AI 1110 Assignment 3

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12.13.6.04 Question: Suppose that 90 % of people are right-handed. What is the probability that atmost 6 of a random sample of 10 people are right-handed.

Answer: 0.0127951893

Solution: Let a binomial random variable be:

$$X \sim Bin(n, p) \tag{1}$$
$$\implies p = \frac{1}{10} = 0.1 \tag{2}$$
$$\implies n = 10 \tag{3}$$

where, p be the probability of a person being left-handed.

n is the number of people.

Proof of Gaussian Approximation on Binomial: By defination,

$$P(X = k) = {}^nC_k p^k (1 - p)^{n-k} \tag{4}$$

By Stirling's formula,

For $m \to \infty$,

$$m! \sim \sqrt{2\pi m} e^{-m} m^m \tag{5}$$

Hence,

$$P(X = k) = \frac{n!}{k!(n-k)!} p^k (1 - p)^{n-k} \tag{6}$$
$$\tag{7}$$

Hence,

$$\sim \frac{\sqrt{2\pi n} e^{-n} n^n p^k (1 - p)^{n-k}}{\sqrt{2\pi k} e^{-k} k^k \sqrt{2\pi(n-k)} e^{-(n-k)} (n-k)^{n-k}} \tag{8}$$
$$= \left(\frac{p}{k}\right)\left(\frac{1-p}{n-k}\right)^{n-k} n^n \sqrt{\frac{n}{2\pi k(n-k)}} \tag{9}$$
$$= \left(\frac{np}{k}\right)^k \left(\frac{n(1-p)}{n-k}\right)^{n-k} \sqrt{\frac{n}{2\pi k(n-k)}} \tag{10}$$

Now by identities,

$$ln\left(\frac{np}{k}\right) = -ln\left(1 + \frac{k - np}{np}\right) \tag{11}$$
$$ln\left(\frac{n(1-p)}{n-k}\right) = -ln\left(1 - \frac{k - np}{n(1-p)}\right) \tag{12}$$

By approximation,

For $y \to 0$

$$ln(1 + y) \sim y - \frac{y^2}{2} + \frac{y^3}{3}... \tag{13}$$

Hence,

$$ln\left(\left(\frac{np}{k}\right)^k \left(\frac{n(1-p)}{n-k}\right)^{n-k}\right) = \tag{14}$$
$$-kln\left(1 + \frac{k - np}{np}\right) + (n-k)ln\left(1 - \frac{k - np}{n(1-p)}\right) \tag{15}$$

$\therefore$,

$$ln\left(\left(\frac{np}{k}\right)^k \left(\frac{n(1-p)}{n-k}\right)^{n-k}\right) \sim -\frac{(k - np)^2)}{2np(1-p)} \tag{16}$$

$\therefore$,

$$\left(\frac{np}{k}\right)^k \left(\frac{n(1-p)}{n-k}\right)^{n-k} \sim e^{-\frac{(k-np)^2)}{2np(1-p)}} \tag{17}$$

By approximating that, $k - np$ is of order $\sqrt{n}$

$$k - np \approx \sqrt{n} \tag{18}$$
$$n - k \approx n(1-p) - \sqrt{n} \tag{19}$$
$$k(n-k) \approx n^2 p(1-p) \tag{20}$$
$$\implies \sqrt{\frac{n}{2\pi k(n-k)}} \sim \frac{1}{\sqrt{2\pi np(n-p)}} \tag{21}$$

$$P(X = k) = \frac{e^{-\frac{(k-np)^2)}{2np(1-p)}}}{\sqrt{2\pi np(n-p)}} \tag{22}$$
$$P(X = k) = \frac{e^{-\frac{(k-\mu)^2)}{2\sigma^2}}}{\sqrt{2\pi}\sigma} \tag{23}$$

By Q-function,

$$P(X \geq 4) = 1 - P(X < 4) \tag{24}$$
$$= 1 - F_X(3) \tag{25}$$

Hence,

$$\begin{aligned}
\mu &= np && (26)\\
\implies \mu &= 10 \times 0.1 = 1 && (27)\\
\sigma^2 &= np(1-p) && (28)\\
\implies \sigma^2 &= 10 \times 0.1 \times 0.9 = 0.9 && (29)\\
\sigma &= \sqrt{np(1-p)} && (30)\\
\implies \sigma &= \sqrt{0.9} && (31)\\
\implies Z &= \frac{X-\mu}{\sigma} && (32)\\
Z &= \frac{3-1}{\sqrt{0.9}} && (33)\\
Z &= 2.108185107 && (34)
\end{aligned}$$

By Z-score table, $P(X \leq 3) = F_X(3) = 0.9854$
Hence,

$$\begin{aligned}
P(X \geq 4) &= 1 - 0.9854 && (35)\\
&= 0.0146 && (36)
\end{aligned}$$

So, The answer differs by,
$0.0146 - 0.0127951893 = 0.001804811$