CS5110: Complexity Theory

Shreyas Havaldar
CS18BTECH11042

October 17, 2020

1 Exercise Set 10

1.1 To show that $\log n$ is space constructible

A function f : $N \longrightarrow N$, where $f(n) \geq \log n$, is called space constructible if there exists a det-TM running in $O(f(n))$ space, the function that that when given the string 1^n on the input tape(assumed read only) writes the binary representation of f(n) and halts.

Note: To show that function f(n) = $\log n$ is space constructible, I have assumed that the input is persistent and we can use multiple tapes. Our input, provided on the input tape can be read at any step during the computation by the det-TM. Our input is provided in unary notation, as a sequence of n ones.

Note: I also assume we can allow a certain loss of precision, that is I can round off the value of fractional results of $\log n$ values to the next lowest integer. And assuming $\log n$ to the base two.

So I create a counter tape that stores integer values in binary notation. It is initialized to 0.

So the det-TM basically aims to count the number of ones on the input tape binary. This is achieved by incrementing the value on our counter tape as long as we see a one on the input tape. So as the TM moves its head along the input tape, for every 1 read, we increment the value on counter tape by 1, until we reach the end of the input tape.

Now on reaching the end of the input tape, our counter tape has the binary value of n stored on it. We know that the length, l of the binary representation of a number is $\lfloor \log n \rfloor + 1$ and thus, by simply traversing the counter tape, and counting the number of symbols on it, thus its length first, and then subtracting one after the entire calculation; we can calculate the value of $\log n$ as $l - 1$. Note: We are actually finding $\lfloor \log n \rfloor$ and thus approximating $\log n$ to it.

This is a simple counting operation and can be performed using a work tape, initialized to 0, that stores the value in binary notation and that increments its value on reading a symbol on the counter tape, until the end of the counter tape is reached. Thus we have the representation in binary notation of the f(n) on thus work tape in finite time by using deterministic choices and thus have shown that $\log n$ is space constructible.

Note: Both our counter tape and work tape are $O(\log n)$ as our counter tape simply stores the binary representation of n, and our work tape stores the binary representation of the length of the binary representation of n, and thus both always take less than $\log n$ bits for representation.

So we have shown that $\log n$ is space constructible.

Hence Proved!