Exercise Set 1
Due February 17, 2020

1. A small sphere of diameter D and mass m is to be dropped through a viscous fluid of density ρ and viscosity μ in a gravitational field with acceleration g.

   (a) Write a simple mathematical model for the motion of the sphere (by simple I mean no partial differential equations), list all assumptions and identify the reliable theories (or "laws") on which the model is based, the solution variables, and quantities that require an embedded model Hint: one embedded model should be needed. This is the beginning of a model document.

   (b) Stokes law will be used for the embedded model (google it if you need to). What are the conditions for which this model can be expected to be accurate.

   (c) Identify all potential sources of uncertainty.

   (d) The model is to be used to predict the time T it will take for beads randomly selected from a bag of nominally equivalent beads to reach the bottom of a large pool of liquid when dropped from a height L from the bottom. This is the quantity of interest (QoI). Based on the model for the motion of the sphere from above, what is the complete statement of the mathematical model needed to make this prediction? What, if any, additional sources of uncertainty are introduced? Among all the sources of uncertainty, identify which are aleatoric and which are epistemic in this use scenario? How would this be different if a single bead is to be repeatedly dropped.

   (e) Another possible use of the model for the motion of a sphere is to measure the viscosity of the fluid and test whether the Newtonian viscous model for internal forces is an accurate characterization of the liquid. How might this be accomplished? How might the observations be arranged to eliminate as many uncertainties as possible?

2. Consider a six-sided die, which is perfectly weighted so that each side has an equal chance of landing up on any role. Each side has been labeled with a number of dots (from 1 to 6) as on a normal die, but instead of each face receiving a different number as on a normal die, the machine that made this die randomly selected a number from 1 to 6, with equal probability, for each face independently.

   (a) What is the probability $P(n|X)$ that the die has one dot on n faces with $n \in \{0, 1, 2, 3, 4, 5, 6\}$, and what is the background information represented by X.

   (b) You are not allowed to look at the die, but the die is thrown N times and the number $m \leq N$ of times that a face with one dot lands up is reported to you. Formulate the Bayesian inference problem to obtain an updated (posterior) probability distribution $P(n|m, N, X)$. Clearly identify the prior, the likelihood, the evidence and the posterior.

   (c) Write a little computer program that simulates this process for each of the possible values of n. For each case (value of n) determine how the posterior probability distribution changes as N increases. About how large does N need to be for you to be 99% certain of the value of n for each value of n.

3. Consider a set of experimental data $D = \{(x_i, y_i), i = 1, 2, \ldots N\}$. Your information about the measurements indicates that the measured x_i are highly accurate, but the measured y_i

have independent normally distributed errors with standard deviation σ_i. You also have background information (e.g. from the literature) about the physical process being modeled that suggests that a linear model

$$y = ax + b \tag{1}$$

is a good representation. Further, your information suggests that the values of a and b should be around a_0 and b_0 respectively, and that your certainty about the values is well represented by independent Normal distributions with standard deviations σ_a and σ_b.

Here we will pursue a Bayesian inference of the parameters.

(a) What is the background information X available for the problem?

(b) What is a joint prior distribution on a and b, $p(a, b|X)$

(c) What is the likelihood $p(D|a, b, X)$

(d) What is the joint posterior distribution $p(a, b|D, X)$

(e) What are the values of a and b at the maximum (the mode) of the posterior distribution. This is the "Maximum A Posteriori" estimate, or MAP.

(f) How does the MAP estimate compare to the result of weighted linear least squares?