EAS 502, Fall 2017, Homework 1

Due: Weds. Sept. 20, 2017, at the beginning of class

Work all problems. Show all work, including any M-files you have written or adapted. Make sure your work is clear and readable - if the TA cannot read what you've written, he will not grade it. All electronic work (m-files, etc.) **must** be submitted through UBLearns (and submitted by the time class starts on the day it's due). Any handwritten work may be submitted in class. Each problem will be graded according to the following scheme: 2 points if the solution is complete and correct, 1 point if the solution is incorrect or incomplete but was using correct ideas, and 0 points if using incorrect ideas.

1. Let the following matrices be defined.

$$A = \begin{bmatrix} 4 & 7 \\ 1 & 2 \\ 5 & 6 \end{bmatrix}, B = \begin{bmatrix} 4 & 3 & 7 \\ 1 & 2 & 7 \\ 2 & 0 & 4 \end{bmatrix}, C = \begin{bmatrix} 3 \\ 6 \\ 1 \end{bmatrix}, D = \begin{bmatrix} 9 & 4 & 3 & -6 \\ 2 & -1 & 7 & 5 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 5 & 8 \\ 7 & 2 & 3 \\ 4 & 0 & 6 \end{bmatrix}, F = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 7 & 3 \end{bmatrix}, G = \begin{bmatrix} 7 & 6 & 4 \end{bmatrix}$$

Answer the following questions regarding these matrices:

(a) What are the dimensions of the matrices?

(b) Identify the square, column, and row matrices.

(c) What are the values of the elements: $a_{12}, b_{23}, d_{32}, e_{22}, f_{12}, g_{12}$.

(d) Perform the following operations or, if not well defined, explain why:

i. $E + B$
ii. $A + F$
iii. $B - E$
iv. $A \times B$
v. $B \times A$
vi. D^T
vii. $\det B$
viii. $\det A$
ix. $E \times B$
x. C^T
xi. $A \times C$
xii. $I \times B$
xiii. $E^T \times E$
xiv. $C^T \times C$

2. Write the following set of equations in matrix form:

$$
\begin{aligned}
&50 = 5x_3 - 7x_2 \\
&4x_2 + 7x_3 + 30 = 0 \\
&x_1 - 7x_3 = 40 - 3x_2 + 5x_1
\end{aligned}
$$

Solve the resulting system of equations using Gaussian Elimination. Check your answer in MATLAB (you do not need to submit checking your answer in MATLAB).

3. Consider the truss shown in Figure 1.

   (a) Assuming the structure is in static equilibrium, determine the equations of equilibrium.

   (b) Rewrite the system of equations in matrix-vector form.

   (c) Solve the system of equations using Gaussian Elimination. You may reorder the equations to facilitate forward elimination.

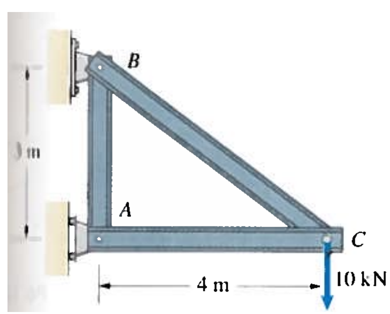

Figure 1: Truss for Problem 3.

4. Consider the truss shown in Figure 2.

   (a) Assuming the structure is in static equilibrium, determine the equations of equilibrium.

   (b) Rewrite the system of equations in matrix-vector form.

   (c) Solve the system of equations using an LU decomposition computed using MATLAB. Write a MATLAB script that computes the LU decomposition and then use it to perform forward and backward substitution to get the solution. You may use the "\" for the forward and backward substitution step.

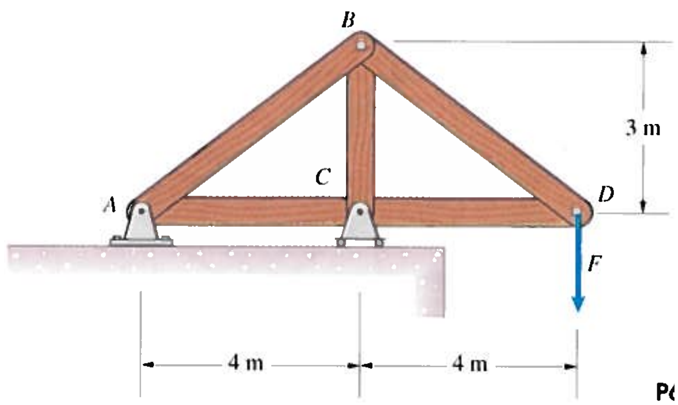

Figure 2: Truss for Problem 4.

5. Three reactors linked by pipes are shown in Figure 3. The rate of mass flow through each pipe is computed as the product of flow (Q) and concentration (c). At steady state, the mass flow into and out of each reactor must be equal. For example, for the second reactor, a *mass balance* can be written as

$$Q_{12}c_1 = Q_{23}c_2 + Q_{21}c_2$$

   (a) Write mass balances for the remaining reactors in Figure 3.
   (b) Express the equations in matrix form.
   (c) Does this linear system have a unique solution? Justify.

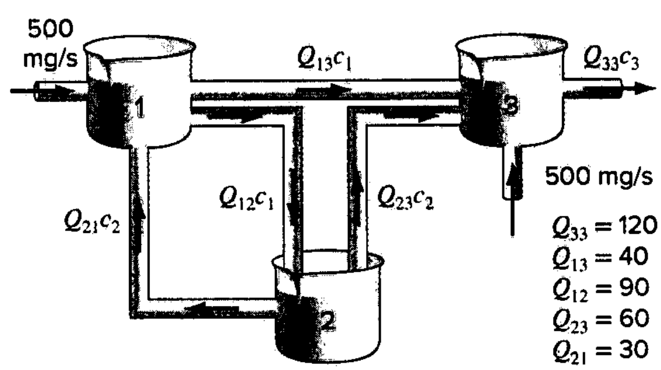

Figure 3: Reactor system for Problem 5.

6. Consider the circuit shown in Figure 4.

   (a) Develop the linear system of equations for the currents in the circuit using Ohm's law and Kirchoff's Laws.

   (b) Write the system in matrix form.

   (c) Solve the linear system, by hand, using the LU decomposition.

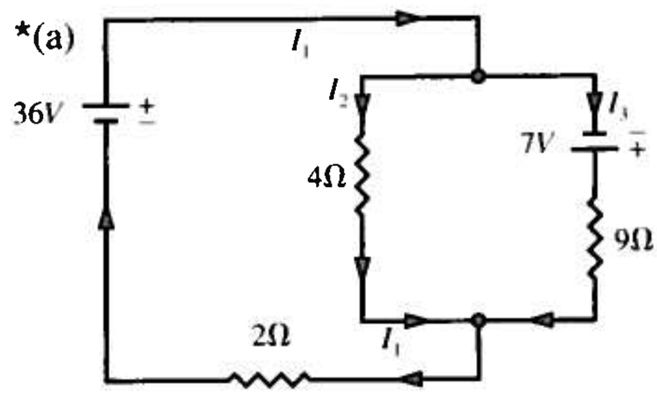

Figure 4: Circuit for Problem 6.

7. Consider the circuit shown in Figure 5.

   (a) Develop the linear system of equations for the currents in the circuit using Ohm's law and Kirchoff's Laws.

   (b) Write the system in matrix form.

   (c) Solve the linear system using the LU decomposition. You may use MATLAB for the LU decomposition and the forward and backward substitution steps.

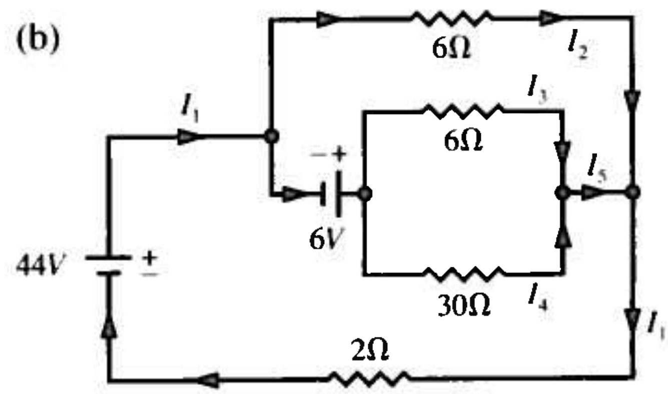

Figure 5: Circuit for Problem 7.

8. Consider the graphs shown in Figure 6.

   (a) For each of the graphs, give the corresponding adjacency matrix.

   (b) Which of these matrices are symmetric?

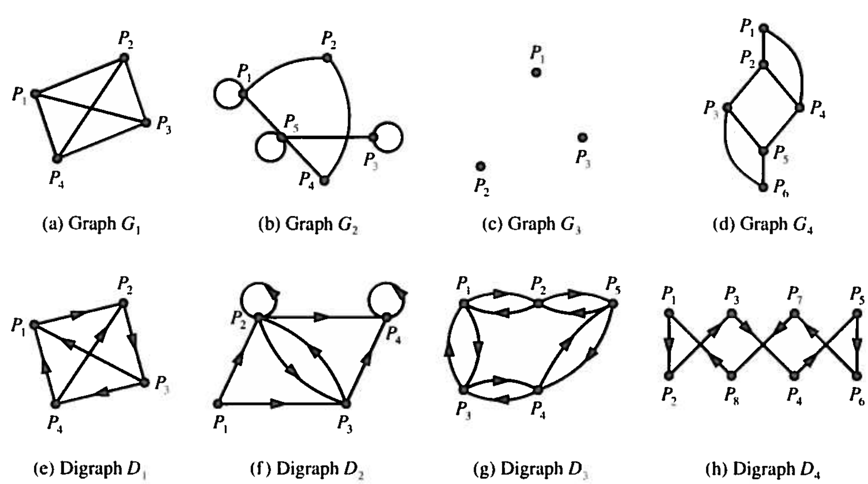

Figure 6: Graphs for Problem 8.

9. Which of the following matrices could be the adjacency matrix for a simple graph or digraph? Draw the corresponding graph and/or digraph when appropriate.

(a) $\mathbf{A} = \begin{bmatrix} -1 & 4 \\ 0 & 1 \\ 6 & 0 \end{bmatrix}$

(b) $\mathbf{B} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

(c) $\mathbf{C} = \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$

(d) $\mathbf{D} = \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$

(e) $\mathbf{E} = \begin{bmatrix} 0 & 0 & 0 & 6 \\ 0 & 0 & 6 & 0 \\ 0 & -6 & 0 & 0 \\ -6 & 0 & 0 & 0 \end{bmatrix}$

(f) $\mathbf{F} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}$

(g) $\mathbf{G} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(h) $\mathbf{H} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

(i) $\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(j) $\mathbf{J} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 1 & 5 & 6 \\ -3 & -5 & 1 & 7 \\ -4 & -6 & -7 & 1 \end{bmatrix}$

(k) $\mathbf{K} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(l) $\mathbf{L} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

(m) $\mathbf{M} = \begin{bmatrix} -2 & 0 & 0 \\ 4 & 0 & 0 \\ -1 & 2 & 3 \end{bmatrix}$

10. Consider the graph shown in Figure 7. Use the adjacency matrix to find the following (you may use MATLAB to assist with computations):

   (a) The number of paths of length 3 from P_2 to P_1.
   (b) The number of paths of length 4 from P_1 to P_5.
   (c) The number of paths of length ≤ 3 from P_3 to P_2.
   (d) The number of paths of length ≤ 4 from P_3 to P_1.
   (e) The length of the shortest path from P_4 to P_5.
   (f) The length of the shortest path from P_4 to P_1.

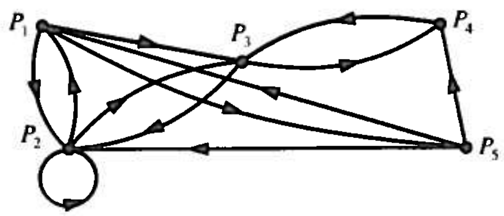

Figure 7: Graphs for Problem 10.

11. Consider the graph shown in Figure 8. Use the adjacency matrix to find the following(you may use MATLAB to assist with computations):

    (a) The number of paths of length 3 from P_2 to P_1.

    (b) The number of paths of length 4 from P_1 to P_5.

    (c) The number of paths of length ≤ 3 from P_3 to P_2.

    (d) The number of paths of length ≤ 4 from P_3 to P_1.

    (e) The length of the shortest path from P_4 to P_5.

    (f) The length of the shortest path from P_4 to P_1.

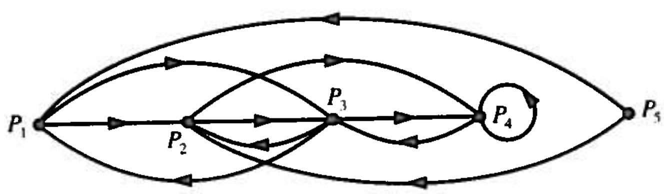

Figure 8: Graphs for Problem 11.

12. Suppose that each of the following represents the transition matrix $\mathbf{M}$ and the initial probability vector $\mathbf{p}$ for a Markov chain. Find the probability vectors $\mathbf{p}_1$ and $\mathbf{p}_2$.

    (a) $\mathbf{M} = \begin{bmatrix} \frac{1}{4} & \frac{1}{3} \\ \frac{3}{4} & \frac{2}{3} \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$

    (b) $\mathbf{M} = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & 0 \\ 0 & \frac{2}{3} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{6} \\ \frac{1}{2} \end{bmatrix}$

    (c) $\mathbf{M} = \begin{bmatrix} \frac{1}{4} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{4} \end{bmatrix}$

13. Suppose that citizens in a certain community tend to switch their votes among political parties, as shown in the following transition matrix. The ordering is Party A, Party B, Party C, and Nonvoting.

$$\begin{bmatrix} 0.7 & 0.2 & 0.2 & 0.1 \\ 0.1 & 0.6 & 0.1 & 0.1 \\ 0.1 & 0.2 & 0.6 & 0.1 \\ 0.1 & 0 & 0.1 & 0.7 \end{bmatrix}$$

(a) Suppose that in the last election 30% of the citizens voted for Party A, 15% voted for Part B, and 45% voted for Party C. What is the likely outcome of the next election? What is the likely outcome of the election after that?

(b) If current trends continue, what percentage of the citizens will vote for Party A one century from now? Party C?

14. Consider the structure in Figure 9. We will study the forces in the bars of this structure for various x. Develop the equations of static equilibrium and write them in matrix form. Take $F = 1$ kN. The matrix $\mathbf{A}$ should be in terms of x.

(a) Compute the force in each of the bars in the structure for values $0 < x \leq 2$ m. You may use the "\" command.

(b) On a single figure, plot the magnitude of the force in each bar, normalized by F, as a function of x. Your figure should be properly labeled with readable fontsizes.

(c) Based on this figure, what value of x is a good choice for this loading?

15. Prove the following theorem: "Let $\mathbf{A}$ be the adjacency matrix for a graph or digraph having vertices $P_1, \ldots, P_n$. Then the total number of paths from P_i to P_j of length k is given by the (i, j) entry in the matrix $\mathbf{A}^k$."

Hint: Use a proof by induction on the length of the path between vertices P_i and P_j. In the inductive step, assume that the given statement is true for paths of length t, and prove that it holds for paths of length $t + 1$. Use the fact that the total number of paths from P_i to P_j of length $t + 1$ is the sum of n products, where each product is the number of paths of length t from P_i to some vertex P_q $(1 \leq q \leq n)$ times the number of paths of length 1 from P_q to P_j. Use the induction

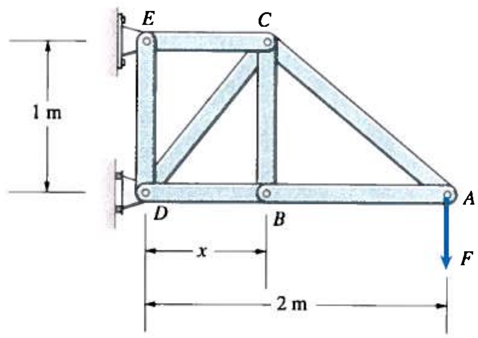

Figure 9: Structure with variable support length.

assumption and the definition of matrix multiplication to show that this sum is the (i, j) entry of $\mathbf{A}^{k+1}$.