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Question : Let Q be the cube with the set of vertices

$$\{(x_1, x_2, x_3) \mid x_1, x_2, x_3 \in \{0, 1\}\} \subset \mathbb{R}^3$$

Let F be the set of all twelve lines containing the diagonals of the six faces of the cube Q. Let S be the set of all four lines containing the main diagonals of the cube Q; for instance, the line passing through the vertices $(0, 0, 0)$ and $(1, 1, 1)$ is in S.

For lines λ_1 and λ_2, let $d(\lambda_1, \lambda_2)$ denote the shortest distance between them. Then the maximum value of $d(\lambda_1, \lambda_2)$, as λ_1 varies over F and λ_2 varies over S, is

Solution : the diagonals of the cube can be written as

Line	Equation
Body diagonal	$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + k_1 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$
Face diagonal	$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Table : diagonals

$$\mathbf{x} = \mathbf{A} + k_1 \mathbf{m}_1, \quad \mathbf{A} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \ \mathbf{m}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \tag{1}$$

$$\mathbf{x} = \mathbf{B} + k_2 \mathbf{m}_2, \quad \mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{m}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \tag{2}$$

$$\mathbf{M} = \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ -1 & 1 \end{pmatrix} \tag{3}$$

$$\begin{pmatrix} \mathbf{B} - \mathbf{A} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \tag{4}$$

$$\begin{pmatrix} \mathbf{M} & \mathbf{B} - \mathbf{A} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & 1 & -1 \end{pmatrix} \tag{5}$$

Performing row operations:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & 1 & -1 \end{pmatrix} \xleftrightarrow{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ -1 & 1 & -1 \end{pmatrix} \xleftrightarrow{R_3 \to R_3 + R_1} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 2 & -1 \end{pmatrix} \xleftrightarrow{R_3 \to R_3 + 2R_2} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{6}$$

Clearly, the rank of this matrix is 3, and therefore, the lines are skew.

From the least squares formulation,

$$\mathbf{M}^\top\mathbf{M}\begin{pmatrix} k_1 \\ -k_2 \end{pmatrix} = \mathbf{M}^\top\left(\mathbf{B} - \mathbf{A}\right) \tag{7}$$

Thus,

$$\mathbf{M}^\top\mathbf{M} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \tag{8}$$

$$\mathbf{M}^\top(\mathbf{B} - \mathbf{A}) = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{9}$$

Therefore,

$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} k_1 \\ -k_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{10}$$

Solving,

$$\begin{pmatrix} k_1 \\ -k_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{2} \end{pmatrix} \tag{11}$$

Hence the closest points are

$$\mathbf{P} = \mathbf{A} + k_1\mathbf{m}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} \tag{12}$$

$$\mathbf{Q} = \mathbf{B} + k_2\mathbf{m}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} \tag{13}$$

The shortest distance is

$$\|\mathbf{P} - \mathbf{Q}\| = \left\|\begin{pmatrix} -\frac{1}{6} \\ \frac{1}{3} \\ \frac{1}{6} \end{pmatrix}\right\| = \frac{1}{\sqrt{6}} \tag{14}$$

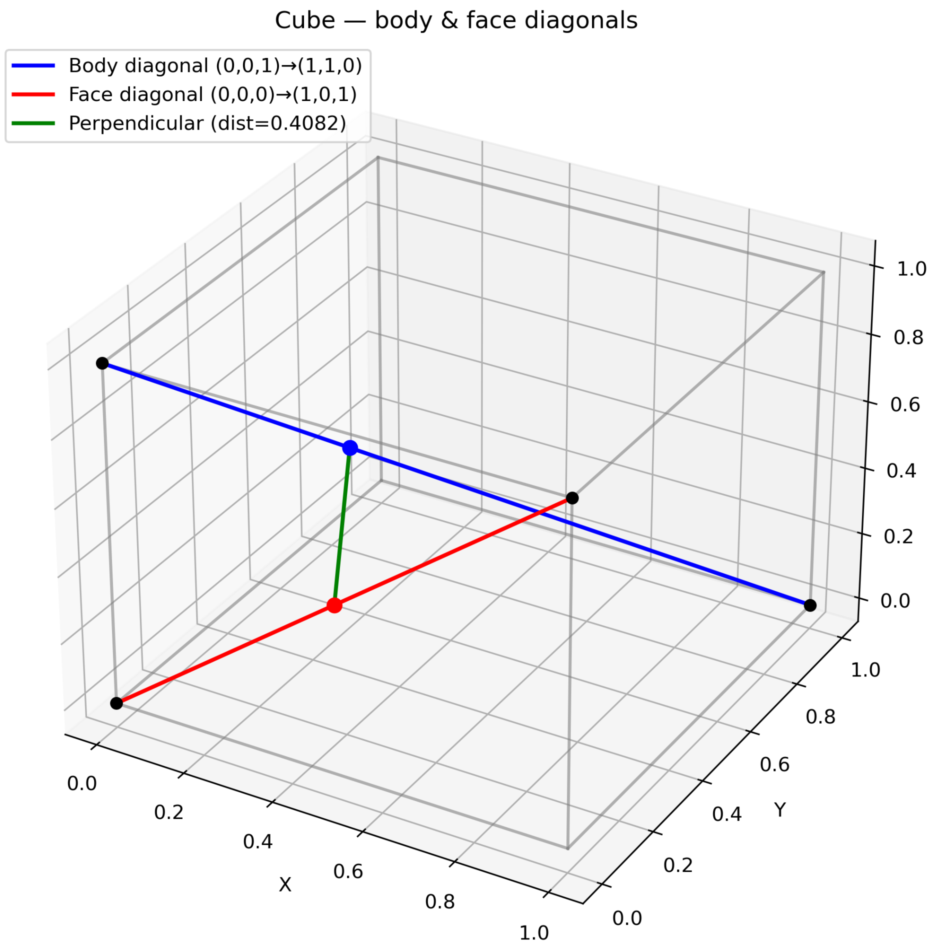

Fig : diagonals