40/40

(1) (a) 10

$p(r_m \in [r, r+dr], \theta_m \in [\theta, \theta+d\theta] | X) = p(r,\theta|X)\, r\,dr\,d\theta$

$p(r_m \in [r, r+dr], \theta_m + s \in [\theta, \theta+d\theta] | X) = p(r,\theta-s|X)\, r\,dr\,d\theta$

$\therefore$ For invariance to rotation,

$p(r,\theta|X)\, r\,dr\,d\theta = p(r,\theta-s|X)\, r\,dr\,d\theta$

$\therefore p(r,\theta|X) = p(r,\theta-s|X) \rightarrow$ True for all s.

$\therefore p(r,\theta|X)$ has to be independent of θ.

$\therefore p(r,\theta|X) = f(r)$

(b) Since we are invariant to translation.
Without loss of generality, translation is in direction $\perp$ to line

$p(r_m \in [r, r+dr], \theta_m \in [\theta, \theta+d\theta] | X) = f(r)\, r\,dr\,d\theta$

$p(r_m + s \in [r, r+dr], \theta_m \in [\theta, \theta+d\theta] | X) = f(\overset{r-s}{\cancel{r}})(r-s)\,dr\,d\theta$

$\therefore f(r-s)(r-s)\,dr\,d\theta = f(r)\, r\,dr\,d\theta$

$\therefore f(r-s)(r-s) = f(r)\, r$

$\therefore$ Pretty similar to Jeffrey distribution,

$\therefore f(r) = \dfrac{c}{r}$

Normalize over a circle of radius R.

$\int_0^{2\pi}\int_0^{R} \frac{c}{r}\, r\,dr\,d\theta = 1 \rightarrow c(2\pi R) = 1 \rightarrow c = \frac{1}{2\pi R}$

Scanned with CamScanner

$\therefore \quad f(r) = \dfrac{1}{2\pi R r}$

(c) We are basically invariant to scaling here.

$\therefore \quad f(\alpha r, \alpha R) \underbrace{dA(\alpha r)}_{\text{Area element}} = f(r, R)\, dA(r) \quad —— (1)$

$\therefore$ We just check if our function satisfies (1),

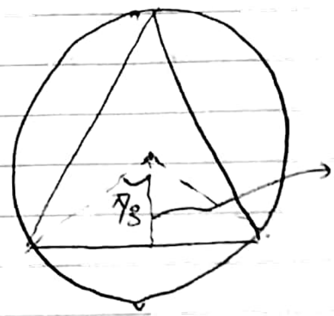
$$\frac{1}{2\pi(\alpha r)(\alpha R)} \alpha r \,(d(\alpha r))\, d\theta = \frac{1}{2\pi r R} r\, dr\, d\theta$$

$$\therefore \quad dr\, d\theta = dr\, d\theta \rightarrow \boxed{LHS = RHS}$$

$\therefore$ Hence, we have proved invariance to the radius of the circle.

Scanned with CamScanner

(d)



We need $r \leq R/2$ for a larger chord

$$\therefore \; p(r \leq R/2) = \int_0^{2\pi} \int_0^{R/2} f(r)\, r\, dr\, d\theta$$

$$= \frac{1}{2\pi R} \int_0^{2\pi} d\theta \int_0^{R/2} dr$$

$$= \frac{1}{2}$$

(2) 10 $g(x) = \frac{c}{x}$ $\quad x \in [\epsilon, L)$ where ϵ is close to zero, L is very large

Normalize $g(x)$ $\quad \int_\epsilon^{L} g(x)\, dx = 1$

$$\therefore \; c(\log L - \ln \epsilon) = 1$$

$$\therefore \; c = \frac{1}{\log(L/\epsilon)}$$

$$\therefore \; g(x) = \frac{1}{x \log(L/\epsilon)}$$

Our constraints are $\int_\epsilon^L p\,dx = 1$, $\int_\epsilon^L x p\,dx = \mu$, $\int_\epsilon^L x^2 p\,dx = \sigma^2$

$$\therefore \; \mathcal{L} = \underbrace{\int_0^\infty p \log p x \log(L/\epsilon)\, dx}_{\text{Maximize relative entropy}} + \lambda_0\left[1 - \int_\epsilon^L p\,dx\right] + \lambda_1\left[\mu - \int_\epsilon^L x p\,dx\right] + \lambda_2\left[\sigma^2 - \int_\epsilon^L x^2 p\,dx\right]$$

$$\delta \mathcal{L} = \int_0^\infty \delta p \underbrace{\left[1 + \log(xp\log(L/\epsilon)) - \lambda_0 - \lambda_1 x - \lambda_2 x^2 \right]}_{=0} dx = 0$$

$$\therefore p = \frac{1}{x\log(L/\epsilon)} e^{(\lambda_0 - 1) + \lambda_1 x + \lambda_2 x^2}$$

Let $L \to \infty$, $\epsilon \to 0$

$$\therefore e^{(\lambda_0 - 1)} \frac{1}{\log(L/\epsilon)} \to A$$

$$\therefore p = \frac{A}{x} e^{\lambda_1 x + \lambda_2 x^2}$$

Now we satisfy the constraints,

① $\int p\,dx = 1 \to A \int_0^\infty \frac{e^{\lambda_1 x + \lambda_2 x^2}}{x} dx = 1$

② $\int x p\,dx = \mu \to A \int_0^\infty e^{\lambda_1 x + \lambda_2 x^2} dx = \mu$

③ $\int x^2 p\,dx = \sigma^2 \Rightarrow A \int_0^\infty x e^{\lambda_1 x + \lambda_2 x^2} dx = \sigma^2$

① indicates that p has to be normalizable. But, since $\frac{e^{\lambda_1 x + \lambda_2 x^2}}{x}$ cannot be normalized, we cannot satisfy this constraint

Let's look at ~~③~~, ②,

OK, but (2) and (3) are now less meaningful

$$A\int_0^{\infty} e^{\lambda_1 x+\lambda_2 x^2}\,dx = \mu$$

Integrate by parts

$$\therefore \frac{A}{\lambda_1}\left[e^{\lambda_2 x^2+\lambda_1 x}\right]_0^{\infty} - A\int_0^{\infty} \frac{2\lambda_2 x}{\lambda_1} e^{\lambda_2 x^2+\lambda_1 x}\,dx = \mu$$

$$\therefore \frac{A}{\lambda_1}(-1) - \frac{2\lambda_2}{\lambda_1}\sigma = \mu$$

$$\therefore \lambda_1 \mu + A + 2\lambda_2 \sigma = 0 \quad — (2)'$$

Solving (3) Gives,

$$A\left[\sqrt{\pi}\,\lambda_1\sqrt{-\lambda_2}\, e^{-\lambda_1^2/4\lambda_2}\, \mathrm{erfc}\left(\frac{-\lambda_1}{2\sqrt{-\lambda_2}}\right) - 2\lambda_2\right] = 4\lambda_2^2\sigma^2 \quad — (3)'$$

Solve for A and λ_1 in terms of λ_2

$$\therefore p = A(\lambda_2)\, e^{\lambda_1(\lambda_2)x+\lambda_2 x^2}$$

Here, $A = A(\lambda_2)$

$\lambda_1 = \lambda_1(\lambda_2)$

(3) Let ϵ be a number that is very close to 0.

10 Let L be a very large number

$\therefore$ $q(x) = \frac{1}{L-\epsilon}$ $x \in [\epsilon, L]$

$\therefore$ $\mathcal{L}$ Our constraints are $\int_{\epsilon}^{L} p\,dx = 1$ $\int_{\epsilon}^{L} xp\,dx = \mu$

$$\mathcal{L} = \int_{\epsilon}^{L} p \log \frac{p}{q}\,dx + \lambda_0\left[1 - \int_{\epsilon}^{L} p\,dx\right] + \lambda_1\left[\mu - \int_{\epsilon}^{L} p\,dx\right]$$

$$\delta\mathcal{L} = \int_{\epsilon}^{L} \delta p \underbrace{\left(1 + \log\frac{p}{q} - \lambda_0 - \lambda_1 x\right)}_{=0} dx = 0$$

$$\therefore\ p = (L-\epsilon)^{-1} e^{(\lambda_0 - 1)} e^{\lambda_1 x}$$

Let $L \to \infty$, $\epsilon \to 0$.

$$\therefore\ (L-\epsilon)^{-1} e^{(\lambda_0 - 1)} \to c$$

$$\therefore\ p = ce^{\lambda_1 x},$$

Now, satisfy the constraints

$$\int_0^{\infty} p\,dx = 1 \rightarrow \frac{c}{\lambda_1}\left[e^{-\infty} - e^{0}\right] = \frac{-c}{\lambda_1} = 1$$

$$\therefore\ c + \lambda_1 = 0$$

$$\therefore\ \lambda_1 = -c$$

$$\int_0^{\infty} xp\,dx = \mu \rightarrow \int x c e^{-cx}\,dx = \mu$$

Scanned with CamScanner

$$\therefore \left[\frac{xc}{-c} e^{-cx}\right]_0^{\infty} - \int_0^{\infty} \frac{c}{-c} e^{-cx} dx$$

$$= \int_0^{\infty} e^{-cx} dx$$

$$= \frac{-1}{c}[-1]$$

$$= \frac{1}{c}$$

$$\therefore \frac{1}{c} = \mu$$

$$\therefore c = 1/\mu$$

$$\therefore \boxed{p(x) = \frac{1}{\mu} e^{-x/\mu}}$$

(4) Let L_1, L_2 be very large numbers s.t.

10

$$q(x) = \frac{1}{4L_1L_2} \quad x \in [-L_1, L_1], y \in [-L_2, L_2]$$

Our constraints are $\int_{-L_2}^{L_2} p(x,y|x)\,dy = \phi(x)$

$$\int_{-L_1}^{L_1} p(x,y|y)\,dx = \psi(y)$$

$$\int_{-L_2}^{L_2}\int_{-L_1}^{L_1} p(x,y|x)\,dx\,dy = 1$$

Scanned with CamScanner

Look at the last 2 constraints.

We write them in the Lagrangian as

$$\int_{-L_2}^{L_2} \lambda_1(y)\left[\psi(y) - \int_{-L_1}^{L_1} p(x,y|Y)\,dx \right] dy$$

Perturb wrt. $\lambda(y)$, and the difference is

$$\int_{-L_2}^{L_2} \delta\lambda_1(y)\left(\psi(y) - \int_{-L_1}^{L_1} p(x,y|Y)\,dx \right) dy = 0$$

This has to be true for all $\delta\lambda_1(y)$. This implies, the bracketed term is 0. ∴ We recover our constraint.

∴ The Lagrange multiplier can be written as

$$\mathcal{L} = \int_{-L_2}^{L_2}\int_{-L_1}^{L_1} p \log\left(\frac{p}{q}\right) dx\,dy + \lambda_0\left[1 - \int_{-L_2}^{L_2}\int_{-L_1}^{L_1} p(x,y)\,dx\,dy\right]$$

$$+ \int_{-L_1}^{L_1} \lambda_1(x)\left[\phi(x) - \int_{-L_2}^{L_2} p(x,y)\,dy\right] dx$$

$$+ \int_{-L_2}^{L_2} \lambda_2(x)\left[\psi(y) - \int_{-L_1}^{L_1} p(x,y)\,dx\right] dy$$

Perturb wrt p

$$\delta L = \int_{-L_2}^{L_2}\int_{-L_1}^{L_1} \delta p\left[1 + \log\frac{p}{q} - \lambda_0 - \lambda_1(x) - \lambda_2(y)\right] dx\,dy = 0$$

Since this has to be true for all δp,

$$p \log \frac{p}{q} = \lambda_0 - 1 + \lambda_1(x) + \lambda_2(y)$$

$$\therefore p = q e^{(\lambda_0 - 1)} e^{\lambda_1(x) + \lambda_2(y)}$$

$$\lim_{L_1 \to \infty, L_2 \to \infty} p(x,y) = c e^{\lambda_1(x) + \lambda_2(y)}$$

Now satisfy the constraints

$$e^{\lambda_2(y)} c \int_{-\infty}^{\infty} e^{\lambda_1(x)} dx = \Psi(y) \quad \text{— ①}$$

$$e^{\lambda_1(x)} c \int_{-\infty}^{\infty} e^{\lambda_2(y)} dy = \phi(x) \quad \text{— ②}$$

$$c \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\lambda_1(x) + \lambda_2(y)} dx\, dy = 1$$

$$\therefore c = \frac{1}{\int e^{\lambda_1(x)} dx \int e^{\lambda_2(y)} dy} = \frac{1}{XY}$$

Put in ① and ②,

$$e^{\lambda_2(y)} = \Psi(y) Y$$

$$e^{\lambda_1(x)} = \phi(x) X$$

$$p = c e^{\lambda_1(x) + \lambda_2(y)} = \frac{1}{XY} X \phi(x) Y \Psi(y)$$

$$= \phi(x) \Psi(y)$$