Date ___/___/___
Page ___
my COMPANION

CS18BTECH11042
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Assignment - 5

Q. Let $G(V,E)$ be a bipartite graph with n vertices and a list $S(v)$ of more than $\log_2 n$ colours associated with each vertex $v \in V$, to prove that there is a proper colouring of G, assigning to each vertex v a colour from its list $S(v)$.

Ans) A proper colouring is defined as one in which there are no two adjacent vertices which have the same colour.

Let $G = A \cup B$.

As G is a bipartite graph, we know there are no edges between i, j such that both vertices $i, j \in A$, or both $i, j \in B$.

Let us define $S_{tot} = \bigcup_{v \in V} S(v)$.

S_{tot} is the set of all colours that can be assigned across all vertices $v \in V$.

Let us also create two sets S_A and S_B that would correspond to the colours we can possibly assign to vertices in A and

B respectively. We'll randomly and for every colour $c \in S_{tot}$, distribute c into S_A or S_B. So we're basically randomly selecting one set from the available two sets to put a colour in.

$$\therefore \Pr(c \in S_A) = \Pr(c \in S_B) = \frac{1}{2}$$

(For all $c \in S_{tot}$)

Now we define two more types of sets.

$\forall v, v \in A, S_A(v) = S_A \cap S(v)$ and

$\forall w, w \in B, S_B(w) = S_B \cap S(w)$

These $|V|$ sets represent the sets of colours that we can assign our vertices after partitioning the colours as defined before.

Because S_A and S_B contains different colours ($S_A \cap S_B = \emptyset$), no two adjacent vertices, say $i \in A$ and $j \in B$ can be assigned the same colour.
And vertices in A cannot be adjacent to each other, thus assigning same colour to them does not violate the proper colouring condition. Likewise for vertices in B.

Now to find the bad probability, i.e. probability that a vertex cannot be assigned any colour

$\forall v \in A$,

$$Pr(S_A(v) = \emptyset) = \left(\frac{1}{2}\right)^{|S(v)|}$$

$\forall w \in B$

$$Pr(S_B(w) = \emptyset) = \left(\frac{1}{2}\right)^{|S(w)|}.$$

This is bad probability because if out of all colours associated with a vertex $v \in A$, if none belongs to S_A, it remains unassigned and thus proper colouring cannot be established. Likewise for $w \in B$.

$\forall v \in V$,

$$\left(\frac{1}{2}\right)^{|S(v)|} < \left(\frac{1}{2}\right)^{\log_2 n} \quad (\text{As } |S(v)| > \log_2 n)$$

$$\Rightarrow Pr(S_A(v) = \emptyset) < \left(\frac{1}{2}\right)^{\log_2 n}$$

$$\Rightarrow Pr(S_A(v) = \emptyset) < \left(\frac{1}{2^{\log_2 n}}\right)$$

$$\Rightarrow Pr(S_A(v) = \emptyset) < \frac{1}{n} \quad \text{—i)}$$

Likewise, $Pr(S_B(w) = \emptyset) < \frac{1}{n}$

Date ___/___/___
Page ___

$\therefore$ Pr(Bad Event) = Pr(Atleast one vertex $v \in V$ is unassigned)

$$Pr(\text{Bad Event}) = Pr\left(\left(\bigvee_{v \in A} S_A(v) = \phi\right) \vee \left(\bigvee_{w \in B} S_B(w) = \phi\right)\right)$$

$\Rightarrow$ Using Linearity of Expectation.

$$Pr(\text{Bad Event}) = \sum_{v \in A} Pr(S_A(v) = \emptyset) + \sum_{w \in B} Pr(S_B(w) = \phi)$$

$$Pr(\text{Bad Event}) < \sum_{v \in A} \frac{1}{n} + \sum_{w \in B} \frac{1}{n} \quad (\text{from } i)$$

$$\therefore Pr(\text{Bad Event}) < \sum_{v \in V} \frac{1}{n}$$

$$\therefore Pr(\text{Bad Event}) < \frac{|V|}{n} \quad (\text{As } |V| = n).$$

$$Pr(\text{Bad Event}) < 1.$$

$\therefore$ Pr(Good Event) > 0 (Complement of Bad Event).

$\therefore$ For some vertex colouring with more than $\log_2 n$ colours associated with every vertex, $\forall v \in A$, v is assigned a colour from S_A, and $\forall w \in B$, w is assigned a colour from S_B. Thus a proper colouring of G under given condition exists

Hence Proved!