The next result is stated without proof.

Theorem 1.172. *Let X be a continuous RV with p.d.f. f_X and support S_X. Suppose $\{x \in \mathbb{R} : f_X(x) > 0\} = \cup_{i=1}^{k}(a_i, b_i)$ and f_X is continuous on each (a_i, b_i). We assume that the intervals (a_i, b_i) are pairwise disjoint.*

Let $h : \mathbb{R} \to \mathbb{R}$ be a function such that on each (a_i, b_i), $h : (a_i, b_i) \to \mathbb{R}$ is strictly monotone and continuously differentiable with inverse function h_i^{-1} for $i = 1, \ldots, k$.

Then $Y = h(X)$ is a continuous RV with support $S_Y = \cup_{i=1}^{k}[c_i, d_i]$, where $c_i = \min\{h(a_i), h(b_i)\}$ and $d_i = \max\{h(a_i), h(b_i)\}$. The p.d.f. is given by

$$f_Y(y) = \sum_{i=1}^{k} f_X\left(h_i^{-1}(y)\right)\left|\frac{d}{dy}h_i^{-1}(y)\right| 1_{(c_i,d_i)}(y), y \in \mathbb{R}$$

where $1_{(c_i,d_i)}(y) = 1$ if $y \in (c_i, d_i)$ and 0 otherwise.

Note 1.173. In Theorem 1.172, the function h may be strictly monotone increasing in some (a_i, b_i) and strictly monotone decreasing in other intervals. Moreover, this monotonicity may be verified by looking at the sign of h'. If $h'(x) > 0, \forall x \in (a_i, b_i)$, then h is strictly monotone increasing on (a_i, b_i). If $h'(x) < 0, \forall x \in (a_i, b_i)$, then h is strictly monotone decreasing on (a_i, b_i).

Example 1.174. Let X be a continuous RV with p.d.f.

$$f_X(x) = \begin{cases} e^{-x}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}$$

and consider $Y = X^2$. Here, $S_X = [0, \infty)$ and the function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) := x^2, \forall x \in \mathbb{R}$ is continuous differentiable on $(0, \infty)$. Moreover, $h'(x) = 2x > 0, \forall x \in (0, \infty)$ and hence h is strictly monotone increasing on $(0, \infty)$. The inverse function is given by $h^{-1}(y) = \sqrt{y}, \forall y \in (0, \infty)$. The p.d.f. f_Y is given by

$$f_Y(y) = \begin{cases} \frac{e^{-\sqrt{y}}}{2\sqrt{y}}, & \text{if } y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

The DF F_Y can now be computed from the p.d.f. f_Y by standard techniques.

Example 1.175. Let X be a continuous RV with p.d.f.

$$f_X(x) = \begin{cases} \frac{|x|}{2}, & \text{if } -1 < x < 1 \\ \frac{x}{3}, & \text{if } 1 \leq x < 2 \\ 0, & \text{otherwise} \end{cases}$$

and consider $Y = X^2$.

Observe that $\{x \in \mathbb{R} : f_X(x) > 0\} = (-1, 0) \cup (0, 2)$. Now, $h(x) = x^2$ is strictly decreasing on $(-1, 0)$ with inverse function $h_1^{-1}(t) = -\sqrt{t}$; and $h(x) = x^2$ is strictly increasing on $(0, 2)$ with inverse function $h_2^{-1}(t) = \sqrt{t}$. Note that $h((-1, 0)) = (0, 1)$ and $h((0, 2)) = (0, 4)$. Then, $Y = X^2$ has p.d.f. given by

$$\begin{aligned} f_Y(y) &= f_X(-\sqrt{y}) \left| \frac{d}{dy}(-\sqrt{y}) \right| 1_{(0,1)}(y) + f_X(\sqrt{y}) \left| \frac{d}{dy}(\sqrt{y}) \right| 1_{(0,4)}(y) \\ &= \begin{cases} \frac{1}{2}, & \text{if } 0 < y < 1 \\ \frac{1}{6}, & \text{if } 1 < y < 4 \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

We can compute the DF of Y and verify that this matches with our earlier computation in Example 1.162.

Let X be a discrete (or continuous) RV defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with DF F_X, support S_X and p.m.f. (or p.d.f.) f_X.

Definition 1.176 (Expectation/Expected value/Mean of the RV X)**.** The Expectation/Expected value/Mean of the RV X, denoted by $\mathbb{E}X$, is defined as the quantity

$$\mathbb{E}[X] := \begin{cases} \sum_{x \in S_X} x f_X(x), & \text{if } \sum_{x \in S_X} |x| f_X(x) < \infty \text{ for discrete } X, \\ \int_{-\infty}^{\infty} x f_X(x)\, dx, & \text{if } \int_{-\infty}^{\infty} |x| f_X(x)\, dx < \infty \text{ for continuous } X. \end{cases}$$

Remark 1.177. If the sum or the integral above converges absolutely, we say that the expectation $\mathbb{E}X$ exists or equivalently, $\mathbb{E}X$ is finite. Otherwise, we shall say that the expectation $\mathbb{E}X$ does not exist.

Note 1.178. Note that it is possible to define the expectation $\mathbb{E}X$ through the law/distribution $\mathbb{P}_X$ of X. However, this is beyond the scope of this course.

Example 1.179. Fix $c \in \mathbb{R}$. Let X be a discrete RV with p.m.f.

$$f_X(x) = \mathbb{P}(X = x) = \begin{cases} 1, & \text{if } x = c \\ 0, & \text{otherwise.} \end{cases}$$

Such RVs are called constant/degenerate RVs. Here, the support is a singleton set $S_X = \{c\}$ and $\sum_{x \in S_X} |x| f_X(x) = |c| < \infty$ and hence $\mathbb{E}X = \sum_{x \in S_X} x f_X(x) = c$.

Example 1.180. Let X be a discrete RV with p.m.f.

$$f_X(x) := \begin{cases} \frac{1}{6}, \forall x \in \{1, 2, 3, 4, 5, 6\} \\ 0, & \text{otherwise.} \end{cases}$$

Here, the support is $S_X = \{1, 2, 3, 4, 5, 6\}$, a finite set with all elements positive and hence $\sum_{x \in S_X} |x| f_X(x) = \sum_{x \in S_X} x f_X(x)$ is finite and

$$\mathbb{E}X = \sum_{x \in S_X} x f_X(x) = \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) = \frac{7}{2}.$$

Example 1.181. Let X be a discrete RV with p.m.f.

$$f_X(x) := \begin{cases} \frac{1}{2^x}, \forall x \in \{1, 2, 3, \cdots\} \\ 0, & \text{otherwise.} \end{cases}$$

Here, the support is $S_X = \{1, 2, 3, \cdots\}$, the set of natural numbers. To check the existence of $\mathbb{E}X$, we need to check the convergence of the series $\sum_{x \in S_X} |x| f_X(x) = \sum_{x=1}^{\infty} x \frac{1}{2^x}$. Now, the x-th term is $\frac{x}{2^x}$ and

$$\lim_{x \to \infty} \frac{\frac{x+1}{2^{x+1}}}{\frac{x}{2^x}} = \frac{1}{2} < 1.$$

By ratio test, we have the required convergence and the existence of $\mathbb{E}X$ follows.
Observe that

$$\mathbb{E}X = \sum_{x=1}^{\infty} x\frac{1}{2^x} = \frac{1}{2} + \sum_{x=2}^{\infty} x\frac{1}{2^x} = \frac{1}{2} + \sum_{x=1}^{\infty}(x+1)\frac{1}{2^{x+1}} = \frac{1}{2} + \frac{1}{2}\sum_{x=1}^{\infty} x\frac{1}{2^x} + \frac{1}{2} = 1 + \frac{1}{2}\mathbb{E}X,$$

which gives $\mathbb{E}X = 2$.

Note 1.182. It is fact that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Example 1.183. Let X be a discrete RV with p.m.f.

$$f_X(x) := \begin{cases} \frac{3}{\pi^2 x^2}, \forall x \in \{\pm 1, \pm 2, \pm 3, \cdots\} \\ 0, \text{ otherwise.} \end{cases}$$

Here, the support is $S_X = \{\pm 1, \pm 2, \pm 3, \cdots\}$. To check the existence of $\mathbb{E}X$, we need to check the convergence of the series $\sum_{x \in S_X} |x| f_X(x) = 2\sum_{n=1}^{\infty} n\frac{3}{\pi^2 n^2} = \frac{6}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n}$. However, this series diverges and hence $\mathbb{E}X$ does not exist.

Example 1.184. Let X be a continuous RV with the p.d.f.

$$f_X(x) = \begin{cases} 1, \text{ if } 0 < x < 1, \\ 0, \text{ otherwise.} \end{cases}$$

To check the existence of $\mathbb{E}X$, we need to check the existence of $\int_{-\infty}^{\infty} |x| f_X(x)\,dx$. Now,

$$\int_{-\infty}^{\infty} |x| f_X(x)\,dx = \int_0^1 x\,dx = \frac{1}{2}$$

and hence $\mathbb{E}X = \frac{1}{2}$.

Example 1.185. Let X be a continuous RV with the p.d.f.

$$f_X(x) = \frac{1}{2}e^{-|x|}, \forall x \in \mathbb{R}.$$

To check the existence of $\mathbb{E}X$, we need to check the existence of $\int_{-\infty}^{\infty} |x| f_X(x)\,dx$. Now,

$$\int_{-\infty}^{\infty} |x| f_X(x)\,dx = \int_{-\infty}^{\infty} |x|\frac{1}{2}e^{-|x|}\,dx = \int_0^{\infty} xe^{-x}\,dx = 1 < \infty$$

and hence $\mathbb{E}X$ exists and

$$\mathbb{E}X = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_{-\infty}^{\infty} x \frac{1}{2} e^{-|x|}\,dx = 0.$$

Example 1.186. Let X be a continuous RV with the p.d.f.

$$f_X(x) = \frac{1}{\pi}\frac{1}{1+x^2}, \forall x \in \mathbb{R}.$$

To check the existence of $\mathbb{E}X$, we need to check the existence of $\int_{-\infty}^{\infty} |x| f_X(x)\,dx$. Now,

$$\int_{-\infty}^{\infty} |x| f_X(x)\,dx = \int_{-\infty}^{\infty} |x| \frac{1}{\pi}\frac{1}{1+x^2}\,dx = \frac{2}{\pi}\int_0^{\infty} \frac{x}{1+x^2}\,dx = \infty$$

and hence $\mathbb{E}X$ does not exist.

Proposition 1.187. *Let X be a discrete or continuous RV such that $\mathbb{E}X$ exists. Then,*

$$\mathbb{E}X = \int_0^{\infty} \mathbb{P}(X > x)\,dx - \int_{-\infty}^{0} \mathbb{P}(X < x)\,dx.$$

Proof. We prove the result when X is continuous. The case for discrete X can be proved in a similar manner. Observe that

$$\begin{aligned}
\mathbb{E}X &= \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_{-\infty}^{0} x f_X(x)\,dx + \int_0^{\infty} x f_X(x)\,dx \\
&= -\int_{x=-\infty}^{0}\int_{y=x}^{0} f_X(x)\,dydx + \int_{x=0}^{\infty}\int_{y=0}^{x} f_X(x)\,dydx \\
&= -\int_{y=-\infty}^{0}\int_{x=-\infty}^{y} f_X(x)\,dxdy + \int_{y=0}^{\infty}\int_{x=y}^{\infty} f_X(x)\,dxdy \\
&= \int_0^{\infty} \mathbb{P}(X > y)\,dy - \int_{-\infty}^{0} \mathbb{P}(X < y)\,dy.
\end{aligned}$$

This completes the proof. □

Remark 1.188. (a) Suppose X is discrete or continuous with $\mathbb{P}(X \geq 0) = 1$. Then $\mathbb{P}(X \leq x) = 0, \forall x < 0$ and hence $\mathbb{E}X = \int_0^{\infty} \mathbb{P}(X > x)\,dx$.

(b) Suppose that X is discrete with $\mathbb{P}(X \in \{0, 1, 2, \cdots\}) = 1$. Then $\mathbb{P}(X > x) = \mathbb{P}(X \geq n+1), \forall x \in [n, n+1), n \in \{0, 1, 2, \cdots\}$ and hence by part (a),

$$\mathbb{E}X = \int_0^\infty \mathbb{P}(X > x)\, dx = \sum_{n=0}^\infty \mathbb{P}(X \geq n+1) = \sum_{n=1}^\infty \mathbb{P}(X \geq n).$$

Note 1.189 (Expectation of functions of RVs)**.** Given a function $h : \mathbb{R} \to \mathbb{R}$ and an RV X, we have already discussed about the distribution of $Y = h(X)$. If the p.m.f./p.d.f. f_Y is known, we can then consider the existence of $\mathbb{E}Y$ through f_Y, as per Definition 1.176. However, to do this, we first need to compute f_Y from X and then check the relevant existence. In what follows, we discuss the computation of $\mathbb{E}Y = \mathbb{E}h(X)$ directly from X, using the p.m.f./p.d.f. f_X.

Proposition 1.190. *(a) Let X be a discrete RV with p.m.f. f_X and support S_X and let $h : \mathbb{R} \to \mathbb{R}$ be a function. Consider the discrete RV $Y := h(X)$. Then $\mathbb{E}Y$ exists provided $\sum_{x \in S_X} |h(x)| f_X(x) < \infty$ and in this case,*

$$\mathbb{E}Y = \mathbb{E}h(X) = \sum_{x \in S_X} h(x) f_X(x).$$

(b) Let X be a continuous RV with p.d.f. f_X and support S_X and let $h : \mathbb{R} \to \mathbb{R}$ be a function. Consider the RV $Y := h(X)$. Then $\mathbb{E}Y$ exists provided $\int_{-\infty}^\infty |h(x)| f_X(x)\, dx < \infty$ and in this case,

$$\mathbb{E}Y = \mathbb{E}h(X) = \int_{-\infty}^\infty h(x) f_X(x)\, dx.$$

Proof. We consider the proof for the case when X is discrete. The other case can be proved by similar arguments.

By Theorem 1.164, $Y = h(X)$ is discrete with support $S_Y = h(S_X)$. Now,

$$\sum_{y \in S_Y} |y| f_Y(y) = \sum_{y \in S_Y} |y| \sum_{\{x \in S_X : h(x) = y\}} f_X(x) = \sum_{y \in S_Y} \sum_{\{x \in S_X : h(x) = y\}} |h(x)| f_X(x) = \sum_{x \in S_X} |h(x)| f_X(x).$$

Therefore, $\mathbb{E}Y$ exists provided $\sum_{x \in S_X} |h(x)| f_X(x) < \infty$ and in this case,

$$\mathbb{E}Y = \sum_{y \in S_Y} y f_Y(y) = \sum_{y \in S_Y} y \sum_{\{x \in S_X : h(x) = y\}} f_X(x) = \sum_{x \in S_X} h(x) f_X(x).$$

This completes the proof. $\square$