COMBINATORICS

⇒ Sunflower

A collection of K sets $S_1, S_2 .. S_k$ is a sunflower with k petals and a core 'y' if

$$\forall i,j \in [k],\ i \neq j .\ S_i \cap S_j = y$$

→ y can be an empty set ϕ

NOTE: if you take a large no of sets, it is guaranteed a sunflower is present.

for a w-uniform family of sets, size of set = w

Sunflower lemma

- Erdos, Rado - 1960s

let F be a w-uniform family of sets if $|F| > w!(K-1)^w$, then F contains a sunflower. (K-sunflower)

Proof: (by Induction on w-set size)

BASE CASE: $w = 1$

$|F| > K-1$

atleast K sets (disjoint) since sets are distinct

⇒ K disjoint sets

⇒ empty core ⇒ ensures sunflower.

INDUCTION STEP: statement true for assumed $w \leq r-1$ where $r > 1$

let $w = r$

$$|F| > r!(K-1)^r$$

if F contains K-pairwise disjoints sets, then done

y → empty core

else let $S_1, S_2 \ldots S_l$ be a maximal subfamily of F which are pairwise disjoint ($l<k$)

$\{S_1, S_2 \ldots S_l\} \rightarrow$ pairwise disjoint subfamily of F with $l \leq k-1$

let $T = S_1 \cup S_2 \cup S_3 \ldots \cup S_l$

↳ T is HITTING SET of F

↳ any set of F will have a non-empty intersection with the hitting set

proof: let $S \in F$, have an empty intersection then it would have been included in T but T was maximal $\Rightarrow$ CONTRADICTION

TWO INFERENCES

1. T is a hitting set
2. $|T| = lr$ (pair wise disjoint)
   $lr \leq (k-1)r$

let T contains elem $\Rightarrow \{x_1, x_2 \ldots x_{|T|}\}$

and $F = \{S_1, S_2, \ldots S_l, S_{l+1} \ldots S_{>r!(k-1)^r}\}$

$\exists\, x \in T$ such that x is present in $> \frac{r!(k-1)^r}{r(k-1)}$ $\frac{\text{sets in F}}{\text{size of T}}$

$= (r-1)!\,(k-1)^{r-1}$

let $F_x = \{S \in F : x \in S\}$

$|F_x| > (r-1)!\,(k-1)^{r-1}$

↳ $F_x' \rightarrow$ after removing x from every set in F_x

↳ $(r-1)$ uniform family

By induction hypo,

F_x' contains k-sunflower.

let $f(k,w)$ denote the minimum number of w-sized sets required to ensure the presence of k-sunflower

$$(k-1)^w < f(k,w) \leq w!\,(k-1)^w + 1$$

To prove: $(k-1)^w < f(k,w)$

Take sets $A_1, A_2 \dots A_w$ with $(k-1)$ elements in each set
take one element from each set
Set Size $= w$
and $|F| = (k-1)^w \Rightarrow (k-1)$ options from each set

Sunflower Conjecture for a fixed k

$$f(k,w) < c^w \quad \text{where } c = c(k)$$

improved bound

$$f(k,w) \leq (\log w)^{w(1+o(1))}$$

↳ depends of k.

$[n] = \{1,2,3,\dots,n\}$
no. of subsets $= 2^n$ (n-element)
smallest subset size $= \phi$ (empty)

INTERSECTING FAMILY

A family of F of subsets of $[n]$ is an intersecting family if $\forall\, A,B \in F$, $A \cap B \neq \phi$

Eg. $F = \{\{1,2,3\}, \{1,2\}, \{2,3\}, \{1,3\}\}$

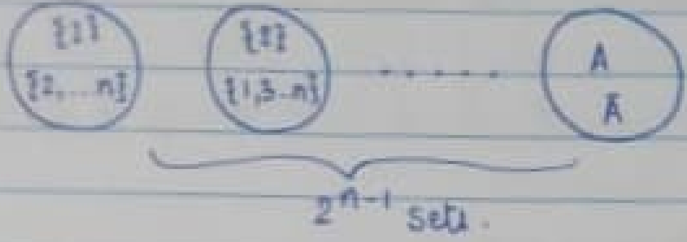
How large can an intersecting family of subsets of $[n]$ be?
if we 1 element
↳ no. of subsets $\Rightarrow 2^{n-1}$ (Intersecting family)

$2^{n-1} \geq$ Size of largest intersecting family of subsets of [n] $\geq 2^{n-1}$

lower bound proved

For upper bound,

Take a set A & its complement (excluding subset {ϕ})



we cant take both A & $\bar{A}$ in intersecting family

thus, upper bound = 2^{n-1}

NOTE: Size of a largest k-uniform family of subsets of [n] that is intersecting.

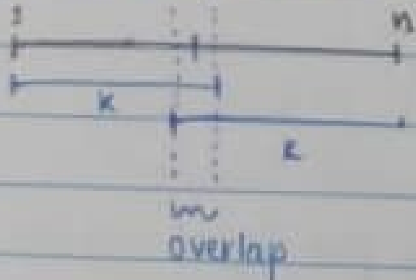
if $k > \frac{n}{2}$

no of subsets of size k : ^nC_k

but since $k > n/_2$ (overlap guaranteed)

thus,

any 2 sets of size k will have common elements



CASE 1 : $n < 2k$ $(k > n/2)$

size of largest k-uniform family of subsets of $[n]$ which are intersecting

$$\Rightarrow {}^{n}C_{k}$$

CASE 2 : $n \geqslant 2k$ $(k \leqslant n/2)$

fix 1 element and take subsets of size $(k-1)$ from $(n-1)$ elements

thus lower bound $\Rightarrow {}^{n-1}C_{k-1}$

* $\boxed{\text{ERDÖS - KO - RADO [1960s]}}$

for $n \geqslant 2k$

$$ {}^{n-1}C_{k-1} \leqslant \begin{matrix}\text{size of largest} \\ \text{k-uniform intersecting} \\ \text{family}\end{matrix} \leqslant {}^{n-1}C_{k-1}$$

let F be a k-uniform intersecting family of subsets of $[n]$ that is intersecting

Further $n \geqslant 2k$

$$|F| \leqslant {}^{n-1}C_{k-1}$$

<u>PROOF</u> (Katona, 1970s)

suppose given a intersecting family of F

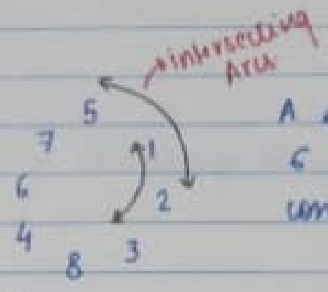
For Example, $k = 3$

$$F = \{ \{1,2,3\} \{2,3,4\} \{1,5,2\}, \{2,3,5\} \}.$$

let

σ : be a circular permutation of $[n]$

let σ :



A set S ∈ F is "present" in σ if elements of S are present contiguously in the permutation (order does not matter).

From the example {1,2,3}, {1,5,2} are present in permutation.

F : Intersecting, k-uniform with σ :

How many sets of F can be present in the circular permutation ??

since all are intersecting sets, we wouldnt have non-intersecting arcs.

we can have atmost k sets of Family F that can appearing contiguously (overlapping : exclude 1 element each time only k arcs would be possible because the $(k+1)^{th}$ arc would be disjoint with the 1st arc.

$G : \{ (S, \sigma) : S \in F$ is present in the circular permutation $\sigma \}$

$$|F| k! (n-k)! \le |G| \le (n-1)! k$$

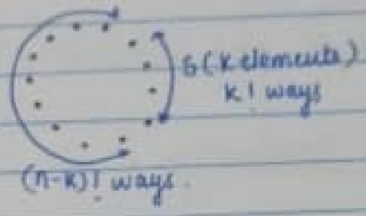
no. of ways to choose S from F (size of family)

S can be present in how many σ

no. of circular permutations

k sets possible

for a set $|S| = k$
elements arranged in $k!$
and rest element in $(n-k)!$



thus for a set S, can be present in
$k!\,(n-k)!$ circular permutations.

Hence,

$$|F|\,k!\,(n-k)! \leq (n-1)!\,k$$

$$|F| \leq \frac{(n-1)!\,k}{(n-k)!\,k!}$$

$$|F| \leq \frac{(n-1)!}{((n-1)-(k-1))!\,(k-1)!}$$

$$|F| \leq {}^{n-1}C_{k-1}$$

* CHAINS AND ANTICHAINS

→ Basic Terminology :

- let $P = (X, \preceq)$ be a tuple where X is a set and $\preceq$ is a binary relation on X

$\preceq \subseteq X \times X$ (cartesian product of X with itself)

we say,

P is a partially ordered set (POSET) if
$\preceq$ is a partial order relation of X
↳ means $\preceq$ is a

* reflexive,	antisymmetric,	transitive
↓	↓	↓
$\forall x \in X$	$\forall$ any $x, y \in X$	for every $x, y, z \in X$
$x \preceq x$	$(x \preceq y$ & $y \preceq x)$	$(x \le y, y \le z)$
	then $x = y$	then $x \le z$

Examples

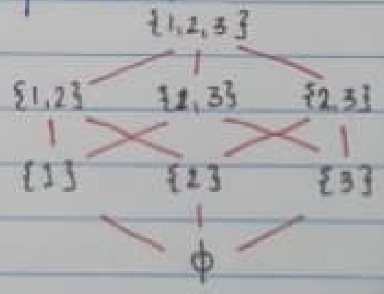
1. let $A = \{1, 2, 3\}$
   X = Powerset of A
   $= \{ \{1\}, \{1,2\}, \{1,2,3\}, \{2\}, \{3\}, \{2,3\}, \{1,3\} \phi \}$.

$P = (X, \subseteq)$
↳ subset relation (containment relation)

⇒ every set is a subset of itself ⇒ reflexive
⇒ if $B \subseteq C$ and $C \subseteq B$, then $B = C$ ⇒ antisymmetric
⇒ if $B \subseteq C$ & $C \subseteq D$, then $B \subseteq D$ ⇒ transitive

P → Partially ordered set.

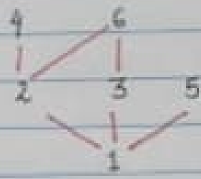
HASSE → DIAGRAM



2 $X = \{1,2,3,4,5,6\}$.
$P = (X, |)$
↳ a divides b.

⇒ each number divides itself $a \% a = 0$ ⇒ reflexive
⇒ if $b \% a == 0$ and $a \% b = 0$
⇒ a divides b & b divides a
⇒ $a = b$ ⇒ antisymmetric
⇒ if a divides b and b divides c ⇒ a divides c ⇒ transitive
P → POSET

HASSE DIAGRAM :



(2,4) are comparable
(3,6) " "

(5,6) uncomparable
(5,4) uncomparable

NOTE :
for any $a, b \in X$, we say a & b are comparable
if $a \preceq b$ or $b \preceq a$
otherwise incomparable.

* CHAIN

given, $P = (X, \preceq)$ → POSET
$X' \subseteq X$ is a chain if
every 2 elements in X' are $\boxed{\text{COMPARABLE}}$
with each other

* ANTICHAIN

$P = (X, \preceq)$ POSET
$Y \subseteq X$ is an antichain if
no 2 elements in Y are comparable
with each other.

Example
1. $X' = \{ \{\phi\}, \{1\}, \{1,2\}, \{1,2,3\} \}$
or $\{ \{2\}, \{2,3\} \}$

$Y = \{ \{1,2\} \{2,3\} , \{1,3$
or $\{ \{3\} \}$.

Note : every single element $x \in X$
$\{ \{x\} \}$ → chain and antichain.

* DILWORTH THEOREM 1

let $P = (X, \alpha) \Rightarrow$ POSET

if the length of the longest chain in P is 'r', then the elements of X can be ~~positioned~~ partitioned into 'r' antichains

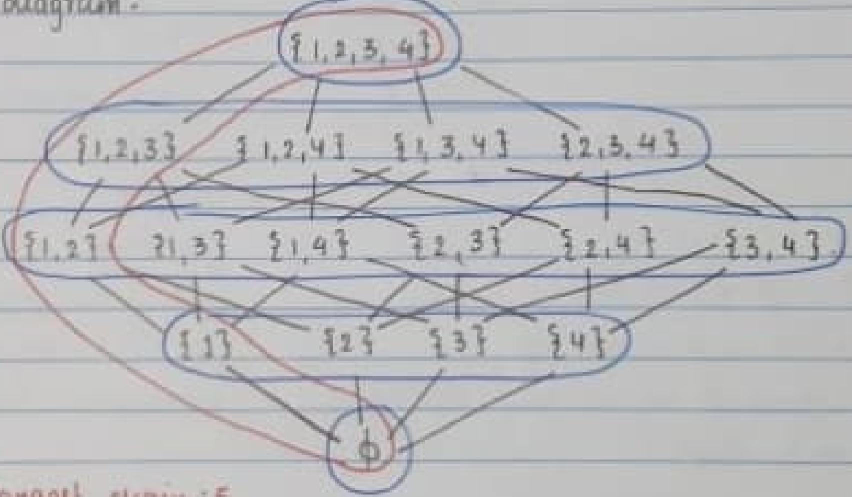
where length of chain : elements in subset X'

* dividing a set
$S = S_1 \cup S_2 \cup S_3 \cdots S_5$
disjoint union
$S_i \cap S_j = \phi$
$\forall i, j$

Example

$P = ($Power set of $\{1,2,3,4\}, \subseteq)$

Hasse diagram:



length of longest chain : 5

↳ also the no. of partitions where each partition is an antichain.

and union of partions provide the set X and they are disjoints

Thus,

minimum no. of antichain into which one can partition X $\geq r$

↑ length of longest chain

Proof by contradiction (for lower bound)
suppose we had $(r-1)$ antichain partitions.

$A_1, A_2, A_3 \ldots A_{r-1}$

we cant take 2 elements from 1 antichain, since they are uncomparable

atmost $(r-1)$ elements can be comparable

but $r \rightarrow$ longest chain (contradiction)

But Dilworth prooves that,

minimum no of antichains of partitions $= r$

Proof: $P = (X, \preceq)$

For any $1 \leq i \leq r$, we define

$A_i = \{ x \in X$; the length of longest chain terminating at x is $i \}$.

For eg.

$A_4 = \{ \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\} \}$.

↳ longest chain ending with $\{1,2,3\}$ is 4

$\{\phi, \{1\}, \{1,2\}, \{1,2,3\}\}$ or $\{\phi, \{2\}, \{2,3\}, \{1,2,3\}\}$ etc.

Claim: for every $1 \leq i \leq r$, A_i is an antichain.

since no 2 elements can be part of 2 sets

thus A_i would be disjoint.

so, thus if we prove A_i is an antichain, we prove the dilworth theorem since there are 'r' antichains

Proof: suppose A_i is not an antichain.

$\Rightarrow x, y \in A_i$ such that $x \preceq y$

But then, length of longest chain ending at y should be $\geq i+1$

because since $x \in A_i$

length of longest chain ending at $x \Rightarrow i$

since

$x \preccurlyeq y$, include y in chain

thus, length of longest chain ending at $y \geqslant i+1$

But this contradicts,

that $y \in A_i$

So, our assumption $x \preccurlyeq y$ is false

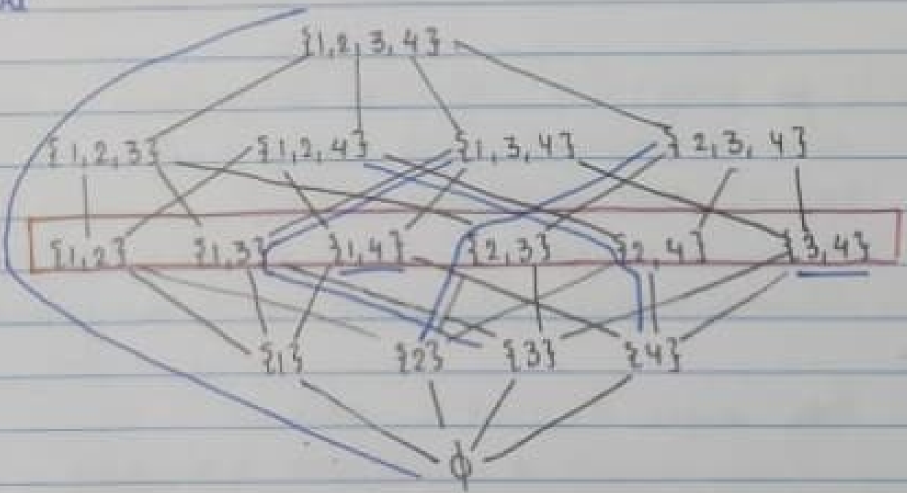
thus for $1 \leq i \leq r$, A_i is an antichain

thus the min no. of partitions which are antichains = r.

✱ **DILWORTH THEOREM 2**

Let $P = (X, \preccurlyeq)$ be a poset, let 'r' be the length of a largest antichain, then the elements of X can be partioned into 'r' chains.

Example:



longest antichain of length = 6

= no. of minimum chain partitions.

chains partitions

: $\{\phi, \{1\}, \{1,2\}, \{1,2,3\}\ \{1,2,3,4\}\}$

: $\{\{3\}, \{1,3\}, \{1,3,4\}\}$

: $\{\{2\}, \{2,3\}, \{2,3,4\}\}$

: $\{1,4\}$

: $\{3,4\}$

Maximum no. of chains = n

↳ each of length 1 where $|X| = n$.

can we have less than r chains

let have $(r-1)$ chains, but this would contradict the fact that 'r' is longest antichain (each element from each chain)

Minimum no. of chains = r.

Proof: Induction on $|X|$.

Base case: $|X| = 1$

length of longest antichain = 1

partitions possible = 1

1 element only → chain and antichain.

Induction step: Assume the statement is true for all posets defined on X with $|X| \leq n-1$

To prove: let $P = (X, \leq)$ be a poset where $|X| = n$.

let $x \in X$ be a maximal element of $P(X, \leq)$

↳ there is no element $y \in X$, $y \neq x$ such that $x \not\leq y$.

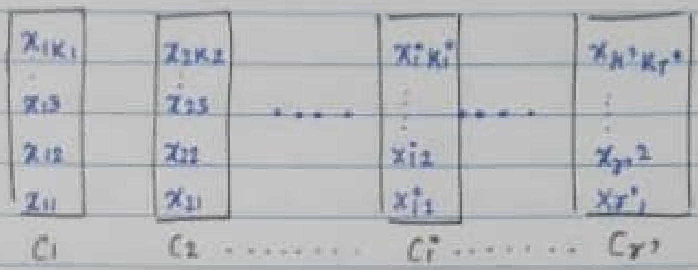
For Example $x \Rightarrow \{1,2,3,4\}$ in prev eg

let $P' = (X \setminus \{x\}, \preceq)$ be a subposet of P ; where $r' \rightarrow$ length of long
By induction hypothesis. anticchain in P'

$|P'| = n-1$

then elements of X' can be partitioned into r' chains.
$\hookrightarrow X \setminus \{x\}$

let these chains be $C_1, C_2, C_3 \ldots C_{r'}$



$X' = C_1 \uplus C_2 \uplus C_3 \cdots \uplus C_i \cdots \uplus C_{r'}$ (disjoint union)

CASE 1 : $r' < r$

then $r' = r-1$ {only removed 1 element}.

Thus length of longest chain would atmost decrease by 1

thus

$X' = C_1 \uplus C_2 \ldots \uplus C_{r-1}$ $(r-1)$ chains

and let $C_r = \{x\}$ (removed element)

And hence

$X = C_1 \uplus C_2 \uplus \ldots \uplus C_{r-1} \uplus C_r$.

$(r$ chains$) \longrightarrow$ proved.

CASE 2 : $r' = r$

thus for X', we used r chains but $\{x\}$ is left out

Subcase 1 :

if for any x_{ik_i} where $1 \le i \le r$

$x_{ik_i} \preceq x$

then, would simply add $\{x\}$ to chain C_i and
X would be partioned in r chains

Subcase 2:

if we dont find any $x_{ik_i} \le x$.
the only way to include x, is to break the existing chains and reorder them to include ~~x~~ $\{x\}$

let us define elements,

$x_{ij_i} \Rightarrow$ highest element in C_i that is present in an antichain of length r in P'

Claim:

$\{x_{1j_1}, x_{2j_2} \ldots x_{rj_r}\}$ is an antichain of length 4 in P'

by contradiction,

suppose it not a antichain.

$\Rightarrow$ 2 elements are related in above set

↳ There exist $x_{ij_i} \underline{\propto} x_{kj_k}$

where $x_{kj_k} \propto x_{ij_i}$ of other chain

this would mean

in an antichain consist of x_{kj_k}

we cant have any elements in the chain C_i which are below x_{ij_i}

but somebody from C_i must be present in that antichain to make a length r.

so, somebody from higher hierarchy in C_i has to be include $\in$ ~~$(x_{ij_i}, x_i$~~k.

but that would contradict the fact that x_{ij_i} was the highest element in C_i in any antichain

$\Rightarrow$ no 2 elements are related $\Rightarrow \{x_{1j_1}, x_{2j_2} \ldots x_{rj_r}\}$ antichain.

let $A' = \{x_{1j_1}, x_{2j_2} \dots x_{ij_i} \dots x_{rj_r}\}$.

Ques: is $A = A' \cup \{x\}$ an antichain in P?

No, this would result in $(r+1)$ length antichain

contradiction.

$\Rightarrow$ one of the x_{ij_i} is comparable to $\{x\}$

since $\{x\}$ is a maximal set

$x_{ij_i} \preccurlyeq x$.

let

$C_i = \{x_{i_1}, x_{i_2} \dots x_{ij_i} \dots x_{ik_i}\}$.

and

$C_i' = \{x_{i_1}, x_{i_2} \dots x_{ij_i}, x\}$.

and $D_i' = \{x_{ij_{i+1}}, \dots x_{ik_i}\}$

let $P'' = ((X \setminus C_i'), \preccurlyeq)$

Claim: length of largest antichain in P'' is $\leq r-1$

because removing C_i', we remove x_{ij_i} → which was taking part in longest antichain of length 'r'

thus antichain length decrease by 1.

By induction hypothesis

X / C_j' can be partioned into $(r-1)$ chains

$$\Downarrow$$

X can be partitioned into r chains

(by addition of C_i')

↳ chain.

* HALL'S THEOREM

proof using Dilworth's theorem 2

let

G: Graph ⇒ (V, E)

V(G) ⇒ vertices set ; E(G) ⇒ edges set

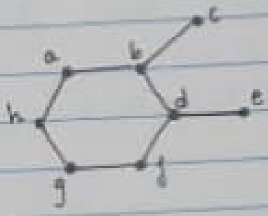
↳ ⊆ V(G) x V(G)

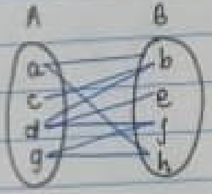
Dealing with simple undirected graphs.

Independent set of vertices: no 2 vertices in the set have an edge between them.

Bipartite graph: A graph G is bipartite if its vertices can be partitioned into 2 parts say A & B, such that there is no edge between any 2 vertices that belong to the same part.

example:



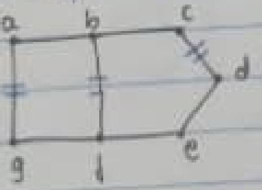


A & B are independent itself.

Matching in a graph.

Subset of the edge set of a graph such that no 2 edges share an endpoint.

example:



M: {ag, bf, cd}.

Match vertex = {a, b, c, d, f, g}.

unmatched vertex = {e}.

wrt a given matching say M, a vertex V is said to be a matched vertex if ∃ some edge in M that has V as an endpoint.

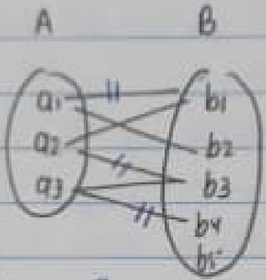
HALL'S THEOREM

let G be a bipartite graph with bipartition $\{A, B\}$.
Then G has a matching that matches all the vertices of A if and only if G satisfies the Hall's condition

Hall's condition

$$\forall\, S \subseteq A, \quad |N_G(S)| \geq |S|$$

↳ neighbourhood of S in G



$M = \{ a_1b_1, a_2b_3, a_3b_4 \}$.

let $S = \{a_1, a_2\}$

$N_G(S) = \{ b_1, b_2, b_3 \}$ Hence $|N_G(S)| \geq |S|$

similarly for $S = \{a_1, a_2, a_3\}$ $\quad$ $S = \{a_3\}$

$N_G(S) = \{ b_1, b_2, b_3, b_4 \}$. $\quad$ $N_G(S) = \{b_3, b_4$

Proof:

FORWARD: if G has matching that matches all vertices in A, then G statisfies Hall's condition.

trivial,

If we a matching ⇒ independent sets

thus for every subset $S \subseteq A$

there would be atleast $|S|$ neighbours in B.

endpoints of the matching edges.

BACKWARD => If Halls condition true, then G has a matching that matches all the vertices in A.

given a bipartite graph,

$\forall S, \ |S| \leq |N_G(S)|$

exercise: proof by dilworth theorem 2.

* Sperner's Theorem (1928)

Let F be a family of subsets of [n]

Further, it is given that F is an antichain under the containment relation (subset relation)

→ $P = (\text{Powerset } [n], \subseteq)$

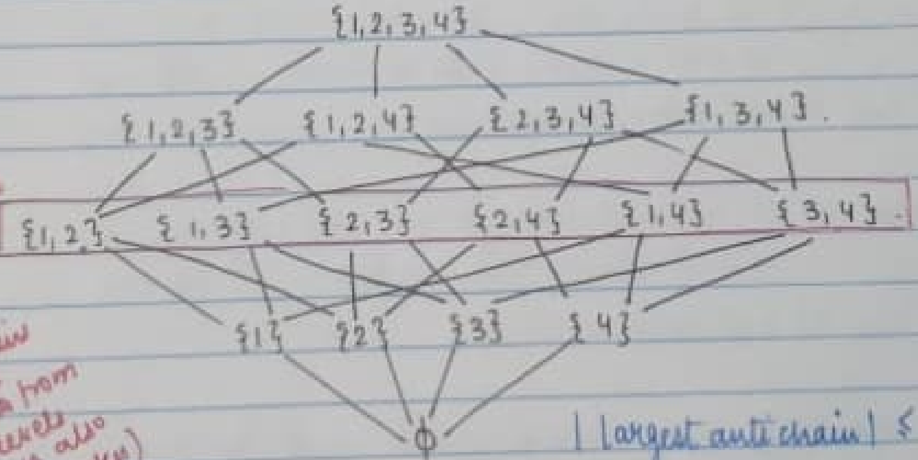
Size of largest antichain $\leq {}^nC_{n/2}$

Then,

$$|F| \leq {}^nC_{\lfloor n/2 \rfloor}$$

For Example: n=4.

Hasse diagram:



one of the longest antichain

↓

many antichains of size 6 (elements from diff levels can also be taken)

$|\text{largest antichain}| \leq {}^4C_2 = 6$

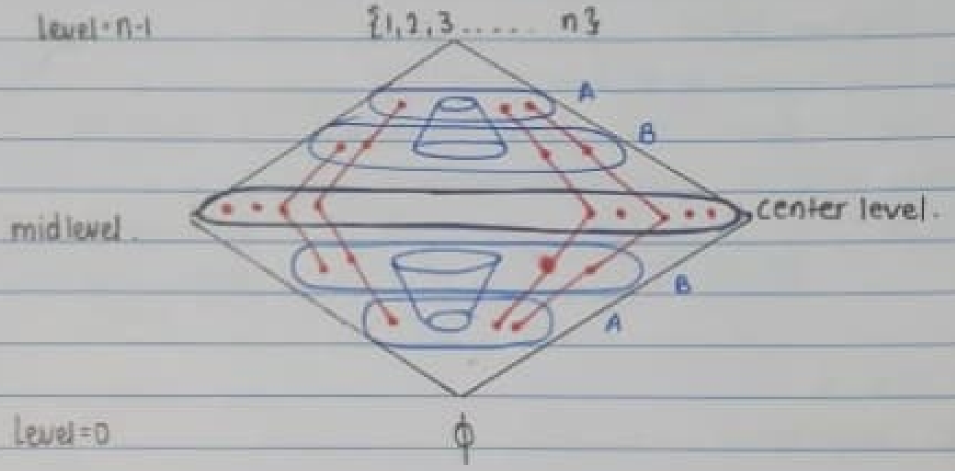
hence using dilworth,

To prove the largest antichain $\leq {}^nC_{n/2}$

we will prove,

the elements of the poset $P = (\text{PowerSet}([n]), \subseteq)$ can be partitioned into $\binom{n}{\lfloor n/2 \rfloor}$ chains.

in general Hasse diagram



for A_i, B_i in below level (going bottom to center)

A_i & B_i satisfy Halls condition $|B| > |A|$

thus A_i & B_i have a matching

Similarly for A_i, B_i in above mid level (top to center)

A_i & B_i satisfy Halls condition $|B| > |A|$

A_i & B_i have matching

Stiching matching from bottom to mid and top to mid.
we form chains partition.

so maximum chains (maximum elements in center level)

$\longrightarrow {}^nC_{\lfloor n/2 \rfloor}$

there would be elements in center level which are not used,
can be taken as single element chains
thus
all elements in center chain level included in chains (separate)
minimum chains = ${}^nC_{n/2}$

LYM Inequality

let F be a family of subsets of [n] Further, F is an antichain under the containment relation (subset).
where, $F = \{A_1, A_2 \ldots A_m\}$.
then

$$\sum_{i=1}^{m} \frac{1}{{}^nC_{|A_i|}} \le 1$$

${}^nC_{|A_i|}$ is maximised for $|A_i| = n/2$ (middle row elements)

thus,

$$\underbrace{m\sum \frac{1}{{}^nC_{[n/2]}}}_{\frac{m}{{}^nC_{n/2}}} \le \sum^{m} \frac{1}{{}^nC_{|A_i|}} \le 1$$

$$\Rightarrow \frac{m}{{}^nC_{n/2}} \le 1 \Rightarrow \boxed{m \le {}^nC_{[n/2]}}$$ Sperner thm.

prooving LYM inequality implies sperner's theorem

Proof: $F = \{A_1, A_2, A_3 \ldots A_m\}$
To show: $\sum^{m} \frac{1}{{}^nC_{|A_i|}} \le 1$

let σ be a linear permutation of [n].
For a set $A_i \in F$
we say A_i is present in σ if

A_i is present in σ if the elements of A_i are precisely the first $|A_i|$ elements of σ

Eg $A_i = \{3, 5, 6\}$

$\sigma_1 : 1\ 2\ 4\ \underline{3\ 5\ 6}\ 7 \ldots$ A_i not present (even though exist not first elements)

$\sigma_2 : \underline{5\ 6\ 3}\ 2\ 1\ 4 \ldots$ A_i present

let $G = \left\{ (\sigma, A_i) : \begin{array}{l} \sigma \text{ is any permutation of } [n] \text{ and} \\ A_i \in F \text{ is 'present' in } \sigma . \end{array} \right\}$

$$|G| \leq n! \cdot 1$$

↳ in a given permutation only 1 subset can be present. (atmost).

↑ no. of permutations of n elements.

no two subsets would be subsets of each other,
⇒ they are antichains
so in one permutation, there will be atmost 1 set that would be present.

For a given set A_i, the no. of permutations that it can be present ⇒ $|A_i|!\,(n-|A_i|)!$

elements of A_i ↳ remaining elements

Hence for each A_i, summation over m

$$\sum_{1}^{m} |A_i|!\,(n-|A_i|)! \leq |G|$$

$$\Rightarrow \quad \sum^{m} |A_i|!\,(n-|A_i|)! \leq n!$$

$$\Rightarrow \quad \sum^{m} \frac{1}{\frac{n!}{|A_i|!\,(n-|A_i|)!}} \leq 1$$

$$\Rightarrow \quad \sum^{m} \frac{1}{{}^{n}C_{|A_i|}} \leq 1$$

* Bollobas theorem

Theorem let $(A_1, A_2 \dots A_m)$ and $(B_1, B_2 \dots B_m)$ be two sequences of sets such that $\forall\, i,j \in [m]$, $A_i \cap B_j = \phi$ if & only if $i = j$ then,

$$\sum_{i=1}^{m} \frac{1}{\binom{a_i + b_i}{a_i}} \leq 1$$

where $a_i = |A_i|$

$b_i = |B_i|$

$\Rightarrow A_i \cap B_j = \phi \;;\; i = j$

$\Rightarrow A_i \cap B_j \neq \phi \;;\; i \neq j$

How bollobas theorem $\Rightarrow$ LYM inequality.

↳ $F = \{A_1 \dots A_m\}$ antichain made of subsets of $[n]$

$$\sum^{m} 1/{}^nC_{|A_i|} \leq 1$$

let $(A_1, A_2, A_3 \dots A_m)$ &

$(\bar{A_1}, \bar{A_2}, \bar{A_3} \dots \bar{A_m})$

Here $A_i \cap \bar{A_j} = \phi$ if $i = j$

thus we use $B = \bar{A}$

intersection would be ϕ of 2 complements

$$\hookrightarrow \sum \frac{1}{\binom{a_i + b_i}{a_i}} \leq 1 \Rightarrow \sum \frac{1}{({}^nC_{a_i})} \leq 1$$

here $a_i = |A_i|$ $b_i = |\bar{A_i}| = n - |A_i|$

Proof ⇒

let $\bigcup_{i=1}^{m} (A_i \cup B_i) = X = \{x_1, x_2, x_3 \ldots\ldots x_n\}$

let ~~A~~ $|X| = n$

let σ be a permutation of $\{x_1 x_2 \ldots x_n\}$ (n! ways)

$\sigma : x_3, \underset{\bullet}{x_2} \; x_7 \; \underset{\bullet}{x_1} \; \underset{\bullet}{x_5} \ldots \underset{\bullet}{x_{11}} \ldots x_n \ldots \underset{\bullet}{x_{10}} \ldots x_{12}$

let

$A_1 = \{x_2, x_1\}$ •

$B_1 = \{x_5, x_{10}, x_{11}\}$ •

The pair (A_i, B_i) is "present" in σ of X if, every element of A_i, appear before every element of B_i

let $G = \left\{ (\sigma, (A_i, B_i)) : \begin{array}{l} \sigma \text{ is a permutation of } X,\ A_i, B_i \text{ are sets} \\ \text{present in the 2 sequences given in thm.} \\ (A_i, B_i) \text{ is present in } \sigma \end{array} \right\}$

let $\sigma : \overset{■}{x_3} \; x_2 \; \overset{■}{\underset{\bullet}{x_{10}}} \; \overset{\bullet}{\underset{■}{x_{11}}} \; \overset{\bullet}{x_1} \; \underset{■}{x_{12}} \; x_9 \; \underset{\bullet}{x_8} \; \underset{\bullet}{x_4} \ldots\ldots$

$A_i = \{x_3, x_{10}\}$ ■ $\quad A_j = \{x_{11}\ x_{12}\}$

$B_i = \{x_{11}, x_1\}$ • $\quad B_j = \{x_{10}\ x_8, x_4\}$

(A_i, B_i) present $\quad (A_j, B_j)$ not present.

How many pairs can be present in 1 permutation ⇒ 1

because if (A_i, B_i) is present A_i appear first then B_i

then $A_i \cap B_j \neq \phi$ & $A_j \cap B_i \neq \phi$ A_j will have elements of B_j & B_j will hav

elements of A_i ⇒ A_j (all) elements will not come before B_j

$$|G| \leq \underset{\text{no of permutations}}{\underset{\uparrow}{n!}} \cdot 1$$

↳ no-of pairs present in each perm.

for a given (A_i, B_i). can be be present in How many perm ??

$$|X \setminus \{A_i \cup B_i\}| = n - a_i - b_i = k \qquad (A_i \cap B_i = \phi)$$

$$\overbrace{\rule{4cm}{0.4pt}}^{a_i}\ \overbrace{\rule{4cm}{0.4pt}}^{b_i}$$

a_i & b_i elements can be arranged in $a!b!$

remainining k elements be $x_1, x_2 \ldots x_k \longrightarrow k!$ ways

take one such arrangement

so How $(a_i + b_i)$ elements can be placd in remaining element arrangenment

$\Rightarrow$ to linear k perm $\Rightarrow$ $k+1$ spaces to be filled.

choose $(a_i + b_i)$ locations from the spaces. Repetition allowed.

(Choosing r element n elements with rep alloweed ${}^{n+r-1}C_r$)

$$\binom{k+1+a_i+b_i-1}{a_i+b_i} \; ; \; k = n - a_i - b_i$$

$$\longrightarrow {}^{n}C_{a_i+b_i}$$

For a given (A_i, B_i)

Total ways:

$$a_i!\, b_i!\, k!\, {}^{n}C_{a_i+b_i} \Rightarrow \frac{a_i!\, b_i!\, (n-a_i-b_i)!\, n!}{(a_i+b_i)!\, (n-a_i-b_i)!}$$

$$\Rightarrow \frac{n!\, a_i!\, b_i!}{(a_i+b_i)!} \Rightarrow \frac{n!}{(a_i+b_i)!\, / a!\, b!}$$

$$\Rightarrow \frac{n!}{\left({}^{a_i+b_i}C_{a_i}\right)}$$

$$\sum_{i=1}^{m} \frac{n!}{\binom{a_i+b_i}{a_i}} \leq G \leq n!$$

$$\Rightarrow \quad \sum_{i=1}^{m} \frac{1}{\binom{a_i+b_i}{a_i}} \leq 1 \xrightarrow{\qquad} \text{prooved}$$

COROLLARY

let $(A_1, A_2 \quad A_m)$ and $(B_1, B_2, B_3 \ldots B_m)$ be two sequences of sets such that $A_i \cap B_j$ Φ iff $i=j$
let $|A_i| \leq a$, $|B_i| \leq b$, $\forall i \in [m]$
them $m \leq \binom{a+b}{a}$

Proof :

$$\sum^{m} \frac{1}{\binom{a_i+b_i}{a_i}} \leq 1$$

and

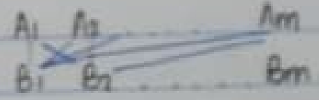
$$\sum^{m} \frac{1}{{}^{a+b}C_a} \leq \sum^{m} \frac{1}{{}^{a_i+b_i}C_{a_i}} \quad (\text{maximising } {}^{a_i+b_i}C_{a_i})$$

thus

$$\frac{m}{{}^{a+b}C_a} \leq 1$$

$$\Rightarrow \quad m \leq \binom{a+b}{a}$$

special case. $|A_i| = a$, $|B_i| = b$ $\forall i \in [m]$



B_1 is a hitting set for $\{B_2, B_3 \quad B_m\}$
B_2 is a hitting set for $\{B_1, B \quad B_m\}$

property of $\{A_1, A_2 \dots A_m\}$ is that if we remove any set from it, then the resulting family has a resulting hitting set of size b

In another words $\{A_1, A_2 \dots A_m\}$ is a minimal family that has no hitting set of size b.

"Skew" version of Bollobas Thm.

let $(A_1, A_2 \dots A_m)$ & $(B_1, B_2 \dots B_m)$ be 2 sequences of sets such that $\forall\ i, j \in [m]$

i) $A_i \cap B_j = \phi$ if $i = j$

ii) $A_i \cap B_j \neq \phi$ if $i \neq j$ & $i < j$

$$\sum^{m} \frac{1}{{}^{a_i + b_i}C_{a_i}} \leq 1 \qquad \text{where } a_i = |A_i|$$

$$b_i = |B_i|$$

also if $|A_i| \leq a$ & $|B_i| \leq b$

$$m < {}^{a+b}C_a.$$

* <u>Application</u> of <u>Bollobas Thm</u>

⇒ system of distinct representations

a system of sets $S_1, S_2, S_3 \ldots S_k$ is the k-tuple $(x_1, x_2, x_3 \ldots x_k)$ such that

distinct ⇒

$$\forall i \in [k] \quad x_i \in S_i$$

$$\forall i, j \in [k] \;\&\; i \neq j \quad x_i \neq x_j$$

Further $(x_1, x_2 \ldots x_k)$ is a system of STRONG system of distinct representations if

i) $\forall i, j \in [k],\ i \neq j \Rightarrow x_i \notin S_j$ (additional property)

ii) $\forall i \in [k],\ x_i \in S_i$

Eg. $S_1 = \{3, 4, 5, 9\}$ $S_2 = \{1, 2, 4, 8\}$ $S_3 = \{1, 4, 7, 5\}$

$(3, 2, 7)$ → strong tuple of S_1, S_2, S_3.

⇒ Theorem [Furo, Tuza, 1985]

let F be a family of size greater than $({}^{r+k}C_k)$. Further every set in F is of size at most r. Then ∃ some "k+1" sets in F that have a strong system of distinct representation.

<u>Proof</u>: Arrange the sets in F in non-increasing order of their sizes let $F = \{S_1, S_2 \ldots S_m\}$.

$$|S_1| \geq |S_2| \geq |S_3| \geq \ldots \geq |S_m|$$

where $m > {}^{r+k}C_k$

Assume for the sake of contradiction that no k+1 sets in F have strong system of dist rep^n.

given $m > \binom{r+k}{k}$ and $|S_i| \leq r$

So,

$|S_1| \geq |S_2| \geq |S_3| \ldots\ldots |S_i| \geq \ldots |S_j| \ldots \geq |S_m|$

let us define another family

$T_1, T_2, T_3 \ldots T_i, T_j \ldots T_k$

where

T_j is a minimal hitting set for

$(S_1/S_j, S_2/S_j, \ldots S_{(j-1)}/S_j)$

all will be no empty because of non-increasing fact

T_j is a minimal set that intersects all

$S_1/S_j, S_2/S_j, S_3/S_j \ldots S_i/S_j \ldots S_{j-1}/S_j$

Property: $\forall e_l \in T_j \; \exists S_l, l<j \Rightarrow S_l \cap T_j = \{e_l\}$.

So for 2 sequences

$(S_1, S_2, S_3 \ldots S_k)$ and $(T_1, T_2, T_3, \ldots T_k)$

$\Rightarrow S_i \cap T_i = \phi$

since T_i = set intersect with $(S_1/S_i, S_2/S_i \ldots S_{i-1}/i)$

↳ wont consist any element of S_i

$\Rightarrow |S_i| \leq r$

$\Rightarrow |T_i| \leq k$ (maximum size when T_k = minimal set $(S_1/k \ldots S_{k-1}/k)$

↳ Claim

Proof: Suppose $\exists j \; |T_j| \geq k+1$

$T_j = \{e_1, e_2 \ldots e_{k+1} \ldots\}$.

let S_1 be a set such that $T_j \cap S_1 = e_1$, $S_2 \ldots T_j \cap S_2 = e_2 \ldots$

$S_{k+1} \ldots T_j \cap S_{k+1} = e_{k+1}$ $\Rightarrow (k+1)$ tuple $(e_1, e_2 \ldots e_k) \rightarrow$ strong system

↳ shows there exist $(k+1)$ sets in F that have strong system

but we assume that no $(k+1)$ sets exist, thus our claim is true.

contradiction.

Thus.

$\Rightarrow S_1, S_2 \ldots\ldots S_m$

$\Rightarrow T_1, T_2 \ldots\ldots T_m$

a) $S_i \cap T_i = \phi$

b) $|S_i| \leq r$

c) $|T_i| \leq k$

Skewed version of Bollobas Theorem.

$$\sum_{1}^{m} \frac{1}{{}^{|S_i|+|T_i|}C_{|S_i|}} \leq 1$$

$$\frac{m}{{}^{r+k}C_r} \leq 1$$

$$\Rightarrow \quad m \leq {}^{r+k}C_r$$

but this contradicts the fact that $m > {}^{r+k}C_r$

Hence our assumption is wrong that no $k+1$ subcollection F has strong system of Representation.

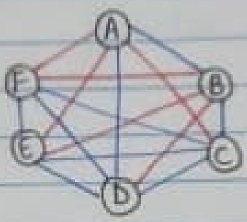
✯ Probabilistic Method in Combinatorics

⇒ Ramsey Numbers

Ques: In a group of 6 people, show that there are either 3 mutual friends or 3 mutual enemies (strangers).
Δ Δ

•F •S

K6
⇒ complete graph
⇒ $^6C_2$ edges

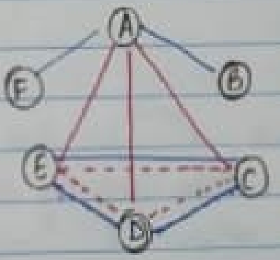


inthis case there is a blue triangle (3 mutual friends)
ΔCDF, ΔDEC, ΔEEC, ΔFED

no matter, how you colour the edges of a K6 with Red or blule colour, you will alway encounter either a BLUE K3 Δ or a RED K3 Δ

Proof: from the vertices, like A, there are 5 edges coming.
By pigeonhole principle, there would be atleast 3 of edges which are of same colour
only 2 colours are there
⇒ ~~6 boxes~~, 2 boxes, 5 pigeons ⇒ $\lceil 5/2 \rceil = 3$ atleast pigeons in 1 box

WLOG, let A has 3 red eges
check for edge CD or DE or CE ⇒ if red ⇒ Red K3 → prooved
if blue
↓
this will lead to a BLUE K3 with the bad edges CD, DE, CE ΔCDE



* Ramsey Number

· symmetric Ramsey no.

$R(k,k)$ is the minimum n such that no matter how we colour the edges of K_n with 2 colours (R/B) we will will surely encounter either a Red K_k or Blue K_k or both.

complete graph K_n

complete graph on k vertices K-clique

eg. $R(3,3) \leq 6$; $R(4,4) = 18$

$\therefore R(3,3) = 6$ (proove)

$\Rightarrow R(k,l) \longrightarrow$ Blue l-clique

↳ Red k-clique

whereas

$43 \leq R(5,5) \leq 48$

$102 \leq R(6,6) \leq 165$

$205 \leq R(7,7) \leq 540$.

Theorem $R(k,k) \leq 2^{2k-3}$

proof:

Using Pigeonhole principle,

given there are 2^{2k-3} vertices, so from 1 vertex no. of edges $= 2^{2k-3}-1$

so,

2 boxes, $2^{2k-3}-1$ pigeons $\Rightarrow$ atleast $\left\lceil \frac{2^{2k-3}-1}{2} \right\rceil$ pigeons in 1 box

$\Rightarrow \lceil 2^{2k-4} - 0.5 \rceil \approx 2^{2k-4}$

assume 2^{2k-4} are red in colour

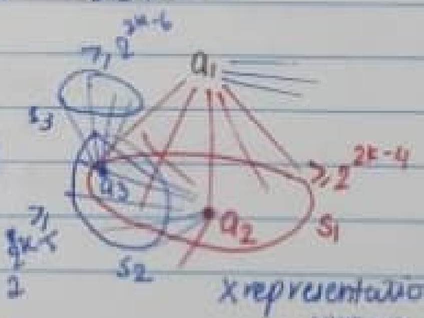
a_1: Type 1 vertex majority red edges.

from a_2, atleast 2^{2k-5} edges would be of same colour (excluding $a_1 - a_2$ edge)

a_2: Type 2 vertex majority blue edges.

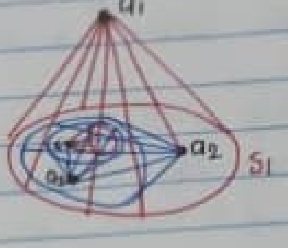
same set

subsets creation



X representation wrong.

So $|S_1| \geq 2^{2k-4}$ Type 1
$|S_2| \geq 2^{2k-5}$ Type 2
$|S_3| \geq 2^{2k-6}$ Type 2
$|S_4| \geq 2^{2k-7}$ Type 1



$a_1 \quad a_2 \quad a_3 \quad a_4 \quad \cdots\cdots \quad a_{2k-3}$

$a_1 \downarrow$ sees 2^{2k-4} red edges

$a_3 \downarrow$ 2^{2k-6} Blue edge

$a_4 \downarrow$ sees 2^{2k-7} red edges

a_{2k-3} — b

$a_{2k-3} \downarrow \geq 2^{2k-4-2k+3+1}$

≥ 1 blue edges.

$\underbrace{a_1 \quad a_2 \quad a_3 \quad a_4 \quad \cdots\cdots \quad a_{2k-3}}_{2k-3 \text{ vertices}}$

atleast $\left\lceil \frac{2k-3}{2} \right\rceil$ are of same type by pigeonhole

$\Downarrow$
$k-1$

WLOG, $(k-1)$ vertices be of type 1

$\hookrightarrow$ together with $b \to k$ red clique.

For Lower bound, Probabilistic method.

<u>Theorem</u> : $R(k,k) > 2^{\lfloor k/2 \rfloor}$ for any $k \geq 3$

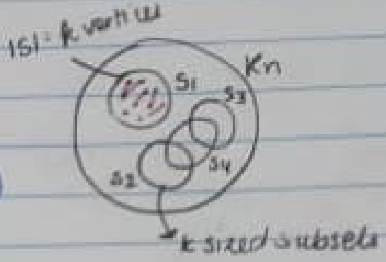
Proof: let $n = 2^{\lfloor k/2 \rfloor}$
in order to prove the theorem we need to show a 2-colouring of edges of K_n which has a neither a red k-clique nor a blue k-clique.

Take K_n. Each edge is colored independently either given a red or blue colour & uniformly at random.

$P(\text{Edge is red}) = P(\text{Edge is blue}) = 1/2$ (independent of other edges)

or at each edge, toss a coin (unbiased)

if $H \Rightarrow$ Red else $T \Rightarrow$ Blue.



$\Pr[S \text{ is Red k-clique}] = (1/2)^{{}^kC_2}$

↳ all edges in S (${}^kC_2$ edges) are red in colour

$\Pr[S \text{ is a Blue k-clique}] = \left(\frac{1}{2}\right)^{{}^kC_2}$

No of k sized Subsets $= {}^nC_k$

$(S_1, S_2 \dots S_{{}^nC_k})$

BAD EVENT : One of $S_1, S_2 \dots S_{{}^nC_k}$ is either a red k-clique or a blue k-clique.

$$\Pr\left[\begin{array}{l} S_1 \text{ is a red k-clique} \cup S_2 \text{ is red} \cdots \cup S_{n_k} \text{ is a red} \\ \cup S_1 \text{ is a Blue} \cdots \cup S^n{}_k \text{ is a blue} \end{array}\right]$$

By union bound $\Rightarrow \Pr[A \cup B] \leq \Pr[A] + \Pr[B]$

$$\Pr(\text{Bad event}) \leq \sum_{i}^{{}^nC_k} \Pr[S_i \text{ is Red k clique}] + \Pr[S_i \text{ is a blue k clique}]$$

$$\leq \sum_{i}^{{}^nC_k} (1/2)^{{}^kC_2} + (1/2)^{{}^kC_2} \leq \sum 2^{1-{}^kC_2}$$

$$\leq \frac{{}^nC_k}{2^{{}^kC_2 - 1}} \longrightarrow Ⓐ$$

if Ⓐ < 1, there is a non-zero probability that a random coloring we did contain a neither a red k-clique nor blue k-clique.

because Pr [Red or blue k clique] < 1 (A)

Pr [neither Red or blue] $\not> 0 \neq 0$

$\Rightarrow$ So we have to show that Ⓐ < 1 $\Rightarrow$ $n = 2^{\lfloor k/2 \rfloor}$

$$\frac{{}^{n}C_{k}}{2^{{}^{k}C_{2}-1}} < \frac{n^{k}}{k!\, 2^{{}^{k}C_{2}-1}}$$

$$\leq \frac{2^{k^2/2}}{k!\, 2^{{}^{k}C_{2}-1}}$$

$$\left({}^{k}C_{2} - 1 = \frac{k(k-1)}{2} - 1 = \frac{k^2}{2} - \frac{k}{2} - 1 \right.$$

$$\leq \frac{2^{k/2+1}\, 2^{k^2/2}}{k!\, 2^{k^2/2}} \leq \boxed{\frac{2^{k/2+1}}{k!}} < 1$$

when $k \geq 3$.

<u>Algo</u> (neither a red / blue).

no. of colorings = $2^{{}^{n}C_{2}}$ (${}^{n}C_{2}$ total edges → either red / blue)

↳ check all (trivial - time complexity).

Randomised : $\frac{2^{k/2+1}}{k!}$ for $k = 20$ $\Rightarrow$ $\frac{2^{11}}{20!} \ll 1$ (very small)

probability of Red or blue k-clique is very small. thus by random colouring may lead to the required colouring if not try again fresh!!

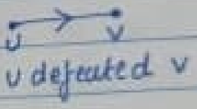
Combinatorics

WEEK4 Onwards

Tournament

A complete graph with directed edges (oriented edges)

Let T be a Tournament on n vertices, We say T satisfies the Property P_k, if for every set S of K players / vertices in T, there is a player / vertex who has defeated everybody in S.



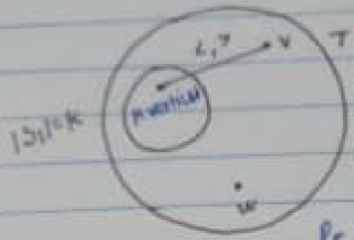
For Every +ve integer k does there always exist a Tournament on n vertices that satisfies property P_k ?? Yes

So given a +ve K what is the minimum 'n' such that satisfies P_K

THEOREM : if $^nC_K \left(1 - \frac{1}{2^k}\right)^{n-k} < 1$, then there is a tournament on n vertices that satisfies Property P_K.

Proof: Construct a random Tournament T on n vertices in the following

- → Take a complete graph on n vertices ; K_n
- → for each edge 'e', independently orient if from > or < prob $\frac{1}{2}$; toss a coin (unbiased) & orient the edge e base on outcome of coin toss.



E: Probability v defeats all players inside S_1

$$Pr[E] = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \ldots \frac{1}{2} = \frac{1}{2^k}$$
$$\underbrace{\qquad\qquad}_{k \text{ vertices}}$$

$Pr[v$ does not defeat all players in $S_1] = 1 - \frac{1}{2^K}$

→ probability of Bad event.

$Pr[w$ does not defeat all players in $S_1] = 1 - \frac{1}{2^k}$.

Thus for $(n-K)$ vertices outside S_1,
Probability that all $(n-K)$ vertices, none of them defeat all vertices in S_1.

$$\Rightarrow \underbrace{\left(1-\frac{1}{2^k}\right)\left(1-\frac{1}{2^k}\right)\cdots\left(1-\frac{1}{2^k}\right)}_{(n-K)\text{ times}} = \frac{1.}{\quad}$$

$$\Rightarrow \left(1-\frac{1}{2^K}\right)^{n-K}$$

E1 ?

↳ Bad event when no player defeats everybody in

For, there are ${}^nC_K$ no. of sets of size k in T

$$Pr[\text{T does not satisfy property } P_K] = Pr[\overset{\text{independent}}{E_1 \cup E_2 \cup E_3 \ldots\ E_{{}^nC_K}}]$$
$$\leq Pr[E_1] + Pr[E_2] \ldots\ Pr[E_{{}^nC_K}]$$

$$Pr[\text{T does not satisfy } P_K] \leq {}^nC_K\left(1-\frac{1}{2^K}\right) < 1$$

↑ given in Ques.

uice Prob. of Bad event < 1

Probability of T satisfying property $P_K \neq 0$

$Pr[T$ satisfy $P_K] = 1 - Pr[$Bad event$] > 0$.

So, what is the minimum 'n' ??
we showed that ∃ a T on n vertices, which satisfy P_k provided

$$ {}^nC_k \left(1-\frac{1}{2^k}\right)^{n-k} < 1 $$

$$ {}^nC_k \le \left(\frac{en}{k}\right)^k $$

$\because 1+x \le e^x$

$$ \Rightarrow \left(1-\frac{1}{2^k}\right) \le e^{-1/2^k} = \frac{1}{e^{1/2^k}} $$

$$ \left(1-\frac{1}{2^k}\right)^{n-k} \le \frac{1}{e^{n-k/2^k}} $$

$$ \Rightarrow \left(\frac{en}{k}\right)^k \cdot \frac{1}{e^{(n-k)/2^k}} < 1 $$ (upper bounds taken).

calculating,

$$ \boxed{n \geq 2\log_e 2 \cdot k^2 \cdot 2^k} $$

let $f(k)$ be the smallest n, such that there is a tournament on n nodes that satisfies P_k.

$$ ck2^k \le f(k) \le 2\ln 2 \cdot k^2 \, 2^k $$

(Szekeres)
c - constant.

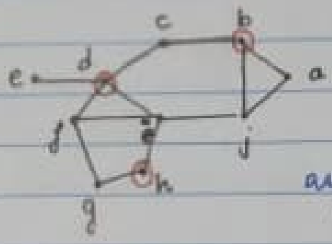
* DOMINATING SET

set of vertices, such that $\{v_1, v_2 \ldots v_k\} \in V$
such that

$$\bigcup_{i=1}^{k} (N(v_i) \cup v_i) = V$$

set of V is itself a dominating set, but we are interested in minimum no of vertices ∈ V that could satisfy the condition.

Eg.



b dominates {b, a, j, c}
d dominates {d, e, f, i}.
h dominates {h, g}

and

$\{b,a,j,c\} \cup \{d,e,f,i\} \cup \{h,g\} = V$

there can be multiple dominating sets,

Here {b,d,h} → dominating set.

Definition let G be a graph with vertex set V(G) & edge set E(G)
A set $S \subseteq V(G)$ is a dominating set for G if every vertex in G is either present in S or is a neighbour
Some vertex in S

$\forall v \in V(G)$, such that $(v \in S)$ or $(\exists v \in S$ and $uv \in E(G))$

itself in S   or a neighbour.

Minimum degree :-

$$\delta = \min(\deg(v) ; v \in V(G))$$

in the above eg. $\delta = 1$ $(\deg(e) = 1)$

Theorem

Let $G = (V,E)$ be a graph on n vertices with minimum degree $\delta > 1$. Then G has a dominating set of size at most

$$\frac{n}{\delta+1}\left(1+\ln(\delta+1)\right)$$

PROBABILITY FUNDAMENTALS

Sample Space Ω : Set of all outcomes

Event : Subset of Ω

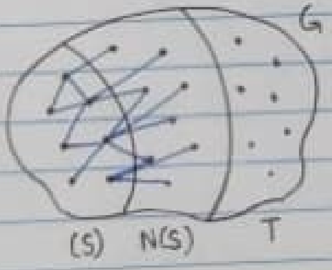
Random Variable $X : \Omega \rightarrow \mathbb{R}$ (function from sample space to $\mathbb{R}$)

Expectation of X : $E[X] = \sum_{X=x} Pr[X=x] \cdot x$

linearity $\Rightarrow E[\alpha X + \beta Y] = \alpha E[X] + \beta E[Y]$.

Proof:

Random Experiment : Construct a set $S \subseteq V(G)$, choose each vertex in G independently with probability p into set S



S dominates S and N(S) but vertices in T are not dominated by S

thus if $T = \phi$ then S is dominati

if $T \neq \phi \Rightarrow$ S is not dominating

Defining Random Var

X_S : denotes size of S

$Y_T = X_T$: denotes size of T

for each vertex $v \in V(G)$

$$X_v = \begin{Bmatrix} 1 & \text{if } v \in S \\ 0 & \text{otherwise} \end{Bmatrix}$$

$$Y_v = \begin{Bmatrix} 1, & \text{if } v \in T \\ 0 & \text{otherwise} \end{Bmatrix}$$

$Pr[X_v = 1] = p$

$E[X_v] = 1.p + 0(1-p) = p.$ $\longrightarrow$ ①

$Pr[Y_v = 1] =$ $v \in T$, i.e. neither v nor $N(v)$ are in S $(\deg(v)+1)$ vertices not in S.

v has $\geq \delta$ neighbours

$(\delta+1)$ are not in S

$Pr[Y_v=1] = (1-p)^{\deg(v)+1}$

$Pr[Y_v=1] \leq (1-p)^{\delta+1}$

$E[Y_v] = (1-p)^{\deg(v)+1} \leq (1-p)^{\delta+1}$ $\longrightarrow$ ②

$\Rightarrow$ $\boxed{X_S = \sum_{v \in V(G)} X_v}$ and $\boxed{Y_T = \sum_{v \in V(G)} Y_v}$

By linearity of expectations.

$$E[X_S] = E[\sum X_v] = \sum E[X_v] = np \longrightarrow ③$$

$$E[Y_T] = E[\sum Y_v] = \sum E[Y_v] \leq n(1-p)^{\delta+1} \longrightarrow ④$$

We Observe that $S \cup T$ is a dominating set for G.

$|S \cup T| \le Z$ (since S & T are disjoint)

where $\Rightarrow |S \cup T| = Z$

$Z = X_S + Y_T$ (Random Variable)

$$E[Z] = E[X_S + Y_T]$$
$$= E[X_S] + E[Y_T]$$
$$\le np + n(1-p)^{\delta+1}$$

$1+x \le e^{x}$

$$E[Z] \le n\left(p + (1-p)^{\delta+1}\right) \le np + \frac{n}{e^{p(\delta+1)}}$$

differentiating to get the minima of E[Z], we get value of p

$$\Rightarrow \frac{d}{dp}\left(p + (1-p)^{\delta+1}\right) = 1 + (\delta+1)(1-p)^{\delta}(-1) = 0$$

$$\Rightarrow (\delta+1)(1-p)^{\delta} = 1$$

$$\ldots \boxed{p = \frac{\ln(\delta+1)}{\delta+1}}$$

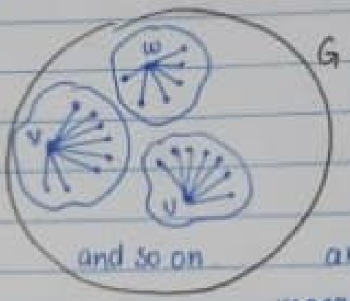
this differentiating and $=0$

$$E[Z] \le \frac{n\ln(\delta+1)}{\delta+1} + \frac{n}{e^{\ln(\delta+1)}} = \frac{n\ln(\delta+1)}{\delta+1} + \frac{n}{\delta+1}$$

$$E[Z] \le \frac{n}{\delta+1}\left(1 + \ln(\delta+1)\right) \longrightarrow \text{proved.}$$

* Deterministic Algo for dominating set of size $\frac{n}{\delta+1}(1+\ln(\delta+1))$

Take vertices whose deg is maximum, and delete all the neighbours of that vertex from G.



Choose a vertex v with max numbers of neighbours and remove $(1+d(v))$ vertices from G.
and includ v in dominating set
repeat until G is empty

$S = \{v, w, u, z \dots\}$.

claim : $|S| \le \frac{n}{\delta+1}(1+\ln(\delta+1))$

let at given point in algo, your graph $G = \{S, N(S), T\}$.
let $|T| = t$

$N(v)$ = neighbours of v open neighbourhood
$N[v] = \{v\} \cup N(v)$ close neighbourhood.
$|N[v]| = |N(v)| + 1$

$$\sum_{i=1}^{t} |N[v_i]| \geq t(\delta+1)$$
↳ atleast degree.

By pigeonhole principle $\exists\, v \in V(G)$, that v present in $\geq \frac{t(\delta+1)}{n}$ such structures

$t(\delta+1)$ structures or baskets and n vertices.

Choose that v in S. This vertex v is dominating $\geq \frac{t(\delta+1)}{n}$ vertices of T

$|\text{old } T| = t$

$|\text{new } T| \leq t - \frac{t(\delta+1)}{n}$

repeating the steps

Initially $|T| = n$

1st round : $|T| \leq n - \frac{n(\delta+1)}{n} := \overbrace{n \cdot \left(1 - \left(\frac{\delta+1}{n}\right)\right)}^{n_1}$

2nd round $|T| \leq n_1 \left(1 - \frac{\delta+1}{n}\right) := n\left(1 - \left(\frac{\delta+1}{n}\right)\right)^2$

$\vdots$ $(1+x \leq e^x)$

K rounds $|T| \leq n\left(1 - \left(\frac{\delta+1}{n}\right)\right)^K \leq \frac{n}{e^{(\delta+1/n)K}}$

We want,

$$\frac{n}{e^{(\delta+1/n)K}} \leq \frac{n}{\delta+1}$$ → add all the remaining vertices to the dominating set.

this happen

$$K = \frac{n\,(\ln(\delta+1))}{\delta+1}$$

So Size

$$\Rightarrow K + \frac{n}{\delta+1} \Rightarrow \frac{n}{\delta+1}(1 + \ln(\delta+1))$$

we show that after k rounds, the no. of vertices left undominated vertices $\leq \frac{n}{\delta+1}$

* Sum-free Sets

A set of integers S, is said to be sum-free if $\forall\ x, y, z \in S$
$x+y \neq z$ [x, y, z may not be distinct]

i.e. $\forall\ x, y \in S$

$x+y \notin S$

example

① $S = \{2, 3, 7, 8\}$ ✓

② $S = \{2, 3, 4\}$ × $(2+2 = 4)$

③ $S = \{-5, -3, 100\}$ ✓

note: Sum free sets → would not have 0 (because sum = element ∵ violates
similarly cant have if $x \in S$ cannot have $2x \notin S$.

Ques Given $B = \{b_1, b_2, b_3 \dots b_n\}$ be a set of n Integers.
How large the subset of B can be, to be a sum free set

THEOREM [Erdos 1965]

Every set $B = \{b_1, b_2, b_3 \dots b_n\}$ of n non-zero integers contains a sum-free subset A of size $|A| > \frac{n}{3}$

Proof Let $P = 3k+2$ (for some k) be a prime number such that
$p > b_1,\ p > b_2 \dots,\ p > b_n$

set addition op modulo p

$(\mathbb{Z}_p, +)$

Let $C = \{k+1, k+2, \dots\dots, 2k+1\}$; $|C| = k+1$

Claim : C is a sum-free subset of the Abelian group $\mathbb{Z}_p$

set

(X, o) Binary op.

is a group if $\forall\ x_1 x_2 \in X$

i) $x_1 o x_2 \in X$ (closure)

ii) $x_1 o (x_2 o x_3) = (x_1 o x_2) o x_3$ (Ass

iii) $\exists\ e$ s.t $x_1 o e = x_1$ (Identity

iv) $\forall\ x \in X,\ \exists\ x^{-1} \in X$ st.

$x o x^{-1} = e$ (Inverse

Abelian group (X, o)

$\forall\ x_1, x_2 \in X,\ x_1 o x_2 = x_2 o x_1$

$(\mathbb{Z}_p, +) \Rightarrow (\{0, 1, 2, \cdots p-1\}, +)$

addition modulo p.

So Here $(\mathbb{Z}_p, +) = (\{0, 1, 2, 3 \ldots, 3k+1\}, +)$

$C = \{k+1, k+2 \ldots 2k+1\}$.

To show that for any x, y where $0 \leq x \leq k$ & $0 \leq y \leq k$

$$(k+1+x) \overset{\text{mod } p}{+} (k+1+y) \notin C$$

$$\Rightarrow \quad 2k+2+\underbrace{(x+y)}_{0 \leq x+y \leq 2k} \bmod (3k+2)$$

for eg for $x = y = 0$

$k+1+k+1 = 2k+2 \notin C$

$x = y = k$

$(2k+1+2k+1) \bmod (3k+2)$

$(4k+2) \bmod (3k+2) = k \notin C$

So for both limits $x+y = 0$

$2k+2 \bmod 3k+2 = 2k+2 \notin C$

$x+y = 2k$

$4k+2 \bmod 3k+2 = k \notin C$

thus <u>C is a sum free subset of $\mathbb{Z}_p$</u>

Given : $B = \{B_1, B_2 \ldots B_n\}$ n-non zero Integer.

To show: size of sumfree set of $B \geq n/3$

Proof : $p = 3k+2$ be a prime no, $p > b_1$, $p > b_2 \ldots p > b_n$

and $C = \{k+1, k+2 \ldots 2k+1\}$ is a sum-free subset of $\mathbb{Z}_p$ $|C| = k$

$$\frac{|C|}{p-1} = \frac{k+1}{p-1} = \frac{k+1}{3k+1} > \frac{1}{3}$$

↳ if exclude 0 from $\mathbb{Z}_p \rightarrow \{1, 2, \cdots p-1\} \rightarrow p-1$ numbers.

So, $|C| > \dfrac{p-1}{3}$

choose an x uniformly at random from $\{1, 2, 3 \ldots p-1\}$

consider any $b_i \in B$

for every $b_i \in B$.

$d_i \equiv b_i x \pmod{p}$ $\longrightarrow$ $b_i < p$ and $x > 0$

So, $b_i > 0$ thus $d_i > 0$

$0 < d_i \leq p-1$

and b_i cannot be a multiple of p

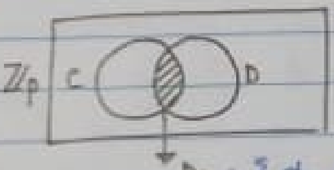
So,

$D = \{d_1, d_2, d_3 \ldots d_n\} \subseteq \{1, 2, 3 \ldots p-1\}$

and we know,

$C = \{k+1, k+2 \ldots 2k+1\} \subseteq \{1, 2, 3 \ldots p-1\}$

↳ sum free subset



$D_c = \{d_1, d_2 \ldots d_k\}$.

clearly D_c would be a sum set free of D

Claim: if $D_c = \{d_1, d_2 \ldots d_k\}$ sum free set of D

$\Rightarrow \{b_1, b_2 \ldots b_k\}$ sumset free of B.

Proof $\Rightarrow$ suppose claim is not true

$\Rightarrow b_1 + b_2 = b_3$ for some elements

$\Rightarrow b_1 x + b_2 x = b_3 x$

$\Rightarrow b_1 x + b_2 x \equiv b_3 x \pmod{p}$

$\Rightarrow d_1 + d_2 \equiv d_3 \pmod{p}$

contradicts $\{d_1, d_2, d_3 \ldots d_k\}$ is a sumfree set

thus assumption wrong

Claim is true.

To show, for some choice of $x \in [p-1]$
The set $D \cap C$ is large $> n/3$.

$B = \{b_1, b_2 \ldots b_n\}$
$p = 3k+2$
$C = \{k+1, k+2 \ldots 2k+1\}$ sumfree subset of $\mathbb{Z}_p$. $|C| > p-1/3$.
$D = \{d_1, d_2 \ldots d_n\}$ was constructed by choosing x randomly from $[p-1]$ and $d_i \equiv b_i x \pmod p$

Take any $b_i \in B$
for any 2 distinct $x, y \in [p-1]$
$b_i x \not\equiv b_i y \pmod p$
if true $\Rightarrow$ $b_i(x-y) \bmod p = 0$
but $x - y \neq 0$ $b_i \neq 0$
and $x, y < p$ so $x-y$ cant be a multiple of p.
similarly $b_i < p$ cant be a multiple of p
and $(x-y)(b_i)$ cant be p (prime number).

Therefore
$\{b_i 1 \pmod p, b_i 2 \pmod p, b_i 3 \pmod p \ldots b_i (p-1) \pmod p\} = [p-1]$

Choosing $x \to$ uniformly randomly from $[p-1]$

$$\Pr[\overbrace{b_i x \pmod p}^{d_i} \in C] = \frac{|C|}{p-1} > \frac{1}{3}$$

↓ elements from $[p-1]$
↳ size $p-1/3$

Random Var $X_i = \begin{cases} 1 & d_i \in C \\ 0 & \text{otherwise} \end{cases}$; $E[X_i] > \frac{1}{3}$ → $d_i \in C$

$E[X]$ = How many d_i will belong to set C

linearity $E[X] = E[X_1 + X_2 + X_3 \ldots X_n] \geq n \cdot \frac{1}{3}$
$E[X] \geq \frac{n}{3}$.

Week-5

* Hypergraph H (V, E) → Edge set
↳ vertex set

E ⊆ Powerset (V)
↳ collection of subsets of V (any-sized subsets of V)

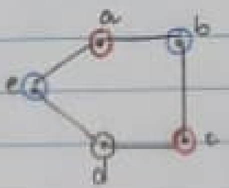
Example

V = {1, 2, 3, 4, 5}.
E = { {1, 2, 3}, {1, 3}, {1, 5} {3, 4, 5} }. {every edge has more than 1 colours}
~ Hyperedges

Ques: Colour the points/vertices in V with as few colours as possible such that every hyperedge in E sees at least 2 colours.
we need to find the minimum colors needs

Example: V = {a, b, c, d, e}
Graph example
E = { {a, b}, {b, c}, {c, d}, {d, e}, {e, a} }

→ in case of graph, same as colouring vertices such that no 2 adjacent vertices have same colour.



3 colours min.

Definition

A hypergraph H (V, E) is K-uniform if every hyperedge is K-sized.

Clearly, graphs are 2-uniform hypergraph.

Theorem

Every k-uniform hypergraph with less than 2^{k-1} hyperedges is 2-colourable.

Proof : let $V = \{1, 2 \ldots n\}$ be the vertex set of H

colour : for each $i \in V$, independently and uniformly at random assign a colour from the set { red, Black }

Consider a hyperedge $e \in E$

$$\text{Prob [all vertices in } \underset{\text{K-sized hyperedge}}{\underline{E \text{ get red colour}}}] = \frac{1}{2^k}$$

$$\text{Prob [" \quad " \quad " \quad " black colour]} = \frac{1}{2^k}$$

$$\text{Prob [all vertices of } \underset{\text{either red or black}}{\underline{e \text{ are monochromatic}}}] = \frac{2}{2^k} = \frac{1}{2^{k-1}}$$

let $e_1, e_2 \ldots e_m$ be hyperedges, $E = \{e_1, e_2 \ldots e_m\}$.

union for all edges to be monochromatic

$$P[(e_1 : \text{monochromatic}) \cup (e_2 : \text{monochromatic}) \ldots \cup (e_m : \overset{\text{mono}}{\text{chromatic}})]$$

$$\leq P[e_1 : \text{monochrom}] + P[e_2 : \text{monochrom}] \ldots + P[e_m : \overset{\text{mono}}{\text{chrom}}]$$

$$\leq m \cdot \frac{1}{2^{k-1}} < 2^{k-1} \cdot \frac{1}{2^{k-1}}$$

$m \to$ no. of hyperedges given : $m < 2^{k-1}$

Bad event : ~~all~~ $\exists$ edges are monochromatic (either of any edges) (union)

Good event : no of the edges are monochromatic (intersection).

$$\Pr(\text{Bad event}) < 2^{k-1} \frac{1}{2^{k-1}} = 1 \quad ; \quad \Pr(\text{Bad event}) < 1$$

Thus taking comp & apply demorgan.

$$\Pr(\text{Good event}) > 0$$

$$\Pr[e_1 \text{ is } \overset{\text{not}}{\text{mono}} \wedge e_2 \text{ is not mono} \ldots \wedge e_m \text{ is not mono}] > 0.$$

✱ Bollobas Thm

(Probabilistic Proof)

Let $(A_1, A_2 \ldots A_m)$ & $(B_1, B_2 \ldots B_m)$ be 2 sequences of sets such that $\forall\, i,j \in [m] \Rightarrow A_i \cap B_j = \phi$ if & only if $i = j$

then

$$m \leq {}^{a+b}C_a,$$

where, $\forall\, i \in [m]$

$$|A_i| = a \quad \& \quad |B_i| = b.$$

let $X = \bigcup_{i=1}^{m} (A_i \cup B_i) = \{x_1, x_2, x_3 \ldots x_n\}.$

σ : some permutations of x

(A_i, B_i) is present in σ if every element of A_i is present before every element of B_i in σ.

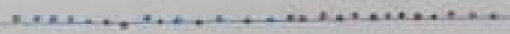
Uniformly at random choose a linear permutation

there are $n!$ permutations, choose 1

$$\Pr[\overbrace{(A_i, B_i) \text{ pair present in } \sigma}^{x_1}] = \frac{1}{\left({}^{a+b}C_a\right)} \longrightarrow ①$$

$$\underset{a}{\downarrow} \quad \underset{b}{\downarrow}$$

$$\text{favourable} = \frac{a_i!\, b_i!}{(a_i + b_i)!} = \frac{1}{{}^{a+b}C_a}$$



Ignore black, focus on only blue & red.

blue before red.

$\forall\, i, j \in [m]\ ,\ i \neq j$

$$Pr[X_i \cup X_j] = Pr[X_i] + Pr[X_j] - \underset{\overset{\shortparallel}{0}}{Pr[X_i \cap X_j]}$$

$$Pr[X_i \cup X_j] = Pr[X_i] + Pr[X_j]$$

(from prev proof we know that atmost only one pair can be present in sigma i.e. (A_i, B_i) cant coexist with (A_j, B_j) ..)

Therefore,

$$Pr[X_1 \cup X_2 \cup \ldots \cup X_m] = \sum_{i=1}^{m} Pr[X_i]$$

$$= \frac{m}{{}^{a+b}C_a} \leq 1$$

(Pr is always ≤ 1).

$$\therefore\ m \leq {}^{a+b}C_a .$$