Numerical Methods - 1

Def: Numerical analysis is a technique to find approx sol^n of the problems using simple operations.

* Numerical methods of finding approx roots of given equation is a repitative process known as iterative process, where in each step the result of the previous step is used and the process is carried out till we get the result to the desired accuracy.

Numerical sol^n of algebraic and transcendental eq^n:

Algebraic equations: Equations involving algebraic quantities like $x, x^2, x^3 \ldots$ are called algebraic eq^n.

eg: $x^3 - 2x + 5 = 0$, $x^4 - x^3 = 0$

Transcendental eq^n: Equations involving non-algebraic quantities like $\log x$, e^x, $\tan x$, $\sinh x \ldots$ are called transcendental eq^n.

eg: $\log x - x \sin x = 0$, $xe^x - 2 = 0$

property: given $f(x) = 0$, if there exists two values a, b such that $f(x)$ at these values has opposite signs, say $f(a) < 0$, $f(b) > 0$, equivalently $f(a) \cdot f(b) < 0$, then there always exists atleast one real root in the interval (a, b)

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Newton Raphson Method:

Formula: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ provided $f'(x_n) \neq 0$

Note: Given $f(x)=0$, find an interval (a,b). If the value of $f(a)$ is close to zero, then the real root lies in the neighbourhood of a, otherwise it lies in the neighbourhood of b.

problems:

① Find the real root of the eq^n $xe^x = 2$ correct to 4 decimal places using Newton Raphson method.

→ Let $f(x) = xe^x - 2 = 0$

put $x = 0, 1, 2 \ldots$

$f(0) = -2 < 0$

$f(1) = 0.7183 > 0$

$\therefore$ Real root lies in $(0,1)$

Since $f(1)$ is close to zero, let $\boxed{x_0 = 1}$

Consider $f(x) = xe^x - 2$

$f'(x) = xe^x + e^x \cdot 1 - 0$

$f'(x) = xe^x + e^x$

formula, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$x_{n+1} = x_n - \frac{(x_n e^{x_n} - 2)}{(x_n e^{x_n} + e^{x_n})}$$

$$x_{n+1} = \frac{x_n^2 e^{x_n} + x_n e^{x_n} - x_n e^{x_n} + 2}{x_n e^{x_n} + e^{x_n}}$$

$$x_{n+1} = \frac{(x_n^2 e^{x_n} + 2)}{(x_n e^{x_n} + e^{x_n})}$$

put $n = 0$

$$x_1 = \frac{x_0^2 e^{x_0} + 2}{x_0 e^{x_0} + e^{x_0}} = \frac{1 \cdot e^1 + 2}{1 \cdot e^1 + e^1} \Rightarrow \boxed{x_1 = 0.8679}$$

put $n = 1$

$$x_2 = \frac{x_1^2 e^{x_1} + 2}{x_1 e^{x_1} + e^{x_1}} = \frac{(0.8679)^2 e^{0.8679} + 2}{0.8679\, e^{0.8679} + e^{0.8679}} \Rightarrow \boxed{x_2 = 0.8528}$$

put $n = 2$

$$x_3 = \frac{(0.8528)^2 e^{0.8528} + 2}{0.8528\, e^{0.8528} + e^{0.8528}} \Rightarrow \boxed{x_3 = 0.8526}$$

put $n = 3$

$$x_4 = \frac{(0.8526)^2 e^{0.8526} + 2}{0.8526\, e^{0.8526} + e^{0.8526}} \Rightarrow \boxed{x_4 = 0.8526}$$

Thus a real root of the given eq^n upto 4 decimal places is 0.8526.

(2) Find the real root of the eq^n $x\sin x + \cos x = 0$ near $x = \pi$ using Newton Raphson method.

Let $f(x) = x\sin x + \cos x$

$f'(x) = x\cos x + \cancel{\sin x} - \cancel{\sin x}$

$f'(x) = x\cos x$

Let $\boxed{x_0 = \pi}$

formula, $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$$x_{n+1} = x_n - \frac{(x_n \sin x_n + \cos x_n)}{x_n \cos x_n}$$

$$x_{n+1} = \frac{x_n^2 \cos x_n - x_n \sin x_n - \cos x_n}{x_n \cos x_n}$$

put $n = 0$

$$x_1 = \frac{\pi^2\cos\pi - \pi\sin\pi - \cos\pi}{\pi\cos\pi} \Rightarrow \boxed{x_1 = 2.8233}$$

put $n = 1$

$$x_2 = \frac{(2.8233)^2\cos(2.8233) - 2.8233 - \sin(2.8233) - \cos(2.8233)}{2.8233\cos(2.8233)} \Rightarrow \boxed{x_2 = 2.7986}$$

put $n = 2$

$$x_3 = \frac{(2.7986)^2\cos(2.7986) - 2.7986 - \sin(2.7986) - \cos(2.7986)}{2.7986\cos(2.7986)} \Rightarrow \boxed{x_3 = 2.7984}$$

$$x_4 = \frac{(2.7984)^2\cos(2.7984) - 2.7984 - \sin(2.7984) - \cos(2.7984)}{2.7984\cos(2.7984)} \Rightarrow \boxed{x_4 = 2.7984}$$

Thus a real root of the given eq^n upto 4 decimal places is 2.7984.

(3) Find the real root of the eq^n $\tan x - x = 0$ near $x = 4.5$

Let $f(x) = \tan x - x$

$f'(x) = \sec^2 x - 1 = \tan^2 x$

Let $\boxed{x_0 = 4.5}$

formula, $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{\tan x_n - x_n}{\tan^2 x_n} \Rightarrow \boxed{\cancel{x_1 = 4.4936}}$$

put $n = 0$

$$x_1 = 4.5 - \left[\frac{\tan(4.5) - 4.5}{\tan^2(4.5)}\right] \Rightarrow \boxed{x_1 = 4.4936}$$

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put $n=1$,

$$x_2 = 4.4936 - \left[\frac{\tan(4.4936) - 4.4936}{\tan^2(4.4936)}\right]$$

$$\boxed{x_2 = 4.4934}$$

put $n=2$,

$$x_3 = 4.4934 - \left[\frac{\tan(4.4934) - 4.4934}{\tan^2(4.4934)}\right]$$

$$\boxed{x_3 = 4.4934}$$

Real Root of eq^n is 4.4934

(4) Find real root of eq^n $x \log_{10} x = 1.2$ upto ~~4~~ 5 decimal places.

Let $f(x) = x \log_{10} x - 1.2$

~~$f'(x) =$~~

put $x = 1, 2, 3 \ldots$

$f(1) = -1.20000 < 0$

$f(2) = -0.59794 < 0$

$f(3) = 0.23136 > 0$

Real root lies in $(2, 3)$

since $f(3)$ is close to zero, let $\boxed{x_0 = 3}$

Consider $f(x) = x \log_{10} x - 1.2$ $\qquad \log_b a = \frac{\log_e a}{\log_e b}$

i.e. $f(x) = x \cdot \frac{\log_e x}{\log_e 10} - 1.2$

$f(x) = 0.43429\, x \log_e x - 1.2$

diff wrt x,

$f'(x) = 0.43429\left[x \cdot \frac{1}{x} + \log_e x \cdot 1\right] - 0$

$f'(x) = 0.43429 + 0.43429 \log_e x$

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formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{0.43429\, x \log_e x - 1.2}{0.43429 + 0.43429 \log_e x}$$

put $n=0$

$$x_1 = 3 - \left[\frac{3\log(3) - 1.2}{0.43429(1+\log_e 3)}\right] \Rightarrow \boxed{x_1 = 2.74615}$$

put $n=1$

$$x_2 = 2.74615 \left[\frac{2.74615\log(2.74615) - 1.2}{0.43429(1+\log_e 2.74615)}\right]$$

$$\boxed{x_2 = 2.74065}$$

put $n=2$

$$x_3 = 2.74065 \left[\frac{2.74065\log(2.74065) - 1.2}{0.43429(1+\log_e 2.74065)}\right]$$

$$\boxed{x_3 = 2.74065}$$

Real Root of eq^n is 2.74065

T.W.

5) $3x - 1 = \cos x$

Let $f(x) = 3x - 1 - \cos x$

put $x = 0, 1, 2$

$f(0) = -2.0000 < 0$

$f(1) = 1.4597 > 0$

Real root lies in $(0, 1)$

Since $f(1)$ is close to zero, let $\boxed{x_0 = 1}$

Consider, $f(x) = 3x - 1 - \cos x$

$f'(x) = 3 + \sin x$

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formula, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$x_{n+1} = x_n - \frac{(3x-1-\cos x)}{3+\sin x}$$

put $n=0$,

$$x_1 = 1 - \frac{(3-1-\cos(1))}{3+\sin(1)} \Rightarrow \boxed{x_1 = 0.6200}$$

put $n=1$,

$$x_2 = 0.6200 - \frac{(3(0.6200)-1-\cos(0.6200))}{3+\sin(0.6200)} \Rightarrow \boxed{x_1 = 0.6071}$$

put $n=2$,

$$x_3 = 0.6071 - \frac{(3(0.6071)-1-\cos(0.6071))}{3+\sin(0.6071)} \Rightarrow \boxed{x_3 = 0.6071}$$

6 $xe^x = \cos x$ near point 1

let $f(x) = xe^x - \cos x$

$f'(x) = x.e^x + e^x + \sin x$

let $\boxed{x_0 = 1}$

formula, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$x_n - \frac{xe^x - \cos x}{xe^x + e^x + \sin x}$$

put $n=0$,

$$x_1 = \frac{x_0 - x_0e^{x_0} - \cos x_0}{x_0 e^{x_0} + e^{x_0} + \sin x_0} \Rightarrow \boxed{x_1 = 0.6531}$$

put $n=1$,

$$x_2 = 0.6531 - \frac{0.6531e^{0.6531} - \cos(0.6531)}{0.6531e^{0.6531} + e^{0.6531} + \sin(0.6531)}$$

$\boxed{x_2 = 0.5313}$

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put $n=2$,

$$x_3 = 0.5313 - \frac{0.5313 e^{0.5313} - \cos(0.5313)}{0.5313 e^{0.5313} + e^{0.5313} + \sin(0.5313)} \Rightarrow \boxed{x_3 = 0.5179}$$

put $n=3$, $$x_4 = 0.5179 - \frac{0.5179 e^{0.5179} - \cos(0.5179)}{0.5179 e^{0.5179} + e^{0.5179} + \sin(0.5179)} \Rightarrow \boxed{x_4 = 0.5178}$$

put $n=4$, $$x_5 = 0.5178 - \frac{0.5178 e^{0.5178} - \cos(0.5178)}{0.5178 e^{0.5178} + e^{0.5178} + \sin(0.5178)} \Rightarrow \boxed{x_5 = 0.5178}$$

7) $x + \log_{10} x = 3.375$ near 2.9

$\rightarrow$ $f(x) = x + \log_{10} x - 3.375$

$$f'(x) = x + \frac{\log_e x}{\log_e 10} - 3.375$$

$$f(x) = x + 0.4343 \log_e x - 3.375$$

$$f'(x) = 1 + 0.4343\left(\frac{1}{x}\right) - 0$$

$$f'(x) = 1 + \frac{0.4343}{x} \Rightarrow \text{let } \boxed{x_0 = 2.9}$$

formula, $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = \frac{x_n - x_n + \log_{10} x_n - 3.375}{1 + \frac{0.4343}{x_n}}$$

put $n=0$, $$x_1 = \frac{x_0 - x_0 + \log_{10} x_0 - 3.375}{1 + \frac{0.4343}{x_0}}$$

$$\boxed{x_1 = 2.9110}$$

put $n=1$, $$x_2 = x_1 - \frac{x_1 + \log_{10} x_1 - 3.375}{1 + \frac{0.4343}{x_1}}$$

$$\boxed{x_2 = 2.9110}$$

(8) $x^3+x^2+3x+4=0$ near $x=-1$

let $f(x) = x^3+x^2+3x+4$

$f'(x) = 3x^2+2x+3$

formula, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$x_{n+1} = x_n - \frac{x_n^3+x_n^2+3x_n+4}{3x_n^2+2x_n+3} \Rightarrow \frac{2x_n^3+x_n^2-4}{3x_n^2+2x_n+3}$

put $n=0$, $x_1 = \frac{2x_0^3+x_0^2-4}{3x_n^2+2x_n+3} \Rightarrow \boxed{x_1 = -1.2500}$

put $n=1$, $x_2 = \frac{2x_1^3+x_1^2-4}{3x_1^2+2x_1+3} \Rightarrow \boxed{x_2 = -1.2229}$

put $n=2$, $x_3 = \frac{2x_2^3+x_2^2-4}{3x_2^2+2x_2+3} \Rightarrow \boxed{x_3 = -1.2225}$

put $n=3$, $x_4 = \frac{2x_3^3+x_3^2-4}{3x_3^2+2x_3+3} \Rightarrow \boxed{x_4 = -1.2225}$

Regular Falsi method (or) the method of False position:

Find real roots of following eq^n ~~for~~ by Regula Falsi method:

Formula: $x = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)}$

(1) $xe^x = \cos x$

Let $f(x) = xe^x - \cos x$

put $x = 0, 1, 2 \ldots$

$f(0) = -1 < 0$

$f(1) = 2.1780 > 0$

$\therefore$ Root lies in $(0,1)$

Since $f(0)$ is close to zero, we excpet the root in the neighbourhood of 0.

$$f(0.1) = -0.8845 < 0$$
$$f(0.2) = -0.7358 < 0$$
$$f(0.3) = -0.5504 < 0$$
$$f(0.4) = -0.3243 < 0$$
$$f(0.5) = -0.0532 < 0$$
$$f(0.6) = 0.2679 > 0$$

$\therefore$ Root lies within $(0.5, 0.6)$

formula, $x = \dfrac{a f(b) - b f(a)}{f(b) - f(a)}$

I. Let $a = 0.5$, $b = 0.6$

$$f(a) = f(0.5) = -0.0532$$
$$f(b) = f(0.6) = 0.2679$$
$$x_1 = \frac{0.5 \times 0.2679 + 0.6 \times 0.0532}{0.2679 + 0.0532}$$
$$\boxed{x_1 = 0.5166}$$
$$f(x_1) = f(0.5166) = -0.0035 < 0$$

$\therefore$ Root lies in $(0.5166, 0.6)$

II. Let $a = 0.5166$, $b = 0.6$

$$f(a) = f(0.5166) = -0.0035$$
$$f(b) = f(0.6) = 0.2679$$
$$x_2 = \frac{0.5166 \times 0.2679 + 0.6 \times 0.0035}{0.2679 + 0.0035}$$
$$\boxed{x_2 = 0.5177}$$
$$f(x_2) = f(0.5177) = -0.0002 < 0$$

$\therefore$ Root lies in $(0.5177, 0.6)$

III Let $a = 0.5177$, $b = 0.6$

$f(a) = f(0.5177) = -0.0002$

$f(b) = f(0.6) = 0.2679$

$$x_3 = \frac{0.5177 \times 0.2679 + 0.6 \times 0.0002}{0.2679 + 0.0002}$$

$$\boxed{x_3 = 0.5178}$$

$f(x_3) = f(0.5178) = 0.0001 > 0$

$\therefore$ Root lies in $(0.5177, 0.5178)$

IV Let $a = 0.5177$, $b = 0.5178$

$f(a) = f(0.5177) = -0.0002$

$f(b) = f(0.5178) = 0.0001$

$$x_4 = \frac{0.5177 \times 0.0001 + 0.5178 \times 0.0002}{0.0001 + 0.0002}$$

$$\boxed{x_4 = 0.5178}$$

Therefore real root for the given eqn is 0.5178

(2) $3x - 1 = \cos x$ lies b/w 0.5 and 1 correct to 3 decimal places.

Let $f(x) = 3x - 1 - \cos x$

$f(0.5) = -0.378 < 0$

$f(1) = 1.460 > 0$

formula, $$x = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

I Let $a = 0.5$, $b = 1$

$f(a) = f(0.5) = -0.378$

$f(b) = f(1) = 1.460$

$$x_1 = \frac{0.5 \times 1.460 + 1 \times 0.378}{1.460 + 0.378} \Rightarrow \boxed{x_1 = 0.603}$$

$f(x_1) = f(0.603) = -0.015$

$\therefore$ Root lies in $(0.603, 1)$

II Let $a = 0.603$, $b = 1$

$f(a) = f(0.603) = -0.015$

$f(b) = f(1) = 1.460$

$$x_2 = \frac{0.603 \times 1.460 + 1 \times 0.015}{1.460 + 0.015} \Rightarrow \boxed{x_2 = 0.607}$$

$f(x_2) = f(0.607) = -0.00036$

Root lies in $(0.607, 1)$

III Let $a = 0.607$, $b = 1$

$f(a) = f(0.607) = -0.00036$

$f(b) = f(1) = 1.460$

$$x_3 = \frac{0.607 \times 1.460 + 1 \times 0.00036}{1.460 + 0.00036} \Rightarrow \boxed{x_3 = 0.607}$$

$\therefore$ Real root for the given eq^n is 0.607

(3) $2x - \log_{10} x = 7$

Let $f(x) = 2x - \log_{10} x - 7$

put $x = 1, 2, 3 \ldots$

$f(1) = -5.0000 < 0$

$f(2) = -3.3010 < 0$

$f(3) = -1.4771 < 0$

$f(4) = 0.3979 > 0$

$\therefore$ Root lies b/w $(3, 4)$

Since $f(4)$ is close to zero, we expect the root in the neighbourhood of 4.

$f(3.9) = 0.2089 > 0$

$f(3.8) = 0.0202 > 0$

$f(3.7) = -0.1682 < 0$

$\therefore$ Root lies in $(3.7, 3.8)$

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I. Let $a = 3.7$, $b = 3.8$

$f(a) = f(3.7) = -0.1682$

$f(b) = f(3.8) = 0.0202$

$$x_1 = \frac{3.7 \times 0.0202 + 3.8 \times 0.1682}{0.0202 + 0.1682} \Rightarrow \boxed{x_1 = 3.7893}$$

$f(x_1) = f(3.7893) = 0.000041$

$\therefore$ Root lies in $(3.7, 3.7893)$

II. Let $a = 3.7$, $b = 3.7893$

$f(a) = f(3.7) = -0.1682$

$f(b) = f(3.7893) = 0.000041$

$$x_2 = \frac{3.7 \times 0.000041 + 3.7893 \times 0.1682}{0.000041 + 0.1682}$$

$$\boxed{x_2 = 3.7893}$$

$\therefore$ Real root for given eq^n is 3.7893.

(4) $x^3 - 2x - 5 = 0$

Let $f(x) = x^3 - 2x - 5$

put $x = 0, 1, 2, 3 \ldots$

$f(0) = -5.0000 < 0$

$f(1) = -6.0000 < 0$

$f(2) = -1.0000 < 0$

$f(3) = 16.0000 > 0$

$\therefore$ Root lies in $(2, 3)$

Since $f(2)$ is close to zero, we expect the root in the neighbourhood of 2

$f(2.1) = 0.0610$

$\therefore$ Root lies in $(2, 2.1)$

formula, $x = \dfrac{af(b) - bf(a)}{f(b) - f(a)}$

I Let $a=2 \quad b=2.1$

$f(a)=f(2)=-1.0000$

$f(b)=f(2.1)=0.0610$

$$x_1=\frac{2\times0.0610+2.1\times1.0000}{0.0610+1.0000} \Rightarrow \boxed{x_1=2.0943}$$

$f(x_1)=f(2.0943)=-0.0028$

$\therefore$ Root lies in $(2.0943, 2.1)$

II let $a=2.0943$, $b=2.1$

$f(a)=f(2.0943)=-0.0028$

$f(b)=f(2.1)=0.0610$

$$x_2=\frac{2.0943\times0.0610+2.1\times0.0028}{0.0610+0.0028}$$

$$\boxed{x_2=2.0946}$$

$f(x_2)=f(2.0946)=0.0005$

$\therefore$ Root lies in $(2.0943, 2.0946)$

III Let $a=2.0943$, $b=2.0946$

$f(a)=f(2.0943)=-0.0028$

$f(b)=f(2.0946)=0.0005$

$$x_3=\frac{2.0943\times0.0005+\cancel{2\times0.0}\ 2.0946\times0.0028}{0.0005+0.0028}$$

$$\boxed{x_3=2.0946}$$

$\therefore$ Real root of the given eq^n is 2.0946.

(5) $x\log_{10}x=1.2$

$\rightarrow$ Let $f(x)=x\log_{10}x-1.2=0$

put $x=1,2,3$

$f(1)=-1.2000<0$

$f(2)=-0.5979<0$

$f(3)=0.2314>0$

$\therefore$ Root lies in $(2,3)$

Since $f(3)$ is close to zero, the root lies in the neighbourhood of 3.

$f(2.9) = 0.1410 > 0$

$f(2.8) = 0.0520 > 0$

$f(2.7) = -0.0353 < 0$

$\therefore$ Root lies in $(2.7, 2.8)$

I Let $a = 2.7$ $b = 2.8$

$f(a) = f(2.7) = -0.0353$

$f(b) = f(2.8) = 0.0520$

$$x_1 = \frac{2.7 \times 0.0520 + 2.8 \times 0.0353}{0.0520 + 0.0353}$$

$$\boxed{x_1 = 2.7404}$$

$f(x_1) = f(2.7404) = +0.00027$

$\therefore$ Root lies in $(2.7404, 2.8)$

II Let $a = 2.7404$ $b = 2.8$

$f(a) = f(2.7404) = -0.0002$

$f(b) = f(2.8) = 0.0520$

$$x_2 = \frac{2.7404 \times 0.0520 + 2.8 \times 0.0002}{0.0520 + 0.0002}$$

$$\boxed{x_2 = 2.7406}$$

(6) $xe^x = 2$

→ Let $f(x) = xe^x - 2$

put $x = 0, 1, 2 \ldots$

$f(0) = -2.0000$

$f(1) = 0.7183$

$\therefore$ Root lies in $(0, 1)$

Since $f(1)$ is close to zero, root lies in the neighbourhood of 1

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$f(0.9) = 0.2136$
$f(0.8) = -0.2196$
$\therefore$ Root lies in $(0.8, 0.9)$

I Let $a = 0.8 \quad b = 0.9$
$f(a) = f(0.8) = -0.2196$
$f(b) = f(0.9) = 0.2136$

$$x_1 = \frac{0.8 \times 0.2136 + 0.9 \times 0.2196}{0.2136 + 0.2196} \Rightarrow \boxed{x_1 = 0.8507}$$

$f(x_1) = f(0.8507) = -0.0083$
$\therefore$ Root lies b/w $(0.8507, 0.9)$

Let $a = 0.8507$, $b = 0.9$
$f(a) = f(0.8507) = -0.0083$
$f(b) = f(0.9) = 0.2136$

$$x_2 = \frac{0.8507 \times 0.2136 + 0.9 \times 0.0083}{0.2136 + 0.0083}$$

$$\boxed{x_2 = 0.8525}$$

$f(x_2) = f(0.8525) = -0.0005$
Root lies in $(0.8525, 0.9)$

Let $a = 0.8525 \quad b = 0.9$
$f(a) = f(0.8525) = -0.0005$
$f(b) = f(0.9) = 0.2136$

$$x_3 = \frac{0.8525 \times 0.2136 + 0.9 \times 0.0005}{0.2136 + 0.0005} \quad \boxed{x_3 = 0.8526}$$

$f(x_3) = f(0.8526) = -0.000023$
Root lies b/w $(0.8526, \cancel{0.2136}\ 0.9)$

let $a = 0.8526 \quad b = 0.9$

$f(a) = f(0.8526) = -0.000023$

$f(b) = f(0.9) = 0.2136$

$$x_4 = \frac{0.8526 \times 0.2136 + 0.9 \times 0.000023}{0.2136 + 0.000023} \Rightarrow \boxed{x_4 = 0.8526}$$

$\therefore$ Real root of the given eqⁿ is 0.8526

(7) $x^2 - \log_e x = 12$ lies b/w 3 and 4

$\longrightarrow$ let $f(x) = x^2 - \log_e x - 12$

$f(3) = -4.0986 < 0$

$f(4) = 2.6137 > 0$

let $a = 3 \quad b = 4$

$f(a) = f(3) = -4.0986$

$f(b) = f(4) = 2.6137$

$$x_1 = \frac{3 \times 2.6137 + 4 \times 4.0986}{2.6137 + 4.0986} \Rightarrow \boxed{x_1 = 3.6106}$$

$f(x_1) = f(3.6106) = -0.2474$

Root lies in $(3.6106, 4)$

let $a = 3.6106 \quad b = 4$

$f(a) = f(3.6106) = -0.2474$

$f(b) = f(4) = 2.6137$

$$x_2 = \frac{3.6106 \times 2.6137 + 4 \times 0.2474}{2.6137 + 0.2474}$$

$\boxed{x_2 = 3.6443}$

$f(x_2) = f(3.6443) = -0.0122$

Root lies in $(3.6443, 4)$

let $a = 3.6443 \quad b = 4$

$f(a) = f(3.6443) = -0.0122$

$f(b) = f(4) = 2.6137$

$$x_3 = \frac{3.6443 \times 2.6137 + 4 \times 0.0122}{2.6137 + 0.0122}$$

$$\boxed{x_3 = 3.6460}$$

$f(x_3) = f(3.6460) = -0.0003$

Root lies in $(3.6460, 4)$

let $a = 3.6460 \quad b = 4$

$f(a) = f(3.6460) = -0.0003$

$f(b) \quad f(4) = 2.6137$

$$x_4 = \frac{3.6460 \times 2.6137 + 4 \times 0.0003}{\cancel{0.000} \; 2.6137 + 0.0003}$$

$$\boxed{x_4 = 3.6460}$$

$\therefore$ Real root of given eq^n is 3.6460

(8) $\tan x + \tanh x = 0$ lies b/w 2 and 3

$f(x) = \tan x + \tanh x$

$f(2) = -1.2210 < 0$

$f(3) = 0.8525 > 0$

Since $f(3)$ is close to zero, it lies in the neighbourhood of 3

$f(2.9) = 0.7476$

$f(2.8) = 0.6341$

$f(2.7) = 0.5183$

$f(2.6) = 0.3874$

$f(2.5) = 0.2396$

$f(2.4) = 0.0677$

$f(2.3) = -0.1391$

$\therefore$ Root lies in $(2.3, 2.4)$

$a = 2.3$, $b = 2.4$

$f(a) = -0.1391$, $f(b) = 0.0677$

$$x = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$x_1 = \frac{2.3 \times 0.0677 + 2.4 \times 0.1391}{0.0677 + 0.1391}$$

$$\boxed{x_1 = 2.3673}$$

$f(x_1) = f(2.3673) = 0.0045$

$\therefore$ Root lies in $(2.3, 2.3673)$

$a = 2.3$, $b = 2.3673$

$f(a) = -0.1391$, $f(b) = 0.0045$

$$x_2 = \frac{2.3 \times 0.0045 + 2.3673 \times 0.1391}{0.0045 + 0.1391}$$

$$\boxed{x_2 = 2.3652}$$

$f(x_2) = f(2.3652) = 0.0004$

$\therefore$ Root lies in $(2.3, 2.3652)$

$a = 2.3$ $b = 2.3652$

$f(a) = -0.1391$ $f(b) = 0.0004$

$$x_3 = \frac{2.3 \times 0.0004 + 2.3652 \times 0.1391}{0.0004 + 0.1391}$$

$$\boxed{x_3 = 2.3650}$$

~~Root lies in~~ $f(2.3650) = -0.000040$

Root lies in $(2.3650, 2.3652)$

$a = 2.3650$ $b = 2.3652$

$f(a) = -0.000040$ $f(b) = 0.0004$

$$x_4 = \frac{2.3650 \times 0.0004 + 2.3652 \times 0.000040}{0.0004 + 0.000040}$$

$$\boxed{x_4 = 2.3650}$$

$\therefore$ Real root of given eq^n is 2.3650

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Finite differences:

Interpolation: Let $y = f(x)$ is an unknown function. The process of finding value of y for any given value of x b/w x_0 and x_n is called interpolation. Also the process of finding the value of y outside the given range of x is called extrapolation. In general the concept of interpolation includes extrapolation also.

Interpolation formulae for equal intervals (equi-distant values of x)

1. Newton Gregory forward interpolation formula / Newton's forward formula.

$$y(x) = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots$$

where, $p = \frac{x - x_0}{h}$, h is the step length, x is the point of interpolation

2. Newton Gregory backward interpolation formula / Newton's backward formula

$$y(x) = y_n + p\,\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \dots$$

where, $p = \frac{x - x_n}{h}$

problems - using suitable interpolation formula find -

(i) $y(38)$ and $y(85)$ for the following data

x	40	50	60	70	80	90
y	184	204	226	250	276	304

→ finite difference table is given by :

x	y	1st diff	2nd diff
40	184		
		20	
50	204		2
		22	
60	226		2
		24	
70	250		2
		26	
80	276		2
		28	
90	304		

from the table,

$\Delta y_0 = 20$, $\Delta^2 y_0 = 2$

$\nabla y_n = 28$, $\nabla^2 y_n = 2$

To find $y(38)$:

$x = 38$ is the point of interpolation

Newton's forward formula,

$$y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0$$

where $p = \dfrac{x - x_0}{h} = \dfrac{38-40}{10} = -0.2$

$$y(38) = 184 + (-0.2)\times 20 - \frac{0.2\times(0.2-1)}{\cancel{2}} \times \cancel{2}$$

$$\boxed{y(38) = 180.24}$$

To find $y(85)$:

$x = 85$ is the point of interpolation

Newton's backward formula,

$$y(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n$$

where, $p = \frac{x - x_n}{h} = \frac{85-90}{10} = -0.5$

$y(85) = 304 - 0.5 \times 28 - \frac{0.5(-0.5+1)}{\cancel{2}} \times \cancel{2}$

$\boxed{y(85) = 289.75}$

(2) find y(8) given that $y(1) = 24$, $y(3) = 120$, $y(5) = 336$, $y(7) = 720$ using suitable interpolation formula.

x	y	1st diff	2nd diff	3rd diff
1	24			
		96		
3	120		120	
		216		48
5	336		168	
		384		
7	720			

To find y(8)

$x = 8$ is the point of interpolation

y(x) = Newton's backward formula is given by:

$y(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n$

where, $p = \frac{x - x_n}{h} = \frac{8-7}{2} = 0.5$

$y(8) = 720 + 0.5 \times 384 + \frac{0.5(0.5+1)}{2} \times 168 + \frac{0.5(0.5+1)(0.5+2)}{3 \times 2} \times 48$

$\boxed{y(8) = 990}$

** (3) The area A of a circle corresponding to the diameter D is given below.

D	80	85	90	95	100
A:	5026	5674	6362	7088	7854

Find the area corresponding to D = 105 using suitable interpolation formula.

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D = x	A = y	1st diff	2nd diff	3rd diff	4th diff
80	5026				
		648			
85	5674		40		
		688		-2	
90	6362		38		4
		726		2	
95	7088		40		
		766			
100	7854				

To find y(105),
Newton's backward formula.

$$y(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n +$$

$$\frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n$$

where $p = \frac{x - x_n}{h} = \frac{105-100}{5} \Rightarrow \boxed{p = 1}$

$$y(105) = 7854 + (1 \times 766) + \left(\frac{1(2)}{2} \times 40\right) + \left(\frac{1\times2\times3}{6} \times 2\right) +$$

$$\left(\frac{1\times2\times3\times4}{24} \times 4\right)$$

$\boxed{y(105) = 8666}$

i.e. when D = 105, area A = 8666

*** (4) Given $\sin 45^\circ = 0.7071$, $\sin 50^\circ = 0.7660$, $\sin 55^\circ = 0.8192$, $\sin 60^\circ = 0.8660$ find $\sin 57^\circ$ using appropriate interpolation formula.

Given:

x :	45°	50°	55°	60°
y :	0.7071	0.7660	0.8192	0.8660

x	y	1st diff	2nd diff	3rd diff
45	0.7071			
		+0.0589		
50	0.7660		-0.0057	
		0.0532		-0.0007
55	0.8192		-0.0064	
		0.0468		
60	0.8660			

To find $y(57)$

Newton's backward formula.

$$y(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n$$

$$+ p(p+1)(p+2)(p+3)\nabla^4 y^n$$

where $p = \frac{x - x_n}{h} = \frac{57-60}{5} = -0.6$

$$y(57) = 0.8660 + (-0.6 \times \overset{0.0468}{0.8660}) + \left[\frac{-0.6 \times 0.4}{2} \times \overset{-0.0064}{0.0468}\right]$$

$$+ \left[\frac{-0.6 \times 0.4 \times 1.4}{6} \times -0.0007\right]$$

$y(57) = 0.8660 - 0.0281 + 0.0008 + 0.0001$

$y(57) = \underline{\underline{0.8388}}$

*** (5) From the table, estimate no of students who obtained marks between 40 and 45:

Marks	30-40	40-50	50-60	60-70	70-80
NO of Students	31	42	51	35	31

→ The cummulative frequency table for the given data is as below:

Marks less than (x):	~~30~~	40	50	60	70	80
No. of Students (y):	~~31~~	~~73~~	73	124	159	190

To find no. of students who have scored marks less than 45 i.e. to find y when $x = 45$

x	y	1^{st} diff	2^{nd}	3^{rd}	4^{th}
40	31				
		42			
50	73		9		
		51		-25	
60	124		-16		37
		35		12	
70	159		-4		
		31			
80	190				

To find $y(45)$:

Newton's forward formula.

$$y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\nabla^2 y_0 + \frac{p(p-1)(p-2)}{3!}\nabla^3 y_0$$

$$+ \frac{p(p-1)(p-2)(p-3)}{4!}\nabla^4 y_0$$

where $p = \frac{x - x_0}{h} = \frac{45 - 40}{10} = 0.5$

$$y(45) = 31 + (0.5 \times 42) + \frac{0.5 \times -0.5}{2} \times 9 \cancel{+ \frac{0.5 \times -0.5 \times -1.5}{6} \times -25}$$

$$+ \frac{0.5 \times -0.5 \times -1.5}{6} \times -25 + \frac{0.5 \times -0.5 \times -1.5 \times -2.5}{24} \times 37$$

$$= 31 + 21 + (-1.1250) + (-1.5625) + (-1.4453)$$
$$= 47.8672$$

$\therefore \boxed{y(45) \simeq 48}$ i.e, 48 students have scored marks less than 45.

Thus, no of students who have scored marks b/w 40 and 45 = 48 − 31 = 17 //

(6) Find the interpolating polynomial satisfying $f(0) = 0$, $f(2) = 4$, $f(4) = 56$, $f(6) = 204$, $f(8) = 496$, $f(10) = 980$. Hence find $f(3)$, $f(5)$

→ finite diff table:

x	y	1st	2nd	3rd
0	0			
		4		
2	4		48	
		52		48
4	56		96	
		148		48
6	204		144	
		292		48
8	496		192	
		484		
10	980			

Newton's forward formula.

$$y(x) = y_0 + p\nabla y_0 + \frac{p(p-1)}{2!}\nabla^2 y_0 + \frac{p(p-1)(p-2)}{3!}\nabla^3 y_0$$

where $p = \frac{x - x_0}{h} = \frac{x-0}{2} \Rightarrow \boxed{p = \frac{x}{2}}$

$$y(x) = 0 + \left(\frac{x}{2}\times 4\right) + \frac{\frac{x}{2}\left(\frac{x}{2}-1\right)}{2}\times 48 + \frac{\frac{x}{2}\left(\frac{x}{2}-1\right)\left(\frac{x}{2}-2\right)}{6}\times \cancel{48}$$

$$y(x) = 2x + \left[\frac{x}{2}\,\frac{(x-2)}{2}\times 24\right] + \left[\frac{x}{2}\,\frac{(x-2)}{2}\,\frac{(x-4)}{2}\times 8\right]$$

$$y(x) = 2x + 6x(x-2) + x(x^2-6x+8)$$

$$= 2x + \cancel{6x^2} - 12x + x^3 - \cancel{6x^2} + 8x$$

$$= -10x + x^3 + 8x$$

$$f(x) = \underline{\underline{x^3 - 2x}}$$

$$\therefore f(3) = 3^3 - 2(3) = 27 - 6 = 21$$

$$f(5) = 5^3 - 2(5) = 125 - 10 = 115$$

Interpolation formulae for unequal intervals:

I. Divided differences:

1^{st} order divided differences are defined as:

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\vdots$$

$$f(x_{n-1}, x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

2^{nd} order divided diff are defined as:

$$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

$$f(x_1, x_2, x_3) = \frac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1}$$

$$\vdots$$

$$f(x_{n-2}, x_{n-1}, x_n) = \frac{f(x_{n-1}, x_n) - f(x_{n-2}, x_{n-1})}{x_n - x_{n-2}}$$

Newton's divided difference formula (or) Newton's general interpolation formula:

$$y = f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3)$$

Note: This is general interpolation formula applicable for both equal and unequal intervals.

① Using Newton's divided difference formula find f(9) from following table

x	5	7	11	13	17
y	150	392	1452	2366	5202

Divided difference table is as below:

	x	$y = f(x)$	1^{st} d.d	2^{nd} d.d	3^{rd} d.d
$x_0 =$	5	$f(x_0) = 150$	$f(x_0, x_1) = 121$	$f(x_0, x_1, x_2) = 24$	$f(x_0, x_1, x_2, x_3) = 1$
$x_1 =$	7	$f(x_1) = 392$	$f(x_1, x_2) = 265$	$f(x_1, x_2, x_3) = 32$	$f(x_1, x_2, x_3, x_4) = 1$
$x_2 =$	11	$f(x_2) = 1452$	$f(x_2, x_3) = 457$	$f(x_2, x_3, x_4) = 42$	
$x_3 =$	13	$f(x_3) = 2366$	$f(x_3, x_4) = 709$		
$x_4 =$	17	$f(x_4) = 5202$	$f(x_4, x_5) =$		

To find $f(9)$:

Newton's d.d formula:

$$y = f(x) = f(x_0) + (x-x_0) f(x_0,x_1) + (x-x_0)(x-x_1) f(x_0,x_1,x_2) + (x-x_0)(x-x_1)(x-x_2) f(x_0,x_1,x_2,x_3)$$

$$y = f(9) = 150 + (9-5)\times 121 + (9-5)(9-7)\times 24 + (9-5)(9-7)(9-11)\times 1$$

$$\underline{f(9) = 810}$$

② Using Newton's d.d formula evaluate $f(8)$ and $f(15)$ given:

x	4	5	7	10	11	13
y	48	100	294	900	1210	2028

x	$y = f(x)$	1^{st} d.d	2^{nd} d.d	3^{rd} d.d
$x_0 = 4$	$f(x_0) = 48$			
		$f(x_0,x_1) = 52$		
$x_1 = 5$	$f(x_1) = 100$		$f(x_0,x_1,x_2) = 15$	
		$f(x_1,x_2) = 97$		$f(x_0,x_1,x_2,x_3) = 1$
$x_2 = 7$	$f(x_2) = 294$		$f(x_1,x_2,x_3) = 21$	
		$f(x_2,x_3) = 202$		$f(x_1,x_2,x_3,x_4) = 1$
$x_3 = 10$	$f(x_3) = 900$		$f(x_2,x_3,x_4) = 27$	
		$f(x_3,x_4) = 310$		$f(x_2,x_3,x_4,x_5) = 1$
$x_4 = 11$	$f(x_4) = 1210$		$f(x_3,x_4,x_5) = 33$	
		$f(x_4,x_5) = 409$		
$x_5 = 13$	$f(x_5) = 2028$			

To find $f(8)$:

Newton's d.d formula:

$$y = f(x) = f(x_0) + (x-x_0) f(x_0,x_1) + (x-x_0)(x-x_1) f(x_0,x_1,x_2) + (x-x_0)(x-x_1)(x-x_2) f(x_0,x_1,x_2,x_3)$$

$$y = f(8) = 48 + (8-4)\times 52 + (8-4)(8-5)\times 15 + (8-4)(8-5)(8-7)\times 1$$

$$\underline{f(8) = 448}$$

$$f(15) = 48 + (15-4)\times 52 + (15-4)(15-5)\times 15 + (15-4)(15-5)(15-7)\times 1$$

$$\underline{f(15) = 3150}$$

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(3) Using divided difference formula, fit an interpolating polynomial for the following data

x	0	1	4	8	10
$f(x)$	-5	-14	-125	-21	355

Divided difference table is as below:

x	$y = f(x)$	1^{st} d.d	2^{nd} d.d	3^{rd} d.d
$x_0 = 0$	$f(x_0) = -5$	$f(x_0, x_1) = -9$	$f(x_0, x_1, x_2) = -7$	
$x_1 = 1$	$f(x_1) = -14$			$f(x_0, x_1, x_2, x_3) = 2$
		$f(x_1, x_2) = -37$		
$x_2 = 4$	$f(x_2) = -125$		$f(x_1, x_2, x_3) = 9$	
		$f(x_2, x_3) = 26$		$f(x_1, x_2, x_3, x_4) = 2$
$x_3 = 8$	$f(x_3) = -21$			
		$f(x_3, x_4) = 188$	$f(x_2, x_3, x_4) = 27$	
$x_4 = 10$	$f(x_4) = 355$			

d.d formula,

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2)$$
$$+ (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3)$$

$$= -5 + (x-0)(-9) + (x-0)(x-1)(-7) + (x-0)(x-1)(x-4)(2)$$
$$= -5 - 9x - 7x(x-1) + 2x(x^2 - 5x + 4)$$
$$= -5 - 9x - 7x^2 + 7x + 2x^3 - 10x^2 + 8x$$
$$= 2x^3 - 17x^2 + 6x - 5$$

(4) Construct the interpolating polynomial for the data given below using Newton's d.d formula

x	0	1	2	3	4	5
$f(x)$	3	2	7	24	59	118

x	$y = f(x)$	1^{st} d.d	2^{nd} d.d	3^{rd} d.d
$x_0 = 0$	$f(x_0) = 3$			
		$f(x_0, x_1) = -1$		
$x_1 = 1$	$f(x_1) = 2$		$f(x_0, x_1, x_2) = 3$	$f(x_0, x_1, x_2, x_3) = 1$
		$f(x_1, x_2) = 5$		
$x_2 = 2$	$f(x_2) = 7$		$f(x_1, x_2, x_3) = 6$	
		$f(x_2, x_3) = 17$		$f(x_1, x_2, x_3, x_4) = 1$
$x_3 = 3$	$f(x_3) = 24$		$f(x_2, x_3, x_4) = 9$	
		$f(x_3, x_4) = 35$		$f(x_2, x_3, x_4, x_5) = 1$
$x_4 = 4$	$f(x_4) = 59$		$f(x_3, x_4, x_5) = 12$	
		$f(x_4, x_5) = 59$		
$x_5 = 5$	$f(x_5) = 118$			

d.d formula,

$$f(x) = f(x_0) + (x-x_0) f(x_0,x_1) + (x-x_0)(x-x_1) f(x_0,x_1,x_2) + (x-x_0)(x-x_1)(x-x_2) f(x_0,x_1,x_2,x_3)$$

$$= 3 + (x-0)(-1) + (x-0)(x-1)(3) + (x-0)(x-1)(x-2)(1)$$

$$= 3 - x + 3x(x-1) + x(x^2-3x+2)$$

$$= 3 - x + \cancel{3x^2} - 3x + x^3 - \cancel{3x^2} + 2x$$

$$= \underline{x^3 - 2x + 3}$$

Lagrange's interpolation formula:

$$y = f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} \cdot y_0$$

$$+ \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} \cdot y_1$$

$$\cdots\cdots\cdots$$

$$+ \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} \cdot y_n$$

① ** Using Lagrange's interpolation formula find f(9) from following data:

x	5	7	11	13	17
$f(x)$	150	392	1452	2366	5202

→ Let $x_0 = 5$ $x_1 = 7$ $x_2 = 11$ $x_3 = 13$ $x_4 = 17$

$y_0 = 150$ $y_1 = 392$ $y_2 = 1452$ $y_3 = 2366$ $y_4 = 5202$

formula,

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} \cdot y_0$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} \cdot y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} \cdot y_2$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} \cdot y_3$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)} \cdot y_4$$

$$f(9) = \frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)} \cdot 150$$

$$+ \frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)} \cdot 392$$

$$+ \frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)} \cdot 1452$$

$$+ \frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)} \cdot 2366$$

$$+ \frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)} \cdot 5202$$

$$= -16.6667 + 209.0667 + 1290.6667 - 788.6667 + 115.6000$$

$$= 810$$

② Use Legrange formula to find y at $x = 10$ given:

x	5	6	9	11
y	12	13	14	16

Let $x_0 = 5$, $x_1 = 6$, $x_2 = 9$, $x_3 = 11$

$y_0 = 12$, $y_1 = 13$, $y_2 = 14$, $y_3 = 16$

Lagrange's interpolation formula.

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \cdot y_0$$

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$$+ \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \cdot y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \cdot y_2$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \cdot y_3$$

$$\Rightarrow y = f(x) = \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)} \cdot 12$$

$$+ \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)} \cdot 13$$

$$+ \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)} \cdot 14$$

$$+ \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)} \cdot 16$$

$$= \overset{2.0000}{\cancel{0.1667}} - 4.3333 + 11.6667 + 5.3333$$

$$= 14.6667$$

(5) Use Legrange's interpolation formula to find interpolating polynomial that fits the following data. Hence find $f(4)$.

x	0	1	2	5
y	2	3	12	147

Let $x_0 = 0$ $x_1 = 1$ $x_2 = 2$ $x_3 = 5$

$y_0 = 2$ $y_1 = 3$ $y_2 = 12$ $y_3 = 147$

$(x-a)(x-b)(x-c) = x^3-(a+b+c)x^2+(ab+bc+ca)x - abc$

formula,

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \cdot y_0$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \cdot y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \cdot y_2$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \cdot y_3$$

$$\Rightarrow y = f(x) = \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)} \cdot 2$$

$$+ \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)} \cdot 3$$

$$+ \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)} \cdot 12$$

$$+ \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)} \cdot 147$$

$1\times -1\times -4$

$5\times 4\times 3$

$2\times 1\times -3$

$$= \frac{x^3-8x^2+17x-10}{-\cancel{10}\,5} \cdot \cancel{2} + \frac{x^3-7x^2+10x}{+4} \cdot 3 +$$

$$\frac{x^3-6x^2+5x}{-\cancel{6}} \cdot \cancel{12}\,2 + \frac{x^3-3x^2+2x}{\cancel{60}\,20} \cdot \cancel{147}\,49$$

-0.2

$$= \left(\frac{-1}{5}+\frac{3}{4}-2+\frac{49}{20}\right)x^3 + \left(\frac{8}{5}-\frac{21}{4}+12-\frac{147}{20}\right)x^2 +$$

$$\left(\frac{-17}{5}+\frac{30}{4} \mp 10+\frac{98}{20}\right)x + (2)$$

$$= x^3+x^2-x+2$$

$$f(4) = 4^3-4^2-4+2 \Rightarrow 78 \,//$$

Numerical integration:

It is a process of obtaining approxiamately the value of definite integral $I = \int_a^b y\,dx$

without actually integrating the function, but by using the values of y at some points of x equally spaced over $[a, b]$

The 3 rules are:

(i) Trapezoidal rule:

$$I = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})\right]$$

(ii) Simpson's $1/3^{rd}$ rule:

$$I = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + \dots) + 2(y_2 + y_4 + \dots)\right]$$

(iii) Simpson's $3/8^{th}$ rule:

$$I = \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5) + \dots) + 2(y_3 + y_6 + \dots)\right]$$

where $h = \frac{b-a}{n}$

Note:- The interval $[a, b]$ is divided into n equal parts so that there will be $n+1$ values of x : $a = x_0, x_1, \dots x_n$ where $x_n = b$. The corresponding values of y are also $n+1$ in number (refered as ordinates). Thus $n+1$ ordinates corresponds to n equal divisions or strips

① Evaluate $\int_0^6 \frac{dx}{1+x^2}$ using Trapezoidal rule
Simpson's $1/3^{rd}$ rule
Simpson's $3/8$ ~~th~~ rule

by taking 7 ordinates.

Given $I = \int_0^6 \frac{1}{1+x^2}\,dx$

Here $a=0$ $b=6$ $y = \frac{1}{1+x^2}$

since $n+1$ ordinates $\Leftrightarrow$ n equal divisions
we have 7 ordinates $\Leftrightarrow$ 6 " "

i.e, $\boxed{n=6}$

$h = \frac{b-a}{n} = \frac{6-0}{6} \Rightarrow \boxed{h=1}$

The points of division for x are,
0, 1, 2, 3, 4, 5, 6

x	0	1	2	3	4	5	6
$y = \frac{1}{1+x^2}$	1	0.5	0.2	0.1	0.0588	0.0385	0.0270
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

(i) Trapezoidal rule.

$I = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+\ldots+y_{n-1})\right]$

$I = \frac{h}{2}\left[(y_0+y_6) + 2(y_1+y_2+y_3+y_4+y_5)\right]$

$I = \frac{1}{2}\left[(1+0.0270) + 2(0.5+0.2+0.1+0.0588+0.0385)\right]$

$\boxed{I = 1.4108}$

(ii) Simpson's $\frac{1}{3}$rd rule

$I = \frac{h}{3}\left[(y_0+y_6) + 4(y_1+y_3+y_5) + 2(y_2+y_4)\right]$

$= \frac{1}{3}\left[(1+0.0270) + 4(0.5+0.1+0.0385) + 2(0.2+0.0588)\right]$

$\boxed{I = 1.3662}$

(iii) Simpson's 3/8 rule:

$$I = \frac{3h}{8}\left[(y_0+y_6) + 3(y_1+y_2+y_4+y_5) + 2(y_3)\right]$$

$$= \frac{3}{8}\left[(1+0.0270) + 3(0.5+0.2+0.0588+0.0385) + 2(0.1)\right]$$

$$\boxed{I = 1.3571}$$

② Using 3 rules find $\int_0^{0.6} e^{-x^2}\,dx$ taking 7 ordinates

Given $I = \int_0^{0.6} e^{-x^2}\,dx$

Here $a=0$, $b=0.6$, $y=e^{-x^2}$, $\boxed{n=6}$

$h = \frac{b-a}{n} = \frac{0.6-0}{6} \Rightarrow \boxed{h=0.1}$

The points of division for x are,
0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6

x	0	0.1	0.2	0.3	0.4	0.5	0.6
$y=e^{-x^2}$	1	0.9900	0.9608	0.9139	0.8521	0.7788	0.6977

(i) Trapezoidal rule:

$$I = \frac{h}{2}\left[(y_0+y_6) + 2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$= \frac{0.1}{2}\left[(1+0.6977) + 2(0.9900+0.9608+0.9139+0.8521+0.7788)\right]$$

$$\boxed{I = 0.5344}$$

(ii) Simpson's 1/3 rule:

$$I = \frac{h}{3}\left[(y_0+y_6) + 4(y_1+y_3+y_5) + 2(y_2+y_4)\right]$$

$$= \frac{0.1}{3}\left[(1+0.6977) + 4(0.9900+0.9139+0.7788) + 2(0.9608+0.8521)\right]$$

$$\boxed{I = 0.5351}$$

(iii) Simpson's 3/8 rule:

$$I = \frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2(y_3)\right]$$

$$= \frac{3\times 0.1}{8}\left[(1+0.6977)+3(0.9900+0.9608+0.8521+0.7788)+2(0.9139)\right]$$

$$\boxed{I = 0.5351}$$

(3) Evaluate $\int_0^1 \frac{x}{1+x^2}\,dx$ using Simpson 1/3 rule taking 6 equal strips. Hence find an approx value of $\log_e\sqrt{2}$

→ given $\boxed{n=6}$

$$h = \frac{b-a}{n} = \frac{1-0}{6} \Rightarrow \boxed{h=\frac{1}{6}}$$

points of division for x are,

$0, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, 1$

x	0	1/6	2/6	3/6	4/6	5/6	1
$y=\frac{x}{1+x^2}$	0	0.1622	0.3	0.4	0.4615	0.4918	0.5

using Simpson's 1/3 rule,

$$I = \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$I = \frac{1}{6\times 3}\left[(0+0.5)+4(0.1622+0.4+0.4918)+2(0.3+0.4615)\right]$$

$$\boxed{I = 0.3466} \quad \text{—— (1)}$$

To find $\log_e 2$:

$$\int_0^1 \frac{x}{1+x^2} = \frac{1}{2}\log_e(1+x^2)\Big]_0^1$$

put $1+x^2=t$
$2x\,dx = dt$
$x\,dx = \frac{dt}{2}$
$\Rightarrow \frac{dt/2}{t} = \frac{1}{2}\log t$
$= \frac{1}{2}\log[1+x^2]$

$$= \frac{1}{2}[\log_e 2 - \log_e 1]$$
$$= \frac{1}{2}[\log_e 2]$$
$$= \log_e 2^{1/2} \Rightarrow \log\sqrt{2} \quad \text{—— (2)}$$

Comparing (1) and (2),

$$\log_e\sqrt{2} = 0.3466$$