Assignment 3

Q: A round robin tournament of $2n$ teams lasted for $(2n-1)$ days, as follows: On every day, every team played one game against another team, with one team winning and one team losing in each of the n games. Over the course of the tournament, each team played every other team exactly once. Can one necessarily choose one winning team from each day without choosing any team more than once?

Ans) Let us create ~~a~~ bipartite graph with bi-partition $\{D, T\}$. Where D is the set of all days, thus $D = \{1, 2 \ldots 2n-1\}$ T is the set of all teams, thus $T = \{1, 2, \ldots 2n\}$.

Thus $|D| = 2n-1$ and $|T| = 2n$

The graph G, has an edge between a team $t \in T$ and a day $d \in D$, iff t won its match on day d of the tournament.

As each day, there are bound be to n matches and each match must have a winning team, there are n edges from every $d \in D$.

$\therefore$ $\forall\ d \in D$, degree of $d = n$.

Note: There are no edges between any i, j such that both $i, j \in D$ or both $i, j \in T$ by our construction.

We need to show that G has a matching that matches all $2n-1$ vertices of D, that is a matching of size $2n-1$ in the Graph $G = \{D \cup T\}$

This is sufficient as if we have a match $\forall\ p \in D, \exists t \in T$, between p and t, then we can assign team t to day p, and we would have a unique winning team for every day of the possible $2n-1$ days. Thus we would not be forced to pick any team more than once.

Using Hall's theorem (proved in assignment 2), we know such a matching exists iff, G satisfies Hall's condition, which in this case would be:

$$\forall S \subseteq D, |N_G(S)| \geq |S|.$$

$\Rightarrow$

Let there exist some set $S \subseteq D$, for which $|N_G(S)| < |S|$. In such a case, for some team $t \in T$, that team could not have won any of its matches in days d, where $\forall\ d$, $d \in S$. Which is equivalent to saying $t \in T - N_G(S)$.

As t did not win even a single match from among the $|S|$ days it played on, it must have necessarily lost all its $|S|$ matches played on all days d, such that $d \in S$.

As every match in the tournament is unique, as each team played every other team only once, there must necessarily be $|S|$ teams such that t lost all its matches on days $\in S$ to those $|S|$ teams.

$\therefore$ These $|S|$ teams won atleast one match on the set of all days $\in S$ and each team won its match on an unique day, the day of its match with t.
Thus there are atleast $|S|$ teams that are inthe neighbourhood of S, as for every day $\in S$, we would have atleast one edge to one of these $|S|$ teams.

$\therefore$ $|N_G(S)| \geq |S|$ which is a contradiction to the assumption that there exists some $S \subseteq D$, such that $|N_G(S)| < |S|$

$\therefore$ $\forall S \subseteq D, |N_G(S)| \geq |S|$ and Hall's condition is satisfied. Thus, the Hall's theorem holds and we have a matching for all $2n-1$ days in D and thus we can indeed choose an unique winning team for every day!