A Primer to Probability distribution and Parameter Estimation

Mean and Variance of Random Variable

- Expected value of a random variable X is the long-run average value of repetitions of the experiment

$$\mathrm{E}[X] = x_1 p_1 + x_2 p_2 + \cdots + x_k p_k \ .$$

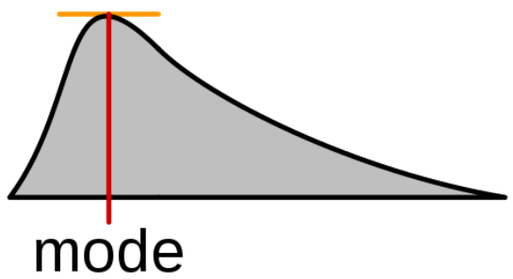
- Variance : Spread of the random variable values

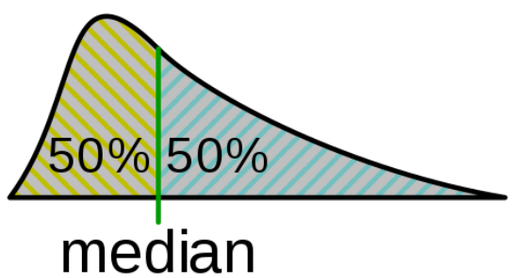
$$\mathrm{Var}(X) = E[(X-\mu)^2] = E[X^2] - (E[X])^2$$

$\sqrt{\mathrm{Var}(X)}$ is called the *standard deviation* of X.

$$W = 0 \quad \text{with probability 1}$$

$$Y = \begin{cases} -1 & \text{with probability } \frac{1}{2} \\ 1 & \text{with probability } \frac{1}{2} \end{cases} \qquad Z = \begin{cases} -100 & \text{with probability } \frac{1}{2} \\ 100 & \text{with probability } \frac{1}{2} \end{cases}$$





Common Discrete Distributions
Bernoulli and Binomial

- Let $X \in \{0, 1\}$ be a binary random variable, with probability of "success" θ, X has a Bernoulli distribution, $X \sim \text{Ber}(\theta)$

  E.g Coin toss, Rain or not

$$\text{Ber}(x|\theta) = \theta^{\mathbf{I}(x=1)}(1-\theta)^{\mathbf{I}(x=0)} \qquad \text{Ber}(x|\theta) = \begin{cases} \theta & \text{if } x = 1 \\ 1-\theta & \text{if } x = 0 \end{cases}$$

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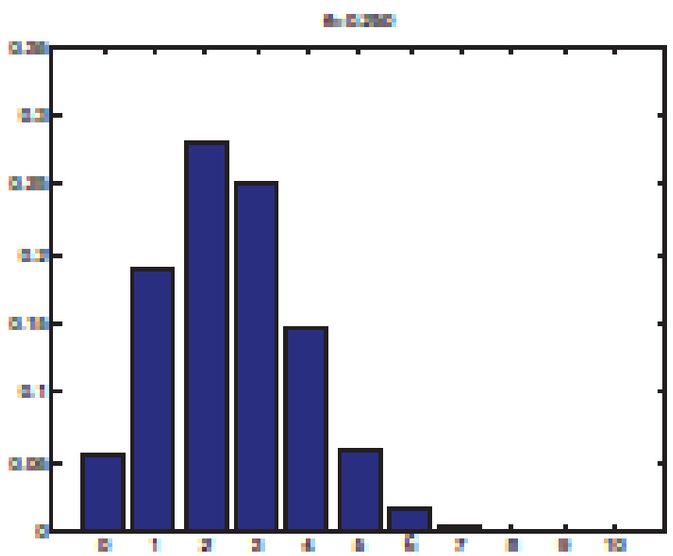
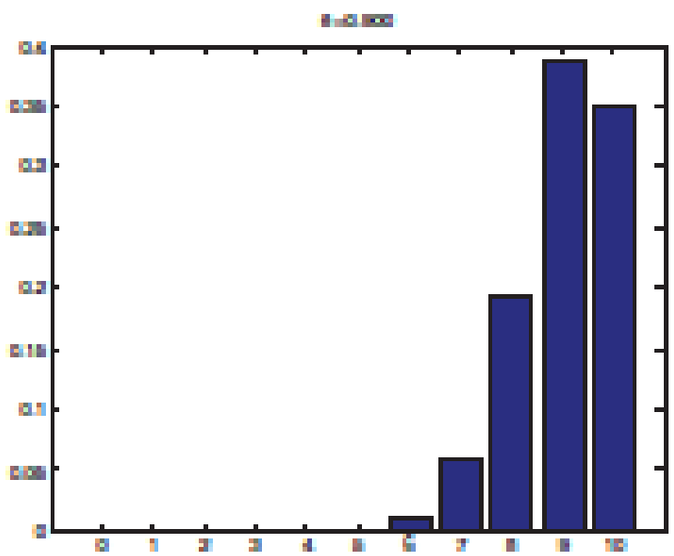
- Suppose we toss a coin n times. Let X ∈ {0, . . . , n} be the number of heads. If the probability of heads is θ , then we say X has a binomial distribution, written as X ~ Bin(n, θ) .

$$\mathrm{Bin}(k|n,\theta) \triangleq \binom{n}{k}\theta^k(1-\theta)^{n-k}$$

$$\text{mean} = \text{n}\,\theta, \quad \text{var} = n\theta(1-\theta)$$

$$\binom{n}{k} \triangleq \frac{n!}{(n-k)!k!}$$





Discrete Distributions : Multinoulli, Multinomial

- Model the outcomes of tossing a K -sided die : categorical/Multinoulli distribution,

  $x \sim \text{Cat}(\boldsymbol{\theta})$, $p(x = j|\boldsymbol{\theta}) = \theta_j$.

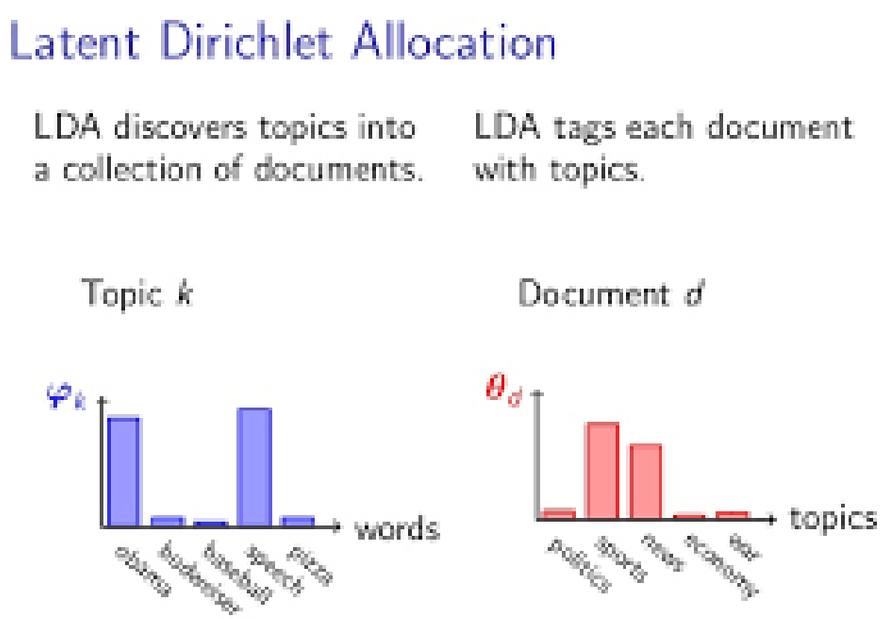
- Multinomial distribution : Models the outcome of n dice rolls, let x = (x1, . . . , xk) be a random vector, where xj number of times side j of the die occurs.

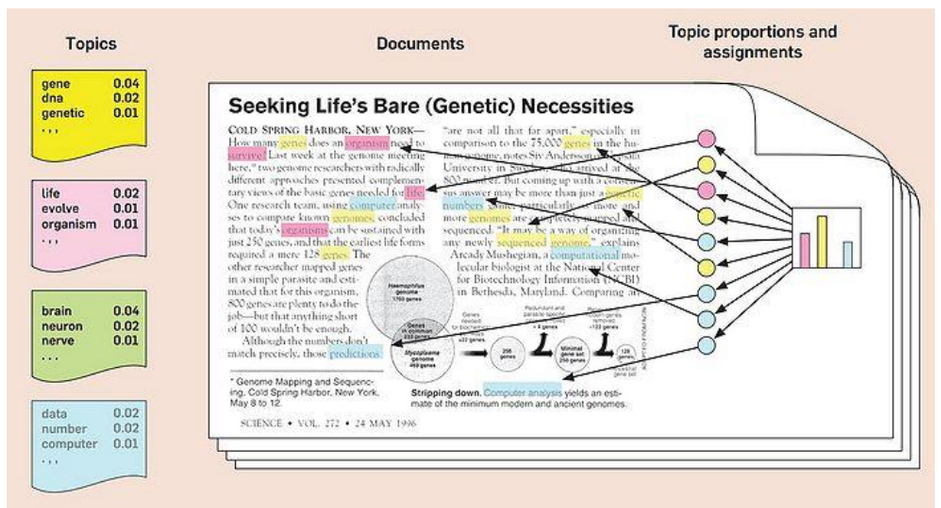
$$\text{Mu}(\mathbf{x}|n,\boldsymbol{\theta}) \triangleq \binom{n}{x_1 \ldots x_K} \prod_{j=1}^{K} \theta_j^{x_j}$$

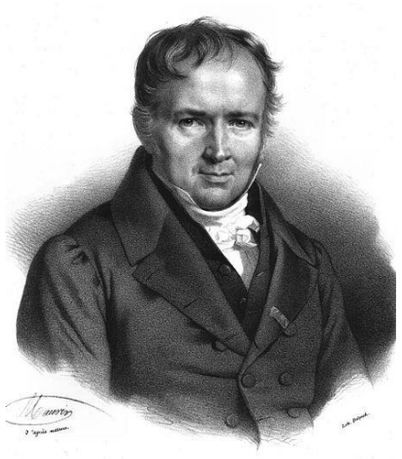
$$\text{Cat}(x|\boldsymbol{\theta}) \triangleq \text{Mu}(\mathbf{x}|1,\boldsymbol{\theta}) \qquad \text{Mu}(\mathbf{x}|1,\boldsymbol{\theta}) = \prod_{j=1}^{K} \theta_j^{\mathbb{I}(x_j=1)}$$

- Probabilistic topic model

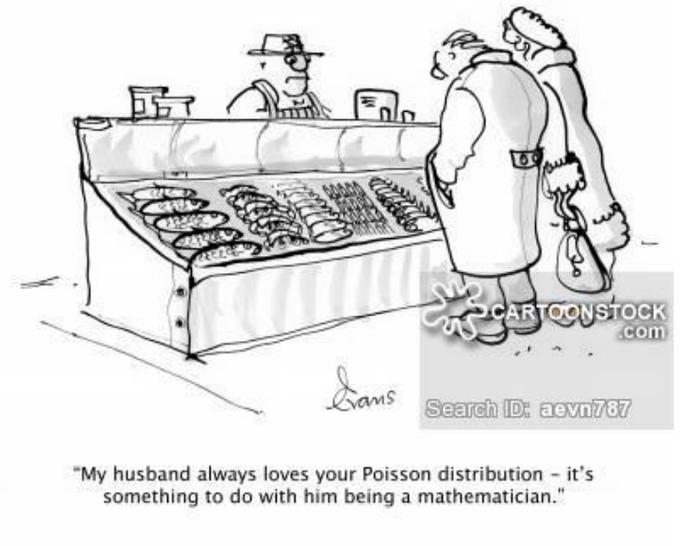
- Text classification





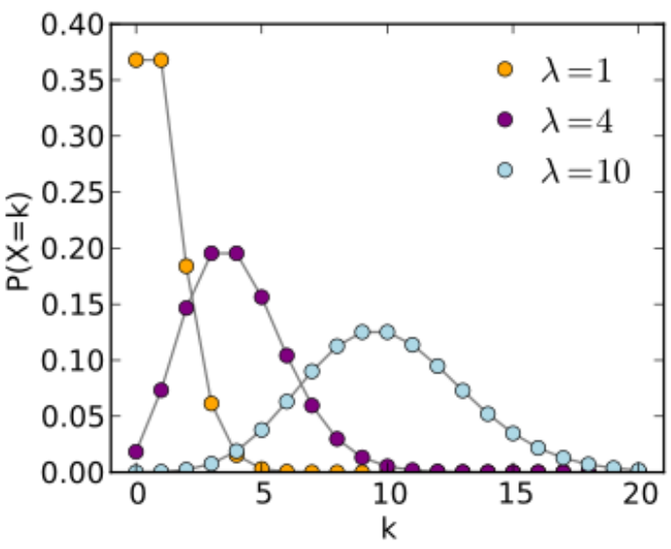


Poisson distribution



- Model number of events occurring in a fixed interval of time/space

$$P(k \text{ events in interval}) = e^{-\lambda}\frac{\lambda^k}{k!}$$



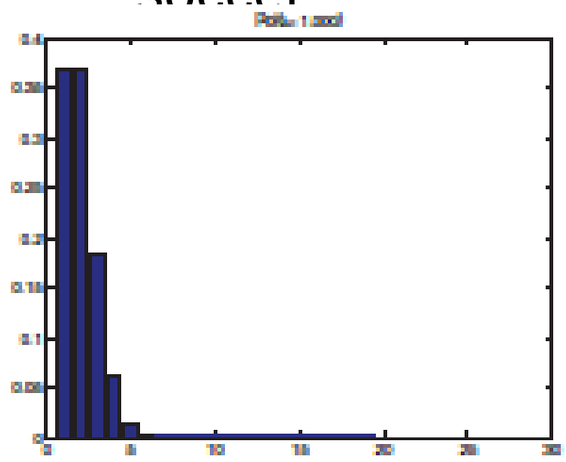
- λ is the average (mean) number of events per interval, k = 0, 1, 2, …, events occur independently, rate is a constant.
- Models rare events
  - Number of misprints on a page of a book.
  - average number of goals in a World Cup match is approximately 2.5 ; λ = 2.5.

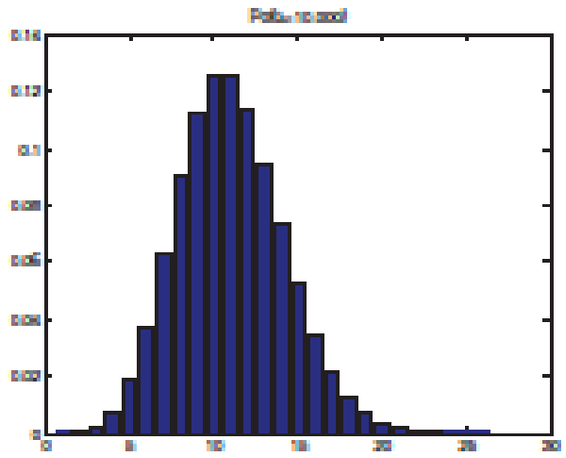
$$P(k \text{ goals in a match}) = \frac{2.5^k e^{-2.5}}{k!}$$

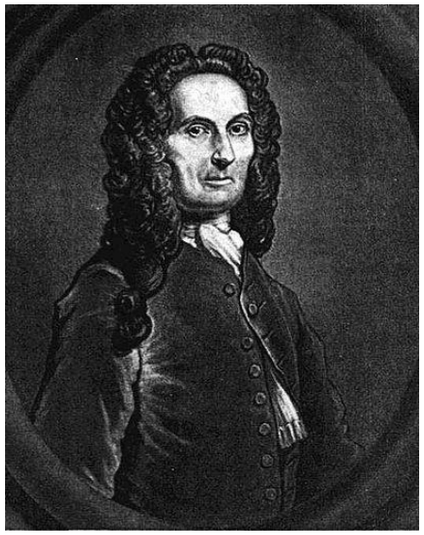
  - Number of wrong telephone numbers

that are dialed in a day.

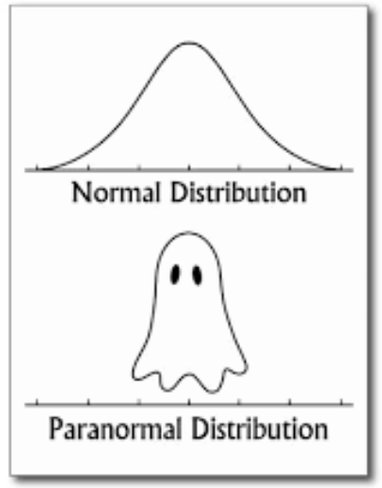
soccer







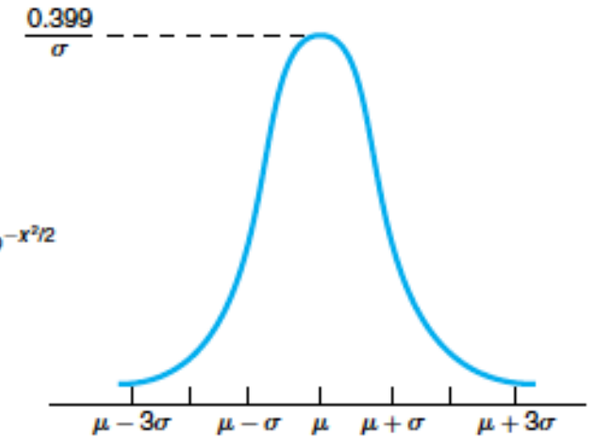
Normal/Gaussian Random Variables



- 1809 Gauss published his monograph "Theoria motus corporum coelestium in sectionibus conicis solem ambientium"
- *All distributions of frequency other than normal are 'abnormal'*- Pearson
- A random variable is said to be normally distributed

  with parameters μ and σ2, X ~ N(μ, σ2)

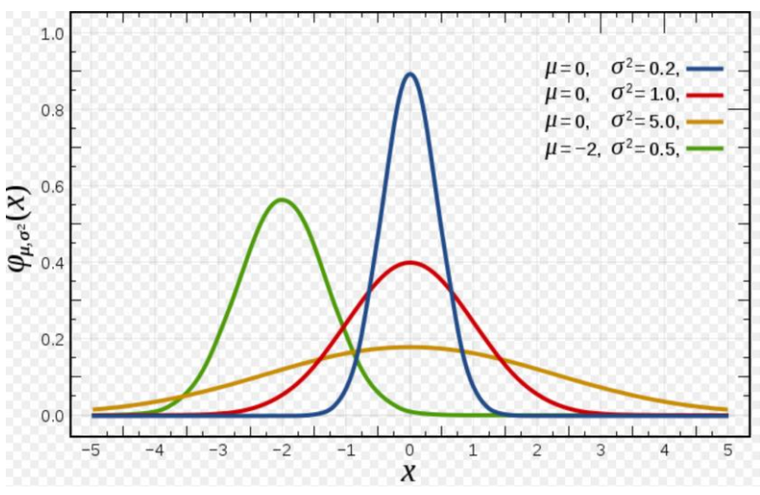
$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}, \qquad -\infty < x < \infty^*$$

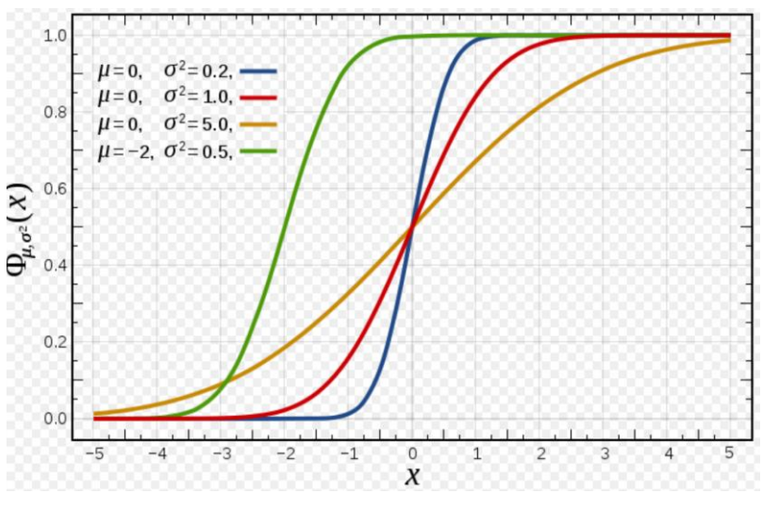


- μ = E [X] is the mean (and mode), and σ2 = var [X] is the variance.

$$\Phi(x;\mu,\sigma^2) \triangleq \int_{-\infty}^{x} \mathcal{N}(z|\mu,\sigma^2)dz$$

- CDF of the Gaussian



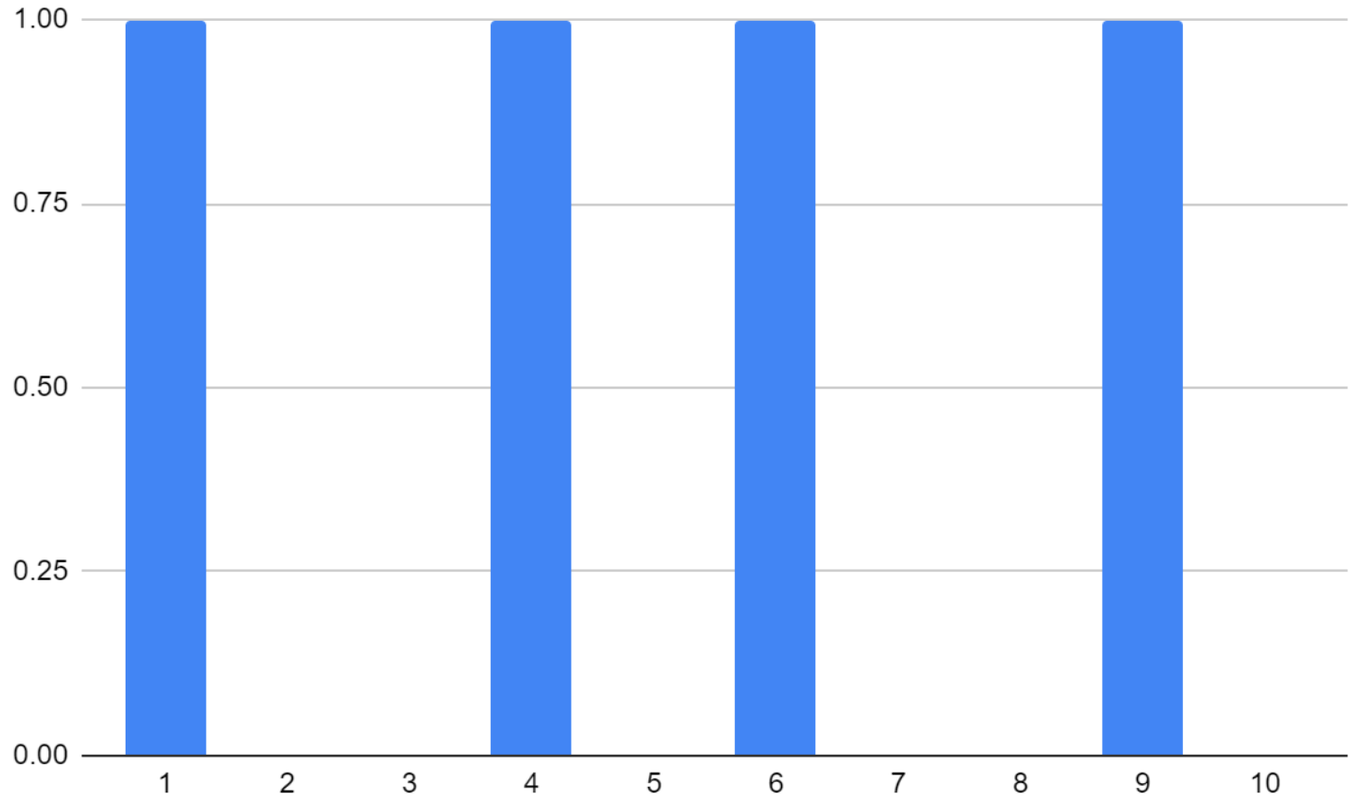


Probability Distribution Summary

- X : Discrete
  - Binary valued scalar (0/1) : Bernoulli
  - Binary valued vector (one of K): Multinoulli/categorical
  - Multivalued scalar (M of N): Binomial
  - Multivalued vector (M1, M2, … MK) : Multinomial
  - Integer valued scalar (1 to infinity) : Poisson
- X : continous, real valued
  - Interval [a,b] : Uniform, Interval [0,1] : Beta
  - non-negative (0,infinity) : Exponential, Gamma
  - real line (-infinity, infinity) : Normal, students, Laplace
  - Vector : Real valued : Gaussian ; Simplex : Dirichlet

Parameter Estimation

- Data points representing the if it has rained or not in last 10 days.
- Whats the probability that it will rain tomorrow ?
- How many days will it rain in next 5 days ?

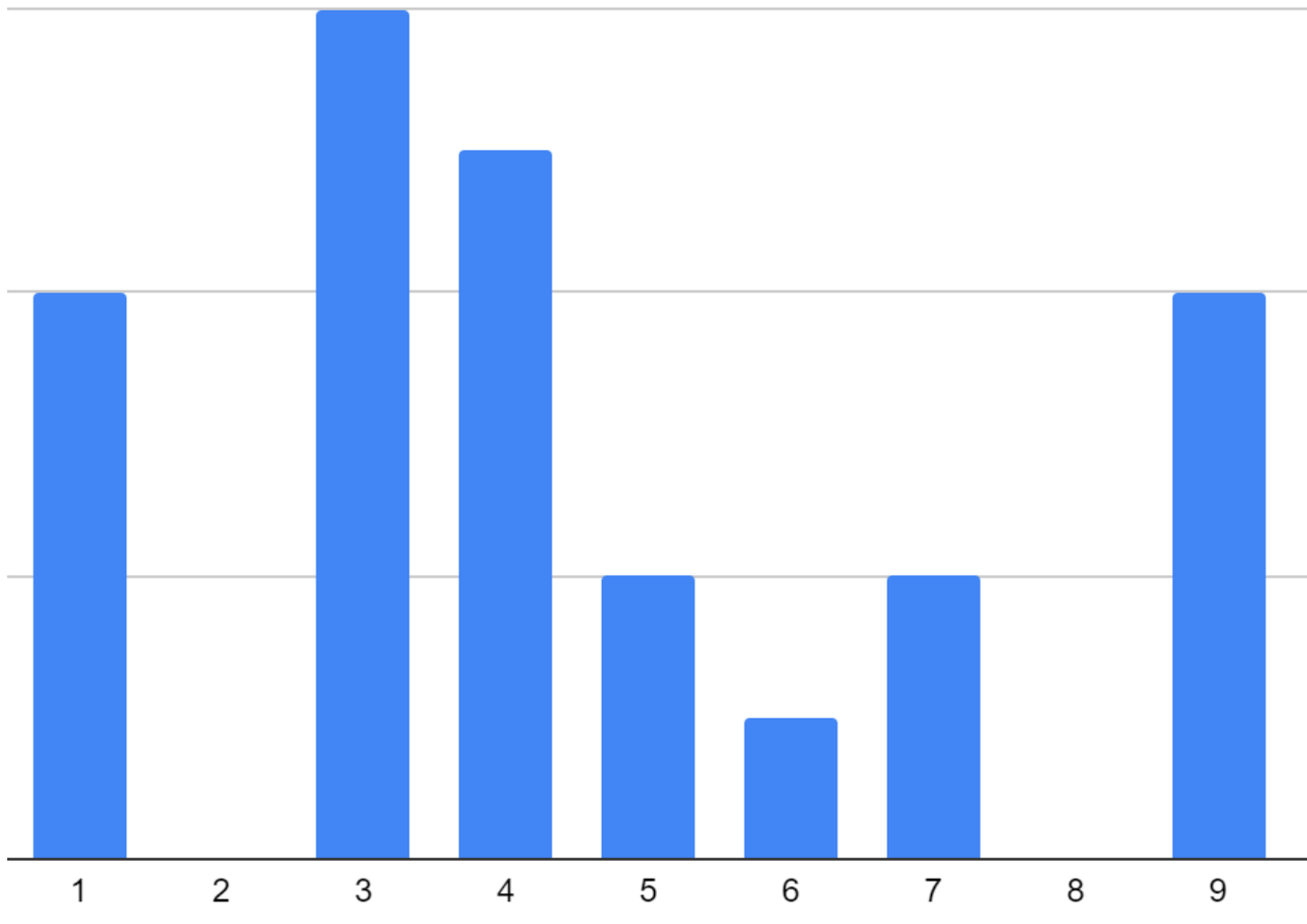


Parameter Estimation

- The number of traffic accidents in Berkeley, California, in 10 randomly chosen nonrainy days in 1998 is as follows:

- 4, 0, 6, 5, 2, 1, 2, 0, 4, 3

- Use these data to estimate the proportion of nonrainy days that had 2 or fewer accidents that year.

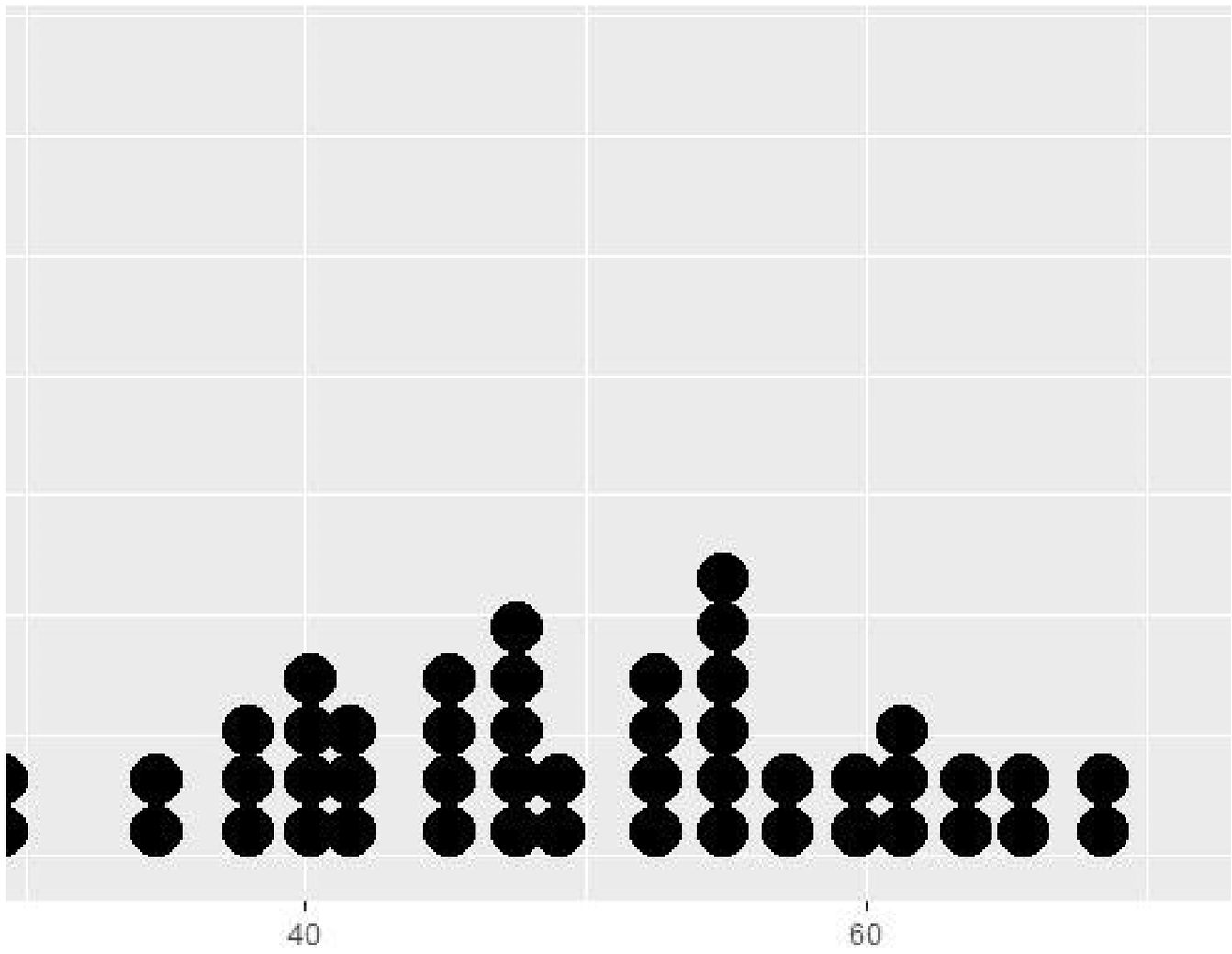


Parameter Estimation

- Data points representing the weight (in kgs) of students in a class.

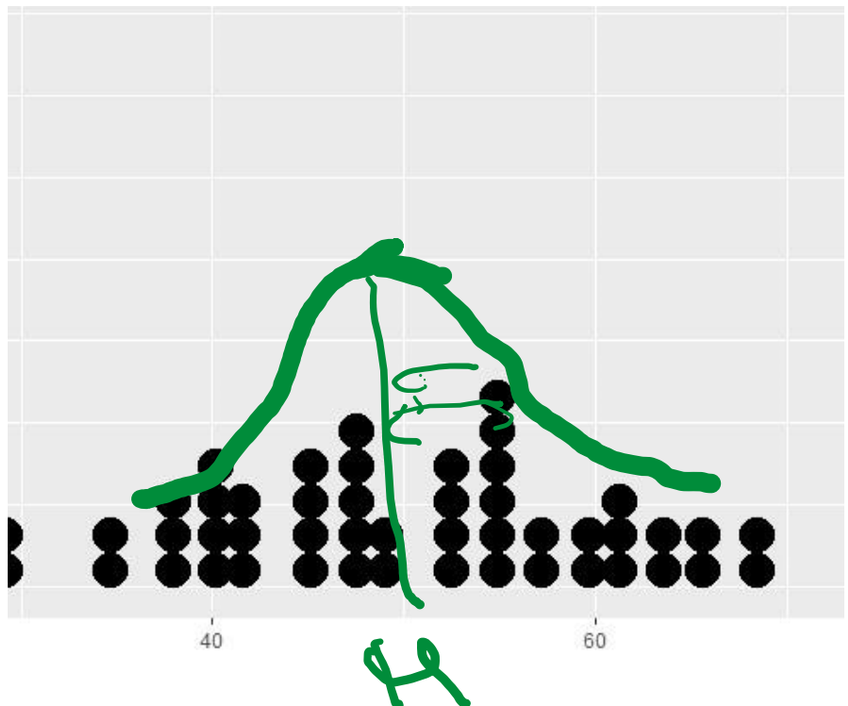
- Whats mean and std deviation of the data ?

- Whats the probability that weight > 60



Parameter Estimation

- Any statistic used to estimate the value of an unknown parameter θ is called an estimator of θ.
  - mean and variance for Normal, rate (lambda) for Poisson, etc.
- **Maximum likelihood** estimator
- MLE can be defined as a method for estimating parameters of a distribution from sample data such that the likelihood of obtaining the observed data is maximized.
- Provides optimal way to fit a distribution to the data



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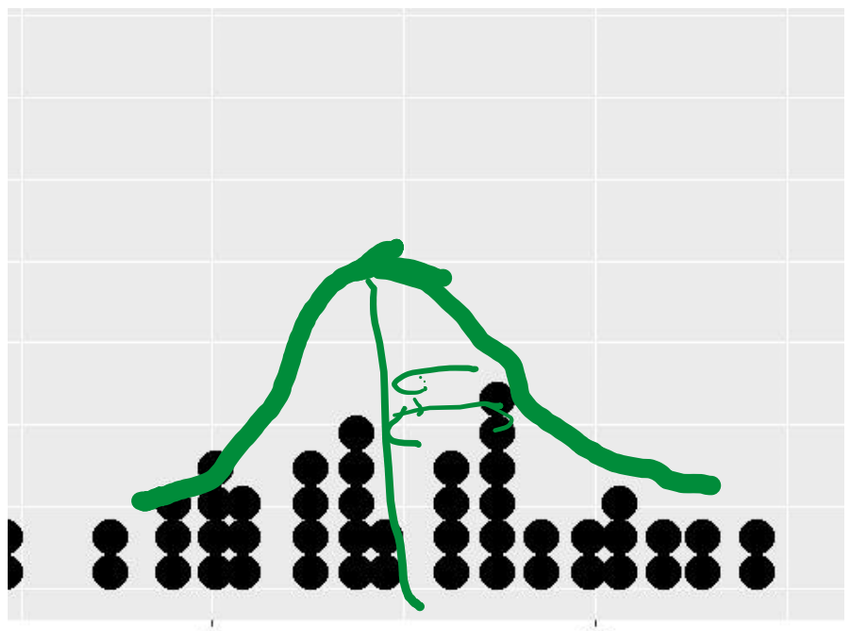


Parameter Estimation

Maximum likelihood estimator

- f(x1, . . . , xn|θ) represents the probability that the values x1, x2, . . . , xn will be observed when θ is the true value of the parameter

- **Maximum Likelihood estimation :** maximum likelihood estimate ˆθ is defined to be that value of θ maximizing L(θ) = f (x1, . . . , xn|θ)

argmax_θ L(θ) = $f(x_1, \ldots, x_n|\theta)$ = argmax_θ $\log[f(x_1, \ldots, x_n|\theta)]$.

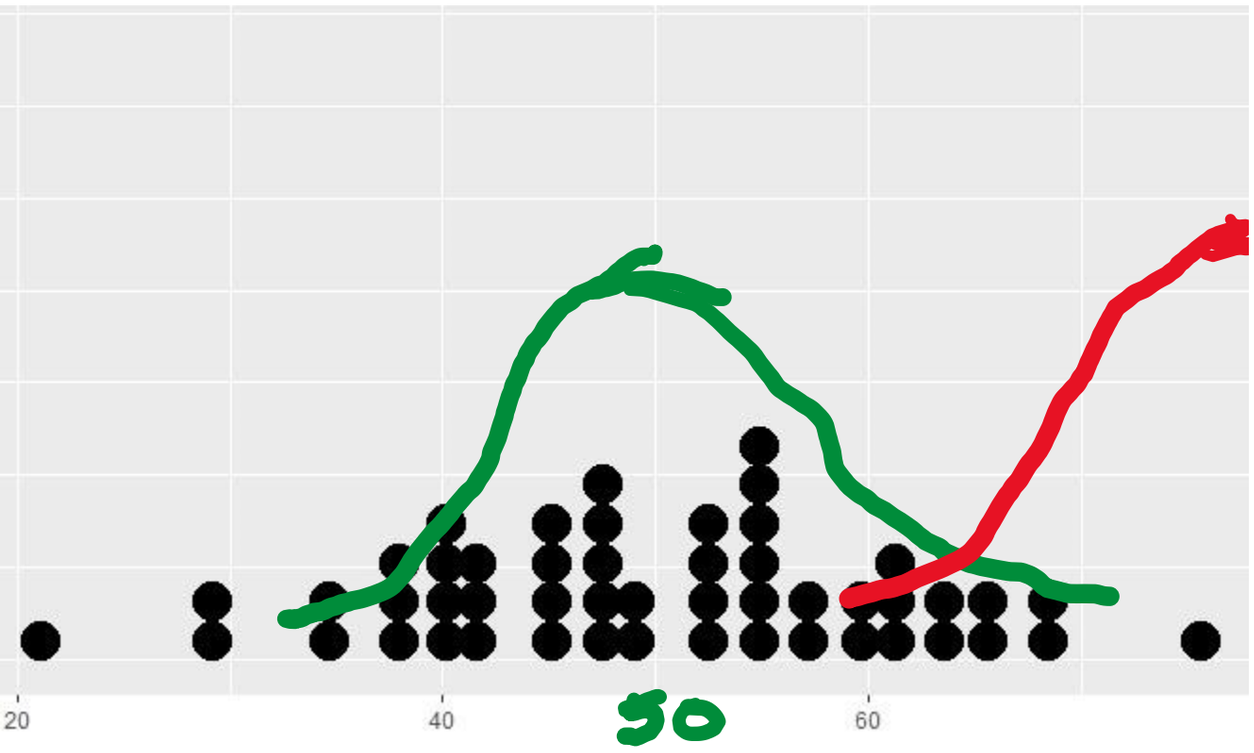
Note that L(θ) is not a distribution over θ but just a function of θ.

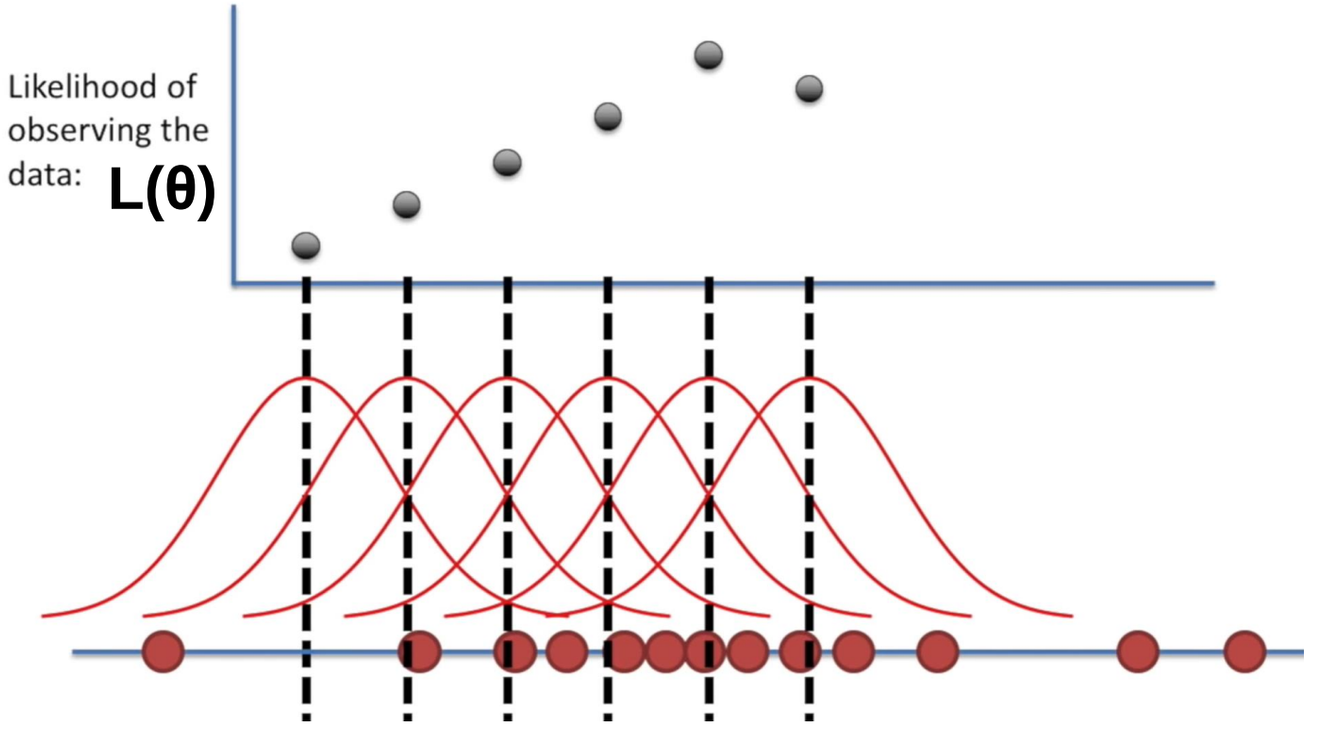
Independent and identically distributed (i.i.d.) assumption

$$f(x_1, x_2, \ldots, x_n) = f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_n}(x_n)$$

Parameter estimation

- which of the following would maximize the probability of observing the data
  - Mean = 100, SD = 10
  - Mean = 50, SD = 10





Maximum likelihood estimation

- (Maximum Likelihood Estimator of a Bernoulli Parameter) Suppose you have data from n independent Bernoulli trials, $X1, \ldots, Xn$. Assuming the success probability is p what is the maximum likelihood estimator of p?

$$X_i = \begin{cases} 1 & \text{if trial } i \text{ is a success} \\ 0 & \text{otherwise} \end{cases}$$

$$P\{X_i = 1\} = p = 1 - P\{X_i = 0\}$$

$$P\{X_i = x\} = p^x(1-p)^{1-x}, \quad x = 0, 1$$

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$$\begin{aligned} f(x_1, \ldots, x_n|p) &= P\{X_1 = x_1, \ldots, X_n = x_n|p\} \\ &= p^{x_1}(1-p)^{1-x_1} \cdots p^{x_n}(1-p)^{1-x_n} \\ &= p^{\Sigma_1^n x_i}(1-p)^{n-\Sigma_1^n x_i}, \quad x_i = 0, 1, \quad i = 1, \ldots, n \end{aligned}$$

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To determine the value of p that maximizes the likelihood,

$$\log f(x_1, \ldots, x_n | p) = \sum_1^n x_i \log p + \left(n - \sum_1^n x_i \right) \log(1 - p)$$

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$$\frac{d}{dp} \log f(x_1, \ldots, x_n|p) = \frac{\sum_1^n x_i}{p} - \frac{\left(n - \sum_1^n x_i\right)}{1-p} \qquad \hat{p} = \frac{\sum_{i=1}^n x_i}{n}$$

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To determine the value of p that maximizes the likelihood,

proportion of the observed trials that result in successes.

$$\frac{d}{dp} \log f(x_1, \ldots, x_n | p) = \frac{\sum_1^n x_i}{p} - \frac{\left(n - \sum_1^n x_i\right)}{1 - p} \qquad \hat{p} = \frac{\sum_{i=1}^n x_i}{n}$$

Suppose that each RAM (random access memory) chip produced by a certain manufacturer is, independently, of acceptable quality with probability *p*. Then if out of a sample of 1,000 tested 921 are acceptable, what is the maximum likelihood estimate of *p* ?

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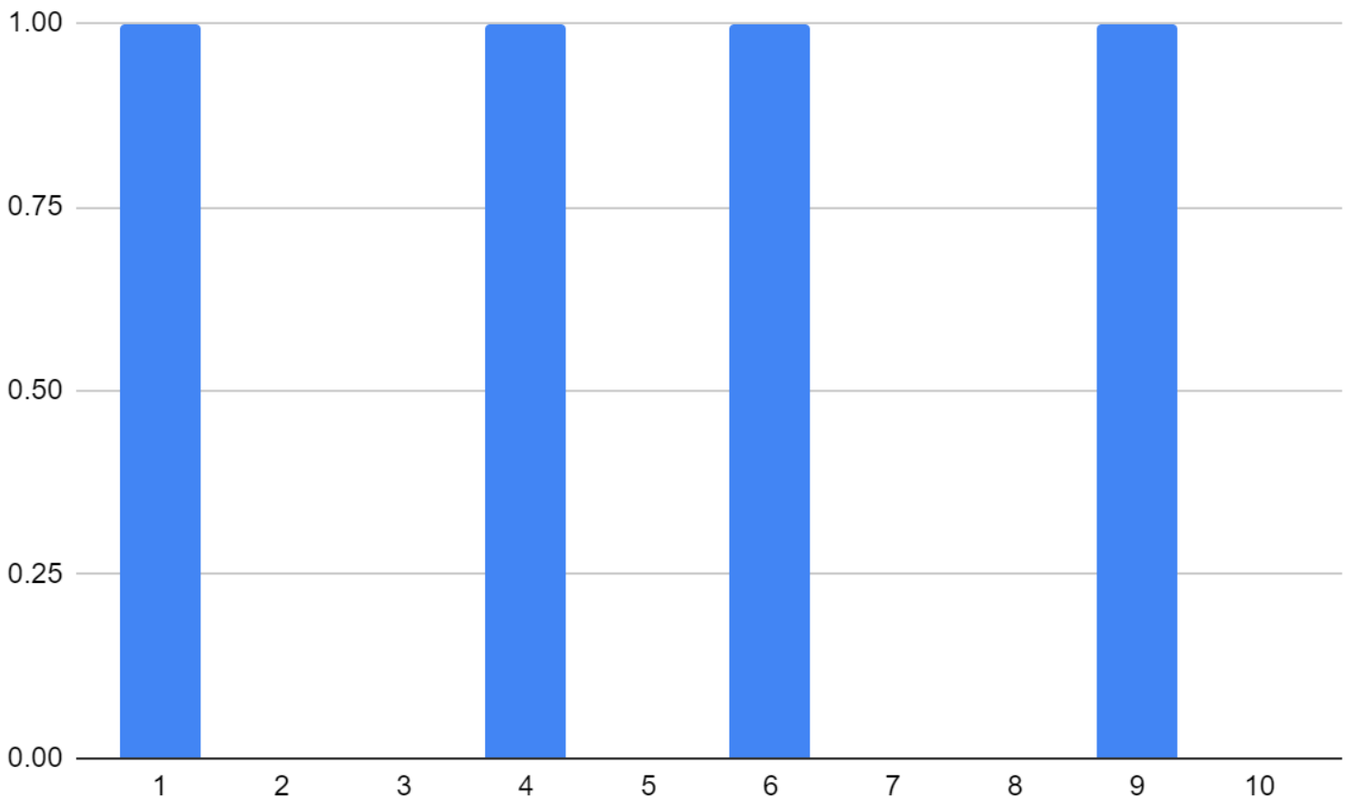

Parameter Estimation

- Multinomial
- 3,1,2,4,3,5,6,1,3,4

$$p(\mathbf{x}|\boldsymbol{\mu}) = \prod_{k=1}^{K} \mu_k^{x_k} \qquad \mathbf{x} = (0, 0, 1, 0, 0, 0)^{\mathrm{T}}.$$

data set $\mathcal{D}$ of N independent observations $\mathbf{x}_1, \ldots, \mathbf{x}_N$.

$$p(\mathcal{D}|\boldsymbol{\mu}) = \prod_{n=1}^{N} \prod_{k=1}^{K} \mu_k^{x_{nk}} = \prod_{k=1}^{K} \mu_k^{\left(\sum_n x_{nk}\right)} = \prod_{k=1}^{K} \mu_k^{m_k}. \qquad m_k = \sum_n x_{nk}$$

Parameter Estimation

- Multinomial

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Parameter Estimation : Multinomial

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$$p(\mathbf{x}|\boldsymbol{\mu}) = \prod_{k=1}^{K}\mu_k^{x_k}$$

Constraint

$$\mu_k^{\mathrm{ML}} = \frac{m_k}{N}$$

ML solution !

MLE : Poisson !

- (Maximum Likelihood Estimator of a Poisson Parameter) Suppose X1, . . . , Xn are independent Poisson random variables each having mean λ. Determine the maximum likelihood estimator of λ.

$$f(x_1, \ldots, x_n|\lambda) = \frac{e^{-\lambda}\lambda^{x_1}}{x_1!} \cdots \frac{e^{-\lambda}\lambda^{x_n}}{x_n!}$$

$$= \frac{e^{-n\lambda}\lambda^{\Sigma_1^n x_i}}{x_1! \ldots x_n!}$$

$$\log f(x_1, \ldots, x_n|\lambda) = -n\lambda + \sum_1^n x_i \log\lambda - \log c$$

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$$\frac{d}{d\lambda} \log f(x_1, \ldots, x_n|\lambda) = -n + \frac{\sum_1^n x_i}{\lambda}$$

$$\hat{\lambda} = \frac{\sum_1^n x_i}{n}$$

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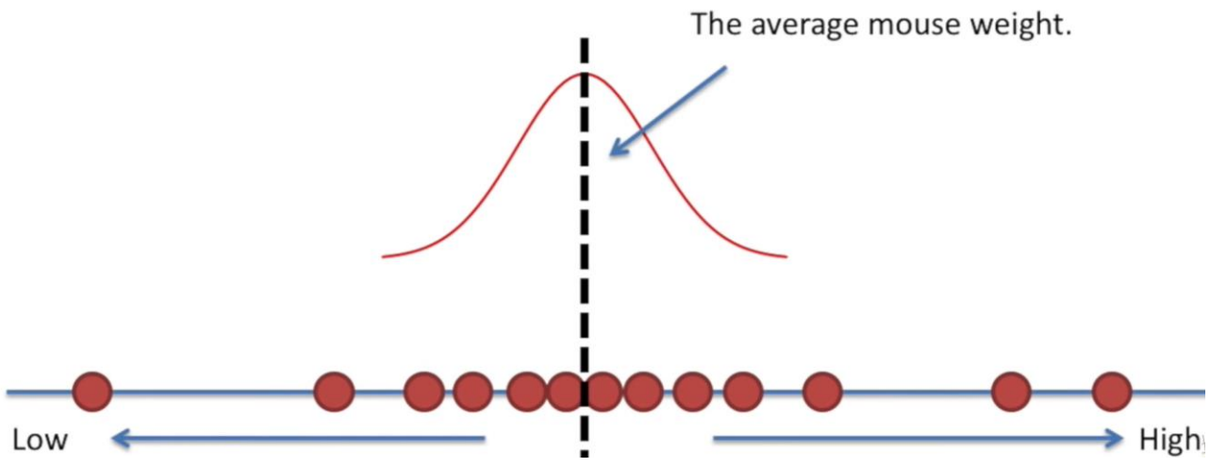
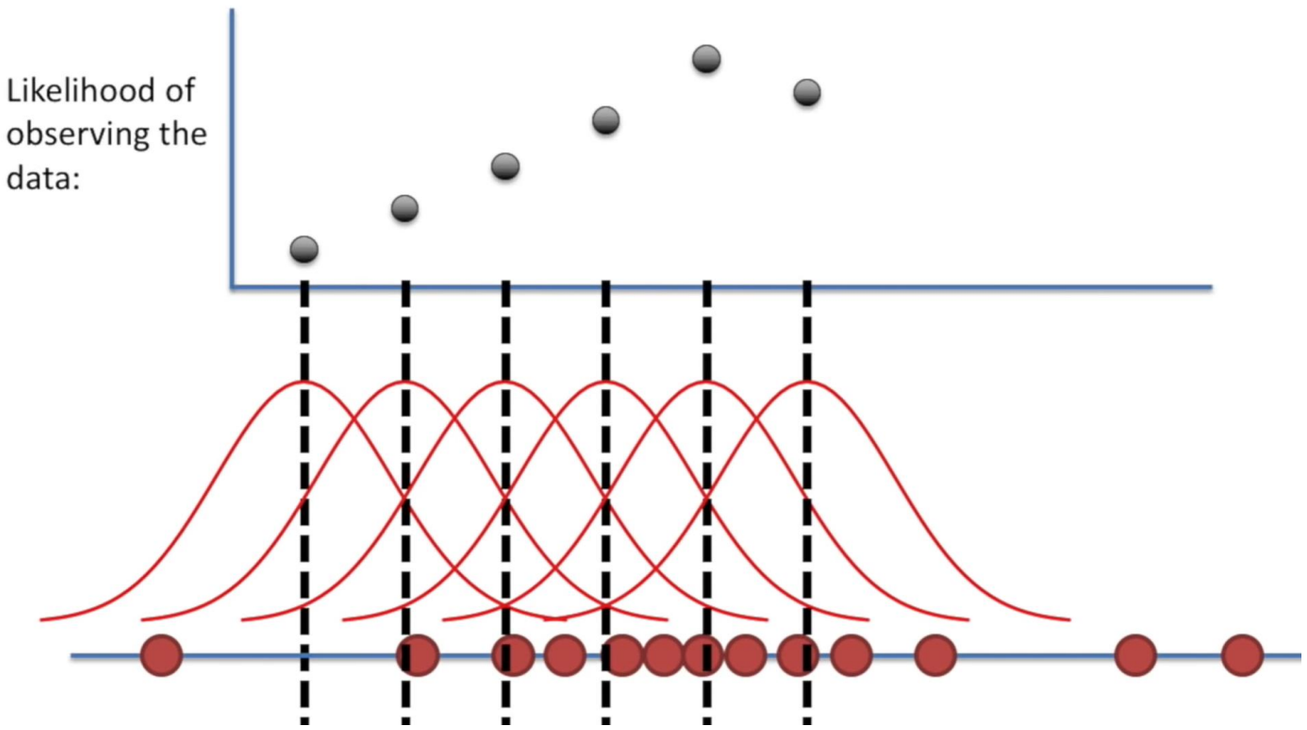
- The number of traffic accidents in Berkeley, California, in 10 randomly chosen nonrainy days in 1998 is as follows:
- 4, 0, 6, 5, 2, 1, 2, 0, 4, 3
- Use these data to estimate the proportion of nonrainy days that had 2 or fewer accidents that year.

MLE : Normal

- (Maximum Likelihood Estimator in a Normal Population) Suppose X1, . . . , Xn are independent, normal random variables each with unknown mean μ and unknown standard deviation σ.

1) We expect most of the measurements (mouse weights) to be close to the mean (average).





MLE : Normal

- (Maximum Likelihood Estimator in a Normal Population) Suppose X1, . . . , Xn are independent, normal random variables each with unknown mean μ and unknown standard deviation σ.

$$f(x_1,\ldots,x_n|\mu,\sigma) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[\frac{-(x_i-\mu)^2}{2\sigma^2}\right]$$

$$= \left(\frac{1}{2\pi}\right)^{n/2} \frac{1}{\sigma^n} \exp\left[\frac{-\sum_{1}^{n}(x_i-\mu)^2}{2\sigma^2}\right]$$

$$\log f(x_1,\ldots,x_n|\mu,\sigma) = -\frac{n}{2}\log(2\pi) - n\log\sigma - \frac{\sum_{1}^{n}(x_i-\mu)^2}{2\sigma^2}$$

MLE : Normal

- (Maximum Likelihood Estimator in a Normal Population) Suppose X1, . . . , Xn are independent, normal random variables each with unknown mean μ and unknown standard deviation σ.

$$\frac{\partial}{\partial \mu} \log f(x_1, \ldots, x_n | \mu, \sigma) = \frac{\sum_{i=1}^{n} (x_i - \mu)}{\sigma^2}$$

$$\frac{\partial}{\partial \sigma} \log f(x_1, \ldots, x_n | \mu, \sigma) = -\frac{n}{\sigma} + \frac{\sum_{1}^{n} (x_i - \mu)^2}{\sigma^3}$$

$$\hat{\mu} = \sum_{i=1}^{n} x_i / n \qquad \hat{\sigma} = \left[\sum_{i=1}^{n} (x_i - \hat{\mu})^2 / n \right]^{1/2}$$

Model Selection

- Given some observations *X1, X2, …, XN*, how do you decide which probability distribution to model it ?


ధన్యవాదాలు धन्यवाद
ధన్యవాదగళు ನಂತಿ धन्यवाद ಧನ್ಯವಾದ ధన్యవాదాలు આભાર நன்றி धन्यवाद
ನಂತಿ ધન્યવાદ ધન્યવાદગళು ನಂತಿ ధన్యవాదాలు நன்றி
न्यवाद ధన్యవాదాలు நன்றி આભાર धन्यवाद ધન્યવાદ
ಂತಿ நன்றி THANK YOU ధన్యవాదగళ
ಯವಾದ ధన్యవాదాలు ధన్యవాదగళు ನಂತಿ ધન્યવાદ ನಂತಿ धन्यवाद ધન્યવ
್ಯವಾದಾಲು धन्यवाद આભાર धन्यवाद ధన్యవాదగళు ನ
ವಾದಗಳು ನಂತಿ ધન્યવાદ ನಂತಿ ధన్యవాదాలు આભાર நன்றி