y^2 = 4ax \quad \text{--- ①}$$

diff w.r.t to 'x'

$$\text{①} \Rightarrow 2y \cdot \frac{dy}{dx} = 4a \cdot 1 \quad (\div 2)$$

$$\Rightarrow y \frac{dy}{dx} = 2a$$

$$\Rightarrow \frac{dy}{dx} = y_1 = \frac{2a}{y} \quad \text{--- ②}$$

diff w.r.t to 'x'

$$\text{②} \Rightarrow \frac{d}{dx} \left(\frac{dy}{dx} \right) = 2a \cdot \frac{d}{dx} \left(\frac{1}{y} \right)$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2a \cdot \left[-\frac{1}{y^2} \right] \frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{2a}{y^2} \left[\frac{2a}{y} \right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{4a^2}{y^3} = y_2$$

WKT,

$$\Rightarrow \rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$\Rightarrow \rho = \frac{\left[1 + \left(\frac{2a}{y}\right)^2\right]^{3/2}}{-\frac{4a^2}{y^3}}$$

$$\Rightarrow \rho = \frac{\left[1 + \frac{4a^2}{y^2}\right]^{3/2} \times y^3}{-4a^2}$$

$$\Rightarrow \rho = \frac{(y^2 + 4a^2)^{3/2} \times y^3}{4a^2}$$

$$\Rightarrow \rho = \frac{(y^2 + 4a^2)^{3/2}}{4a^2}$$

3. Find the radius of curvature at any point of the curve, $y = c \cosh(x/c)$

$$\Rightarrow y = c \cosh\left[\frac{x}{c}\right] \quad \text{--- ①}$$

diff w.r.t x

$$\Rightarrow \text{① } \frac{dy}{dx} = c \sinh\left[\frac{x}{c}\right] \left[\frac{1}{c}\right]$$

$$\Rightarrow y_1 = \sinh\left[\frac{x}{c}\right] \quad \text{--- ②}$$

diff w.r.t x

$$\textcircled{2} \Rightarrow \frac{d}{dx}(y_1) = \cosh\left[\frac{x}{c}\right] \cdot \frac{1}{c}$$

$$\Rightarrow y_2 = \frac{1}{c} \cosh\left[\frac{x}{c}\right]$$

WKT,

$$\Rightarrow \rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$\Rightarrow \rho = \frac{\left[1 + \sinh^2\left(\frac{x}{c}\right)\right]^{3/2}}{\frac{1}{c} \cosh\left(\frac{x}{c}\right)}$$

$$\Rightarrow \rho = \frac{c [\cosh^2\left(\frac{x}{c}\right)]^{3/2}}{\cosh\left(\frac{x}{c}\right)}$$

$$\Rightarrow \rho = \frac{c \cosh^3\left(\frac{x}{c}\right)}{\cosh\left(\frac{x}{c}\right)}$$

$$\Rightarrow \rho = c \cosh^2\left(\frac{x}{c}\right)$$

$$\Rightarrow \rho = c (y/c)^2$$

$$\Rightarrow \rho = \frac{c \cdot y^2}{c^2}$$

$$\Rightarrow \rho = \frac{y^2}{c}$$

4. Find the radius of curvature of $x^3 + y^3 = 3axy$
at the point $\left[\frac{3a}{2}, \frac{3a}{2}\right]$

$$\Rightarrow x^3 + y^3 = 3axy \quad \text{--- ①}$$

$$\text{let, } P = \left[\frac{3a}{2}, \frac{3a}{2}\right]$$

diff w.r.t 'x'

$$\Rightarrow 3x^2 + 3y^2 y_1 = 3a [1 \cdot y + xy_1] \quad (\div 3)$$

$$\Rightarrow x^2 + y^2 y_1 = ay + axy_1$$

$$\Rightarrow y^2 y_1 - axy_1 = ay - x^2$$

$$\Rightarrow y_1(y^2 - ax) = ay - x^2$$

$$\Rightarrow y_1 = \frac{ay - x^2}{y^2 - ax} \quad \text{--- (2)}$$

$$\Rightarrow (y_1)_P = \frac{a\left(\frac{3a}{2}\right) - \left(\frac{3a}{2}\right)^2}{\left(\frac{3a}{2}\right)^2 - a\left[\frac{3a}{2}\right]}$$

$$\Rightarrow (y_1)_P = \frac{\left[\frac{3a}{2}\right]^2 - a\left[\frac{3a}{2}\right]}{\left[\frac{3a}{2}\right]^2 - a\left[\frac{3a}{2}\right]}$$

$$\Rightarrow (y_1)_P = -1$$

diff w.r.t to 'x'

$$\textcircled{2} \Rightarrow y_2 = \frac{(y^2 - ax)(ay_1 - 2x) - (ay - x^2)(2yy_1 - a)}{y^2 - ax^2}$$

$$\left[\left(\frac{3a}{2} \right)^2 - a\left(\frac{3a}{2} \right) \right] \left[a(-1) - 2\left[\frac{3a}{2} \right] - \left[a\left(\frac{3a}{2} - \frac{3a}{2} \right)^2 \right] \right]$$

$$\left[2\left[\frac{3a}{2}(-1) - a \right] \right]$$

$$\Rightarrow y_2 P = \frac{\left[\left(\frac{3a}{2} \right)^2 - 4\left(\frac{3a}{2} \right) \right]^2}{\left[\left(\frac{3a}{2} \right)^2 - 4\left(\frac{3a}{2} \right) \right]^2}$$

$$\Rightarrow (y_2)_P = \frac{\left[\frac{9a^2}{4} - \frac{3a^2}{2} \right] [-4a] - \left[-\frac{3a^2}{2} + \frac{9a^2}{4} \right] [4a]}{\left[\frac{9a^2}{4} - \frac{3a^2}{2} \right]^2}$$

$$\Rightarrow (y_2)_P = \frac{\left[\frac{9a^2}{4} - \frac{3a^2}{4} \right] [-4a - 4a]}{\left[\frac{9a^2}{4} - \frac{3a^2}{2} \right]^2}$$

$$\Rightarrow y_2 = \frac{-8a}{\frac{9a^2 - 6a^2}{4}}$$

$$\Rightarrow y^2 = \left| \frac{8a \times 4}{3a^2} \right|$$

$$\Rightarrow y^2 = \frac{32}{3a}$$

WKT,

$$\Rightarrow \rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$\Rightarrow \rho = \frac{[1+(-1)^2]^{3/2}}{\frac{32}{3a}}$$

$$\Rightarrow \rho = \frac{2^{3/2} \times 3a}{3a}$$

$$\Rightarrow \rho = \frac{2\sqrt{2} \times 3a}{8 \times 2\sqrt{2} \times \sqrt{2}}$$

$$\Rightarrow \rho = \frac{3a}{8\sqrt{2}}$$

5. Find the radius of curvature of a curve

$$x^2y = a(x^2 + y^2) \text{ at the point } P(-2a, 2a)$$

$$\Rightarrow x^2y = a(x^2 + y^2) \quad \text{--- ①}$$

$$\Rightarrow \text{where, } P = (-2a, 2a)$$

diff w.r.t 'x'

$$\textcircled{1} \Rightarrow x^2(1) + y \cdot 2x \cdot x_1 = a(2x \cdot x_1 + 2ay)$$

$$\Rightarrow x^2 + 2xy \cdot x_1 = 2ax \cdot x_1 + 2ay$$

$$\Rightarrow 2xyx_1 - 2ax^2 = 2ay - x^2$$

$$\Rightarrow x_1 (\partial xy - \partial ax) = \partial ay - x^2$$

$$\Rightarrow x_1 = \frac{\partial ay - x^2}{\partial xy - \partial ax} \quad \text{--- } ②$$

$$\therefore (x_1)p = \frac{\partial a(\partial a) - (-\partial a^2)}{2(-\partial a)(\partial a) - \partial a(-\partial a)}$$

$$(x_1)p = \frac{4a^2 - 4a^2}{-8a^2 + 4a^2}$$

$$(x_1)p = -\frac{0}{4a^2}$$

$$\Rightarrow (x_1)p = 0$$

diff w.r.t 'y'

$$② \Rightarrow x_2 = \frac{(\partial xy - \partial ax)(\partial a - \partial x x_1) - (\partial ay - x^2)(\partial x + \partial xy - \partial ax_1)}{(\partial xy - \partial ax)^2}$$

$$\Rightarrow (x_2)p = \frac{[\partial(-\partial a)(\partial a) - \partial a(-\partial a)][\partial a - \partial(-\partial a)(0)] - [\partial a(\partial a) - (-\partial a)^2]}{[\partial[-\partial a] - \partial(0) - 0]}$$

$$[\partial[-\partial a][\partial a] - \partial a[-\partial a]]^2$$

$$\Rightarrow (x_2)p = \frac{(-8a^2 + 4a^2) - 0}{(-8a^2 + 4a^2)^2}$$

$$\Rightarrow (x_2)p = \frac{(2a) - (4a^2)}{(-4a^2)^2}$$

$$\Rightarrow x_2 = \frac{\partial a}{-4a^2} = \frac{1}{-2a}$$

$$\Rightarrow p = \frac{(1 + x_1^2)^{3/2}}{x_2} = \frac{(1 + (0)^2)^{3/2}}{1/-2a} = \frac{(1+0)^{3/2}}{1/-2a} = \frac{(1)^{3/2} \times (-2a)}{1}$$

$$\Rightarrow \rho = \frac{x\sqrt{2} \times (-3a)}{r}$$

$$\Rightarrow \rho = \sqrt{2}(-3a)$$

6. Find the radius of curvature the curve $a^2y = x^3 - a^3$ at the point where the curve cuts 'x' axis.

\Rightarrow Given:-

$$a^2y = x^3 - a^3 \quad \text{--- 1}$$

Let, p be a point cuts the x-axis ($y=0$)

$$\text{1} \Rightarrow 0 = x^3 - a^3$$

$$\Rightarrow x^3 = a^3$$

$$\Rightarrow x = a$$

$$\therefore p(a, 0)$$

diff 1 w.r to 'x'

$$\text{1} \Rightarrow a^2y_1 = 3x^2 - 0$$

$$\Rightarrow y_1 = \frac{3x^2}{a^2} \quad \text{--- 2}$$

$$\Rightarrow (y_1)_p = \frac{3a^2}{a^2}$$

$$\Rightarrow (y_1)_p = 3$$

$$\Rightarrow y_2 = \frac{3}{a^2}(2x)$$

$$\Rightarrow y_2 = \frac{6x}{a^2}$$

$$\Rightarrow (y_2)_p = \frac{6a}{a^2}$$

$$\Rightarrow y_2 = \frac{6}{a}$$

$$\therefore \rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$\Rightarrow \rho = \frac{(1+9)^{3/2}}{6/a}$$

$$\Rightarrow \rho = \frac{a(10)^{3/2}}{6}$$

$$\Rightarrow \rho = \frac{a \times 10 \times \sqrt{10}}{6^3}$$

$$\Rightarrow \rho = \frac{5\sqrt{10} \cdot a}{3}$$

7. For the curve $y = \frac{ax}{a+x}$, prove that

$\left[\frac{2\rho}{a} \right]^{2/3} = \left(\frac{y}{x} \right)^2 + \left(\frac{x}{y} \right)^2$, where, ρ is the radius of curvature at its point $P(x, y)$

$$\Rightarrow y = \frac{ax}{a+x} \quad \text{--- ①}$$

diff w.r.t x'

$$\Rightarrow y_1 = \frac{(a+x)(a) - ax(1)}{(a+x)^2}$$

$$\Rightarrow y_1 = \frac{a^2 + ax - ax}{(a+x)^2}$$

$$\Rightarrow y_1 = \frac{a^2}{(a+x)^2} \quad \text{--- ②}$$

diff w.r.t x'

$$\Rightarrow y_2 = \frac{(a+x)^2(0) - a^2 \cdot 2(a+x)(1)}{(a+x)^4}$$

$$\Rightarrow y_2 = \frac{-2a^2(a+x)}{(a+x)^4}$$

$$\Rightarrow y_2 = \frac{-2a^2}{(a+x)^3}$$

\therefore WKT;

$$\Rightarrow \rho = \left| \frac{(1+y_1^2)^{3/2}}{y_2} \right|$$

$$\Rightarrow \rho = \left| \frac{\left[1 + \frac{a^4}{(a+x)^4} \right]^{3/2}}{\frac{-2a^2}{(a+x)^3}} \right|$$

$$\Rightarrow \rho = \frac{(a+x)^3}{2a^2} \left[1 + \frac{a^4}{(a+x)^4} \right]^{3/2}$$

$$\Rightarrow \frac{\partial \rho}{a} = \frac{2}{a} \frac{(a+x)^3}{2a^2} \left[1 + \frac{a^4}{(a+x)^4} \right]^{3/2}$$

$$\Rightarrow \frac{\partial \rho}{a} = \frac{(a+x)^3}{a^3} \left[1 + \frac{a^4}{(a+x)^4} \right]^{3/2}$$

$$\Rightarrow \left[\frac{\partial \rho}{a} \right]^{2/3} = \frac{[(a+x)^3]^{2/3}}{(a^3)^{2/3}} \left[1 + \frac{a^4}{(a+x)^4} \right]^{3/2 \times 2/3}$$

$$\Rightarrow \left[\frac{\partial \rho}{a} \right]^{2/3} = \frac{(a+x)^2}{a^2} \left[1 + \frac{a^4}{(a+x)^4} \right]$$

$$\Rightarrow \left[\frac{\partial \rho}{a} \right]^{2/3} = \frac{(a+x)^2}{a^2} + \frac{a^2}{(a+x)^2}$$

$$\Rightarrow \left[\frac{\partial \rho}{a} \right]^{2/3} = \left(\frac{a+x}{a} \right)^2 + \left(\frac{a}{a+x} \right)^2$$

$$\Rightarrow \left[\frac{2\theta}{a} \right]^{2/3} = \left[\frac{x}{y} \right]^2 + \left[\frac{y}{x} \right]^2$$

8. Find the radius of curvature at any point of the cycloid.

$$x = a(\theta + \sin\theta) \quad y = a(1 - \cos\theta)$$

$$\Rightarrow x - a(\theta + \sin\theta) \quad \text{--- ①} \quad \Rightarrow y = a(1 - \cos\theta) \quad \text{--- ②}$$

diff w.r.t. θ

$$\Rightarrow \frac{dx}{d\theta} = a[1 + \cos\theta] \quad \Rightarrow \frac{dy}{d\theta} = a[\sin\theta]$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin\theta}{1 + \cos\theta}$$

$$\therefore y_1 = \frac{dy/d\theta}{dx/d\theta}$$

$$\Rightarrow y_1 = \frac{a\sin\theta}{a(1 + \cos\theta)} = \frac{\sin\theta}{1 + \cos\theta} \quad \text{--- ③}$$

diff eqn ③ w.r.t x

$$③ \Rightarrow \frac{d}{dx}(y_1) = \frac{d}{dx} \left[\frac{\sin\theta}{1 + \cos\theta} \right]$$

$$\Rightarrow \frac{d}{dx}(y_1) = \frac{d}{d\theta} \left[\frac{\sin\theta}{1 + \cos\theta} \right] \frac{d\theta}{dx}$$

$$= \frac{(1 + \cos\theta)\cos\theta - \sin\theta(-\sin\theta)}{(1 + \cos\theta)^2} \frac{d\theta}{dx}$$

$$= \frac{\cos\theta + (\cos^2\theta + \sin^2\theta)}{(1 + \cos\theta)^2} \cdot \frac{1}{\frac{d\theta}{dx}}$$

$$= \frac{1 + \cos\theta}{(1 + \cos\theta)^2} \times \frac{1}{a(1 + \cos\theta)}$$

$$\Rightarrow y_2 = \frac{1}{a(1+\cos\theta)^2}$$

WKT,

$$\Rightarrow \rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$\left[1 + \frac{\sin^2\theta}{(1+\cos\theta)^2} \right]^{3/2}$$

$$\Rightarrow \rho = \frac{1}{\frac{1}{a + [\cos\theta + 1]^2}}$$

$$\Rightarrow \rho = a(1+\cos\theta)^2 \left[\frac{[(1+\cos\theta)^2 + \sin^2\theta]^{3/2}}{[(1+\cos\theta)^2]^{3/2}} \right]$$

$$\Rightarrow \rho = \frac{a(1+\cos\theta)^2}{(1+\cos\theta)^3} [1+2\cos\theta + \cos^2\theta + \sin^2\theta]^{3/2}$$

$$\Rightarrow \rho = \frac{a}{1+\cos\theta} [1+2\cos\theta+1]^{3/2}$$

$$\Rightarrow \rho = \frac{a}{1+\cos\theta} [2+2\cos\theta]^{3/2}$$

$$\Rightarrow \rho = \frac{2^{3/2}a}{(1+\cos\theta)} (1+\cos\theta)^{3/2}$$

9. Find the radius of curvature of curve

$$\sqrt{x} + \sqrt{y} = \sqrt{a} \text{ at the point } \left[\frac{a}{4}, \frac{a}{4} \right]$$

$$\Rightarrow \sqrt{x} + \sqrt{y} = \sqrt{a} \quad \text{--- (1)} \quad P \left[\frac{a}{4}, \frac{a}{4} \right]$$

diff w.r.t x

$$(1) \Rightarrow \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \frac{dy}{dx} = 0.$$

$$\Rightarrow \frac{1}{\sqrt{y}} \frac{dy}{dx} = -\frac{1}{\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}} = y_1 \quad \text{--- ②}$$

$$\therefore (y_1)_P = \frac{-\sqrt{a/x}}{\sqrt{a/x}} = -1$$

diff w.r.t x

$$\Rightarrow \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{\sqrt{x} \cdot \frac{1}{2\sqrt{y}} \cdot y_1 - \sqrt{y} \cdot \frac{1}{2\sqrt{x}}}{(\sqrt{x})^2}$$

$$\Rightarrow y_2 = \left| -\frac{1}{2} \frac{y_1 \sqrt{x}/\sqrt{y} - \sqrt{y}/\sqrt{x}}{x} \right|$$

$$\Rightarrow (y_2)_P = \left[\frac{\frac{(-1)\sqrt{a/4} - \sqrt{a/4}}{\sqrt{a/4} - \sqrt{a/4}}}{a/4} \right]$$

$$\Rightarrow (y_2)_P = -\frac{1}{2} \left[\frac{-1-1}{a/4} \right]$$

$$\Rightarrow y_2 = -\frac{1}{2} \left[\frac{-2}{a/4} \right]$$

$$\Rightarrow y_2 = \frac{4}{a} \neq 0$$

\therefore WKT,

$$\Rightarrow P = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$\Rightarrow P = \frac{(1+(-1)^2)^{3/2}}{4/a}$$

$$\Rightarrow p = \frac{18\sqrt{2}}{24/a}$$

$$\Rightarrow p = \frac{\sqrt{2}}{2} a$$

$$\Rightarrow p = \frac{\sqrt{2} a}{\sqrt{2}\sqrt{2}}$$

$$\Rightarrow p = \frac{a}{\sqrt{2}}$$

10. Find the radius of curvature for the curve

$$r^n = a^n \cos n\theta$$

\Rightarrow Given:-

$$\Rightarrow r^n = a^n \cos n\theta \quad \text{--- ①}$$

diff w.r.t 'θ'

$$\textcircled{1} \Rightarrow nr^{n-1} \frac{dr}{d\theta} = a^n [-\sin n\theta] [n]$$

$$\Rightarrow \frac{r^n}{r} \frac{dr}{d\theta} = -a^n \sin n\theta$$

$$\Rightarrow \frac{1}{r} \cdot \frac{dr}{d\theta} = -\frac{a^n \sin n\theta}{r^n}$$

$$\Rightarrow \frac{1}{r} \cdot \frac{dr}{d\theta} = -\frac{a^n \sin n\theta}{a^n \cos n\theta}$$

$$\Rightarrow \cot \phi = -\tan n\theta \quad \text{--- ②}$$

\therefore WKT

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} [1 + \cot^2 \phi]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} [1 + \tan^2 n\theta]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} [\sec^2 n\theta]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{\gamma^2 \cos^2 n\theta}$$

$$\Rightarrow p^2 = \gamma^2 \cos^2 n\theta$$

$$\Rightarrow p = \gamma \cos n\theta$$

$$\Rightarrow p = \gamma \frac{\gamma^n}{a^n}$$

$$\Rightarrow a^n p = \gamma^{n+1} \quad \text{--- ②}$$

diff w.r.t to 'p'

$$② \Rightarrow a^n (1) = (n+1) \gamma^n \cdot \frac{d\gamma}{dp}$$

$$\Rightarrow \gamma^n \frac{d\gamma}{dp} = \frac{a^n}{n+1}$$

$$\Rightarrow \gamma^{n-1} \cdot \gamma \frac{d\gamma}{dp} = \frac{a^n}{n+1}$$

$$\Rightarrow \gamma \frac{d\gamma}{dp} = \frac{a^n}{n+1} \cdot \frac{1}{\gamma^{n-1}}$$

$$\Rightarrow \rho = \left[\frac{a^n}{n+1} \right] \frac{1}{\gamma^{n-1}}$$

$$\Rightarrow \rho \propto \frac{1}{\gamma^{n-1}}$$

11. $\gamma^n = a^n \sin n\theta$

$$\Rightarrow \gamma^n = a^n \sin n\theta \quad \text{--- ①}$$

diff w.r.t. 'θ'

$$\Rightarrow n \gamma^{n-1} \frac{d\gamma}{d\theta} = a^n (\cos n\theta) (\cancel{a})$$

$$\Rightarrow \frac{\gamma^n}{\gamma} \frac{d\gamma}{d\theta} = a^n \cos n\theta$$

$$\Rightarrow \frac{1}{\gamma} \frac{d\gamma}{d\theta} = \frac{\alpha^n \cos n\theta}{\gamma^n}$$

$$\Rightarrow \frac{1}{\gamma} \frac{d\gamma}{d\theta} = \frac{\alpha^n \cos n\theta}{\alpha^n \sin n\theta}$$

$$\Rightarrow \cot \phi = \cot n\theta \quad \text{--- (2)}$$

WKT;

$$\Rightarrow \frac{1}{p^2} = \frac{1}{\gamma^2} [1 + \cot^2 \phi]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{\gamma^2} [1 + \cot^2 n\theta]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{\gamma^2} (\cosec^2 n\theta)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{\gamma^2 \sin^2 n\theta}$$

$$\Rightarrow p^2 = \gamma^2 \sin^2 n\theta$$

$$\Rightarrow p = \gamma \sin n\theta$$

$$\Rightarrow p = \gamma \frac{\gamma^n}{\alpha^n}$$

$$\Rightarrow \alpha^n p = \gamma^{n+1} \quad \text{--- (3)}$$

diff w.r.t to 'p'

$$(3) \Rightarrow \alpha^n \cdot 1 = (n+1) \gamma^n \cdot \frac{d\gamma}{dp}$$

$$\Rightarrow \gamma^n \frac{d\gamma}{dp} = \frac{\alpha^n}{n+1}$$

$$\Rightarrow \frac{\gamma d\gamma}{dp} = \frac{\alpha^n}{n+1} \cdot \frac{1}{\gamma^{n-1}}$$

$$\Rightarrow p = \left(\frac{\alpha^n}{n+1} \right) \frac{1}{\gamma^{n-1}}$$

$$\Rightarrow p \propto \frac{1}{\gamma^{n-1}}$$

12. Find $\frac{2a}{\gamma} = 1 + \cos\theta$ and hence show that
 $p^2 \propto \gamma^3$ (or) $\frac{p^2}{\gamma^3}$ be a constant.

$$\Rightarrow \frac{2a}{\gamma} = 1 + \cos\theta$$

$$\Rightarrow \gamma(1 + \cos\theta) = 2a \quad \text{--- ①}$$

diff w.r.t. θ

$$\Rightarrow (1 + \cos\theta) \frac{d\gamma}{d\theta} + \gamma(0 - \sin\theta) = 0$$

$$\Rightarrow (1 + \cos\theta) \frac{d\gamma}{d\theta} = \gamma \sin\theta$$

$$\Rightarrow \frac{1}{\gamma} \frac{d\gamma}{d\theta} = \frac{\sin\theta}{1 + \cos\theta}$$

$$\Rightarrow \cot\phi = \frac{\cancel{\gamma} \sin \theta/2 \cos \theta/2}{\cancel{\gamma} \cos^2 \theta/2}$$

$$\Rightarrow \cot\phi = \tan(\theta/2)$$

WKT;

$$\Rightarrow \frac{1}{p^2} = \frac{1}{\gamma^2} (1 + \cot^2\phi)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{\gamma^2} (1 + \tan^2(\theta/2))$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{\gamma^2} (\sec^2 \theta/2)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{\gamma^2 \cos^2 \theta/2}$$

$$\Rightarrow p^2 = \gamma^2 \cos^2 \theta/2$$

$$\Rightarrow p^2 = \gamma^2 \left[\frac{1 + \cos\theta}{2} \right]$$

$$\Rightarrow p^2 = \frac{\gamma^2}{2} [1 + \cos\theta]$$

$$\Rightarrow p^2 = \frac{\gamma^2}{\alpha} \left[\frac{2a}{\gamma} \right]$$

$$\Rightarrow p^2 = ar \quad \text{--- ②}$$

diff w.r.t to 'p'

$$② \Rightarrow 2p = a \frac{dr}{dp}$$

$$\Rightarrow \frac{d\gamma}{dp} = \frac{2}{a} p$$

$$\Rightarrow \gamma \frac{dr}{dp} = \frac{2}{a} pr$$

$$\Rightarrow p = \frac{2}{a} pr$$

$$\Rightarrow p^2 = \frac{4}{a^2} p^2 r^2$$

$$\Rightarrow p^2 = \frac{4}{a^2} (ar)(r^2)$$

$$\Rightarrow p^2 = \frac{4}{a} r^3$$

$$\Rightarrow p^2 = K r^3 \quad \therefore K = 4/a$$

$$\Rightarrow p^2 \propto r^3$$

(or)

$$\Rightarrow \frac{p^2}{r^3} = K = \text{constant.}$$

$$13. \frac{2a}{\gamma} = (1 - \cos\theta)$$

$$\Rightarrow 2a = \gamma(1 - \cos\theta) \quad \text{--- ①}$$

diff w.r.t to 'θ'

$$\Rightarrow 0 = (1 - \cos\theta) \frac{dr}{d\theta} - (-r\sin\theta)$$

$$\Rightarrow -r \sin \theta = (1 - \cos \theta) \frac{dr}{d\theta}$$

$$\Rightarrow \frac{-\sin \theta}{1 - \cos \theta} = r \frac{dr}{d\theta}$$

$$\Rightarrow \frac{-\gamma \sin \theta/2 \cos \theta/2}{\gamma \sin^2 \theta/2} = \cot \phi$$

$$\Rightarrow -\cot \theta/2 = \cot \phi$$

$$\Rightarrow \cot \phi = \cot \left[-\frac{\theta}{2} \right]$$

WKT;

$$\Rightarrow \frac{1}{P^2} = \frac{1}{\gamma^2} [1 + \cot^2 \phi]$$

$$\Rightarrow \frac{1}{P^2} = \frac{1}{\gamma^2} [1 + \cot^2 \left(-\frac{\theta}{2} \right)]$$

$$\Rightarrow \frac{1}{P^2} = \frac{1}{\gamma^2} [\cosec^2 \frac{\theta}{2}]$$

$$\Rightarrow \frac{1}{P^2} = \frac{1}{\gamma^2 \sin^2 \theta/2}$$

$$\Rightarrow P^2 = \gamma^2 \sin^2 \frac{\theta}{2}$$

$$\Rightarrow P^2 = \gamma^2 \sin^2 \theta/2$$

$$\Rightarrow P^2 = \gamma^2 \left[\frac{1 - \cos \theta}{2} \right]$$

$$\Rightarrow P^2 = \frac{\gamma^2}{2} [1 - \cos \theta]$$

$$\Rightarrow P^2 = \frac{\gamma^2}{2} \left(\frac{\theta a}{\pi} \right)$$

$$\Rightarrow P^2 = ar \quad \text{--- (2)}$$

$$\Rightarrow 2p = \gamma \frac{dr}{dp}$$

$$\Rightarrow \frac{dr}{dp} = \frac{2}{\alpha} p$$

$$\Rightarrow \gamma \frac{dr}{dp} = \frac{2}{\alpha} pr$$

$$\Rightarrow p = \frac{\alpha}{2} pr$$

$$\Rightarrow p^2 = \frac{4}{\alpha^2} p^2 r^2$$

$$\Rightarrow p^2 = \frac{4}{\alpha^2} (dr)(r^2)$$

$$\Rightarrow p^2 = \frac{4}{\alpha^2} r^3$$

$$\Rightarrow p^2 = Kr^3 \quad \therefore K = 4/\alpha$$

$$\Rightarrow p^2 \propto r^3$$

(or)

$$\frac{p^2}{r^3} = K = \text{constant.}$$

14. $r = a(1 - \cos \theta)$

$$\Rightarrow r = a(1 - \cos \theta) \quad \text{--- } ①$$

diff w.r.t θ

$$\Rightarrow \frac{dr}{d\theta} = a \sin \theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a \sin \theta}{r}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a \sin \theta}{a(1 - \cos \theta)}$$

$$\Rightarrow \cot \phi = \frac{2 \sin^2 \frac{\theta}{2} \cos \frac{\theta}{2}}{2 \sin^2 \frac{\theta}{2}}$$

$$\Rightarrow \cot \phi = \cot \frac{\theta}{2}$$

WKT,

$$\Rightarrow \frac{1}{P^2} = \frac{1}{\gamma^2} \left[1 + \cot^2 \phi \right]$$

$$\Rightarrow \frac{1}{P^2} = \frac{1}{\gamma^2} \left[1 + \cot^2 \frac{\theta}{2} \right]$$

$$\Rightarrow \frac{1}{P^2} = \frac{1}{\gamma^2} \cosec^2 \frac{\theta}{2}$$

$$\Rightarrow \frac{1}{P^2} = \frac{1}{\gamma^2 \sin^2 \frac{\theta}{2}}$$

$$\Rightarrow P^2 = \gamma^2 \sin^2 \frac{\theta}{2}$$

$$\Rightarrow P^2 = \gamma^2 \left[\frac{1 - \cos \frac{\theta}{2}}{2} \right]$$

$$\Rightarrow P^2 = \frac{\gamma^2}{2} \left[1 - \cos \frac{\theta}{2} \right]$$

$$\Rightarrow P^2 = \frac{\gamma^2}{2} \left[\frac{\gamma}{a} \right]$$

$$\Rightarrow P^2 = \frac{\gamma^3}{2a}$$

$$\Rightarrow 2ap^2 = \gamma^3 \quad \text{--- (2)}$$

diff (2) wrt to 'p'

$$(2) \Rightarrow 4ap = 3\gamma^2 \cdot \frac{dr}{dp}$$

$$\Rightarrow \gamma^2 \cdot \frac{dr}{dp} = \frac{4a}{3} p$$

$$\Rightarrow \gamma \cdot \frac{dr}{dp} = \frac{4a}{3} \cdot \frac{p}{\gamma}$$

$$\Rightarrow p = \frac{4a}{3} \cdot \frac{p}{\gamma}$$

$$\Rightarrow p^2 = \frac{16a^2}{9} \cdot \frac{p^2}{\gamma^2}$$

$$\Rightarrow p^2 = \frac{16a^2}{9} \cdot \frac{1}{\gamma^2} \left[\frac{\gamma^3}{2a} \right]$$

$$\Rightarrow p^2 = \frac{8a}{9} \gamma$$

$$\Rightarrow p^2 \propto \gamma$$

15. Find the radius of curvature of the curve

$$\theta = \frac{\sqrt{\gamma^2 - a^2}}{a} - \cos^{-1} \left[\frac{a}{\gamma} \right]$$

$$\Rightarrow \theta = \frac{\sqrt{\gamma^2 - a^2}}{a} - \cos^{-1} \left[\frac{a}{\gamma} \right] \quad \textcircled{1}$$

diff w.r.t γ

$$\Rightarrow \frac{d\theta}{d\gamma} = \frac{1}{a} \frac{1}{2\sqrt{\gamma^2 - a^2}} (2\gamma) - \frac{(-1)}{\sqrt{1 - \left[\frac{a}{\gamma} \right]^2}} \cdot \frac{d}{d\gamma} \left[\frac{a}{\gamma} \right]$$

$$\Rightarrow \frac{d\theta}{d\gamma} = \frac{\gamma}{a\sqrt{\gamma^2 - a^2}} + \frac{1}{\sqrt{1 - \frac{a^2}{\gamma^2}}} \left[\frac{-a}{\gamma^2} \right]$$

$$\Rightarrow \frac{d\theta}{d\gamma} = \frac{\gamma}{a\sqrt{\gamma^2 - a^2}} - \frac{a}{\sqrt{\gamma^2 - a^2}} \left[\frac{1}{\gamma^2} \right]$$

$$\Rightarrow \frac{d\theta}{d\gamma} = \frac{\gamma}{a\sqrt{\gamma^2 - a^2}} - \frac{a}{\gamma\sqrt{\gamma^2 - a^2}}$$

$$\Rightarrow \frac{d\theta}{dr} = \frac{1}{\sqrt{r^2-a^2}} \left[\frac{r}{a} - \frac{a}{r} \right]$$

$$\Rightarrow \frac{d\theta}{dr} = \frac{1}{\sqrt{r^2-a^2}} \left[\frac{r^2-a^2}{ar} \right]$$

$$\Rightarrow \frac{d\theta}{dr} = \frac{1}{\sqrt{r^2-a^2}} \cdot \frac{\sqrt{r^2-a^2} \cdot \sqrt{r^2-a^2}}{ar}$$

$$\Rightarrow \frac{d\theta}{dr} = \frac{\sqrt{r^2-a^2}}{ar}$$

$$\Rightarrow \frac{dr}{d\theta} = \frac{ar}{\sqrt{r^2-a^2}}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a}{\sqrt{r^2-a^2}}$$

$$\Rightarrow \cot \phi = \frac{a}{\sqrt{r^2-a^2}}$$

$$\Rightarrow \cot^2 \phi = \frac{a^2}{r^2-a^2}$$

$$\Rightarrow 1 + \cot^2 \phi = 1 + \frac{a^2}{r^2-a^2}$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{r^2-a^2+a^2}{r^2-a^2} = \frac{r^2}{r^2-a^2}$$

WKT,

$$\Rightarrow \frac{1}{P^2} = \frac{1}{r^2} [1 + \cot^2 \phi]$$

$$\Rightarrow \frac{1}{P^2} = \frac{1}{r^2} \times \frac{r^2}{r^2-a^2}$$

$$\Rightarrow \frac{1}{P^2} = \frac{1}{r^2-a^2}$$

$$\Rightarrow P^2 = r^2 - a^2$$

$$\Rightarrow P = \sqrt{r^2-a^2} \quad (2)$$

diff w.r.t 'P'

$$(2) \Rightarrow 1 = \frac{1}{2\sqrt{r^2-a^2}} \cdot 2r \cdot \frac{dr}{dp} \Rightarrow r \frac{dr}{dp} = \sqrt{r^2-a^2}$$

$$\Rightarrow P = \sqrt{r^2-a^2}$$