AI 1110 Assignment 3

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12.13.6.04 Question: Suppose that 90 % of people are right-handed. What is the probability that atmost 6 of a random sample of 10 people are right-handed.

Answer: 0.0127951893

Solution: Let a binomial random variable be:

$$X \sim Bin(n, p) \tag{1}$$

$$\implies p = \frac{1}{10} = 0.1 \tag{2}$$

$$\implies n = 10 \tag{3}$$

where, p be the probability of a person being left-handed.

n is the number of people.

Proof of Gaussian Theorem:

Let two independent random variables be X and Y with PDF's $f_X(x)$ and $f_Y(y)$

$$\therefore\ f_{XY}(x, y) = f_X(x) f_Y(y) \tag{4}$$

Since the distribution is identical and symmetric,

$$f_{XY}(x, y) = g(x^2 + y^2) \tag{5}$$

$$f_X(x) f_Y(y) = g(x^2 + y^2) \tag{6}$$

For y=0,

$$f_X(x) \propto g(x^2) \tag{7}$$

For x=0,

$$f_X(y) \propto g(y^2) \tag{8}$$

Multiplying the above two equations,

$$f_X(x) f_X(y) \propto g(x^2) g(y^2) \tag{9}$$

$$\implies f_{XY}(x, y) \propto g(x^2) g(y^2) \tag{10}$$

$$\implies g(x^2 + y^2) \propto g(x^2) g(y^2) \tag{11}$$

$g(x)$ can be Ae^{-Bx}

$$\therefore,\ f_X(x) \propto g(x^2) = Ae^{-Bx^2} \tag{12}$$

$$f_X(x) = Ae^{-Bx^2} \tag{13}$$

By axioms of PDF:

$$\int_{-\infty}^{\infty} f_X(x) dx = 1 \tag{14}$$

$$A \int_0^{\infty} e^{-Bx^2} dx = \frac{1}{2} \tag{15}$$

Similarly,

$$A \int_0^{\infty} e^{-By^2} dy = \frac{1}{2} \tag{16}$$

Multiplying the above two equations,

$$\int_0^{\infty} \int_0^{\infty} e^{-Bx^2} e^{-By^2} dx dy = \frac{1}{4A^2} \tag{17}$$

$$\int_0^{\infty} \int_0^{\infty} e^{-B(x^2+y^2)} dx dy = \frac{1}{4A^2} \tag{18}$$

By polar co-ordinates,

$$\int_0^{\infty} \int_0^{\infty} e^{-Bx^2} e^{-By^2} dx dy = \int_0^{\frac{\pi}{2}} \int_0^{\infty} e^{-Br^2} r\, dr\, d\theta \tag{19}$$

$$\int_0^{\infty} \int_0^{\infty} e^{-B(x^2+y^2)} dx dy = \int_0^{\frac{\pi}{2}} \frac{d\theta}{2B} \tag{20}$$

$$\implies \frac{1}{4A^2} = \frac{\pi}{4B} \tag{21}$$

$$\implies A = \sqrt{\frac{B}{\pi}} \tag{22}$$

Since,

$$\int_{-\infty}^{\infty} x^2 f(x) dx = \sigma^2 \tag{23}$$

$$\implies 2\sqrt{\frac{B}{\pi}} \int_0^{\infty} x^2 e^{-Bx^2} dx = \sigma^2 \tag{24}$$

Now by IBP,

$$u = x \tag{25}$$

$$dv = xe^{-Bx^2} \tag{26}$$

Hence the values,

$$B = \frac{1}{2\sigma^2} \text{ and } A = \frac{1}{\sqrt{2\pi\sigma^2}} \tag{27}$$

$$\implies f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} \tag{28}$$

For mean μ,

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{29}$$

By Q - function,

$$P(X \geq 4) = 1 - P(X < 4) \tag{30}$$
$$= 1 - F_X(3) \tag{31}$$

Hence,

$$\mu = np \tag{32}$$
$$\implies \mu = 10 \times 0.1 = 1 \tag{33}$$
$$\sigma^2 = np(1-p) \tag{34}$$
$$\implies \sigma^2 = 10 \times 0.1 \times 0.9 = 0.9 \tag{35}$$
$$\sigma = \sqrt{np(1-p)} \tag{36}$$
$$\implies \sigma = \sqrt{0.9} \tag{37}$$
$$\implies Z = \frac{X - \mu}{\sigma} \tag{38}$$
$$Z = \frac{3-1}{\sqrt{0.9}} \tag{39}$$
$$Z = 2.108185107 \tag{40}$$

By Z-score table, $P(X \leq 3) = F_X(3) = 0.9854$
Hence,

$$P(X \geq 4) = 1 - 0.9854 \tag{41}$$
$$= 0.0146 \tag{42}$$

So, The answer differs by,
$0.0146 - 0.0127951893 = 0.001804811$