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In this problem, we use measurements of the time it takes for a small sphere to fall through a viscous silicone oil to infer the viscosity of the fluid. A graduated cylinder of diameter D was filled with the oil, spheres were dropped from the free surface at the center of the cylinder, and the time t it took for the sphere to fall between two of the markings on the cylinder (usually the 700ml mark and the 300ml mark) was measured. The distance h between the marks was measured with a caliper. The measurement interval was determined to be sufficiently far from the top for the terminal velocity to have been reached.

The measurements were taken in Urbana Illinois on three different days. On each day, the specific gravity $S = \rho_f/\rho_w$ (the ratio of the fluid density to that of water) was measured with a hydrometer. Before being dropped into the oil, the diameter d of each sphere was measured, and it was weighed to determine its mass m. Spheres of two different nominal diameters and of two different materials (teflon and steel) were used. Using the relationship $m = V\rho$ (mass = volume $\times$ density), and assuming the objects to be perfectly spherical, we compute ρ for each sphere and compare it to the ballpark estimates of $\rho_{\text{teflon}} \approx 2.2\text{g/cm}^3$ and $\rho_{\text{steel}} \approx 8.05\text{g/cm}^3$ to classify the sphere as teflon or steel.

We have the following data from the above measurements, including an estimate of the uncertainty in the measurements of d, m, t, and S expressed as the standard deviation σ. The uncertainties in D and h are very small and are not reported.

Trial	Date	d [mm]	m [g]	h [mm]	t [s]	S	material
1	10 July 2015	6.33	0.29	67.83	3.45	0.97	teflon
2	10 July 2015	6.33	0.28	138.08	7.00	0.97	teflon
3	10 July 2015	6.32	0.28	67.83	3.52	0.97	teflon
4	10 July 2015	6.32	0.28	138.08	6.87	0.97	teflon
5	10 July 2015	6.31	0.28	138.08	6.97	0.97	teflon
6	10 July 2015	6.30	0.28	138.08	7.07	0.97	teflon
7	13 July 2015	6.32	0.29	138.08	6.94	0.98	teflon
8	13 July 2015	6.35	0.29	138.08	7.12	0.98	teflon
9	13 July 2015	6.32	0.29	138.08	6.99	0.98	teflon
10	13 July 2015	6.31	0.28	138.08	7.28	0.98	teflon
11	13 July 2015	6.32	0.28	138.08	7.28	0.98	teflon
12	13 July 2015	4.74	0.43	138.08	1.96	0.98	steel
13	13 July 2015	4.74	0.44	138.08	1.99	0.98	steel
14	13 July 2015	4.74	0.44	138.08	2.08	0.98	steel
15	14 July 2015	4.75	0.44	138.08	1.97	0.98	steel
16	14 July 2015	4.75	0.43	138.08	1.97	0.98	steel
17	14 July 2015	4.75	0.44	138.08	1.99	0.98	steel
18	14 July 2015	4.75	0.44	138.08	1.97	0.98	steel
19	14 July 2015	4.75	0.44	138.08	2.10	0.98	steel
σ		0.01	0.01		0.01	0.01	

Table 1: Experimental data for the problem. Before computing, we convert each column to SI units (meters, kilograms, seconds). The diameter of the cylinder is assumed to be $D = 68.15$mm, with no uncertainty.

1. To mathematically model this experiment, we appeal to Newton's second law, $F = ma$. Consider an inertial frame of reference with the x-axis parallel to the gravitational force, but pointing in the opposite direction (i.e., x points up and gravity points down). Within this framework, the acceleration a of the sphere is given by $\ddot{x}(t) = \frac{d^2x}{dt^2}(t)$; the velocity of the sphere, given by $\dot{x}(t) = \frac{dx}{dt}(t)$, is negative when the sphere is falling. The force is the sum of gravitational effects (pushing the sphere down), drag induced by the viscous fluid (resisting the motion of the sphere), and the buoyancy of the sphere (pushing the sphere up), i.e., $F = f_g + f_d + f_b$. By another application of Newton's law, the gravitational force is $f_g = -mg$, where $g = 9.81\text{m/s}^2$ is the standard gravitational constant. We use Stokes' law as the embedded model for the drag force f_d,

$$f_d = 6\pi\mu \left(\frac{d}{2}\right) \mathbf{v} = 3\pi\mu d\mathbf{v},$$

where $\mathbf{v}$ is the flow velocity relative to the object. In this case, since we assume the motion is one-dimensional the velocity of the sphere is simply $\mathbf{v} = -\dot{x}(t)$ (in the opposite direction of the motion of the sphere). Finally, for the buoyancy force f_b, we appeal to Archimedes' principle that buoyancy is equal to the weight of the displaced fluid, i.e.,

$$f_b = \rho_f V g = \rho_w S g \frac{4}{3}\pi \left(\frac{d}{2}\right)^3 = \frac{\pi}{6}\rho_w g d^3 S,$$

where $\rho_w = 997.07\text{kg/m}^3$ is the fluid density of water at 25° C. Thus, writing $F = ma$ for the entire system, we obtain the ODE

$$m\ddot{x}(t) = -mg - 3\pi\mu d\dot{x}(t) + \frac{\pi}{6}\rho_w g d^3 S, \qquad x(0) = x_0, \qquad \dot{x}(0) = \dot{x}_0.$$

Since we assume that the terminal velocity has been reached before the measurements are taken, we have $\dot{x} \equiv v$ is constant and $\ddot{x} = 0$ over the time interval of interest. This allows us to directly solve for the velocity of the object:

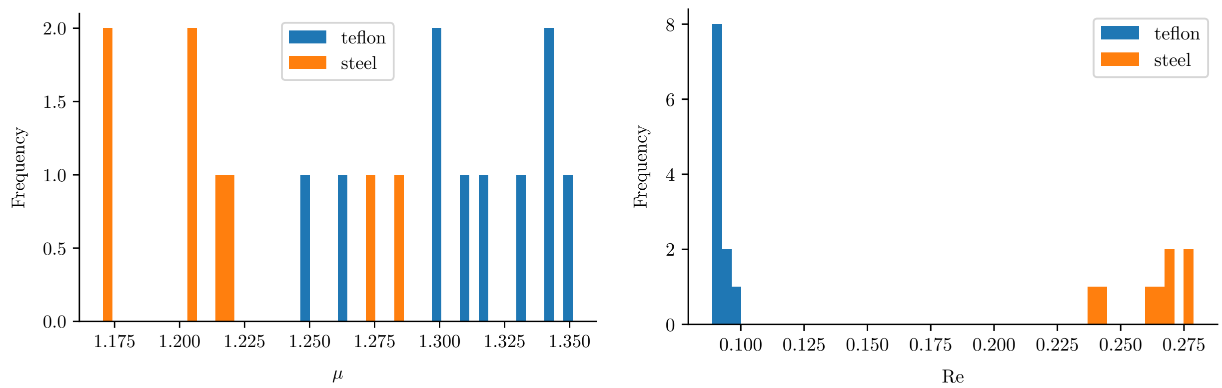
$$\cancelto{0}{m\ddot{x}(t)} = g \cdot \left(\frac{\pi\rho_w g d^3 S}{6} - m\right) - 3\pi\mu d\cancelto{v}{\dot{x}(t)} \quad \Longrightarrow \quad v = \frac{g}{3\pi\mu d}\left(\frac{\pi\rho_w d^3 S}{6} - m\right).$$

Therefore, the displacement of the sphere after t seconds is given by

$$\int_0^t v\, ds = tv = t\frac{g}{3\pi\mu d}\left(\frac{\pi\rho_w d^3 S}{6} - m\right).$$

We have data for the amount of time t it takes for the spheres to be displaced by $-h$ (i.e., the ball drops from an initial height $x(0) = x_0$ to the height $x(t) = x_0 - h$), together with measurements for d, m, and S, reported in Table 1. To compute the viscosity μ,

$$-h = t\frac{g}{3\pi\mu d}\left(\frac{\pi\rho_w d^3 S}{6} - m\right) \quad \Longrightarrow \quad \boxed{\mu = \frac{gt}{3\pi d h}\left(m - \frac{\pi\rho_w d^3 S}{6}\right)}. \tag{1}$$



(a) Distribution of viscosities, calculated by (1).

(b) Distribution of Reynolds numbers.

Figure 1: Distributions of estimated fluid viscosity μ (left) and Reynolds number Re (right), grouped by sphere material. On average, $\mathrm{Re} \approx 0.263$ for steel and $\mathrm{Re} \approx 0.092$ for teflon.

Figure 1a reveals a systemic variation in the measurement of μ: the two kinds of spheres used in the experiment give different results. To see why this might be the case, we compute the *Reynolds number* $\mathrm{Re} = \rho_w v d S/\mu$ for each experiment, shown in Figure 1b. The Stokes law model is only applicable for very small Reynolds numbers ($\mathrm{Re} \ll 1$), so the model appears to be more trustworthy for the teflon spheres than the steel spheres. We thus have an initial guess of $\mu \approx 1.3$, the mean calculated viscosity for the teflon spheres.

2. We seek a probabilistic formulation based on the Stokes law model (1) to determine the viscosity μ of the fluid from the observed drop times t, (i.e., determine $P(\mu|t, X)$). There are a few ways to do this; we start with a simple, direct propagation strategy that is easy to compute with, then explore the Bayesian inference setting.

   Since we have measurements and estimated uncertainties for d, m, t, and S, it is natural to treat each of these variables as Gaussian random variables. Specifically, let $d \sim \mathcal{N}(\bar{d}, \sigma_d^2)$, $m \sim \mathcal{N}(\bar{m}, \sigma_m^2)$, $t \sim \mathcal{N}(\bar{t}, \sigma_t^2)$, and $S \sim \mathcal{N}(\bar{S}, \sigma_S^2)$, where the means $\bar{d}$, $\bar{m}$, $\bar{t}$, and $\bar{S}$ and standard deviations σ_d, σ_m, σ_t, and σ_S are taken from the data (i.e., $\bar{\mathbf{x}} = (\bar{d}, \bar{m}, \bar{h}, \bar{t}, \bar{S})$ are the measurements from a single trial). These random variables are all independent from each other **except** for d and m, which are related via the mass-volume-density equation

$$m = \rho_{\text{sphere}} V = \rho_{\text{sphere}} \frac{4\pi}{3} \left(\frac{d}{2}\right)^3 = \frac{\pi \rho_{\text{sphere}} d^3}{6}, \tag{2}$$

   assuming the objects are perfectly spherical (a reasonable assumption).

   Next, we use a first-order Taylor expansion of (1) about a data point $\bar{\mathbf{x}} = (\bar{d}, \bar{m}, \bar{h}, \bar{t}, \bar{S})$:

$$\mu(d, m, h, t, S) = \mu(\mathbf{x}) \approx \mu(\bar{\mathbf{x}}) + \frac{\partial \mu}{\partial d}(\bar{\mathbf{x}})\delta d + \frac{\partial \mu}{\partial m}(\bar{\mathbf{x}})\delta m + \frac{\partial \mu}{\partial h}(\bar{\mathbf{x}})\delta h^{\nearrow 0} + \frac{\partial \mu}{\partial t}(\bar{\mathbf{x}})\delta t + \frac{\partial \mu}{\partial S}(\bar{\mathbf{x}})\delta S,$$

where

$$\delta d = (d - \bar{d}) \sim \mathcal{N}(0, \sigma_d^2), \qquad \delta m = (m - \bar{m}) \sim \mathcal{N}(0, \sigma_m^2),$$
$$\delta t = (t - \bar{t}) \sim \mathcal{N}(0, \sigma_t^2), \qquad \delta S = (S - \bar{S}) \sim \mathcal{N}(0, \sigma_S^2),$$

and $\delta h \equiv 0$ since there is no uncertainty in h. For convenience, define

$$C_d = \frac{\partial \mu}{\partial d}(\bar{\mathbf{x}}) = -\frac{g\bar{t}}{3\bar{h}}\left(\frac{m}{\pi d^2} + \frac{\rho_w \bar{d}\bar{S}}{3}\right), \qquad C_m = \frac{\partial \mu}{\partial m}(\bar{\mathbf{x}}) = \frac{g\bar{t}}{3\pi\bar{d}\bar{h}},$$
$$C_t = \frac{\partial \mu}{\partial t}(\bar{\mathbf{x}}) = \frac{g}{3\bar{h}}\left(\frac{\bar{m}}{\pi\bar{d}} - \frac{\rho_w \bar{d}^2\bar{S}}{6}\right), \qquad C_S = \frac{\partial \mu}{\partial S}(\bar{\mathbf{x}}) = -\frac{g\rho_w \bar{d}^2\bar{t}}{18\bar{h}},$$

so that the Taylor approximation can be written as

$$\mu(\mathbf{x}) \approx \mu(\bar{\mathbf{x}}) + C_d\delta d + C_m\delta m + C_t\delta t + C_S\delta S. \tag{3}$$

Now recall that expectation is linear: for two random variables X and Y and $a, b \in \mathbb{R}$, we have $\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y]$. Then using the Taylor approximation 3,

$$\begin{aligned}
\mathbb{E}[\mu] &\approx \mathbb{E}\left[\mu(\bar{\mathbf{x}}) + C_d\delta d + C_m\delta m + C_t\delta t + C_S\delta S\right] \\
&= \mathbb{E}[\mu(\bar{\mathbf{x}})] + C_d\cancelto{0}{E[\delta d]} + C_m\cancelto{0}{E[\delta m]} + C_t\cancelto{0}{E[\delta t]} + C_S\cancelto{0}{E[\delta S]} \\
&= \mu(\bar{\mathbf{x}}) = \frac{\bar{t}g}{3\pi\bar{d}\bar{h}}\left(\bar{m} - \frac{\pi\rho_w\bar{d}^3\bar{S}}{6}\right).
\end{aligned} \tag{4}$$

Note that $\delta\mu = \mu(\mathbf{x}) - \mu(\bar{\mathbf{x}})$ has mean 0 since $\mathbb{E}[\mu] = \mu(\bar{\mathbf{x}})$. Furthermore,

$$\delta\mu = \mu(\mathbf{x}) - \mu(\bar{\mathbf{x}}) \approx C_d\delta d + C_m\delta m + C_t\delta t + C_S\delta S,$$

so $\delta\mu$ is (approximately) a linear combination of Gaussian random variables, each with mean 0, which implies that $\delta\mu$ is also Gaussian. To calculate its variance, recall the property $\mathbb{V}\mathrm{ar}[aX + bY] = a^2\mathbb{V}\mathrm{ar}[X] + b^2\mathbb{V}\mathrm{ar}[Y] + 2ab\mathbb{C}\mathrm{ov}[X, Y]$, and that for independent X and Y, $\mathbb{C}\mathrm{ov}[X, Y] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0$. Then

$$\begin{aligned}
\mathbb{V}\mathrm{ar}[\delta\mu] &= \mathbb{V}\mathrm{ar}[C_d\delta d + C_m\delta m + C_t\delta t + C_S\delta S] \\
&= C_d^2\mathbb{V}\mathrm{ar}[\delta d] + C_m^2\mathbb{V}\mathrm{ar}[\delta m] + C_t^2\mathbb{V}\mathrm{ar}[\delta t] + C_S^2\mathbb{V}\mathrm{ar}[\delta S] + 2C_mC_d\mathbb{C}\mathrm{ov}(\delta m, \delta d) \\
&= C_d^2\sigma_d^2 + C_m^2\sigma_m^2 + C_t^2\sigma_t^2 + C_S^2\sigma_S^2 + 2C_dC_m\mathbb{C}\mathrm{ov}(\delta d, \delta m).
\end{aligned}$$

To compute the covariance term $\mathbb{C}\mathrm{ov}(\delta d, \delta m)$, we must use the relation (2). Let $\rho = \rho_{\text{sphere}}$ for notational convenience. As $\mathbb{C}\mathrm{ov}[aX, Y] = a\mathbb{C}\mathrm{ov}[X, Y]$ and $\mathbb{C}\mathrm{ov}[X + a, Y + b] = \mathbb{C}\mathrm{ov}[X, Y]$,

$$\begin{aligned}
\mathbb{C}\mathrm{ov}[\delta d, \delta m] &= \mathbb{C}\mathrm{ov}[d - \bar{d}, m - \bar{m}] \\
&= \mathbb{C}\mathrm{ov}[d, m] \qquad \text{(This is designed to look like “COVID”)} \\
&= \mathbb{C}\mathrm{ov}\left[d, \tfrac{\pi\rho d^3}{6}\right] \\
&= \frac{\pi\rho}{6}\mathbb{C}\mathrm{ov}\left[d, d^3\right] = \frac{\pi\rho}{6}\left(\mathbb{E}[d^4] - \mathbb{E}[d]\mathbb{E}[d^3]\right).
\end{aligned}$$

Recall that we assumed $d \sim \mathcal{N}(\bar{d}, \sigma_d^2)$. Then from Lemma 1 (see Appendix B),

$$\mathbb{E}[d] = \bar{d}, \qquad \mathbb{E}[d^3] = \bar{d}^3 + 3\bar{d}\sigma_d^2, \qquad \mathbb{E}[d^4] = \bar{d}^4 + 6\bar{d}^2\sigma_d^2 + 3\sigma_d^4,$$

and so

$$\begin{aligned}\mathbb{C}\mathrm{ov}[d, m] &= \frac{\pi\rho}{6}\left(\mathbb{E}[d^4] - \mathbb{E}[d]\mathbb{E}[d^3]\right) \\ &= \frac{\pi\rho}{6}\left(3\sigma_d^4 + 6\sigma_d^2\bar{d}^2 + \bar{d}^4 - \bar{d}(3\sigma_d^2\bar{d} + \bar{d}^3)\right) = \frac{\pi\rho\sigma_d^2}{2}\left(\sigma_d^2 + \bar{d}^2\right).\end{aligned}$$

For the material density ρ, we calculate the average densities of the trial spheres for each material, $\rho_{\text{teflon}} \approx 2145\text{kg/m}^3$ and $\rho_{\text{steel}} \approx 7815\text{kg/m}^3$. We will neglect the epistemic uncertainty introduced by these density estimates. Putting it all together, we have

$$\sigma_\mu^2(\bar{\mathbf{x}}) = \mathbb{V}\mathrm{ar}[\delta\mu] = C_d^2\sigma_d^2 + C_m^2\sigma_m^2 + C_t^2\sigma_t^2 + C_S^2\sigma_S^2 + C_d C_m \pi\rho\sigma_d^2\left(\sigma_d^2 + \bar{d}^2\right), \tag{5}$$

and so $\delta\mu \sim \mathcal{N}(0, \sigma_\mu^2)$. Since $\mu(\mathbf{x}) = \delta\mu + \mu(\bar{\mathbf{x}})$, we have $\mu \sim \mathcal{N}(\mu(\bar{\mathbf{x}}), \sigma_\mu^2(\bar{\mathbf{x}}))$.

Finally, all of this analysis only incorporates measurements $\bar{\mathbf{x}}$ from one experiment; if we have a group of (independent) experiments with measurements $\{\bar{\mathbf{x}}^{(i)}\}_{i=1}^N$, we can calculate means and variances $\mu(\bar{\mathbf{x}}^{(i)})$ and $\sigma_\mu^2(\bar{\mathbf{x}}^{(i)})$ for each $i = 1, 2, \dots, N$, and then set

$$\mu = \frac{1}{N}\sum_{i=1}^N \mathcal{N}\big(\mu(\bar{\mathbf{x}}^{(i)}), \sigma_\mu^2(\bar{\mathbf{x}}^{(i)})\big), \tag{6}$$

which is a Gaussian distribution with mean and variance

$$\bar{\mu} = \frac{1}{N}\sum_{i=1}^N \mu(\bar{\mathbf{x}}^{(i)}), \qquad \bar{\sigma}_\mu^2 = \frac{1}{N^2}\sum_{i=1}^N \sigma_\mu^2(\bar{\mathbf{x}}^{(i)}).$$

We will use this approximation for our computations.

Remark. *For a more precise estimate of the mean of $\mu(\mathbf{x})$, we can directly compute the expected value based on* (1). *Since $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$ if X and Y are independent, and as the only variables of (d, m, t, S) that are mutually dependent are d and m,*

$$\begin{aligned}\mathbb{E}[\mu(d, m, h, t, S)] = \mathbb{E}\left[\frac{gt}{3\pi dh}\left(m - \frac{\pi\rho_w d^3 S}{6}\right)\right] &= \frac{g}{3\pi}\mathbb{E}\left[\frac{mt}{dh} - \frac{\pi\rho_w d^2 tS}{6h}\right] \\ &= \frac{g\mathbb{E}[t]}{3\pi\bar{h}}\left(\mathbb{E}\left[\frac{m}{d}\right] - \frac{\pi\rho_w\mathbb{E}[d^2]\mathbb{E}[S]}{6}\right).\end{aligned}$$

We have $\mathbb{E}[d^2] = \bar{d}^2 + \sigma_d^2$ from Lemma 1 (see Appendix B) and, using (2),

$$\mathbb{E}\left[\frac{m}{d}\right] = \mathbb{E}\left[\frac{\pi\rho d^3}{6d}\right] = \frac{\pi\rho}{6}\mathbb{E}\left[d^2\right] = \frac{\pi\rho}{6}\left(\bar{d}^2 + \sigma_d^2\right).$$

Therefore,

$$\bar{\mu} = \mathbb{E}[\mu] = \frac{g\bar{t}}{3\pi\bar{h}}\left(\frac{\pi\rho_{sphere}}{6}(\bar{d}^2+\sigma_d^2) - \frac{\pi\rho_w(\bar{d}^2+\sigma_d^2)\bar{S}}{6}\right) = \frac{g\bar{t}(\bar{d}^2+\sigma_d^2)}{18\bar{h}}\left(\rho_{sphere} - \rho_w\bar{S}\right). \quad (7)$$

We can attempt the same strategy with the variance, but we have no reason to expect that the resulting variable μ should be Gaussian, so we'll stick with the Taylor approximation strategy.

Before continuing, we explore the Bayesian setting for this problem. First, Bayes' rule gives

$$P(\mu|t,X) = \frac{P(t|\mu,X)P(\mu|X)}{P(t|X)} \propto P(t|\mu,X)P(\mu|X). \quad (8)$$

The prior $P(\mu|X)$ can be chosen by selecting reasonable bounds for μ for a silicone oil, such as $0.5 \le \mu \le 2$, and setting $P(\mu|X) = \mathcal{U}(0.5, 2)$. The likelihood $P(t|\mu,X)$ is tougher to deal with: using the law of total probability,

$$P(t|\mu,X) = \int_d\int_m\int_S P(t|\mu,d,m,S,X)P(d,m,S|X)\,dd\,dm\,dS.$$

We can define $P(t|\mu,d,m,h,S,X)$ as a Gaussian by using (1) to solve for t in terms of the other variables,

$$t = \frac{3\pi dh\mu}{g\left(m - \frac{\pi\rho_w d^3 S}{6}\right)}, \quad (9)$$

and using the reported uncertainty of $\sigma_t = 0.01$s for t. That is, we set

$$P(t|\mu,d,m,S,X) = \mathcal{N}\left(\frac{3\pi dh\mu}{g\left(m - \frac{\pi\rho_w d^3 S}{6}\right)}, 0.01^2\right) \quad (10)$$

Next, since the specific gravity S is a property of the fluid, it is clearly independent from the characteristics d and m of the sphere, so we have

$$P(d,m,S|X) = P(d,m|X)P(S|X). \quad (11)$$

For the joint distribution $P(d,m|X)$, we previously computed

$$\mathbb{C}\text{ov}[d,m] = \frac{\pi\rho\sigma_d^2}{2}\left(\bar{d}^2+\sigma_d^2\right),$$

so the covariance matrix for d and m is

$$\Sigma_{d,m}(\rho) = \begin{bmatrix} \sigma_d^2 & \frac{\pi\rho\sigma_d^2}{2}\left(\bar{d}^2+\sigma_d^2\right) \\ \frac{\pi\rho\sigma_d^2}{2}\left(\bar{d}^2+\sigma_d^2\right) & \sigma_m^2 \end{bmatrix},$$

and we therefore set $P(d,m|X) = \mathcal{N}\left(\begin{bmatrix}\bar{d} & \bar{m}\end{bmatrix}^\mathsf{T}, \Sigma_{d,m}\right)$ (a function of d and m).

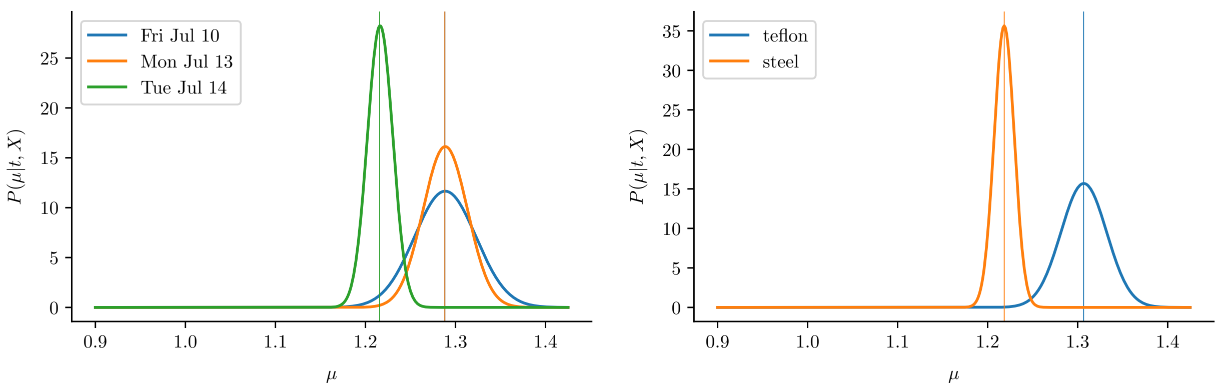
The specific gravity S is measured only once per day, so the background information X tells us which S value to expect. We therefore model $P(S|X)$ as a Gaussian with mean $\bar{S}$ from the data and (given) standard deviation $\sigma_S = 0.01$, i.e., $P(S|X) = \mathcal{N}(\bar{S}, 0.01^2)$ (a function of S).

We can now restate the Bayes formulation (8) as

$$P(\mu|t,X) \propto \iiint_{d,m,S} \mathcal{N}\left(\frac{3\pi d h \mu}{g\left(m - \frac{\pi \rho_w d^3 S}{6}\right)}, 0.01^2\right) \mathcal{N}\left(\begin{bmatrix}\bar{d} & \bar{m}\end{bmatrix}^\mathsf{T}, \Sigma_{d,m}\right) \mathcal{N}(\bar{S}, 0.01^2)\, dd\, dm\, dS \times \mathcal{U}(0.5, 2).$$

The trouble with this setup is the obviously the integral. We may be able to do some analytical work with it since the integrand is a product of Gaussian distributions, but the Taylor series framework is much more convenient.

3. Using the model described in (4)–(6), we determine posterior distributions for the viscosity on each day (Figure 2a) and by material (Figure 2b).



(a) Posterior distributions for μ by day. (b) Posterior distributions for μ by material.

Figure 2: "Posterior" distributions for μ using data grouped by date (left) and by material (right), calculated with the Stokes model (1) and the Taylor series-based model (4)–(6). The vertical points indicate the MAP estimate (which in this case is also the mean of each distribution).

While the two materials clearly give different approximations for μ (not unlike the raw predictions from Figure 1a), it is interesting to note how the distributions from trials grouped by sphere material correlate with the distributions grouped by experiment date: on Friday, when only teflon spheres were used, the distribution looks like the one derived from all telfon experiments; likewise, the distribution from Tuesday's trials is similar (but not identical) to the distribution derived from all steel experiments. However, Monday's distribution—the day when both teflon and steel spheres were used—seems to match the distribution derived from teflon. We report the precise statistics of each distribution in Table 2.

Grouping	$\bar{\mu}$	$\bar{\sigma}_\mu^2$
Fri Jul 10	1.2886	0.00117
Mon Jul 13	1.2887	0.00061
Tue Jul 14	1.2162	0.00020
Teflon	1.3068	0.00065
Steel	1.2184	0.00013

Table 2: Statistics of the estimated distribution for μ, derived from portions of the data.

From our current model, it appears that there are variations in the fluid viscosity across different days. There are a few possible explanations: 1) the liquid is left out over time and the properties of the oil is changing because of the exposure to moisture in the air, or 2) the teflon spheres, which have a much larger diameter than the steel spheres, may suffer from boundary effects of the cylinder that we have not accounted for. Either of these could lead to variations in the predicted viscosity.

4. Stokes' law is valid in the case of a sphere falling in an infinite fluid medium. While the ratio of the sphere diameter d to the cylinder diameter D is small, there could still be an effect of the cylinder walls. To test whether this is important, we group the data by material (telfon or steel), infer the viscosity from the teflon balls using the procedure described in Problem 2, and use that distribution of μ to predict the fall time of the steel spheres with (9). That is, for each data point, we calculate

$$t = \frac{3\pi d h}{g\left(m - \frac{\pi \rho_w d^3 S}{6}\right)} \mu_{\text{teflon}} \qquad \text{where} \qquad \mu_{\text{teflon}} = \frac{1}{N_{\text{teflon}}} \sum_{i \in \mathcal{I}_{\text{teflon}}} \mathcal{N}\left(\mu(\bar{\mathbf{x}}^{(i)}), \sigma_\mu^2(\bar{\mathbf{x}}^{(i)})\right). \tag{12}$$

The result is a Gaussian distribution with statistics

$$\mathbb{E}[t] = \mathbb{E}\left[\frac{3\pi d h}{g\left(m - \frac{\pi \rho_w d^3 S}{6}\right)} \mu_{\text{teflon}}\right] = \frac{3\pi d h}{g\left(m - \frac{\pi \rho_w d^3 S}{6}\right)} \mathbb{E}\left[\mu_{\text{teflon}}\right],$$

$$\mathbb{V}\text{ar}[t] = \mathbb{V}\text{ar}\left[\frac{3\pi d h}{g\left(m - \frac{\pi \rho_w d^3 S}{6}\right)} \mu_{\text{teflon}}\right] = \left(\frac{3\pi d h}{g\left(m - \frac{\pi \rho_w d^3 S}{6}\right)}\right)^2 \mathbb{V}\text{ar}\left[\mu_{\text{teflon}}\right].$$

We likewise use the steel data to get μ_{steel} and predict the fall times for the teflon spheres. The results are in Table 3. Here, we are using the observed values from the data for d, m, h, and S, but we could treat each as random variables as in Problem 2. This would give us a slightly refined value for $\mathbb{V}\text{ar}[t]$, but it requires a Taylor series expansion and isn't quite worth the effort for this consistency check.

Trial	Material	$\bar{t}$	$\mathbb{E}[t]$	$\mathbb{V}\mathrm{ar}[t]$	σ-distance
1	teflon	3.45	3.11	0.000817	11.86
2	teflon	7.00	6.75	0.003847	4.02
3	teflon	3.52	3.30	0.000918	7.34
4	teflon	6.87	6.71	0.003804	2.54
5	teflon	6.97	6.68	0.003762	4.80
6	teflon	7.07	6.64	0.003721	7.07
7	teflon	6.94	6.35	0.003404	10.10
8	teflon	7.12	6.45	0.003517	11.21
9	teflon	6.99	6.35	0.003404	10.96
10	teflon	7.28	6.73	0.003827	8.83
11	teflon	7.28	6.77	0.003871	8.17
12	steel	1.96	2.19	0.001814	5.36
13	steel	1.99	2.13	0.001721	3.41
14	steel	2.08	2.13	0.001721	1.24
15	steel	1.97	2.14	0.001732	4.04
16	steel	1.97	2.19	0.001825	5.26
17	steel	1.99	2.14	0.001732	3.55
18	steel	1.97	2.14	0.001732	4.04
19	steel	2.10	2.14	0.001732	0.91

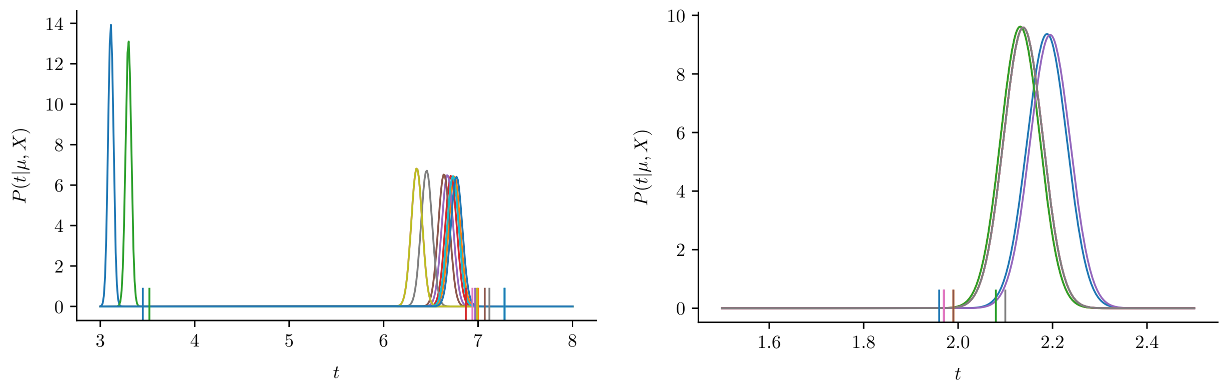
Table 3: Values of t from the data—denoted $\bar{t}$—and the mean and variance of the distribution for t (12). For teflon balls, we use μ_{steel} to compute t, and for steel balls, we use μ_{teflon}. The final column indicates the number of standard deviations that $\bar{t}$ is from $\mathbb{E}[t]$.

The predicted times appear to be consistently lower than the observed times for teflon and higher than the observed times for steel. This suggests that the estimated viscosity from the teflon spheres may be an overestimate, i.e., that the effect of the cylinder walls is nontrivial. Indeed, some of the predictions for t are significantly different from the original measurements (11 standard deviations off in one case!). Since the predicted values are so far off from the experimental data (in terms of the number of standard deviations), it is safe to say that predictions are not consistent with the observed fall times.

5. Brenner and Brenner & Happel developed asymptotic analysis to account for the effects of the walls for a small sphere moving in a highly viscous fluid in a cylinder (as well as other situations). They claim that for a sphere moving axially in a cylinder,

$$f_d = \frac{3\pi\mu d\mathbf{v}}{C(\alpha, \beta)}, \tag{13}$$

where f_d is the drag force on the sphere, $\mathbf{v}$ is its axial velocity, $\alpha = d/D$ is the ratio of sphere to cylinder diameters, and $\beta = 2b/D$ is the ratio of the offset b of the center of the



(a) Fall times for the teflon spheres, calculated from the steel sphere trials.

(b) Fall times for the steel spheres, calculated from the teflon sphere trials.

Figure 3: Distributions for calculated fall times t for each data point, based on a distribution of viscosity μ. The vertical lines on the x-axis indicate the data points.

sphere from the cylinder axis to the radius of the cylinder, and the denominator in (13) is a correction factor to the Stokes law drag given by

$$C(\alpha, \beta) = 1 - [2.1044 - 0.6977\beta^2 + O(\beta^4)]\alpha + O(\alpha^3).$$

We will incorporate (13) into our model and reproduce our experiments. Starting again with $F = ma$,

$$m\ddot{x}(t) = -mg + f_d + \frac{\pi}{6}\rho_w g d^3 S.$$

Using the new embedded model (13) for f_d and the assumption of terminal velocity being reached (so $\mathbf{v} = -\dot{x}(t) \equiv -v$ and $\ddot{x}(t) \equiv 0$),

$$\cancelto{0}{m\ddot{x}(t)} = g \cdot \left(\frac{\pi\rho_w g d^3 S}{6} - m\right) - \frac{3\pi\mu d \cancelto{v}{\dot{x}(t)}}{C(\alpha, \beta)} \quad \Longrightarrow \quad v = \frac{gC(\alpha, \beta)}{3\pi\mu d}\left(\frac{\pi\rho_w d^3 S}{6} - m\right).$$

Therefore, the displacement of the sphere after t seconds is given by

$$\int_0^t v\, ds = tv = \frac{gC(\alpha, \beta)}{3\pi\mu d}\left(\frac{\pi\rho_w d^3 S}{6} - m\right),$$

so when the displacement is $-h$, we have

$$-h = \frac{gC(\alpha, \beta)}{3\pi\mu d}\left(\frac{\pi\rho_w d^3 S}{6} - m\right) \quad \Longrightarrow \quad \boxed{\mu = \frac{gtC(\alpha, \beta)}{3\pi d h}\left(m - \frac{\pi\rho_w d^3 S}{6}\right)}. \tag{14}$$

Likewise, the fall time predicted from μ is given by

$$t = \frac{3\pi d h}{gC(\alpha, \beta)\left(m - \frac{\pi\rho_w d^3 S}{6}\right)}\mu. \tag{15}$$

For now, we assume $b = 0$ (i.e., the spheres are dropped in the exact center of the cylinder), hence $\beta = 0$. Using the refined estimates (14) for μ and (15) for t, we revisit the consistency check from Problem 4. The only adjustments to be made are that now,

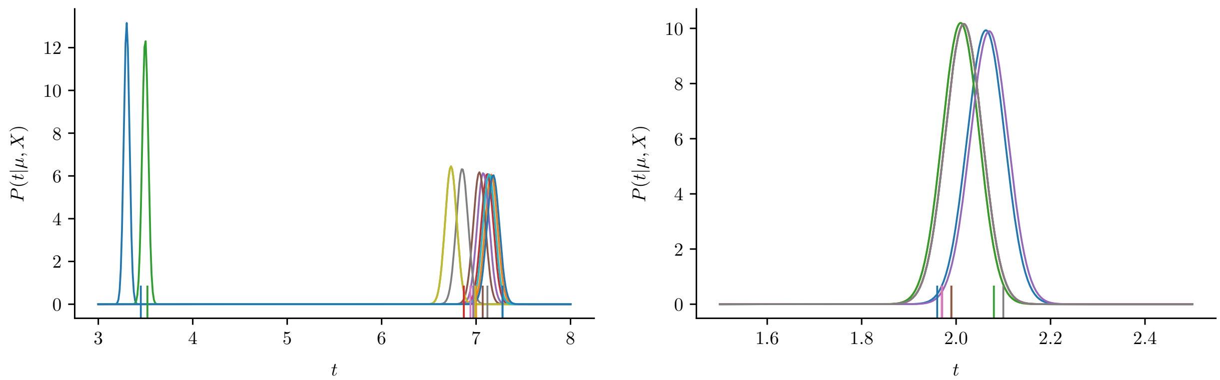
$$\mathbb{E}[t] = \frac{3\pi dh}{gC(\alpha,\beta)\left(m - \frac{\pi\rho_w d^3 S}{6}\right)}\mathbb{E}\left[\mu_{\text{teflon}}\right], \quad \mathbb{V}\text{ar}[t] = \left(\frac{3\pi dh}{gC(\alpha,\beta)\left(m - \frac{\pi\rho_w d^3 S}{6}\right)}\right)^2 \mathbb{V}\text{ar}\left[\mu_{\text{teflon}}\right],$$

with the correction that the mean for an individual trial must be multiplied by $C(\alpha, \beta)$, and the corresponding variance must be multiplied by $C(\alpha, \beta)^2$. We again ignore uncertainties in m, d, and S in this calculation (consistent with our procedure in Problem 4), though we are confident that this process yields at least a very good ballpark estimate. See Table 4 and Figure 4 for the results.

Trial	Material	$\bar{t}$	$\mathbb{E}[t]$	$\mathbb{V}\text{ar}[t]$	σ-distance	old σ-distance
1	teflon	3.45	3.30	0.000919	4.95	11.86
2	teflon	7.00	7.16	0.004329	2.45	4.02
3	teflon	3.52	3.50	0.001032	0.72	7.34
4	teflon	6.87	7.12	0.004277	3.80	2.54
5	teflon	6.97	7.08	0.004227	1.63	4.80
6	teflon	7.07	7.03	0.004177	0.55	7.07
7	teflon	6.94	6.73	0.003828	3.33	10.10
8	teflon	7.12	6.85	0.003964	4.25	11.21
9	teflon	6.99	6.73	0.003828	4.14	10.96
10	teflon	7.28	7.14	0.004300	2.17	8.83
11	teflon	7.28	7.18	0.004352	1.51	8.17
12	steel	1.96	2.06	0.001613	2.57	5.36
13	steel	1.99	2.01	0.001530	0.50	3.41
14	steel	2.08	2.01	0.001530	1.80	1.24
15	steel	1.97	2.02	0.001541	1.18	4.04
16	steel	1.97	2.07	0.001624	2.48	5.26
17	steel	1.99	2.02	0.001541	0.67	3.55
18	steel	1.97	2.02	0.001541	1.18	4.04
19	steel	2.10	2.02	0.001541	2.13	0.91

Table 4: Values of t from the data versus the value predicted by (15). For teflon balls, we use μ_{steel} to predict t, and for steel balls, we use μ_{teflon} in the prediction. The non-corrected σ distances—taken from Table 3—are larger in most cases.

In almost every case, the new predicted t distributions match the data better than the predictions without the correction in (13). In all cases, the data lies within 4 standard deviations of $\mathbb{E}[t]$, which gives us more confidence that our model is self-consistent with the



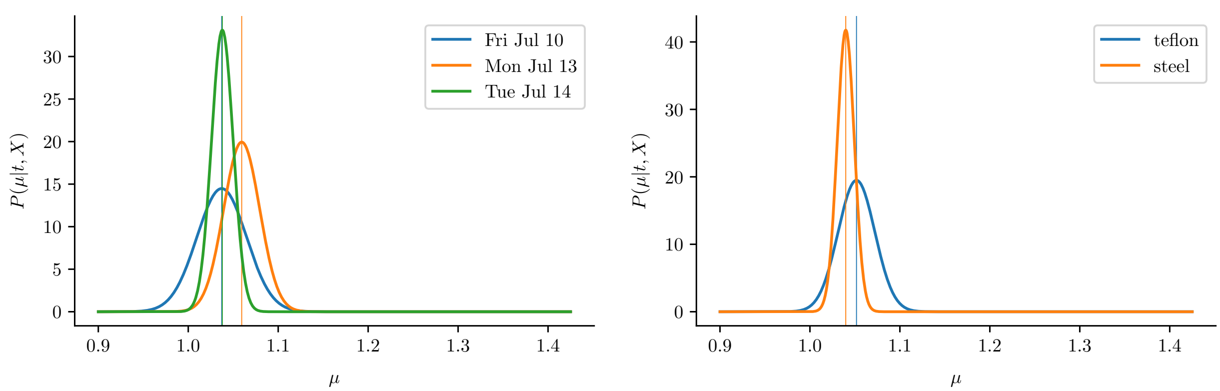
(a) Fall times for the teflon spheres, calculated from the steel sphere trials.

(b) Fall times for the steel spheres, calculated from the teflon sphere trials.

Figure 4: Distributions for calculated fall times t for each data point, based on a distribution of viscosity μ. The vertical lines on the x-axis indicate the data points. Compare to Figure 3.

data, and that the discrepancies between $\bar{t}$ and $\mathbb{E}[t]$ are due to uncertainties in the data (and not the model).

6. With the correction introduced in (13), we repeat our inference of the viscosity using the model from (4)–(6). See Figures 5 and Table 5.



(a) Posterior distributions for μ by day.

(b) Posterior distributions for μ by material.

Figure 5: "Posterior" distributions for μ using data grouped by date (left) and by material (right), calculated with the Stokes model with corrected drag force (13). Compare to Figure 2.

The drag force correction has the general effect of shifting the distributions to the left, resulting in much closer estimates for μ among the trials than before. The variances have also slightly decreased, indicating a decrease in the uncertainty of the estimate. This gives

Grouping	$\bar{\mu}$	$\bar{\sigma}_{\mu}^2$
Fri Jul 10	1.0372	0.00076
Mon Jul 13	1.0594	0.00040
Tue Jul 14	1.0379	0.00015
Teflon	1.0517	0.00042
Steel	1.0398	0.00009

Table 5: Statistics of the estimated distribution for μ, derived from portions of the data and using the drag correction (13). Compare to Table 2.

us confidence that the viscosity of the fluid is constant (or nearly constant) across the trials, and that the true value resides somewhere in the vicinity of $\mu \approx 1.04$.

Finally, we asses the importance of dropping the sphere exactly on the axis of the cylinder—in other words, what happens if $b \neq 0$ in (13)? Consider again the corrected Stokes estimate for the viscosity (14),

$$\begin{aligned}
\mu &= \frac{gtC(\alpha,\beta)}{3\pi dh}\left(m - \frac{\pi\rho_w d^3 S}{6}\right) \\
&\approx \frac{gt\left(1 - [2.1044 - 0.6977\beta^2]\alpha\right)}{3\pi dh}\left(m - \frac{\pi\rho_w d^3 S}{6}\right) \\
&= \frac{gt\left(1 - [2.1044\frac{d}{D} - 0.6977(\frac{4b^2 d}{D^3})]\right)}{3\pi dh}\left(m - \frac{\pi\rho_w d^3 S}{6}\right).
\end{aligned}$$

Then

$$\frac{\partial\mu}{\partial b} = \frac{gt\left(0.6977(\frac{8bd}{D^3})\right)}{3\pi dh}\left(m - \frac{\pi\rho_w d^3 S}{6}\right) = \frac{5.5816gtb}{3\pi D^3 h}\left(m - \frac{\pi\rho_w d^3 S}{6}\right).$$

To account for uncertainty in b, let $b \sim \mathcal{U}(\frac{d-D}{2}, \frac{D-d}{2})$ (uniform over the valid range of values), which has mean $\bar{b} = 0$. The linearization step (3) gains an extra term $\frac{\partial\mu}{\partial b}(\bar{\mathbf{x}})\delta b$, but since the linearization is around the point $\bar{b} = 0$,

$$\frac{\partial\mu}{\partial b}(\bar{\mathbf{x}}) = \frac{5.5816\bar{g}\bar{t}\overset{0}{\bar{b}}}{3\pi D^3 \bar{h}}\left(\bar{m} - \frac{\pi\rho_w \bar{d}^3 \bar{S}}{6}\right) = 0.$$

Therefore, the only place that b affects our drag-corrected model is in the value of $C(\alpha,\beta)$. Recall that if $\bar{\mu}_{\text{old}}$ is the mean of the non-corrected model (6), the corrected model has mean $\bar{\mu} = C(\alpha,\beta)\bar{\mu}_{\text{old}}$. Our current estimates use $C(\alpha,0) = 1 - 2.1044\alpha$, but

$$0 < C(\alpha,\beta) - C(\alpha,0) = 0.6977\beta^2\alpha = 0.6977\left(4\frac{b^2 d}{D^3}\right) < 0.6977\frac{(D-d)^2 d}{D^3},$$

since $|b| < \frac{D-d}{2}$ (equality would mean the sphere is rubbing against the cylinder wall). Evaluating this last constant for each data point $\bar{d}$, we have on average $C(\alpha,\beta) - C(\alpha,0) <$

0.0533 for teflon and $C(\alpha, \beta) - C(\alpha, 0) < 0.0421$ for steel. This means that dropping the sphere somewhere other than the axis of the cylinder can affect the error no more that 5.4%; in the worst case, the average estimates for $\bar{\mu}$ would rise to about $\bar{\mu}_{\text{teflon}} \approx 1.1077$ and $\bar{\mu}_{\text{steel}} \approx 1.0835$. If we want this error to be unimportant, say 1% or less, then we need

$$0.01 > C(\alpha, \beta) - C(\alpha, 0) = 0.6977\beta^2\alpha \qquad \Longrightarrow \qquad |\beta| < \sqrt{\frac{0.01d}{0.6977D}} \leq 0.3922,$$

where this last number is the sharpest bound afforded from the data. That is, in order to get less than 1% error due to drop placement, the sphere must be dropped in approximately the central 40% of the cylinder, which is not very hard for a human to do.

Appendix A: Code

```
import numpy as np
import pandas as pd
from scipy import stats
from matplotlib import pyplot as plt


pi = np.pi
D = .06815                          # Diameter of cylinder [m]
G = 9.81                            # Gravitational constant [m/s^2]
RHO_S = 7870                        # Guess for density of steel [kg/m^3]
RHO_T = 2200                        # Guess for density of teflon [kg/m^3]
RHO_W = 997.07                      # Fluid density of water at 25C [kg/m^3]

# Problem 1 --------------------------------------------------------------------
def read_and_process_data(filename="SphereDrop.xlsx"):
    """Extract the excel spreadsheet data using pandas."""
    # Read the data and clear out extra rows/columns.
    df = pd.read_excel(filename, header=2, index_col=0)
    df = df.dropna(how="all")

    # Convert all units to SI.
    df["d [m]"] = df["d [mm]"] * 0.001          # mm -> m
    df["m [kg]"] = df["m [g]"] * 0.001          # g -> kg
    df = df.drop(["d [mm]", "m [g]"], axis=1)

    # Separate trial data and uncertainty measures.
    data = df.drop("Sigma")
    sigma = df.loc["Sigma"].drop(["Date"])
    d, m, t, h, S = data[["d [m]", "m [kg]", "t [s]", "h [m]", "S"]].values.T

    # Calculate the volume and density of each sphere.
    data["V [m^3]"] = pi * d**3 / 6
    data["rho [kg/m^3]"] = m / data["V [m^3]"]

    # Classify the material of the ball for each trial.
    global RHO_S, RHO_T
    data["material"] = ["teflon" if abs(r - RHO_T) < abs(r - RHO_S)
                                 else "steel" for r in data["rho [kg/m^3]"]]

    # Overwrite estimates for solid densities.
    RHO_S, RHO_T = data.groupby("material")["rho [kg/m^3]"].mean()
    data["rhobar [kg/m^3]"] = [RHO_T if m == "teflon" else RHO_S
                                               for m in data["material"]]

    # Calculate viscosities from Stokes law (also mubar from Taylor series).
    data["mu (stokes)"] = G * t * (m - (pi * RHO_W * d**3 * S)/6) / (3*pi*d*h)
    mu = data["mu (stokes)"]

    # Calculate Reynolds numbers from Stokes law.
    data["velocity"] = G * ((pi * RHO_W * d**3 * S)/6 - m) / (3*pi*mu*d)
    data["Reynolds"] = RHO_W * data["velocity"].abs() * d * S / mu
```

```python
    # Calculate estimated mu variance.
    Cd = -G * t * ((m / (pi * d**2)) + (RHO_W * d * S) / 3) / (3 * h)
    Cm = G * t / (3 * pi * d * h)
    Ct = G * ((m / (pi * d)) - (RHO_W * d**2 * S) / 6) / (3 * h)
    Cs = - G * RHO_W * d**2 * t / (18 * h)
    sd, sm, st, ss = sigma[["d [m]", "m [kg]", "t [s]", "S"]]
    s2 = Cd**2*sd**2 + Cm**2*sm**2 + Ct**2*st**2 + Cs**2*ss**2 \
       + Cd*Cm*pi*data["rhobar [kg/m^3]"]*sd**2*(sd**2 + d**2)  # Cov(d,m) term
    data["variance (stokes)"] = s2

    # Calculate the Stokes drag correction factor.
    b = 0
    data["drag correction"] = 1 - (2.1044 - 0.6977*(2*b/D)**2)*(d / D)

    return data

def viscosity_naive(data, verbose=False):
    """Plot viscosities from the Stokes model (Figure 1a)."""
    mu = data["mu (stokes)"]
    murange = [mu.min(), mu.max()]
    gb = mu.groupby(data["material"])
    for lbl in ["teflon", "steel"]:
        gb.get_group(lbl).plot(kind="hist", bins=50, range=murange, label=lbl)
    plt.xlabel(r"$\mu$")
    plt.legend()

def reynolds_initial(data, verbose=False):
    """Plot Reynolds numbers from the Stokes model (Figure 1b)."""
    re = data["Reynolds"]
    rerange = [re.min(), re.max()]
    gb = re.groupby(data["material"])
    for lbl in ["teflon", "steel"]:
        gb.get_group(lbl).plot(kind="hist", bins=50, range=rerange, label=lbl)
    plt.xlabel(r"Re")
    plt.legend()

    return re

# Problem 3 -----------------------------------------------------------------
def posterior(mu, df, correct=False):
    """Calculate the posterior for mu over the given domain."""
    if correct:
        mubar = np.mean(df["mu (stokes)"] * df["drag correction"])
        sbar = np.sqrt(np.sum(df["variance (stokes)"]\
                          *df["drag correction"]**2)) / df.shape[0]
    else:
        mubar = np.mean(df["mu (stokes)"])
        sbar = np.sqrt(np.sum(df["variance (stokes)"])) / df.shape[0]

    post = stats.norm(loc=mubar, scale=sbar).pdf(mu)
    assert post.shape == mu.shape
    return post, mubar, sbar

def posterior_by_group(data, how="Date", correct=False):
    """Plot the posteriors on by some grouping (Figure 2)."""
    mu = np.linspace(.9, 1.425, 800)
```

```
    gb = data.groupby(how)
    keys = list(gb.groups.keys()) if how != "material" else ["teflon", "steel"]

    for i,cat in enumerate(keys):
        group = gb.get_group(cat)
        post, mubar, sbar = posterior(mu, group, correct)

        lbl = cat.ctime()[:10] if not isinstance(cat, str) else cat
        plt.plot(mu, post, label=lbl)
        plt.axvline(mubar, color=f"C{i}", lw=.5)

    plt.xlabel(r"$\mu$")
    plt.ylabel(r"$P(\mu|t,X)$")
    plt.legend(loc="upper right" if correct else "upper left")

# Problem 4 ---------------------------------------------------------------
def predict_fall_times(data, ref="teflon", target="steel", correct=False):
    """Get mu from telfon data and predict steel fall times (or vice versa)."""
    # Get mean and variance from data.
    gb = data.groupby("material")
    group1 = gb.get_group(ref)
    if correct:
        mu = np.mean(group1["mu (stokes)"] * group1["drag correction"])
        var = np.mean(group1["variance (stokes)"] \
                      * group1["drag correction"]**2) / len(group1)
    else:
        mu = group1["mu (stokes)"].mean()
        var = group1["variance (stokes)"].mean() / len(group1)

    # Unpack target data.
    group2 = gb.get_group(target)
    d, m, h, S = group2[["d [m]", "m [kg]", "h [m]", "S"]].values.T

    # Predict target fall times and compute the relative error.
    df = group2[["t [s]"]].copy()
    D = 3 * pi * d * h  / (m - (pi * RHO_W * d**3 * S)/6) / G

    df["E[t]"] = D * mu
    df["Var[t]"] = D**2 * var
    if correct:
        df["E[t]"] /= group2["drag correction"]
        df["Var[t]"] /= group2["drag correction"]**2

    # Plot distributions and data points.
    ts = np.linspace(3,8,400) if target=="teflon" else np.linspace(1.5,2.5,400)
    for loc, sig in zip(df["E[t]"], np.sqrt(df["Var[t]"])):
        plt.plot(ts, stats.norm(loc=loc, scale=sig).pdf(ts), lw=1)
    for i in range(len(df["E[t]"])):
        plt.plot(df["t [s]"].values[i], 0, '|', color=f'C{i}', ms=20)
    plt.xlabel(r"$t$")
    plt.ylabel(r"$P(t|\mu,X)$")

    df["sigma dist"] = np.abs(df["E[t]"] - df["t [s]"]) / np.sqrt(df["Var[t]"])
    return df
```

Appendix B: Proofs

Lemma 1. *Let $X \sim \mathcal{N}(\mu, \sigma^2)$. The kth* non-central moment *of X is defined by*

$$\mathbb{E}[X^k] = \int_{-\infty}^{\infty} x^k \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \, dx.$$

The first four non-central moments of X are given by

$$\mathbb{E}[X] = \mu, \qquad \mathbb{E}[X^2] = \mu^2 + \sigma^2, \qquad \mathbb{E}[X^3] = \mu^3 + 3\mu\sigma^2, \qquad \mathbb{E}[X^4] = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4.$$

Proof. Obviously, $\mathbb{E}[X] = \mu$ by definition. For $k > 1$, let $Y = (X - \mu) \sim \mathcal{N}(0, \sigma^2)$. Since $\mathbb{E}[Y] = 0$, we have $\mathbb{E}[Y^2] = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 = \mathbb{V}\text{ar}[Y] = \sigma^2$. Furthermore, for k odd,

$$\mathbb{E}[Y^k] = \int_{-\infty}^{\infty} y^k \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y}{\sigma}\right)^2} \, dy = 0,$$

since the integrand is an odd function and the domain of integration is symmetric about 0. In addition, we can directly compute

$$\mathbb{E}[Y^4] = \int_{-\infty}^{\infty} y^4 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y}{\sigma}\right)^2} \, dy = 3\sigma^4.$$

Using these tools,

$$\begin{aligned}
\mathbb{E}[X^2] = \mathbb{E}[(\mu + Y)^2] &= \mathbb{E}[\mu^2 + 2\mu Y + Y^2] \\
&= \mathbb{E}[\mu^2] + 2\mu\cancelto{0}{\mathbb{E}[Y]} + \mathbb{E}[Y^2] = \mu^2 + \sigma^2,
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[X^3] = \mathbb{E}[(\mu + Y)^3] &= \mathbb{E}[\mu^3 + 3\mu^2 Y + 3\mu Y^2 + Y^3] \\
&= \mathbb{E}[\mu^3] + 3\mu^2\cancelto{0}{\mathbb{E}[Y]} + 3\mu\mathbb{E}[Y^2] + \cancelto{0}{\mathbb{E}[Y^3]} = \mu^3 + 3\mu\sigma^2,
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[X^4] = \mathbb{E}[(\mu + Y)^4] &= \mathbb{E}[\mu^4 + 4\mu^3 Y + 6\mu^2 Y^2 + 4\mu Y^3 + Y^4] \\
&= \mathbb{E}[\mu^4] + 4\mu^3\cancelto{0}{\mathbb{E}[Y]} + 6\mu^2\mathbb{E}[Y^2] + 4\mu\cancelto{0}{\mathbb{E}[Y^3]} + \mathbb{E}[Y^4] = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4,
\end{aligned}$$

as desired. ∎