EAS 502, Fall 2017, Homework 2

Due: Thurs. Oct. 26, 2017, at the beginning of class

Work all problems. Show all work, including any M-files you have written or adapted. Make sure your work is clear and readable - if the TA cannot read what you've written, he will not grade it. All electronic work (m-files, etc.) **must** be submitted through UBLearns (and submitted by the time class starts on the day it's due). Any handwritten work may be submitted in class. Each problem will be graded according to the following scheme: 2 points if the solution is complete and correct, 1 point if the solution is incorrect or incomplete but was using correct ideas, and 0 points if using incorrect ideas.

1. Run the MATLAB script `problem_1.m` that internally calls the Gaussian elimination with partial pivoting function included with the assignment. At the end, a plot will be produced. Note: the script could take a few minutes to run.

   (a) Examine the script and the resulting plot. What is the graph demonstrating?

   (b) Based on this graph, how long would you estimate a 10000 x 10000 linear system would take to solve (don't bother actually trying, the computer probably doesn't have enough memory)? Show your work - a good engineering estimate will suffice (i.e. order of magnitude).

2. Consider the data in following table:

$$x = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}$$
$$y = \begin{bmatrix} -0.02 & 1.1 & 1.98 & 3.05 & 3.95 & 5.1 & 6.02 \end{bmatrix}$$

   (a) Use a linear regression analysis to fit the data using a linear function $f(x) = a_0 + a_1 x$. Plot the data and the resulting regression line.

   (b) Now use a linear regression analysis to fit the data using a quadratic polynomial $f(x) = a_0 + a_1 x + a_2 x^2$. Plot the data and the resulting regression line.

   (c) Which of the two functions do you think is the more appropriate fit of the data?

3. This problem illustrates how Gaussian elimination can be unstable without pivoting. Take the following matrix:

$$\mathbf{A} = \begin{bmatrix} 10^{-20} & 1 \\ 1 & 1 \end{bmatrix}$$

   (a) Compute the condition number of this matrix in MATLAB using the `cond` command.

   (b) Compute the LU factorization of this matrix, by hand and *without* partial pivoting, using *exact* arithmetic.

   (c) Compute the solution to $\mathbf{Ax} = \mathbf{b}$ with $\mathbf{b} = [1, 0]^T$, again with *exact* arithmetic.

   (d) Now repeat the factorization and solution computation using *finite-precision* arithmetic.

   (e) What will the solution be once partial pivoting is enabled using *finite-precision* arithmetic?

4. This problem explores the trade-off between the truncation error in the finite difference approximation of the derivative and floating point truncation error incurred when using finite precision arithmetic. Using a simple Taylor series, it is easy to construct a *forward difference* approximation of the derivative:

$$f'(x) = \frac{f(x+h) - f(x)}{h} + \mathcal{O}(h)$$

where h is some defined step-size.

   (a) Write a MATLAB script that computes the forward finite difference approximation of the derivative of $\sin(x)$ for $x = \pi/3$. Use step sizes, h, ranging from 10^{-1} to 10^{-16}. You may find the `logspace` command useful. Plot the error, using a log-log scale, between your approximation and the exact derivative for each of your step sizes.

   (b) Based on your plot, what is the best step size to use for the finite difference step size?

   (c) Explain the behavior you see in the graph.

5. Prove that the set $\mathcal{P}$ of all polynomials with real coefficients forms a vector space under the usual operations of polynomial addition and scalar multiplication.

6. Determine whether each of the following subsets of $\mathcal{M}_{nn}$ are subspaces of $\mathcal{M}_{nn}$ under the usual matrix operations. Justify.

   (a) The set of $n \times n$ symmetric matrices

   (b) The set of $n \times n$ singular matrices

   (c) The set of $n \times n$ nonsingular matrices

7. Let $\mathbf{B}$ be a fixed nonsingular matrix in $\mathcal{M}_{nn}$. Show that the mapping $f : \mathcal{M}_{nn} \to \mathcal{M}_{nn}$ given by $f(\mathbf{A}) = \mathbf{B}^{-1}\mathbf{A}\mathbf{B}$ is a linear operator.

8. Suppose that $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ orthogonal matrices.

   (a) Show that $\mathbf{A}^{-1}$ is an orthogonal matrix.

   (b) Prove that $\mathbf{AB}$ is an orthogonal matrix.