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CS18BTECH11042
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Assignment - 6

Q. To prove that every three-uniform hypergraph with n vertices and $m \geq n/3$ edges contains an independent set of size at least

$$\frac{2n^{3/2}}{3\sqrt{3}\cdot\sqrt{m}}$$

Ans) Let the 3-uniform hypergraph be $H=(V,E)$ where $|V|=n$ and $|E|=m$.

Creating a set S, and choosing vertices from V, independently and at random to put them into set S, with probability p. $(Pr(v \in S)=p)(\forall v \in V)$

$$X_v = \begin{cases} 1, & \text{if } v \text{ is chosen into } S \\ 0, & \text{otherwise} \end{cases}$$

$Pr[X_v]=p$, $E[X_v]=p\cdot 1+0=p$ - i)

(X_v is a random variable defined for every vertex v in V, to indicate whether $v \in S$ or not)

Defining another random variable X, to denote the size of S. (number of vertices in S)

$X = |S| = \sum_{v \in V} X_v$, $E[X] = E\left[\sum_{v \in V} X_v\right]$

$E[X] = \sum_{v \in V} E[X_v]$, using linearity of expectation.
by

$\therefore E[X] = \sum_{v \in V} p$ (from i)

$\therefore E[X] = np$ (as $|V| = n$) — ii)

Therefore the expected size of set $S = np$.

Defining another random variable Y_e, for every edge $e \in E$, such that

$$Y_e = \begin{cases} 1, & \text{if all 3 vertices of edge } e \text{ are chosen into } S. \\ 0, & \text{otherwise.} \end{cases}$$

$\therefore$ If $Y_e = 1$, (for any e) an edge $e = \{u, v, w\}$ exists in the edge set of S and it can't be an independent set.

Pr[All 3 vertices of edge e chosen into S]

$= Pr(u \in S) \cdot Pr(v \in S) \cdot Pr(w \in S)$

$= p \cdot p \cdot p = p^3$.

$E[Y_e] = p^3 \cdot 1 + 0 = p^3$ — iii)

Defining another random variable Y, to denote the number of edges present in S.

$$Y = \sum_{e \in E} Y_e \;, \quad E[Y] = E\left[\sum_{e \in E} Y_e\right]$$

$$E[Y] = \sum_{e \in E} E[Y_e]$$, By using linearity of expectation.

$$E[Y] = \sum_{e \in E} p^3 \quad (\text{from iii})$$

$$E[Y] = mp^3 \quad (\text{as } |E| = m) \text{ — iv)}$$

Now we know S contains non-zero number of edges (for non-empty S, H), and thus S is not an independent S. If we remove one vertex each from all the edges present in S, no three vertices constituting an edge would still be present in S, thus we would create an independent set.

Defining another random variable Z, to denote the size of the independent set formed after the above alteration operation

$$Z = \underbrace{X}_{\text{no. of vertices in S}} - \underbrace{Y}_{\text{no. of edges in S.}}$$

$$E[Z] = E[X - Y]$$

$$\Rightarrow E[Z] = E[X] - E[Y]$$, using linearity of expectation

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$E[Z] = np - mp^3$ (from ii) & iv)) — v)

To find the value of 'p' for which, $E[Z]$, that is the size of the independent set is maximised, differentiating v) wrt p, and equating to 0.

$\Rightarrow n - 3mp^2 = 0$

$\Rightarrow n = 3mp^2 \Rightarrow p^2 = \dfrac{n}{3m}$

As we know $m \geq n/3$, $(p^2 \leq 1)$ so no contradiction

$\Rightarrow p = \sqrt{\dfrac{n}{3m}}$ (Only positive square root possible as probability value is necessarily positive)

(Maxima at $p = \sqrt{n/3m}$) as 2nd derivative negative at this value.

Substituting in v), $E[Z] = n\sqrt{\dfrac{n}{3m}} - m\left(\sqrt{\dfrac{n}{3m}}\right)^3$

$$E[Z] = \frac{n^{3/2}}{\sqrt{3}\cdot\sqrt{m}} - \frac{\cancel{m}\cdot n^{3/2}}{3^{3/2}\cdot m^{\cancel{3}/2}} = \frac{n^{3/2}}{\sqrt{3}\cdot\sqrt{m}}\left(1 - \frac{1}{3}\right)$$

$$E[Z] = \frac{2n^{3/2}}{\sqrt{3}\cdot\sqrt{m}} \quad \text{— vi)}$$

As expected value of size of independent set in H is $\dfrac{2n^{3/2}}{\sqrt{3}\cdot\sqrt{m}}$, there exists some independent set of size atleast $\dfrac{2n^{3/2}}{\sqrt{3}\cdot\sqrt{m}}$ (as otherwise expected value would have been smaller) under the given conditions

Hence Proved!