Bayesian Inversion Over Long Time Scales for the Greenland Ice Sheet

Shreyas Gaikwad

Oden Institute for Computational Engineering and Sciences, The University of Texas at Austin

Supervising Committee

Patrick Heimbach (Advisor)
Rachel Ward
Clint Dawson
Omar Ghattas
Ginny Catania

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Abstract

Ice sheets are grounded ice masses of continental size, with areas greater than 50,000 sq. km. The two contemporary ice sheets are Greenland and Antarctica. They are dynamic entities whose evolution is governed by non-linear Partial Differential Equations (PDEs) that describe the conservation of mass, momentum, and energy. Ice sheet evolution is driven by its present state and forcings such as surface temperature, precipitation, basal sliding, geothermal heat flux, etc. The Greenland Ice Sheet (GrIS) is the second-largest present-day ice sheet on Earth. Its ice volume above floatation is equivalent to 7.42 meters of sea-level rise [17]. It outpaces the Antarctic Ice Sheet, which is the largest present-day ice sheet on Earth, in its present contribution to Global Mean Sea Level (GMSL) rise, contributing approximately 15% to the total sea-level budget [7]. Given its substantial contribution to the GMSL, it is thus important to predict with increased confidence the future behavior of the GrIS.

Computational models have been essential in understanding the impact of various forcings on ice sheets. Despite recent advances in numerical and physical modeling of ice sheets, the ice sheet models differ substantially in the forecasts they make for various climate scenarios [3]. Many of the forecasts are made starting from ad-hoc initializations, and using partially ad-hoc surface climate forcings, e.g. [3, 6], leading to steady states that differ greatly from the present-day GrIS configuration. Other forecasts use accurate snapshot-based initializations, e.g. [12]. These forecasts, while they start from a representative state of the GrIS, suffer from model drift, since they do not reflect accurately the past history of the ice sheet from which they arrived.

These shortcomings can be overcome by leveraging data for model calibration. In the past, work of this nature has been limited by a lack of data, a lack of robust techniques for model calibration, and a lack of computational resources. The model calibration is an exercise in PDE-constrained gradient-based optimization under uncertainty. These inverse problems can be naturally formulated within the Bayesian framework. The Bayesian framework inherently requires the quantification of uncertainties in data, surface forcings, and model parameters by characterizing them as probability density functions (pdfs). This framework elegantly breaks down the problem into three steps - (1) Deterministic inversion given prior uncertainties in the parameter space using second-order non-linear optimization techniques to retrieve the maximum a posteriori (MAP) point, (2) Quantifying posterior uncertainties of the parameters. The typical approach for sampling the posterior pdf is to employ Markov Chain Monte Carlo methods, which are prohibitive in this case since the parameter space has $\mathcal{O}(10^6)$ dimensions. Instead, this study assumes a Laplace approximation, which in essence is a linearization that reduces the posterior to a Gaussian distribution. The important modes of this linearized posterior can be evaluated using randomized methods as in [8, 23], and (3) Projection of posterior uncertainties from the parameter space to the quantity of interest (QoI), such as contributions to GMSL. This projection is evaluated using the posterior covariance of the parameters and the gradient of the parameter-to-observable map. The model calibration broadly follows techniques as described in [25, 1, 11].

The model calibration requires an adjoint model to evaluate the gradients and the linearized hessian. Most of the existing work uses analytically derived adjoint models, for example, [11, 14]. To evaluate the adjoints of complex forward models, which are manifested in the form of huge codebases, we leverage source-to-source transformation algorithmic differentiation (AD) using an open-source AD tool. Using AD tools allows the generation of adjoint models that are up-to-date with the latest developments in the forward model. Recently, AD-derived adjoints have been widely used for model sensitivity studies across the climate sciences, e.g. [9, 13, 21, 24]. A search of the literature revealed only a few studies that used adjoint models for transient model calibration in the climate sciences, e.g. [4, 5] for ice-ocean interactions and [18] for the Artic subpolar gyres.

Thus, the proposed work will generally consist of four main components: (1) Developing an automatic differentiation (AD) based adjoint model for finite differences based ice-sheet model SICOPOLIS using the open-source AD tool TAPENADE, (2) Recover an optimal transient state of the ice sheet

that is reconstructed over the Holocene period ($\sim$ last 10,000 years), which is in close agreement with Operation IceBridge (OIB) age-layer data [15] for the GrIS. The long period of reconstruction justifies the use of SICOPOLIS, which uses the shallow-ice approximation but is very efficient for paleoclimate simulations, (3) Recover an optimal set of model parameters and reconstructed time-varying surface boundary conditions required to produce a consistent ice sheet evolution and achieve close fit to the observations, and (4) Quantifying uncertainties in predicting the contribution of the GrIS to the GMSL in the next century.

Recovering the optimal set of time-varying surface climate conditions will be a step toward characterizing with greater accuracy the two major climate events in the Holocene epoch - Little Ice Age (LIA) and Holocene Climatic Optimum (HCO). These two events are of importance since they allow us to ascertain the response of the GrIS to drastic climate events, which might help improve predictions of the response of the GrIS to sharp increases in the global average temperature. The proposed work will also provide constrained spatio-temporal maps of surface climate forcings, which will allow modelers to initialize models that are in harmony with the past flow history of the GrIS. This will reduce model drift, as well as preclude ad-hoc initialization. Research of this kind may be extended to numerous studies using other paleo proxy data types, for ice sheets such as Greenland, Antarctica, the Laurentide Ice Sheet, the Fennoscandia ice sheet, and even Mars.

1 Introduction

2 The Forward Model

2.1 Constitutive models for ice

Ice occurring in ice sheets is comprised of crystallites that are oriented in random directions. Macroscopically, the local anisotropy averages out and the behavior is thus isotropic.

Ice is modeled as a non-Newtonian fluid. Its viscosity is much larger than fluids such as water, and its fluid nature is apparent only over longer time scales.

2.1.1 Glen's Flow Law

Glacier ice has a highly non-linear rheology. Creep is the deformation of ice crystals, in response to applied stress. The relationship between the stress and strain is given by *Nye's generalization to Glen's flow law* or simply, *Glen's flow law* ([2, 19]).

$$\mathbf{D} = A(T')\sigma_e^{n-1}\mathbf{t}^{\mathrm{D}} \tag{2.1}$$

where $\mathbf{D}$ is the strain-rate tensor, $\mathbf{t}^{\mathrm{D}}$ is the deviatoric stress tensor, and σ_e is the effective stress. σ_e is modeled as the second invariant of the deviatoric stress tensor. Therefore,

$$\sigma_e = \sqrt{\frac{1}{2}\mathrm{tr}(\mathbf{t}^{\mathrm{D}})^2}$$

n is the stress exponent. The value $n = 3$ is almost universally accepted. However, different processes can have different values for the stress exponent, and $n = 3$ is an aggregate value. For example, $n = 4.1 \pm 0.4$ has been found to be a better fit for fast-flowing ice streams [16].

$A(T') = A_0 \exp(-Q/RT')$ is the rate factor expressed as an Arrhenius law. T' is the temperature relative to the pressure melting point, $T' = T + \beta p$. It is defined so that the melting point always corresponds to $T' = 0°\mathrm{C}$. A_0 is the pre-exponential constant. Q is the activation energy, R is the ideal gas constant. A_0, Q are step functions of T', around $T' = -10°\mathrm{C}$ [20].

The corresponding viscosity is therefore given by,

$$\eta = \frac{1}{2A(T')\sigma_e^{n-1}} \tag{2.2}$$

The analytical inversion of Glen's flow is given by,

$$\mathbf{t}^{\mathrm{D}} = B(T')d_e^{-\left(1-\frac{1}{n}\right)}\mathbf{D} \tag{2.3}$$

where $B(T') = A(T')^{-1/n}$ and d_e is the effective strain rate, given by

$$d_e = \sqrt{\frac{1}{2}\mathrm{tr}(\mathbf{D})^2}$$

As $\sigma_e \to 0$, the infinite viscosity limit can cause numerical issues. Therefore, a small regularization term is added, and the corresponding flow law is known as *Regularized Glen's flow law.*

$$\eta = \frac{1}{2A(T')\left[\sigma_e^{n-1} + \sigma_0^{n-1}\right]} \tag{2.4}$$

An analytical inversion cannot be explicitly specified for *Regularized Glen's flow law*, only an implicit relation can be stated and solved iteratively.

2.1.2 Flow Enhancement Factor

All flow laws stated above are only valid for regimes of secondary creep. In regions with relatively high temperatures and stresses, tertiary creep may prevail. This regime is characterized by higher strain rates compared to secondary creep. This is because ice crystals recrystallize in a way that promotes deformation. We can model tertiary creep by considering a flow enhancement factor $E > 1$ [10], such that $A(T') \to EA(T')$, or equivalently, $B(T') \to E^{-1/n}B(T')$. Previously, the enhancement factor has been modeled as varying within the ice sheet based on the age of the ice as a step function in [6], where $E = 1$ for the Holocene interglacial ice and $E = 3$ for the Wisconsin glacial ice. Flow enhancement factors can also be modeled to account for anisotropy, as in [22].

2.1.3 Heat flux and internal energy

The heat flux within glacier ice is modeled by Fourier's law of heat conduction,

$$\mathbf{q} = -\kappa(T)\nabla T \tag{2.5}$$

where the thermal conductivity increases with decreasing temperature.

$$\kappa(T) = 9.828\exp(-0.0057\,T[K])\ \mathrm{Wm^{-1}K^{-1}} \tag{2.6}$$

The internal energy is characterized by a temperature-dependent specific heat, that decreases with decreasing temperature, given by

$$c(T) = (146.3 + 7.253T[K])\ \mathrm{J\ kg^{-1}K^{-1}} \tag{2.7}$$

2.2 Conservation of mass

The general conservation of mass equation is given by,

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{v} = 0$$

Here ρ is the density of ice, $\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + \nabla\cdot(\mathbf{v}\rho)$ is the total derivative of the density field, and $\mathbf{v}$ is the velocity field. Ice is assumed to be an incompressible material. We can thus state a simplified form for mass conservation.

$$\nabla\cdot\mathbf{v} = 0 \tag{2.8}$$

Under the shallow ice approximation described below, the vertical direction is essentially decoupled from the horizontal direction. The vertical velocity is computed as a diagnostic quantity using the continuity equation and appropriate boundary conditions.

2.3 Conservation of momentum

The equation for the conservation of momentum is given by

$$\rho\frac{D\mathbf{v}}{Dt} = -\nabla p + \eta\Delta\mathbf{v} + (\nabla\mathbf{v} + \nabla\mathbf{v}^T)\cdot\nabla\eta + \rho\mathbf{g} - 2\rho\mathbf{\Omega}\times\mathbf{v}$$

Here, $\mathbf{v}$ is the three-dimensional velocity field, $\nabla(\cdot)$ is the gradient operator, $\Delta(\cdot)$ is the Laplacian operator, $\frac{D\mathbf{v}}{Dt} = \frac{\partial\mathbf{v}}{\partial t} + \nabla\cdot(\mathbf{v}\mathbf{v})$ is the total derivative of the velocity field, ($\nabla\cdot[\cdot]$ is the divergence operator), $\mathbf{\Omega}$ is the angular velocity of the Earth, and $-2\rho\mathbf{\Omega}\times\mathbf{v}$ is the Coriolis force term.

2.3.1 Full Stokes Flow Problem

We can use dimensional considerations to significantly simplify the momentum equation (see Appendix A), by eliminating the Coriolis force term as well as the term of the inertial force on the LHS. As discussed in 2.1.1, $\eta = \eta(T', d_e)$, i.e., viscosity is a non-linear function of temperature and effective strain rate, which means our system is thermo-mechanically coupled.

$$-\nabla p + \eta \Delta \mathbf{v} + (\nabla \mathbf{v} + \nabla \mathbf{v}^T) \cdot \nabla \eta + \rho \mathbf{g} = 0 \tag{2.9}$$

2.3.2 Shallow Ice Approximation

These equations can be further simplified, under the following assumptions -

Assumption 2.1 ***Hydrostatic Approximation****: In all parts of an ice sheet, the shear stresses t_{xz} and t_{yz} are small compared to the vertical normal stress t_{zz}, which is approximately equal to the pressure p, so that $t_{zz} = -\rho g(h - z)$.*

Assumption 2.2 ***First Order Approximation****: Horizontal derivatives of vertical velocity are negligible compared to vertical derivatives of horizontal velocity. The solution of the momentum equation is fully decoupled from the vertical velocity v_z via the continuity equation.*

Assumption 2.3 ***Shallow Ice Approximation****: The shear stress t_{xz}, t_{yz} are supported by basal drag and are dominant in comparison to the deviatoric normal stresses $t^D_{xx}, t^D_{yy}, t^D_{zz}$ as well as shear stress in the vertical plane t_{xy}.*

The equations below are stated under all these assumptions.

All normal stress components are equal to negative of pressure, $t_{xx} = t_{yy} = t_{zz} = -p$. The vertical momentum equation reduces to

$$\frac{\partial p}{\partial z} = -\rho g \tag{2.10}$$

This equation can be integrated, using appropriate boundary conditions (see subsection 2.5),

$$p = \rho g(h - z) \tag{2.11}$$

The horizontal components of the momentum equation reduce to

$$\frac{\partial t_{xz}}{\partial z} = \frac{\partial p}{\partial x} = \rho g \frac{\partial h}{\partial x} \tag{2.12}$$

$$\frac{\partial t_{yz}}{\partial z} = \frac{\partial p}{\partial y} = \rho g \frac{\partial h}{\partial y} \tag{2.13}$$

Since the right-hand side in these equations does not depend on z, we can readily integrate them with respect to z (using the stress-free boundary conditions in subsection 2.5).

$$t_{xz} = \rho g(h - z)\frac{\partial h}{\partial x} \tag{2.14}$$

$$t_{yz} = \rho g(h - z)\frac{\partial h}{\partial y} \tag{2.15}$$

The effective stress σ_e is given by,

$$\sigma_e = \sqrt{t_{xz}^2 + t_{yz}^2} = \rho g(h - z)|\nabla h| \tag{2.16}$$

Using these relations in Glen's flow law, and neglecting the derivatives of the vertical velocity based on the first-order approximation yields,

$$\frac{\partial v_x}{\partial z} = -2A(T')\left[\rho g(h - z)\right]^n |\nabla h|^{n-1} \frac{\partial h}{\partial x} \tag{2.17}$$

$$\frac{\partial v_y}{\partial z} = -2A(T')\left[\rho g(h - z)\right]^n |\nabla h|^{n-1} \frac{\partial h}{\partial y} \tag{2.18}$$

Again, we can integrate both sides with respect to z,

$$v_x = v_{bx} - 2(\rho g)^n |\nabla h|^{n-1} \frac{\partial h}{\partial x} \int_b^z A(T')\left[(h - \bar{z})\right]^n \, d\bar{z} \tag{2.19}$$

$$v_x = v_{by} - 2(\rho g)^n |\nabla h|^{n-1} \frac{\partial h}{\partial y} \int_b^z A(T')\left[(h - \bar{z})\right]^n \, d\bar{z} \tag{2.20}$$

Here v_{bx}, v_{by} are basal velocities that are modeled based on discussion in 2.5.2. Based on their forms, and by defining $\mathbf{v}_h = (v_x, v_y)^T$, the above relationship can be reduced to,

$$\mathbf{v}_h = -C\nabla h \tag{2.21}$$

with C being a scalar function given by,

$$C = \begin{cases} -2(\rho g)^n |\nabla h|^{n-1} \int_b^z A(T')\left[(h - \bar{z})\right]^n \, d\bar{z} & \text{if } T_b < T_m \\ C_b(\rho g H)^{p-q} |\nabla h|^{p-1} - 2(\rho g)^n |\nabla h|^{n-1} \int_b^z A(T')\left[(h - \bar{z})\right]^n \, d\bar{z} & \text{if } T_b = T_m \end{cases}$$

These cases represent cold and temperate ice at the base respectively, which affects sliding and thus the dynamics significantly.

Finally, the decoupled vertical velocity can be calculated by integrating the continuity equation,

$$v_z = v_z|_{z=b} - \int_b^z \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right) d\bar{z} \tag{2.22}$$

$v_z|_{z=b}$ is determined by the kinematic condition, defined in 2.5.2.

2.4 Equation for temperature evolution

The general equation for energy conservation is as follows -

$$\rho \frac{Du}{Dt} = -\nabla \cdot \mathbf{q} + \mathrm{tr}(\mathbf{t} \cdot \mathbf{D}) + \rho r$$

Here u is the internal energy, which we model as $u = \int_{T_0}^{T} c(\bar{T}) d\bar{T}$, where $T_0 = 273.15K$ and $c(T)$ is given by 2.7. $\frac{Du}{Dt} = \frac{\partial u}{\partial t} + \nabla \cdot (\mathbf{v}u)$ is the total derivative of the internal energy field. $\mathbf{q}$ is the heat flux, modeled by Fourier's law, as in 2.6. $\mathrm{tr}(\mathbf{t} \cdot \mathbf{D})$ is the viscous dissipation power term and ρr is the radiation term, which is assumed to be negligible. The equation specifically for ice is thus given by,

$$\rho c \frac{DT}{Dt} = \nabla \cdot (\kappa \nabla T) + 4\eta d_e^2$$

This equation is subject to the condition that the temperature of ice does not exceed the pressure melting point, i.e. $T < T_m$. Based on dimensional analysis in Appendix B, we can neglect horizontal heat conduction terms. Furthermore, using equations 2.1, 2.2, 2.3 further reduces the equation to its final form.

$$\rho c \frac{DT}{Dt} = \frac{\partial}{\partial z}\left(\kappa \frac{\partial T}{\partial z}\right) + 2A(T')\sigma_e^{n-1} \tag{2.23}$$

2.5 Boundary conditions

We assume the shallow ice approximation (SIA), as described in 2.3.2. This means that the horizontal gradients of the surface and ice base are assumed to be negligible (they are $\mathcal{O}(\epsilon)$, ϵ being the aspect ratio as defined in Appendix A), making the normal approximately vertical.

2.5.1 Free surface

The kinematic boundary condition at the surface is then given by

$$\frac{\partial h}{\partial t} + v_x \frac{\partial h}{\partial x} + v_y \frac{\partial h}{\partial y} - v_z = a_s \tag{2.24}$$

Here a_s is the accumulation-ablation function.

If we neglect the small advective flux term across the boundary as well as the atmospheric stress, the dynamic boundary condition is simply the stress-free condition. Since the normal vector is approximately vertical, this implies

$$t_{xz} = t_{yz} = t_{zz} = 0 \tag{2.25}$$

The thermodynamic boundary condition is given by prescribing the surface temperature.

$$T = T_s \tag{2.26}$$

T_s can be assumed to be the mean annual surface air temperature as long as it is less than or equal to 0°C.

2.5.2 Ice base

The kinematic boundary condition at the base is then given by

$$\frac{\partial b}{\partial t} + v_x \frac{\partial b}{\partial x} + v_y \frac{\partial b}{\partial y} - v_z = a_b \tag{2.27}$$

Here a_b is the basal melting rate. It is computed using an energy jump condition at the base. Note that $a_b = 0$ for a cold base. The equation is only valid for a temperate base. $\mathbf{n} \approx [0, 0, -1]^T$ is the normal vector of the ice base. Unlike a_s on the free surface, a_b is a computed quantity, while $q_g eo$ is prescribed.

$$a_b = \frac{q_{\text{geo}} - \kappa \nabla T \cdot \mathbf{n} - \mathbf{v}_b \cdot \mathbf{t} \cdot \mathbf{n}}{\rho L} \tag{2.28}$$

If we neglect the small advective flux term across the boundary, the dynamic boundary condition is given by (the normal vector is approximately vertical),

$$\mathbf{t} \cdot \mathbf{n} = \mathbf{t}_{\text{lith}} \cdot \mathbf{n} \tag{2.29}$$

Since no information is available about the lithospheric stress term, we use an empirical relationship instead. For a cold base, i.e. $T_b < T_m$, no-slip conditions prevail since the ice base is frozen. If temperate ice exists at the base, the velocity is expressed as a function of basal drag $\boldsymbol{\tau}_b$ and normal stress N_b in the form of a *Weertman-type sliding law.*

$$\mathbf{v}_b = \begin{cases} \mathbf{0} & \text{if } T_b < T_m \\ -C_b \frac{\tau_b^p}{N_b^q} \mathbf{e}_t, & \text{if } T_b = T_m \end{cases}$$

Commonly used values of basal sliding coefficients (p, q) are $(3, 1), (3, 2)$ for sliding on hard rock, and $(1, 0)$ for sliding on soft sediments.

Since a cold base has no basal melting and no-slip conditions, the thermodynamic boundary condition is simply given by,

$$\kappa \nabla T \cdot \mathbf{n} = q_{\text{geo}} \tag{2.30}$$

For a temperate base, the temperature is at the pressure melting point.

$$T = T_m \tag{2.31}$$

2.6 Ice thickness equation

The ice thickness equation cannot be derived by simply subtracting equations 2.24 and 2.27 since the velocity fields are not the same. Instead, it is derived by integrating the continuity equation 2.8 in the z-direction and expanding the x, y terms using the Leibniz rule to expose the boundary conditions, and then substituting in the kinematic boundary conditions 2.24 and 2.27.

$$\frac{\partial H}{\partial t} = -\nabla \cdot \mathbf{Q} + a_s - a_b \tag{2.32}$$

where $\mathbf{Q}$ is the volume flux, calculated by vertically integrating the horizontal velocity $\mathbf{v}_h$.

$$\mathbf{Q} = \int_b^h \mathbf{v}_h \, dz$$

2.7 Age equation

The age equation is based on a tautological relationship that the age of any ice parcel increases identically with time.

$$\frac{DA}{Dt} = 1$$

Here, $\frac{DA}{Dt} = \frac{\partial A}{\partial t} + \nabla \cdot (\mathbf{v}A)$ is the total derivative of the age field. Since this is a purely advective relationship, numerical diffusivity in the z-direction is introduced on the RHS for reasons of numerical stability [6]. The final equation is thus,

$$\frac{\partial A}{\partial t} + \nabla \cdot (\mathbf{v}A) = 1 + D_A \frac{\partial^2 A}{\partial z^2} \tag{2.33}$$

2.8 Modeling polythermal ice

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5 Timeline

A Dimensional Analysis for Momentum Equation

The equation for the conservation of momentum is given by,

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \eta \Delta \mathbf{v} + (\nabla \mathbf{v} + \nabla \mathbf{v}^T) \cdot \nabla \eta + \rho \mathbf{g} - 2\rho \mathbf{\Omega} \times \mathbf{v}$$

The typical values for the horizontal and vertical extent of an ice sheet, the horizontal and vertical flow velocities, the pressure, and the time are as follows,

Typical horizontal extent $[L] = 1000\,\mathrm{km}$,
Typical vertical extent $[H] = 1\,\mathrm{km}$,
Typical horizontal velocity $[U] = 100\,\mathrm{m/a}$,
Typical vertical velocity $[W] = 0.1\,\mathrm{m/a}$,
Typical pressure $[P] = \rho g [H] \approx 10\,\mathrm{MPa}$,
Typical time-scale $[t] = [L]/[U] = [H]/[W] = 10^4\,\mathrm{a}$.

We define the following non-dimensional quantities -

Aspect Ratio $\epsilon = \frac{[H]}{[L]} = \frac{[W]}{[U]} = 10^{-3}$,
Rossby Number $Ro = \frac{[U]}{2\Omega[L]} \approx 2 \times 10^{-8}$,
Froude Number $Fr = \frac{\rho[U]/[t]}{[P]/[L]} = \frac{\rho[U]^2/[L]}{\rho g[H]/[L]} \approx 2 \times 10^{-15}$.

The aspect ratio is the ratio of the typical vertical and horizontal length/velocity scales. The Rossby number is the ratio of the horizontal inertial terms to the Coriolis terms. The ratio of the horizontal inertial terms and horizontal pressure gradient is the Froude number which is very small, so the horizontal inertial terms can be neglected.

The ratio of the vertical inertial terms and vertical pressure gradient is given by,

$$\frac{\rho[W]/[t]}{[P]/[H]} = \frac{[W]}{[U]} \frac{[H]}{[L]} \frac{\rho[U]/[t]}{[P]/[L]} = \epsilon^2 Fr \approx 2 \times 10^{-21}$$

This implies that all inertial terms are negligible compared to the pressure gradients.

Similarly, the ratio of the Coriolis term to the pressure gradient given by

$$\frac{2\rho\Omega U}{[P]/[L]} = \frac{2\rho\Omega[U]}{\rho g[H]/[L]} = \frac{Fr}{Ro} \approx 5 \times 10^{-8}$$

which is negligible too. Hence, the Coriolis term and the inertial terms can be neglected. This gives us the *Stokes equation.*

$$-\nabla p + \eta \Delta \mathbf{v} + (\nabla \mathbf{v} + \nabla \mathbf{v}^T) \cdot \nabla \eta + \rho \mathbf{g} = 0$$

B Dimensional Analysis for Temperature Evolution Equation

The equation for temperature evolution is given by,

$$\rho c \frac{DT}{Dt} = \nabla \cdot (\kappa \nabla T) + 4\eta d_e^2$$

For this analysis, we use the scales and dimensionless parameters as defined in Appendix A. The ratio of the horizontal to vertical heat conduction is given by,

$$\frac{\frac{\partial}{\partial x}\left(\kappa\frac{\partial T}{\partial x}\right)}{\frac{\partial}{\partial z}\left(\kappa\frac{\partial T}{\partial z}\right)}, \frac{\frac{\partial}{\partial y}\left(\kappa\frac{\partial T}{\partial y}\right)}{\frac{\partial}{\partial z}\left(\kappa\frac{\partial T}{\partial z}\right)} = \frac{\kappa\frac{[\Delta T]}{[L]^2}}{\kappa\frac{\Delta T}{[H]^2}} = \frac{[H]^2}{[L]^2} = \epsilon^2 = 10^{-6}$$

Thus the horizontal conduction terms can be neglected and the equation reduces to,

$$\rho c\frac{DT}{Dt} = \frac{\partial}{\partial z}\left(\kappa\frac{\partial T}{\partial z}\right) + 4\eta d_e^2$$

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