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Exercise-1
UQ -

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Nice Job

(1)(a) Model - 10

$$ma = F_g - F_d - F_B = mg - F_d - \rho\left(\frac{4}{3}\pi\left(\frac{D}{2}\right)^3\right)g$$

drag (F_d), gravity (F_g), Buoyancy (F_B)

not for the above, but for applicability of stokes

Assumptions - Continuum mechanics, not close to walls, no rotations on the sphere, no roughness, (differential roughness might cause a lift force),

Drag force needs an embedded model. x is the solution variable and $a = \ddot{x}$.

(b) Stokes law

$$F_d = 6\pi\mu \frac{D}{2}\dot{x} = 3\pi\mu D\dot{x}$$

Assumptions - Laminar flow, truly spherical particles, homogenous material, smooth surfaces, no other particle interferes with this one. (This particle is not in the wake of some other particle)

(c) Sources of uncertainty

(i) Measurement errors - Diameter of sphere

(ii) Property variations (parameter) - Density and viscosity of fluid, homogeneity of sphere and fluid, smoothness of surface

(iii) Model inadequacy - For F_{drag}

(iv) Process variations (initial conditions) - drop height, velocity and spin imparted to sphere.

(d) $m\ddot{x} = mg - \rho\left(\frac{4}{3}\pi\left(\frac{D}{2}\right)^3\right)g - 3\pi\mu D\dot{x}$

$\dot{x}(0) = 0,\ x(0) = L,\ x(T) = 0$

Our mathematical Statement

As stated in (c)(iv), process variations are the additional sources of uncertainty introduced.

Aleatoric – Dimensional variability, measurement errors, [(c)(i)] process uncertainty [(c)(iv)]

Epistemic – Model inadequacy, property/parameter uncertainties [(c)(ii)]

If a single bead is repeatedly dropped, dimensional variability becomes epistemic.

(e) We can wait till the terminal velocity is reached, mark two lines after that and measure the time to calculate the viscosity

$\therefore\ \Delta X = V_{terminal}\,\Delta t$ where $V_{terminal} = \left[\dfrac{mg - \rho\left(\frac{4}{3}\pi\left(\frac{D}{2}\right)^3\right)g}{3\pi\mu D}\right]$

$\therefore\ \mu = \dfrac{\Delta t}{\Delta X}\left[\dfrac{mg - \rho\left(\frac{4}{3}\pi\left(\frac{D}{2}\right)^3\right)g}{3\pi D}\right]$

You should arrange your observations far away from walls with consistent initial conditions and polish your sphere surfaces to eliminate some uncertainties.

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(2) (a) X – (i) Each side of the die has an equal chance of landing up.

(ii) Each side has ~~0~~ 1 to 6 dots, equally likely, independently

$$P(n|X) = {}^6C_n \left(\frac{1}{6}\right)^n \left(\frac{5}{6}\right)^{6-n}$$

$$= {}^6C_n \frac{5^{6-n}}{6^6} \rightarrow \text{Prior}$$

(b) $$P(n|m,N,X) = \frac{p(m|N,n,X)\ \overbrace{p(n|N,X)}^{\text{Prior}}}{p(m|N,X)}$$

$$p(m|N,n,X) = {}^NC_m \left(n/6\right)^m \left(1-n/6\right)^{N-m}$$

$$p(m|N,X) = \sum_n p(m|N,n,X)\,p(n|N,X)$$

$$\therefore P(n|m,N,X) = \frac{\dfrac{{}^6C_n\, {}^NC_m\, 5^{6-n} (n/6)^m \left(\frac{6-n}{6}\right)^{N-m}}{6^6}}{\sum_n \dfrac{{}^6C_n\, {}^NC_m\, 5^{6-n} (n/6)^m \left(\frac{6-n}{6}\right)^{N-m}}{6^6}}$$

$$= \frac{{}^6C_n\, 5^{6-n}\, n^m (6-n)^{N-m}}{\sum_{n=0}^{6} {}^6C_n\, 5^{6-n}\, n^m (6-n)^{N-m}}$$

(c) The code is attached.

for 99% certainty

Here is a table for N, for various simulations

good.

n \ N	N_1	N_2	N_3	N_4	N_5
0	27	27	27	27	27
1	30	30	22	24	33
2	76	74	77	79	77
3	89	92	90	96	94
4	87	87	86	81	80
5	64	49	69	50	61
6	44	44	44	44	44

always has to be same

For each n, as N increases, the posterior certainty increases. As expected it seems to be the most difficult to be certain for n=3 and n=4. The plots of the (posterior distribution for various n) vs (N) are attached at the end.

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(3) (a) X –

(i) f is linear $f(x) = ax+b$

(ii) $a = \mathcal{N}(a_0, \sigma_a^2)$

$b = \mathcal{N}(b_0, \sigma_b^2)$

(iii) The measurements have independent normally distributed errors with s.d. σ_i.

$e_i = \mathcal{N}(0, \sigma_i^2)$

(b) $p(a,b|X) = p(a|X)\, p(b|X)$ → independent

$$= \frac{1}{2\pi\sigma_a\sigma_b} e^{-\frac{1}{2}\left[\frac{(a-a_0)^2}{\sigma_a^2} + \frac{(b-b_0)^2}{\sigma_b^2}\right]}$$

(c) Since all the observations are independent

$$p(D|a,b,X) = \prod_i p(f_i \,|\, a,b,X)$$

↓

$\mathcal{N}(ax_i+b, \sigma_i^2)$

$$= \prod_i \frac{1}{\sqrt{2\pi}\sigma_i} e^{-\frac{(f_i-(ax_i+b))^2}{2\sigma_i^2}}$$

(d) $$p(a,b|D,X) = \frac{p(D|a,b,X)\, p(a,b|X)}{p(D|X)}$$

$$= c\, p(D|a,b,X)\, p(a,b|X)$$

↗ normalizer

Plug in $\bar{f}_i, x_i, \sigma_i, a_0, b_0, \sigma_a, \sigma_b$

$$\therefore p(a,b|D,X) = c\, f(a,b)$$

$$\therefore c = \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(a,b)\, da\, db\right]^{-1}$$

where
$$f(a,b) = p(a,b|X)\, p(D|a,b,X)$$
$$= N(a_0,\sigma_a^2)\, N(b_0,\sigma_b^2) \prod_i \frac{1}{\sqrt{2\pi}\,\sigma_i} e^{-\frac{(\bar{f}_i-(ax_i+b))^2}{2\sigma_i^2}}$$

$$\therefore p(a,b|D,X) = \frac{f(a,b)}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(a,b)\, da\, db}$$

(e) To find the maximum

$$\frac{\partial p(a,b|D,X)}{\partial a} = 0 \rightarrow \frac{\partial f(a,b)}{\partial a} = 0$$

$$\frac{\partial p(a,b|D,X)}{\partial b} = 0 \rightarrow \frac{\partial f(a,b)}{\partial b} = 0$$

$$f(a,b) = \frac{1}{(2\pi)^{(N+2)/2}} \left(\prod_{i=1}^{N} \frac{1}{\sigma_i}\right) \frac{1}{\sigma_a\sigma_b}\, e^{-\frac{1}{2}\underbrace{\left[\frac{(a-a_0)^2}{\sigma_a^2} + \frac{(b-b_0)^2}{\sigma_b^2} + \sum_i \frac{(\bar{f}_i-(ax_i+b))^2}{\sigma_i^2}\right]}_{\Phi}}$$

To maximize $f(a,b)$ you minimize the Φ.

$$\therefore \text{Minimize } \frac{(a-a_0)^2}{\sigma_a^2} + \frac{(b-b_0)^2}{\sigma_b^2} + \sum_i \frac{(\bar{f}_i-(ax_i+b))^2}{\sigma_i^2}$$

Differentiating wrt a and b,

$$\frac{(a-a_0)}{\sigma_a^2} + \sum_{i=1}^{N}\left[\frac{\bar{f}_i-(ax_i+b)}{\sigma_i^2}\right](-x_i) = 0$$

$$\frac{(b-b_0)}{\sigma_b^2} + \sum_{i=1}^{N}\left[\frac{\bar{f}_i-(ax_i+b)}{\sigma_i^2}\right](-1) = 0$$

$$\therefore b\underbrace{\left(\frac{1}{\sigma_b^2}+\sum_{i=1}^{N}\frac{1}{\sigma_i^2}\right)}_{\beta_1} + a\underbrace{\left(\sum_{i=1}^{N}\frac{x_i}{\sigma_i^2}\right)}_{\beta_2} = \underbrace{\sum_{i=1}^{N}\frac{\bar{f}_i}{\sigma_i^2}+\frac{b_0}{\sigma_b^2}}_{\gamma_2}$$

$$b\underbrace{\left(\sum_{i=1}^{N}\frac{x_i}{\sigma_i^2}\right)}_{\alpha_1} + a\underbrace{\left(\frac{1}{\sigma_a^2}+\sum_{i=1}^{N}\frac{x_i^2}{\sigma_i^2}\right)}_{\alpha_2} = \underbrace{\sum_{i=1}^{N}\frac{\bar{f}_i x_i}{\sigma_i^2}+\frac{a_0}{\sigma_a^2}}_{\gamma_1}$$

$$a = \frac{\beta_1\gamma_1-\alpha_1\gamma_2}{\beta_1\alpha_2-\beta_2\alpha_1} \qquad b = \frac{\alpha_2\gamma_2-\beta_2\gamma_1}{\beta_1\alpha_2-\beta_2\alpha_1}$$

(f) Following a similar procedure for least squares, we will get a similar result but

$$\beta_1 = \sum_{i=1}^{N}\frac{1}{\sigma_i^2} \quad \beta_2 = \sum_{i=1}^{N}\frac{x_i}{\sigma_i^2}, \quad \gamma_2 = \sum_{i=1}^{N}\frac{\bar{f}_i}{\sigma_i^2}$$

$$\alpha_1 = \sum_{i=1}^{N}\frac{x_i}{\sigma_i^2} \quad \alpha_2 = \sum_{i=1}^{N}\frac{x_i^2}{\sigma_i^2}, \quad \gamma_1 = \sum_{i=1}^{N}\frac{\bar{f}_i x_i}{\sigma_i^2}$$

To compare the results, compare the minimization problem

$$\text{MAP} \rightarrow \underbrace{\frac{(a-a_0)^2}{\sigma_a^2} + \frac{(b-b_0)^2}{\sigma_b^2}}_{\text{Regularization term}} + \underbrace{\sum_{i=1}^{N} \frac{(f_i - (ax_i + b))^2}{\sigma_i^2}}_{\text{Weighted Least Squares}}$$

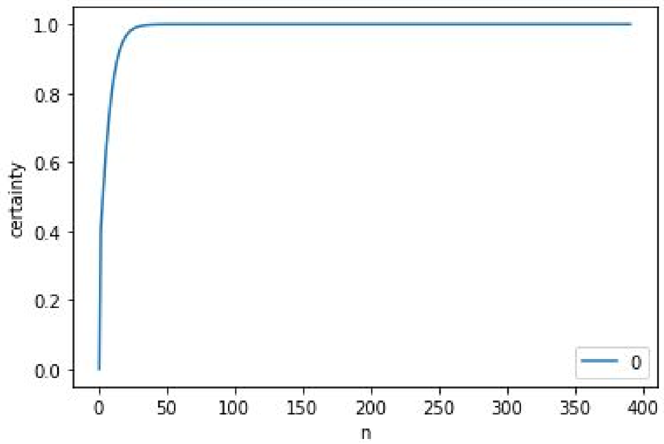
Regularization term

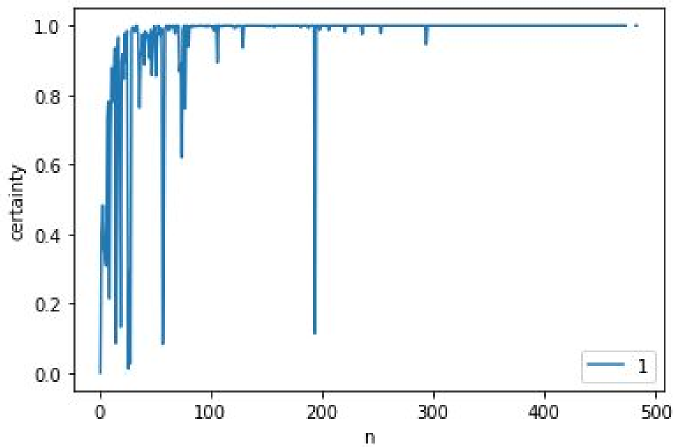
Due to this term, MAP estimate of (a,b) will be closer to (a_0, b_0).

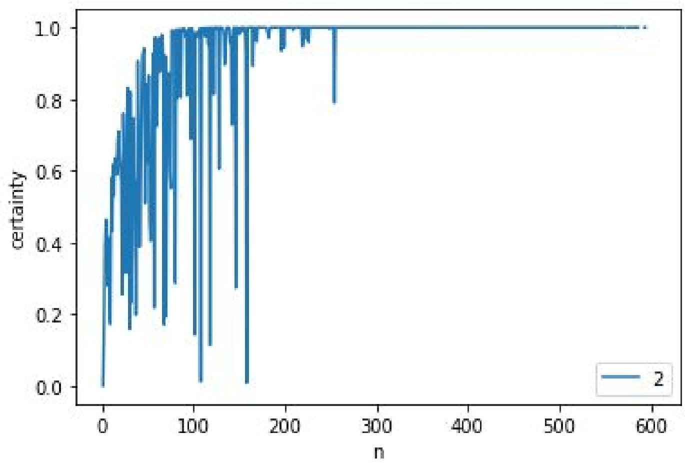
Here is the code used for the simulation.

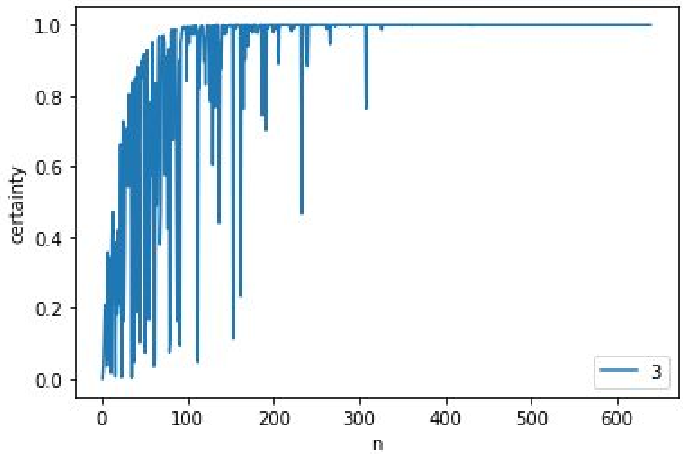
```
import numpy as np
import matplotlib.pyplot as plt

def numerator(n, m, N):
    return ((np.math.factorial(6)/
(np.math.factorial(n)*np.math.factorial(6-n)))*n**m*(6-n)**(N-m)*5**(6-n))

def denominator(m, N):
    S = 0
    for n in range(7):
        S = S + numerator(n, m, N)
    return S

def prior(n):
    return ((np.math.factorial(6)/
(np.math.factorial(n)*np.math.factorial(6-n)))*5**(6-n)/6**6)

num_experiments = 1000
m = np.zeros((7,num_experiments))
pn_m = np.zeros((7,num_experiments))
lowest_N_99_certainty = np.zeros((7,1))

for n in range(7):
    for N in range(1, num_experiments):
        experiments = 6*np.random.rand(N,1)
        m[n][N] = (experiments <= n).sum()
        pn_m[n][N] = numerator(n, m[n][N], N)/denominator(m[n][N], N)

    for N in range(1, num_experiments):
        if(pn_m[n][N]>=0.99):
            lowest_N_99_certainty[n] = N
            break

print(lowest_N_99_certainty)
plt.plot(np.arange(1,num_experiments+1), np.transpose(pn_m))
plt.legend(('0','1','2','3','4','5','6'))
```

Here are the results of ‘certainty’ (posterior probability) vs N (Number of experiments). The graphs say ‘n’ on the X-axis but it is ‘N’. The observations are noted in the handwritten notes. It should also be noted that n=1 and n=6 have no fluctuations because the experiments have a deterministic result.

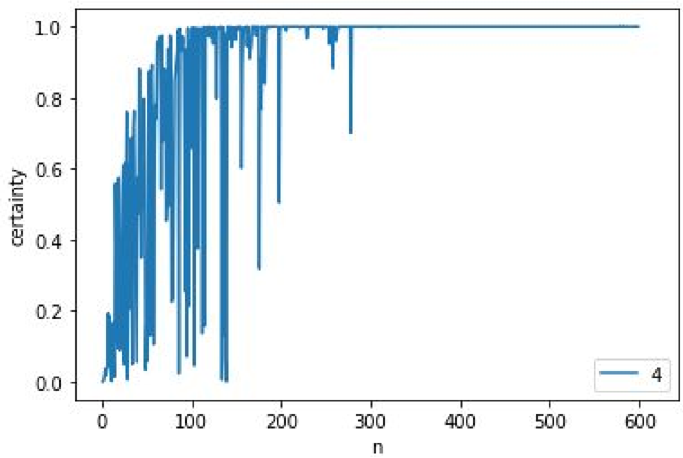

1.0
0.8
0.6
0.4
0.2
0.0
certainty
0
50
100
150
200
250
300
350
400
n
0


1.0
0.8
0.6
0.4
0.2
0.0
certainty
0
100
200
300
400
500
n
1

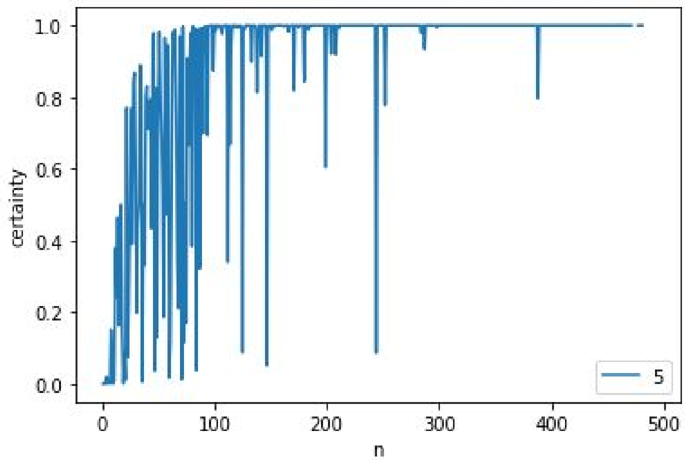

1.0
0.8
0.6
0.4
0.2
0.0
certainty
0
100
200
300
400
500
600
n
2

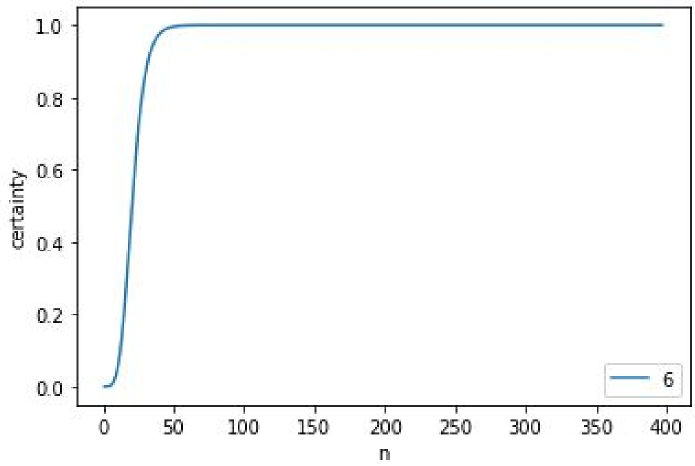


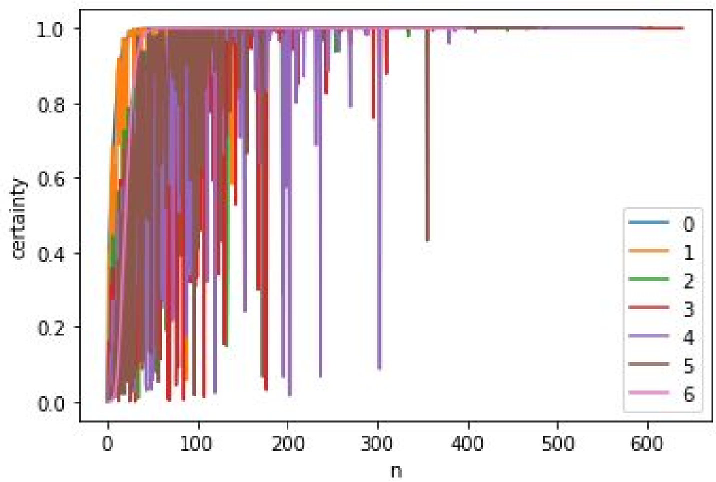
certainty
n
3



certainty
n
4


certainty
n
5


certainty
n
6


certainty
n
0
1
2
3
4
5
6