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EE25BTECH11031 - Sai Sreevallabh

Question:

The perpendicular bisector of the line segment joining the points $\mathbf{A}(1,5)$ and $\mathbf{B}(4,6)$ cuts the y-axis at ________.

Solution:

Given points are

$$\mathbf{A} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 4 \\ 6 \end{pmatrix} \tag{0.1}$$

Let the perpendicular bisector of line segment $\mathbf{B} - \mathbf{A}$ intersect the y-axis at point $\mathbf{P}$.

$$\mathbf{P} = \begin{pmatrix} 0 \\ y \end{pmatrix} \tag{0.2}$$

All points on the perpendicular bisector of line segment are equidistant from the end points.
Hence, $\mathbf{P}$ is equidistant from both $\mathbf{A}$ and $\mathbf{B}$. So, the norms of vectors $\mathbf{P} - \mathbf{B}$ and $\mathbf{P} - \mathbf{A}$ are equal.

$$||\mathbf{P} - \mathbf{B}|| = ||\mathbf{P} - \mathbf{A}|| \tag{0.3}$$

$$\implies ||\mathbf{P} - \mathbf{B}||^2 = ||\mathbf{P} - \mathbf{A}||^2 \tag{0.4}$$

$$\implies ||\mathbf{P}||^2 - 2\mathbf{P}^\top \mathbf{A} + \mathbf{A}^2 = ||\mathbf{P}||^2 - 2\mathbf{P}^\top \mathbf{B} + \mathbf{B}^2 \tag{0.5}$$

Simplification of the above results in:

$$(\mathbf{A} - \mathbf{B})^\top \mathbf{P} = \frac{||A||^2 - ||B||^2}{2} \tag{0.6}$$

$$\because \mathbf{P} = y\mathbf{e_2} \tag{0.7}$$

$$y = \frac{||A||^2 - ||B||^2}{2(\mathbf{A} - \mathbf{B})^\top \mathbf{e_2}} \tag{0.8}$$

Substituting the values of $\mathbf{A}$ and $\mathbf{B}$:

$$y = \frac{\left\|\begin{pmatrix}1\\5\end{pmatrix}\right\|^2 - \left\|\begin{pmatrix}4\\6\end{pmatrix}\right\|^2}{2\begin{pmatrix}-3 & -1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}} \tag{0.9}$$

$$y = 13 \tag{0.10}$$

$\therefore$ The point where the perpendicular bisector of $\mathbf{B}-\mathbf{A}$ intersects the y-axis is the point $\mathbf{P} = \begin{pmatrix}0\\13\end{pmatrix}$.