CS5110: Complexity Theory

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1 Exercise Set 16, Lecture 25

1.1 Protocol for AVG

Without loss of generality let Alice calculate the sum of the elements in the set S denoted by SUM_S. The maximum possible value of this SUM_S is $1 + 2 + \ldots + n$, that is when S contains all the elements of the set it derives its elements from. $Max(SUM_S) = \frac{n(n+1)}{2}$, therefore the maximum number of bits required to represent SUM_S is $\log\left(\frac{n(n+1)}{2}\right)$ Alice also sends to Bob the number of elements present in the set S as NUM_S. As S can contain maximum n elements, the maximum number of bits required to represent NUM_S is $\log n$.

Alice communicates total $\log\left(\frac{n(n+1)}{2}\right) + \log n$ bits of information as SUM_S and NUM_S respectively to Bob.

Now, Bob has already calculated the SUM_T, sum of all the elements belonging to the set T and NUM_T, number of elements belonging to the set T.

Now the mean for the multi-set $S \cup T$ the average would be calculated by using the bits communicated by Alice and his own calculations as:

$$AVG = \frac{SUM_S + SUM_T}{NUM_S + NUM_T}$$

Now note that the AVG calculated cannot be more than n as all elements belonging to both sets are atmost n, thus the AVG calculated can be represented by atmost $\log n$ bits and would be communicated in $\log n$ bits by Bob to Alice.

Thus total number of bits communicated thus in this protocol is $\log\left(\frac{n(n+1)}{2}\right) + \log n + \log n = O(\log n)$

1.2 GT_n

The protocol for finding the greater of two numbers x and y, represented in binary, is to find which of the two numbers has 1 on the left most position that they differ on. This simple protocol takes atmost $n + 1$ bits of communication using trivial method, n bits sent by Alice to Bob to convey the value of x, and then Bob by performing the comparison between x and y, send either 0 or 1, thus 1 bit is communicated to Alice as the result. Thus $D \leq n + 1$

Now for the lower bound on the complexity. We use the fooling set argument where we define:

$$S = \{(x, x) \mid x \in \{0, 1\}^n\}$$

$|S| = 2^n$, as each of the input bits for x can take 2 values. $\forall (x, x) \in S, GT_n(x, x) = 0$ Therefore for distinct x and y, we know $GT_n(x, x) = 0 = GT_n(y, y)$. But now as we've established $x \neq y$, either $x > y$ OR $y > x$. Thus either $GT_n(x, y) = 1$ OR $GT_n(y, x) = 1$ Therefore (x, x) and (y, y) cannot be in the same combinatorial rectangle as $R \subseteq X \times Y$ is a rectangle iff $(x_1, y_1) \in R$ and $(x_2, y_2) \in R \implies (x_1, y_2) \in R$

Thus each distinct element of S, that is all 2^n elements of S must belong to a different monochromatic combinatorial rectangle. Thus in the partition of $X \times Y$ into monochromatic rectangles there must exist atleast 2^n rectangles. Using the theorem from class.

Thus

$$D(GT_n) \geq \log(|S|)$$

$$D(GT_n) \geq n$$