# Exercise Set 2

Due February 28, 2020

1. Consider the intersection of randomly drawn lines with a circle. For example, straws could be randomly tossed on a surface on which a circle has been drawn. For those lines that do intersect the circle, characterize that intersection by the polar coordinates $r$ and $\theta$ of the mid-point between the intersection points. We are interested in the joint probability distribution $p(r, \theta|X)$

   (a) Use invariance of our knowledge of this problem to rotation of the coordinate system to show that $p(r, \theta|X) = f(r)$, i.e. is independent of $\theta$

   (b) Use invariance of our knowledge of this problem to translation of the circle to show that
   $$P(r, \theta|X) = f(r) = \frac{1}{2\pi Rr} \tag{1}$$
   where $R$ is the radius of the circle, and $r < R$.

   (c) Show that the distribution in (b) is consistent with the invariance of our knowledge of this problem to the size of the circle.

   (d) The distribution determined in (b) provides a solution to Betrand's paradox (posed by Joseph Bertrand in 1889): What is the probability that a chord of a circle chosen at random is longer than the side of an inscribed equilateral triangle? It was considered a paradox because there were several naive ways to determine this probability, which gave different results. What is the unique value of this probability determined by the invariance arguments?

2. The maximum entropy distribution, relative to a uniform distribution, when the mean $\mu$ and standard deviation $\sigma$ are known, is a Gaussian. Derive the maximum entropy distribution relative to a Jeffrey distribution ($x \in (0, \infty)$, $p(x|X) \sim 1/x$) when $\mu > 0$ and $\sigma$ are specified.

3. Derive the maximum entropy distribution, relative to a uniform distribution on $[0, \infty)$ when the mean $\mu$ is specified.

4. Show that, relative to a uniform distribution on $(-\infty, \infty)^2$, the maximum entropy distribution $p(x, y|X)$ is given by $p(x, y, |X) = \phi(x)\psi(y)$, when it is known that
   $$\int_{-\infty}^{\infty} p(x, y|X)\, dy = \phi(x) \qquad \int_{-\infty}^{\infty} p(x, y|X)\, dx = \psi(y) \tag{2}$$