ASSIGNMENT - 2

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DESIGN AND ANALYSIS
OF ALGORITHM

(12) Demonstrate Binary Search method to search key=23 from the array a[]={2,5,8,12,16,23,38,56,72,91}

$mid = \frac{l+h}{2}$

a[mid] > key

h=mid-1

a[mid] < key

l=mid+1

l=0

h=9

$a[] = \{\overset{0}{2}, \overset{1}{5}, \overset{2}{8}, \overset{3}{12}, \overset{4}{16}, \overset{5}{23}, \overset{6}{38}, \overset{7}{56}, \overset{8}{72}, \overset{9}{91}\}$

$mid = \frac{0+9}{2} = \frac{9}{2} = 4.5$

a[4] < key

~~mid=~~ l=4+1

l=5

$mid = \frac{5+9}{2} = \frac{14}{2} = 7$

a[7] = 56

56 > key

h = mid - 1

= 7-1

h = 6

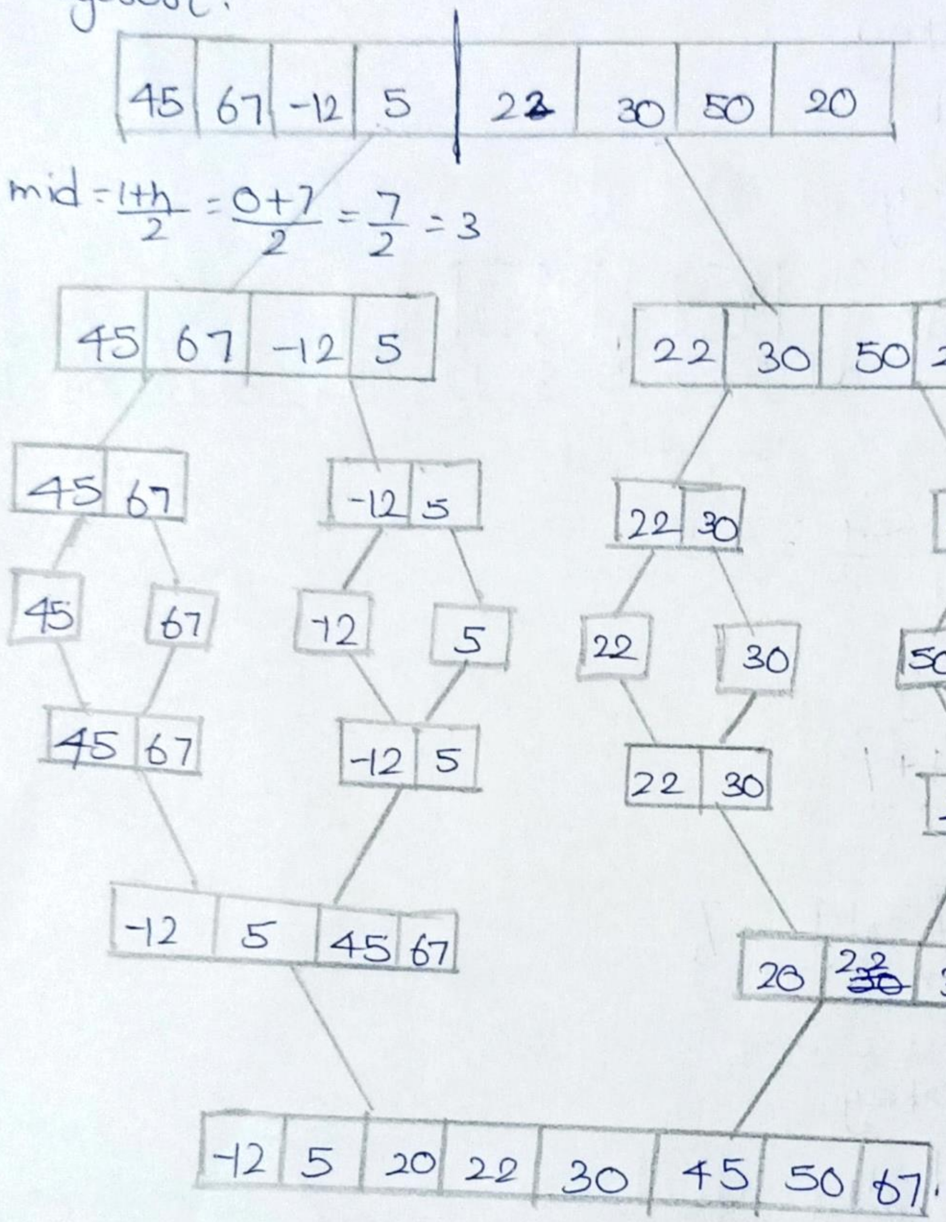
$mid = \frac{5+6}{2} = \frac{11}{2} = 5.5$ or 5

a[5] = 23

a[mid] = key

Index = 5

(13) Apply merge sort and order the list of 8 elements. Data $d=(45, 67, -12, 5, 22, 30, 50, 20)$. Set up a recurrence relation for the number of key comparison made by mergesort.



$mid = \frac{l+h}{2} = \frac{0+7}{2} = \frac{7}{2} = 3$

$T(n)$ is the number of comparisons

Recurrence Relation

$T(n) = 2T\left(\frac{n}{2}\right) + n - 1$

When $n=1$ no comparison needed to sort the array.

The list is divided into two halves so, $2T\left(\frac{n}{2}\right)$ and for merging $(n-1)$

(14) Find the no. of times to perform swapping for selection sort. Also estimate the time complexity for the order of notation Set S(12, 7, 5, -2, 18, 6, 13, 4)

12	7	5	-2	18	6	13	4

min = -2

Swap 12 and -2

-2	7	5	12	18	6	13	4

min = 4

Swap 7 and 4

-2	4	5	12	18	6	13	7

min = 5

No swap

-2	4	5	12	18	6	13	7

min = 6

Swap 12 and 6

-2	4	5	6	18	12	13	7

min = 7

Swap 18 and 7

-2	4	5	6	7	12	13	18

Sorted Array

Total Comparisons

$(n-1)+(n-2)+\dots+1 = \frac{n(n-1)}{2} = O(n^2)$

Time complexity $O(n^2)$

$n = 8$ Swaps = 4 times

(15) Find the index of the target value 10 using binary search from the following list of elements [2, 4, 6, 8, 10, 12, 14, 16, 18, 20]

$$\begin{array}{cccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \end{array}$$
$a = [2, 4, 6, 8, 10, 12, 14, 16, 18, 20]$

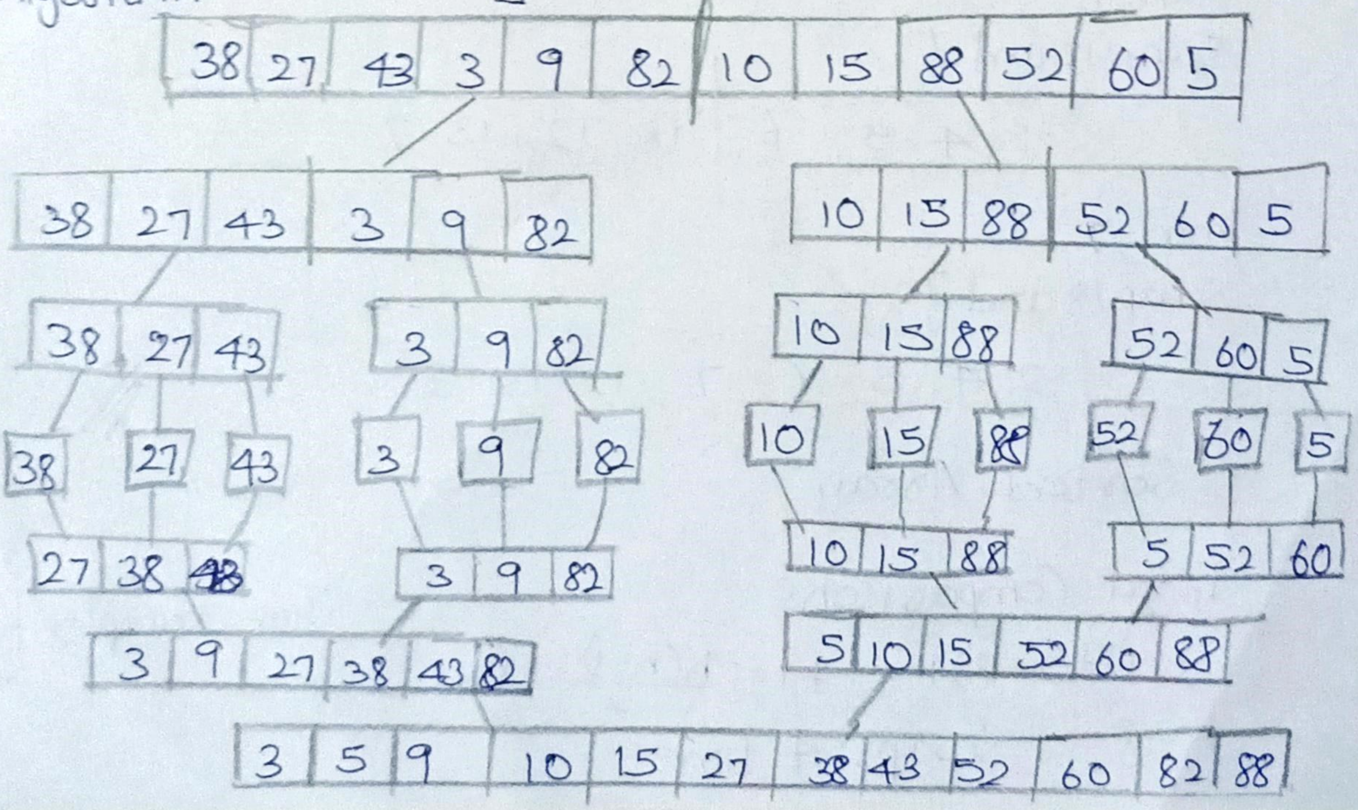
$mid = \frac{0+9}{2} = \frac{9}{2} = 4.5$ or 4

$a[mid] = 10$

$a[mid] = Target$

Index $= 4$

(16) Sort the following elements using merge sort divide and conquer strategy [38, 27, 43, 3, 9, 82, 10, 15, 88, 52, 60, 5] and analyze the complexity of Algorithm



(17) Sort the array 64, 34, 25, 12, 22, 11, 90 using bubble Sort

64	34	25	12	22	11	90

Best Case: $O(n^2)$

Average Case: $O(n^2)$

Worst Case: $O(n^2)$

34	64	25	12	22	11	90

Pass I

34	25	64	12	22	11	90
34	25	12	64	22	11	90
34	25	12	22	64	11	90
34	25	12	22	11	64	90

Pass II

25	34	12	22	11	64	90
25	12	34	22	11	64	90
25	12	22	34	11	64	90
25	12	22	11	34	64	90

Pass III

12	25	22	11	34	64	90
12	22	25	11	34	64	90
12	22	11	25	34	64	90

Pass IV

12	22	11	25	34	64	90
12	11	22	25	34	64	90

Pass V

11	12	22	25	34	64	90

(18) Sort the array 64, 34, 25, 12, 22, 11 using Selection Sort

64	34	25	12	22	11

min = 11

Swap 64 and 11

11	34	25	12	22	64

min = 12

Swap 34 and 12

11	12	25	34	22	64

min = 22

Swap 25 and 22

11	12	22	34	25	64

min = 25

Swap 34 and 25

11	12	22	25	34	64

Sorted Array

Time Complexity

Best Case: $O(n^2)$

Average Case: $O(n^2)$

Worst Case: $O(n^2)$

(19) Given an array of [4, -2, 5, 3, 10, -5, 2, 8, -3, 6, 7, -4, 1, 9, -1, 0, -6, -8, 11, -9] integers sort the following elements using insertion sort using brute force approach.

4	-2	5	3	10	-5	2	8	-3	6	7	-4	1	9	-1	0	-6	-8	11	-9

key = 4

-2	4	5	3	10	-5	2	8	-3	6	7	-4	1	9	-1	0	-6	-8	11	-9

key = 5

-2	4	5	3	10	-5	2	8	-3	6	7	-4	1	9	-1	0	-6	-8	11	-9

key = 3

-2	3	4	5	10	-5	2	8	-3	6	7	-4	1	9	-1	0	-6	-8	11	-9

key = -5

-5	-2	3	4	5	10	2	8	-3	6	7	-4	1	9	-1	0	-6	-8	11	-9

key = 2

-5	-2	2	3	4	5	10	8	-3	6	7	-4	1	9	-1	0	-6	-8	11	-9

key = 8

-5	-2	2	3	4	5	8	10	-3	6	7	-4	1	9	-1	0	-6	-8	11	-9

key = -3

-5	-3	-2	2	3	4	5	8	10	6	7	-4	1	9	-1	0	-6	-8	11	-9

key = 6

-5	-3	-2	2	3	4	5	6	8	10	7	-4	1	9	-1	0	-6	-8	11	-9

key = 7

-5	-3	-2	2	3	4	5	6	7	8	10	-4	1	9	-1	0	-6	-8	11	-9

key = -4

3

-5	-4	-3	-2	2	4	5	6	7	8	10	1	9	-1	0	-6	-8	11	-9	

key = 1

(20) Sort the following elements using insertion sort using brute force approach strategy [38, 27, 43, 3, 9, 82, 10, 15, 88, 52, 60, 5]

38	27	43	3	9	82	10	15	88	52	60	5

Key = 38

27	38	43	3	9	82	10	15	88	52	60	5

Key = 43

3	27	38	43	9	82	10	15	88	52	60	5

Key = 9

3	9	27	38	43	82	10	15	88	52	60	5

Key = 10

3	9	10	27	38	43	82	15	88	52	60	5

Key = 15

3	9	10	15	27	38	43	82	88	52	60	5

Key = 52

3	9	10	15	27	38	43	52	82	88	60	5

Key = 60

3	9	10	15	27	38	43	52	60	82	88	5

Key = 5

3	5	9	10	15	27	38	43	52	60	82	88

Sorted

Best case: $O(n)$ Worst Case: $O(n^2)$

Average Case: $O(n^2)$

$\boxed{3}$

-5	-4	-3	-2	1	2	4	5	6	7	8	10	9	-1	0	-6	-8	11	-9

key = 9

-5	-4	-3	-2	1	2	3	4	5	6	7	8	9	10	-1	0	-6	-8	11	-9

key = -1

-5	-4	-3	-2	-1	1	2	3	4	5	6	7	8	9	10	0	-6	-8	11	-9

key = 0

-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	-6	-8	11	-9

key = -6

-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	-8	11	-9

key = -8

-8	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	-9

key = -9

-9	-8	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11

Sorted

Best Case: $O(n)$

Average Case: $O(n^2)$

Worst Case: $O(n^2)$

(10) Solve the recurrence relation

$T(n) = 4T\left(\frac{n}{2}\right) + n^2$, $T(1) = 1$

Case 2: $\log_b^a = k$

$T(n) = aT\left(\frac{n}{b}\right) + f(n)$

$p = 1$

$p > -1$ $\theta(n^k \log_n^{p+1})$

$a = 4$ $b = 2$ $k = 2$

$\theta(n^2 \log_n^{1+1})$

$\log_b^a = \log_2^4 = 2$

$\theta(n^2 \cdot \log n)$

The order of growth solution to the recurrence relation is $\theta(n^2 \log n)$

① If $t_1(n) \in O(g1(n))$ and $t_2(n) \in O(g^2(n))$ prove that $t_1(n)+t_2(n)$
$\in O(\max\{g_1(n), g_2(n)\})$

For any four arbitrary real numbers a_1, b_1, a_2, b_2
such that $a_1 \le b_1$ and $a_2 \le b_2$
we have $a_1 + a_2 \le 2\max\{b_1, b_2\}$

Since $t_1(n) \in O(g_1(n))$, then there exists some constant c_1 and non-negative integer n_1 such that

$$t_1(n) \le c_1 g_1(n) \text{ for all } n \ge n_1$$

since $t_2(n) \in O(g_2(n))$, then there exists integer n_2 such that

$$t_2(n) \le c_2 g_2(n) \text{ for all } n \ge n_2$$

Let $c_3 = \max\{c_1, c_2\}$ and $n_0 = \max\{n_1, n_2\}$

$$\begin{aligned} t_1(n) + t_2(n) &\le c_1 g_1(n) + c_2 g_2(n) \\ &\le c_3 g_1(n) + c_3 g_2(n) \\ &= c_3\{g_1(n) + g_2(n)\} \\ &\le 2c_3 \max\{g_1(n), g_2(n)\} \end{aligned}$$

Hence, $t_1(n) + t_2(n) \in O(\max\{g_1(n), g_2(n_2)\}$ with constants c and n_0

② Find the time complexity of the below recurrence equation

③ $T(n) = \begin{cases} 2T\left(\frac{n}{2}\right) + 1 & \text{if } n > 1 \\ 1 & \text{otherwise} \end{cases}$

$T(n) = aT\left(\frac{n}{b}\right) + f(n)$

$a = 2$

$b = 2$

$\log_b^a = \log_2^2 = 1$

$k = 0$

$\log_b^a > k$

Case: 1

$\theta(n \cdot \log_b^a)$

$\theta(n \cdot \log_2^2)$

$\theta(n \cdot 1)$

$\theta(n)$

④ $T(n) = \begin{cases} 2T(n-1) & \text{if } n > 0 \\ 1 & \text{otherwise} \end{cases}$

$T(n) = 2T(n-1) \rightarrow ①$ $\quad T(0) = 0$

$n = n-1$

$T(n-1) = 2T((n-1)-1)$

$= 2T(n-2) \rightarrow ②$

Sub ② in ①

$T(n) = 2[2T(n-2)]$

$= 2^2 T(n-2) \rightarrow ③$

$n = n-2$

$T(n-2) = 2T((n-2)-1)$

$= 2T(n-3) \rightarrow (4)$

Sub (4) in (3)

$T(n) = 2^2[2T(n-3)]$

$T(n) = 2^3 T(n-3) \rightarrow (5)$

$n = n-3$

$T(n-3) = 2T((n-3)-1)$

$T(n-3) = 2T(n-4) \rightarrow (6)$

Sub (6) in (5)

$T(n) = 2^3 [2T(n-4)]$

$= 2^4 T(n-4) \rightarrow (7)$

$T(n) = 2^k T(n-k)$

$n-k=0 \Rightarrow n=k$

if $T(0)=1$

$T(n) = 2^k \cdot T(0)$

$T(n) = 2^k \cdot 1$

$T(n) = 2^k$

$\therefore n = k$

$T(n) = O(2^n)$

(5) Big O Notation: show that $f(n) = n^2+3n+5$ is $O(n^2)$

To prove that $f(n) = n^2+3n+5$ if $O(n^2)$ we need to find constants c and n_0 such that $f(n) \leq c \cdot n^2$ for all $n \geq n_0$

$f(n) = n^2+3n+5$

For $n \geq 1$, $n^2 \geq n$ .... so on

$f(n) = n^2 + 3n + 5 \le n^2 + 3n^2 + 5n^2$

$f(n) = n^2 + 3n + 5 \le 9n^2$ for $n \ge 1$

So, for $c = 9$ and $n_0 = 1$

$f(n) \le c \cdot n^2$ for all $n \ge n_0$

that proves $f(n)$ is $O(n^2)$

(6) Big omega notation: prove that $g(n) = n^3 + 2n^2 + 4n$ is $\Omega(n^3)$

To prove that $g(n) = n^3 + 2n^2 + 4n$ is $\Omega(n^3)$ we need to find constants c and n_0 such that

$g(n) \ge c \cdot n^3$ for all $n \ge n_0$

$g(n) = n^3 + 2n^2 + 4n$

For $n \ge 1$,

$g(n) = n^3 + 2n^2 + 4n \ge n^3$

since $2n^2$ and $4n$ are both less than n^3 when $n \ge 1$

so, for $c = 1$ and $n_0 = 1$

$g(n) \ge c \cdot n^3$ for all $n \ge n_0$

that proves $g(n)$ is $\Omega(n^3)$

⑦ Big Theta notation: Determine whether $h(n) = 4n^2 + 3n$ is $\Theta(n^2)$ or not

1. $h(n) = 4n^2 + 3n$ is $\Theta(n^2)$

For $n \geq 1$, $h(n) \leq 4n^2 + 3n^2$

(Since $3n$ is less than n^2 when $n \geq 1$)

For this simplifies to $h(n) \leq 7n^2$ for $n \geq 1$

Therefore, $h(n)$ is $O(n^2)$

2. $h(n) = 4n^2 + 3n$ is $\Omega(n^2)$

For $n \geq 1$, $h(n) \geq 4n^2$

(Since $3n$ is positive)

Therefore $h(n)$ is $\Omega(n^2)$

Since $h(n)$ is both $O(n^2)$ and $\Omega(n^2)$, it is $\Theta(n^2)$

⑧ Let $f(n) = n^3 - 2n^2 + n$ and $g(n) = -n^2$ show whether $f(n) = \Omega(g(n))$ is true or false and justify your answer

$n = 1$

$f(1) = 1^3 - 2(1)^2 + 1$ $\quad$ $g(1) = -(1)^2$

$= 1 - 2 + 1$ $\quad$ $= (-1)^2$

$= 0$ $\quad$ $= 1$

$n = 2$

$f(2) = 2^3 - 2(2)^2 + 2$ $\quad$ $g(2) = (-2)^2$

$= 8 - 8 + 2$ $\quad$ $= 4$

$= 2$

$n = 3$ $\quad$ $g(3) = (-3)^2$

$f(3) = 3^3 - 2(3)^2 + 3$ $\quad$ $= 9$

$= 27 - 9 + 3 = 21$

$n=4$

$f(4) = 4^3 - 2(4)^2 + 4$
$= 64 - 32 + 4$
$= 32 + 4$
$= 36$

$g(4) = (-4)^2$
$= 16$

$n=5$

$f(5) = 5^3 - 2(5)^2 + 5$
$= \cancel{75}\,125 - 50 + 5$
$= 85$

$g(5) = (-5)^2$
$= 25$

$f(n) \geq g(n)$

So it is best case according to asymptotic notations

$f(n) = \Omega(g(n))$

(9) Determine whether $h(n) = n \log n + n$ is in $\Theta(n \log n)$ prove a rigorous proof for your conclusion

1. Upper Bound (O notation)

we need to find c_1 and n_0 such that

$h(n) \leq c_1 \cdot n \log n$ for all $n \geq n_0$

$h(n) = n \log n + n$
$\leq n \log n + n \log n$ (since $\log n$ is increasing)
$= 2n \log n$

now, let $c_1 = 2$, then $h(n) \leq 2n \log n$ for all $n \geq 1$

So, $h(n)$ is $O(n \log n)$

2. Lower Bound (Ω notation)

we need to find c_2 and n_0 such that

$h(n) \geq c_2 \cdot n \log n$ for all $n \geq n_0$

$h(n) = n \log n + n$

$\geq \frac{1}{2} \cdot n \log n$ (for $n \geq 2$)

(11) Given an array of [4, -2, 5, 3, 10, -5, 2, 8, -3, 6, 7, -4, 18, 9, -1, 0, -6, -8, 11, -9] integers find the maximum and minimum product that can be obtained by multiplying 2 integers

Sort

a: [-9, -8, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 3, 4, 5, 6, 7, 8, 9, 10, 11]

Maximum product

$\Rightarrow 10 \times 11 = 110$

$\Rightarrow -9 \times -8 = 72$

Minimum product

$\Rightarrow -9 \times -8 = 72$

$\Rightarrow -9 \times 11 = -99$

$\Rightarrow -8 \times 11 = -88$

Maximum product = 110

Minimum product = -99