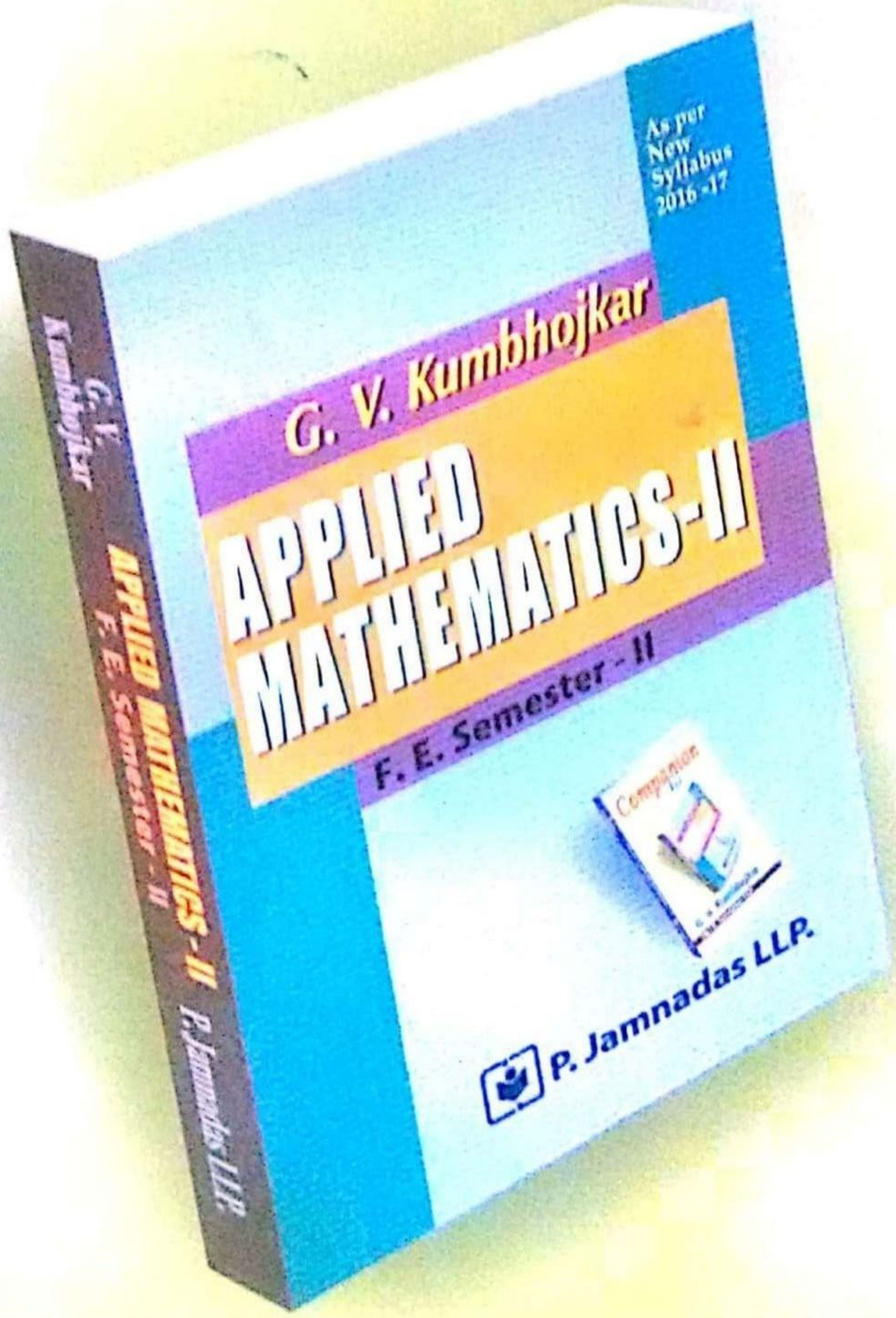
210

Companion to



G. V. Kumbhojkar

P. Jamnadas LLP.

Scanned by CamScanner

CONTENTS

1.	Differential Equations of The First Order And of The First Degree	1-1 to 1-32
2.	Applications of Differental Equations	2-1 to 2-4
3.	Linear Differential Equations With Constant Coefficients	3-1 to 3-50
4.	Cauchy's Homogeneous Linear Differential Equations	4-1 to 4-14
5.	Numerical Solutions of Ordinary Differental Equations	5-1 to 5-19
6.	Beta, Gamma Functions	6-1 to 6-29
7.	Differentiation Under Integral Sign	7-1 to 7-7
8.	Rectification	8-1 to 8-10
9.	Double Integral	9-1 to 9-22
10.	Areas	10-1 to 10-6
11.	Mass of Lamina	11-1 to 11-6
12.	Triple Integration	12-1 to 12-8
13.	Volumes of Solids	13-1 to 13-7
14.	Numerical Integration	14-1 to 14-10
	Total	224

CHAPTER 1

Differential Equations of the First Order and of the First Degree

Exercise I (Page 1-13)

Solve the following equations : Class (a) : 4 Marks

Example 1 : $(\sin x \cos y + e^{2x})\, dx + (\cos x \sin y + \tan y)\, dy = 0$

Sol. : $\int M\, dx = -\cos x \cos y + \frac{1}{2} e^{2x}$; $\int N\, dy = \log \sec y$ ■

Example 2 : $(2x^2 + 3y^2 - 7)\, x\, dx + (3x^2 + 2y^2 - 8)\, y\, dy = 0$

Sol. : $\int M\, dx = \frac{x^4}{2} + \frac{3x^2y^2}{2} - \frac{7x^2}{2}$; $\int N\, dy = \frac{y^4}{2} - \frac{8y^2}{2}$ ■

Example 3 : $\frac{dy}{dx} = \frac{y+1}{(y+2)e^y - x}$

Sol. : $\int M\, dx = xy + x = x(y+1)$

$\int dy = -\int e^y (y+2)\, dy = -y e^y + e^y - 2e^y$ ■

Example 4 : $(y^2 e^{xy^2} + 4x^3)\, dx + (2xy\, e^{xy^2} - 3y^2)\, dy = 0$

Sol. : $\int M\, dx = e^{xy^2} + x^4$; $\int N\, dy = -y^3$ ■

Example 5 : $(4x + 3y - 4)\, dx + (2x - 7y - 3)\, dy = 0.$

Sol. : $\int M\, dx = 2x^2 + 3xy - 4x$; $\int N\, dy = -\frac{7y^2}{2} - 3y$ ■

Example 6 : $\frac{dy}{dx} = \frac{y}{2y \log y + y - x}$

Sol. : $y\, dx + (x - y - 2y \log y)\, dy = 0$

$\int M\, dx = xy$; $\int N\, dy = \int (-y - 2y \log y)\, dy$

$\therefore \int N\, dy = -\frac{y^2}{2} - \log y \cdot y^2 + \int \frac{y^2}{y}\, dy = -\frac{y^2}{2} - y^2 \log y + \frac{y^2}{2}$ ■

Example 7 : $\dfrac{dy}{dx} = \dfrac{a^2 - 2xy - y^2}{(x+y)^2}$

Sol. : $(a^2 - 2xy - y^2)\,dx - (x+y)^2\,dy = 0$

$\therefore \int M\,dx = a^2x - x^2y - xy^2 \;;\; \int N\,dy = \int -y^2 dy = -\dfrac{y^3}{3}$ ■

Example 8 : $\dfrac{y\,dx - x\,dy}{(x+y)^2} = \dfrac{dx}{2\sqrt{1-x^2}}$

Sol. : $\left[\dfrac{y}{(x-y)^2} - \dfrac{1}{2\sqrt{1-x^2}}\right]dx - \dfrac{x}{(x-y)^2}\,dy = 0$

$\therefore \dfrac{\partial M}{\partial y} = \dfrac{(x-y)^2 - y\cdot 2(x-y)(-1)}{(x-y)^4}$

$\therefore \dfrac{\partial N}{\partial x} = -\left[\dfrac{(x-y)^2\cdot 1 - x\cdot 2(x-y)}{(x-y)^4}\right]$

$\therefore \int M\,dx = \int\left[\dfrac{y}{(x-y)^2} - \dfrac{1}{2\sqrt{1-x^2}}\right]dx = -\dfrac{y}{x-y} - \dfrac{1}{2}\sin^{-1}x$

$\therefore \int N\,dy = 0$ ■

Example 9 : $(ye^{xy} - 2y^3)\,dx + (xe^{xy} - 6xy^2 - 2y)\,dy = 0$

Sol. : $\int M\,dx = e^{xy} - 2xy^3 \;;\; \int N\,dy = -y^2$ ■

Example 10 : $(x^2 - x\tan^2 y + \sec^2 y)\,dy = (\tan y - 2xy - y)\,dx$

Sol. : $\int M\,dx = xy + x^2y - x\tan y \;;\; \int N\,dy = \tan y$ ■

Example 11 : $\left(\dfrac{y^2}{1+x^2} - 2y\right)dx + (2y\tan^{-1}x - 3x + \sin hy)\,dy = 0$

Sol. : $\int M\,dx = \int\left(\dfrac{y^2}{1+x^2} - 2y\right)dx = y^2\tan^{-1}x - 2xy$

$\int N\,dy = \int \sin hy\,dy = \cos hy$ ■

Example 12 : $[\,y\sin(xy) + xy^2\cos(xy)\,]\,dx$
$+ [\,x\sin(xy) + x^2y\cos(xy)\,]\,dy = 0$

Sol. : $\dfrac{\partial M}{\partial y} = \sin xy + xy\cos xy - x^2y^2\sin xy + 2xy\cos xy$

$\dfrac{\partial N}{\partial x} = \sin xy + xy\cos xy + 2xy\cos xy - x^2y^2\sin xy$

$\int M\,dx = -\cos xy + y^2\int x\cos xy\,dx$

$\therefore \int M\,dx = -\cos xy + y^2\left[\dfrac{x}{y}\sin(xy) - \int\dfrac{\sin xy}{y}\,dx\right]$

$\therefore \int M\,dx = -\cos xy + xy\sin y + \cos xy \;;\; \int N\,dy = 0$ ■

Example 13 : $(2x^2y + 4x^3 - 12xy^2 + 3y^2 + xe^y + e^{2x})\,dy$
$+ (12x^2y + 2xy^2 + 4x^3 - 4y^3 + 2ye^{2x} + e^y)\,dx = 0$

Sol. : $\int M\,dx = 4x^3y + x^2y^2 + x^4 - 4y^3x + ye^{2x} + xe^y$

$\int N\,dy = \int 3y^2 dy = y^3.$ ■

Example 14 : $\left[\dfrac{y^2}{(y-x)^2} - \dfrac{1}{x}\right]dx + \left[\dfrac{1}{y} - \dfrac{x^2}{(x-y)^2}\right]dy = 0$

Sol. : $\dfrac{\partial M}{\partial y} = \dfrac{(y-x)^2\cdot 2y - y^2\cdot 2(y-x)}{(y-x)^4}$

$\dfrac{\partial N}{\partial x} = -\left[\dfrac{(x-y)^2\cdot 2x - x^2\cdot 2\cdot(x-y)}{(x-y)^4}\right]$

$\int M\,dx = \int\left[\dfrac{y^2}{(y-x)^2} - \dfrac{1}{x}\right]dx = \dfrac{y^2}{y-x} - \log x$

$\int N\,dy = \int\dfrac{1}{y}\,dy = \log y.$ ■

Example 15 : $(2xy\cos x^2 - 2xy + 1)\,dx + (\sin x^2 - x^2)\,dy = 0.$

Sol. : $\int M\,dx = \int(2x\cos x^2 - 2xy + 1)\,dx = y\sin x^2 - x^2y + x$

$\int N\,dy = \int 0\,dy = 0.$ ■

Exercise II (Page 1-16)

Solve the following equations : Class (b) : 6 Marks

Example 1 : $x\,dy - y\,dx + 2x^3dx = 0$

Sol. : Divide by x^2, $\dfrac{x\,dy - y\,dx}{x^2} + 2x\,dx = 0 \quad \therefore \dfrac{y}{x} + x^2 = c.$ ■

Example 2 : $y(1+x)\,dx + x(1-y)\,dy = 0.$

Sol. : Divide by xy, $\dfrac{dx}{x} + dx + \dfrac{dy}{y} - dy = 0$

$\therefore \log x + x + \log y - y = c.$ ■

Example 3 : $x\,dy - y\,dx = (x^2 + y^2)(dx + dy)$

Sol. : Divide by $x^2 + y^2$,

$$\frac{x\,dy - y\,dx}{x^2 + y^2} = dx + dy \quad \therefore \quad \tan^{-1}\frac{y}{x} = x + y + c$$

Example 4 : $(1 + 2xy)\,y\,dx + (1 - 2xy)\,x\,dy = 0$

Sol. : Divide by x^2y^2, $\quad \dfrac{1}{x^2y}dx + \dfrac{1}{xy^2}dy + \dfrac{2dx}{x} - \dfrac{2dy}{y} = 0$

$$\therefore \quad \frac{y\,dx + x\,dy}{x^2y^2} + \frac{2dx}{x} - \frac{2dy}{y} = 0 \quad \therefore \quad -\frac{1}{xy} + \log\left(\frac{x}{y}\right)^2 = c.$$

Example 5 : $(1 - xy + x^2y^2)\,dx + (x^3y - x^2)\,dy = 0$

Sol. : Divide by x, $\quad \dfrac{dx}{x} - y\,dx - x\,dy + xy^2dx + x^2y\,dy = 0$

$$\therefore \quad \log x - xy + \frac{x^2y^2}{2} = c.$$

Example 6 : $x\,dy - y\,dx = a(x^2 + y^2)\,dy$

Sol. : Divide by $x^2 + y^2$, $\quad \dfrac{x\,dy - y\,dx}{x^2 + y^2} = a\,dy$

$$\therefore \quad \tan^{-1}\left(\frac{y}{x}\right) = ay + c.$$

Example 7 : $a(x\,dy + 2y\,dx) = xy\,dy.$

Sol. : Divide by xy, $\quad a\left(\dfrac{dy}{y} + 2\dfrac{dx}{x}\right) = dy$

$$\therefore \quad a(\log y + 2\log x) = y + c.$$

Exercise III (Page 1-20)

Solve the following equations : Class (b) : 6 Marks

Example 1 : $(x^2 + y^2)\,dx - 2xy\,dy = 0$

Sol. : $\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)\Big/N = -\dfrac{2}{x}$; $\text{I.F.} = e^{\int -\frac{2}{x}dx} = \dfrac{1}{x^2}$

$$\therefore \quad \left(1 + \frac{y^2}{x^2}\right)dx - \frac{2y}{x}dy = 0$$

$$\int M\,dx = \int\left(1 + \frac{y^2}{x^2}\right)dx = x - \frac{y^2}{x} \; ; \; \int N\,dy = 0$$

Example 2 : $\left(y + \dfrac{1}{3}y^3 + \dfrac{1}{2}x^2\right)dx + \dfrac{1}{4}(x + xy^2)\,dy = 0$

Sol. : $\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)\Big/N = \dfrac{12 + 12y^2 - 3 - 3y^2}{3x(1 + y^2)} = \dfrac{9(1 + y^2)}{3x(1 + y^2)} = \dfrac{3}{x}$

$$\therefore \quad \text{I.F.} = e^{\int \frac{3}{x}dx} = e^{3\log x} = x^3$$

$$\therefore \quad \int M\,dx = \int(12x^3y + 4x^3y^3 + 6x^5)\,dx = 3x^4y + x^4y^3 + x^6$$

$$\therefore \quad \int N\,dy = 0.$$

Example 3 : $(x^2 + y^2 + 2x)\,dx + 2y\,dy = 0$

Sol. : $\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)\Big/N = \dfrac{2y}{2y} = 1 \quad \therefore \quad \text{I.F.} = e^x$

$$\therefore \quad \int M\,dx = \int e^x(x^2 + 2x + y^2)\,dx = e^xx^2 + e^xy^2 \; ; \; \int N\,dy = 0.$$

Example 4 : $(x^3e^x - my^2)\,dx + m\,xy\,dy = 0$

Sol. : $\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)\Big/N = -\dfrac{3}{x} \quad \therefore \quad \text{I.F.} = e^{\int -\frac{3}{x}dx} = \dfrac{1}{x^3}$

$$\therefore \quad \int M\,dx = \int\left(e^x - \frac{my^2}{x^3}\right)dx = e^x + \frac{my^2}{2x^2} \; ; \; \int N\,dy = 0$$

Example 5 : $(y - 2x^2)\,dx - x(1 - xy)\,dy = 0$

Sol. : $\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)\Big/N = -\dfrac{2}{x} \quad \therefore \quad \text{I.F.} = e^{-\int \frac{2}{x}dx} = \dfrac{1}{x^2}$

$$\therefore \quad \int M\,dx = \int\left(\frac{y}{x^2} - 2\right)dx = -\frac{y}{x} - 2x \; ; \; \int N\,dy = \int y\,dy = \frac{y^2}{2}.$$

Example 6 : $(x^4e^x - 2m\,xy^2)\,dx + 2\,mx^2y\,dy = 0$

Sol. : $\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)\Big/N = -\dfrac{4}{x} \quad \therefore \quad \text{I.F.} = \dfrac{1}{x^4}$

$$\therefore \quad \int M\,dx = \int\left(e^x - \frac{2my^2}{x^3}\right)dx = e^x + \frac{my^2}{x^2} \; ; \; \int N\,dy = 0.$$

Exercise IV (Page 1-23)

Solve the following equations : Class (b) : 6 Marks

Example 1 : $(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\,dy = 0$

Sol. : $\left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right)\Big/M = -\dfrac{3}{y} \quad \therefore \quad \text{I.F.} = e^{-\int \frac{3}{y}dy} = \dfrac{1}{y^3}$

$$\therefore \left(y+\frac{2}{y^2}\right)dx+\left(x+2y-\frac{4x}{y}\right)dy=0$$

$$\therefore \int M\,dx = xy+\frac{2x}{y^2} \;;\; \int N\,dy = y^2.$$

Example 2 : $(3x^2y^4+2xy)\,dx+(2x^3y^3-x^2)\,dy=0$

Sol. : $\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)\Big/M=-\frac{2}{y} \quad \therefore \text{ I.F.}=\frac{1}{y^2}$

$$\therefore \left(3x^2y^2+\frac{2x}{y}\right)dx+\left(2x^2y-\frac{x^2}{y^2}\right)dy=0$$

$$\therefore \int M\,dx = x^3y^2+\frac{x^2}{y} \;;\; \int N\,dy = 0$$

Example 3 : $(2xy^2-y)\,dx+x\,dy=0$

Sol. : $\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)\Big/M=-\frac{2}{y} \quad \therefore \text{ I.F.}=\frac{1}{y^2}$

$$\therefore \left(2x-\frac{1}{y}\right)dx+\frac{x}{y^2}dy=0 \quad \therefore \int M\,dx = x^2-\frac{x}{y},\ \int N\,dy = 0.$$

Example 4 : $y(x^2y+e^x)\,dx-e^x\,dy=0$

Sol. : $\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)\Big/M=-\frac{2}{y} \quad \therefore \text{ I.F.}=\frac{1}{y^2}$

$$\therefore \int M\,dx = \int\left(x^2+\frac{e^x}{y}\right)dx=\frac{x^3}{3}+\frac{e^x}{y} \;;\; \int N\,dy = 0.$$

Example 5 : $(2x^2y+e^x)\,y\,dx-(e^x+y^3)\,dy=0$

Sol. : $\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)\Big/M=-\frac{2}{y} \quad \therefore \text{ I.F.}=\frac{1}{y^2}$

$$\therefore \int M\,dx = \int\left(2x^2+\frac{e^x}{y}\right)dy=\frac{2x^3}{3}+\frac{e^x}{y}.$$

$$\therefore \int N\,dy = -\int y\,dy=-\frac{y^2}{2}$$

Example 6 : $xe^x(dx-dy)+e^x\,dx+ye^y\,dy=0$

Sol. : $\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)\Big/M=-1 \quad \therefore \text{ I.F.}=e^{-y}$

$$\therefore \int M\,dx = \int(xe^{x-y}+e^{x-y})\,dx=\int e^{(x-y)}\cdot(x+1)\,dx$$

$$=(x+1)\cdot e^{(x-y)}-\int e^{(x-y)}\cdot 1\,dx=(x+1)e^{(x-y)}-e^{(x-y)}$$

$$\therefore \int M\,dx = x\cdot e^{x-y} \;;\; \int N\,dy=\int y\,dy=\frac{y^2}{2}.$$

Exercise V (Page 1-26)

Solve the following equations : Class (b) : 6 Marks

Example 1 : $y(1+xy+x^2y^2)\,dx+x(1-xy+x^2y^2)\,dy=0$

Sol. : $Mx-Ny=2x^2y^2$. Dividing by $2x^2y^2$

$$\therefore \int M\,dx=\int\left(\frac{1}{2x^2y}+\frac{1}{2x}+\frac{y}{2}\right)dx=-\frac{1}{2}\cdot\frac{1}{xy}+\frac{1}{2}\log x+\frac{1}{2}xy$$

$$\int -\frac{1}{2y}\,dy=-\frac{1}{2}\log y$$

Example 2 : $y(1+xy)\,dx+x(1-xy)\,dy=0$

Sol. : $Mx-Ny=2x^2y^2$. Dividing by $2x^2y^2$.

$$\therefore \int M\,dx=\int\left(\frac{1}{2x^2y}+\frac{1}{2x}\right)dx=-\frac{1}{2xy}+\frac{1}{2}\log x$$

$$\int N\,dy=\int -\frac{1}{2y}\,dy=-\frac{1}{2}\log y$$

Example 3 : $y(2xy+1)\,dx+x(1+2xy-x^3y^3)\,dy=0$

Sol. : $Mx-Ny=x^4y^4$. Dividing by x^4y^4.

$$\therefore \int M\,dx=\int\left(\frac{2}{x^3y^2}+\frac{1}{x^4y^3}\right)dx=-\frac{1}{x^2y^2}-\frac{1}{3x^3y^3}$$

$$\therefore \int -\frac{1}{y}\,dy=-\log y.$$

Example 4 : $y(xy+2x^2y^2)\,dx+x(xy+x^2y^2)\,dy=0$

Sol. : $Mx-Ny=x^3y^3$. Dividing by x^3y^3.

$$\therefore \int M\,dx=\int\left(\frac{1}{x^2y}+\frac{2}{x}\right)dx=-\frac{1}{xy}+2\log x$$

$$\therefore \int N\,dy=\int\frac{1}{y}\,dy=\log y.$$

Example 5 : $y(\sin xy+xy\cos xy)\,dx+x(xy\cos xy-\sin xy)\,dy=0$

Sol. : $Mx-Ny=2\,xy\sin xy$. Dividing by $2xy\sin xy$

$$\therefore \int M\,dx=\int\left(\frac{1}{2x}+\frac{y}{2}\cdot\frac{\cos xy}{\sin xy}\right)dx=\frac{1}{2}\log x+\frac{1}{2}\log\sin xy$$

$$\therefore \int N\,dy=-\int\frac{1}{2y}\,dy=-\frac{1}{2}\log y.$$

Exercise VI (Page 1-29)

Solve the following equations : Class (b) : 6 Marks

Example 1 : $x^2y\,dx - (x^3 + y^3)\,dy = 0$

Sol. : $Mx + Ny = -y^4$. $\quad -\dfrac{x^2}{y^3}dx + \left(\dfrac{x^3}{y^4} + \dfrac{1}{y}\right)dy = 0$

$$\therefore \int M\,dx = -\int \frac{x^2}{y^3}dx = -\frac{x^3}{3y^3} \;;\; \int N\,dy = \int \frac{1}{y}dy = \log y.$$

Example 2 : $(3xy^2 - y^3)\,dx - (2x^2y - xy^2)\,dy = 0$

Sol. : $Mx + Ny = x^2y^2 \quad \therefore \left(\dfrac{3}{x} - \dfrac{y}{x^2}\right)dx - \left(\dfrac{2}{y} - \dfrac{1}{x}\right)dy = 0$

$$\therefore \int M\,dx = 3\log x + \frac{y}{x} \;;\; \int N\,dy = \int -\frac{2}{y}dy = -2\log y.$$

Example 3 : $(3xy - 2ay^2)\,dx + (x^2 - 2axy)\,dy = 0$

Sol. : $Mx + Ny = 4xy\,(x - ay)$

$$\therefore \int M\,dx = \int \frac{3xy - 2ay^2}{4xy\,(x - ay)}dx = \frac{3}{4}\int \frac{1}{x - ay}dx - \frac{ay}{2}\int \frac{1}{x\,(x - ay)}dx$$

$$= \frac{3}{4}\int \frac{1}{x - ay}dx - \frac{1}{2}\int\left(\frac{1}{x - ay} - \frac{1}{x}\right)dx$$

$$= \frac{3}{4}\log(x - ay) - \frac{1}{2}\log(x - ay) + \frac{1}{2}\log x$$

$$= \frac{1}{4}\log(x - ay) + \frac{1}{4}\log x^2 = \frac{1}{4}\log(x - ay)\,x^2.$$

Now, $N = \dfrac{x^2 - 2axy}{4xy\,(x - ay)} = \dfrac{(x^2 - axy) - axy}{4xy\,(x - ay)} = \dfrac{1}{4y} - \dfrac{a}{4\,(x - ay)}$

$$\therefore \int (\text{Terms in } N \text{ free from } x)\,dy = \int \frac{1}{4y}dy = \frac{1}{4}\log y.$$

Example 4 : $(x^3 + y^3)\,dx - xy^2\,dy = 0$

Sol. : $Mx + Ny = x^4$

$$\therefore \int M\,dx = \int\left(\frac{1}{x} + \frac{y^3}{x^4}\right)dx = \log x - \frac{y^3}{3x^3} \;;\; \int N\,dy = 0$$

Example 5 : $x\,(x - y)\,dy + y^2\,dx = 0$

Sol. : $Mx - Ny = -x^2y \quad \therefore -\dfrac{y}{x^2}dx + \left(-\dfrac{1}{y} + \dfrac{y}{x}\right)dy = 0$

$$\therefore \int M\,dx = -\int \frac{y}{x^2}dx = \frac{y}{x} \;;\; \int N\,dy = -\int \frac{dy}{y} = -\log y.$$

Example 6 : $(x^4 + y^4)\,dx - xy^3\,dy = 0$

Sol. : $Mx - Ny = x^5$.

$$\therefore \int M\,dx = \int\left(\frac{1}{x} + \frac{y^4}{x^5}\right)dx = \log x - \frac{y^4}{4x^4} \;;\; \int N\,dy = 0$$

Exercise VII (Page 1-33)

Solve the following equations : Class (b) : 6 Marks

Example 1 : $\dfrac{dy}{dx} - \dfrac{1}{x\,(x - 1)}\cdot y = x\,(x - 1)$

Sol. : $e^{\int P\,dx} = e^{-[\log(x-1) - \log x]} = \dfrac{x}{x - 1}$

$$\therefore y\cdot\frac{x}{x - 1} = \int \frac{x}{x - 1}\cdot x\,(x - 1)\,dx + c = \int x^2\,dx + c$$

Example 2 : $\dfrac{dy}{dx} + \dfrac{2x}{x^2 + 1}\cdot y = \dfrac{1}{(x^2 + 1)^2}$ given that $y = 0$ when $x = 1$.

Sol. : $e^{\int P\,dx} = e^{\int \frac{2x}{x^2+1}dx} = x^2 + 1$

$$\therefore y\cdot(x^2 + 1) = \int \frac{1}{x^2 + 1}dx + c = \tan^{-1}x + c.$$

Example 3 : $\dfrac{dy}{dx} + \dfrac{2x}{x^2 + 1}\cdot y = \dfrac{4x^2}{(x^2 + 1)}$

Sol. : $y\cdot(x^2 + 1) = \int 4x^2dx + c = \dfrac{4x^3}{3} + c.$

Example 4 : $x\log x\cdot\dfrac{dy}{dx} + y = 2\log x$

Sol. : $\dfrac{dy}{dx} + \dfrac{1}{x\log x}\cdot y = \dfrac{2}{x} \;;\; e^{\int P\,dx} = e^{\log\log x} = \log x$

$$\therefore y\cdot\log x = \int 2\cdot\frac{\log x}{x}dx + c = (\log x)^2 + c.$$

Example 5 : $\cos x\,\dfrac{dy}{dx} + y\sin x = \sec^2 x$

Sol. : $\dfrac{dy}{dx} + \tan x\cdot y = \sec^3 x$

$$\therefore e^{\int P\,dx} = e^{\int \tan x\,dx} = e^{\log\sec x} = \sec x$$

$$\therefore y\sec x = \int \sec^4 x\,dx + c = \int (1 + \tan^2 x)\cdot\sec^2 x\,dx + c$$

$$\therefore y\sec x = \tan x + \frac{\tan^3 x}{3} + c.$$

Example 6 : $\cos^2 x \dfrac{dy}{dx} + y = \tan x$

Sol. : $\dfrac{dy}{dx} + \sec^2 x \cdot y = \tan x \sec^2 x \quad \therefore \; e^{\int P\,dx} = e^{\tan x}$

$\therefore \; y \cdot e^{\tan x} = \int e^{\tan x} \cdot \tan x \sec^2 x \, dx + c$

$= \int e^t \cdot t \, dt + c \qquad [\text{Put } \tan x = t]$

$= e^t (t-1) + c = e^{\tan x}(\tan x - 1) + c$ ■

Example 7 : $dr + (2r \cot\theta + \sin 2\theta)\, d\theta = 0$

Sol. : $\dfrac{dr}{d\theta} + 2\cot\theta \cdot r = -\sin 2\theta$

$\therefore \; e^{\int 2\cot\theta\, d\theta} = e^{\log \sin^2\theta} = \sin^2\theta$

$\therefore \; r \sin^2\theta = \int -2\sin\theta\cos\theta \cdot \sin^2\theta \, d\theta + c$

$= -2\int \sin^3\theta \cos\theta \, d\theta + c = -\dfrac{\sin^4\theta}{2} + c.$ ■

Exercise VIII (Page 1-36)

Solve the following equations : Class (b) : 6 Marks

Example 1 : $(1+y^2)\, dx = (\tan^{-1} y - x)\, dy$

Sol. : $\dfrac{dx}{dy} + \dfrac{1}{1+y^2} x = \dfrac{\tan^{-1} y}{1+y^2} \quad \therefore \; e^{\int \frac{dy}{1+y^2}} = e^{\tan^{-1} y}$

$\therefore \; x \cdot e^{\tan^{-1} y} = e^{\int \tan^{-1} y} \cdot \dfrac{\tan^{-1} y}{1+y^2} dy + c \qquad [\text{Put } \tan^{-1} y = t]$

$= \int e^t \, t \, dt + c = e^t (t-1) + c.$ ■

Example 2 : $dx + x\, dy = e^{-y} \sec^2 y \, dy$

Sol. : $\dfrac{dx}{dy} + x = e^{-y} \sec^2 y \quad \therefore \; e^{\int P' dy} = e^y$

$\therefore \; x e^y = \int e^y e^{-y} \sec^2 y \, dy + c = \tan y + c.$ ■

Example 3 : $(x+y+1) \dfrac{dy}{dx} = 1$

Sol. : $\dfrac{dx}{dy} = x + y + 1 \quad \therefore \; \dfrac{dx}{dy} - x = y + 1 \quad \therefore \; e^{\int P' dy} = e^{-y}$

$\therefore \; x e^{-y} = \int e^{-y} (y+1) dy + c \qquad [\text{By parts}]$

$\therefore \; x = -y - 1 - 1 + c e^y$ ■

Example 4 : $\sin 2y \, dx = (\tan y - x)\, dy$

Sol. : $\sin 2y \dfrac{dx}{dy} + x = \tan y \quad \therefore \; \dfrac{dx}{dy} - \dfrac{1}{\sin 2y} \cdot x = \dfrac{\tan y}{\sin 2y}$

$\therefore \; e^{\int P' dy} = e^{\int -\frac{1}{2\sin y \cos y} dy} = e^{\frac{1}{2}\int \frac{\sec^2 y}{\tan y} dy}$

$= e^{\log \sqrt{\tan y}} = \sqrt{\tan y}$

$\therefore \; x\sqrt{\tan y} = \int \sqrt{\tan y} \cdot \dfrac{\tan y}{2\sin y \cos y} dy + c$

$= \int \sqrt{\tan y} \cdot \dfrac{\tan y}{2\tan y} \sec^2 y \, dy + c$

$= \dfrac{1}{2} \int \sqrt{\tan y} \cdot \sec^2 y \, dy + c$

$= \dfrac{1}{3} (\tan y)^{3/2} + c.$ ■

Exercise IX (Page 1-41)

Solve the following equations : Class (b) : 6 Marks

Example 1 : $\sec y \dfrac{dy}{dx} + 2x \sin y = 2x \cos y$

Sol. : $\sec^2 y \dfrac{dy}{dx} + 2\tan y \cdot x = 2x.$

Put $\tan y = v, \qquad \therefore \; \sec^2 y \dfrac{dy}{dx} = \dfrac{dv}{dx}$

$\therefore \; \dfrac{dv}{dx} + 2x \cdot v = 2x \quad \therefore \; e^{\int P dx} = e^{x^2}$

$\therefore \; v e^{x^2} = \int e^{x^2} 2x \, dx + c = e^{x^2} + c.$ ■

Example 2 : $\tan y \dfrac{dy}{dx} + \tan x (1 - \cos y) = 0$

Sol. : $\tan y \dfrac{dy}{dx} + \tan x = \tan x \cos y$

$\therefore \; \sec y \tan y \dfrac{dy}{dx} + \sec y \tan x = \tan x$

Put $\sec y = v, \qquad \therefore \; \sec y \tan y \dfrac{dy}{dx} = \dfrac{dv}{dx}$

$\therefore \; \dfrac{dv}{dx} + \tan x \cdot v = \tan x \quad \therefore \; e^{\int P dx} = e^{\int \tan x \, dx} = \sec x$

$\therefore \; v \sec x = \int \sec x \tan x + c = \sec x + c$ ■

Example 3 : $\dfrac{dy}{dx} + \dfrac{1}{x}\tan y = \dfrac{1}{x^2}\tan y \sin y$

Sol. : $\cot y \operatorname{cosec} y \dfrac{dy}{dx} + \operatorname{cosec} y \cdot \dfrac{1}{x} = \dfrac{1}{x^2}$

Put $\operatorname{cosec} y = v$, $\therefore -\operatorname{cosec} y \cot y \dfrac{dy}{dx} = \dfrac{dv}{dx}$

$\therefore \dfrac{dv}{dx} - \dfrac{1}{x} \cdot v = -\dfrac{1}{x^2}$ $\quad \therefore e^{\int -\frac{1}{x}dx} = e^{-\log x} = \dfrac{1}{x}$

$\therefore \dfrac{1}{x} v = \int -\dfrac{1}{x^3} dx + c = \dfrac{1}{2x^2} + c.$ ■

Example 4 : $\sec^2 y \dfrac{dy}{dx} + 2\tan x \tan y = \sin x$

Sol. : Put $\tan y = v$, $\therefore \sec^2 y \dfrac{dy}{dx} = \dfrac{dv}{dx}$ $\quad \therefore \dfrac{dv}{dx} + 2\tan x v = \sin x$

$\therefore e^{\int 2\tan x dx} = e^{\log \sec^2 x} = \sec^2 x$

$\therefore v \sec^2 x = \int \sec^2 x \sin x \, dx + c$

$= \int \sec x \tan x \, dx + c = \sec x + c$ ■

Example 5 : $x \dfrac{dy}{dx} - 1 = x e^{-y}$

Sol. : $e^y \dfrac{dy}{dx} - \dfrac{1}{x} e^y = 1$. Put $e^y = v$ $\quad \therefore e^y \dfrac{dy}{dx} = \dfrac{dv}{dx}$

$\therefore \dfrac{dv}{dx} - \dfrac{1}{x} \cdot v = 1$ $\quad \therefore e^{\int -\frac{1}{x}dx} = e^{-\log x} = \dfrac{1}{x}$

$\therefore v \cdot \dfrac{1}{x} = \int \dfrac{1}{x} dx + c = \log x + c.$ ■

Example 6 : $x \cos y \dfrac{dy}{dx} - \sin y = x \sin^2 y$

Sol. : Divide by $\sin^2 y$, $\quad x \operatorname{cosec} y \cot y \dfrac{dy}{dx} - \operatorname{cosec} y = x$

$\therefore \operatorname{cosec} y \cot y \dfrac{dy}{dx} - \dfrac{1}{x} \operatorname{cosec} y = 1$

Put $\operatorname{cosec} y = v$, $\therefore -\operatorname{cosec} y \cot y \dfrac{dy}{dx} = \dfrac{dv}{dx}$

$\therefore -\dfrac{dv}{dx} - \dfrac{1}{x} v = 1$ $\quad \therefore e^{\int -\frac{1}{x}dx} = e^{\log x} = \dfrac{1}{x}$

$\therefore -v \cdot \dfrac{1}{x} = \int \dfrac{1}{x} dx + c = \log x + c$ ■

Example 7 : $3y^2 \dfrac{dy}{dx} + 2y^3 x = 4x^3 e^{x^2}$.

Sol. : Put $y^3 = v$, $\therefore 3y^2 \dfrac{dy}{dx} = \dfrac{dv}{dx}$

$\therefore \dfrac{dv}{dx} + 2x \cdot v = 4x^3 e^{x^2}$ $\quad \therefore e^{\int 2x dx} = e^{x^2}$

$\therefore v \cdot e^{x^2} = \int e^{x^2} \cdot e^{x^2} \cdot 4 \cdot (x^2) x \, dx + c = \int e^{2t} \cdot 2t \cdot dt + c$ [Put $x^2 = t$]

$= 2\left[t \dfrac{e^{2t}}{2} - \int \dfrac{e^{2t}}{2} dt\right] + c = 2\left[t e^{2t} - \dfrac{e^{2t}}{4}\right] + c.$ ■

Exercise - X (Page 1-42)

Solve the following equations : Class (b) : 6 Marks

Example 1 : $e^x(x+1)\,dx + (y^2 e^{2y} - x e^x)\,dy = 0$

Sol. : $e^x(x+1)\dfrac{dx}{dy} - x e^x = -y^2 e^{2y}$. Put $x e^x = v$

$\therefore e^x(x+1)\dfrac{dx}{dy} = \dfrac{dv}{dy}$ $\quad \therefore \dfrac{dv}{dy} - v = y^2 e^{2y}$ $\quad \therefore e^{\int P dy} = e^{-y}$

$\therefore v \cdot e^y = -\int y^2 e^{2y} \cdot e^{-y} dy + c = -\int e^y \cdot y^2 dy + c$

$\therefore \int e^y \cdot y^2 dy = y^2 e^y - \int e^y \cdot 2y \, dy = y^2 e^y - 2[e^y \cdot y - \int e^y dy]$

$= (y^2 - 2y + 1)e^y = (y-1)^2 e^y.$ ■

Example 2 : $y \dfrac{dx}{dy} = x + y x^2 \log y$

Sol. : $\dfrac{dx}{dy} - \dfrac{x}{y} = x^2 \log y$ $\quad \therefore \dfrac{1}{x^2} \cdot \dfrac{dy}{dx} - \dfrac{1}{x} \cdot \dfrac{1}{y} = \log y$

Put $-\dfrac{1}{x} = v$ $\quad \therefore \dfrac{1}{x^2} \cdot \dfrac{dx}{dy} = \dfrac{dv}{dy}$ $\quad \therefore \dfrac{dv}{dx} + \dfrac{1}{y} \cdot v = \log y$

$\therefore e^{\int P dy} = e^{\int \frac{1}{y} dy} = e^{\log y} = y$

$\therefore y \cdot v = \int y \log y + c = \log y \cdot \dfrac{y^2}{2} - \int \dfrac{y^2}{2} \cdot \dfrac{1}{y} + c$

$= \dfrac{y^2}{2} \log y - \dfrac{y^2}{4} + c.$ ■

Example 3 : $\dfrac{dx}{dy} = e^{y-x}(e^y - e^x)$

Sol. : $e^x \dfrac{dx}{dy} + e^x \cdot e^y = e^{2y}$. Put $e^x = v$ $\quad \therefore e^x \dfrac{dx}{dy} = \dfrac{dv}{dy}$

$\therefore \dfrac{dv}{dy} + e^y \cdot v = e^{2y} \quad \therefore e^{\int P\,dy} = e^{e^y}$

$\therefore v e^{e^y} = \int e^{e^y} \cdot e^{2y}\,dy + c = \int e^{e^y} \cdot e^y \cdot e^y dy + c \qquad [\text{Put } e^y = t]$

$= \int e^t t\,dt + c = e^t (t-1) + c.$ ■

Example 4 : $y \sin x \dfrac{dx}{dy} - \cos x = 2y^3 \cos^2 x$

Sol. : Divide by $\cos^2 x$, $\quad y \sec x \tan x \dfrac{dx}{dy} - \sec x = 2y^3$

Put $\sec x = v$, $\quad \therefore \sec x \tan x \dfrac{dx}{dy} = \dfrac{dv}{dy}$

$\therefore \dfrac{dv}{dy} - \dfrac{1}{y} v = 2y^2 \quad \therefore e^{-\int \frac{1}{y} dy} = e^{-\log y} = \dfrac{1}{y}$

$\therefore v \cdot \dfrac{1}{y} = \int \dfrac{2y^2}{y} dy + c = y^2 + c.$ ■

Exercise XI (Page 1-50)

Solve the following equations : Class (b) : 6 Marks

Example 1 : $\dfrac{dy}{dx} = 2y(1 - 2xy)$

Sol. : $\dfrac{1}{y^2} \cdot \dfrac{dy}{dx} - \dfrac{2}{y} = -4x.$ Put $-\dfrac{1}{y} = v \quad \therefore \dfrac{1}{y^2} \cdot \dfrac{dy}{dx} = \dfrac{dv}{dx}$

$\therefore \dfrac{dv}{dx} + 2v = -4x \quad \therefore e^{\int P dx} = e^{2x}$

$\therefore v e^{2x} = -4 \int e^{2x} x\,dx + c = -4\left[x \dfrac{e^{2x}}{2} - \int \dfrac{e^{2x}}{2} - 1 dx + c\right]$

$= -4\left[x \dfrac{e^{2x}}{2} - \dfrac{e^{2x}}{4} + c\right] = -2x e^{2x} + e^{2x} + c.$ ■

Example 2 : $2x\,dx - y^2(y^3 + x^2)\,dy = 0$

Sol. : $2x \cdot \dfrac{dx}{dy} - y^2 x^2 = y^5.$ Put $x^2 = v \quad \therefore 2x \dfrac{dx}{dy} = \dfrac{dv}{dy}$

$\therefore \dfrac{dv}{dy} - y^2 v = y^5 \quad \therefore e^{\int -y^2 dy} = e^{-y^3/3}$

$\therefore v e^{-y^3/3} = \int e^{-y^3/3} \cdot y^5\,dy + c$

Put $-\dfrac{y^3}{3} = t. \quad \therefore -y^2 \cdot dy = dt$

$= 3 \int e^t t \cdot dt + c = 3[e^t(t-1) + c]$ ■

Example 3 : $3x(1 - x^2) y^2 \dfrac{dy}{dx} + (2x^2 - 1) y^3 = ax^3$

Sol. : $3y^2 \dfrac{dy}{dx} + \dfrac{2x^2 - 1}{x(1 - x^2)} y^3 = \dfrac{a x^3}{x(1 - x^2)}$

Put $y^3 = v$, $\quad \therefore 3y^2 \dfrac{dy}{dx} = \dfrac{dv}{dx} \quad \therefore \dfrac{dv}{dx} + \dfrac{2x^2 - 1}{x(1 - x^2)} v = \dfrac{ax^2}{1 - x^2}$

$\therefore \int \dfrac{2x^2}{x(1 - x^2)} dx = \int -\dfrac{dx}{x} + \dfrac{1}{2} \int \dfrac{dx}{1 - x} - \dfrac{1}{2} \int \dfrac{dx}{1 + x}$

$= -\log x - \dfrac{1}{2} \log(1 - x) - \dfrac{1}{2} \log(1 + x)$

$= \log\left(\dfrac{1}{x\sqrt{1 - x^2}}\right) \quad \therefore e^{\int P dx} = \dfrac{1}{x\sqrt{1 - x^2}}$

$\therefore v \cdot \dfrac{1}{x\sqrt{1 - x^2}} = \int \dfrac{ax^2}{x\sqrt{1 - x^2}\,(1 - x^2)} dx + c$

$= a \int \dfrac{x}{(1 - x^2)^{3/2}} dx + c = \dfrac{a}{\sqrt{1 - x^2}} + c.$ ■

Example 4 : $x \dfrac{dy}{dx} + y = y^3 x^{n+1}$

Sol. : $\dfrac{1}{y^3} \cdot \dfrac{dy}{dx} + \dfrac{1}{y^2} \cdot \dfrac{1}{x} = x^n$

Put $\dfrac{1}{y^2} = v \quad \therefore -\dfrac{2}{y^3} \cdot \dfrac{dy}{dx} = \dfrac{dv}{dx}$

$\therefore \dfrac{dv}{dx} - \dfrac{2}{x} v = -2x^n \quad \therefore e^{\int -\frac{2}{x} dx} = e^{\log \frac{1}{x^2}} = \dfrac{1}{x^2}$

$\therefore v \cdot \dfrac{1}{x^2} = \int -2x^{n-2} dx + c = -\dfrac{2x^{n-1}}{n - 1} + c.$ ■

Example 5 : $\dfrac{dz}{dx} + \dfrac{z}{x} \log z = \dfrac{z}{x^2} (\log z)^3$

Sol. : $\dfrac{1}{z} \cdot \dfrac{dz}{dx} + \log z \cdot \dfrac{1}{x} = \dfrac{(\log z)^3}{x^2}$

Put $\log z = v$, $\quad \therefore \dfrac{1}{z} \cdot \dfrac{dz}{dx} = \dfrac{dv}{dx} \quad \therefore \dfrac{dv}{dx} + \dfrac{1}{x} \cdot v = \dfrac{v^3}{x^2}$

Dividing by v^3, $\quad \dfrac{1}{v^3} \cdot \dfrac{dv}{dx} + \dfrac{1}{x} \cdot \dfrac{1}{v^2} = \dfrac{1}{x^2}$

Put $\dfrac{1}{v^2} = u \quad \therefore -\dfrac{2}{v^3} \cdot \dfrac{dv}{dx} = \dfrac{du}{dx}$

$$\therefore\ -\frac{1}{2}\cdot\frac{du}{dx}+\frac{1}{x}u=\frac{1}{x^2}\quad\therefore\ \frac{du}{dx}-\frac{2}{x}u=-\frac{2}{x^2}$$

$$\therefore\ \int e^{-(2/x)dx}=e^{\log(1/x^2)}=\frac{1}{x^2}$$

$$\therefore\ \frac{1}{x^2}\cdot u=\int-\frac{2}{x^4}dx+c=\frac{2}{3}\cdot\frac{1}{x^3}+c.$$

Example 6 : $\dfrac{dy}{dx}-xy=y^2e^{-(x^2/2)}\cdot\log x$

Sol. : $\dfrac{1}{y^2}\cdot\dfrac{dy}{dx}-\dfrac{1}{y}\cdot x=e^{-x^2/2}\log x$

Put $-\dfrac{1}{y}=v\quad\therefore\ \dfrac{1}{y^2}\cdot\dfrac{dy}{dx}=\dfrac{dv}{dx}\quad\therefore\ \dfrac{dv}{dx}+xv=e^{-x^2/2}\log x$

$$\therefore\ e^{\int P\,dx}=e^{x^2/2}$$

$$\therefore\ v\cdot e^{x^2/2}=\int e^{x^2/2}\cdot e^{-x^2/2}\cdot\log x\,dx+c$$

$$=\int\log x\,dx+c=x\log x-x+c.$$

Example 7 : $x\dfrac{dy}{dx}+2y=y^2x^3$

Sol. : $\dfrac{1}{y^2}\cdot\dfrac{dy}{dx}+\dfrac{2}{x}\cdot\dfrac{1}{y}=x^2$

Put $\dfrac{1}{y}=v\quad\therefore\ -\dfrac{1}{y^2}\cdot\dfrac{dy}{dx}=\dfrac{dv}{dx}\quad\therefore\ \dfrac{dv}{dx}-\dfrac{2}{x}\cdot v=-x^2$

$$\therefore\ e^{\int-(2/x)dx}=e^{\log(1/x^2)}=\frac{1}{x^2}$$

$$\therefore\ v\cdot\frac{1}{x^2}=\int\frac{1}{x^2}(-x^2)\,dx=\int-dx=-x+c.$$

Example 8 : $x\dfrac{dy}{dx}+y=y^2\log x$

Sol. : $\dfrac{1}{y^2}\cdot\dfrac{dy}{dx}+\dfrac{1}{x}\cdot\dfrac{1}{y}=\dfrac{\log x}{x}$

Put $\dfrac{1}{y}=v\quad\therefore\ -\dfrac{1}{y^2}\cdot\dfrac{dy}{dx}=\dfrac{dv}{dx}\quad\therefore\ \dfrac{dv}{dx}-\dfrac{1}{x}\cdot v=-\dfrac{\log x}{x}$

$$\therefore\ e^{\int-(1/x)dx}=e^{\log(1/x)}=\frac{1}{x}\quad\therefore\ v\cdot\frac{1}{x}=\int-\frac{\log x}{x^2}dx.$$

Put $\log x=t,\quad\therefore\ \dfrac{1}{x}dx=dt\quad\therefore\ x=e^t$

$$\therefore\ \int\frac{\log x}{x}\cdot\frac{dx}{x}=\int te^{-t}\,dt=t(-e^{-t})-\int(-e^{-t})\cdot 1dt$$

$$=-te^{-t}-e^{-t}=-e^{-t}(t+1)=-\frac{1}{x}(1+\log x).$$

Example 9 : $y\dfrac{dx}{dy}=x-yx^2\cos y$

Sol. : $\dfrac{1}{x^2}\cdot\dfrac{dx}{dy}-\dfrac{1}{x}\cdot\dfrac{1}{y}=-\cos y$. Put $-\dfrac{1}{x}=v,\ \dfrac{1}{x^2}\cdot\dfrac{dx}{dy}=\dfrac{dv}{dy}$

$$\therefore\ \frac{dv}{dy}+v\cdot\frac{1}{y}=-\cos y\quad\therefore\ e^{\int(1/y)dy}=e^{\log y}=y$$

$$\therefore\ v\cdot y=\int-y\cos y\,dy+c'=-\left[y\cdot\sin y-\int\sin y\cdot 1\cdot dy\right]+c'$$

Example 10 : $xy-\dfrac{dy}{dx}=y^3e^{-x^2}$

Sol. : $-\dfrac{1}{y^3}\cdot\dfrac{dv}{dx}+\dfrac{1}{y^2}\cdot x=e^{-x^2}$. Put $\dfrac{1}{y^2}=v\quad\therefore\ -\dfrac{2}{y^3}\cdot\dfrac{dy}{dx}=\dfrac{dv}{dx}$

$$\therefore\ \frac{dv}{dx}+2xv=2e^{-x^2}\quad\therefore\ e^{\int P\,dx}=e^{\int 2x\,dx}=e^{x^2}$$

$$\therefore\ v\cdot e^{x^2}=\int 2\cdot e^{x^2}\cdot e^{-x^2}dx+c=\int 2\,dx+c=2x+c.$$

Example 11 : $\dfrac{dy}{dx}+\dfrac{y}{x}=y^3x^n$

Sol. : $\dfrac{1}{y^3}\cdot\dfrac{dy}{dx}+\dfrac{1}{y^2}\cdot\dfrac{1}{x}=x^n$. Put $\dfrac{1}{y^2}=v\quad\therefore\ -\dfrac{2}{y^3}\cdot\dfrac{dy}{dx}=\dfrac{dv}{dx}$

$$\therefore\ \frac{dv}{dx}-\frac{2}{x}\cdot v=-2x^n\quad\therefore\ e^{\int-(2/x)dx}=e^{\log(1/x^2)}=\frac{1}{x^2}$$

$$\therefore\ v\cdot\frac{1}{x^2}=\int-2x^{n-2}dx+c\quad\therefore\ \frac{v}{x^2}=-2\cdot\frac{x^{n-1}}{n-1}+c.$$

Miscellaneous Exercise (Page 1-57)

Solve the following equations : Class (b) : 6 Marks

Example 1 : $(2x^2+3y^2-7)\,x\,dx+(3x^2+2y^2-8)\,y\,dy=0$

Sol. : $(2x^3+3xy^2-7x)\,dx+(3x^2y+2y^3-8y)\,dy=0$

$$\frac{\partial M}{\partial y}=6xy=\frac{\partial N}{\partial x}\qquad\therefore\ \text{Exact.}$$

$$\int M\,dx=\frac{2x^4}{4}+\frac{3x^2}{2}y^2-\frac{7x^2}{2}\ ;\ \int N\,dy=\frac{2y^4}{4}-\frac{8y^2}{2}$$

Example 2 : $\sin y\dfrac{dy}{dx}=(1-x\cos y)\cos y$

Sol. : Divide by $\cos^2 y$, $\therefore\ \sec y\tan y\dfrac{dy}{dx}-\sec y=-x$

Put $\sec y=v\quad\therefore\ \dfrac{dv}{dx}-v=-x\quad\therefore\ e^{\int P\,dx}=e^{-\int dx}=e^{-x}$

$\therefore\ ve^{-x} = -\int e^{-x} x\,dx + c$ [Integrate by parts]

Example 3 : $xy - \dfrac{dy}{dx} = y^3 e^{-x^2}$

Sol. : $-\dfrac{1}{y^3}\cdot\dfrac{dy}{dx} + \dfrac{1}{y^2}x = e^{-x^2}$.

Put $\dfrac{1}{y^2} = v \quad \therefore\ \dfrac{dv}{dx} + 2xv = 2e^{-x^2} \quad \therefore$ I.F. $= e^{2\int x\,dx} = e^{x^2}$

$\therefore\ v e^{x^2} = \int e^{-x^2}\cdot e^{x^2}dx + c = x + c$

Example 4 : $x\dfrac{dy}{dx} = 2x^2y + y\log y$

Sol. : $\dfrac{1}{y}\cdot\dfrac{dy}{dx} - \log y\cdot\dfrac{1}{x} = 2x.$

Put $\log v = v, \quad \therefore\ \dfrac{dv}{dx} - \dfrac{1}{x}\cdot v = 2x \quad \therefore\ e^{-\int\frac{1}{x}dx} = e^{-\log x} = \dfrac{1}{x}$

$\therefore\ v\cdot\dfrac{1}{x} = \int 2\,dx + c = 2x + c.$

Example 5 : $x\,dy - [\,y + xy^3(1+\log x)\,]\,dx = 0$

Sol. : Divide by xy^3, $\quad \therefore\ \dfrac{1}{y^3}\cdot\dfrac{dy}{dx} - \dfrac{1}{y^2}\cdot\dfrac{1}{x} = (1+\log x)$

Put $-\dfrac{1}{y^2} = v \quad \therefore\ \dfrac{2}{y^3}\cdot\dfrac{dy}{dx} = \dfrac{dv}{dx}$

$\therefore\ \dfrac{dv}{dx} + \dfrac{2}{x}v = 2(1+\log x) \quad \therefore\ e^{\int P\,dx} = e^{2\log x} = x^2$

$\therefore\ v\cdot x^2 = \int(2x^2 + 2x^2\log x)\,dx$

$= \dfrac{2x^3}{3} + 2\left[\log x\,\dfrac{x^3}{3} - \int\dfrac{x^3}{3}\cdot\dfrac{1}{x}dx\right]$

$= \dfrac{2x^3}{3}(1+\log x) - \dfrac{2}{9}x^3 + c.$

Example 6 : $(2x - y)\,dx - (x + 2y - 5)\,dy = 0$

Sol. : $\dfrac{\partial M}{\partial y} = -1 = \dfrac{\partial N}{\partial x}$

$\therefore\ \int M\,dx = x^2 - xy\ ;\ \int N\,dy = \int(-2y+5)\,dy = -y^2 + 5y.$

Example 7 : $x\dfrac{dy}{dx} + \dfrac{y^2}{x} = y$

Sol. : $Mx - Ny = xy^2 - x^2y + x^2y = xy^2 \quad \therefore\ \left(\dfrac{1}{x} - \dfrac{1}{y}\right)dx + \dfrac{x}{y^2}dy = 0$

$\therefore\ \int M\,dx = \int\left(\dfrac{1}{x} - \dfrac{1}{y}\right)dx = \log x - \dfrac{x}{y}\ ;\ \int N\,dy = 0$

Example 8 : $\dfrac{dy}{dx} = x^3(x+y)^3 - x(x+y) - 1$

Sol. : Put $x + y = v, \quad \therefore\ 1 + \dfrac{dy}{dx} = \dfrac{dv}{dx} \quad \therefore\ \dfrac{dv}{dx} = v^3x^3 - vx$

$\therefore\ \dfrac{dv}{dx} + vx = v^3x^3 \quad \therefore\ \dfrac{1}{v^3}\dfrac{dv}{dx} + \dfrac{1}{v^2}x = x^3$

Put $\dfrac{1}{v^2} = z \quad \therefore\ -\dfrac{1}{2}\cdot\dfrac{dz}{dx} + zx = x^3$

$\therefore\ \dfrac{dz}{dx} - 2xz = -2x^3 \quad \therefore\ e^{\int -2x\,dx} = e^{-x^2}$

$\therefore\ z e^{-x^2} = -2\int e^{-x^2}x^3dx + c$

Put $-x^2 = t \quad \therefore\ -2x\,dx = dt$

$\therefore\ I = -\int e^{-t}t\,dt = -\left[t e^t - \int e^t\right] = -\left[e^t(t-1)\right]$

Example 9 : $x\dfrac{dy}{dx} + y\log y = xy\,e^x$

Sol. : $\dfrac{1}{y}\cdot\dfrac{dy}{dx} + \log y\cdot\dfrac{1}{x} = e^x.$

Put $\log y = v, \quad \therefore\ \dfrac{dv}{dx} + v = e^x \quad \therefore\ e^{\int\frac{1}{x}dx} = e^{\log x} = x$

$\therefore\ v e^x = \int e^x x\,dx + c = e^x(x-1) + c.$

Example 10 : $(12y + 4y^3 + 6x^2)\,dx + 3(x + xy^2)\,dy = 0$

Sol. : $\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)\Big/N = \dfrac{9(1+y^2)}{3x(1+y^2)} = \dfrac{3}{x}$

$\therefore\ e^{\int f(x)dx} = e^{3\log x} = x^3$

$\therefore\ (12x^3y + 4x^3y^3 + 6x^5)\,dx + 3x^3(x + xy^2)\,dy = 0$

$\therefore\ \int M\,dx = 3x^4y + x^4y^3 + x^6\ ;\ \int N\,dy = 0$

Example 11 : $x\dfrac{dy}{dx} + y = y^2\log x$

Sol. : $\dfrac{1}{y^2}\cdot\dfrac{dy}{dx} + \dfrac{1}{y}\cdot\dfrac{1}{x} = \dfrac{1}{x}\log x.$

Put $\dfrac{1}{y} = v, \quad \therefore\ \dfrac{dv}{dx} - \dfrac{1}{x}\cdot v = -\dfrac{1}{x}\log x \quad \therefore\ e^{-\int\frac{1}{x}dx} = \dfrac{1}{x}$

$\therefore\ v\cdot\dfrac{1}{x} = \int -\dfrac{1}{x^2}\log x + c = \log x\left(\dfrac{1}{x}\right) - \int\dfrac{1}{x}\cdot\dfrac{1}{x}dx + c.$

Example 12 : $(x^2e^x - my)\,dx + mx\,dy = 0$

Sol. : $\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\Big/N = \frac{-m-m}{mx} = -\frac{2}{x} \quad \therefore\ e^{\int -\frac{2}{x}dx} = \frac{1}{x^2}$

$\therefore\ \int M\,dx = \int\left(e^x - \frac{my}{x^2}\right)dx = e^x + \frac{my}{x}\ ;\ \int N\,dy = 0.$ ■

Example 13 : $(x^2 + y^2 + 2x)\,dx + 2y\,dy = 0$

Sol. : $\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\Big/N = \frac{2y-0}{2y} = 1 \quad \therefore\ \text{I.F.} = e^x$

$\therefore\ \int M\,dx = \int e^x\,[(x^2+2x)+y^2]\,dx = e^x x^2 + e^x y^2\ ;\ \int N\,dy = 0$ ■

Example 14 : $\left(xy^2 - e^{1/x^3}\right)dx - x^2y\,dy = 0$

Sol. : $\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\Big/N = \frac{2xy+2xy}{-x^2y} = -\frac{4}{x} \quad \therefore\ \text{I.F.} = e^{-\int\frac{4}{x}dx} = \frac{1}{x^4}$

$\therefore\ \int M\,dx = \int\left[\frac{y^2}{x^3} - e^{1/x^3}\cdot\frac{1}{x^4}\right]dx = -\frac{y^2}{2x^2} + \frac{1}{3}e^{1/x^3}\ ;\ \int N\,dy = 0$ ■

Example 15 : $(1+y^2)\,dx = \left(\sqrt{1+y^2}\cdot\sin y - xy\right)dy$

Sol. : $\frac{dx}{dy} + \frac{y}{1+y^2}x = \frac{\sin y}{\sqrt{1+y^2}} \quad \therefore\ e^{\int\frac{y}{1+y^2}} = e^{\frac{1}{2}\log(1+y^2)} = \sqrt{1+y^2}$

$\therefore\ x\cdot\sqrt{1+y^2} = \int \sin y\,dy + c$ ■

Example 16 : $\frac{dy}{dx} = \frac{x^2+y^2}{xy}$

Sol. : $(x^2+y^2)\,dx - xy\,dy = 0 \quad \therefore\ Mx - Ny = x^3 + xy^2 - xy^2 = x^3$

$\therefore\ \left(\frac{1}{x} + \frac{y^2}{x^3}\right)dx - \frac{y\,dy}{x^2} = 0 \quad \therefore\ \int M\,dx = \log x - \frac{y^2}{2x^2}\ ;\ \int N\,dy = 0.$ ■

Example 17 : $y\,dx - x\,dy + \log x\,dx = 0$

Sol. : $(y + \log x)\,dx - x\,dy = 0$

$\therefore\ \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\Big/N = \frac{1+1}{-x} = -\frac{2}{x} \quad \therefore\ \text{I.F.} = e^{-\int\frac{2}{x}dx} = \frac{1}{x^2}$

$\therefore\ \int M\,dx = \int\left[\frac{y}{x^2} + \frac{\log x}{x^2}\right]dx = -\frac{y}{x} + \log\left(-\frac{1}{x}\right) - \int -\frac{1}{x}\cdot\frac{1}{x}\,dx$

$= -\frac{y}{x} - \frac{1}{x}\log x - \frac{1}{x^2}\ ;\ \int N\,dy = 0.$ ■

Example 18 : $(1-x^2)\frac{dy}{dx} - xy = (1-x^2)^{3/2}\cdot\sin x$

Sol. : $\frac{dy}{dx} - \frac{x}{1-x^2}\cdot y = \sqrt{1-x^2}\cdot\sin x$

$\therefore\ \text{I.F.} = e^{\int P\,dx} = e^{\frac{1}{2}\int\frac{-2x}{1-x^2}dx} = e^{\log\sqrt{1-x^2}} = \sqrt{1-x^2}$

$\therefore\ y\cdot\sqrt{1-x^2} = \int(1-x^2)\sin x\,dx + c$ [By parts]

$= (1-x^2)(-\cos x) - \int -\cos x\,(-2x)\,dx + c$

$= -(1-x^2)\cos x - 2x\sin x + \int \sin x\,(2)\,dx + c$ ■

Example 19 : $[\cos x\tan y + \cos(x+y)]\,dx + [\sin x\sec^2 y + \cos(x+y)]\,dy = 0$

Sol. : $\frac{\partial M}{\partial y} = \cos x\sec^2 y - \sin(x+y) = \frac{\partial N}{\partial x}$

$\int M\,dx = \sin x\tan y + \sin(x+y)\ ;\ \int N\,dy = 0$ ■

Example 20 : $x\frac{dy}{dx} - y = x^3\cos x$, given $y = 0$ when $x = \pi$.

Sol. : $\frac{dy}{dx} - \frac{1}{x}y = x^2\cos x \quad \therefore\ e^{\int P\,dx} = e^{-\int\frac{1}{x}dx} = e^{-\log x} = \frac{1}{x}$

$\therefore\ \frac{y}{x} = \int x\cos x\,dx = x(\sin x) - \int \sin x\,dx + c.$ ■

Example 21 : $(2x + 3y - 5)\,dy + (3x + 2y - 5)\,dx = 0$

Sol. : Exact. $\therefore\ \int M\,dx = \frac{3x^2}{2} + 2xy - 5x\ ;\ \int N\,dy = \frac{3y^2}{2} - 5y.$ ■

Example 22 : $x\,dx + y\,dy + \frac{x\,dy - y\,dx}{x^2+y^2} = 0$

Sol. : Exact. $\frac{x^2}{2} + \frac{y^2}{2} + \tan^{-1}\frac{y}{x} = c$ ■

Example 23 : $x(x^2+1)\frac{dy}{dx} = y(1-x^2) + x^3\log x$

Sol. : $\frac{dy}{dx} - \frac{(1-x^2)}{x(1+x^2)}y = \frac{x^3\log x}{x(x^2+1)}$

$\therefore\ \int P\,dx = \int\left(-\frac{1}{x} + \frac{2x}{1+x^2}\right)dx = \log\frac{1+x^2}{x} \quad \therefore\ \text{I.F.} = e^{\int P\,dx} = \frac{1+x^2}{x}$

$\therefore\ y\left(\frac{x^2+1}{x}\right) = \int\frac{x^2+1}{x}\cdot\frac{x^3\log x}{x(x^2+1)}\,dx + c$

$= \int x\log x\,dx + c = \log x\cdot\frac{x^2}{2} - \int\frac{x^2}{2}\cdot\frac{1}{x}\,dx + c$ ■

Example 24 : $dy + (2x\tan^{-1}y - x^3)(1+y^2)\,dx = 0$

Sol. : $\dfrac{1}{1+y^2}\cdot\dfrac{dy}{dx} + 2\tan^{-1}y\cdot x = x^3$. Put $\tan^{-1}y = v$

$\therefore\ \dfrac{dv}{dx} + 2x\cdot v = x^3 \quad \therefore$ I.F. $= e^{\int 2x\,dx} = e^{x^2}$

$\therefore\ v\cdot e^{x^2} = \int e^{x^2}x^3\,dx + c \qquad [\text{Put } x^2 = t]$

$= \dfrac{1}{2}\int e^t\cdot t\,dt + c = \dfrac{1}{2}e^t(t-1) + c.$ ■

Example 25 : $(x^3 + 3xy^2)\,dx + (y^3 + 3x^2y)\,dy = 0$

Sol. : $\dfrac{\partial M}{\partial y} = 6xy = \dfrac{\partial N}{\partial x} \quad \therefore \int M\,dx = \dfrac{x^4}{4} + \dfrac{3x^2y^2}{2}\ ;\ \int N\,dy = \dfrac{y^4}{4}.$ ■

Example 26 : $xy - \dfrac{dy}{dx} = y^3e^{-x^2}$

Sol. : $\dfrac{1}{y^3}\cdot\dfrac{dy}{dx} - \dfrac{1}{y^2}\cdot x = -e^{x^2}$. Put $-\dfrac{1}{y^2} = v \quad \therefore \dfrac{2}{y^3}\cdot\dfrac{dy}{dx} = \dfrac{dv}{dx}$

$\therefore\ \dfrac{dv}{dx} + 2x\cdot v = -2e^{-x^2} \quad \therefore\ e^{\int 2x\,dx} = e^{x^2}$

$\therefore\ v\cdot e^{x^2} = \int e^{x^2}(-2)e^{-x^2}\,dx + c = -2x + c.$ ■

Example 27 : $(y^2e^{xy^2} + 4x^3)\,dx + (2xy\,e^{xy^2} - 3y^2)\,dy = 0$

Sol. : $\dfrac{\partial M}{\partial y} = 2xy^3e^{xy^2} = \dfrac{\partial N}{\partial x}$

$\therefore\ \int M\,dx = e^{xy^2} + x^4\ ;\ \int N\,dy = -y^3.$ ■

Example 28 : $(2x^3y + 3)\,dy + (3x^2y^2 + 2x)\,dx = 0$

Sol. : $\dfrac{\partial M}{\partial y} = 6x^2y = \dfrac{\partial N}{\partial x}$

$\therefore\ \int M\,dx = \int(3x^2y^2 + 2x)\,dx = x^3y^2 + x^2\ ;\ \int N\,dy = 3y.$ ■

Example 29 : $\dfrac{dy}{dx} = \dfrac{\tan y - 2xy - y}{x^2 - x\tan^2 y + \sec^2 y}$

Sol. : $(\tan y - 2xy - y)\,dx - (x^2 - x\tan^2 y + \sec^2 y)\,dy = 0$

$\therefore\ \dfrac{\partial M}{\partial y} = \sec^2 y - 2x - 1\ ;\ \dfrac{\partial N}{\partial x} = -2x + \tan^2 y = -2x + \sec^2 y - 1$

$\therefore\ x\tan y - x^2y - xy + \tan y = c$ ■

Example 30 : $(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\,dy = 0$

Sol. : $\left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right)\Big/M = \dfrac{3(y^2+2)}{y(y^2+2)} = -\dfrac{3}{y} \quad \therefore$ I.F. $= e^{\int -\frac{3}{y}dy} = \dfrac{1}{y^3}$

$\therefore\ \left(y + \dfrac{2}{y^2}\right)dx + \left(x + 2y - \dfrac{4x}{y^3}\right)dy = 0 \quad \therefore\ xy + \dfrac{2x}{y^2} + y^2 = c.$ ■

Example 31 : $\sin x\,\dfrac{dy}{dx} + 2y = \tan^3\left(\dfrac{x}{2}\right)$

Sol. : $\dfrac{dy}{dx} + 2\operatorname{cosec} x\cdot y = \dfrac{\tan^3(x/2)}{\sin x}$

$\therefore\ e^{\int P\,dx} = e^{\int 2\operatorname{cosec} x\,dx} = e^{2\log\tan(x/2)} = \tan^2\left(\dfrac{x}{2}\right)$

$\therefore\ y\tan^2\dfrac{x}{2} = \int\dfrac{\tan^3(x/2)}{\sin x}\cdot\tan^2\left(\dfrac{x}{2}\right)dx + c = \int\dfrac{\tan^5(x/2)}{2\sin(x/2)\cos(x/2)}\,dx$

$= \int\dfrac{\tan^5(x/2)/\cos^2(x/2)}{2\sin(x/2)\cos(x/2)/\cos^2(x/2)}\,dx$

$= \dfrac{1}{2}\int\tan^4\left(\dfrac{x}{2}\right)\cdot\sec^2\left(\dfrac{x}{2}\right)dx = \dfrac{\tan^5(x/2)}{5} + c.$ ■

Example 32 : $y\log y\,dx + (x - \log y)\,dy = 0$

Sol. : $\left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right)\Big/M = -\dfrac{\log y}{y\log y} = -\dfrac{1}{y} \quad \therefore\ e^{\int -\frac{1}{y}dy} = \dfrac{1}{y}$

$\therefore\ \int M\,dx = \int\log y\,dx = x\log y$

$\therefore\ \int N\,dy = -\int\dfrac{1}{y}\log y\,dy = -\dfrac{1}{2}(\log y)^2.$ ■

Example 33 : $x^2\,\dfrac{dy}{dx} = 3x^2 - 2xy + 1$

Sol. : $\dfrac{dy}{dx} + \dfrac{2}{x}\cdot y = 3 + \dfrac{1}{x^2} \quad \therefore\ e^{\int P\,dx} = e^{2\int\frac{1}{x}dx} = e^{2\log x} = x^2$

$\therefore\ y\cdot x^2 = \int x^2\left(3 + \dfrac{1}{x^2}\right)dx + c = \int(3x^2 + 1)\,dx + c = x^3 + x + c$ ■

Example 34 : $\cos x\,\dfrac{dy}{dx} + y\sin x = \sqrt{y\sec x}$

Sol. : Divide by $2\sqrt{y}\cdot\cos x$,

$\therefore\ \dfrac{1}{2\sqrt{y}}\cdot\dfrac{dy}{dx} - \dfrac{1}{2}\sqrt{y}\cdot\dfrac{\sin x}{\cos x} = \dfrac{1}{2}\sec^{3/2}x$

Put $\sqrt{y} = v \quad \therefore\ \dfrac{1}{2\sqrt{y}}\cdot\dfrac{dy}{dx} = \dfrac{dv}{dx} \quad \therefore\ \dfrac{dv}{dx} - \dfrac{1}{2}\dfrac{\sin x}{\cos x}v = \dfrac{1}{2}\sec^{3/2}x$

$\therefore\ e^{\int -\frac{1}{2}\frac{\sin x}{\cos x}dx} = e^{-\frac{1}{2}\log\cos x} = e^{\log\sqrt{\sec x}} = \sqrt{\sec x}.$

$\therefore\ v\cdot\sqrt{\sec x} = \dfrac{1}{2}\int\sec^2 x\,dx + c = \dfrac{1}{2}\tan x + c.$ ■

Example 35 : $(y + e^y - e^{-x})\,dx + x(1+e^y)\,dy = 0$

Sol. : $\dfrac{\partial M}{\partial y} = 1 + e^y = \dfrac{\partial N}{\partial x}$ $\quad\therefore\; xy + xe^y + e^{-x} = c.$ ■

Example 36 : $\sin x \dfrac{dy}{dx} + y\cos x = 2\sin^2 x\cos x$

Sol. : $\dfrac{dy}{dx} + \cot x \cdot y = 2\sin x\cos x$ $\quad\therefore\; e^{\int \frac{\cos x}{\sin x}dx} = e^{\log \sin x} = \sin x$

$\therefore\; y\sin x = \int 2\sin^2 x\cos x\,dx + c.$ ■

Example 37 : $[(e^y+1)2x + e^x y^2]\,dx + [\,x^2 e^y + (e^x+1)2y\,]\,dy = 0$

Sol. : $\dfrac{\partial M}{\partial y} = 2x\,e^y + 2e^x y = \dfrac{\partial N}{\partial x}$

$\therefore\; \int M\,dx = (e^y+1)x^2 + e^x y^2 \;;\; \int N\,dy = \int 2y\,dy = y^2$ ■

Example 38 : $\dfrac{dy}{dx} - \dfrac{\tan y}{x+1} = (x+1)\,e^x \sec y$

Sol. : Multiply by $\cos y$, $\therefore\; \cos y\dfrac{dy}{dx} - \dfrac{\sin y}{x+1} = (x+1)\,e^x.$

Put $\sin y = v$, $\therefore\; \dfrac{dv}{dx} - \dfrac{1}{x+1}v = (x+1)\,e^x$

$\therefore\; \text{I.F.} = e^{-\int \frac{dx}{x+1}} = \dfrac{1}{x+1}$ $\quad\therefore\; \sin y\left(\dfrac{1}{x+1}\right) = \int e^x\,dx = e^x + c.$ ■

Example 39 : $\dfrac{dy}{dx} - y = y^2(\sin x + \cos x)$

Sol. : $\dfrac{1}{y^2}\cdot\dfrac{dy}{dx} - \dfrac{1}{y} = \sin x + \cos x$

Put $-\dfrac{1}{y} = v$ $\quad\therefore\; \dfrac{1}{y^2}\cdot\dfrac{dy}{dx} = \dfrac{dv}{dx}$ $\quad\therefore\; \dfrac{dv}{dx} + v = \sin x + \cos x$

$\therefore\; v\cdot e^x = \int e^x(\sin x + \cos x)\,dx = e^x\sin x + c.$

$$\left[\int e^x[f(x) + f'(x)]\,dx = e^x f(x)\right]$$ ■

Example 40 : $(x^3 + xy^4)\,dx + 2y^3\,dy = 0$

Sol. : $\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)\Big/N = \dfrac{4xy^3}{2y^3} = 2x$ $\quad\therefore\; \text{I.F.} = e^{\int 2x\,dx} = e^{x^2}$

$\therefore\; \int e^{x^2}x^3\,dx = \dfrac{1}{2}\int e^t\cdot t\,dt = \dfrac{1}{2}e^t(t-1)$ $\quad[\text{Put } x^2 = t]$

$\therefore\; y^4\int e^{x^2}x\,dx = \dfrac{1}{2}e^{x^2}y^4$

$\therefore\; \int M\,dx = \dfrac{1}{2}e^{x^2}(x^2-1) + \dfrac{1}{2}e^{x^2}y^4 \;;\; \int N\,dy = 0.$ ■

Example 41 : $(xy^3 + y)\,dx + 2(x^2y^2 + x + y^4)\,dy = 0$

Sol. : $\left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right)\Big/N = \dfrac{1}{y}$ $\quad\therefore\; \text{I.F.} = e^{\int (1/y)dy} = y$

$\therefore\; \int M\,dx = \int (xy^4 + y^2)\,dx = \dfrac{x^2}{2}y^4 + xy^2$

$\therefore\; \int N\,dy = 2\int y^5\,dy = \dfrac{y^6}{3}.$ ■

Example 42 : $(1 + xy^2)\,dx + (1 + x^2y)\,dy = 0$

Sol. : $\dfrac{\partial M}{\partial y} = 2xy = \dfrac{\partial N}{\partial x}$ $\quad\therefore\; \int M\,dx = x + \dfrac{x^2y^2}{2} \;;\; \int N\,dy = y$ ■

Example 43 : $(x - y^2)\,dx + 2xy\,dy = 0$

Sol. : $\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)\Big/N = \dfrac{-2y - 2y}{2xy} = -\dfrac{2}{x}$

$\therefore\; \text{I.F.} = e^{-2\int \frac{dx}{x}} = \dfrac{1}{x^2}$

$\therefore\; \int\left(\dfrac{1}{x} - \dfrac{y^2}{x^2}\right)dx = \log x + \dfrac{y^2}{x} \;;\; \int N\,dy = 0$ ■

Example 44 : $x\dfrac{dy}{dx} + \sin y\cos y - x^3\cos^2 y = 0$

Sol. : $\sec^2 y\dfrac{dy}{dx} + \tan y\dfrac{1}{x} = x^2.$ Put $\tan y = v$ $\quad\therefore\; \dfrac{dv}{dx} + \dfrac{1}{x}v = x^2$

$\therefore\; \text{I.F.} = e^{\int \frac{1}{x}dx} = x$ $\quad\therefore\; vx = \int x^3\,dx + c = \dfrac{x^4}{4} + c.$ ■

Example 45 : $\sin y\dfrac{dy}{dx} = (1 - x\cos y)\cos y$

Sol. : $\sin y\dfrac{dy}{dx} - \cos y = -x\cos^2 y.$ $\qquad$ Put $-\cos y = v.$

$\therefore\; \sin y\dfrac{dy}{dx} = \dfrac{dv}{dx}$ $\quad\therefore\; \dfrac{dv}{dx} + v = -xv^2$

$\therefore\; -\dfrac{1}{v^2}\cdot\dfrac{dv}{dx} - \dfrac{1}{v} = x.$ Put $\dfrac{1}{v} = w$

$\therefore\; \dfrac{dw}{dx} - w = x$ $\quad\therefore\; e^{\int -dx} = e^{-x}$

$\therefore\; we^{-x} = \int e^{-x}x\,dx + c = x(-e^{-x}) - \int -e^{-x}\,dx + c$

$= -xe^{-x} - e^{-x} + c.$ ■

Example 46 : $\frac{dy}{dx} + \frac{1}{x}\tan y = \frac{1}{x^2}\tan y \cdot \sin y$

Sol. : $\cot y \operatorname{cosec} y \frac{dy}{dx} + \frac{1}{x}\operatorname{cosec} y = \frac{1}{x^2}$

Put $-\operatorname{cosec} y = v$, $\therefore \frac{dv}{dx} - \frac{1}{x}\cdot v = \frac{1}{x}$

$\therefore e^{\int -\frac{1}{x}dx} = e^{\log\frac{1}{x}} = \frac{1}{x} \quad \therefore \frac{1}{x}v = \int \frac{1}{x^3}dx + c = -\frac{1}{2x^2} + c.$ ■

Example 47 : $(x+1)\frac{dy}{dx} - y = e^x(x+1)^2$

Sol. : $\frac{dy}{dx} - \frac{1}{x+1}\cdot y = e^x(x+1) \quad \therefore e^{\int -\left(\frac{1}{x+1}\right)dx} = e^{-\log(x+1)} = \frac{1}{x+1}$

$\therefore y\cdot\frac{1}{x+1} = \int e^x(x+1)\cdot\frac{1}{x+1}dx + c = \int e^x dx + c = e^x + c.$ ■

Example 48 : $\left[\frac{\log(\log y)}{x} + \frac{2}{3}xy^3\right]dx + \left[\frac{\log x}{y\log y} + x^2y^2\right]dy = 0.$

Sol. : $\frac{\partial M}{\partial y} = \frac{1}{x\cdot y\log y} + 2xy^2 = \frac{\partial N}{\partial x}$. Exact.

$\int M dx = \int\left[\frac{\log(\log y)}{x} + \frac{2}{3}xy^3\right]dx$

$= \log(\log y)\cdot\log x + \frac{x^2y^3}{3}$ ■

$\int N dy = 0.$

Example 49 : $x(x+y)\,dy - y^2\,dx = 0$

Sol. : Homogeneous.

$\text{I.F.} = \frac{1}{Mx + Ny} = \frac{1}{x^2y}$

$\therefore \int M dx = \int -\frac{y}{x^2}dx = \frac{y}{x}; \quad \int N dy = \int \frac{1}{y}dy = \log y.$ ■

Exercise XII (Page 1-60)

(A) Solve the following differential equations :

Class (a) : 3 or 4 Marks

Example 1 : $(\sec x \tan x \cdot \tan y - e^x)\,dx + \sec x \sec^2 y\,dy = 0$

Sol. : $\frac{\partial M}{\partial y} = \sec x \tan x \sec^2 y = \frac{\partial N}{\partial x}$

$\int M dx = \int[(\sec x \tan x)\cdot\tan y - e^x]\,dx = \sec x \tan y - e^x$

$\int N dy = 0.$ ■

Example 2 : $(2x - y + 1)\,dx + (2y - x - 1)\,dy = 0$

Sol. : $\frac{\partial M}{\partial y} = -1 = \frac{\partial N}{\partial x}$

$\int M dx = \int(2x - y + 1)dx = x^2 - xy + x$

$\int N dy = \int(2y - 1)\,dx = y^2 - y$ ■

Example 3 : $(2xy\cos x^2 - 2xy + 1)\,dx + (\sin x^2 - x^2)\,dy = 0$

Sol. : $\frac{\partial M}{\partial y} = 2x\cos^2 x - 2x = \frac{\partial N}{\partial x}$

$\int M dx = \int(y\cdot 2x\cos x^2 - y\cdot 2x + 1)\,dx$

$= y\sin x^2 - x^2y + x$

$\int N dy = 0$ ■

Example 4 : $(e^x + 2xy^2 + y^3)\,dx + (2x^2y + 3xy^2)\,dy = 0$

Sol. : $\frac{\partial M}{\partial y} = 4xy + 3y^2 = \frac{\partial N}{\partial x}$

$\int M dx = \int(e^x + 2xy^2 + y^3)\,dx = e^x + x^2y^2 + xy^3$

$\int N dy = 0$ ■

Example 5 : $\frac{dy}{dx} + 2y\tan x = \sin x$

Sol. : $\int P dx = \int 2\tan x\,dx = \log\sec^2 x$

$\therefore e^{\int P dx} = e^{\log\sec^2 x\,dx} = \sec^2 x$

$y\cdot\sec^2 x = \int \sec^2 x\cdot\sin x\,dx + c$

$= \int \sec x \tan x\,dx + c = \sec x + c$ ■

Example 6 : $\frac{dy}{dx} - \frac{1}{x}\cdot y = 1$

Sol. : $\int P dx = -\int\frac{dx}{x} = \log x^{-1} = \log\frac{1}{x}$

$e^{\int P dx} = e^{\log(1/x)} = \frac{1}{x}$

$y\cdot\frac{1}{x} = \int\frac{1}{x}dx + c = \log x + c.$ ■

(B) Find the integrating factor : Class (a) : 3 or 4 Marks

Example 1 : $(12y + 4y^3 + 6x^2)\,dx + 3(x + xy^2)\,dy = 0$

Sol. : $$\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \Big/ N = \frac{12 + 12y^2 - 3 - 3y^2}{3(x + xy^2)} = \frac{9(1 + y^2)}{3x(1 + y^2)} = \frac{3}{x}$$

$$\text{I.F.} = e^{\int \frac{3}{x} dx} = e^{3\log x} = e^{\log x^3} = x^3.$$ ■

Example 2 : $(2x \log x - xy)\,dy + 2y\,dx = 0$

Sol. : $$\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \Big/ N = \frac{2 - 2\log x - 2 + y}{2x\log x - xy} = -\frac{2\log x - y}{x(2\log x - y)} = -\frac{1}{x}$$

$$\text{I.F.} = e^{\int -\frac{1}{x} dx} = e^{-\log x} = e^{\log(1/x)} = \frac{1}{x}$$ ■

Example 3 : $(xy^3 + y)\,dx + 2(x^2y^2 + x + y^4)\,dy = 0$

Sol. : $$\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \Big/ M = \frac{2(2xy^2 + 1) - 3xy^2 - 1}{y(xy^2 + 1)} = \frac{xy^2 + 1}{y(xy^2 + 1)} = \frac{1}{y}$$

$$\text{I.F.} = e^{\int \frac{1}{y} dy} = e^{\log y} = y$$ ■

Example 4 : $(2xy^4e^y + 2xy^3 + y)\,dx + (x^2y^4e^y - x^2y^2 - 3x)\,dy = 0$

Sol. :

$$\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \Big/ M = \frac{(2xy^4e^y - 2xy^2 - 3) - (8xy^3e^y + 2xy^4e^y + 6xy^2 + 1)}{x^2y^4e^y - x^2y^2 - 3x}$$

$$= \frac{-8xy^3e^y - 8xy^2 - 4}{y(2xy^3e^y + xy^2 + 1)} = -\frac{4}{y}$$

$$\text{I.F.} = e^{\int -\frac{4}{y} dy} = e^{-4\log y} = \frac{1}{y^4}$$ ■

Example 5 : $y(1 + xy)\,dx + x(1 + xy + x^2y^2)\,dy = 0$

Sol. : $$Mx - Ny = (xy + x^2y^2) - (xy + x^2y^2 + x^3y^3) = -x^3y^3$$

$$\text{I.F.} = -\frac{1}{x^3y^3}$$ ■

Example 6 :

$y(1 + xy + x^2y^2 + x^3y^3)\,dx + x(1 - xy - x^2y^2 + x^3y^3)\,dy = 0$

Sol. : $$Mx - Ny = xy + x^2y^2 + x^3y^3 + x^4y^4 - xy + x^2y^2 + x^3y^3 - x^4y^4$$

$$\text{I.F.} = \frac{1}{2x^2y^2(1 + xy)}$$ ■

Example 7 : $(xy \sin xy + \cos xy)\,y\,dx + (xy \sin xy - \cos xy)\,x\,dy = 0$

Sol. : $$Mx - Ny = x^2y^2 \sin xy + xy \cos xy - x^2y^2 \sin x + xy \cos xy$$
$$= 2xy \cos xy$$

$$\text{I.F.} = \frac{1}{2xy \cos xy}.$$ ■

Example 8 : $(3xy - 2ay^2)\,dx + (x^2 - 2axy)\,dy = 0$

Sol. : $$Mx + Ny = 3x^2y - 2a\,xy^2 + x^2y - 2a\,xy^2$$
$$= 4x^2y - 4a\,xy^2 = 4xy(x - ay)$$

$$\text{I.F.} = \frac{1}{4xy(x - ay)}.$$ ■

Example 9 : $(x^2 - xy + y^2)\,dx - xy\,dy = 0$

Sol. : $$Mx + Ny = x^3 - x^2y + xy^2 - xy^2$$

$$\text{I.F.} = \frac{1}{x^2(x - y)}.$$ ■

Example 10 : $x(x - 1)\dfrac{dy}{dx} - (x - 2)y = x^3(2x - 1)$

Sol. : $$\frac{dy}{dx} - \frac{x - 2}{x(x - 1)}y = \frac{x^2(2x - 1)}{x - 1}$$

$$\text{I.F.} = e^{\int P dx} = e^{-\int \frac{x-2}{x(x-1)} dx} = e^{-\int \left[\frac{2}{x} - \frac{1}{x-1}\right] dx}$$

$$= e^{-(2\log x - \log(x-1))} = e^{-\log\left(\frac{x^2}{x-1}\right)} = \frac{x - 1}{x^2}$$ ■

Example 11 : $(1 + x + xy^2)\,dy + (y + y^3)\,dx = 0.$

Sol. : $$\frac{dx}{dy} + \frac{x}{y} = -\frac{1}{y(1 + y^2)}$$

$$\text{I.F.} = e^{\int P' dy} = e^{\int \frac{1}{y} dy} = e^{\log y} = y.$$ ■

Example 12 : $[\,2y \cos y - x(\sec y + \tan y)\,]\,dy = (1 + \sin y)\,dx$

Sol. : $$\frac{dx}{dy} + \frac{\sec y + \tan y}{1 + \sin y} = \frac{2y \cos y}{1 + \sin y}$$

$$\therefore \frac{dx}{dy} + \sec y \frac{(1 + \sin y)}{(1 + \sin y)} = \frac{2y \cos y}{1 + \sin y}$$

$$\int P' dy = \int \sec y\, dy = \log(\sec y + \tan y)$$

$$\text{I.F.} = e^{\int P' dy} = e^{\log(\sec y + \tan y)} = \sec y + \tan y.$$ ■

Example 13 : $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$

Sol. : $\sec^2 y \frac{dy}{dx} + 2 \tan y \cdot x = x^3$

Put $\tan y = v$, $\frac{dv}{dx} + 2x \cdot v = x^3$

$\therefore$ I.F. $= e^{\int P dx} = e^{\int 2x dx} = e^{x^2}$.

Example 14 : $\frac{dy}{dx} + (2x \tan^{-1} y - x^3)(1 + y^2) = 0$

Sol. : $\frac{1}{1+y^2} \cdot \frac{dy}{dx} + \tan^{-1} y \cdot 2x = x^3$

Put $\tan^{-1} y = v$ $\therefore$ $\frac{dv}{dx} + 2x \cdot v = x^3$

$\therefore$ I.F. $= e^{\int P dx} = e^{\int 2x dx} = e^{x^2}$.

Example 15 : $e^x (x+1) dx + (y e^y - x e^x) dy = 0$

Sol. : $e^x (x+1) \frac{dx}{dy} - x e^x = -y e^y$

Put $x e^x = v$ $\therefore$ $e^x (x+1) \frac{dx}{dy} = \frac{dv}{dy}$ $\therefore$ $\frac{dv}{dy} - v = -y e^y$

I.F. $= e^{\int P dy} = e^{\int -dy} = e^{-y}$.

Example 16 : $(1 + y^2) dx = (\tan^{-1} y - x) dy$

Sol. : $(1+y^2) \frac{dx}{dy} + x = \tan^{-1} y$

Put $\tan^{-1} y = v$ $\therefore$ $\frac{1}{1+y^2} \cdot \frac{dy}{dx} = \frac{dv}{dx}$

$\therefore$ $(1+y^2) \frac{dx}{dy} = \frac{dx}{dv}$ $\therefore$ $\frac{dx}{dv} + x = v$

$\therefore$ I.F. $= e^{\int P dv} = e^{\int 1 \cdot dv} = e^v = e^{\tan^{-1} y}$.

Example 17 : $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}} \right) \frac{dx}{dy} = 1$

Sol. : $\frac{dy}{dx} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}$; $\frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$

I.F. $= e^{\int \frac{1}{\sqrt{x}} dx} = e^{2\sqrt{x}}$.

(C) Reduce the following equations to linear form : Class (a) : 3 or 4 Marks

Example 1 : $\frac{dy}{dx} = x^3 y^3 - xy$

Sol. : $\frac{dy}{dx} + xy = x^3 y^3$. Divide by y^3. $\therefore$ $\frac{1}{y^3} \cdot \frac{dy}{dx} + \frac{1}{y^2} \cdot x = x^3$

Put $\frac{1}{y^2} = v$ $\therefore$ $-\frac{2}{y^3} \cdot \frac{dy}{dx} = \frac{dv}{dx}$ $\therefore$ $\frac{dv}{dx} - 2xv = -2x^3$.

Example 2 : $y - \cos x \frac{dy}{dx} = y^2 (1 - \sin x) \cos x$

Sol. : $\cos x \frac{dy}{dx} - y = -y^2 (1 - \sin x) \cos x$

Divide by $-y^2 \cos x$, $-\frac{1}{y^2} \frac{dy}{dx} + \frac{\sec x}{y} = 1 - \sin x$

Put $\frac{1}{y} = v$, $-\frac{1}{y^2} \cdot \frac{dy}{dx} = \frac{dv}{dx}$

$\therefore$ $\frac{dv}{dx} + \sec x \cdot v = 1 - \sin x$.

Example 3 : $\frac{dr}{d\theta} = r \tan\theta - \frac{r^2}{\cos\theta}$

Sol. : $\frac{1}{r^2} \cdot \frac{dr}{d\theta} - \frac{1}{r} \tan\theta = -\sec\theta$

Put $-\frac{1}{r} = v$ $\therefore$ $\frac{1}{r^2} \cdot \frac{dr}{d\theta} = \frac{dv}{d\theta}$

$\therefore$ $\frac{dv}{d\theta} + \tan\theta \cdot v = -\sec\theta$.

Example 4 : $x^3 \frac{dy}{dx} + 2x^2 y = y^3$

Sol. : Divide by $x^3 y^3$, $\frac{1}{y^3} \cdot \frac{dy}{dx} + \frac{2}{y^2} \cdot \frac{1}{x} = \frac{1}{x^3}$

Put $\frac{1}{y^2} = v$ $\therefore$ $-\frac{2}{y^3} \cdot \frac{dy}{dx} = \frac{dv}{dx}$

$-\frac{1}{2} \frac{dv}{dx} + \frac{2}{x} \cdot v = \frac{1}{x^3}$ $\therefore$ $\frac{dv}{dx} - \frac{4}{x} \cdot v = -\frac{2}{x^3}$.

Example 5 : $xy(1 + xy^2) \frac{dy}{dx} = 1$

Sol. : $\frac{dx}{dy} = xy + x^2 y^3$ $\therefore$ $\frac{dx}{dy} - xy = x^2 y^3$

Divide by x^2, $\quad \frac{1}{x^2} \cdot \frac{dx}{dy} - \frac{1}{x} \cdot y = y^3$

Put $-\frac{1}{x} = v \quad \therefore \quad \frac{1}{x^2} \cdot \frac{dx}{dy} = \frac{dv}{dy} \quad \therefore \quad \frac{dv}{dy} + vy = y^3.$

Example 6 : $\frac{dy}{dx} = 1 - x(y-x) - x^3(y-x)^2$

Sol. : Put $y - x = v \quad \therefore \quad \frac{dy}{dx} - 1 = \frac{dv}{dx}$

$\therefore \quad \frac{dv}{dx} = -xv - x^3v^2 \quad \therefore \quad \frac{dv}{dx} + xv = -x^3v^2$

Dividing by $-v^2$, $\quad -\frac{1}{v^2} \cdot \frac{dv}{dx} - \frac{1}{v} \cdot x = x^3$

Put $\frac{1}{v} = t \quad \therefore \quad -\frac{1}{v^2} \cdot \frac{dv}{dx} = \frac{dt}{dx} \quad \therefore \quad \frac{dt}{dx} - t \cdot x = x^3.$

Example 7 : $\frac{dy}{dx} = \frac{e^y}{x^2} - \frac{1}{x}$

Sol. : Dividing by e^y, $\quad e^{-y}\frac{dy}{dx} + e^{-y} \cdot \frac{1}{x} = \frac{1}{x^2}$

Put $e^{-y} = v \quad \therefore \quad -e^{-y}\frac{dy}{dx} = \frac{dv}{dx}$

$\therefore \quad \frac{dv}{dx} - \frac{1}{x} \cdot v = -\frac{1}{x^2}.$

Example 8 : $4x^2y\frac{dy}{dx} = 3x(3y^2+2) + (3y^2+2)^3$

Sol. : Put $3y^2 + 2 = v \quad \therefore \quad 6y\frac{dy}{dx} = \frac{dv}{dx}$

$\therefore \quad 4x^2 \cdot \frac{1}{6}\frac{dv}{dx} = 3xv + v^3$

Divide by $\frac{2x^2}{3}$, $\quad \frac{dv}{dx} - \frac{9}{2x}v = \frac{3}{2x^2}v^3$

Divide by v^3, $\quad \frac{1}{v^3} \cdot \frac{dv}{dx} - \frac{9}{2x} \cdot \frac{1}{v^2} = \frac{3}{2x^2}$

Put $-\frac{1}{v^2} = t \quad \therefore \quad \frac{2}{v^3} \cdot \frac{dv}{dx} = \frac{dt}{dx} \quad \therefore \quad \frac{dt}{dx} + \frac{9}{x} \cdot t = \frac{3}{x^2}.$

CHAPTER 2

Applications of Differential Equations

Exercise - I (Page 2-7)

Solve the following examples : Class (b) : 6 Marks

Example 1 : The differential equation of a body in motion is given by $\frac{dv}{dt} = k\left(1 - \frac{t}{T}\right)$ where k and T are constants. Find the maximum speed and the distance travelled when the maximum speed is attained. (At $t = 0$, $s = 0$, $v = 0$)

Sol. : $\quad \frac{dv}{dt} = k - \frac{kt}{T} \quad$ ..... (1)

For maximum speed $\frac{dV}{dt} = 0 \quad \therefore \quad kT - kt = 0 \quad \therefore \quad t = T$

By integrating (1), $\quad V = kt - \frac{kt^2}{2T} + c$

But $V = 0$ when $t = 0 \quad \therefore \quad c = 0$

$$V = kt - \frac{kt^2}{2T} \quad \text{..... (2)}$$

When $t = T$, $\quad V = kT - \frac{kT^2}{2T} = \frac{kT}{2}$

Integrating $\quad \frac{ds}{dt} = V = kt - \frac{kt^2}{2T} \quad \therefore \quad s = \frac{kt^2}{2} - \frac{kt^3}{6T} + c$

When $t = 0$, $s = 0 \quad \therefore \quad c = 0 \quad \therefore \quad s = \frac{kt^2}{2} - \frac{kt^3}{6T}$

When $t = T$, $\quad s = \frac{kT^2}{2} - \frac{kT^3}{6T} = \frac{kT^2}{6}.$

Example 2 : The velocity of a bullet fired in a sand tank is given by $\frac{dv}{dt} = -k\sqrt{v}$. If at $t = 0$, $v = V$, find how long it will take to come to rest.

Sol. : $\quad \frac{dv}{dt} = -k\sqrt{v} \quad \therefore \quad \frac{dv}{\sqrt{v}} = -k\,dt$

$\therefore\ 2v^{1/2} = -kt + c.$ At $t = 0,\ v = V$ $\therefore\ c = 2V^{1/2}$

$\therefore\ 2\sqrt{v} = -kt + 2\sqrt{v}.$

The bullet comes to rest when $v = 0.$ $\therefore\ t = \dfrac{2\sqrt{V}}{K}.$ ■

Example 3 : A chain coiled up near the edge of a smooth table starts to fall over the edge. The velocity v when a length x has fallen is given by $xv\dfrac{dv}{dx} + v^2 = gx$. Show that $v = 8\sqrt{x/3}$.

Sol. : $xv\dfrac{dv}{dx} + v^2 = gx.$ Put $v^2 = y$ $\therefore\ 2v\dfrac{dv}{dx} = \dfrac{dy}{dx}$

$\therefore\ \dfrac{x \cdot 1}{2} \cdot \dfrac{dy}{dx} + y = gx$ $\therefore\ \dfrac{dy}{dx} + \dfrac{2}{x}y = 2g$

$$y \cdot e^{\int \frac{2}{x}dx} = \int 2g \cdot e^{\int \frac{2}{x}dx} + c$$

$$y \cdot x^2 = \int 2gx^2 + c = 2g\frac{x^3}{3} + c$$

When $x = 0,\ v = 0$ $\therefore\ y = 0$ $\therefore\ c = 0$

$y = v^2 = \dfrac{2gx}{3} = \dfrac{64x}{3}$ $\therefore\ v = 8\sqrt{\dfrac{x}{3}}.$ ■

Example 4 : The equation of motion of a particle moving in a straight line is given by $v\dfrac{dv}{dx} = -\dfrac{k}{x^3}$. If initially the particle was at rest at a distance a from the origin, show that it will be at a distance $\dfrac{a}{2}$ from the origin at $t = \dfrac{a^2}{2}\sqrt{\dfrac{3}{k}}$.

Sol. : $v\dfrac{dv}{dx} = -\dfrac{k}{x^3}$ $\therefore\ v\,dv = -\dfrac{k}{x^3}dx$ $\therefore\ \dfrac{v^2}{2} = \dfrac{k}{2x^2} + c$

At $t = 0,\ v = 0,\ x = a$ $\therefore\ c = -\dfrac{k}{2a^2}$

$\therefore\ v^2 = \dfrac{k}{x^2} - \dfrac{k}{a^2} = k\dfrac{(a^2 - x^2)}{a^2x^2}$

$\therefore\ v = \dfrac{\sqrt{k}}{a} \cdot \dfrac{\sqrt{a^2 - x^2}}{x}$ $\dfrac{dx}{dt} = \dfrac{\sqrt{k}}{a} \cdot \dfrac{\sqrt{a^2 - x^2}}{x}$

$\dfrac{x\,dx}{\sqrt{a^2 - x^2}} = \dfrac{\sqrt{k}}{a}dt$ $\therefore\ -\sqrt{a^2 - x^2} = \dfrac{\sqrt{k}}{a}t + c$

At $t = 0,\ x = a$ $\therefore\ c = 0$

$\therefore\ -\sqrt{a^2 - x^2} = \dfrac{\sqrt{k}}{a} \cdot t$ $\therefore\ a^2 - x^2 = \dfrac{k}{a^2} \cdot t^2$

When $x = \dfrac{a}{2},$ $\dfrac{3a^2}{4} = \dfrac{k}{a^2}t^2$

$\therefore\ t = \dfrac{3a^4}{4} \cdot \dfrac{1}{k}$ $\therefore\ t = \dfrac{a^2}{2}\sqrt{\dfrac{3}{K}}.$

Example 5 : The equation of motion of a particle moving in a straight line in a resisting medium is given by $\dfrac{dv}{ds} = -kv^2$. If u is the initial velocity, prove that $v = \dfrac{u}{1 + ksu}$, $t = \dfrac{s}{u} + \dfrac{ks^2}{2}$.

Sol. : $\dfrac{dv}{ds} = -kv^2$ $\therefore\ \dfrac{dv}{v^2} = -k\,ds$ $\therefore\ -\dfrac{1}{v} = -ks + c$

At $t = 0,\ s = 0,\ v = u.$ $\therefore\ c = -\dfrac{1}{u}$ $\therefore\ -\dfrac{1}{v} = -ks - \dfrac{1}{u}$

$\therefore\ \dfrac{1}{v} = \dfrac{ksu + 1}{u}$ $\therefore\ v = \dfrac{u}{1 + ksu}$

$\therefore\ \dfrac{ds}{dt} = \dfrac{u}{1 + ksu}$ $\therefore\ (1 + ksu)\,ds = u\,dt$ $\therefore\ s + ku\dfrac{s^2}{2} = ut + c$

At $t = 0,\ s = 0$ $\therefore\ c = 0$ $\therefore\ t = \dfrac{s}{u} + \dfrac{ks^2}{2}.$ ■

Example 6 : The differential equation of a body falling from rest subjected to the force of gravity and air resistance is given by

$$v\frac{dv}{dx} + \frac{n^2}{g} \cdot v^2 = g.$$

Prove that the velocity is given by $v^2 = \dfrac{g^2}{n^2}(1 - e^{-2n^2x/g})$. ($v = 0$ at $x = 0$)

Sol. : $v\dfrac{dv}{dx} + \dfrac{n^2}{g}v^2 = g$

Put $v^2 = y,$ $\therefore\ 2v\dfrac{dv}{dx} = \dfrac{dy}{dx},$ $\dfrac{dy}{dx} + \dfrac{2n^2}{g} \cdot y = 2g$

$$\therefore\ y \cdot e^{\int (2n^2/g)\,dx} = 2g\int e^{\int (2n^2/g)\,dx} + c$$

$$\therefore\ y \cdot e^{(2n^2/g)x} = 2g\int e^{(2n^2/g)x}\,dx + c = 2g \cdot \frac{e^{(2n^2/g)x}}{2n^2/g} + c$$

$$y \cdot e^{(2n^2/g)x} = \frac{g^2}{n^2} \cdot e^{(2n^2/g)x} + c$$

When $t = 0,\ v = 0$ $\therefore\ y = 0$ $\therefore\ c = -\dfrac{g^2}{n^2}$

$$\therefore\ y + e^{(2n^2/g)x} = \frac{g^2}{n^2}\left(e^{(2n^2/g)x} - 1\right)$$

$$\therefore\ y = \frac{g^2}{n^2}\left(1 - e^{-(2n^2/g)x}\right)$$

Exercise II (Page 2-14)

Solve the following examples : Class (b) : 6 Marks

Example 1 : A circuit contains a resistance R and a condenser of capacity C farads and connected to a constant e.m.f. E is given by $Ri + \frac{q}{c} = \frac{E}{R}$. Find q, given that $q = 0$ when $t = 0$.

Sol. : $Ri + \frac{q}{c} = \frac{E}{R}$ $\quad\therefore\ \frac{dq}{dt} + \frac{1}{CR}\cdot q = \frac{E}{R}$

$$\therefore\ q e^{t/CR} = \int e^{t/CR}\frac{E}{R}\,dt + c = EC\,e^{t/CR} + c'$$

Example 2 : When the inner of two concentric spheres of radius r_1 and r_2 $(r_1 < r_2)$ carries an electric charge, the differential equation for the potential v at a distance r from the common centre is

$$\frac{d^2v}{dr^2} + \frac{2}{r}\frac{dv}{dr} = 0. \qquad \text{Find } v \text{ in terms or } r.$$

Sol. : Put $\frac{dv}{dr} = u$ $\quad\therefore\ \frac{du}{dr} + \frac{2u}{r} = 0$; $\quad \log u + 2\log r = \log c$.

$$\therefore\ \frac{dv}{dr}\cdot r^2 = c_1 \quad \therefore\ dv = c_1\frac{dr}{r^2} \quad \therefore\ vr = c_2 r + c_1.$$

Example 3 : When a switch is closed, the current built up in an electric circuit is given by $E = Ri + L\frac{di}{dt}$. If $L = 640$, $R = 250$, $E = 500$ and $i = 0$ when $t = 0$ show that the current will approach 2 amp. when $t \to \infty$.

Sol. : As in solved Ex. 1, page 2-9 of the Text Book, $i = \frac{E}{R}\left(1 - e^{-Rt/L}\right)$

As $t \to \infty$, $\quad i = \frac{E}{R} = \frac{500}{250} = 2.$

Example 4 : A resistance of 100 ohms and inductance of 0.5 henries are connected in series with a battery of 20 volts. Find the current at any instant if the relation between L, R, E is $L\frac{di}{dt} + Ri = E$.

Sol. : $\frac{di}{dt} + 200i = 40$. Linear $\quad\therefore\ i e^{200t} = \int e^{200t}\cdot 40\,dt + k$

***

CHAPTER 3

Linear Differential Equations with Constant Coefficients

Exercise I (Page 3-4)

Solve the following equations : Class (a) : 3 Marks

Example 1 : $2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 12y = 0$

Sol. : $2D^2 + 5D - 12 = 0 \quad\therefore\ (2D-3)(D+4) = 0$

$$\therefore\ y = c_1 e^{(3/2)x} + c_2 e^{-4x}.$$

Example 2 : $9\frac{d^2y}{dx^2} + 18\frac{dy}{dx} - 16y = 0$

Sol. : $9D^2 + 18D - 16 = 0 \quad\therefore\ (3D-2)(3D+8) = 0$

$$\therefore\ y = c_1 e^{(2/3)x} + c_2 e^{-(8/3)x}.$$

Example 3 : $6\frac{d^3x}{dt^3} + 23\frac{d^2x}{dt^2} + 29\frac{dx}{dt} + 12x = 0$

Sol. : $6D^3 + 23D^2 + 29D + 12 = 0 \therefore\ (D+1)(3D+4)(2D+3) = 0$

$$\therefore\ y = c_1 e^{-x} + c_2 e^{-(3/2)x} + c_3 e^{-(4/3)x}.$$

Example 4 : $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - 5\frac{dy}{dx} - 6y = 0$

Sol. : $D^3 + 2D^2 - 5D - 6 = 0 \quad\therefore\ (D+1)(D+3)(D-2) = 0$

$$\therefore\ y = c_1 e^{-x} + c_2 e^{2x} + c_3 e^{-3x}.$$

Example 5 : $\frac{d^3y}{dx^3} + 4\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$

Sol. : $D^3 + 4D^2 + D - 6 = 0 \quad\therefore\ (D-1)(D+2)(D+3) = 0$

$$\therefore\ y = c_1 e^{x} + c_2 e^{-2x} + c_3 e^{-3x}.$$

Example 6 : $\frac{d^4y}{dx^4} - 5\frac{d^3y}{dx^3} + 5\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 6y = 0$

Sol. : $D^4 - 5D^3 + 5D^2 + 5D - 6 = 0$

$\therefore\ (D-1)(D+1)(D-2)(D-3)=0$

$\therefore\ y = c_1 e^{-x} + c_2 e^{x} + c_3 e^{2x} + c_4 e^{3x}.$ ■

Exercise II (Page 3-5)

Solve the following equations : Class (a) : 3 Marks

Example 1 : $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4y = 0$

Sol. : $D^2 + 4D + 4 = 0 \quad \therefore\ (D+2)^2 = 0 \quad \therefore\ y = (c_1 + c_2 x)e^{-2x}.$ ■

Example 2 : $\dfrac{d^3y}{dx^3} + \dfrac{d^2y}{dx^2} = 0$

Sol. : $D^3 + D^2 = 0 \quad \therefore\ D^2(D+1) = 0 \quad \therefore\ y = (c_1 + c_2 x)c_3 e^{-x}.$ ■

Example 3 : $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - y = 0$

Sol. : $D^3 - 3D^2 + 3D - 1 = 0 \quad \therefore\ (D-1)^3 = 0$

$\therefore\ y = (c_1 + c_2 x + c_3 x^2)e^{x}.$ ■

Example 4 : $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 4y = 0$

Sol. : $D^3 - 3D^2 + 4 = 0 \quad \therefore\ (D+1)(D-2)^2 = 0$

$\therefore\ y = c_1 e^{-x} + (c_2 + c_3 x)e^{2x}.$ ■

Example 5 : $\dfrac{d^4y}{dx^4} - 18\dfrac{d^2y}{dx^2} + 81y = 0$

Sol. : $D^4 - 18D^2 + 81 = 0 \quad \therefore\ (D-3)^2(D+3)^2 = 0$

$\therefore\ y = (c_1 + c_2 x)e^{3x} + (c_3 + c_4 x)e^{-3x}.$ ■

Example 6 : $\dfrac{d^4y}{dx^4} - 2\dfrac{d^3y}{dx^3} + \dfrac{d^2y}{dx^2} = 0$

Sol. : $D^4 - 2D^3 + D^2 = 0 \quad \therefore\ D^2(D-1)^2 = 0$

$\therefore\ y = (c_1 + c_2 x) + (c_3 + c_4 x)e^{x}.$ ■

Exercise III (Page 3-8)

Solve the following equations : Class (a) : 3 Marks

Example 1 : $\dfrac{d^3y}{dx^3} + 8y = 0$

Sol. : $D^3 + 8 = 0 \quad \therefore\ (D+2)(D^2 - 2D + 4) = 0$

$\therefore\ y = c_1 e^{-2x} + e^{x}(c_2 \cos\sqrt{3}\cdot x + c_3 \sin\sqrt{3}\cdot x).$ ■

Example 2 : $\dfrac{d^4y}{dx^4} + 2\dfrac{d^2y}{dx^2} + 1 = 0$

Sol. : $D^4 + 2D^2 + 1 = 0 \quad \therefore\ (D^2+1)^2 = 0$

$\therefore\ y = (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x.$ ■

Example 3 : $\dfrac{d^4y}{dx^4} - m^4 y = 0$

Sol. : $D^4 - m^4 = 0 \quad \therefore\ (D^2 - m^2)(D^2 + m^2) = 0$

$\therefore\ y = c_1 e^{mx} + c_2 e^{-mx} + (c_3 \cos mx + c_4 \sin mx).$ ■

Example 4 : $\dfrac{d^4y}{dx^4} + 13\dfrac{d^2y}{dx^2} + 36y = 0$

Sol. : $D^4 + 13D^2 + 36 = 0 \quad \therefore\ (D^2+4)(D^2+9) = 0$

$\therefore\ y = (c_1 \cos 2x + c_2 \sin 2x) + (c_3 \cos 3x + c_4 \sin 3x)$ ■

Example 5 : $\dfrac{d^4y}{dx^4} + y = 0$

Sol. : $D^4 + 1 = 0 \quad \therefore\ (D^2+1)^2 - (\sqrt{2}\cdot D)^2 = 0$

$D^2 - \sqrt{2}\cdot D + 1 = 0$ or $D^2 + \sqrt{2}\cdot D + 1 = 0$

$\therefore\ D = \dfrac{1}{\sqrt{2}} \pm i\dfrac{1}{\sqrt{2}},\quad D = -\dfrac{1}{\sqrt{2}} \pm i\dfrac{1}{\sqrt{2}}$

$\therefore\ y = e^{x/\sqrt{2}}[c_1 \cos(x/\sqrt{2}) + c_2 \sin(x/\sqrt{2})]$

$\qquad + e^{-x/\sqrt{2}}[c_3 \cos(x/\sqrt{2}) + c_4 \sin(x/\sqrt{2})].$ ■

Example 6 : $\dfrac{d^4y}{dx^4} + 4\dfrac{d^3y}{dx^3} + 8\dfrac{d^2y}{dx^2} + 8\dfrac{dy}{dx} + 4y = 0$

Sol. : $D^4 + 4D^3 + 8D^2 + 8D + 4 = 0$

$\therefore\ (D^2 + 2D + 2)^2 = 0 \quad \therefore\ D = -1 \pm 2i,\ -1 \pm 2i$

$\therefore\ y = e^{-x}[(c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x].$ ■

Example 7 : $\dfrac{d^4y}{dx^4} + 2\dfrac{d^3y}{dx^3} + 3\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 0$

Sol. : $D^4 + 2D^3 + 3D^2 + 2D + 1 = 0 \quad \therefore\ (D^2 + D + 1)^2 = 0$

$\therefore\ D = -\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2},\quad -\dfrac{1}{2} + \dfrac{i\sqrt{3}}{2}$ repeated.

$\therefore\ y = e^{-x/2}[(c_1 + c_2 x)\cos(\sqrt{3}/2)\cdot x + (c_3 + c_4)\sin(\sqrt{3}/2)\cdot x]$ ■

Example 8 : $\dfrac{d^6y}{dx^6} - 64y = 0$

Sol.: $(D^3)^2 - 8^2 = 0 \quad \therefore (D^3 - 8)(D^3 + 8) = 0$

$\therefore (D-2)(D^2+2D+4) = 0 \quad \therefore (D+2)(D^2-2D+4) = 0$

$\therefore y = c_1 e^{2x} + c_2 e^{-2x} + e^{-x}(c_3 \cos\sqrt{3}\cdot x + c_4 \sin\sqrt{3}\cdot x)$

$+ e^{x}(c_5 \cos\sqrt{3}\cdot x + c_6 \sin\sqrt{3}\cdot x).$ ■

Example 9 : $(D^3 - D^2 + D - 1)^2 y = 0$

Sol.: $D^3 - D^2 + D - 1 = 0 \quad \therefore (D^2+1)(D-1) = 0$ repeated twice.

$\therefore y = (c_1 + c_2 x)e^x + (c_3 + c_4 x)\cos x + (c_5 + c_6 x)\sin x.$ ■

Example 10 : $\{(D-1)^4(D^2+2D+2)^2\} y = 0$

Sol.: $(D-1)^4 = 0 \quad \therefore D = 1$, repeated 4 times.

$\therefore (D^2+2D+2)^2 = 0 \quad \therefore D = \dfrac{-2 \pm \sqrt{4-8}}{2}$ repeated twice.

$\therefore y = (c_1 + c_2 x + c_3 x^2 + c_4 x^3)e^x$

$+ e^{-x}[(c_5 + c_6 x)\cos x + (c_7 + c_8 x)\sin x].$ ■

Example 11 : $\{(D^2+1)^3(D^2+D+1)^2\} y = 0$

Sol.: $D^2 + 1 = 0 \quad \therefore D = \pm i$ repeated thrice.

$\therefore D^2 + D + 1 = 0 \quad \therefore D = -\dfrac{1}{2} \pm i\dfrac{\sqrt{3}}{2}$ repeated twice.

$\therefore y = (c_1 + c_2 x + c_3 x^2)\cos x + (c_3 + c_4 x + c_5 x^2)\sin x$

$+ e^{-x/2}[(c_7 + c_8 x)\cos(\sqrt{3}x/2) + (c_9 + c_{10} x)\sin(\sqrt{3}x/2)]$ ■

Example 12 : $(D^4 + 8D^2 + 16) y = 0$

Sol.: $D^4 + 8D^2 + 16 = 0 \quad \therefore (D^2+4)^2 = 0$

$\therefore D = \pm 2i$ repeated twice.

$\therefore y = (c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x.$ ■

Exercise IV (Page 3-12)

Find the particular integrals of the following differential equations : Class (a) : 3 Marks

Example 1 : $D(y) = \sin x$

Sol.: $\dfrac{1}{D}\sin x = \int \sin x\, dx = -\cos x$ ■

Example 2 : $D^2(y) = e^{2x}$.

Sol.: $\dfrac{1}{D^2}e^{2x} = \dfrac{1}{D}\int e^{2x}\, dx = \int \dfrac{e^{2x}}{2}\, dx = \dfrac{e^{2x}}{4}.$ ■

Example 3 : $D(y) = x^3$

Sol.: $\dfrac{1}{D}x^3 = \int x^3\, dx = \dfrac{x^4}{4}$ ■

Example 4 : $D^2(y) = x^3$

Sol.: $\dfrac{1}{D^2}x^3 = \dfrac{1}{D}\int x^3 dx = \int \dfrac{x^4}{4}\, dx = \dfrac{x^5}{20}.$ ■

Example 5 : $(D-1)\, y = \sin x$

Sol.: $\dfrac{1}{D-1}\sin x = e^x \int e^{-x}\sin x\, dx$

$= e^x\left[\dfrac{e^{-x}}{1+1}(-\sin x - \cos x)\right] = -\dfrac{\sin x + \cos x}{2}$ ■

Example 6 : $(D+2)\, y = x^2$.

Sol.: $\dfrac{1}{D+2}\cdot x^2 = e^{-2x}\int e^{2x}\cdot x^2\, dx$

$= e^{-2x}\left[x^2 \cdot \dfrac{e^{2x}}{2} - x \cdot \dfrac{e^{2x}}{2} + \dfrac{e^{2x}}{4}\right]$

$= \dfrac{x^2}{2} - \dfrac{x}{2} + \dfrac{1}{4}.$ ■

Example 7 : $(D-2)\, y = \sin x$

Sol.: $\dfrac{1}{D-2}\cdot \sin x = e^{2x}\int e^{-2x}\sin x\, dx = -\dfrac{1}{5}(2\sin x + \cos x).$ ■

Example 8 : $(D+2)\, y = \cos x$

Sol.: $\dfrac{1}{D+2}\cdot \cos x = e^{-2x}\int e^{2x}\cos x\, dx$

$= e^{-2x}\left[e^{2x}\cdot \dfrac{1}{4+1}(2\cos x + \sin x)\right]$ ■

Example 9 : $(D^3 - D^2)\, y = x$

Sol.: $\dfrac{1}{D^3 - D^2}\cdot x = \dfrac{1}{D^2(D-1)}\cdot x = \dfrac{1}{D^2}e^x \int e^{-x}\, x\, dx$

$= -\dfrac{1}{D^2}(x+1) = -\dfrac{1}{D}\int (x+1)\, dx$

$= -\int\left(\dfrac{x^2}{2} + x\right)dx = -\left(\dfrac{x^3}{6} + \dfrac{x^2}{2}\right)$ ■

Example 10 : $(D^4 - 4D^2)\, y = 5e^{3x}$

Sol.: $\dfrac{1}{D^4 - 4D^2}\cdot(5e^{3x}) = 5\cdot\dfrac{1}{D^2(D-2)(D+2)}e^{3x}$

$$= \frac{5}{D^2(D+2)} e^{2x} \int e^{-2x} \cdot e^{3x}\, dx$$

$$= \frac{5}{D^2(D+2)} e^{2x} \int e^{x}\, dx = \frac{5}{D^2(D+2)} e^{3x}$$

$$= \frac{5}{D^2} e^{-2x} \int e^{2x} \cdot e^{3x}\, dx = \frac{1}{D} \cdot \int e^{3x}\, dx$$

$$= \frac{1}{3} \int e^{3x}\, dx = \frac{1}{9} e^{3x}.$$

Exercise V (Page 3-12)

Solve the following differential equations : Class (b) : 6 Marks

Example 1 : $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = e^{2x}$

Sol. : $(D-3)(D-4) = 0 \quad \therefore$ C.F. $y = c_1 e^{3x} + c_2 e^{4x}$.

$$\therefore \text{P.I.} = \frac{1}{(D-3)(D-4)} e^{2x} = \frac{1}{D-3} e^{4x} \int e^{-4x} \cdot e^{2x}\, dx$$

$$= \frac{1}{D-3} \cdot \frac{e^{2x}}{(-2)} = -\frac{1}{2} \cdot e^{3x} \int e^{-3x} \cdot e^{2x}\, dx = \frac{1}{2} e^{2x}.$$

Example 2 : $\frac{d^2y}{dx^2} - y = 2 + 5x$

Sol. : $(D-1)(D+1) = 0 \quad \therefore$ C.F. $y = c_1 e^{x} + c_2 e^{-x}$.

$$\therefore \text{P.I.} = \frac{1}{(D-1)(D+1)} (2+5x) = \frac{1}{D-1} e^{-x} \int e^{x} (2+5x)\, dx$$

$$= \frac{1}{D-1} (5x-3) = e^{x} \int e^{-x} (5x-3)\, dx = -5x - 2.$$

Example 3 : $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 2e^{2x}$

Sol. : $(D+1)(D+2) = 0 \quad \therefore$ C.F. $y = c_1 e^{-x} + c_2 e^{-2x}$.

$$\therefore \text{P.I.} = \frac{1}{(D+1)} \cdot \frac{1}{(D+2)} 2e^{2x} = \frac{2}{D+1} \cdot e^{-2x} \int e^{2x} \cdot e^{2x}\, dx$$

$$= \frac{1}{2} \cdot \frac{1}{D+1} e^{2x} = \frac{1}{2} e^{-x} \int e^{x} e^{2x} dx = \frac{e^{2x}}{6}.$$

Example 4 : $\frac{d^2y}{dx^2} - \frac{3dy}{dx} + 2y = 3 + x$

Sol. : $D^2 - 3D + 2 = 0 \quad \therefore$ C.F. $y = c_1 e^{x} + c_2 e^{2x}$.

$$\therefore \text{P.I.} = \frac{1}{(D-1)(D-2)} (3+x) = \frac{1}{(D-1)} e^{2x} \int e^{-2x} (3+x)\, dx$$

$$= \frac{-1}{D-1} \left(\frac{x}{2} + \frac{7}{4} \right) = -e^{x} \int e^{-x} \left(\frac{x}{2} + \frac{7}{4} \right) dx = \frac{x}{2} + \frac{9}{4}.$$

Example 5 : $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 2x$

Sol. : $(D^2 + D - 2) = 0 \quad \therefore$ C.F. $y = c_1 e^{x} + c_2 e^{-2x}$.

$$\therefore \text{P.I.} = \frac{1}{(D+2)(D-1)} 2x = \frac{2}{D+2} e^{x} \int e^{-x}\, x\, dx$$

$$= -\frac{2}{D+2} (x+1) = 2e^{-2x} \int e^{2x} (x+1) = 2\left(\frac{x}{2} + \frac{1}{4} \right) = x + \frac{1}{2}.$$

Example 6 : $(D^2 - 2D + 1)\, y = x^2 - 1$

Sol. : $(D-1)^2 = 0 \quad \therefore$ C.F. $y = (c_1 + c_2 x) e^{x}$.

$$\therefore \text{P.I.} = \frac{1}{(D-1)(D-1)} (x^2 - 1) = \frac{1}{D-1} e^{x} \int e^{-x} (x^2 - 1)\, dx$$

$$= \frac{-1}{D-1} (x^2 + 2x - 1) = -e^{-x} \int e^{-x} (x^2 + 2x - 1)\, dx$$

$$= x^2 + 4x + 1.$$

Exercise VI (Page 3-17)

Solve the following differential equations : Class (b) : 6 Marks

Example 1 : $(D-1)\, y = e^{3x}$.

Sol. : $D - 1 = 0 \quad \therefore D = 1 \quad \therefore y = c_1 e^{x}$.

$$\text{P.I.} = \frac{1}{D-1} e^{3x} = \frac{1}{(3-1)} e^{3x} = \frac{1}{2} \cdot e^{2x}.$$

Example 2 : $(D^3 - 1)\, y = (e^{x} + 1)^2$

Sol. : $(D-1)(D^2 + D + 1) = 0$

$$\therefore \text{C.F. } y = c_1 e^{x} + c_2 e^{-x/2} \left[\cos(\sqrt{3}/2)\, x + i \sin(\sqrt{3}/2)\, x \right]$$

$$\therefore \text{P.I.} = \frac{1}{D^3 - 1} (e^{2x} + 2e^{x} + 1)$$

$$= \frac{1}{D^3 - 1} e^{2x} + 2 \cdot \frac{1}{(D-1)(D^2 + D + 1)} e^{x} + \frac{1}{D^3 - 1} e^{0x}$$

Example 3 : $(D^2 + 4D + 4)\, y = \cos h\, 2x$

Sol. : $(D+2)^2 = 0 \quad \therefore$ C.F. $y = (c_1 + c_2 x) e^{-2x}$.

$$\therefore \text{ P.I.} = \frac{1}{(D+2)^2} \cdot \left(\frac{e^{2x} + e^{-2x}}{2} \right) = \frac{1}{2} \cdot \frac{1}{16} e^{2x} + \frac{1}{2} \cdot \frac{x^2}{2} \cdot e^{-2x}$$

Example 4 : $(D^2 - D - 6)\, y = e^x \cosh 2x$

Sol. : $(D-3)(D+2) = 0. \quad \therefore$ C.F. $y = c_1 e^{-2x} + c_2 e^{3x}$.

$$\therefore \text{ P.I.} = \frac{1}{(D-3)(D+2)} e^x \left(\frac{e^{2x} + e^{-2x}}{2} \right)$$

$$= \frac{1}{2} \left[\frac{1}{(D-3)(D+2)} \right] (e^{3x} + e^{-x}) = \frac{1}{2} \cdot \frac{x}{5} e^{3x} + \frac{1}{2 \cdot (-4)} e^x.$$

Example 5 : $\dfrac{d^2y}{dx^2} + 2a \dfrac{dy}{dx} + (a^2 + b^2)\, y = e^{mx}$

Sol. : $D^2 + 2aD + (a^2 + b^2) = 0$

$$\therefore D = \frac{-2a \pm \sqrt{4a^2 - 4a^2 - 4b^2}}{2} = -a \pm bi$$

$$\therefore \text{ C.F. } y = c_1 e^{-ax} (\cos bx + i \sin bx)$$

$$\therefore \text{ P.I.} = \frac{1}{D^2 + 2aD + (a^2 + b^2)} e^{mx} = \frac{1}{(m^2 + 2am + a^2 + b^2)} e^{mx}.$$

Example 6 : $\dfrac{d^2y}{dx^2} + 3 \dfrac{dy}{dx} + 2y = e^{-x}$

Sol. : $D^2 + 3D + 2 = 0 \quad \therefore (D+1)(D+2) = 0$

$$\therefore \text{ C.F. } y = c_1 e^{-x} + c_2 e^{-2x}$$

$$\therefore \text{ P.I.} = \frac{1}{(D+1)(D+2)} e^{-x} = x \cdot \frac{e^{-x}}{1}.$$

Example 7 : $(2D+1)^2 y = 4e^{-x/2}$

Sol. : $(2D+1)^2 = 0 \quad \therefore$ C.F. $y = (c_1 + c_2 x) e^{-x/2}$.

$$\therefore \text{ P.I.} = \frac{1}{(2D+1)^2} \cdot 4e^{-x/2} = 4 \cdot \frac{x^2}{2!} e^{-x/2}.$$

Example 8 : $(D^4 + 1)\, y = \cosh 4x \sinh 3x$

Sol. : $(D^4 + 2D^2 + 1) - 2D^2 = 0 \quad \therefore (D^2 + 1)^2 - (\sqrt{2} D)^2 = 0$

$$\therefore D^2 + \sqrt{2} \cdot D + 1 = 0 \text{ and } D^2 - \sqrt{2} \cdot D + 1 = 0$$

$$\therefore D = -\frac{\sqrt{2} + \sqrt{2}\, i}{2} \text{ and } D = \frac{\sqrt{2} \pm \sqrt{2}\, i}{2}$$

$$\therefore \text{ C.F. } y = e^{-x/\sqrt{2}} [c_1 \cos(x/\sqrt{2}) + c_2 \sin(x/\sqrt{2})]$$

$$+ e^{x/\sqrt{2}} [c_3 \cos(x/\sqrt{2}) + c_4 \sin(x/\sqrt{2})]$$

$$\therefore \text{ P.I.} = \frac{1}{D^4 + 1} \left[\left(\frac{e^{4x} + e^{-4x}}{2} \right) \left(\frac{e^{3x} - e^{-3x}}{2} \right) \right]$$

$$= \frac{1}{4} \cdot \frac{1}{D^4 + 1} (e^{7x} - e^x + e^{-x} - e^{-7x})$$

$$\therefore \text{ P.I.} = \frac{1}{9608} (e^x - e^{-7x}) - \frac{1}{8} (e^x - e^{-x})$$

Example 9 : $(D^3 - 4D^2 + 5D - 2)\, y = e^x$

Sol. : $D^3 - 4D^2 + 5D - 2 = 0 \quad \therefore (D-1)(D-1)(D-2) = 0$

$$\therefore \text{ C.F. } y = (c_1 + c_2 x) e^x + c_3 e^{2x}.$$

$$\therefore \text{ P.I.} = \frac{1}{(D-1)^2} \cdot \frac{1}{(D-2)} e^{-x} = \frac{x^2}{2} \cdot \left(-\frac{1}{1} \right) e^{-x}.$$

Example 10 : $(D^2 - 9)\, y = e^{-3x} + 1 + e^{3x}$

Sol. : $(D-3)(D+3) = 0 \quad \therefore$ C.F. $y = c_1 e^{3x} + c_2 e^{-3x}$.

$$\therefore \frac{1}{(D-3)(D+3)} e^{-3x} = -\frac{x}{6} e^{-3x} \quad \therefore \frac{1}{(D-3)(D+3)} e^0 x = \frac{1}{-9}$$

$$\therefore \frac{1}{(D-3)(D+3)} e^{3x} = x \cdot \frac{e^{3x}}{6} \quad \therefore \text{ P.I.} = -\frac{x}{6} e^{-3x} - \frac{1}{9} + \frac{x}{6} e^{3x}.$$

Example 11 : $(D^2 - 2D + 1)\, y = e^x + 1$

Sol. : $(D-1)^2 = 0 \quad \therefore$ C.F. $y = (c_1 + c_2 x) e^x$.

$$\therefore \text{ P.I.} = \frac{1}{(D-1)^2} (e^x + e^{0x}) = \frac{x^2}{2!} e^x + 1.$$

Example 12 : $(D^3 - 4D)\, y = 2 \cosh 2x$

Sol. : $D(D^2 - 4) = 0 \quad \therefore$ C.F. $y = c_1 + c_2 e^{2x} + c_3 e^{-2x}$.

$$\therefore \text{ P.I.} = \frac{2}{D^3 - 4D} \cosh \cdot 2x = \frac{1}{D(D-2)(D+2)} (e^{2x} + e^{-2x})$$

$$= \frac{x}{D(D+2)} e^{2x} + \frac{x}{D(D-2)} e^{-2x} = \frac{x}{8} (e^{2x} + e^{-2x}).$$

Example 13 : $(D^3 - a^2 D) = 2 \cosh ax$

Sol. : $D(D^2 - a^2) = 0 \quad \therefore$ C.F. $y = c_1 + c_2 e^{ax} + c_3 e^{-ax}$.

$$\therefore \text{ P.I.} = \frac{x}{D(D+a)} e^{ax} + \frac{x}{D(D-a)} e^{-ax} = \frac{x}{2a^2} e^{ax} + \frac{x}{2a^2} e^{-ax}.$$

Example 14 : $(D^4 - 4D^3 + 8D^2 - 8D + 4)\, y = e^x + 1.$

Sol. : $(D^2 - 2D^2 + 2)^2\, y = e^x + 1.$ A.E. $D^2 - 2D + 2 = 0$

$\therefore\ D = \dfrac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$ (repeated twice)

$\therefore$ C.F. $y = e^x\,[(c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x]$

$$\text{P.I.} = \frac{1}{(D^2 - 2D + 2)}(e^x + e^{0x})$$

$$= \frac{1}{(1-2+2)^2} \cdot e^x + \frac{1}{(0-0+2)^2} \cdot e^{0x} = e^x + \frac{1}{4}.$$

Exercise VII (Page 3-25)

Solve the following differential equations : Class (b) : 6 Marks

Example 1 : $(D^2 - D + 1)\, y = \cos 2x$

Sol. : $D^2 - D + 1 = 0 \quad \therefore\ D = \dfrac{1 \pm \sqrt{3}\, i}{2}$

$\therefore$ C.F. $y = e^{x/2}\,[c_1 \cos(\sqrt{3}/2)x + c_2 \sin(\sqrt{3}/2)x]$

$$\therefore\ \text{P.I.} = \frac{1}{D^2 - D + 1}\cos 2x = \frac{1}{-D-3}\cos 2x = -\frac{D-3}{D^2-9}\cos 2x$$

$$= \frac{1}{13}(-2\sin 2x - 3\cos 2x).$$

Example 2 : $\dfrac{d^3y}{dx^3} + y = \cos ax$

Sol. : $(D+1)(D^2 - D + 1) = 0$

$\therefore$ C.F. $y = c_1 e^{-x} + e^{x/2}\,[c_2 \cos(\sqrt{3}/2)x + c_3 \sin(\sqrt{3}/2)x]$

$$\therefore\ \text{P.I.} = \frac{1}{D^3+1}\cos ax = \frac{1}{-a^2 D + 1}\cos ax = \frac{1 + a^2 D}{1 - a^4 D^2}\cos ax$$

$$= \frac{\cos ax - a^3 \sin ax}{1 + a^6}.$$

Example 3 : $(D^3 + D)\, y = \cos x$

Sol. : $D(D^2 + 1) = 0 \quad \therefore\ D = 0,\ +i,\ -i$

$\therefore$ C.F. $y = c_1 + (c_2 \cos x + c_3 \sin x)$

$$\therefore\ \text{P.I.} = \frac{1}{D(D^2+1)}\cos x = \frac{1}{D}\cdot\left(\frac{x}{2D}\right)\cos x$$

$$= \frac{x}{2D^2}\cos x = -\frac{x}{2}\cos x.$$

Example 4 : $\operatorname{cosec} x \dfrac{d^4y}{dx^4} + \operatorname{cosec} x \cdot y = \sin 2x$

Sol. : $D^4 + 1 = (D^2 + 1)^2 - 2D^2 = 0$

$\therefore\ (D^2 + \sqrt{2}\,D + 1)(D^2 - \sqrt{2}\,D + 1) = 0$

$\therefore$ C.F. $y = e^{x/\sqrt{2}}\,[c_1 \cos(x/\sqrt{2}) + c_2 \sin(x/\sqrt{2})]$

$\qquad + e^{-x/\sqrt{2}}\,[c_3 \cos(x/\sqrt{2}) + c_4 \sin(x/\sqrt{2})]$

$$\therefore\ \text{P.I.} = \frac{1}{D^4+1}\left[-\frac{1}{2}(\cos 3x - \cos x)\right]$$

$$= -\frac{1}{2}\cdot\frac{1}{82}\cos 3x + \frac{1}{2}\cdot\frac{1}{2}\cos x$$

Example 5 : $\dfrac{d^4y}{dx^4} - a^4 y = \sin ax$

Sol. : $D^4 - a^4 = 0 \quad \therefore\ (D^2 + a^2)(D^2 - a^2) = 0$

$\therefore\ D = ai,\ -ai,\ a,\ -a.$

$\therefore$ C.F. $y = c_1 e^{ax} + c_2 e^{-ax} + c_3 \cos ax + c_4 \sin ax$

$$\therefore\ \text{P.I.} = \frac{1}{D^4 - a^4}\sin ax = \frac{1}{(D^2 + a^2)}\cdot\frac{1}{(D^2 - a^2)}\sin ax$$

$$= -\frac{1}{2a^2}\cdot\frac{1}{D^2 + a^2}\sin ax = -\frac{1}{2a^2}\cdot\frac{x}{2D}\sin ax$$

$$= -\frac{1}{2a^2}\cdot\frac{1}{2}\int \sin ax\, dx = -\frac{1}{4a^2}\cdot\frac{-x}{a}\cos ax = \frac{x}{4a^3}\cos ax.$$

Example 6 : $(D^4 + 8D^2 + 16)\, y = \cos^2 x$

Sol. : $D^4 + 8D^2 + 16 = 0 \quad \therefore\ (D^2 + 4)^2 = 0$

$\therefore$ C.F. $y = (c_1 + c_2 x)(c_3 \cos 2x + c_4 \sin 2x)$

$$\therefore\ \text{P.I.} = \frac{1}{(D^2+4)^2}\cdot\left(\frac{1 + \cos 2x}{2}\right)$$

Now, $\dfrac{1}{(D^2+4)^2}\cdot\dfrac{1}{2}e^0 x = \dfrac{1}{32}$

and $\dfrac{1}{(D^2+4)^2}\cdot\dfrac{1}{2}\cos 2x = \dfrac{1}{2}\cdot\dfrac{x}{2(D^2+4)\cdot 2D}\cos 2x$

$$= \frac{x}{8(D^2+4)}\cdot\frac{1}{D}\cos 2x = \frac{x}{8}\cdot\frac{1}{(D^2+4)}\int \cos 2x\, dx$$

$$= \frac{x}{8}\cdot\frac{1}{D^2+4}\cdot\frac{\sin 2x}{2}\, dx = \frac{x}{16}\cdot\frac{x}{2D}\sin 2x\, dx$$

$$= \frac{x^2}{32}\int \sin 2x\, dx = -\frac{x^2}{64}\cos 2x.$$

Example 7 : $(D-1)^2(D^2+1)^2 y = \sin^2\frac{x}{2} + e^x$

Sol. : $(D-1)^2(D^2+1)^2 = 0$

$\therefore$ C.F. $y = (c_1 + c_2 x)e^x + (c_3 + c_4 x)(c_5 \cos x + c_6 \sin x)$

$$\therefore \text{ P.I.} = \frac{1}{(D-1)^2(D^2+1)^2}\cdot\left[\frac{(1-\cos x)}{2} + e^x\right].$$

Now, $\dfrac{1}{(D-1)^2(D^2+1)^2}\cdot\dfrac{1}{2}e^{0x} = \dfrac{1}{(-1)^2(1)^2} = \dfrac{1}{2}$

And $\dfrac{1}{(D-1)^2(D^2+1)^2}\cos x = \dfrac{1}{(D^2+1)^2}\cdot\dfrac{1}{(D^2-2D+1)}\cos x$

$$= \frac{1}{(D^2+1)^2}\cdot\frac{1}{-2D}\cos x = -\frac{1}{2}\cdot\frac{1}{(D^2+1)^2}\int\cos x\,dx$$

$$= -\frac{1}{2}\cdot\frac{1}{(D^2+1)^2}\sin x = -\frac{1}{2}\cdot\frac{x}{2(D^2+1)\cdot 2D}\sin x$$

$$= -\frac{1}{8}\cdot\frac{x}{D^2+1}(-\cos x) = \frac{x}{8}\cdot\frac{x}{2D}\cos x$$

$$= \frac{x^2}{16}\int\cos x\,dx = \frac{x^2}{16}\sin x.$$

And $\dfrac{1}{(D-1)^2(D^2+1)^2}e^x = \dfrac{x^2}{2\cdot 4}e^x.$ ■

Example 8 : $(D^4 + 10D^2 + 9)y = 96\sin 2x\cos x$

Sol. : $(D^2+1)(D^2+9) = 0$

$\therefore$ C.F. $y = c_1\cos x + c_2\sin x + c_3\cos 3x + c_4\sin 3x$

$$\therefore \text{ P.I.} = \frac{1}{D^4+10D^2+9}\cdot 96\sin 2x\cos x$$

$$= \frac{48}{(D^2+1)(D^2+9)}(\sin 3x + \sin x)$$

Now, $\dfrac{1}{(D^2+1)(D^2+9)}\sin 3x = -\dfrac{1}{8}\cdot\dfrac{x}{2D}\sin 3x$

$$= -\frac{1}{8}\cdot\frac{x}{2}\int\sin 3x\,dx = \frac{1}{16}\cdot\frac{x}{3}\cos 3x$$

And $\dfrac{1}{(D^2+1)(D^2+9)}\sin x = \dfrac{1}{8}\cdot\dfrac{1}{D^2+1}\sin x$

$$= \frac{1}{8}\cdot\frac{x}{2D}\sin x = \frac{x}{16}\int\sin x\,dx = -\frac{x}{16}\cos x.$$ ■

Example 9 : $D^2(D^2+1)y = \sin x + e^{-x}$

Sol. : $D^2(D^2+1) = 0$

$\therefore$ C.F. $y = c_1 + c_2 x + c_3\cos x + c_4\sin x$

$$\therefore \text{ P.I.} = \frac{1}{D^2(D^2+1)}\sin x = \frac{1}{(-1)}\cdot\frac{1}{D^2+1}\sin x$$

$$= -\frac{1}{2D}\sin x = -\frac{x}{2}\int\sin x\,dx = \frac{x}{2}\cos x$$

And $\dfrac{1}{D^2(D^2+1)}e^{-x} = \dfrac{1}{1}\cdot\dfrac{1}{2}e^{-x}.$ ■

Example 10 : $D^2(D^2+9)y = \cos 3x + 5$

Sol. : $D^2(D^2+9) = 0.$ $\therefore$ C.F. $y = c_1 + c_2 x + c_3\cos x + c_4\sin x$

$$\therefore \text{ P.I.} = \frac{1}{D^2(D^2+9)}\cos 3x = \frac{1}{(-9)}\cdot\frac{1}{D^2+9}\cos 3x$$

$$= -\frac{1}{9}\cdot\frac{x}{2D}\cos 3x = -\frac{1}{9}\cdot\frac{x}{2}\int\cos 3x\,dx$$

$$= -\frac{x}{18}\cdot\frac{\sin 3x}{3} = -\frac{x}{54}\sin 3x$$

And $\dfrac{1}{D^2(D^2+9)}5e^{0x} = 5\cdot\dfrac{x^2}{2}\cdot\dfrac{1}{9} = \dfrac{5}{18}x^2.$ ■

Example 11 : $(D^4 + 10D^2 + 9)y = \cos(2x+3)$

Sol. : $D^4 + 10D^2 + 9 = 0$ $\therefore$ $(D^2+9)(D^2+1) = 0$

$\therefore$ C.F. $y = c_1\cos x + c_2\sin x + c_3\cos 3x + c_4\sin 3x$

$$\therefore \text{ P.I.} = \frac{1}{D^4+10D^2+9}[\cos(2x+3)] = \frac{1}{16-40+9}\cos(2x+3).$$ ■

Example 12 : $(D^2-4)y = \sin^2 x.$

Sol. : $D^2 - 4 = 0$ $\therefore$ C.F. $y = c_1 e^{2x} + c_2 e^{-2x}.$

$$\therefore \text{ P.I.} = \frac{1}{D^2-4}\sin^2 x = \frac{1}{D^2-4}\cdot\left(\frac{1-\cos 2x}{2}\right)$$

$$= \frac{1}{2}\cdot\frac{1}{D^2-4}\cdot(e^{0x}) - \frac{1}{2}\cdot\frac{1}{D^2-4}\cos 2x = -\frac{1}{8} - \frac{1}{2}\cdot\frac{x}{2\cdot 2}\sin 2x.$$ ■

Exercise VIII (Page 3-29)

Solve the following differential equations : Class (b) : 6 Marks

Example 1 : $(D^3 + 2D^2 + D)y = x^2 + x$

Sol. : $D(D+1)^2 = 0$ $\therefore$ C.F. $y = c_1 + (c_2 + c_3 x)e^{-x}.$

$$\therefore \text{ P.I.} = \frac{1}{D(D+1)^2}(x^2+x) = \frac{1}{D}[1 + (2D + D^2)]^{-1}(x^2+x)$$

$$= \frac{1}{D}[1 - 2D + 3D^2](x^2+x) = \frac{1}{D}[(x^2+x) - 2(2x+1) + 3\cdot 2]$$ ■

Example 2 : $(D^3 + 3D^2 + 2D)\, y = x^2$

Sol. : $D(D+1)(D+2) = 0 \quad \therefore$ C.F. $y = c_1 + c_2 e^{-x} + c_3 e^{-2x}$.

$$\therefore \text{ P.I.} = \frac{1}{D} \cdot \frac{1}{2[1 + (3D + D^2)/2]} \cdot x^2$$

$$= \frac{1}{2} \cdot \frac{1}{D}\left[1 - \frac{3D}{2} + \frac{7}{4}D^2\right] x^2 = \frac{1}{2} \cdot \frac{1}{D}\left[x^2 - 3x + \frac{7}{2}\right]$$

Example 3 : $(D^4 - 2D^3 + D^2)\, y = x^3$

Sol. : $D^2 (D-1)^2 = 0 \quad \therefore$ C.F. $y = c_1 + c_2 x + (c_3 + c_4 x)\, e^x$

$$\therefore \text{ P.I.} = \frac{1}{D^2} \cdot \frac{1}{1 - 2D + D^2} x^2 = \frac{1}{D^2}[1 - (2D - D^2)]^{-1} x^3$$

$$= \frac{1}{D^2}[1 + 2D + 3D^2 - 4D^3 + \ldots] x^3 = \frac{1}{D^2}[x^3 + 6x^2 + 18x - 24]$$

$$= \frac{1}{D}\left[\frac{x^4}{4} + 2x^3 + 9x^2 - 24x\right] = \frac{x^5}{20} + \frac{x^4}{2} + 3x^2 - 12x^2$$

Example 4 : $(D^2 + 2D + 3)\, y = x - x^2$

Sol. : $D^2 + 2D + 3 = 0 \quad \therefore D = -1 + \sqrt{2} \cdot i$

$\therefore$ C.F. $y = e^{-x}(c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}\, x)^2$

$$\therefore \text{ P.I.} = \frac{1}{3 + 2D + D^2}(x - x^2) = \frac{1}{3} \cdot \left[1 + \frac{2D + D^2}{3}\right]^{-1} (x - x^2)$$

$$= \frac{1}{3}\left[1 - \frac{2D}{3} + \frac{1}{9}D^2 \ldots\right](x - x^2) = \frac{1}{3}\left[(x - x^2) - \frac{2}{3}(1 - 2x) + \frac{1}{9}(-2)\right]$$

Example 5 : $(D^3 - D^2 + 6D)\, y = x^2 + \sin x$

Sol. : $D^3 - D^2 - 6D = 0 \quad \therefore D(D-3)(D+2) = 0$

$\therefore$ C.F. $y = c_1 + c_2 e^{-2x} + c_3 e^{-3x}$.

Now, $$\frac{1}{D(D^2 - D - 6)} = \frac{1}{-6D[1 + (D - D^2)/6]} x^2$$

$$= -\frac{1}{6D} \cdot \left[1 - \frac{D}{6} + \frac{7}{36}D^2 \ldots\right] x^2 = -\frac{1}{6D}\left(x^2 - \frac{x}{3} - \frac{14}{36}\right)$$

And $$\frac{1}{D^3 - D^2 - 6D} \sin x = \frac{1}{-D + 1 - 6D} \sin x$$

$$= \frac{1 + 7D}{1 - 49D^2} \sin x = \frac{\sin x + 7 \cos x}{50}.$$

Example 6 : $(D^2 + D - 2)\, y = x + e^{-2x}$

Sol. : $(D+2)(D-1) = 0 \quad \therefore$ C.F. $y = c_1 e^x + c_2 e^{-2x}$.

Now, $$\frac{1}{D^2 + D - 2} x = -\frac{1}{2}\left[1 + \frac{D}{2} \ldots\right] x = -\frac{1}{2}\left(x + \frac{1}{2}\right)$$

And $$\frac{1}{D^2 + D - 2} e^{-2x} = \frac{1}{(D+2)(D-1)} = -\frac{x}{3} e^{-2x}.$$

Example 7 : $(D^2 - 4D + 4)\, y = 8(x^2 + \sin 2x + e^{2x})$

Sol. : $(D-2)^2 = 0 \quad \therefore$ C.F. $y = (c_1 + c_2 x) e^{2x}$.

Now, $$\frac{1}{4 - 4D + D^2} 8x^2 = \frac{8}{4[1 - (4D - D^2)/4]} x^2$$

$$= \frac{8}{4}\left[1 + \frac{4D - D^2}{4} + D^2\right] x^2 = \frac{8}{4}\left[x^2 + 2x + \frac{3}{2}\right].$$

And $$\frac{1}{D^2 - 4D + 4} 8 \sin 2x = \frac{8}{-4D} \sin 2x = -2 \int \sin 2x = \cos 2x$$

And $$\frac{1}{(D^2 - 2)^2} \cdot 8 \cdot e^{3x} = 8 \cdot \frac{x^2}{2} e^{2x} = 4x^2 e^{2x}.$$

Example 8 : $(D^3 - D)\, y = 2e^x + 2x + 1 - 4 \cos x$

Sol. : $D(D-1)(D+1) = 0 \quad \therefore$ C.F. $y = c_1 + c_2 e^x + c_3 e^{-x}$.

Now, $$\frac{1}{D(D+1)(D-1)} 2 \cdot e^x = \frac{2}{1 \cdot 2} \cdot \frac{x}{1} \cdot e^x = x e^x$$

And $$\frac{1}{D^3 - D}(2x + 1) = -\frac{1}{D}(1 - D^2)^{-1}(2x + 1)$$

$$= -\frac{1}{D}(2x + 1) = -(x^2 + x)$$

And $$\frac{1}{D^3 - D}(-4) \cos x = -\frac{4}{-2D} \cos x = 2 \int \cos x\, dx = 2 \sin x$$

Example 9 : $(D^2 + 4D + 4)\, y = x^2$

Sol. : $(D+2)^2 = 0 \quad \therefore$ C.F. $y = (c_1 + c_2 x) e^{-2x}$.

$$\therefore \text{ P.I.} = \frac{1}{4 + 4D + 4D^2} = \frac{1}{4}\left[1 + \frac{4D + D^2}{4}\right]^{-1} x^2$$

$$= \frac{1}{4}\left[1 - \frac{4D + D^2}{4} + D^2\right] x^2 = \frac{1}{4}\left[1 - D + \frac{3}{4}x^2\right]$$

Example 10 : $(D^2 + 4)\, y = x^2 + \sin 2x$

Sol. : $D^2 + 4 = 0 \quad \therefore$ C.F. $y = c_1 \cos 2x + c_2 \sin 2x$.

Now, $$\frac{1}{D^2 + 4} \sin x = \frac{1}{3} \sin x \quad \text{and} \quad \frac{1}{D^2 + 4} e^x = \frac{1}{5} e^x.$$

Example 11 : $(D^2+2D+2)\,y = x^2+1$

Sol. : $D^2+2D+2=0 \quad \therefore$ C.F. $y = e^{-x}(c_1\cos x + c_2 \sin x)$.

$$\therefore \text{ P.I.} = \frac{1}{2[1+D+D^2/2]}(x^2+1) = \frac{1}{2}\left[1+\left(D+\frac{D^2}{2}\right)\right]^{-1}(x^2+1)$$

$$= \frac{1}{2}\left[1-D+\frac{D^2}{2}\right](x^2+1) = \frac{1}{2}\left[(x^2+1)-2x+1\right].$$

Exercise IX (Page 3-36)

Solve the following differential equations : Class (b) : 6 Marks

Example 1 : $(D^2-4D+4)\,y = e^{2x}x^2$

Sol. : $(D-2)^2=0 \quad \therefore$ C.F. $y = (c_1+c_2 x)\,e^{2x}$.

$$\therefore \text{ P.I.} = \frac{1}{(D-2)^2}\cdot e^{2x}x^2 = \frac{e^{2x}}{(D+2-2)^2}\cdot x^2$$

$$= e^{2x}\cdot\frac{1}{D^2}x^2 = e^{2x}\cdot\frac{1}{D}\cdot\frac{x^3}{3} = e^{2x}\cdot\frac{x^4}{12}.$$

Example 2 : $(D^2+3D+2)\,y = e^{2x}\sin x$

Sol. : $(D+1)(D+2)=0 \quad \therefore$ C.F. $y = c_1e^{-x}+c_2e^{-2x}$.

$$\therefore \text{ P.I.} = \frac{1}{D^2+3D+2}e^{2x}\sin x = e^{2x}\cdot\frac{1}{(D+2)^2+3(D+2)+2}\sin x$$

$$= e^{2x}\cdot\frac{1}{D^2+7D+12}\sin x = e^{2x}\cdot\frac{1}{7D+11}\sin x$$

$$= e^{2x}\cdot\frac{7D-11}{49D^2-11^2}\sin x = e^{2x}\cdot\frac{(7\cos x-11\sin x)}{-170}$$

Example 3 : $(D^2-4D+1)\,y = e^{2x}\sin 2x$

Sol. : $D^2-4D+1=0 \quad \therefore$ C.F. $y = c_1e^{(2+\sqrt{3})x}+c_2e^{(2-\sqrt{3})x}$.

$$\therefore \text{ P.I.} = e^{2x}\cdot\frac{1}{(D+2)^2-4(D+2)+1}\sin 2x$$

$$= e^{2x}\cdot\frac{1}{D^2-3}\sin 2x = \frac{-e^{2x}}{7}\sin 2x.$$

Example 4 : $(D^3-7D-6)\,y = e^{2x}(x+1)$

Sol. : $D^3-7D-6=0 \quad \therefore (D+1)(D+2)(D-3)=0$

$\therefore$ C.F. $y = c_1e^{-x}+c_2e^{-2x}+c_3e^{3x}$.

$$\therefore \text{ P.I.} = e^{2x}\cdot\frac{1}{(D+2)^3-7(D+2)-6}(x+1)$$

$$= e^{2x}\cdot\frac{1}{D^3+6D^2+12D+8-7D-14-6}(x+1)$$

$$= e^{2x}\cdot\frac{1}{-12+5D+6D^2+D^3}(x+1)$$

$$= -\frac{e^{2x}}{12}\left[1+\frac{5D}{12}\right](x+1) = -\frac{e^{2x}}{12}\left(x+\frac{17}{12}\right).$$

Example 5 : $(D^3-7D-6)\,y = (1+x^2)\,e^{2x}$

Sol. : $D^3-7D-6=0 \quad \therefore (D+1)(D+2)(D-3)=0$

$\therefore$ C.F. $y = c_1e^{-x}+c_2e^{-2x}+c_3e^{3x}$.

$$\therefore \text{ P.I.} = \frac{1}{-6-7D+D^3}e^{2x}(1+x^2) = e^{2x}\cdot\frac{1}{-6-7(D+2)+(D+2)^3}(1+x^2)$$

$$= e^{2x}\cdot\frac{1}{-12+5D+6D^2+D^3}(1+x^2) = \frac{e^{2x}}{-12}\cdot\left(1-\frac{5D+6D^2}{12}\right)^{-1}(1+x^2)$$

$$= -\frac{e^{-2x}}{11}\left[1+\frac{5D+6D^2}{12}+\frac{25D^2}{144}\right](1+x^2)$$

$$= -\frac{e^{-2x}}{11}\left[1+x^2+\frac{5}{12}\cdot 2x+1+\frac{25}{72}\right]$$

Example 6 : $(D^2+4D+4)\,y = \dfrac{e^{-2x}}{x^5}$

Sol. : $D^2+4D+4=0 \quad \therefore$ C.F. $y = (c_1+c_2x)\,e^{-2x}$.

$$\therefore \text{ P.I.} = \frac{1}{D^2+4D+4}e^{-2x}\cdot x^{-5} = e^{-2x}\cdot\frac{1}{(D+2-2)^2}x^{-5}$$

$$= e^{-2x}\frac{1}{D^2}x^{-5} = e^{-2x}\frac{1}{D}\int x^{-5}\,dx$$

$$= e^{-2x}\cdot\frac{1}{D}\cdot\frac{x^{-4}}{-4} = -\frac{1}{4}\int x^{-4}\,dx = \frac{e^{2x}}{12}x^{-3}.$$

Example 7 : $(D^2+D-6)\,y = e^{2x}\sin 3x$

Sol. : $(D+3)(D-2)=0 \quad \therefore$ C.F. $y = c_1e^{2x}+c_2e^{-3x}$.

$$\therefore \text{ P.I.} = e^{2x}\cdot\frac{1}{(D+2)^2+(D+2)-6}\sin 3x = e^{2x}\cdot\frac{1}{D^2+5D}\sin 3x$$

$$= e^{2x}\cdot\frac{1}{5D-9}\sin 3x = e^{3x}\frac{5D+9}{25D^2-81}(\sin 3x)$$

$$= -\frac{1}{102}\cdot e^{2x}(5\cos 3x+3\sin 3x).$$

Example 8 : $(D^2-4)\,y = x^2e^{3x}$

Sol. : $D^2-4=0 \quad \therefore$ C.F. $y = c_1e^{2x}+c_2e^{-2x}$.

$$\therefore \text{ P.I.} = \frac{1}{D^2-4}e^{3x}x^2 = e^{3x}\frac{1}{(D+3)^2-4}x^2 = \frac{e^{3x}}{5}\cdot\left[1+\frac{6D+D^2}{5}\right]^{-1}x^2$$

$$= \frac{e^{3x}}{5}\left[1-\frac{6D}{5}-\frac{D^2}{5}+\frac{36}{25}D^2+\ldots\right]x^2$$

Example 9 : $(D^2-1)\,y = x\sinh x$

Sol. : $(D^2-1)=0 \quad \therefore$ C.F. $y = c_1e^{x}+c_2e^{-x}$.

$$\therefore \text{ P.I.} = \frac{1}{D^2-1}\cdot x\left(\frac{e^x-e^{-x}}{2}\right)$$

$$= \frac{1}{2}\cdot\frac{1}{D^2-1}xe^x - \frac{1}{2}\cdot\frac{1}{D^2-1}\cdot xe^{-x}$$

$$= \frac{e^x}{2}\cdot\frac{1}{(D+1)^2-1}\cdot x - \frac{e^{-x}}{2}\cdot\frac{1}{(D-1)^2-1}\cdot x$$

$$= \frac{e^x}{2}\cdot\frac{1}{D^2+2D}\cdot x - \frac{e^{-x}}{2}\cdot\frac{1}{D^2-2D}\cdot x$$

$$= \frac{e^x}{2}\cdot\frac{1}{2D}\left[1+\frac{D}{2}\right]^{-1}x - \frac{e^{-x}}{2(-2D)}\left[1-\frac{D}{2}\right]^{-1}x$$

$$= \frac{e^x}{4}\int\left(x-\frac{1}{2}\right)dx + \frac{e^{-x}}{4}\int\left(x+\frac{1}{2}\right)dx$$

$$= \frac{e^x}{4}\cdot\left(\frac{x^2}{2}-\frac{x}{2}\right)+\frac{e^{-x}}{4}\left(\frac{x^2}{2}+\frac{x}{2}\right)$$

$$= \frac{x^2}{4}\left[\frac{e^x+e^{-x}}{2}\right]-\frac{x}{4}\cdot\left[\frac{e^x-e^{-x}}{2}\right]$$

$$= \frac{x^2}{4}\cosh x - \frac{x}{4}\sinh x.$$

Example 10 : $(D^2-2D+1)\,y = \dfrac{3e^x}{x^2}$

Sol. : $(D-1)^2=0 \quad \therefore$ C.F. $y = (c_1+c_2x)e^x$.

$$\therefore \text{ P.I.} = \frac{1}{(D-1)^2}\cdot\frac{3e^x}{x^2} = 3e^x\cdot\frac{1}{(D+1-1)^2}\cdot\frac{1}{x^2} = 3\frac{e^x}{D^2}\cdot\frac{1}{x^2}$$

$$= 3e^x\cdot\frac{1}{D}\int\frac{1}{x^2}dx = 3e^x\cdot\frac{1}{D}\left(-\frac{1}{x}\right)$$

$$= 3e^x\int -\frac{1}{x}dx = -3e^x\log x.$$

Example 11 : $(D^4+D^2+1)\,y = ax^2+be^{-x}\sin 2x$

Sol. : $(D^4+2D^2+1)-D^2=0 \quad \therefore (D^2+1)^2-D^2=0$

$\therefore D^2+D+1=0 \quad \therefore D^2-D+1=0$

$$\therefore \text{ C.F. } y = e^{-x/2}[c_1\cos(\sqrt{3}/2)x + c_2\sin(\sqrt{3}/2)x]$$
$$+ e^{x/2}[c_3\cos(\sqrt{3}/2)x + c_4\sin(\sqrt{3}/2)x]$$

Now, $\dfrac{1}{1+D^2+D^4}ax^2 = [1+D^2+\ldots]^{-1}ax^2$

$$= (1-D^2\ldots)ax^2 = ax^2-2ax.$$

And $b\cdot e^{-x}\dfrac{1}{1+(D-1)^2+(D-1)^4}\sin 2x$

$$= be^{-x}\cdot\frac{1}{10D-9}\sin 2x = be^{-x}\cdot\frac{10D+9}{100D^2-81}(\sin 2x).$$

Example 12 : $(D^2-1)\,y = e^x\sin 3x$

Sol. : $D=\pm 1 \quad \therefore$ C.F. $y = c_1e^x+c_2e^{-x}$.

$$\therefore \text{ P.I.} = \frac{1}{D^2-1}e^x\sin 3x = e^x\cdot\frac{1}{(D+1)^2-1}\sin 3x$$

$$= e^x\cdot\frac{1}{D^2+2D}e^x\sin 3x = e^x\frac{1}{2D-9}\sin 3x$$

$$= e^x\frac{2D+9}{4D^2-81}\sin 3x.$$

Example 13 : $(D^2-2D+4)\,y = e^x\cos^2 x$

Sol. : $D^2-2D+4=0$

$\therefore$ C.F. $y = e^x(c_1\cos\sqrt{3}\,x + c_2\sin\sqrt{3}\,x)$.

$$\therefore \text{ P.I.} = \frac{1}{D^2-2D+4}e^x\cos^2 x = \frac{1}{D^2-2D+4}e^x\left(\frac{1+\cos 2x}{2}\right)$$

Now, $\dfrac{1}{D^2-2D+4}\cdot\dfrac{1}{2}e^x = \dfrac{1}{8}e^x$

And $\dfrac{1}{D^2-2D+4}\cdot\dfrac{1}{2}\cdot e^x\cos 2x = \dfrac{e^x}{2}\cdot\dfrac{1}{(D+1)^2-2(D+1)+4}\cos 2x$

$$= \frac{e^x}{2}\cdot\frac{1}{3+D^2}\cos 2x = -\frac{e^x}{2}\cos 2x.$$

Example 14 : $(D^2-4D+3)\,y = e^x\cos 2x+\cos 3x$

Sol. : $(D-3)(D-1)=0 \quad \therefore$ C.F. $y = c_1e^x+c_2e^{3x}$.

Now, $\dfrac{1}{D^2-4D+3}e^x\cos 2x = e^x\cdot\dfrac{1}{(D+1)^2-4(D+1)+3}\cos 2x$

$$= e^x \cdot \frac{1}{D^2 - 2D} \cos 2x = e^x \cdot \frac{1}{-4-2D} \cos 2x$$

$$= -\frac{e^x}{2} \cdot \frac{D-2}{D^2-4} \cos 2x$$

And $$\frac{1}{D^2 - 4D + 3} \cos 3x = \frac{1}{-6-4D} \cos 3x$$

$$= -\frac{1}{2} \cdot \frac{3-2D}{9-4D^2} \cos 3x$$

Example 15 : $(D^2+1)\,y = \sin x \sin hx$

Sol. : $D^2 + 1 = 0 \quad \therefore$ C.F. $y = c_1 \cos x + c_2 \sin x$.

$$\therefore \text{ P.I.} = \frac{1}{D^2+1} \sin x \sin hx = \frac{1}{D^2+1} \cdot \left(\frac{e^x - e^{-x}}{2}\right) \sin x$$

$$\therefore \text{ P.I.} = \frac{1}{2}\left[e^x \cdot \frac{1}{(D+1)^2+1} \sin x - e^{-x} \cdot \frac{1}{(D-1)^2+1} \sin x\right]$$

$$= \frac{1}{2}\left[e^x \cdot \frac{1}{2D+1} \sin x + e^{-x} \cdot \frac{1}{2D-1} \sin x\right]$$

$$= \frac{1}{2}\left[e^x \cdot \frac{2D-1}{4D^2-1} \sin x + e^{-x} \cdot \frac{2D+1}{4D^2-1} \sin x\right]$$

$$= \frac{1}{2}\left[-\frac{e^x}{5}(2\cos x - \sin x) - \frac{e^{-x}}{5}(2\cos x + \sin x)\right]$$

$$= \frac{1}{5}\left[-2\cos x\left(\frac{e^x + e^{-x}}{2}\right) + \sin x\left(\frac{e^x - e^{-x}}{2}\right)\right]$$

$$= \frac{1}{5}[-2\cos x \cos hx + \sin x \sin hx].$$

Example 16 : $(D^2 - 3D + 2)\,y = 2e^x \sin\left(\frac{x}{2}\right)$

Sol. : $(D-2)(D-1) = 0 \quad \therefore$ C.F. $y = c_1 e^x + c_2 e^{2x}$.

$$\therefore \text{ P.I.} = \frac{1}{D^2 - 3D + 2} 2 \cdot e^x \sin\left(\frac{x}{2}\right)$$

$$= 2 \cdot e^x \frac{1}{D^2 - D} \sin\left(\frac{x}{2}\right) = 2e^x \frac{1}{-(1/4) - D} \sin\left(\frac{x}{2}\right)$$

$$= -2e^x \cdot \frac{4}{4D+1} \cdot \frac{4D-1}{4D-1} \sin\left(\frac{x}{2}\right)$$

Example 17 : $(D^4 - 1)\,y = \cos x \cos hx$.

Sol. : $(D^2 - 1)(D^2 + 1) = 0$

$$\therefore \text{ C.F. } y = c_1 e^x + c_2 e^{-x} + (c_3 \cos x + c_4 \sin x)$$

$$\therefore \text{ P.I.} = \frac{1}{D^4 - 1} \cos x \left(\frac{e^x + e^{-x}}{2}\right)$$

$$= \frac{1}{2} \cdot \frac{1}{D^4 - 1} e^x \cos x + \frac{1}{2} \cdot \frac{1}{D^4 - 1} e^{-x} \cdot \cos x$$

$$= \frac{e^x}{2} \cdot \frac{1}{(D+1)^4 - 1} \cos x + \frac{e^{-x}}{2} \cdot \frac{1}{(D-1)^4 - 1} \cos x$$

$$= \frac{e^x}{2} \cdot \frac{1}{(D^4 + 4D^3 + 6D^2 + 4D + 1) - 1} \cos x$$

$$+ \frac{e^{-x}}{2} \cdot \frac{1}{D^4 - 4D^3 + 6D^2 - 4D + 1 - 1} \cos x$$

$$\therefore \text{ P.I.} = \frac{e^x}{2} \cdot \frac{1}{-5} \cos x + \frac{e^{-x}}{2} \cdot \frac{1}{-5} \cos x.$$

Exercise X (Page 3-42)

Solve the following differential equations : Class (b) : 6 Marks

Example 1 : $(D^2 - 2D + 1)\,y = x \sin x$

Sol. : $(D-1)^2 = 0 \quad \therefore$ C.F. $y = c_1 e^x (c_1 + c_2)$.

$$\therefore \text{ P.I.} = \frac{1}{D^2 - 2D + 1} x \sin x$$

$$= \left[x - \frac{1}{D^2 - 2D + 1}(2D - 2)\right] \cdot \frac{1}{D^2 - 2D + 1} \sin x$$

$$= \left[x - \frac{1}{D^2 - 2D + 1} 2(D-1)\right] \cdot \left(-\frac{1}{2D} \sin x\right)$$

$$= \left[x - \frac{1}{D^2 - 2D + 1} 2(D-1)\right] \cdot \frac{\cos x}{2}$$

$$= \left[\frac{x \cos x}{2} - \frac{1}{D^2 - 2D + 1}(-\sin x - \cos x)\right]$$

$$= \frac{x}{2} \cos x + \frac{1}{D^2 - 2D + 1} \sin x + \frac{1}{D^2 - 2D + 1} \cos x.$$

Example 2 : $(D^2 + 2D + 1)\,y = x \cos x$

Sol. : $(D+1)^2 = 0 \quad \therefore$ C.F. $y = e^{-x}(c_1 + c_2 x)$.

$$\therefore \text{ P.I.} = \frac{1}{D^2 + 2D + 1} x \cos x$$

$$= \left[x - \frac{1}{D^2 + 2D + 1}(2D + 2)\right] \cdot \frac{1}{D^2 + 2D + 1} \cos x$$

$$= \left[x - \frac{1}{D^2+2D+1} 2(D+1)\right] \cdot \frac{1}{2D} \cos x$$

$$= \left[x - \frac{1}{D^2-2D+1} 2(D+1)\right] \cdot \frac{\sin x}{2}$$

$$= \frac{x \sin x}{2} - \frac{1}{D^2+2D+1} (\cos x + \sin x).$$

Example 3 : $(D^2+1)\,y = x^2 \sin 2x$

Sol. : $D = \pm i$. $\therefore$ C.F. $y = c_1 \cos x + c_2 \sin x$.

$$\therefore \text{ P.I.} = \frac{1}{D^2+1} x^2 \sin 2x = \text{I.P.} \left[\frac{1}{D^2+1} x^2 e^{2ix}\right]$$

$$= \text{I.P.} e^{2ix} \cdot \frac{1}{(D+2i)^2+1} x^2 = \text{I.P.} e^{2ix} \cdot \frac{1}{D^2+4iD-4+1} x^2$$

$$= \text{I.P.} \cdot e^{2ix} \frac{1}{D^2+4D-3} x^2 = \text{I.P.} \frac{e^{2ix}}{-3} \cdot \left[1 - \frac{4iD+D^2}{3}\right]^{-1} x^2$$

$$= \text{I.P.} \; -\frac{e^{2ix}}{3}\left[1 + \frac{4iD}{3} - \frac{13}{9} D^2\right] x^2$$

$$= \text{I.P.} \; -\frac{e^{2ix}}{3}\left[x^2 + \frac{8ix}{3} - \frac{26}{9}\right]$$

$$= \text{I.P.} \; -\frac{1}{3}(\cos 2x + i \sin 2x)\left[\left(\frac{9x^2-26}{9}\right) + \frac{8ix}{3}\right]$$

Example 4 : $(D^2-1)\,y = x^2 \cos 2x$

Sol. : $D^2 - 1 = 0$ $\therefore$ C.F. $y = c_1 e^x + c_2 e^{-x}$.

$$\therefore \text{ P.I.} = \frac{1}{D^2-1} x^2 \cos x = \text{R.P.} \frac{1}{D^2-1} x^2 e^{ix}$$

$$= \text{R.P. } e^{ix} \cdot \frac{1}{(D+i)^2-1} x^2 = \text{R.P. } e^{ix} \cdot \frac{1}{D^2+2iD-2} x^2$$

$$= \text{R.P.} e^{ix} \cdot \frac{1}{-2\left[1 - \frac{2iD+D^2}{2}\right]} x^2$$

$$= \text{R.P.} \frac{e^{ix}}{-2}\left[1 + \frac{2iD+D^2}{2} - D^2\right] x^2$$

$$= \text{R.P.} -\frac{1}{2} e^{ix}\left[x^2 + 2ix - 1\right]$$

Example 5 : $(D^2-1)\,y = x \sin x + (1+x^2)\,e^x$

Sol. : $D^2 - 1 = 0$ $\therefore$ C.F. $y = c_1 e^x + c_2 e^{-x}$.

Now, $$\frac{1}{D^2-1} x \sin x = \left[x - \frac{1}{D^2-1} \cdot 2D\right] \cdot \frac{1}{D^2-1} \sin x$$

$$= \left[x - \frac{1}{D^2-1} 2D\right]\left(-\frac{1}{2}\right) \sin x$$

$$= \left[-\frac{x}{2} \sin x - \frac{1}{D^2-1}(-\cos x)\right]$$

$$= \left[-\frac{x}{2} \sin x - \frac{1}{2} \cos x\right]$$

And $$\frac{1}{D^2-1}(1+x^2)\,e^x = e^x \cdot \frac{1}{(D+1)^2-1} \cdot (1+x^2)$$

$$= e^x \cdot \frac{1}{D^2+2D}(x^2+1) = e^x \cdot \frac{1}{2D}\left[1 + \frac{D}{2}\right]^{-1} (x^2+1)$$

$$= e^x \cdot \frac{1}{2D}\left[1 - \frac{D}{2} + \frac{D^2}{4}\right](x^2+1)$$

$$= e^x \cdot \frac{1}{2D}\left[x^2 + 1 - x + \frac{1}{2}\right] = \frac{e^x}{2}\left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{3}{2}x\right).$$

Example 6 : $(D^4 + 2D^2 + 1)\,y = x^2 \sin x$.

Sol. : See solved Ex. 6, page 3-40 of the Text Book.

P.I. = Imaginary part.

Exercise XI (Page 3-50)

Solve the following differential equations : Class (b) : 6 Marks

Example 1 : $(D^2 + a^2)\,y = \operatorname{cosec} ax$

Sol. : $D^2 + a^2 = 0$ $\therefore$ C.F. $y = c_1 \cos ax + c_2 \sin ax$.

$$\therefore \text{ P.I.} = \frac{1}{D^2+a^2} \operatorname{cosec} ax = \frac{1}{2ai} \cdot \left[\frac{1}{D-ai} - \frac{1}{D+ai}\right] \operatorname{cosec} x$$

Now, $$\frac{1}{D-ai} \operatorname{cosec} x = e^{aix} \int e^{-aix} \operatorname{cosec} ax$$

$$= e^{aix} \int \operatorname{cosec} ax (\cos ax - i \sin ax)\, dx$$

$$= e^{aix} \int (\cot ax - i)\, dx = e^{aix}\left[\frac{\log \sin ax}{a} - ix\right]$$

And $$\frac{1}{D+ai} \operatorname{cosec} ax = e^{-iax}\left[\frac{\log \sin ax}{a} + ix\right]$$

Example 2 : $(D^2 + 2D + 1)\,y = 4e^{-x} \log x$

Sol. : $D^2 + 2D + 1 = 0$ $\therefore$ C.F. $y = (c_1 + c_2 x)\,e^{-x}$.

$$\therefore \text{ P.I.} = \frac{1}{(D+1)(D+1)} 4e^{-x}\log x = \frac{1}{D+1} e^{-x}\int 4e^{x}\cdot e^{-x}\log x\,dx$$

$$= \frac{4e^{-x}}{D+1}\int \log x\,dx = \frac{4e^{-x}}{D+1}[x\log x - x]$$

$$= 4e^{-x}\int e^{x}\cdot e^{-x}(x\log x - x)\,dx = 4e^{-x}\int (x\log x - x)\,dx$$

$$= 4e^{-x}\left[\log x\cdot\frac{x^2}{2} - \int \frac{x^2}{2}\cdot\frac{1}{x}\,dx - \int x\,dx\right]$$

$$= 4e^{-x}\left[\frac{x^2}{2}\log x - \int \frac{x}{2}\,dx - \int x\,dx\right]$$

$$= 4e^{-x}\left[\frac{x^2}{2}\log x - \frac{x^2}{4} - \frac{x^2}{2}\right].$$

Example 3 : $(D^2 - 3D + 2)\,y = \dfrac{1}{e^{e-x}} + \cos\left(\dfrac{1}{e^{x}}\right)$

Sol. : $(D-2)(D-1) = 0 \quad \therefore$ C.F. $y = c_1 e^{x} + c_2 e^{2x}$.

Now, $\dfrac{1}{(D-2)(D-1)}\cdot\dfrac{1}{e^{e}}\cdot e^{x} = \dfrac{1}{e^{e}}(-1)\cdot x\,e^{-x} = \dfrac{-x\cdot 1}{e^{e-x}}$

And $\dfrac{1}{D-2}\cdot e^{x}\int e^{-x}\cos(e^{-x})\,dx \quad [\text{Put } e^{-x} = t]$

$$= -\frac{1}{D-2}\cdot e^{x}\sin e^{-x} = -e^{2x}\int e^{-2x}e^{x}\sin e^{-x}\,dx$$

$$= -e^{2x}\int e^{-x}\sin(e^{-x})\,dx = -e^{2x}\cos e^{-x} \quad [\text{Put } e^{-x} = t]$$

Example 4 : $(D^2 + 6D + 9)\,y = \dfrac{e^{-3x}}{1+x^2}$

Sol. : $(D+3)^2 = 0 \quad \therefore$ C.F. $y = (c_1 + c_2 x)e^{-3x}$.

$$\therefore \text{ P.I.} = \frac{1}{(D+3)(D+3)}\cdot\frac{e^{-3x}}{1+x^2} = \frac{1}{D+3}e^{-3x}\int \frac{e^{3x}\cdot e^{-3x}}{1+x^2}\,dx$$

$$= \frac{1}{D+3}e^{-3x}\int \frac{dx}{1+x^2} = \frac{1}{D+3}e^{-3x}\cdot\tan^{-1}x$$

$$= e^{-3x}\int e^{3x}\cdot e^{-3x}\tan^{-1}x\,dx = e^{-3x}\int \tan^{-1}x\,dx \quad [\text{By parts}]$$

Example 5 : $(D^2 - 1)\,y = e^{-x}\sin(e^{-x}) + \cos(e^{-x})$

Sol. : $D^2 - 1 = 0 \quad \therefore$ C.F. $y = c_1 e^{x} + c_2 e^{-x}$.

$$\therefore \text{ P.I.} = \frac{1}{(D-1)(D+1)}\int (\cos e^{-x} + e^{-x}\sin e^{-x})\,dx$$

$$= \frac{1}{D-1}\cdot e^{-x}\int e^{x}(\cos e^{-x} + e^{-x}\sin e^{-x})\,dx$$

$$= \frac{1}{D-1}\cdot e^{-x}[e^{x}\cos e^{-x}] \quad \left[\because \int e^{x}[f(x) + f'(x)]\,dx = e^{x}f(x)\right]$$

$$= \frac{1}{D-1}\cos e^{-x} = e^{x}\int e^{-x}\cos e^{-x}\,dx \quad [\text{Put } e^{-x} = t]$$

$$= -e^{x}\sin e^{-x}.$$

Example 6 : $(D^2 - 1)\,y = \dfrac{2}{1+e^{x}}$

Sol. : $D^2 - 1 = 0 \quad \therefore$ C.F. $y = c_1 e^{x} + c_2 e^{-x}$.

$$\therefore \text{ P.I.} = \frac{1}{(D-1)(D+1)}\cdot\frac{2}{1+e^{x}} = \frac{2}{D-1}\cdot e^{-x}\int \frac{e^{x}}{1+e^{x}}\,dx$$

$$= \frac{2}{D-1}e^{-x}\log(1+e^{x}) = 2e^{x}\int e^{-2x}\log(1+e^{x})\,dx$$

$$= 2e^{x}\left[-\log(1+e^{x})\cdot\frac{e^{-2x}}{2} - \int -\frac{e^{-2x}}{2}\cdot\frac{e^{x}}{1+e^{x}}\,dx\right]$$

$$= -e^{-x}\log(1+e^{x}) + e^{x}\int \frac{e^{-x}}{1+e^{x}}\,dx$$

$$= -e^{-x}\log(1+e^{x}) + e^{x}\int \frac{e^{-x}\cdot e^{-x}}{e^{-x}+1}\,dx$$

Put $(e^{-x} + 1) = t \quad \therefore -e^{-x}\,dx = dt$

$$\therefore \text{ P.I.} = -e^{-x}\log(1+e^{x}) - e^{x}\int\left(\frac{t-1}{t}\right)dt$$

$$= -e^{-x}\log(1+e^{x}) - e^{x}[t - \log t]$$

Exercise XII (Page 3-62)

Solve the following differential equations by the method of variation of parameters : Class (c) : 8 Marks

Example 1 : $\dfrac{d^2y}{dx^2} + k^2 y = \tan kx$

Sol. : $D^2 + k^2 = 0 \quad \therefore D = ki,\ -ki$

$\therefore$ C.F. $y = c_1\cos kx + c_2\sin kx$, $X = \tan kx$

$\therefore$ P.I. $= uy_1 + uy_2$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos kx & \sin kx \\ -k\sin kx & k\cos kx \end{vmatrix} = k$$

$$u = -\int \frac{y_2 X}{W}\,dx = -\int \frac{\sin kx\cdot\tan kx}{k}\,dx$$

$$u = -\frac{1}{k}\int \frac{\sin^2 kx}{\cos kx}\,dx = -\frac{1}{k}\int \frac{1-\cos^2 kx}{\cos kx}\,dx$$

$$= -\frac{1}{k}\left[\int \sec kx\,dx - \int \cos kx\,dx\right]$$

$$= -\frac{1}{k}\left[\frac{1}{k}\log(\sec kx + \tan kx) - \frac{\sin kx}{k}\right]$$

$$v = \int \frac{y_1 X}{w}\,dx = \int \frac{\cos kx \cdot \tan kx}{k}$$

$$= \frac{1}{k}\int \sin kx\,dx = -\frac{1}{k^2}\cos kx.$$

Example 2 : $\dfrac{d^2y}{dx^2} + 4y = \tan 2x$

Sol. : Put $k = 2$ in the above example and write all the steps.

Example 3 : $\dfrac{d^2y}{dx^2} + y = x \sin x$

Sol. : $D^2 + 1 = 0$ $\quad\therefore$ C.F. $y = c_1 \cos x + c_2 \sin x$

$\therefore$ P.I. $= uy_1 + vy_2$; $X = x \sin x$.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$\therefore\ u = -\int \frac{y_2 X}{W}\,dx = -\int x \sin^2 x\,dx = -\int x\left(\frac{1-\cos 2x}{2}\right)dx$$

$$\therefore\ u = -\frac{1}{2}\left[\int x\,dx - \int x\cos 2x\,dx\right]$$

$$= -\frac{1}{2}\left[\frac{x^2}{2} - \left\{x\frac{\sin 2x}{2} - \int \frac{\sin 2x}{2}\cdot 1\cdot dx\right\}\right]$$

$$= -\frac{1}{2}\left[\frac{x^2}{2} - \frac{x\sin 2x}{2} - \frac{\cos 2x}{4}\right]$$

$$\therefore\ v = \int \frac{y_1 X}{W}\,dx = \int x \sin x \cos x\,dx = \frac{1}{2}\int x \sin 2x\,dx$$

$$= \frac{1}{2}\left[x\left(-\frac{\cos 2x}{2}\right) - \int -\frac{\cos 2x}{2}\,dx\right]$$

$$= \frac{1}{2}\left[-\frac{x}{2}\cos 2x + \frac{\sin 2x}{4}\right] = -\frac{x}{4}\cos 2x + \frac{\sin 2x}{8}.$$

Example 4 : $(D^2 + 1)\,y = \operatorname{cosec} x$

Sol. : C.F. $y = c_1 \cos x + c_2 \sin x$; $X = \dfrac{1}{\sin x}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$\therefore\ u = -\int \frac{y_2 X}{W}\,dx = -\int \frac{\sin x}{1}\cdot\frac{1}{\sin x}\,dx = -\int dx = -x$$

$$\therefore\ v = \int \frac{y_1 X}{W}\,dx = \int \cos x\left(\frac{1}{\sin x}\right)dx = \log \sin x.$$

Example 5 : $(D^2 + 2D + 5)\,y = 4e^{-x}\tan 2x + 5e^{-x}$

Sol. : $D^2 + 2D + 5 = 0$

$\therefore$ C.F. $y = e^{-x}(c_1 \cos 2x + c_2 \sin 2x)$; $X = 4e^{-x}\tan 2x + 5e^{-x}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^{-x}\cos 2x & e^{-x}\sin 2x \\ -e^{-x}(\cos 2x + 2\sin 2x) & e^{-x}(-\sin 2x + 2\cos 2x) \end{vmatrix} = 2e^{-2x}$$

$$\therefore\ u = -\frac{1}{2}\int \frac{y_2 X}{W}\,dx = -\int \frac{e^{-x}\sin 2x\cdot(4e^{-x}\tan 2x + 5e^{-x})}{2e^{-2x}}\,dx$$

$$= -\frac{1}{2}\int (4\tan 2x \sin 2x + 5\sin 2x)\,dx$$

$$= -2\int \frac{\sin^2 2x}{\cos 2x}\,dx - \frac{5}{2}\int \sin 2x\,dx$$

$$= -2\int \left(\frac{1-\cos^2 2x}{\cos 2x}\right)dx - \frac{5}{2}\int \sin 2x\,dx$$

$$= -2\int (\sec 2x - \cos 2x)\,dx - \frac{5}{2}\int \sin 2x\,dx$$

$$\therefore\ v = \int \frac{y_1 X}{W}\,dx = \int \frac{e^{-x}\cos 2x\cdot(4e^{-x}\tan 2x + 5e^{-x})}{2e^{-2x}}\,dx$$

$$= \frac{1}{2}\int (4\sin 2x\,dx + 5\cos 2x)\,dx.$$

Example 6 : $(D^2 - 1)\,y = \operatorname{sec} h\, x$

Sol. : C.F. $y = c_1 e^x + c_2 e^{-x}$; $X = \sec hx = \dfrac{1}{\cos hx} = \dfrac{2}{e^x + e^{-x}}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2$$

$$\therefore\ u = \frac{1}{2}\int \frac{e^{-x}\cdot 2}{e^x + e^{-x}}\,dx \quad \left[\text{Put } e^{-x} = t,\ e^x = \frac{1}{t},\ -e^{-x}\,dx = dt\right]$$

$$= -\int \frac{dt}{(1/t) + t} = -\frac{1}{2}\int \frac{2t\,dt}{t^2 + 1} = -\frac{1}{2}\log(t^2 + 1) = -\frac{1}{2}\log(e^{-2x} + 1)$$

$$\therefore\ v = \int \frac{y_1 X}{W}\,dx = 2\int \frac{e^x}{e^x + e^{-x}}\,dx \quad \left[\text{Put } e^x = t,\ e^{-x} = \frac{1}{t},\ e^x\,dx = dt\right]$$

$$= 2\int \frac{dt}{t + (1/t)} = 2\int \frac{t}{t^2+1}\,dt = \log(t^2+1) = \log(e^{2x}+1).$$

Alternatively : Let $y_1 = c_1 \cos hx + c_2 \sin hx$

$y = uy_1 + vy_2$; $y_1 = \cos hx,\ y_2 = \sin hx,\ X = \sec hx.$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos hx & \sin hx \\ \sin hx & \cos hx \end{vmatrix} = 1$$

$$\therefore\ u = -\int \frac{y_2 X}{W}\,dx = -\int \sin hx \cdot \sec hx\,dx$$

$$= -\int \tan hx\,dx = -\log \cos hx$$

$$\therefore\ v = \int \frac{y_1 X}{W}\,dx = \int \cos hx \cdot \sec hx\,dx = \int dx = x.$$

Example 7 : $(D^2+1)\,y = 3x - 8\cot x$

Sol. : $D^2 + 1 = 0 \quad \therefore$ C.F. $y = c_1 \cos x + c_2 \sin x$

$y_1 = \cos x,\ y_2 = \sin x,\ X = 3x - 8\cot x.$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$\therefore\ u = -\int \frac{y_2 X}{W}\,dx = -\int (\sin x \cdot 3x - \sin x \cdot 8\cot x)\,dx$$

$$= -3\left[\int x \sin x\,dx\right] + \int 8\cos x\,dx$$

$$\therefore\ v = \int \frac{y_1 X}{W}\,dx = \int \cos x\,(3x - 8\cot x)\,dx$$

$$3\int x\cos x\,dx = 3\left[x\sin x - \int \sin x\,dx\right] = 3x\sin x + 3\cos x$$

$$8\int \frac{\cos^2 x}{\sin x}\,dx = 8\int \left(\frac{1-\sin^2 x}{\sin x}\right)dx$$

$$= 8\int \operatorname{cosec} x\,dx - 8\int \sin x\,dx$$

$$= 8\log(\operatorname{cosec} x - \cot x) + 8\cos x.$$

$y = c_1 \cos x + c_2 \sin x + 3x - 8 \sin x \log(\operatorname{cosec} x - \cot x).$ ■

Example 8 : $(D^2 + D)\,y = \dfrac{1}{1+e^x}$

Sol. : $D(D+1) = 0 \quad \therefore$ C.F. $y = c_1 + c_2 e^{-x}.$

$y_1 = 1,\ y_2 = e^{-x},\ X = \dfrac{1}{1+e^x}.$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} 1 & e^{-x} \\ 0 & -e^{-x} \end{vmatrix} = -e^{-x}$$

$$\therefore\ u = -\int \frac{y_2 X}{W}\,dx = -\int \frac{e^{-x}}{-e^{-x}} \cdot \frac{1}{1+e^x} = \int \frac{dx}{1+e^x}$$

$$= \int \frac{e^{-x}}{1+e^{-x}}\,dx = -\log(1+e^{-x})$$

$$\therefore\ v = \int \frac{y_1 X}{W}\,dx = \int \frac{1}{e^x} \cdot \frac{1}{(1+e^x)}\,dx = \int \frac{e^x}{e^x+1}\,dx = \log(1+e^x).$$ ■

Example 9 : $(D^2 - 1)\,y = e^x \sin 2x$

Sol. : $D^2 - 1 = 0 \quad \therefore$ C.F. $y = c_1 e^x + c_2 e^{-x}.$

$y_1 = e^x,\ y_2 = e^{-x},\ X = e^x \sin 2x.$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2e^0 = -2$$

$$\therefore\ u = -\int \frac{y_2 X}{W}\,dx = -\frac{1}{2}\int e^{-x} e^x \sin 2x\,dx$$

$$= -\frac{1}{2}\int \sin 2x\,dx = -\frac{1}{4}\cos 2x.$$

$$\therefore\ v = \int \frac{y_1 X}{W}\,dx = -\frac{1}{2}\int e^x \cdot e^x \sin 2x\,dx$$

$$= -\frac{1}{2}\int e^{2x} \sin 2x\,dx = -\frac{1}{2}\cdot\frac{1}{8}\left[e^{2x}(2\sin 2x - 2\cos 2x)\right]$$ ■

Example 10 : $(D^2 - 3D + 2)\,y = \dfrac{e^x}{1+e^x}$

Sol. : $(D-1)(D-2) = 0 \quad \therefore$ C.F. $y = c_1 e^x + c_2 e^{2x}$

$$y_1 = e^x,\ y_2 = e^{2x}\ ;\ X = \frac{e^x}{1+e^x}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^{3x}$$

$$\therefore\ u = -\int \frac{y_2 X}{W}\,dx = -\int \frac{e^{2x}}{e^{3x}} \cdot \frac{e^x}{1+e^x} = -\int \frac{dx}{1+e^x}$$

$$= -\int \frac{e^{-x}}{e^{-x}+1} = \log(e^{-x}+1) \qquad \text{..... (1)}$$

$$\therefore\ v = \int \frac{y_1 X}{W}\,dx = \int \frac{e^x \cdot e^x}{e^{3x}(1+e^x)}\,dx = \int \frac{1}{e^x(1+e^x)}\,dx$$

$$= \int \left(\frac{1}{e^x} - \frac{1}{1+e^x}\right)dx = -e^{-x} + \log(e^{-x}+1) \text{ as in (1).}$$ ■

Example 11 : $(D^2 - 4D + 4)\, y = \dfrac{e^{2x}}{x}$

Sol. : $(D-2)^2 = 0 \quad \therefore$ C.F. $y = (c_1 + c_2 x)\, e^{2x}$.

$$y_1 = e^{2x},\quad y_2 = x e^{2x},\quad X = \frac{e^{2x}}{x}.$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & x e^{2x} \\ 2e^{2x} & e^{2x} + 2x e^{2x} \end{vmatrix} = e^{4x}$$

$$\therefore\ u = -\int \frac{y_2 X}{W}\, dx = -\int \frac{x e^{2x}}{e^{4x}} \cdot \frac{e^{2x}}{x}\, dx = -\int dx = -x$$

$$v = \int \frac{y_1 X}{W}\, dx = \int \frac{e^{2x}}{e^{4x}} \cdot \frac{e^{2x}}{x}\, dx = \int \frac{dx}{x} = \log x$$

Example 12 : $(D^2 + 2D + 1)\, y = e^{-x} \log x$

Sol. : $(D+1)^2 = 0 \quad \therefore$ C.F. $y = c_1 e^{-x} + c_2 x e^{-x}$.

$$y_1 = e^{-x},\quad y_2 = x e^{-x},\quad X = e^{-x} \log x$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & x e^{-x} \\ -e^{-x} & e^{-x} - x e^{-x} \end{vmatrix} = e^{-2x}$$

$$\therefore\ u = -\int \frac{y_2 X}{W}\, dx = -\int \frac{e^{-x} \log x \cdot x e^{-x}}{e^{-2x}}\, dx = -\int x \log x\, dx$$

$$\therefore\ v = \int \frac{y_1 X}{W}\, dx = \int \frac{e^{-x} \cdot e^{-x} \log x}{e^{-2x}}\, dx = \int \log x\, dx$$

Example 13 : $(D^2 + 6D + 5)\, y = 16\, e^{3x}$

Sol. : $(D+5)(D+1) = 0 \quad \therefore$ C.F. $y = c_1 e^{-x} + c_2 e^{-5x}$.

$$y_1 = e^{-x},\quad y_2 = e^{-5x},\quad X = 16 e^{3x}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-5x} \\ -e^{-x} & -5e^{-5x} \end{vmatrix} = -4\, e^{-6x}$$

$$\therefore\ u = -\int \frac{y_2 X}{W}\, dx = -\int \frac{16 e^{-5x} \cdot e^{3x}}{-4 e^{-6x}} = 4 \int e^{4x}\, dx = e^{4x}$$

$$\therefore\ v = \int \frac{y_1 X}{W}\, dx = \int \frac{e^{-x}\, 16 e^{3x}}{-4 e^{-6x}}\, dx = -4 \int e^{8x} = -\frac{e^{8x}}{2}.$$

Example 14 : $(D^2 + 1)\, y = x - \cot x$

Sol. : $D^2 + 1 = 0 \quad \therefore$ C.F. $y = c_1 \cos x + c_2 \sin x$.

$$y_1 = \cos x,\quad y_2 = \sin x,\quad X = x - \cot x$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$\therefore\ u = -\int \frac{y_2 X}{W}\, dx = -\int \frac{\sin x\,(x - \cot x)}{1}\, dx$$

$$= -\int x \sin x\, dx + \int \cos x\, dx = x \cos x - \sin x + \sin x$$

$$\therefore\ v = \int \frac{y_1 X}{W} = \int \cos x\,(x - \cot x)\, dx$$

$$= \int x \cos x\, dx - \int \frac{\cos^2 x}{\sin x}\, dx = x \sin x - \int \sin x\, dx - \int \frac{1 - \sin^2 x}{\sin x}\, dx$$

$$= x \sin x + \cos x - \int \operatorname{cosec} x + \int \sin x\, dx$$

$$= x \sin x + \cos x - \log(\operatorname{cosec} x - \cot x) - \cos x.$$

Example 15 : $(D^2 + 3D + 2)\, y = \sin(e^x)$

Sol. : $(D+1)(D+2) = 0 \quad \therefore$ C.F. $y = c_1 e^{-x} + c_2 e^{-2x}$.

$$y_1 = e^{-x},\quad y_2 = e^{-2x},\quad X = \sin e^x.$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -e^{-3x}$$

$$\therefore\ u = -\int \frac{y_2 X}{W}\, dx = -\int \frac{e^{-2x} \sin e^x}{-e^{-3x}}\, dx = \int e^x \sin e^x\, dx = \cos e^x.$$

$$\therefore\ v = \int \frac{y_1 X}{W}\, dx = \int \frac{e^{-x} \sin e^x}{e^{-3x}}\, dx$$

$$= \int e^{2x} \sin e^x\, dx = \int t \sin t\, dt \qquad [\text{Put } e^x = t]$$

$$= t(-\cos t) - \int -\cos t\, dt.$$

Example 16 : $(D^2 + a^2)\, y = a^2 \sec^2 ax$

Sol. : $D^2 + a^2 = 0 \quad \therefore$ C.F. $y = c_1 \cos ax + c_2 \sin ax$.

$$y_1 = \cos ax,\quad y_2 = \sin ax,\quad X = a^2 \sec^2 ax.$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a \sin ax & a \cos ax \end{vmatrix} = a$$

$$\therefore\ u = -\int \frac{y_2 X}{W}\, dx = -\frac{1}{a} \cdot a^2 \int \sin ax \cdot \sec^2 ax\, dx$$

$$= -a \int \sin ax \cdot \frac{1}{\cos ax} \cdot \sec ax\, dx$$

$$= -a \int \sec ax \tan ax\, dx = -\sec ax$$

$$\therefore\ v = \int \frac{y_1 X}{W}\, dx = \frac{a^2}{a} \int \cos ax \cdot \sec^2 ax\, dx$$

$$= a \int \sec ax\, dx = \log(\sec ax + \tan ax)$$

Example 17 : $(D^2 - 4D + 4)\,y = e^{2x}\sec^2 x$

Sol. : $(D-2)^2 = 0 \quad \therefore$ C.F. $y = (c_1 + c_2 x)\,e^{2x}$.

$y_1 = e^{2x},\ \ y_2 = x\,e^{2x},\ \ X = e^{2x}\sec^2 x.$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & x\,e^{2x} \\ 2e^{2x} & 2x\,e^{2x} + e^{2x} \end{vmatrix} = e^{4x}$$

$$\therefore\ u = -\int \frac{y_2 X}{W}\,dx = -\int \frac{x\,e^{2x}}{e^{4x}} \cdot e^{2x}\sec^2 x\,dx$$

$$= -\int x\sec^2 x\,dx = -\left[x \cdot \tan x - \int \tan x\,dx\right]$$

$$= -x\tan x + \log sex$$

$$\therefore\ v = \int \frac{y_1 X}{W}\,dx = \int \frac{e^{2x} \cdot e^{2x}\sec^2 x}{e^{4x}}\,dx = \int \sec^2 x\,dx = \tan x.$$

Example 18 : $(D^2 + 3D + 2)\,y = \dfrac{1}{1+e^x}$

Sol. : $(D+1)(D+2) = 0 \quad \therefore$ C.F. $y = c_1\,e^{-x} + c_2\,e^{-2x}$

$\therefore\ y_1 = e^{-x},\ \ y_2 = e^{-2x},\ \ X = \dfrac{1}{1+e^x}.$

Let P.I. $= y = uy_1 + vy_2$

$$w = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -e^{-3x}$$

$$u = -\int \frac{y_2 X}{W} = -\int \frac{e^{-2x}}{-e^{-3x}} \cdot \frac{1}{(1+e^x)}\,dx$$

$$= \int \frac{e^x}{1+e^x}\,dx = \log(1+e^x)$$

$$v = \int \frac{y_1 X}{W}\,dx = \int \frac{e^{-x}}{-e^{-3x}} \cdot \frac{1}{1+e^x}$$

$$v = -\int \frac{e^{2x}}{1+e^x}\,dx \qquad \text{Put } e^x = t$$

$$\therefore\ v = -\int \frac{t}{1+t}\,dt = -\int \frac{t+1-1}{t+1}\,dt$$

$$= -\int dt + \int \frac{dt}{t+1} = -e^x + \log(1+e^x).$$

$$\therefore\ y = c_1 e^x + c_2 e^{-x} + (e^{-x} - e^{-2x})\log(1+e^x) + e^{-2x}\log(1+e^x).$$

Miscellaneous Exercise (Page 3-67)

Solve the following differential equations : Class (b) : 6 Marks

Example 1 : $(D^4 + 4)\,y = 0$

Sol. : $(D^4 + 4D^2 + 4) - (4D^2) = 0 \qquad \therefore\ (D^2+2)^2 - (2D)^2 = 0$

$\therefore\ y = e^{-x}(c_1 \cos x + c_2 \sin x) + e^{x}(c_3 \cos x + c_4 \sin x)$

Example 2 : $(D^4 - 2D^3 + 3D^2 - 2D + 1)\,y = 0$

Sol. : $(D^2 - D + 1)^2 = 0 \qquad \therefore\ D = \dfrac{1 \pm \sqrt{3}\,i}{2}$ repeated.

$$\therefore\ y = e^{x/2}\left[(c_1 + c_2 x)\cos\frac{\sqrt{3}}{2}x + (c_3 + c_4 x)\sin\frac{\sqrt{3}}{2}x\right]$$

Example 3 : $(D^4 + 2D^3 + 3D^2 + 2D + 1)\,y = 0$

Sol. : $D^4 + 2D^3 + 3D^2 + 2D + 1 = 0 \qquad \therefore\ (D^2 + D + 1)^2 = 0$

$\therefore\ D = \dfrac{-1 \pm \sqrt{3}\,i}{2}$ repeated.

$$\therefore\ y = e^{-x/2}\left[(c_1 + c_2)\cos\frac{\sqrt{3}}{2}x + (c_3 + c_4 x)\sin\frac{\sqrt{3}}{2}x\right]$$

Example 4 : $(D^2 - 4D + 4)\,y = 4\,(e^{2x} - \cos 2x)$

Sol. : $(D-2)^2 = 0 \quad \therefore$ C.F. $y = (c_1 + c_2 x)\,e^{2x}$.

Now $\dfrac{1}{(D-2)^2}\,4\,e^{2x} = 4 \cdot \dfrac{x^2}{2}\,e^{2x}$

And $\dfrac{1}{D^2 - 4D + 4}\,4\cos 2x = -\int \cos 2x = \dfrac{1}{2}\sin 2x$

Example 5 : $(D^2 + 1)\,y = \sin x \sin 2x$

Sol. : $D^2 + 1 = 0 \quad \therefore$ C.F. $y = c_1 \cos x + c_2 \sin x$.

$$\therefore\ \text{P.I.} = \frac{1}{D^2+1}\left(-\frac{1}{2}\right)[\cos 3x - \cos x]$$

$$\text{P.I.} = -\frac{1}{2} \cdot \frac{1}{D^2+1}\cos 3x + \frac{1}{2} \cdot \frac{1}{D^2+1}\cos x$$

$$= \frac{1}{16}\cos 3x + \frac{1}{2}x \cdot \frac{1}{2D}\cos x.$$

Example 6 : $(D^2 - (a+b)D + ab)\,y = e^{ax} + e^{bx}$

Sol. : $(D-a)(D-b) = 0 \quad \therefore$ C.F. $y = c_1 e^{ax} + c_2\,e^{bx}$.

$$\frac{1}{D^2 - (a+b) + ab}\,e^{ax} = \frac{1}{(D-a)(D-b)}\,e^{ax} = x \cdot \frac{1}{a-b} \cdot e^{ax}$$

$$\frac{1}{(D-a)(D-b)} \cdot e^{bx} = x \cdot \frac{1}{b-a} \cdot e^{bx}$$

$$\therefore \; y = c_1 e^{ax} + c_2 e^{bx} + \frac{x}{a-b}[e^{ax} - b^{bx}]$$

Example 7 : $(D^4 - 2D^3 + D^2)\, y = x^3$

Sol. : $D^2(D^2 - 2D + 1) = 0 \quad \therefore$ C.F. $y = c_1 + c_2 x + (c_3 + c_4 x)e^x$

$$\therefore \text{ P.I.} = \frac{1}{D^2(1 - 2D + D^2)} x^3 = \frac{1}{D^2}[1 - (2D - D^2)]^{-1} x^3$$

$$= \frac{1}{D^2}[1 + 2D + 3D^2 + 4D^3]\, x^3$$

$$= \frac{1}{D^2}[x^3 + 6x^2 + 18x + 24] \qquad \text{[Integrate twice]}$$

$$= \frac{x^5}{20} + \frac{x^4}{2} + 3x^3 + 12x^2$$

Example 8 : $(D^2 - 5D + 6)\, y = x(x + e^x)$

Sol. : $D^2 - 5D + 6 = 0 \qquad \therefore$ C.F. $y = c_1 e^{2x} + c_2 e^{3x}$.

$$\frac{1}{D^2 - 5D + 6} x^2 = \frac{1}{6}\left[1 - \frac{5D - D^2}{6}\right]^{-1} x^2 = \frac{1}{6}\left[1 + \frac{5D}{6} + \frac{19}{36}D^2\right] x^2$$

$$\frac{1}{D^2 - 5D + 6} x e^x = e^x \cdot \frac{1}{(D+1)^2 - 5(D+1) + 6} x$$

$$= e^x \cdot \frac{1}{D^2 - 3D + 2} x = e^x \cdot \frac{1}{2}\left[1 - \frac{3D - D^2}{2}\right]^{-1} x$$

$$= e^x \cdot \frac{1}{2}\left[1 + \frac{3D}{2}\right] x$$

Example 9 : $(D^3 - 3D^2 + 4D - 2)\, y = e^x + \cos x$

Sol. : $(D - 1)(D^2 - 2D + 2) = 0$

$\therefore$ C.F. $y = c_1 e^x + e^x (c_2 \cos x + c_3 \sin x)$

Now, $\dfrac{1}{(D-1)(D^2 - 2D + 2)} e^x = x \cdot e^x$

And $\dfrac{1}{D^3 - 3D^2 + 4D - 2} \cos x = \dfrac{1}{3D + 1} \cos x = \dfrac{(3D - 1)}{9D^2 - 1} \cos x$

Example 10 : $(D^2 + 1)^2 y = \sin x + \cos x$

Sol. : $(D^2 + 1)^2 = 0 \quad \therefore$ C.F. $y = (c_1 + c_2 x)(c_3 \cos x + c_4 \sin x)$

$$\therefore \text{ P.I.} = \frac{1}{D^4 + 2D^2 + 1} \sin x = \frac{x^2}{\Phi''(D^2)} \sin x$$

$$\Phi'(D) = 4D^3 + 4D + 1 \;;\quad \Phi''(D^2) = 12D^2 + 4$$

Now, $\dfrac{1}{D^4 + 2D^2 + 1} \sin x = \dfrac{x^2}{12D^2 + 4} \sin x = \dfrac{x^2}{-8} \sin x$

And $\dfrac{1}{D^4 + 2D^2 + 1} \cos x = \dfrac{x^2}{12D^2 + 4} \cos x = \dfrac{x^2}{-8} \cos x$ ■

Example 11 : $(D^2 - 2D + 1)\, y = e^x + \sin(\sqrt{3})\, x$

Sol. : $(D - 1)^2 = 0 \quad \therefore$ C.F. $y = (c_1 + c_2 x)\, e^x$.

Now, $\dfrac{1}{(D-1)^2} \cdot e^x = \dfrac{x^2}{2!} e^x$

And $\dfrac{1}{D^2 - 2D + 1} \sin\sqrt{3}\, x = \dfrac{1}{-2 - 2D} \sin\sqrt{3}\, x$

$$= -\frac{1}{2} \cdot \frac{D - 1}{(D^2 - 1)} \sin\sqrt{3}\, x$$ ■

Example 12 : $(D^2 - 2D + 1)\, y = e^x$

Sol. : $(D - 1)^2 = 0 \quad \therefore$ C.F. $y = (c_1 + c_2) e^x$ as above.

$$\therefore \text{ P.I.} = \frac{1}{(D-1)^2} e^x = \frac{x^2}{2} e^x$$ ■

Example 13 : $(D^2 + 2D + 1)\, y = \cos^2 x$

Sol. : $(D + 1)^2 = 0 \quad \therefore$ C.F. $y = (c_1 + c_2 x)\, e^{-x}$

$$\therefore \text{ P.I.} = \frac{1}{D^2 + 2D + 1} \cdot \left(\frac{1 + \cos 2x}{2}\right)$$

Now, $\dfrac{1}{2} \cdot \dfrac{1}{D^2 + 2D + 1} \cdot e^{0x} = \dfrac{1}{2}$

And, $\dfrac{1}{2} \cdot \dfrac{1}{D^2 + 2D + 1} \cos 2x = \dfrac{1}{2} \cdot \dfrac{1}{2D - 3} \cos 2x$

$$= \frac{1}{2} \cdot \frac{2D + 3}{4D^2 - 9} \cos 2x$$ ■

Example 14 : $(D^2 + D + 1)\, y = (1 + \sin x)^2$

Sol. : $D^2 + D + 1 = 0$

$\therefore$ C.F. $y = e^{-x/2}[c_1 \cos(\sqrt{3}/2)x + c_2 \sin(\sqrt{3}/2)x]$

$$\therefore \text{ P.I.} = \frac{1}{D^2 + D + 1}(1 + 2\sin x + \sin^2 x)$$

$$= \frac{1}{D^2 + D + 1}\left(\frac{3}{2} + 2\sin x - \frac{1}{2}\cos 2x\right)$$

Now, $\dfrac{1}{D^2 - D + 1} \cdot \dfrac{3}{2} e^{0x} = \dfrac{3}{2}$

And $\dfrac{1}{D^2+D+1}2\sin x = 2\cdot\dfrac{1}{D}\sin x = -2\cos x$

And $\dfrac{1}{D^2+D+1}\left(-\dfrac{1}{2}\cos 2x\right) = -\dfrac{1}{2}\cdot\dfrac{1}{D-3}\cos 2x.$

$$= -\frac{1}{2}\cdot\frac{D+3}{D^2-9}\cos 2x$$

Example 15 : $(D^2-2D+4)\,y = e^x\cos^2 x$

Sol. : $D^2-2D+4=0$ $\quad\therefore$ C.F. $y = e^x(c_1\cos\sqrt{3}\,x + c_2\sin\sqrt{3}\,x)$.

$\therefore$ P.I. $= \dfrac{1}{D^2-2D+4}e^x\left(\dfrac{1+\cos 2x}{2}\right)$

Now, $\dfrac{1}{2}\cdot\dfrac{1}{D^2-2D+4}\cdot e^x = \dfrac{1}{2}\cdot\dfrac{1}{3}e^x = \dfrac{1}{6}e^x.$

And $\dfrac{1}{D^2-2D+4}\cdot\dfrac{1}{2}e^x\cos 2x = \dfrac{e^x}{2}\cdot\dfrac{1}{(D+1)^2-2(D+1)+4}\cos 2x$

$$= \frac{e^x}{2}\cdot\frac{1}{D^2+3}\cos 2x.$$

Example 16 : $(D^3+3D)\,y = \cosh 2x\ \sinh 3x$

Sol. : $D(D^2+3)=0$

$\therefore$ C.F. $y = c_1 + c_2\cos\sqrt{3}\,x + c_2\sin\sqrt{3}\,x$

$\therefore$ P.I. $= \dfrac{1}{D(D^2+3)}\cdot\left(\dfrac{e^{2x}+e^{-2x}}{2}\right)\left(\dfrac{e^{3x}-e^{-3x}}{2}\right)$

$$= \frac{1}{4}\cdot\frac{1}{D^3+3D}\cdot(e^{5x}-e^{-x}+e^{x}-e^{-5x})$$

Example 17 : $(D^4-1)\,y = e^x\cos x$

Sol. : $(D^2-1)(D^2+1)=0$

$\therefore$ C.F. $y = c_1e^x + c_2e^{-x} + c_3\cos x + c_4\sin x$

$\therefore$ P.I. $= \dfrac{1}{D^4-1}e^x\cos x = e^x\cdot\dfrac{1}{(D+1)^4-1}\cos x$

$$= e^x\cdot\frac{1}{1-4D-6+4D}\cos x = -\frac{e^x}{5}\cos x.$$

Example 18 : $(D^2+D-6)\,y = e^{2x}\sin 3x$

Sol. : $(D+3)(D-2)=0 \therefore$ C.F. $y = c_1e^{2x} + c_2e^{-3x}$.

Now, $\dfrac{1}{D^2+D-6}e^{2x}\sin 3x = e^{2x}\cdot\dfrac{1}{(D+2)^2+(D+2)-6}\sin 3x$

$$= e^{2x}\cdot\frac{1}{5D-9}\sin 3x = e^{2x}\cdot\frac{5D+9}{25D^2-81}\sin 3x$$

Example 19 : $(D^2+3D+2)\,y = e^{2x}\sin x$

Sol. : $(D+1)(D+2)=0$ $\quad\therefore$ C.F. $y = c_1e^{-x} + c_2e^{-2x}$.

$\therefore$ P.I. $= \dfrac{1}{D^2+3D+2}e^{2x}\sin x = e^{2x}\cdot\dfrac{1}{(D+2)^2+3(D+2)+2}\sin x$

$$= e^{2x}\cdot\frac{1}{D^2+7D+12}\sin x$$

$$= e^{2x}\cdot\frac{1}{7D+11}\sin x = e^{2x}\,\frac{7D-11}{49D^2-121}\sin x.$$

Example 20 : $(D^3-3D-2)\,y = 540\,x^3e^{-x}$

Sol. : $(D+1)(D-2)(D+1)=0$

$\therefore$ C.F. $y = (c_1+c_2x)e^{-x} + c_3e^{2x}$.

$\therefore$ P.I. $= \dfrac{1}{D^3-3D-2}540\,x^3e^{-x} = 540\cdot e^{-x}\cdot\dfrac{1}{(D-1)^3-3(D-1)-2}x^3$

$$= 540\cdot e^{-x}\cdot\frac{1}{D^3-3D^2}x^3 = 540\cdot e^{-x}\cdot\left(-\frac{1}{3D^2}\right)\left[1-\frac{D}{3}\right]^{-1}x^3$$

$$= -180\,\frac{e^{-x}}{D^2}\left[1+\frac{D}{3}+\frac{D^2}{9}+\frac{D^3}{27}\right]x^3$$

$$= -180\,\frac{e^{-x}}{D^2}\left[x^3+x^2+\frac{2x}{3}+\frac{2}{9}\right]$$ [Now, integrate twice]

Example 21 : $(D^2+2D+2)\,y = 2e^{-x}\sin x$

Sol. : $D^2+2D+2=0$ $\quad\therefore$ C.F. $y = e^{-x}(c_1\cos x + c_2\sin x)$

$\therefore$ P.I. $= \dfrac{1}{D^2+2D+2}2e^{-x}\sin x$

$$= 2\cdot e^{-x}\cdot\frac{1}{(D-1)^2+2(D-1)+2}\sin x = 2e^{-x}\cdot\frac{1}{D^2+1}\sin x$$

$\therefore$ P.I. $= 2\cdot e^{-x}\,\dfrac{x}{2D}\sin x = e^{-x}\cdot x\int\sin x\,dx = -e^{-x}x\cos x.$

Example 22 : $(D^2+a^2)\,y = x\cos ax$

Sol. : $D^2+a^2=0$ $\quad\therefore$ C.F. $y = c_1\cos ax + c_2\sin ax$.

$\therefore$ P.I. $=$ R.P. $\dfrac{1}{D^2+a^2}(x\,e^{iax}) =$ R.P. $e^{iax}\cdot\dfrac{1}{(D+ia)^2+a^2}x$

$$= \text{R.P. } e^{iax}\cdot\frac{1}{D^2+2aiD}\cdot x = \text{R.P. } e^{iax}\cdot\frac{1}{2aiD[1+(D/2ai)]}\cdot x$$

$$= \text{R.P. } e^{iax}\cdot\frac{1}{2aiD}\left[1-\frac{D}{2ai}\right]\cdot x = \text{R.P. } e^{iax}\cdot\frac{1}{2aiD}\left[x-\frac{1}{2ai}\right]$$

$$= \text{R.P. } e^{iax} \cdot \frac{1}{2ai}\left[\frac{x^2}{2} - \frac{x}{2ai}\right]$$

Example 23 : $(D^3 - 2D^2 + 5D + 26)\, y = e^x \cos^2 2x$

Sol. : $(D+2)(D^2 - 4D + 13) = 0$

$\therefore$ C.F. $y = c_1 e^{-2x} + e^{2x}(c_2 \cos 3x + c_3 \sin 3x)$

$$\therefore \text{ P.I.} = \frac{1}{D^3 - 2D^2 + 5D + 26} \cdot e^x \left(\frac{1 + \cos 4x}{2}\right)$$

$$= \frac{1}{2} \cdot \frac{e^x}{(1 - 2 + 5 + 26)} - \frac{e^x}{2} \cdot \frac{1}{(D+1)^3 - 2(D+1)^2 + 5(D+1) + 26} \cos 4x$$

$$= \frac{e^x}{60} - \frac{e^x}{2} \cdot \frac{1}{D^3 + D^2 + 4D + 30} \cos 4x$$

$$= \frac{e^x}{60} - \frac{e^x}{2} \cdot \frac{1}{-16D - 16 + 4D + 30} \cos 4x$$

$$= \frac{e^x}{60} - \frac{e^x}{2} \cdot \frac{1}{(-12D + 14)} \cos 4x$$

$$= \frac{e^x}{60} - \frac{e^x}{2} \cdot \frac{7 + 6D}{(49 - 36D^2)} \cos 4x$$

Example 24 : $(D^2 - 1)\, y = \cos hx \cos x + a^x$

Sol. : $D^2 - 1 = 0 \therefore$ C.F. $y = c_1 e^x + c_2 e^{-x}$.

Now, $$\frac{1}{D^2 - 1} \cdot \left(\frac{e^x + e^{-x}}{2}\right) \cos x$$

$$= \frac{1}{2}\left[e^x \cdot \frac{1}{(D+1)^2 - 1} \cos x + e^{-x} \cdot \frac{1}{(D-1)^2 - 1} \cos x\right]$$

$$= \frac{1}{2}\left[e^x \cdot \frac{1}{D^2 + 2D} \cos x + e^{-x} \cdot \frac{1}{D^2 - 2D} \cos x\right]$$

$$= \frac{1}{2}\left[e^x \cdot \frac{1}{2D - 1} \cos x + e^{-x} \frac{1}{-2D - 1} \cos x\right]$$

$$= \frac{1}{2}\left[e^x \cdot \frac{(2D+1)}{4D^2 - 1} \cos x - e^{-x} \cdot \frac{(2D-1)}{4D^2 - 1} \cos x\right]$$

$$= -\frac{1}{5}\left[\cos x \left(\frac{e^x + e^{-x}}{2}\right)\right] + \frac{2}{5} \sin x \left(\frac{e^x - e^{-x}}{2}\right)$$

And $$\frac{1}{D^2 - 1} \cdot a^x = \frac{1}{D^2 - 1} \cdot e^{x \log a}$$

Example 25 : $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 5y = e^x \cos 3x$

Sol. : $(D+1)(D^2 - 2D + 5) = 0$

$\therefore$ C.F. $y = c_1 e^{-x} + e^x(c_2 \cos 2x + c_3 \sin 2x)$

$$\therefore \text{ P.I.} = \frac{1}{D^3 - D^2 + 3D + 5} e^x \cos 3x$$

$$\therefore \text{ P.I.} = e^x \cdot \frac{1}{D^3 + 2D^2 + 4D + 8} \cos 3x = -\frac{e^x}{5} \cdot \frac{1}{D+2} \cos 3x.$$

Example 26 : $\dfrac{d^3y}{dx^3} - y = (1 + e^x)^2$

Sol. : $(D-1)(D^2 + D + 1) = 0$

$\therefore$ C.F. $y = c_1 e^x + e^{-x/2}[c_2 \cos(\sqrt{3}/2)x + c_3 \sin(\sqrt{3}/2)x]$

$$\therefore \text{ P.I.} = \frac{1}{D^3 - 1}(1 + 2e^x + e^{2x})$$

Now, $\dfrac{1}{D^3 - 1} \cdot e^{0x} = -1.$ And $\dfrac{1}{(D-1)(D^2 + D + 1)} \cdot 2e^x = \dfrac{2}{3} x \cdot e^x$

And $\dfrac{1}{D^3 - 1} \cdot e^{2x} = \dfrac{1}{7} e^{2x}.$

Example 27 : $\dfrac{d^2y}{dx^2} + 2y = x^2 e^{3x} + e^x \cos 3x$

Sol. : $D^2 + 2 = 0 \therefore$ C.F. $y = c_1 \cos \sqrt{2}\, x + c_2 \sin \sqrt{2}\, x.$

Now, $$\frac{1}{D^2 + 2} x^2 e^{3x} = e^{3x} \cdot \frac{1}{11 + 6D + D^2} x^2$$

$$= e^{3x} \frac{1}{11 + 6D + D^2} x^2 = \frac{e^{3x}}{11}\left[1 + \frac{6D + D^2}{11}\right]^{-1} x^2$$

$$= \frac{e^{3x}}{11}\left[1 - \frac{6D + D^2}{11} + \frac{36}{121} D^2\right] x^2 = \frac{e^{3x}}{11}\left[x^2 - \frac{12x}{11} + \frac{50}{121}\right]$$

And $$\frac{1}{D^2 + 2} e^x \cos 3x = e^x \cdot \frac{1}{(D+1)^2 + 2} \cos 3x$$

$$= e^x \cdot \frac{1}{-9 + 2D + 3} \cos 3x = e^x \cdot \frac{1}{2D - 6} \cos 3x$$

$$= e^x \cdot \frac{1}{2} \cdot \frac{D + 3}{D^2 - 9} \cos 3x.$$

Example 28 : $(D^2 + 3D + 2)\, y = xe^{-x} \sin x$

Sol. : $(D+1)(D+2) = 0 \therefore$ C.F. $y = c_1 e^{-x} + c_2 e^{-2x}$.

$$\therefore \text{ P.I.} = e^{-x} \cdot \frac{1}{(D-1)^2 + 3(D-1) + 2} x \sin x = e^{-x} \cdot \frac{1}{D^2 + D} x \sin x$$

$$= e^{-x}\left[x - \frac{1}{D^2+D}(2D+1)\right] \cdot \frac{1}{D^2+D} \sin x.$$

$$= e^{-x}\left[x - \frac{1}{D^2+D}(2D+1)\right] \cdot \frac{D+1}{D^2-1} \sin x$$

$$= \frac{e^{-x}}{-2}\left[x - \frac{1}{D^2+D}(2D+1)\right](\sin x + \cos x)$$

$$= \frac{e^{-x}}{-2}\left[x(\sin x + \cos x) - \frac{2\cos x - 2\sin x + \sin x + \cos x}{D^2+D}\right]$$

$$= \frac{e^{-x}}{-2}\left[x(\sin x + \cos x) - \frac{1}{D^2+D}(3\cos x - \sin x)\right]$$

$$= \frac{e^{-x}}{-2}\left[x(\sin x + \cos x) - \frac{D+1}{D^2+1}(3\cos x - \sin x)\right]$$

$$= \frac{e^{-x}}{-2}\left[x(\sin x + \cos x) + \frac{1}{2}(-3\sin x - \cos x + 3\cos x - \sin x)\right]$$

Alternatively

$$\text{P.I.} = e^{-x} \cdot \frac{1}{D^2+D} \sin x \cdot x = \text{I.P. of } e^{-x} \cdot \frac{1}{D^2+D} \cdot e^{ix} \cdot x$$

$$= \text{I.P. of } e^{-x} \cdot e^{ix} \frac{1}{(D+i)^2 + (D+i)} \cdot x$$

$$= \text{I.P. of } e^{-x} \cdot e^{ix} \frac{1}{D^2 + 2iD - 1 + D + i} x$$

$$= \text{I.P. of } e^{-x} \cdot e^{ix} \cdot \frac{1}{-1 + i + (2i+1)D + D^2} x$$

$$= \text{I.P. of } e^{-x} \cdot e^{ix} \cdot \frac{1}{(-1+i)}\left[1 + \left(\frac{2i+1}{-1+i}\right)D \ldots\ldots\right]^{-1} \cdot x$$

$$= \text{I.P. of } e^{-x} \cdot e^{ix} \cdot \frac{1+i}{-2}\left[1 - \frac{(2i+1)(i+1)}{2}\right] x$$

$$\text{P.I.} = \text{I.P. of } \frac{e^{-x} \cdot e^{ix}}{2}(1+i)\left[x + \frac{(-1+3i)}{2}\right]$$ ■

Example 29 : $(D^2 - 8D + 16)\, y = \dfrac{e^{4x}}{x^2}$

Sol. : $(D-4)^2 = 0 \therefore$ C.F. $y = c_1 \cos 4x + c_2 \sin 4x.$

$$\therefore \text{ P.I.} = \frac{1}{(D-4)^2} e^{4x} \cdot \frac{1}{x^2} = e^{4x} \cdot \frac{1}{(D+4-4)^2} x^{-2}$$

$$= e^{4x} \cdot \frac{1}{D^2} x^{-2} = \frac{e^{4x}}{D} \int x^{-2}\, dx$$ ■

Example 30 : $(D^2 - 6D + 9)\, y = \dfrac{e^{3x}}{x^2}$

Sol. : $(D-3)^2 = 0 \therefore$ C.F. $y = (c_1 + c_2 x)\, e^{3x}.$

$$\therefore \text{ P.I.} = e^{3x} \cdot \frac{1}{(D+3-3)^2} \cdot \frac{1}{x^2} = e^{3x} \cdot \frac{1}{D^2} x^{-2}$$ [Integrate twice] ■

Example 31 : $(D^2 - 6D + 13)\, y = 8e^{3x} \sin 4x + 2^x$

Sol. : $D^2 - 6D + 13 = 0 \quad \therefore$ C.F. $y = e^{2x}(c_1 \cos 3x + c_2 \sin 3x)$

Now, $$\frac{8}{D^2 - 6D + 13} e^{3x} \sin 4x = 8e^{3x} \cdot \frac{1}{(D+3)^2 - 6(D+3) + 13} \sin 4x$$

$$= e^{3x} \cdot \frac{8}{D^2 + 4} \sin 4x$$

And $$\frac{1}{D^2 - 6D + 13} e^{x \log 2} = \frac{e^{x \log 2}}{(\log 2)^2 - 6 \log 2 + 13}$$ ■

Example 32 : $(D^2 + 6D + 9)\, y = \dfrac{1}{x^3} e^{-3x} + 2^x$

Sol. : $(D+3)^2 = 0 \quad \therefore$ C.F. $y = (c_1 + c_2 x)\, e^{-3x}$

Now, $$\frac{1}{D^2 + 6D + 9} e^{3x} \cdot x^{-3} = e^{-3x} \cdot \frac{1}{(D-3+3)^2} \cdot \frac{1}{x^3}$$

$$= e^{-3x} \cdot \frac{1}{D^2} x^{-3}$$

And $$\frac{1}{(D+3)^2} e^{x \log 2} = \frac{1}{(\log 2 + 3)^2} \cdot 2^x.$$ ■

Example 33 : $(D^2 + 5D + 6)\, y = e^{-2x} \sin 2x + 4x^2 e^x$

Sol. : $(D+2)(D+3) = 0 \quad \therefore$ C.F. $y = c_1 e^{-2x} + c_2 e^{-3x}.$

Now, $$\frac{1}{D^2 + 5D + 6} e^{-2x} \sin 2x = e^{-2x} \cdot \frac{1}{D^2 + D} \cdot \sin 2x$$

$$= e^{-2x} \cdot \frac{1}{D-4} \sin 2x = e^{-2x} \cdot \frac{D+4}{D^2 - 16} \sin 2x$$

And $$\frac{1}{D^2 + 5D + 6} 4x^2 e^x = 4e^x \cdot \frac{1}{D^2 + 2D + 1 + 5D + 5 + 6} x^2$$

$$= \frac{4}{12} e^x \left[1 + \frac{7D + D^2}{12}\right] \cdot x^2$$

$$= \frac{4}{12} e^x \cdot \left[1 - \frac{7D + D^2}{12} + \frac{49}{144} D^2\right] \cdot x^2.$$ ■

Example 34 : $(D^2+9)y = 4[\cos(\pi/3)+x]$

Given that $y=0$ at $x=0$ and $y=2$ at $x=\pi/6$.

Sol. : $D^2+9=0$ $\therefore$ C.F. $y = c_1\cos 3x + c_2 \sin 3x$.

$$\therefore \text{ P.I.} = \frac{1}{D^2+9}4\cos(x+60^\circ) = \frac{1}{D^2+9}\left(\frac{4}{2}\cos x - \frac{4\sqrt{3}}{2}\sin x\right)$$

$$= 2\cdot\frac{1}{8}\cos x - \frac{2\sqrt{3}}{8}\sin x$$

Example 35 : $(D^2+4D+4)y = xe^{-x}\sinh x$

Sol. : $(D+2)^2=0$ $\therefore$ C.F. $y=(c_1+c_2x)e^{-2x}$

$$\therefore \text{ P.I.} = \frac{1}{(D+2)^2}\cdot e^{-x}x\left(\frac{e^x-e^{-x}}{2}\right) = \frac{1}{(D+2)^2}\cdot\frac{1}{2}\cdot(1-e^{-2x})x$$

$$= \frac{1}{2}\left[\frac{1}{D^2+4D+4}x - e^{-2x}\cdot\frac{1}{(D-2+2)^2}\cdot x\right]$$

$$= \frac{1}{2}\cdot\frac{1}{4}\left[1+\frac{4D+D^2}{4}\right]^{-1}x - e^{-2x}\cdot\frac{1}{D^2}x$$

$$= \frac{1}{8}x - \frac{1}{8} - \frac{e^{-2x}}{2}\cdot\frac{x^3}{6}.$$

Example 36 : $(D^4-1)y = \cos x \cosh x$

Sol. : $D=\pm 1, \pm i$ $\therefore$ C.F. $y = c_1e^x + c_2e^{-x} + c_3\cos x + c_4\sin x$

$$\therefore \text{ P.I.} = \frac{1}{D^4-1}\cos x\left(\frac{e^x+e^{-x}}{2}\right)$$

$$= \frac{1}{2}\left[e^x\cdot\frac{1}{D^4+4D^3+6D^2+4D}\cos x + e^{-x}\cdot\frac{1}{D^4-4D^3+6D^2-4D}\cos x\right]$$

$$= \frac{1}{2}\cdot\left[e^x\cdot\frac{\cos x}{-5} + e^{-x}\cdot\frac{\cos x}{-5}\right].$$

Example 37 : $(D^2-2D+2)y = e^x(x+\sin x)$

Sol. : $D^2-2D+2=0$ $\therefore$ C.F. $y = e^x(c_1\cos x + c_2\sin x)$

Now, $$\frac{1}{D^2-2D+2}e^x\cdot x = e^x\cdot\frac{1}{(D+1)^2-2(D+1)+2}x$$

$$= e^x[1-D^2] = e^x x.$$

And $$\frac{1}{D^2+2D+1}e^x\sin x = e^x\cdot\frac{1}{D^2+1}\sin x = e^x\frac{1}{2D}\sin x.$$

Example 38 : Find $\dfrac{1}{D^2-2D+2}e^x(x+\sin x)$

Sol. : $$\text{P.I.} = e^x\cdot\frac{1}{(D+1)^2-2(D+1)+2}[x+\sin x]$$

$$= e^x\cdot\left[\frac{1}{D^2+1}\cdot x + \frac{1}{D^2+1}\sin x\right] = e^x\cdot\left[(1+D^2)^{-1}x - \frac{x}{2}\cos x\right]$$

$$= e^x\cdot\left[(1-D^2+\ldots)x - \frac{x}{2}\cos x\right].$$

Example 39 : Find $\dfrac{1}{D^2+a^2}(\sin ax + \cos ax)$

Sol. : $$\text{P.I.} = \frac{1}{D^2+a^2}\sin ax + \frac{1}{D^2+a^2}\cos ax$$

$$= -\frac{x}{2a}\cos ax + \frac{x}{2a}\sin ax.$$

Example 40 : $(D^2-4D+4)y = \dfrac{e^{2x}}{1+x^2}$.

Sol. : $(D-2)^2=0$ $\therefore$ C.F. $y=(c_1+c_2x)e^{2x}$.

$$\therefore \text{ P.I.} = e^{2x}\cdot\frac{1}{(D+2-2)^2}\cdot\frac{1}{1+x^2} = e^{2x}\cdot\frac{1}{D^2}\cdot\frac{1}{1+x^2}$$

$$= e^{2x}\cdot\frac{1}{D}\int\frac{1}{1+x^2}dx = e^{2x}\cdot\frac{1}{D}\tan^{-1}x = e^{2x}\int\tan^{-1}x\,dx$$

$$= e^{2x}\cdot\left[\tan^{-1}x\cdot x - \int x\cdot\frac{1}{1+x^2}dx\right]$$

$$= e^{2x}\cdot\left[x\cdot\tan^{-1}x - \frac{1}{2}\int\frac{2x}{1+x^2}dx\right]$$

$$= e^{2x}\cdot\left[x\cdot\tan^{-1}x - \frac{1}{2}\log(1+x^2)\right].$$

Example 41 : $(D^2+6D+9)y = \sinh 3x$

Sol. : $(D+3)^2=0$ $\therefore$ C.F. $y=(c_1+c_2x)e^{-3x}$.

$$\therefore \text{ P.I.} = \frac{1}{(D+3)^2}\sinh 3x = \frac{1}{(D+3)^2}\left[\frac{e^{3x}+e^{-3x}}{2}\right]$$

$$= \frac{1}{2}\cdot\left[\frac{e^{3x}}{36} + \frac{x^2}{2}\cdot e^{-3x}\right].$$

Example 42 : $(D-2)^2y = 8[e^{2x}+\sin 2x + x^2]$

Sol. : $(D-2)^2=0$ $\therefore$ C.F. $y=(c_1+c_2x)e^{2x}$.

Now, $$\frac{1}{(D-2)^2}\cdot e^{2x} = \frac{x^2}{2}\cdot e^{2x}$$

$$\frac{1}{(D-2)^2}\sin 2x = \frac{1}{D^2-4D+4}\sin 2x = -\frac{1}{4D}\sin 2x$$

And $$\frac{1}{(D-2)^2}x^2 = \frac{1}{4}\left[1-\frac{4D-D^2}{4}\right]^{-1}x^2$$

$$= \frac{1}{4}\left[1+D+\frac{3D^2}{4}+\ldots\right]x^2 = \frac{1}{4}\left[x^2+2x+\frac{3}{4}\cdot 2\right]$$

Example 43 : $(D^3-7D-6)\,y = (1+x^2)\,e^{2x}$

Sol. : $(D+1)(D+2)(D-3)=0 \quad \therefore$ C.F. $y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x}$

$$\therefore \text{ P.I.} = e^{2x}\cdot\frac{1}{(D+2)^3-7(D+2)-6}\cdot(1+x^2)$$

$$= -\frac{e^{2x}}{12}\left[1-\frac{5D+6D^2+D^3}{12}\right]^{-1}(1+x)^2$$

$$= -\frac{e^{2x}}{12}\left[1+\frac{5D+6D^2}{12}+\frac{25D^2}{144}+\ldots\right](1+x^2).$$

Example 44 : $(D^2-4D+4)\,y = e^{2x}+x^3+\cos 2x.$

Sol. : $(D-2)^2=0 \quad \therefore$ C.F. $y=(c_1+c_2x)\,e^{2x}$.

Now, $$\frac{1}{(D-2)^2}\cdot e^{2x} = \frac{x^2}{2}\cdot e^{2x}$$

And $$\frac{1}{D^2-4D+4}x^3 = \frac{1}{4\left[1-\frac{4D-D^2}{4}\right]}\cdot x^3$$

$$= \frac{1}{4}\left[1+D+\frac{3D^2}{4}+\frac{D^3}{2}\ldots\ldots\right]x^3$$

And $$\frac{1}{D^2-4D+4}\cos 2x = \frac{1}{-4D}\cos 2x = -\frac{1}{4}\cdot\frac{\sin 2x}{2}.$$

Example 45 : Solve $(D^2+4)\,y = \cos x\cos 2x\cos 3x.$

Sol. : A.E. is $D^2+4=0 \quad \therefore D=2i,-2i.$

C.F. is $y = c_1\cos 2x + c_2\sin 2x.$

$$\therefore \text{ P.I.} = \frac{1}{D^2+4}\cos x\cos 2x\cos 3x$$

$$= \frac{1}{D^2+4}\cos x\left[\frac{1}{2}(\cos x+\cos 5x)\right]$$

$$\therefore \text{ P.I.} = \frac{1}{2(D^2+4)}[\cos^2 x+\cos x\cos 5x]$$

$$= \frac{1}{2(D^2+4)}\left[\frac{1+\cos 2x}{2}+\frac{1}{2}(\cos 6x+\cos 4x)\right]$$

$$= \frac{1}{4(D^2+4)}[1+\cos 2x+\cos 4x+\cos 6x]$$

$$= \frac{1}{4(D^2+4)}e^{0x}+\frac{1}{4(D^2+4)}\cos 2x$$

$$+\frac{1}{4(D^2+4)}\cos 4x+\frac{1}{4(D^2+4)}\cos 6x$$

$$= \frac{1}{16}+\frac{x}{16}\sin 2x-\frac{1}{48}\cos 4x-\frac{1}{128}\cos 6x.$$

Example 46 : Solve $(D^2-3D+2)\,y = \sin e^{-x}.$

Sol. : A.E. $\quad D^2-3D+2=0 \quad \therefore D=1, 2.$

C.F. $\quad y = c_1 e^x + c_2 e^{2x}.$

We shall use the method of variation of parameters.

Here, $y_1 = e^x,\ y_2 = e^{2x},\ X = \sin e^{-x}.$

Let P.I. $y = uy_1 + vy_2$

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^{3x}$$

$$\therefore\ u = -\int\frac{y_2 X}{W}dx = -\int\frac{e^{2x}\cdot\sin e^{-x}}{e^{3x}}dx$$

$$= -\int e^{-x}\sin e^{-x}\,dx = -\cos e^{-x}.$$

$$v = \int\frac{y_1 X}{W}dx = \int\frac{e^x\cdot\sin e^{-x}}{e^{3x}}dx$$

$$= \int e^{-2x}\cdot\sin e^{-x}\,dx = \int e^{-x}\cdot(e^{-x}\sin e^{-x})\,dx$$

Integrating by parts,

$$v = e^{-x}(\cos e^{-x}) - \int -\cos e^{-x}\cdot e^{-x}\,dx = e^{-x}\cos e^{-x} - \sin e^{-x}.$$

Alternatively the problem can also be solved by mere integration as shown below.

$$\text{P.I.} = \frac{1}{D^2-3D+2}\sin e^{-x} = \left(\frac{1}{D-2}-\frac{1}{D-1}\right)\sin e^{-x}$$

$$I_1 = e^{2x}\int e^{-2x}\sin e^{-x}\,dx \quad \text{Put } e^{-x}=t \quad \therefore\ -e^{-x}dx = dt$$

$$= e^{-2x}\int t\sin t\,dt = -e^{2x}[t(-\cos t)-\int -\cos t\,dt]$$

$$I_1 = -e^{2x}[-t\cos t + \sin t] = e^{2x}(e^{-x}\cos e^{-x} - \sin e^{-x})$$

$$I_2 = e^x \int e^{-x}\sin e^{-x}\,dx = -e^x \int \sin t\,dt = -e^x \cos e^{-x}.$$

Exercise - XIII (Page 3-71)

(A) Solve the following differential Equations : Class (a) : 3 marks

Example 1 : Find the solution of $(D^3 - 3D^2 + 3D - 1)\,y = 0$.

Sol.: $(D-1)^3 = 0 \quad \therefore D = 1, 1, 1 \quad \therefore y = (c_1 + c_2 x + c_3 x^2)\,e^x.$

Example 2 : Solve $(D^4 - 2D^3 + 3D^2 - 2D + 1)\,y = 0$.

Sol.: $(D^2 - D + 1)^2 = 0 \quad \therefore D = \dfrac{1 \pm \sqrt{3}\cdot i}{2}, \dfrac{1 \pm \sqrt{3}\cdot i}{2}$

$$y = e^{x/2}\left[(c_1 + c_2 x)\cos\frac{\sqrt{3}}{2}\cdot x + (c_3 + c_4 x)\sin\frac{\sqrt{3}}{2}\cdot x\right]$$

Example 3 : Solve $[(D-1)^4 (D^2+1)^2]\,y = 0$.

Sol.: $(D-1)^4 = 0 \quad \therefore D = 1, 1, 1, 1$

$(D^2+1)^2 = 0 \quad \therefore D = \pm i, \pm i$

$$y = (c_1 + c_2 x + c_3 x^2 + c_4 x^3)e^x + (c_5 + c_6)\cos x + (c_7 + c_8)\sin x$$

Example 4 : Solve $(D^4 - 2D^3 + D^2)\,y = 0$.

Sol.: $[D^2(D-1)^2]\,y = 0, \quad D^2 = 0 \quad \therefore D = 0, 0$

$(D-1)^2 = 0 \quad \therefore D = 1, 1$

$y = c_1 + c_2 x + (c_3 + c_4 x)\,e^x.$

Example 5 : Solve $(D^6 + 2D^4 + D^2)\,y = 0$.

Sol.: $D^2(D^4 + 2D^2 + 1) = 0, \quad D^2 = 0 \quad \therefore D = 0, 0$

$(D^2+1)^2 = 0 \quad \therefore D = \pm i, \pm i.$

$\therefore y = c_1 + c_2 x + (c_3 + c_4 x)\cos x + (c_5 + c_6 x)\sin x.$

(B) Find the Particular Integrals of : Class (a) : 3 Marks

Example 1 : $(D-4)\,y = x$.

Sol.: $$\frac{1}{D-4}x = e^{4x}\int e^{-4x}\,x\,dx$$

$$= e^{4x}\left[x\frac{e^{-4x}}{-4} + \frac{e^{-4x}}{4(-4)}\right] = -\frac{x}{4} - \frac{1}{16}$$

Example 2 : $(D^2 - 5D + 6)\,y = e^{2x}$.

Sol.: $$\frac{1}{D^2 - 5D + 6}\cdot e^{2x} = \left[\frac{1}{D-3} - \frac{1}{D-2}\right]e^{2x}$$

$$= e^{3x}\int e^{-3x}\cdot e^{2x}\,dx - e^{2x}\int e^{-2x}\cdot e^{2x}\,dx$$

$$= -e^{2x} - e^{2x}\cdot x.$$

Example 3 : $(D+2)\,y = \sin x$.

Sol.: $$\frac{1}{D+2}\sin x = e^{-2x}\int e^{2x}\sin x\,dx$$

$$= e^{-2x}\left[\frac{1}{4+1}e^{2x}(2\sin x - \cos x)\right]$$

Example 4 : $(D+1)y = \left(\log x + \dfrac{1}{x}\right)$.

Sol.: $$\frac{1}{D+1}\left(\log x + \frac{1}{x}\right) = e^{-x}\int e^x\left(\log x + \frac{1}{x}\right)dx$$

$$= e^{-x}e^x \log x = \log x$$

Example 5 : $(D+1)\,y = \left[\dfrac{1-x}{x^2}\right]$.

Sol.: $$\frac{1}{D+1}\left[-\frac{1}{x} + \frac{1}{x^2}\right] = e^{-x}\int e^x\left(-\frac{1}{x} + \frac{1}{x^2}\right)dx$$

$$= e^{-x}\cdot e^x\left(-\frac{1}{x}\right) = -\frac{1}{x}.$$

Example 6 : $(D^3 - 3D^2)\,y = e^{4x}$.

Sol.: $$\frac{1}{D^3 - 3D^2}\cdot e^{4x} = \frac{1}{4^3 - 3\cdot 4^2}\cdot e^{4x}$$

$$= \frac{1}{64 - 48}\cdot e^{4x} = \frac{1}{16}\cdot e^{4x}$$

Example 7 : $(D^3 - D^2)\,y = x$.

Sol.: $$\frac{1}{D^3 - D^2}x = \frac{1}{D^2}\cdot\frac{1}{D-1}x = \frac{1}{D^2}\cdot e^x\int e^{-x}\,x\,dx$$

$$= \frac{1}{D^2}\cdot e^x\left[x\cdot\frac{e^{-x}}{-1} - \int\frac{e^{-x}}{-1}\cdot 1\,dx\right]$$

$$= \frac{1}{D^2}\cdot e^x\left[-xe^{-x} - e^{-x}\right] = -\frac{1}{D^2}(x+1)$$

$$= -\frac{1}{D}\int(x+1)\,dx = -\int\left(\frac{x^2}{2} + x\right)dx = -\left(\frac{x^3}{6} + \frac{x^2}{2}\right)$$

Example 8 : $(D^3 - 3D^2 + 4)\, y = e^{2x}$.

Sol. : $$\text{P.I.} = \frac{1}{(D+1)(D-2)^2} \cdot e^{2x} = \frac{1}{2+1} \cdot \frac{x^2}{2} e^{2x} = \frac{1}{6} \cdot e^{2x}.$$

Example 9 : $(D+1)^3\, y = e^{-x}$.

Sol. : $$\frac{1}{(D+1)^3} \cdot e^{-x} = \frac{x^3}{3!} \cdot e^{-x}.$$

Example 10 : $\left(\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y\right) = 2e^{3x}$.

Sol. : $$\text{P.I.} = \frac{1}{D^2 + 2D + 1} \cdot 2e^{3x} = 2 \cdot \frac{1}{9+6+1} \cdot e^{3x} = \frac{1}{8} e^{3x}.$$

Example 11 : $(D^3 - D)\, y = 2 \cos h\, x$.

Sol. : $$\text{P.I.} = 2 \cdot \frac{1}{D(D-1)(D+1)} \cdot \left(\frac{e^x + e^{-x}}{2}\right)$$

$$= \frac{1}{2 \cdot 1} \cdot \frac{1}{D-1} e^x + \frac{1}{(-2)(-1)} \cdot \frac{1}{D+1} e^{-x}$$

$$= \frac{1}{2} \cdot xe^x + \frac{1}{2} \cdot xe^{-x} = x\left(\frac{e^x + e^{-x}}{2}\right) = x \cos hx.$$

Example 12 : $(D^4 - 81)\, y = \sin 3x$.

Sol. : $$\text{P.I.} = \frac{1}{(D^2 - 9)} \cdot \frac{1}{(D^2 + 9)} \sin 3x = -\frac{1}{18} \cdot \frac{1}{D^2 + 9} \sin 3x$$

$$= -\frac{1}{18} \cdot \left(-\frac{x}{9}\right) \cos 3x = \frac{x}{162} \cos 3x.$$

Example 13 : $(D^2 - D + 1)\, y = \cos x$.

Sol. : $$\text{P.I.} = \frac{1}{D^2 - D + 1} \cos x = \frac{1}{-D} \cos x = -\int \cos x = -\sin x.$$

Example 14 : $(D^4 - 16)\, y = \sin 2x$.

Sol. : $$\text{P.I.} = \frac{1}{(D^2 - 4)} \cdot \frac{1}{(D^2 + 4)} \sin 2x = -\frac{1}{8} \cdot \frac{1}{D^2 + 4} \sin 2x$$

$$= -\frac{1}{8}\left(-\frac{x}{4} \cos 2x\right) = \frac{x}{32} \cos 2x$$

Example 15 : $(D^2 - D + 1)\, y = \sin 2x$.

Sol. : $$\text{P.I.} = \frac{1}{D^2 - D + 1} \sin 2x = -\frac{1}{D+3} \sin 2x$$

$$= -\frac{D-3}{D^2 - 9} \sin 2x = -\frac{(D-3)}{-13} \sin 2x$$

$$\text{P.I.} = \frac{1}{13}(2 \cos 2x - 3 \sin 2x).$$

Example 16 : $(D-1)^2 (D^2 + 1)^2\, y = \sin^2 \frac{x}{2}$.

Sol. : $$\text{P.I.} = \frac{1}{(D-1)^2} \cdot \frac{1}{(D^2+1)^2} \cdot \frac{(1 - \cos x)}{2}$$

$$= \frac{1}{(D-1)^2} \cdot \frac{1}{(D^2+1)^2} \cdot \frac{e^{0x}}{2} - \frac{1}{(D-1)^2 (D^2+1)^2} \cdot \frac{\cos x}{2}$$

$$= \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{(D^2 - 2D + 1)(D^2 + 1)^2} \cdot \cos x$$

$$= \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{(D^2+1)^2} \cdot \frac{1}{(-2D)} \cdot \cos x$$

$$= \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{(D^2+1)^2} \int \cos x\, dx = \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{(D^2+1)^2} \sin x$$

$$= \frac{1}{2} + \frac{1}{4} \cdot \frac{x}{2(D^2+1) \cdot 2D} \sin x = \frac{1}{2} + \frac{1}{16} \cdot \frac{x}{D^2+1} \int \sin x$$

$$= \frac{1}{2} + \frac{1}{16} \cdot \frac{x}{D^2+1}(-\cos x) = \frac{1}{2} + \frac{1}{16} \cdot x(-x \sin x)$$

$$= \frac{1}{2} - \frac{x^2}{16} \sin x.$$

Example 17 : $(D^2 + 1)^2\, y = \sin x$.

Sol. : $$\text{P.I.} = \frac{1}{(D^2+1)^2} \sin x = \frac{1}{D^4 + 2D^2 + 1} \sin x = \frac{x^2}{\Phi''(D^2)} \sin x$$

$$\Phi'(D^2) = 4D^3 + 4D = 12D^2 + 4$$

$$\therefore \text{P.I.} = \frac{x^2}{12D^2 + 4} \sin x = -\frac{x^2}{8} \sin x.$$

Example 18 : $(D^3 - D)\, y = 2x + 1$.

Sol. : $$\text{P.I.} = -\frac{1}{D(1 - D^2)}(2x+1) = -\frac{1}{D}[1 + D^2 + \ldots](2x+1)$$

$$= -\frac{1}{D}[2x+1] = -(x^2 + x).$$

Example 19 : $(D^4 - 2D^3 + D^2)\, y = x^2$.

Sol. : $$\text{P.I.} = \frac{1}{D^2(D^2 - 2D + 1)} x^2 = \frac{1}{D^2}[1 - (2D - D^2)]^{-1} x^2$$

$$= \frac{1}{D^2}[1 + 2D - D^2 + 4D^2 - 4D^3 + \ldots] x^2$$

$$= \frac{1}{D^2}[1+2D+3D^2+\ldots]x^2 = \frac{1}{D^2}[x^2+4x+6]$$

$$= \frac{1}{D}\left[\frac{x^3}{3}+2x^2+6x\right] = \frac{x^4}{12}+\frac{2x^3}{3}+3x^2$$

Example 20 : $(D-1)^3 y = x e^x$.

Sol. : $$\text{P.I.} = \frac{1}{(D-1)^3}\cdot e^x x = e^x \frac{1}{(D+1-1)^3} x = e^x \cdot \frac{1}{D^3}\cdot x$$

$$= e^x \cdot \frac{1}{D^2}\int x\,dx = e^x \cdot \frac{1}{D^2}\cdot\frac{x^2}{2} = \frac{e^x}{2}\cdot\frac{1}{D}\int x^2\,dx$$

$$= \frac{e^x}{2}\cdot\frac{1}{D}\cdot\frac{x^3}{3} = \frac{e^x}{6}\int x^3\,dx = \frac{e^x}{6}\cdot\frac{x^4}{4} = \frac{e^x \cdot x^4}{24}.$$

Example 21 : $(D^2+D-6)\,y = e^{2x}\sin 3x$.

Sol. : $$\text{P.I.} = \frac{1}{D^2+D-6}\cdot e^{2x}\sin 3x = e^{2x}\cdot\frac{1}{(D+2)^2+(D+2)-6}\sin 3x$$

$$= e^{2x}\cdot\frac{1}{D^2+5D}\sin 3x = e^{2x}\cdot\frac{1}{5D-9}\sin 3x$$

$$= e^{2x}\cdot\frac{5D+9}{25D^2-81}\sin 3x = -\frac{1}{102}\cdot e^{2x}(5\cos 3x + 3\sin 3x)$$

Example 22 : $(D^2-2D+1)\,y = \dfrac{e^x}{x^2}$.

Sol. : $$\text{P.I.} = \frac{1}{(D-1)^2}\cdot\frac{e^x}{x^2} = e^x\cdot\frac{1}{(D+1-1)^2}\cdot\frac{1}{x^2}$$

$$= e^x\cdot\frac{1}{D^2}\cdot\frac{1}{x^2} = e^x\cdot\frac{1}{D}\int\left(\frac{1}{x^2}\right)dx$$

$$= e^x\cdot\frac{1}{D}\cdot\left(-\frac{1}{x}\right) = e^x\int\left(-\frac{1}{x}\right)dx = -e^x\log x.$$

Example 23 : $(D+1)\,y = e^{e^x}$.

Sol. : $$\text{P.I.} = \frac{1}{D+1}e^{e^x} = e^{-x}\int e^x e^{e^x}\,dx$$

$$= e^{-x}\int e^t\,dt \qquad \text{Put } e^x = t$$

$$= e^{-x}\cdot e^t = e^{-x}\cdot e^{e^x}.$$

***

CHAPTER 4

Cauchy's Homogeneous Linear Differential Equations

Exercise I (Page 4-17)

Solve the following differential equations : Class (c) : 8 Marks

Example 1 : $x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = x^{-1}$.

Sol. : A.E. $D^2 - 3D + 2 = 0 \quad \therefore\ (D-1)(D-2) = 0$

$\therefore$ C.F. $y = c_1 e^z + c_2 e^{2z}$.

$$\therefore\ \text{P.I.} = \frac{1}{(D-1)(D-2)}\cdot e^{-z} = \frac{1}{(-2)(-3)} = \frac{1}{6}\cdot e^{-z}$$

Example 2 : $x^3\dfrac{d^3y}{dx^3} + 6x^2\dfrac{d^2y}{dx^2} + 8x\dfrac{dy}{dx} + 2y = x^2 + 3x - 4$

Sol. : A.E. $D^3 + 3D^2 + 4D + 2 = 0 \quad \therefore\ (D+1)(D^2+2D+2) = 0$

$\therefore$ C.F. $y = c_1 e^{-z} + e^{-z}(c_2\cos z + c_3\sin z)$

$$\therefore\ \text{P.I.} = \frac{1}{30}e^{2z} + \frac{3}{10}e^z - 2$$

Example 3 : $x^4\dfrac{d^4y}{dx^4} + 2x^3\dfrac{d^3y}{dx^3} + x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = x + \log x$

Sol. : A.E. $D^4 - 4D^3 + 6D^2 - 4D + 1 = 0 \quad \therefore\ (D-1)^4 = 0$

$\therefore$ C.F. $y = (c_1 + c_2 z + c_3 z + c_4 z^2)e^z$

$$\therefore\ \text{P.I.} = \frac{1}{(D-1)^4}e^z + \frac{1}{1-4D+6D^2}z$$

$$= \frac{z^4}{4!}e^z + [1+4D+\ldots]z = \frac{z^4}{4!}e^z + z + 4.$$

Example 4 : $x^3\dfrac{d^3y}{dx^3} + x^2\dfrac{d^2y}{dx^2} = x^2 + x + 1$

Sol. : A.E. $D^3 - 2D^2 + D = 0 \quad \therefore\ D(D-1)^2 = 0$

$\therefore$ C.F. $y = c_1 + (c_2 + c_3 z)e^z$

Now, $\dfrac{1}{D(D-1)^2}e^{2z} = \dfrac{1}{2}e^{2z}$. And $\dfrac{1}{D(D-1)^2}e^{z} = \dfrac{1}{1}\cdot\dfrac{z^2}{2!}e^{z}$

And $\dfrac{1}{D(D-1)^2}\cdot e^{0z} = \dfrac{1}{1\cdot D} = \int dz = z$

Example 5 : $x^3\dfrac{d^3y}{dx^3} + 2x^2\dfrac{d^2y}{dx^2} + 2y = 10\left(x + \dfrac{1}{x}\right)$

Sol. : A.E. $D^3 - D^2 + 2 = 0 \quad \therefore\ D = -1,\ 1 \pm i$

$\therefore$ C.F. $y = c_1 e^{-z} + e^{z}(c_2 \cos z + c_3 \sin z)$

$\therefore$ P.I. $= \dfrac{10}{(D+1)(D^2-2D+2)}(e^{z} + e^{-z})$

$= 10\left[\dfrac{1}{2}\cdot\dfrac{1}{1}\cdot e^{z} + \dfrac{z}{1}\cdot\dfrac{1}{5}e^{-z}\right] = 5e^{z} + 2e^{-z}.$ ■

Example 6 : $[x^2D^2 - (2m-1)xD + (m^2+n^2)]y = n^2x^m \log x$

Sol. : A.E. $D^2 - 2mD + (m^2+n^2) = 0$

$\therefore\ D = \dfrac{2m \pm \sqrt{4m^2 - 4m^2 - 4n^2}}{2} = m \pm ni$

$\therefore$ C.F. $y = e^{mz}(c_1 \cos nz + c_2 \sin nz)$

$\therefore$ P.I. $= \dfrac{1}{D^2 - 2mD + (m^2+n^2)} n^2 e^{mz} z$

$= n^2 e^{mz}\cdot\dfrac{1}{(D+m)^2 - 2m(D+m) + m^2 + n^2} z$

$= n^2\cdot e^{mz}\cdot\dfrac{1}{D^2+n^2} z = n^2\cdot e^{mz}\dfrac{1}{n^2}\left[1 - \dfrac{n^2}{D^2} + \ldots\right] z = e^{mz}\cdot z.$ ■

Example 7 : $(x^4D^4 + 6x^3D^3 + 9x^2D^2 + 3xD + 1)y = (1 + \log x)^2$

Sol. : A.E. $D^4 + 2D^2 + 1 = 0 \quad \therefore\ (D^2+1)^2 = 0$

$\therefore$ C.F. $y = (c_1 + c_2 z)\cos z + (c_3 + c_4 z)\sin z$

$\therefore$ P.I. $= \dfrac{1}{(D^2+1)^2}(1 + 2z + z^2) = (1+D^2)^{-2}(1 + 2z + z^2)$

$= (1 - 2D^2 + 3D^4 \ldots)(z^2 + 2z + 1)$ ■

Example 8 : $\dfrac{d^2y}{dx^2} + \dfrac{1}{x}\dfrac{dy}{dx} = \dfrac{12 \log x}{x^2}$

Sol. : $(x^2D^2 + xD)y = 12 \log x \quad \therefore$ A.E. $D^2 - D + D = 0 \quad \therefore\ D = 0, 0$

$\therefore$ C.F. $y = c_1 + c_2 z.$

$\therefore$ P.I. $= \dfrac{12}{D^2} z = 12\cdot\dfrac{1}{D}\cdot\int\dfrac{z^2}{2}dz = 6\int z^3\,dz = 6\cdot\dfrac{z^3}{3} = 2z^3.$ ■

Example 9 : $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = (\log x)^2 + x\sin(\log x)$

Sol. : A.E. $D^2 + 1 = 0 \quad \therefore\ D = i, -i$

$\therefore$ C.F. $y = c_1 \cos z + c_2 \sin z.$

Now, $\dfrac{1}{1+D^2} z^2 = (1 - D^2) z^2 = z^2 - 2$

And $\dfrac{1}{D^2+1}\cdot e^{z}\sin z = e^{z}\cdot\dfrac{1}{(D+1)^2+1}\sin z$

$= e^{z}\cdot\dfrac{1}{D^2+2D+2}\sin z = e^{z}\cdot\dfrac{1}{2D+1}\sin z$

$= e^{z}\cdot\dfrac{2D-1}{4D^2-1}\sin z = e^{z}\dfrac{(2\cos z - \sin z)}{-5}.$ ■

Example 10 : $x^3\dfrac{d^2y}{dx^2} + 3x^2\cdot\dfrac{dy}{dx} + xy = \sin\log x$

Sol. : $(x^2D^2 + 3xD + 1)y = \dfrac{1}{x}\sin\log x$

$\therefore$ A.E. $D^2 + 2D + 1 = 0 \quad \therefore$ C.F. $y = (c_1 + c_2 z)e^{-z}.$

$\therefore$ P.I. $= \dfrac{1}{(D+1)^2}\cdot e^{-z}\sin z = e^{-z}\cdot\dfrac{1}{D^2}\sin z$ [Integrate twice] ■

Example 11 : $x^2\dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = e^{x}$

Sol. : A.E. $(D^2 + 3D + 2) = 0 \quad \therefore\ (D+1)(D+2) = 0$

$\therefore$ C.F. $y = c_1 e^{-z} + c_2 e^{-2z}.$

$\therefore$ P.I. $= \dfrac{1}{(D+2)(D+1)}\cdot e^{e^z} = \dfrac{1}{D+2}e^{-z}\int e^{z}\cdot e^{e^z}dz$ [Put $e^z = t$]

$= \dfrac{1}{D+2}\cdot e^{-z}\cdot e^{e^z} = e^{-2z}\cdot\int e^{2z}\cdot e^{-z}\cdot e^{e^z}dz$

$= e^{-2z}\cdot\int e^{z}\cdot e^{e^z}dz = e^{-2z}\cdot e^{e^z}.$ ■

Example 12 : $x^3\dfrac{d^3y}{dx^3} + 3x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = \sin\log x$

Sol. : A.E. $D^3 + 1 = 0 \quad \therefore\ (D-1)(D^2 - D + 1) = 0$

$\therefore$ C.F. $y = c_1 e^{-z} + e^{z/2}[c_2 \cos(\sqrt{3}/2)z + c_3 \sin(\sqrt{3}/2)z].$

$\therefore$ P.I. $= \dfrac{1}{D^3+1}\sin z = \dfrac{1}{-D+1}\sin z = \dfrac{1+D}{1-D^2}\sin z$

$= -\dfrac{1}{2}(\sin z + \cos z).$ ■

Example 13 : $(x^3D^3 + 2x^2D^2 - xD + 1)\,y = x^3 + x$

Sol. : A.E. $D^3 - D^2 - D + 1 = 0 \quad \therefore\ (D-1)(D+1)(D-1) = 0$

$\therefore$ C.F. $y = (c_1 + c_2 z)e^z + c_3 e^{-z}$

$$\therefore\ \text{P.I.} = \frac{1}{(D+1)(D-1)^3}(e^{3z} + e^z)$$

$$= \frac{1}{(D+1)(D-1)^2}e^{3z} + \frac{1}{(D+1)(D-1)^2}e^z$$

$$= \frac{1}{16}e^{3z} + \frac{z^2}{2}\cdot\frac{1}{2}\cdot e^z.$$

Example 14 : $x^2\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} - 6y = x^3 \log x$

Sol. : A.E. $(D^2 + D - 6) = 0 \quad \therefore\ (D-2)(D+3) = 0$

$\therefore$ C.F. $y = c_1 e^{2z} + c_2 e^{-3z}$

$$\therefore\ \text{P.I.} = \frac{1}{D^2 + D - 6}\cdot e^{3z}\cdot z = e^{3z}\cdot\frac{1}{(D+3)^2 + (D+3) - 6}\cdot z$$

$$= e^{3z}\cdot\frac{1}{D^2 + 7D + 6}\cdot z = e^{3z}\cdot\frac{1}{6}\left[1 - \frac{7}{6}D\right]z.$$

Example 15 : $x^2\dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} + 5y = 5 + \dfrac{4}{x}$

Sol. : A.E. $D^2 + 2D + 5 = 0 \quad \therefore\ D = -1 \pm 2i$

$\therefore$ C.F. $y = e^{-z}(c_1 \cos 2z + c_2 \sin 2z).$

$$\therefore\ \text{P.I.} = \frac{1}{D^2 + 2D + 5}\cdot 5e^{0z} + \frac{4}{(D^2 + 2D + 5)}e^{-z}$$

$$= \frac{5}{5}\cdot e^{0z} + \frac{4}{4}\cdot e^{-z}.$$

Example 16 : $\dfrac{d^2y}{dx^2} + \dfrac{1}{x}\dfrac{dy}{dx} = 8 + 4\log x$

Sol. : $(x^2D^2 + xD)\,y = 8x^2 + 4x^2 \log x \quad \therefore$ A.E. $D^2 = 0$

$\therefore\ D = 0,\ 0 \quad \therefore$ C.F. $y = c_1 + c_2 z.$

$$\therefore\ \text{P.I.} = \frac{1}{D^2}e^{2z}(8 + 4z) = e^{2z}\cdot\frac{1}{(D+2)}(8 + 4z)$$

$$= \frac{e^{2z}}{4}\left[1 + \frac{D}{2}\right]^{-2}(8 + 4z) = \frac{e^{2z}}{4}[1 - D](8 + 4z).$$

Example 17 : $x^2\dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} + y = \dfrac{1}{(1+x)^2}$

Sol. : A.E. $D^2 + 2D + 1 = 0 \quad \therefore\ (D+1)^2 = 0 \quad \therefore$ C.F. $y = (c_1 + c_2 z)\,e^{-z}$

$$\therefore\ \text{P.I.} = \frac{1}{(D+1)}\cdot\frac{1}{(D+1)}\cdot\frac{1}{(1+e^z)^2}dz = \frac{1}{D+1}e^{-z}\cdot\int\frac{e^z}{(1+e^z)^2}dz$$

$$= \frac{1}{D+1}e^{-z}\cdot\left(-\frac{1}{1+e^z}\right) = -e^{-z}\int e^z\cdot e^{-z}\cdot\frac{1}{(1+e^z)}dz$$

$$= -e^{-z}\int\frac{dz}{1+e^z} = -e^{-z}\int\frac{e^{-z}}{e^{-z}+1}dz = e^{-z}\log(e^{-z}+1).$$

Example 18 : $x^2\dfrac{d^3y}{dx^3} + 3x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + \dfrac{y}{x} = 4\log x$

Sol. : $(x^3D^3 + 3x^2D^2 + xD + 1)\,y = 4x\log x$

$\therefore$ A.E. $D^3 + 1 = 0 \quad \therefore\ (D-1)(D^2 - D + 1) = 0$

$\therefore$ C.F. $y = c_1 e^{-z} + e^{z/2}[c_2 \cos(\sqrt{3}/2)\cdot z + c_3 \sin(\sqrt{3}/2)\cdot z]$

$$\therefore\ \text{P.I.} = \frac{4}{D^2+1}\cdot e^z z = 4e^z\cdot\frac{1}{(D+1)^3+1}\cdot z$$

$$= 4e^z\cdot\frac{1}{D^3 + 3D^2 + 3D + 2}z = \frac{4}{2}e^z\left[1 - \frac{3D}{2}\right]z.$$

Example 19 : $\dfrac{d^2y}{dx^2} + \dfrac{1}{x}\dfrac{dy}{dx} = A + B\log x$

Sol. : $(x^2D^2 + xD)\,y = Ax^2 + Bx^2\log x \quad \therefore$ A.E. $D^2 = 0$

$\therefore\ D = 0,\ 0 \quad \therefore$ C.F. $y = c_1 + c_2 z.$

Now, $\dfrac{1}{D^2}Ae^{2z} = \dfrac{A}{4}e^{2z}$

And $B\cdot\dfrac{1}{D^2}e^{2z}\cdot z = B\cdot e^{2z}\cdot\dfrac{1}{(D+2)^2}\cdot z$

$$= B\cdot\frac{e^{2z}}{4}[1 - D + \ldots]z$$

Example 20 : $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + y = \dfrac{\log x \sin\log x}{x} + \dfrac{1}{x}$

Sol. : A.E. $D^2 - 4D + 1 = 0 \quad \therefore\ D = 2 \pm \sqrt{3}$

$\therefore$ C.F. $y = c_1 e^{(2+\sqrt{3})z} + c_2 e^{(2-\sqrt{3})z}$

Now, $\dfrac{1}{D^2 - 4D + 1}\cdot e^{-z} = \dfrac{1}{6}\cdot e^{-z}$

And $\dfrac{1}{D^2 - 4D + 1}\cdot e^{-z}\cdot z\sin z = e^{-z}\cdot\dfrac{1}{(D-1)^2 - 4(D-1) + 1}z\sin z$

$$= e^{-z}\cdot\frac{1}{D^2 - 6D + 6}z\sin z$$

$$= e^{-z}\left[z - \frac{1}{D^2 - 6D + 6}(2D - 6)\right]\cdot\frac{1}{D^2 - 6D + 6}\sin z$$

$$= e^{-z}\left[z - \frac{1}{D^2-6D+6}(2D-6)\right]\cdot\frac{1}{5-6D}\sin z$$

$$= e^{-z}\left[z - \frac{1}{D^2-6D+6}(2D-6)\right]\cdot\frac{5+6D}{25-36D^2}\sin z$$

$$= e^{-z}\left[z - \frac{1}{D^2-6D+6}(2D-6)\right]\cdot\frac{5\sin z+6\cos z}{61}$$

$$= e^{-z}\left[\frac{z(5\sin z+6\cos z)}{61} - \frac{1}{(D^2-6D+6)61}\right.$$

$$\left.\times 10\cos z - 12\sin z - 30\sin z - 36\cos z\right]$$

$$= e^{-z}\left[\frac{z}{61}(5\sin z+6\cos z) - \frac{1}{61}\cdot\frac{1}{D^2-6D+6}(-42\sin z - 26\cos z)\right]$$

Now, second term

$$= e^{-z}\left[\frac{1}{61}\cdot\frac{42}{5-6D}\sin z + \frac{26}{61}\cdot\frac{1}{5-6D}\cos z\right]$$

$$= e^{-z}\left[\frac{42}{61}\cdot\frac{5+6D}{25-36D^2}\sin z + \frac{26}{61}\cdot\frac{5+6D}{25-36D^2}\cos z\right]$$

$$= e^{-z}\left[\frac{42}{61}\cdot\frac{5\sin z+6\cos z}{61} + \frac{26}{61}\cdot\frac{5\cos z-6\sin z}{61}\right]$$

$$= e^{-z}\left[\frac{210\sin z+252\cos z+130\cos z-156\sin z}{3721}\right]$$

$$= e^{-z}\left[\frac{382\cos z+54\sin z}{3721}\right]$$

Example 21 : $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 2\sin\log x$

Sol. : A.E. $D^2+1=0$, $D=\pm i$ $\therefore$ C.F. $y = c_1\cos z + c_2\sin z$

$$\therefore \text{ P.I.} = \frac{1}{D^2+1}2\sin z = 2\cdot\left(-\frac{z}{2}\right)\cos z = -z\cos z.$$

Example 22 : $x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} - 4y = x^2 + 2\log x$

Sol. : A.E. $(D-4)(D+1)=0$ $\therefore$ C.F. $y = c_1e^{-z} + c_2e^{4z}$.

$$\therefore \text{ P.I.} = \frac{1}{D^2-3D-4}(e^{2z}+2z)$$

$$= \frac{1}{D^2-3D-4}\cdot e^{2z} - \frac{1}{4}\left[1-\frac{3D}{4}+\ldots\right]22.$$

Example 23 : $(x^3D^3 + x^2D^2 - 2)y = x + \dfrac{1}{x^3}$

Sol. : A.E. $D^3 - 2D^2 + D - 2 = 0$ $\therefore$ $(D^2+1)(D-2)=0$

$\therefore$ C.F. $y = c_1e^{2z} + c_2\cos z + c_3\sin z$

$$\therefore \text{ P.I.} = \frac{1}{(D^2+1)(D-2)}(e^z + e^{-3}z) = -\frac{e^z}{2} - \frac{1}{50}e^{-3z}.$$

Example 24 : $\dfrac{d^2y}{dx^2} + \dfrac{1}{x}\dfrac{dy}{dx} = 2 + 5\log x$

Sol. : A.E. $D = 0, 0$ $\therefore$ C.F. $y = c_1 + c_2 z$.

$$\therefore \text{ P.I.} = \frac{1}{D^2}[2e^z + 5ze^z] = 2\cdot\frac{1}{D^2}e^z + 5\left[z - \frac{1}{D^2}\cdot 2D\right]\cdot\frac{1}{D^2}e^z$$

$$= 2\cdot e^z + 5\left[z - \frac{2}{D}\right]e^z = 2e^z + 5[ze^z - 2e^z].$$

Example 25 : $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = x\log x$

Sol. : A.E. $(D-1)^2 = 0$ $\therefore$ C.F. $y = (c_1 + c_2 z)e^z$.

$$\therefore \text{ P.I.} = \frac{1}{(D-1)^2}ze^z = e^z\cdot\frac{1}{D^2}\cdot z$$

$$= e^z\cdot\frac{1}{D}\int z\,dz = e^z\cdot\frac{1}{D}\cdot\frac{z^2}{2} = e^z\cdot\int\frac{z^2}{2}dz.$$

Example 26 : $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - 3y = x^2\log x$

Sol. : A.E. $(D-3)(D+1)=0$ $\therefore$ C.F. $y = c_1e^{-z} + c_2e^{3z}$.

$$\therefore \text{ P.I.} = \frac{1}{D^2-2D-3}z\cdot e^{2z} = e^{2z}\cdot\frac{1}{(D+2)^2-2(D+2)-3}\cdot z$$

$$= e^{2z}\cdot\frac{1}{D^2+2D-3}z = e^{2z}\cdot\left(-\frac{1}{3}\right)\left[1-\frac{2D+D^2}{z}\right]^{-1}z$$

$$= -\frac{e^{2z}}{3}\left[1-\frac{2D}{3}-\ldots\right]z.$$

Example 27 : $x^2\dfrac{d^3y}{dx^3} + 3x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 1 + \dfrac{1}{x^2}$

Sol. : $(x^3D^3 + 3x^2D^2 + xD)y = x + \dfrac{1}{x}$

$\therefore$ A.E. $D^3 = 0$ $\therefore$ C.F. $y = c_1 + c_2 z + c_3 z^2$.

$$\therefore \text{ P.I.} = \frac{1}{D^3}(e^z + e^{-z}) = e^z - e^{-z}.$$

Example 28 : $(x^3D^3 + x^2D^2 - xD + 1)\,y = x \log x$

Sol. : A.E. $(D-1)(D^2 - D - 1) = 0$

$$\therefore \text{ C.F. } y = c_1 x + \sqrt{x}\left[c_1 \cos h\left(\frac{\sqrt{5}}{2}\log x\right) + c_2 \sin h\left(\frac{\sqrt{5}}{2}\log x\right)\right]$$

$$\therefore \text{ P.I.} = \frac{1}{D^3 - 2D^2 + 1} \cdot e^z \cdot z = e^z \cdot \frac{1}{(D+1)^3 - 2(D+1)^2 + 1} \cdot z$$

$$= e^z \cdot \frac{1}{D^3 + D^2 - D} z = -\frac{e^z}{D}[1 + (D + D^2) + \ldots] z$$

$$= -\frac{e^z}{D}[z+1] = -e^z \int (1+z)\, dz.$$

Example 29 : Find the equation of the curve which satisfies the differential equation

$$4x^2 \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + y = 0$$

and crosses the x-axis at an angle of 60° at $x = 1$.

Sol. : A.E. $4D^2 - 8D + 1 = 0$

$$\therefore \text{ C.F. } \quad y = c_1 e^{[(2+\sqrt{3})/2]z} + c_2 e^{[(2-\sqrt{3})/2]z}$$

$$= c_1 x^{(2+\sqrt{3})/2} + c_2 x^{(2-\sqrt{3})/z}$$

$y = 0$ at $x = 1$ $\quad \therefore\ 0 = c_1 + c_2 \quad \therefore\ c_1 = -c_2.$

$$\frac{dy}{dx} = \tan 60^\circ = \sqrt{3} = c_1\left(\frac{2+\sqrt{3}}{2}\right) + c_2\left(\frac{2-\sqrt{3}}{2}\right) \quad \text{at } x = 1$$

$$\therefore\ \sqrt{3} = (c_1 + c_2) + \frac{\sqrt{3}}{2}(c_1 - c_2) = 0 + \frac{\sqrt{3}}{2}(c_1 - c_2)$$

$\therefore\ c_1 - c_2 = 2 \quad \therefore\ -2c_2 = 2.$

Exercise II (Page 4-23)

Solve the following differential equations : Class (c) : 8 Marks

Example 1 : $(5+2x)^2 \dfrac{d^2y}{dx^2} - 6(5+2x)\dfrac{dy}{dx} + 8y = 0$

Sol. : Put $5 + 2x = v$, $\quad \therefore\ v^2 \cdot \dfrac{d^2y}{dv^2} - 3\dfrac{dy}{dv} + 2y = 0.$

Put $v = e^z$. $\quad$ A.E. $D^2 - 4D + 2 = 0 \quad \therefore\ D = 2 \pm \sqrt{2}$

$\therefore$ C.F. $y = e^{2z}(c_1 e^{\sqrt{2}\cdot z} + c_2 e^{-2\sqrt{2}\cdot z}).$

Example 2 : $(2x-1)^3 \dfrac{d^3y}{dx^3} + (2x-1)\dfrac{dy}{dx} - 2y = 0$

Sol. : Put $2x - 1 = v$. $\quad \therefore\ 4v^3 \dfrac{d^3y}{dv^3} + v\dfrac{dy}{dv} - y = 0.$

Put $v = e^z$. A.E. $4D^3 - 12D^2 + 9D - 1 = 0$

$\therefore\ (D-1)(4D^2 - 8D + 1) = 0 \quad \therefore\ D = 1,\ 1 \pm \dfrac{\sqrt{3}}{2}.$

$\therefore$ C.F. $y = c_1 e^z + e^z (c_2 e^{(\sqrt{3}/2)z} + c_2 e^{-(\sqrt{3}/2)z}).$

Example 3 : $(1+2x)^2 \dfrac{d^2y}{dx^2} - 6(1+2x)\dfrac{dy}{dx} + 16y = 8(1+2x)^2$

Sol. : Put $2x + 1 = v$, $\quad v^2 \dfrac{d^2y}{dv^2} - 3v\dfrac{dy}{dv} + 4y = 2v^2$

Put $v = e^z$. $\quad \therefore$ A.E. $(D-2)^2 = 0 \quad \therefore\ D = 2, 2.$

$\therefore$ C.F. $y = (c_1 + c_2 z)e^{2z}.$

$$\therefore \text{ P.I.} = \frac{2}{(D-2)^2} e^{2z} = 2 \cdot \frac{z^2}{2!} e^{2z}.$$

Example 4 : $(x+2)^2 \dfrac{d^2y}{dx^2} - 4(x+2)\dfrac{dy}{dx} + 6y = x$

Sol. : Put $x + 2 = v$. $\quad v^2 \dfrac{d^2y}{dv^2} - 4v\dfrac{dy}{dv} + 6y = (v-2)$

A.E. $D^2 - 5D + 6 = 0 \quad \therefore\ (D-3)(D-2) = 0 \quad \therefore\ D = 2, 3.$

$\therefore$ C.F. $y = c_1 e^{2z} + c_2 e^{3z}.$

$$\therefore \text{ P.I.} = \frac{1}{D^2 - 5D + 6}(e^z - 2e^{0z}) = \frac{1}{2}e^z - \frac{2}{6}.$$

Example 5 : $(x+a)^2 \dfrac{d^2y}{dx^2} - 4(x+a)\dfrac{dy}{dx} + 6y = x$

Sol. : Put $x + a = v$. $\quad$ A.E. $D^2 - 5D + 6 = 0$

$\therefore\ (D-2)(D-3) = 0 \quad \therefore\ D = 2, 3.$

$\therefore$ C.F. $y = c_1 e^{2z} + c_2 e^{3z}.$

$$\therefore \text{ P.I.} = \frac{1}{D^2 - 5D + 6}(e^z - a e^{0z}) = \frac{1}{2}e^z - \frac{1}{6}a.$$

Example 6 : $(3x+2)^2 \dfrac{d^2y}{dx^2} + 5(3x+2)\dfrac{dy}{dx} - 3y = x^2 + x + 1$

Sol. : Put $3x + 2 = v$. $\quad$ A.E. $\quad (3D^2 + 2D - 1) = 0$

$\therefore\ (3D-1)(D+1) = 0 \quad \therefore$ C.F. $y = c_1 e^{z/3} + c_2 e^{-z}.$

$$\therefore \text{ P.I.} = \frac{1}{27} \cdot \frac{1}{3D^2 + 2D - 1} [e^{2z} - e^z + 7]$$

$$= \frac{1}{27}\left[\frac{1}{15} e^{2z} + \frac{1}{4} e^z - 7\right].$$

Example 7 : $(1+x)^2 \dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = 2\cos\log(1+x)$

Sol. : Put $1 + x = v$. A.E. $D^2 + 1 = 0$ $\therefore$ $D = i, -i$

$\therefore$ C.F. $y = c_1 \cos z + c_2 \sin z$.

$$\therefore \text{ P.I.} = \frac{4}{D^2+1} \cos z = 4 \cdot \frac{z}{2} \sin z$$

Example 8 : $(1+x)^2 \dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = 2\sin\log(1+x)$

Sol. : Put $1 + x = v$. A.E. $D^2 + 1 = 0$

$\therefore$ C.F. $y = c_1 \cos z + c_2 \sin z$.

$$\therefore \text{ P.I.} = \frac{2\sin z}{D^2+1} = \frac{2z}{2} \cos z$$

Example 9 : $(1+x)^2 \dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = \sin[2\log(1+x)]$

Sol. : Put $1 + x = v$. A.E. $D^2 + 1 = 0$

$\therefore$ C.F. $y = c_1 \cos z + c_2 \sin z$.

$$\therefore \text{ P.I.} = \frac{1}{D^2+1} \sin 2z = -\frac{1}{3} \sin 2z.$$

Example 10 : $(3x+1)^2 \dfrac{d^2y}{dx^2} - 3(3x+1)\dfrac{dy}{dx} - 12y = 9x$

Sol. : Put $3x + 1 = v$.

$\therefore$ A.E. $3D^2 - 6D - 4 = 0$ $\therefore$ $D = 1 \pm \sqrt{7/3}$.

$\therefore$ C.F. $y = c_1 e^{z + \sqrt{7/3} \cdot z} + c_2 e^{z - \sqrt{7/3} \cdot z}$

$$\therefore \text{ P.I.} = \frac{1}{3D^2 - 6D - 4} 3(e^z - e^{0z})$$

$$= 3\left(-\frac{1}{7} e^z + \frac{1}{4}\right) = -3\left(\frac{e^z}{7} - \frac{1}{4}\right).$$

Example 11 :

$$(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = [\log(1+x)]^4 + \cos\log(1+x)$$

Sol. : As in Ex. 9. $\therefore$ C.F. $y = c_1 \cos z + c_2 \sin z$.

$y = c_1 \cos[\log(1+x)] + c_2 \sin[\log(1+x)]$

$$\therefore \text{ P.I.} = \frac{1}{D^2+1} z^4 + \frac{1}{D^2+1} \cos z$$

$$= [1 - D^2 + D^4 \ldots] z^4 + \frac{z}{2} \sin z.$$

Example 12 : $(x+1)^2 \dfrac{d^2y}{dx^2} - 3(x+1)\dfrac{dy}{dx} + 4y = x^2$

Sol. : Put $x + 1 = v$. $\therefore$ A.E. $(D-2)^2 = 0$ $\therefore$ $D = 2, 2$

$\therefore$ C.F. $y = (c_1 + c_2 z) e^z$.

Now, $\dfrac{1}{(D-2)^2} e^{2z} = \dfrac{z^2}{2} e^{2z}$, $\dfrac{1}{(D-2)^2} e^z = \dfrac{e^z}{1}$,

$$\frac{1}{(D-2)^2} e^{0z} = \frac{1}{4}$$

$$\therefore \text{ P.I.} = \frac{z^2}{2} e^{2z} + 2e^z + \frac{1}{4}.$$

Example 13 : $(3x+2)^2 \dfrac{d^2y}{dx^2} + 3(3x+2)\dfrac{dy}{dx} - 36y = 3x^2 + 4x + 1$

Sol. : Put $3x + 2 = v$. $\therefore$ A.E. $27(D^2 - 4) = 0$ $\therefore$ $D = 2, -2$.

$\therefore$ C.F. $y = c_1 e^{2z} + c_2 e^{-2z}$.

$$\therefore \text{ P.I.} = \frac{1}{27} \cdot \frac{1}{D^2 - 4} \cdot (e^{2z} - 1)$$

$$= \frac{1}{27} \cdot \frac{1}{(D-2)(D+2)} e^{2z} - \frac{1}{27} \cdot \frac{1}{D^2 - 4} e^{0z}$$

$$= \frac{1}{27} \cdot z \cdot \frac{1}{4} e^{2z} - \frac{1}{27} \cdot \left(-\frac{1}{4}\right).$$

Example 14 : $(2x+1)^2 \dfrac{d^2y}{dx^2} - 2(2x+1)\dfrac{dy}{dx} - 12y = x^2$

Sol. : Put $2x + 1 = v$. $\therefore$ A.E. $D^2 - 2D - 3 = 0$

$\therefore$ C.F. $y = c_1 e^{3z} + c_2 e^{-z}$.

$$\therefore \text{ P.I.} = \frac{1}{D^2 - 2D - 3} \cdot \frac{1}{16}(e^{2z} - 2e^z + 1).$$

Exercise - III (Page 4-24)

(A) Solve the following differential equations : Class (a) : 3 Marks

Example 1 : Solve $(x^2 D^2 - xD)\, y = 0$.

Sol. : Put $z = \log x$, $e^z = x$.

$\therefore [D(D-1)-D]y=0 \quad \therefore (D^2-2D)y=0$

$\therefore$ A.E. $D(D-2)=0 \quad \therefore D=0, 2$

$\therefore y=c_1+c_2e^{2z} \quad \therefore y=c_1+c_2x^2.$

Example 2 : Solve $x\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}=0.$

Sol. : $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}=0.$

$[D(D-1)+D]y=0$

$\therefore D^2y=0 \quad \therefore D=0, 0$

$\therefore y=c_1+c_2z=c_1+c_2\log x.$

Example 3 : Solve $(x^2D^2-xD+1)y=0.$

Sol. : $D(D-1)-D+1=0$

$\therefore (D-1)^2=0 \quad \therefore D=1, 1$

$\therefore y=(c_1+c_2z)e^z=(c_2+c_2\log x)x.$

Example 4 : Solve $(x^2D^2-xD+2)y=0.$

Sol. : $D(D-1)-D+2=0 \quad \therefore D^2-2D+2=0$

$\therefore D=\dfrac{2\pm\sqrt{-2}}{2}=1\pm i$

$\therefore y=e^z(c_1\cos z+c_2\sin z)$

$=x(c_1\cos\log x+c_2\sin\log x).$

Example 5 : Solve $(x^2D^2-4xD+6)y=0.$

Sol. : $D(D-1)-4D+6=0 \quad \therefore D^2-5D+6=0$

$(D-2)(D-3)=0 \quad \therefore D=2, 3$

$\therefore y=c_1e^{2z}+c_2e^{3z}=c_1x^2+c_2x^3.$

Example 6 : Solve $(1+x)^2\dfrac{d^2y}{dx^2}+(1+x)\dfrac{dy}{dx}+y=0.$

Sol. : $1+x=v \quad \therefore \dfrac{dy}{dx}=\dfrac{dv}{dx}, \dfrac{d^2y}{dx^2}=\dfrac{d^2v}{dx^2}$

$(v^2D^2+vD+1)y=0 \quad \therefore \log v=z$

$D(D-1)+D+1=0$

$\therefore D^2+1=0 \therefore D=i, -i.$

$y=c_1\cos z+c_2\sin z$

$=c_1\cos\log(1+x)+c_2\sin\log(1+x).$

Example 7 : Solve $(1+x)^2\dfrac{d^2y}{dx^2}-4(1+x)\dfrac{dy}{dx}+6=0.$

Sol. : Put $1+x=v \quad \therefore v^2\dfrac{d^2x}{dv^2}-4v\dfrac{dy}{dv}+6=0$

$D(D-1)-4D+6=0 \quad \therefore D^2-5D+6=0$

$\therefore (D-2)(D-3)=0 \quad \therefore D=2, D=3.$

$\therefore y=c_1e^{2z}+c_2e^{3z}=c_1v^2+c_2v^3$

$=c_1(1+x)^2+c_2(1+x)^3.$ ■

Example 8 : Solve $(x+3)^2\dfrac{d^2y}{dx^2}-2(x+3)\dfrac{dy}{dx}-4y=0.$

Sol. : Put $x+3=v, \quad v^2\dfrac{d^2y}{dv^2}-2v\dfrac{dy}{dv}-4=0$

$D(D-1)-2D-4=0 \quad \therefore D^2-3D-4=0$

$\therefore (D-4)(D+1)=0 \quad \therefore D=4, -1$

$\therefore y=c_1e^{4z}+c_2e^{-z}=c_1(x+3)^4+c_2(x+3)^{-1}.$ ■

(B) Find the complementary functions of the solution of the following differential equations : Class (a) : 3 Marks

Example 1 : $(x^2D^2-3xD+4)y=2x^2.$

Sol. : $D(D-1)-3D+4=0 \quad \therefore D^2-4D+4=0$

$\therefore (D-2)^2=0 \quad \therefore D=2, 2$

$\therefore y=(c_1+c_2z)e^{2z}=(c_1+c_2\log x)x^2.$ ■

Example 2 : $(x^2D^2-3xD+3)y=\sin\log x.$

Sol. : $D(D-1)-3D+3=0 \quad \therefore D^2-4D+3=0$

$\therefore D=\dfrac{4\pm 2i}{2}=2\pm i$

$\therefore y=(c_1\cos z+c_2\sin z)e^{2z}$

$=(c_1\cos\log x+c_2\sin\log x)x^2.$ ■

(C) Find the particular integrals of the following differential equations : Class (a) : 3 Marks

Example 1 : $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}+y=(\log x)^2.$

Sol. : $D(D-1)+D+1=0 \quad \therefore D^2+1=0$

$\text{P.I.}=\dfrac{1}{1+D^2}z^2=(1-D^2+D^4)z^2=z^2-2=(\log x)^2-2$ ■

Example 2 : $(x^2D^2-3xD+4)y=2x^2.$

Sol. : $D(D-1)-3D+4=0 \quad \therefore D^2-4D+4=0$

$$\text{P.I.} = \frac{1}{D^2 - 4D + 4} 2 \cdot e^{2z} = \frac{2}{(D-2)^2} \cdot e^{2z}$$

$$= 2 \cdot \frac{z^2}{2!} e^{2z} = x^2 (\log x)^2.$$

Example 3 : $(x^2 D^2 - xD + 1)\, y = 2x.$

Sol. : $D(D-1) - D + 1 = 0 \qquad \therefore\ D^2 - 2D + 1 = 0$

$$\text{P.I.} = \frac{2}{(D-1)^2} e^z = \frac{2z^2}{2!} e^z = (\log x)^2 \cdot x.$$

Example 4 : $(x^2 D^2 - xD + 1)\, y = \log x.$

Sol. : $\text{P.I.} = \dfrac{1}{(D-1)^2} \cdot z = \dfrac{1}{1 - (2D - D^2)} \cdot z$

$$= (1 + 2D \ldots.)\, z = z + 2 = \log x + 2$$

Example 5 : $(x^2 D^2 + 2x D)\, y = x^{-2}.$

Sol. : $D(D-1) + 2D = 0, \qquad D^2 + D = 0$

$$\text{P.I.} = \frac{1}{D^2 + D} \cdot e^{-2z} = \frac{1}{4-2} \cdot e^{-2z} = \frac{1}{2} \cdot e^{-2z} = \frac{1}{2x^2}$$

Example 6 : $x^2 \dfrac{d^2y}{dx^2} - 3x \dfrac{dy}{dx} + 5y = \sin \log x.$

Sol. : $D(D-1) - 3D + 5 = 0 \qquad \therefore\ D^2 - 4D + 5 = 0$

$$\therefore\ \text{P.I.} = \frac{1}{D^2 - 4D + 5} \sin z = \frac{1}{4 - 4D} \sin z$$

$$= \frac{1}{4(1-D)} \sin z = \frac{1}{4} \cdot \frac{(1+D)}{(1-D^2)} \sin z$$

$$= \frac{1}{4} \cdot \frac{(\sin z + \cos z)}{2} = \frac{1}{8} (\sin z + \cos z).$$

$$= \frac{1}{8} (\sin \log x + \cos \log x)$$

***

CHAPTER 5

Numerical Solutions of Ordinary Differential Equations

Exercise I (Page 5-8)

Using Euler's method find the approximate value of y from the following : Class (b) : 6 Marks

Example 1 : $\dfrac{dy}{dx} = y^2 - \dfrac{y}{x}$ **and** $y(1) = 1$, **taking** $h = 0{\cdot}1$, **at** $x = 1{\cdot}3, 1{\cdot}5.$

Sol. :

x_n	y_n	$f(x_n, y_n) = y_n^2 - \dfrac{y_n}{x^n}$	$y_{n+1} = y_n + h\, f(x_n, y_n)$
1	1	$1 - 1 = 0$	$1 + 0{\cdot}1\,(0) = 1$
1·1	1	$(1)^2 - \dfrac{1}{1{\cdot}1} = 0{\cdot}0909$	$1 + 0{\cdot}1(0{\cdot}0909) = 1{\cdot}00909$
1·2	1·00909	$(1{\cdot}00909)^2 - \dfrac{1{\cdot}00909}{1{\cdot}2} = 0{\cdot}1773$	$1{\cdot}00909 + 0{\cdot}1(0{\cdot}1773) = 1{\cdot}0268$
1·3	1·0268	$(1{\cdot}0268)^2 - \dfrac{1{\cdot}0268}{1{\cdot}3} = 0{\cdot}2648$	$1{\cdot}0268 + 0{\cdot}1(0{\cdot}2648) = 1{\cdot}05328$
1·4	1·05328	$(1{\cdot}05328)^2 - \dfrac{1{\cdot}05328}{1{\cdot}4} = 0{\cdot}3570$	$1{\cdot}05328 + 0{\cdot}1(0{\cdot}3570) = 1{\cdot}08898$
1·5	1·08898		

Example 2 : $\dfrac{dy}{dx} = x + y,\ y(0) = 1$, **taking** $h = 0{\cdot}2$, **at** $x = 1.$

Sol. :

x_n	y_n	$f(x_n, y_n) = x_n + y_n$	$y_{n+1} = y_n + h\, f(x_n, y_n)$
0	1	$0 + 1 = 1$	$1 + 0{\cdot}2\,(1) = 1{\cdot}2$
0·2	1·2	$0{\cdot}2 + 1{\cdot}2 = 1{\cdot}4$	$1{\cdot}2 + 0{\cdot}2\,(1{\cdot}4) = 1{\cdot}48$
0·4	1·48	$0{\cdot}4 + 1{\cdot}48 = 1{\cdot}88$	$1{\cdot}48 + 0{\cdot}2\,(1{\cdot}48) = 1{\cdot}856$
0·6	1·856	$0{\cdot}6 + 1{\cdot}856 = 2{\cdot}456$	$1{\cdot}856 + 0{\cdot}2\,(2{\cdot}456) = 2{\cdot}3472$
0·8	2·3472	$0{\cdot}8 + 2{\cdot}3472 = 3{\cdot}1472$	$2{\cdot}3472 + 0{\cdot}2\,(3{\cdot}1472) = 2{\cdot}97664$
1·0	2·97664		

Example 3 : $\frac{dy}{dx} = x + y^2$, $y(0) = 1$, taking $h = 0{\cdot}01$, at $x = 0{\cdot}05$.

Sol. :

x_n	y_n	$f(x_n, y_n) = x_n + y_n^2$	$y_{n+1} = y_n + h\, f(x_n, y_n)$
0	1	$0 + (1)^2 = 1$	$1 + 0{\cdot}01\,(1) = 1{\cdot}01$
0·01	1·01	$0{\cdot}01 + (1{\cdot}01)^2 = 1{\cdot}0301$	$1{\cdot}01 + 0{\cdot}01\,(1{\cdot}0301) = 1{\cdot}020301$
0·02	1·020301	$0{\cdot}02 + (1{\cdot}0020301)^2 = 1{\cdot}0610$	$1{\cdot}02301 + 0{\cdot}01\,(1{\cdot}0610) = 1{\cdot}03362$
0·03	1·03362	$0{\cdot}03 + (1{\cdot}03362)^2 = 1{\cdot}09834$	$1{\cdot}03362 + 0{\cdot}01\,(1{\cdot}09834) = 1{\cdot}0446$
0·04	1·0446	$0{\cdot}04 + (1{\cdot}0446)^2 = 1{\cdot}1312$	$1{\cdot}0446 + 0{\cdot}001\,(1{\cdot}1312) = 1{\cdot}0559$
0·05	1·0559		

Example 4 : $\frac{dy}{dx} = 1 - 2xy$, $y(0) = 0$, taking $h = 0{\cdot}2$, at $x = 0{\cdot}6$.

Sol. :

x_n	y_n	$f(x_n, y_n) = 1 - 2\,x_n\,y_n$	$y_{n+1} = y_n + h\, f(x_n, y_n)$
0	0	$1 - 0 = 1$	$0 + 0{\cdot}2\,(1) = 0{\cdot}2$
0·2	0·2	$1 - 2\,(0{\cdot}2)\,(0{\cdot}2) = 0{\cdot}92$	$0{\cdot}2 + 0{\cdot}2\,(0{\cdot}92) = 0{\cdot}384$
0·4	0·384	$1 - 2\,(0{\cdot}4)\,(0{\cdot}384) = 0{\cdot}6928$	$0{\cdot}384 + 0{\cdot}2\,(0{\cdot}6928) = 0{\cdot}52256$
0·6	0·52256		

Example 5 : $\frac{dy}{dx} = 2 + \sqrt{xy}$, $y(1) = 1$, taking $h = 0{\cdot}2$, at $x = 2$.

Sol. :

x_n	y_n	$f(x_n, y_n) = 2 + \sqrt{x_n\,y_n}$	$y_{n+1} = y_n + h\, f(x_n, y_n)$
1	1	$2 + \sqrt{1 \times 1} = 3$	$1 + 0{\cdot}2\,(3) = 1{\cdot}6$
1·2	1·6	$2 + \sqrt{(1{\cdot}2)(1{\cdot}6)} = 3{\cdot}3856$	$1{\cdot}6 + 0{\cdot}2\,(3{\cdot}3856) = 2{\cdot}2771$
1·4	2·2771	$2 + \sqrt{(1{\cdot}4)(2{\cdot}2771)} = 3{\cdot}7855$	$2{\cdot}2771 + 0{\cdot}2\,(3{\cdot}7855) = 3{\cdot}0342$
1·6	3·0342	$2 + \sqrt{(1{\cdot}6)(3{\cdot}0342)} = 4{\cdot}2033$	$3{\cdot}0342 + 0{\cdot}2\,(4{\cdot}2033) = 3{\cdot}87487$
1·8	3·87487	$2 + \sqrt{(1{\cdot}8)(3{\cdot}87487)} = 4{\cdot}64097$	$3{\cdot}87487 + 0{\cdot}2\,(4{\cdot}64097) = 4{\cdot}8030$
2·0	4·8030		

Example 6 : $\frac{dy}{dx} = x + \sqrt{y}$, $y(2) = 4$, taking $h = 0{\cdot}2$, at $x = 3$.

Sol. :

x_n	y_n	$f(x_n, y_n) = x + \sqrt{y}$	$y_{n+1} = y_n + h\, f(x_n, y_n)$
2	4	$2 + \sqrt{4} = 4$	$4 + 0{\cdot}2\,(4) = 4{\cdot}8$
2·2	4·8	$2{\cdot}2 + \sqrt{4{\cdot}8} = 4{\cdot}39089$	$4{\cdot}8 + 0{\cdot}2\,(4{\cdot}39089) = 5{\cdot}67818$
2·4	5·67818	$2{\cdot}4 + \sqrt{5{\cdot}67818} = 4{\cdot}78289$	$5{\cdot}67818 + 0{\cdot}2\,(4{\cdot}78289) = 6{\cdot}63476$
2·6	6·63476	$2{\cdot}6 + \sqrt{6{\cdot}63476} = 5{\cdot}1758$	$6{\cdot}63476 + 0{\cdot}2\,(5{\cdot}1758) = 7{\cdot}66992$
2·8	7·6692	$2{\cdot}8 + \sqrt{7{\cdot}6692} = 5{\cdot}5693$	$7{\cdot}6692 + 0{\cdot}2\,(5{\cdot}5693) = 8{\cdot}7831$
3·0	8·7831		

Example 7 : $\frac{dy}{dx} = x^2 + y^2$, $y(0) = 0$, taking $h = 0{\cdot}1$, at $x = 0{\cdot}5$.

Sol. :

x_n	y_n	$f(x_n, y_n) = x^2 + y^2$	$y_{n+1} = y_n + h\, f(x_n, y_n)$
0	0	$0 + 0 = 0$	$0 + 0{\cdot}1\,(0) = 0$
0·1	0	$(0{\cdot}1)^2 + 0^2 = 0{\cdot}01$	$0 + 0{\cdot}1\,(0{\cdot}01) = 0{\cdot}001$
0·2	0·001	$(0{\cdot}2)^2 + (0{\cdot}001)^2 = 0{\cdot}040001$	$0{\cdot}001 + 0{\cdot}1\,(0{\cdot}040001) = 0{\cdot}005$
0·3	0·005	$(0{\cdot}3)^2 + (0{\cdot}005)^2 = 0{\cdot}090025$	$0{\cdot}005 + 0{\cdot}1\,(0{\cdot}09005) = 0{\cdot}014$
0·4	0·014	$(0{\cdot}4)^2 + (0{\cdot}014)^2 = 0{\cdot}16019$	$0{\cdot}014 + 0{\cdot}1\,(0{\cdot}16019) = 0{\cdot}03$
0·5	0·03		

Example 8 : $\frac{dy}{dx} = xy$, $y(0) = 1$, taking $h = 0{\cdot}1$, at $x = 0{\cdot}5$.

Sol. :

x_n	y_n	$f(x_n, y_n) = x_n\,y_n$	$y_{n+1} = y_n + h\, f(x_n, y_n)$
0	1	$(0)\,(1) = 0$	$1 + 0{\cdot}1\,(0) = 1$
0·1	1	$(0{\cdot}1)\,(1) = 0{\cdot}1$	$1 + 0{\cdot}1\,(0{\cdot}1) = 1{\cdot}01$
0·2	1·01	$(0{\cdot}2)\,(1{\cdot}01) = 0{\cdot}202$	$1{\cdot}01 + 0{\cdot}1\,(0{\cdot}202) = 1{\cdot}0302$
0·3	1·0302	$(0{\cdot}3)\,(1{\cdot}0302) = 0{\cdot}30906$	$1{\cdot}0302 + 0{\cdot}1\,(0{\cdot}30906) = 1{\cdot}0611$
0·4	1·0611	$(0{\cdot}4)\,(1{\cdot}0611) = 0{\cdot}42444$	$1{\cdot}0611 + 0{\cdot}1\,(0{\cdot}42444) = 1{\cdot}1035$
0·5	1·1035		

Example 9 : $\frac{dy}{dx} = 1 - y^2$, $y(0) = 0$, taking $h = 0{\cdot}2$, at $x = 1$.

Sol. :

x_n	y_n	$f(x_n, y_n) = 1 - y_n^2$	$y_{n+1} = y_n + h\,f(x_n, y_n)$
0	0	$1 - 0 = 1$	$0 + 0{\cdot}2\,(1) = 0{\cdot}2$
0·2	0·2	$1 - (0{\cdot}2)^2 = 0{\cdot}96$	$0{\cdot}2 + 0{\cdot}2\,(0{\cdot}96) = 0{\cdot}392$
0·4	0·392	$1 - (0{\cdot}392)^2 = 0{\cdot}8463$	$0{\cdot}392 + 0{\cdot}2\,(0{\cdot}8463) = 0{\cdot}56126$
0·6	0·56126	$1 - (0{\cdot}56126)^2 = 0{\cdot}68499$	$0{\cdot}56126 + 0{\cdot}2\,(0{\cdot}68499) = 0{\cdot}69826$
0·8	0·69826	$1 - (0{\cdot}69826)^2 = 0{\cdot}5124$	$0{\cdot}69826 + 0{\cdot}2\,(0{\cdot}5124) = 0{\cdot}80074$
1·0	0·80074		

Example 10 : $\frac{dy}{dx} = x + y + xy$, $y(0) = 1$, taking $h = 0{\cdot}025$, at $x = 0{\cdot}1$.

Sol. :

x_n	y_n	$f(x_n, y_n) = x + y + xy$	$y_{n+1} = y_n + h\,f(x_n, y_n)$
0	1	$0 + 1 + 0 = 1$	$1 + 0{\cdot}025\,(1) = 1{\cdot}025$
0·025	1·025	$0{\cdot}025 + 1{\cdot}025 + (0{\cdot}025) \times (1{\cdot}025) = 1{\cdot}075625$	$1{\cdot}025 + 0{\cdot}025\,(1{\cdot}075625) = 1{\cdot}0519$
0·05	1·0519	$0{\cdot}05 + 1{\cdot}0519 + (0{\cdot}05) \times (1{\cdot}0519) = 1{\cdot}15449$	$1{\cdot}0519 + 0{\cdot}025\,(1{\cdot}15449) = 1{\cdot}0808$
0·75	1·0808	$0{\cdot}75 + 1{\cdot}0808 + (0{\cdot}75) \times (1{\cdot}0808) = 2{\cdot}6414$	$1{\cdot}0808 + 0{\cdot}025\,(2{\cdot}6414) = 1{\cdot}1468$
0·1	1·1468		

Example 11 : $\frac{dy}{dx} = -y$ with $y(0) = 1$, taking $h = 0{\cdot}01$, at $x = 0{\cdot}04$.

Compare it with the exact value.

Sol. :

x_n	y_n	$f(x_n, y_n) = -y_n$	$y_{n+1} = y_n + h\,f(x_n, y_n)$
0·00	1	$= -1$	$= 1 + 0{\cdot}01\,(-1) = 0{\cdot}99$
0·01	0·99	$= -0{\cdot}99$	$= 0{\cdot}99 + 0{\cdot}01\,(-0{\cdot}99) = 0{\cdot}9801$
0·02	0·980	$= -0{\cdot}9801$	$= 0{\cdot}9801 + 0{\cdot}01\,(-0{\cdot}9801) = 0{\cdot}97029$
0·03	0·97029	$= -0{\cdot}97029$	$= 0{\cdot}97029 + 0{\cdot}01\,(-0{\cdot}97029) = 0{\cdot}9606$
0·04	0·9606		

$$\frac{dy}{y} = -dx \quad \therefore \log y = -x + \log c \quad \therefore y = ce^{-x},\ c = 0$$

When $x = 0{\cdot}04$, Exact $y = 0{\cdot}9608$.

Example 12 : $\frac{dy}{dx} = x + y$ with $y(0) = 0$, taking $h = 0{\cdot}2$, at $x = 0{\cdot}6$.

Sol. :

x_n	y_n	$f(x_n, y_n) = x_n + y_n$	$y_{n+1} = y_n + h\,f(x_n, y_n)$
0	0	$= 0 + 0 = 0$	$= 0 + 0{\cdot}2\,(0) = 0$
0·2	0	$= 0{\cdot}2 + 0 = 0{\cdot}2$	$= 0 + 0{\cdot}2\,(0{\cdot}2) = 0{\cdot}04$
0·4	0·04	$= 0{\cdot}4 + 0{\cdot}04 = 0{\cdot}44$	$= 0 + 0{\cdot}2\,(0{\cdot}44) = 0{\cdot}128$
0·6	0·128		

Example 13 : $\frac{dy}{dx} = 1 + y^2$ with $y(0) = 0$, taking $h = 0{\cdot}2$, at $x = 1$.

Sol. :

x_n	y_n	$f(x_n, y_n) = 1 + y_n^2$	$y_{n+1} = y_n + h\,f(x_n, y_n)$
0	0	$1 + 0 = 1$	$0 + 0{\cdot}2\,(1) = 0{\cdot}2$
0·2	0·2	$1 + (0{\cdot}2)^2 = 1{\cdot}04$	$0{\cdot}2 + 0{\cdot}2\,(1{\cdot}04) = 0{\cdot}408$
0·4	0·408	$1 + (0{\cdot}408)^2 = 1{\cdot}1665$	$0{\cdot}408 + 0{\cdot}2\,(1{\cdot}1665) = 0{\cdot}6413$
0·6	0·6413	$1 + (0{\cdot}6413)^2 = 1{\cdot}411$	$0{\cdot}6413 + 0{\cdot}2\,(1{\cdot}411) = 0{\cdot}9235$
0·8	0·9235	$1 + (0{\cdot}9235)^2 = 1{\cdot}8529$	$0{\cdot}9235 + 0{\cdot}2\,(1{\cdot}8529) = 1{\cdot}2941$
1·0	1·2941		

Exercise II (Page 5-18)

Use Euler's modified method *i.e.* Runge-Kutta Method of Second Order to find the value of y upto 4 places of decimals satisfying the following equations : Class (b) : 6 Marks

Example 1 : $\frac{dy}{dx} = x - y^2$, $y(0) = 1$, for $x = 0{\cdot}4$ by taking $h = 0{\cdot}2$.

Sol. : (a) $x_0 = 0,\ y_0 = 1,\ h = 0{\cdot}2,\ x_1 = 0{\cdot}2$

$f(x, y) = x - y^2, \qquad f(x_0, y_0) = 0 - 1^2 = 1$

$y_1 = y_0 + h\,f(x_0, y_0) = 1 + 0{\cdot}2\,(-1) = 0{\cdot}8$

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$$

$$= 1 + \frac{0{\cdot}2}{2}[(-1) + \{0{\cdot}2 - (0{\cdot}8)^2\}] = 0{\cdot}856$$

$y_1^{(2)} = 0{\cdot}4867,\quad y_1^{(3)} = 0{\cdot}8483,\quad y_1^{(4)} = 0{\cdot}8480,\quad y_1^{(5)} = 0{\cdot}8480.$

(b) $x_0 = 0{\cdot}2,\ y_0 = 0{\cdot}8480,\ h = 0{\cdot}2,\ x_1 = 0{\cdot}4$

$f(x_0, y_0) = 0{\cdot}2 - 0{\cdot}8480^2 = -0{\cdot}5191$

$y_1 = y_0 + h\,f(x_0, y_0) = 0{\cdot}8480 + 0{\cdot}2\,(-0{\cdot}5191) = 0{\cdot}7444$

$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$

$= 0{\cdot}8480 + \frac{0{\cdot}2}{2}\left[-0{\cdot}5191 + (0{\cdot}2 - (0{\cdot}7444))^2\right] = 0{\cdot}76068$

$y_1^{(2)} = 0{\cdot}7583,\ \ y_1^{(3)} = 0{\cdot}7587,\ \ y_1^{(4)} = 0{\cdot}7586,\ \ y_1^{(5)} = 0{\cdot}7586.$ ∎

Example 2 : $\frac{dy}{dx} = x^2 + y,\ y(0) = 0{\cdot}94$ **for** $x = 0{\cdot}1$.

Sol. : $f(x, y) = x^2 + y,\ x_0 = 0,\ y_0 = 0{\cdot}94,\ h = 0{\cdot}1,\ x_1 = 0{\cdot}1$

$f(x_0, y_0) = 0^2 + (0{\cdot}94) = 0{\cdot}94$

$y_1 = y_0 + h\,f(x_0, y_0) = 0{\cdot}94 + 0{\cdot}1\,(0{\cdot}94) = 1{\cdot}034$

$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)] = 1{\cdot}0392$

$y_1^{(2)} = 1{\cdot}03946,\ \ y_1^{(3)} = 1{\cdot}03947,\ \ y_1^{(4)} = 1{\cdot}03947.$ ∎

Example 3 : $\frac{dy}{dx} = x + y,\ y(0{\cdot}1) = 1{\cdot}1104$ **for** $x = 0{\cdot}15$ **and** $x = 0{\cdot}2$.

Sol. : (a) $f(x, y) = x + y,\ x_0 = 0{\cdot}1,\ y_0 = 1{\cdot}1104,\ h = 0{\cdot}15,\ x_1 = 0{\cdot}15$

$f(x_0, y_0) = 0{\cdot}1 + 1{\cdot}1104 = 1{\cdot}2104$

$y_1 = y_0 + h\,f(x_0, y_0) = 1{\cdot}1104 + 0{\cdot}15\,(1{\cdot}2104) = 1{\cdot}29196$

$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)] = 1{\cdot}3093$

$y_1^{(2)} = 1{\cdot}3106,\ \ y_1^{(3)} = 1{\cdot}3107,\ \ y_1^{(4)} = 1{\cdot}3107.$

(b) $x_0 = 0{\cdot}15,\ y_0 = 1{\cdot}3107,\ h = 0{\cdot}05,\ x_1 = 0{\cdot}2$

$f(x_0, y_0) = 0{\cdot}15 + 1{\cdot}3107 = 1{\cdot}4607$

$y_1 = y_0 + h\,f(x_0, y_0) = 1{\cdot}3107 + 0{\cdot}05\,(1{\cdot}4607) = 1{\cdot}3837$

$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)] = 1{\cdot}3868$

$y_1^{(2)} = 1{\cdot}3869,\ \ y_1^{(3)} = 1{\cdot}3869.$ ∎

Example 4 : $\frac{dy}{dx} = 1 - y,\ y(0) = 0$ **for** $x = 0{\cdot}2$ **by taking** $h = 0{\cdot}1$.

Sol. : (a) $f(x, y) = 1 - y,\ x_0 = 0,\ y_0 = 0,\ h = 0{\cdot}1,\ x_1 = 0{\cdot}1$

$f(x_0, y_0) = 1 - 0 = 1$

$y_1 = y_0 + h\,f(x_0, y_0) = 0 + 0{\cdot}1\,(1) = 0{\cdot}1$

$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)] = 0{\cdot}095,\ \ y_1^{(2)} = 0{\cdot}095.$

(b) $x_0 = 0{\cdot}1,\ y_0 = 0{\cdot}095,\ h = 0{\cdot}1,\ x_1 = 0{\cdot}2$

$f(x_0, y_0) = 1 - y_0 = 1 - 0{\cdot}095 = 0{\cdot}905$

$y_1 = y_0 + h\,f(x_0, y_0) = 0{\cdot}095 + 0{\cdot}1\,(0{\cdot}905) = 0{\cdot}1855$

$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)] = 0{\cdot}18097$

$y_1^{(2)} = 0{\cdot}1812,\ \ y_1^{(3)} = 0{\cdot}1812.$ ∎

Example 5 : $\frac{dy}{dx} = x + y,\ y(0) = 1$ **for** $x = 0{\cdot}5$.

Sol. : (a) $f(x, y) = x + y,\ x_0 = 0,\ y_0 = 1,\ h = 0{\cdot}5,\ x_1 = 0{\cdot}5$

$f(x_0, y_0) = 0 + 1 = 1\ \ ;\ \ y_1 = y_0 + h\,f(x_0, y_0) = 1 + 0{\cdot}5\,(1) = 1{\cdot}5$

$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$

$= 1 + 0{\cdot}25[1{\cdot}5 + (0{\cdot}5 + 1{\cdot}5)] = 1{\cdot}875$

$y_1^{(2)} = 1{\cdot}9687,\ \ y_1^{(3)} = 1{\cdot}9922,\ \ y_1^{(4)} = 1{\cdot}998,$

$y_1^{(5)} = 1{\cdot}9995,\ \ y_1^{(6)} = 1{\cdot}999.$

Example 6 : $\frac{dy}{dx} = -xy^2,\ y(0) = 2$ **for** $x = 0{\cdot}2$ **by taking** $h = 0{\cdot}1$.

Sol. : (a) $f(x, y) = -xy^2,\ x_0 = 0,\ y_0 = 2,\ h = 0{\cdot}1,\ x_1 = 0{\cdot}1$

$f(x_0, y_0) = -0 \times 2^2 = 0\ \ ;\ \ y_1 = y_0 + h\,f(x_0, y_0) = 2 + 0{\cdot}5\,(0) = 2$

$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$

$= 2 + 0{\cdot}05\,[0 + (-0{\cdot}1 \times 2^2)] = 1{\cdot}98$

$y_1^{(2)} = 1{\cdot}9804,\ \ y_1^{(3)} = 1{\cdot}9804.$

(b) $x_0 = 0{\cdot}1,\ y_0 = 1{\cdot}9804,\ h = 0{\cdot}1,\ x_1 = 0{\cdot}2$

$f(x_0, y_0) = -0{\cdot}1 \times 1{\cdot}904^2 = -0{\cdot}3922$

$y_1 = y_0 + h\,f(x_0, y_0) = 1{\cdot}9804 + 0{\cdot}1\,(-0{\cdot}1 \times 1{\cdot}9804^2) = 1{\cdot}5882$

$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$

$= 1{\cdot}980 + 0{\cdot}05\,[-0{\cdot}3922 - (0{\cdot}2 \times 1{\cdot}5882^2)] = 1{\cdot}9352$

$y_1^{(2)} = 1{\cdot}9229,\ \ y_1^{(3)} = 1{\cdot}923,\ \ y_1^{(4)} = 1{\cdot}9234.$ ∎

Example 7 : $\frac{dy}{dx} = 1 + \frac{y}{x},\ y(1) = 2$ **for** $x = 1{\cdot}2$ **and compare it with its exact value.**

Sol. : $f(x, y) = 1 + \frac{y}{x},\ x_0 = 1,\ y_0 = 2,\ h = 0{\cdot}2,\ x_1 = 1{\cdot}2$

$y_1 = y_0 + h\,f(x_0, y_0) = 2 + 0{\cdot}2\left[1 + \frac{2}{1}\right] = 2{\cdot}6$

$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$

$= 2 + 0{\cdot}1\left[3 + 1 + \frac{2{\cdot}6}{1{\cdot}2}\right] = 2{\cdot}6167$

$y_1^{(2)} = 2{\cdot}6180,\ \ y_1^{(3)} = 2{\cdot}6182,\ \ y_1^{(4)} = 2{\cdot}6182$

Exact solution : $\frac{dy}{dx} - \frac{y}{x} = 1 \quad \therefore \text{ I.F.} = e^{\int -\frac{1}{x}dx} = \frac{1}{x}$

$\therefore \frac{y}{x} = \int \frac{1}{x} dx + c = \log x + c \quad \therefore c = 2$

$\therefore y = 2x + x \log x.$ When $x = 1{\cdot}2$, $y = 2{\cdot}6188$.

Example 8 : $\frac{dy}{dx} = 2 + \sqrt{xy}$, $y(1) = 1$ for $x = 1{\cdot}2$.

Sol. : $f(x, y) = 2 + \sqrt{xy}$, $x_0 = 1$, $y_0 = 1$, $h = 0{\cdot}2$, $x_1 = 1{\cdot}2$

$y_1 = y_0 + h\, f(x_0, y_0) = 1 + 0{\cdot}2\,[2 + \sqrt{1}] = 1{\cdot}6$

$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)] = 1 + 0{\cdot}1\left[3 + 2\sqrt{1{\cdot}2 \times 1{\cdot}6}\right]$

$\therefore y_1^{(1)} = 1{\cdot}6386, \quad y_1^{(2)} = 1{\cdot}6402, \quad y_1^{(3)} = 1{\cdot}6402.$

Example 9 : $\frac{dy}{dx} = x + \sqrt{y}$, $y(0) = 1$, for $x = 0{\cdot}2$.

Sol. : $f(x, y) = x + \sqrt{y}$, $x_0 = 0$, $y_0 = 1$, $h = 0{\cdot}2$, $x_1 = 0{\cdot}2$

$y_1 = y_0 + h\, f(x_0, y_0) = 1 + 0{\cdot}2\,[0 + 1] = 1{\cdot}2$

$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$

$\therefore y_1^{(1)} = 1{\cdot}2295, \quad y_1^{(2)} = 1{\cdot}2309, \quad y_1^{(3)} = 1{\cdot}2309.$

Example 10 : $\frac{dy}{dx} = y - x$, $y(0) = 2$ for $x = 0{\cdot}2$. **Also compare it with exact value.**

Sol. : $f(x, y) = y - x$, $x_0 = 0$, $y_0 = 2$, $h = 0{\cdot}2$, $x_1 = 0{\cdot}2$

$y_1 = y_0 + h\, f(x_0, y_0) = 2 + 0{\cdot}2\,[2 - 0] = 2{\cdot}4$

$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$

$\therefore y_1^{(1)} = 2{\cdot}42, \quad y_1^{(2)} = 2{\cdot}4222, \quad y_1^{(3)} = 2{\cdot}4222.$

Exact sol. : $\frac{dy}{dx} - y = -x$

$\therefore y \cdot e^{-x} = \int -x e^{-x} dx + c = e^{-x}(x + 1) + c \quad \therefore c = 1$

$\therefore y = (x + 1) + e^{x}.$ When $x = 0{\cdot}2$, $y = 0{\cdot}2 + 1 + e^{0{\cdot}2} = 2{\cdot}4214$

Example 11 : $\frac{dy}{dx} = 2x - y$, $y(0) = -1$ for $x = 0{\cdot}2$. **Also compare with exact value.**

Sol. : $f(x, y) = 2x - y$, $x_0 = 0$, $y_0 = -1$, $h = 0{\cdot}2$, $x_1 = 0{\cdot}2$

$y_1 = y_0 + h\, f(x_0, y_0) = -1 + 0{\cdot}2\,[0 - 1] = -1{\cdot}2$

$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)] = -0{\cdot}74$

$y_1^{(2)} = -0{\cdot}786, \quad y_1^{(3)} = -0{\cdot}7814, \quad y_1^{(4)} = -0{\cdot}7818, \quad y_1^{(5)} = -0{\cdot}7818.$

Exact sol. : $\frac{dy}{dx} + y = 2x \quad \therefore y e^{x} = 2 \cdot x e^{x} + c = 2e^{x}(x + 1) + c$

$\therefore y = 2(-0{\cdot}8) - 3e^{-0{\cdot}2} = -0{\cdot}7813$

Example 12 : $\frac{dy}{dx} = x^2 + y$, $y(0) = 1$ for $x = 0{\cdot}02, 0{\cdot}04, 0{\cdot}06$.

Sol. : (a) $f(x, y) = x^2 + y$, $x_0 = 0$, $y_0 = 1$, $h = 0{\cdot}02$, $x_1 = 0{\cdot}02$

$y_1 = y_0 + h\, f(x_0, y_0) = 1 + 0{\cdot}02\,[0 + 1] = 1{\cdot}02$

$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$

$= 1 + 0{\cdot}01[1^2 + 0 + 0{\cdot}02^2 + 1{\cdot}02] = 1{\cdot}020$

$y_1^{(2)} = 1{\cdot}020$

(b) $x_0 = 0{\cdot}02$, $y_0 = 1{\cdot}02$, $x_1 = 0{\cdot}04$, $h = 0{\cdot}02$

$y_1 = y_0 + h\, f(x_0, y_0) = 1{\cdot}02 + 0{\cdot}02\,[0{\cdot}02^2 + 1{\cdot}02] = 1{\cdot}02$

$y_1^{(1)} = 1{\cdot}02 + 0{\cdot}01\,[\{(0{\cdot}02)^2 + 1{\cdot}02\} + \{(0.04)^2 + 1{\cdot}02\}]$

$= 1{\cdot}0404$

$y_1^{(2)} = 1{\cdot}0406, \quad y_1^{(3)} = 1{\cdot}0406.$

(c) $x_0 = 0{\cdot}04$, $y_0 = 1{\cdot}0406$, $x_1 = 0{\cdot}06$, $h = 0{\cdot}02$

$y_1 = y_0 + h\, f(x_0, y_0)$

$= 1{\cdot}0406 + 0{\cdot}02\,[0{\cdot}04^2 + 1{\cdot}0406] = 1{\cdot}0614$

$y_1^{(1)} = 1{\cdot}0406 + 0{\cdot}01\,[\{(0{\cdot}04)^2 + 1{\cdot}0406\} + \{(0{\cdot}06)^2 + 1{\cdot}0406\}]$

$= 1{\cdot}0616$

$y_1^{(2)} = 1{\cdot}0616$

Example 13 : $\frac{dy}{dx} = y^2 - \frac{y}{x}$, $y(1) = 1$ for $x = 1{\cdot}1$ taking $h = 0{\cdot}05$.

Sol. : $f(x, y) = y^2 - \frac{y}{x}$, $x_0 = 1$, $y_0 = 1$, $h = 0{\cdot}05$, $x_1 = 1{\cdot}05$

$y_1 = y_0 + h\, f(x_0, y_0) = 1 + 0{\cdot}05\,[1 - 1] = 1$

$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$

$\therefore y_1^{(1)} = 1{\cdot}0012, \quad y_1^{(2)} = 1{\cdot}0012.$

(b) $y_1 = y_0 + h\, f(x_0, y_0)$

$= 1{\cdot}0012 + 0{\cdot}05\left[1{\cdot}0012^2 - \frac{1{\cdot}0012}{1{\cdot}05}\right] = 1{\cdot}0036$

$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$

$\therefore y_1^{(1)} = 1{\cdot}0072, \quad y_1^{(2)} = 1{\cdot}0073.$

Example 14 : $\dfrac{dy}{dx} = x + y,\ x_0 = 0,\ y_0 = 1$ at $x = 1$.

Sol. : $f(x, y) = x + y,\ x_0 = 0,\ y_0 = 1,\ h = 0{\cdot}5,\ x_1 = 0{\cdot}5$

$f(x_0, y_0) = 0 + 1 = 1 \quad ; \quad y_1 = y_0 + h\, f(x_0, y_0) = 1 + 0{\cdot}5\,[\,0 + 1\,] = 1{\cdot}5$

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$$

$$= 1 + 0{\cdot}25\,[\,1 + 0{\cdot}5 + (1{\cdot}5)\,] = 1{\cdot}75$$

$y_1^{(2)} = 1{\cdot}8125,\ \ y_1^{(3)} = 1{\cdot}8281,\ \ y_1^{(4)} = 1{\cdot}8320,\ \ y_1^{(5)} = 1{\cdot}833.$

(b) $x_0 = 0{\cdot}5,\ y_0 = 1{\cdot}833,\ h = 0{\cdot}5,\ x_1 = 1$

$f(x_0, y_0) = 0{\cdot}5 + 1{\cdot}833 = 2{\cdot}333$

$y_1 = y_0 + h\, f(x_0, y_0) = 1{\cdot}833 + 0{\cdot}25\,(2{\cdot}333) = 2{\cdot}41625$

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$$

$$= 1{\cdot}833 + 0{\cdot}25\,[\,2{\cdot}333 + 1 + 2{\cdot}61625\,] = 3{\cdot}2703$$

$y_1^{(2)} = 3{\cdot}4838,\ \ y_1^{(3)} = 3{\cdot}5372,\ \ y_1^{(4)} = 3{\cdot}55055,$

$y_1^{(5)} = 3{\cdot}5538,\ \ y_1^{(6)} = 3{\cdot}5547.$

Example 15 : $\dfrac{dy}{dx} = 1 + xy$ **with** $y = 1$ **at** $x = 0$ **for** $x = 0{\cdot}1$ **and** $x = 0{\cdot}2$

Sol. : $f(x, y) = 1 + xy,\ x_0 = 0,\ y_0 = 1,\ h = 0{\cdot}1,\ x_1 = 0{\cdot}1$

$f(x_0, y_0) = 1 + 0 = 1$

$y_1 = y_0 + h\, f(x_0, y_0) = 1 + 0{\cdot}1\,(1) = 1{\cdot}1$

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$$

$$= 1 + \frac{0{\cdot}1}{2}[1 + 1 + 0{\cdot}1 \times (1{\cdot}1)] = 1{\cdot}1055, \qquad y_1^{(2)} = 1{\cdot}10553.$$

(b) $x_0 = 0{\cdot}1,\ y_0 = 1{\cdot}1055,\ h = 0{\cdot}1,\ x_1 = 0{\cdot}2$

$f(x_0, y_0) = 1 + 0{\cdot}1 \times 1{\cdot}1055 = 1{\cdot}11055$

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$$

$$= 1{\cdot}11055 + \frac{0{\cdot}1}{2}[1{\cdot}11055 + 1 + 0{\cdot}1 \times (1{\cdot}11055)] = 1{\cdot}2216$$

$y_1^{(2)} = 1{\cdot}2221,\ \ y_1^{(3)} = 1{\cdot}2222.$

Exercise III (Page 5-29)

Using Taylor's series method obtain the solutions of the following differential equations (correct to four places of decimals) : Class (b) : 6 Marks

Example 1 : $\dfrac{dy}{dx} = 3x + y^2$ **with** $x_0 = 0,\ y_0 = 1$ **at** $x = 0{\cdot}1$.

Sol. :

$$y = y_0 + xy'_0 + \frac{x^2}{2!}y'' + \frac{x^3}{3!}y_0''' + \ldots$$

$y' = 3x + y^2, \qquad y_0' = 1$

$y'' = 3 + 2yy', \qquad y_0'' = 5$

$y''' = 2y'^2 + 2yy'', \qquad y_0''' = 12$

$y^{iv} = 4y'y'' + 2y'y'' + 2yy''' = 6y'y'' + 2yy'' = 54$

$$y = 1 + x + \frac{x^2}{2!}5 + \frac{x^3}{3!}\cdot 12 + \frac{x^4}{4!}\cdot 54 + \ldots = 1{\cdot}127225$$

Example 2 : $\dfrac{dy}{dx} = -xy$ **with** $x_0 = 0,\ y_0 = 1$.

Sol. :

$y' = -xy, \qquad y_0' = 0$

$y'' = -y - xy', \qquad y_0'' = -1$

$y''' = -y' - y' - xy'', \qquad y_0''' = 0$

$y^{iv} = -y'' - y'' - y'' - xy''' = -3y'' - xy''', \qquad y_0^{iv} = -3$

$y^{v} = -3y''' - y''' - xy^{iv}, \qquad y_0^{v} = 0.$

$$\therefore\ y = 1 - \frac{x^2}{2} - \frac{x^4}{8} + \ldots$$

Example 3 : $\dfrac{dy}{dx} = y - xy$ **with** $x_0 = 0,\ y_0 = 1$.

Sol. :

$y' = y - xy, \qquad y_0' = 1$

$y'' = y' - y - xy', \qquad y_0'' = 0$

$y''' = y'' - y' - y' - xy'', \qquad y_0''' = -2$

$y^{iv} = y''' - 2y'' - y'' - xy''', \qquad y_0^{iv} = -2$

$y^{v} = y^{iv} - 3y''' - y''' - xy^{iv}, \qquad y_0^{v} = 6.$

$$\therefore\ y = 1 + xy_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' + \frac{x^4}{4!}y^{iv} + \frac{x^5}{5!}y^{v} + \ldots$$

$$= 1 + x - \frac{x^3}{3} - \frac{x^4}{12} + \frac{x^5}{20}.$$

Example 4 : $\dfrac{dy}{dx} = x - y^2$ **with** $x_0 = 0,\ y_0 = 1$ **at** $0{\cdot}1$.

Sol. :

$y' = x - y^2, \qquad y_0' = -1$

$y'' = 1 - 2yy', \qquad y_0'' = 3$

$y''' = -2y'^2 - 2yy'', \qquad y_0''' = -8$

$y^{iv} = -4y'y'' - 2y'y'' - 2yy''', \qquad y_0^{iv} = 34$

$\therefore\ y = 0.9138.$

Example 5 : $\dfrac{dy}{dx} = (0.1)(x^3 + y^2)$ **with** $x_0 = 0$, $y_0 = 1$.

Sol. :

$y' = 0{\cdot}1\,(x^3 + y^2)$, $\quad y_0' = 0{\cdot}1$

$y'' = 0{\cdot}1\,(3x^2 + 2yy')$, $\quad y_0'' = 0{\cdot}02$

$y''' = 0{\cdot}1\,(6x + 2y'^2 + 2yy'')$, $\quad y_0''' = 0{\cdot}006$

$y^{iv} = 0{\cdot}1\,(6 + 4y'y'' + 2y'y'' + 2yy''')$, $\quad y_0^{iv} = 0{\cdot}6024$

$$\therefore\ y = y + xy_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' + \frac{x^4}{4!}y_0^{iv}$$

$$= 1 + 0{\cdot}1\,x + 0{\cdot}01\,x^2 + 0{\cdot}001\,x^3 + 0{\cdot}0251\,x^4 + \ldots..$$

Example 6 : $\dfrac{dy}{dx} = y\sin x + \cos x$ **with** $x_0 = 0$, $y_0 = 0$.

Sol. :

$y' = y\sin x + \cos x$, $\quad y_0' = 1$

$y'' = y'\sin x + y\cos x - \sin x$, $\quad y_0'' = 0$

$y''' = y''\sin x + 2y'\cos x - y\sin x - \cos x$, $\quad y_0''' = 1$

$y^{iv} = y'''\sin x + 3y''\cos x - 3y'\sin x - y\cos x + \sin x$,

$\therefore\ y_0^{iv} = 0$

$y^{v} = y^{iv}\sin x - 4y'''\cos x - 6y''\sin x - 4y'\cos x + y'\sin x + \cos x$

$\therefore\ y_0^{v} = 1$

$$\therefore\ y = y_0 + xy_0' + \frac{x^2}{2}y_0'' + \frac{x^3}{3!}y_0''' + \frac{x^4}{4!}y_0^{iv} + \frac{x^5}{5!}y_0^{v} + \ldots.$$

$$= x + \frac{x^3}{6} + \frac{x^5}{120} + \ldots..$$

Example 7 : $\dfrac{dy}{dx} = xy - 1$ **with** $x_0 = 1$, $y_0 = 2$ **at** $x = 1{\cdot}02$.

Sol. :

$y' = xy - 1$, $\quad y_0' = 1$

$y'' = xy' + y$, $\quad y_0'' = 3$

$y''' = xy'' + 2y'$, $\quad y_0''' = 5$

$y^{iv} = xy''' + 3y''$, $\quad y_0^{iv} = 14$

$\therefore\ y = 2{\cdot}02061.$

Example 8 : $\dfrac{dy}{dx} = x^2 + y^2$ **with** $x_0 = 1$, $y_0 = 0$ **at** $x = 1{\cdot}3$.

Sol. :

$y' = x^2 + y^2$, $\quad y_0' = 1$

$y'' = 2x + 2yy'$, $\quad y_0'' = 2$

$y''' = 2 + 2y'^2 + 2yy''$, $\quad y_0''' = 4$

$y^{iv} = 4y'y'' + 2y'y'' + 2yy'''$, $\quad y_0^{iv} = 12$

$\therefore\ y = 0{\cdot}41205.$

Example 9 : $\dfrac{dy}{dx} = 4 + y^2$, $x_0 = 0$, $y_0 = 0$ **at** $x = 0{\cdot}2$ **and** $0{\cdot}3$.

Sol. :

$y' = 4 + y^2$, $\quad y_0' = 4$

$y'' = 2yy'$, $\quad y_0'' = 0$

$y''' = 2y'^2 + 2yy''$, $\quad y_0''' = 32$

$y^{iv} = 6y'y'' + 2y'y''$, $\quad y_0^{iv} = 0$

$y^{v} = 6y''^2 + 8y'y''' + 2yy^{iv} = 1024$, $\quad y_0^{v} = \ldots..$

$$\therefore\ y = 4x + \frac{x^3}{3!}\cdot 32 + \frac{x^5}{5!}\cdot 1024$$

$\therefore$ At $x = 0{\cdot}2$, $y = 0{\cdot}8454$ and $x = 0{\cdot}3$, $y = 1{\cdot}3648$.

Example 10 : $\dfrac{dy}{dx} = xy - 1$, $x_0 = 1$, $y_0 = 2$ **at** $x = 1{\cdot}02$.

Sol. :

$y' = xy - 1$, $\quad y_0' = 1$

$y'' = y + xy'$, $\quad y_0'' = 3$

$y''' = 2y' + xy''$, $\quad y_0''' = 5$

$y^{iv} = 3y'' + y'''$, $\quad y_0^{iv} = 14$ $\quad \therefore\ y = 2{\cdot}02061.$

Example 11 : $\dfrac{dy}{dx} = 1 + xy$, $x_0 = 0$, $y_0 = 1$ **at** $x = 0{\cdot}1$.

Sol. :

$y' = 1 + xy$, $\quad y_0' = 1$

$y'' = y + xy'$, $\quad y_0'' = 1$

$y''' = 2y' + xy''$, $\quad y_0''' = 2$

$y^{iv} = 3y'' + xy'''$, $\quad y_0^{iv} = 3$ $\quad \therefore\ y = 1{\cdot}1053.$

Example 12 : $\dfrac{dy}{dx} = x + y^2$, $x_0 = 0$, $y_0 = 0$ **at** $x = 0{\cdot}2$.

Sol. :

$y' = x + y^2$, $\quad y_0' = 0$

$y'' = 1 + 2yy'$, $\quad y_0'' = 1$

$y''' = 2y'^2 + 2yy''$, $\quad y_0''' = 0$

$y^{iv} = 6y'y'' + 2yy'''$, $\quad y_0^{iv} = 0$

$y^{v} = 6y''^2 + 6y'y''' + 2y'y''' + 2yy^{iv}$, $\quad y_0^{v} = 6$

$\therefore\ y = 0{\cdot}2002.$

Example 13 : $\dfrac{dy}{dx} = \dfrac{1}{x^2 + y^2}$, $x_0 = 4$, $y_0 = 4$ **at** $x = 4{\cdot}1, 4{\cdot}2$.

Sol. :

$$y' = \frac{1}{x^2 + y^2} = \frac{1}{32} = 0{\cdot}03125$$

$$y'' = -\frac{2x}{(x^2 + y^2)^2} = -\frac{8}{32^2} = -0{\cdot}0078$$

$\therefore\ y = 4{\cdot}0031,\ 4{\cdot}00609.$

Exercise IV (Page 5-37)

Apply Runge-Kutta Method of Fourth Order to find an approximate value of y given that : Class (c) : 8 Marks

Solutions to these problems are not given as the problems are based on simple arithmetical calculations.

Exercise - V (Page 5-38)

Solve the following examples : Class (a) : 3 or 4 Marks

Example 1 : If $\frac{dy}{dx} = y - xy$ with $x_0 = 0$, $y_0 = 2$ then find the first three terms of the Taylor's series for y.

Sol. : $y = y_0 + x y_0' + \frac{x^2}{2!} y_0'' + \ldots, \quad x_0 = 0, \; y_0 = 2.$

$y' = y - xy \qquad \therefore \; y_0' = 2$

$y'' = y' - xy' - y \qquad \therefore \; y_0'' = 2 - 0 - 2 = 0$

$y''' = y'' - xy'' - 2y' \qquad \therefore \; y_0''' = 0 - 0 - 4 = -4$

$\therefore \; y = 2 + 2x - \frac{2}{3}x^3.$

■

Example 2 : If $\frac{dy}{dx} = -xy^2$ with $x_0 = 0$, $y_0 = 2$, find the first two terms of the Taylor's series for y.

Sol. : $y = y_0 + xy_0' + \frac{x^2}{2!} y_0'' + \ldots, \quad x_0 = 0, \; y_0 = 2$

$y' = -xy^2 \qquad \therefore \; y_0' = 0$

$y'' = -y^2 - 2xy\,y' \qquad \therefore \; y_0'' = -4$

$\therefore \; y = 2 + \frac{x^2}{2}(-4)$

■

Example 3 : If $\frac{dy}{dx} = x^2 y - 1$, with $x_0 = 0$, $y_0 = 1$, find the first three terms of Taylor's series for y.

Sol. : $y = y_0 + xy_0' + \frac{x^2}{2!} y_0'' + \ldots$

$y' = x^2 y - 1 \qquad \therefore \; y_0' = -1$

$y'' = 2xy + x^2 y' \qquad \therefore \; y_0'' = 0$

$y''' = 2y + 2xy' + 2xy' + x^2 y'' \qquad \therefore \; y_0''' = 2$

$\therefore \; y = 1 - x + \frac{x^3}{3}.$

■

Example 4 : If $\frac{dy}{dx} = 3x + y^2$ with $x_0 = 1$, $y_0 = 1$, find the first four terms of Taylor's series for y.

Sol. : $y = y_0 + xy_0' + \frac{x^2}{2!} y_0'' + \frac{x^3}{3!} y_0''' + \ldots$

$y' = 3x + y^2 \qquad \therefore \; y_0' = 1$

$y'' = 3 + 2y\,y' \qquad \therefore \; y_0'' = 3 + 2 = 5$

$y''' = 2y\,y'' + 2y'^2 \qquad \therefore \; y_0''' = 10 + 2 = 12$

$\therefore \; y = 1 + x + \frac{5}{2}x^2 + 2x^3.$

■

Example 5 : If $\frac{dy}{dx} = x - y^2$ with $x_0 = 0$, $y_0 = 1$, find the first four terms of Taylor's series for y.

Sol. : $y = y_0 + xy_0' + \frac{x^2}{2!} y_0'' + \frac{x^3}{3!} y_0''' + \ldots$

$y' = x - y^2 \qquad \therefore \; y_0' = -1$

$y'' = 1 - 2y\,y' \qquad \therefore \; y_0'' = 1 + 2 = 3$

$y''' = -2y'^2 - 2y\,y'' \qquad \therefore \; y_0''' = -2 - 6 = -8$

$\therefore \; y = 1 - x + \frac{3x^2}{2} - \frac{4x^3}{3}.$

■

Example 6 : Using Euler's method find the approximate value of y at $x = 0{\cdot}04$ in two steps, given that $\frac{dy}{dx} = \frac{y - x}{y + x}$ and $y(0) = 1$.

Sol. : Divide the interval (0, 0·04) in two steps and take $h = 0{\cdot}02$.

$f(x, y) = \frac{y - x}{y + x}, \quad f(x_0, y_0) = \frac{1 - 0}{1 + 0} = 1$

At $x_1 = 0{\cdot}02$, $\quad y_1 = y_0 + h\,f(x_0, y_0)$

$= 1 + 0{\cdot}02\,(1) = 1{\cdot}02$

Now, $\quad f(x_1, y_1) = \frac{y_1 - x_1}{y_1 + x_1} = \frac{1{\cdot}02 - 0{\cdot}02}{1{\cdot}02 + 0{\cdot}02} = 0{\cdot}9615$

At $x_2 = 0{\cdot}04$, $\quad y_2 = y_1 + h\,f(x_1, y_1)$

$= 1{\cdot}02 + 0{\cdot}02\,(0{\cdot}9615) = 1{\cdot}0392$

■

Example 7 : Using Euler's method find the approximate value of y at 1·2 in two steps, given that $\frac{dy}{dx} = y^2 - \frac{y}{x}$ and $y(1) = 1$.

Sol. : Divide the interval in two steps and take $h = 0{\cdot}1$.

$$f(x, y) = y^2 - \frac{y}{x}, \qquad f(x_0, y_0) = y_0^2 - \frac{y_0}{x_0} = 1 - 1 = 0.$$

At $x_1 = 1{\cdot}1$, $\quad y_1 = y_0 + h\, f(x_0, y_0)$

$$= 1 + 0{\cdot}1\,(0) = 1$$

Now, $\quad f(x_1, y_1) = y_1^2 - \dfrac{y_1}{x_1} = 1 - \dfrac{1}{1{\cdot}1} = 0{\cdot}0909$

At $x_1 = 1{\cdot}2$, $\quad y_2 = y_1 + h\, f(x_1, y_1)$

$$= 1 + 0{\cdot}1\,(0{\cdot}0909) = 1{\cdot}00909$$

Example 8 : Using Euler's method find the approximate value of y at $x = 0{\cdot}02$ in two steps given that $\dfrac{dy}{dx} = x + y^2$ and $y(0) = 1$.

Sol. : Take $h = 0{\cdot}01$. $f(x_0, y_0) = x_0 + y_0^2 = 0 + 1^2 = 1$.

At $x_1 = 0{\cdot}01$, $\quad y_1 = y_0 + h\, f(x_0, y_0)$

$$= 1 + 0{\cdot}01\,(1) = 1{\cdot}01$$

$$f(x_1, y_1) = x_1 + y_1^2 = 0{\cdot}01 + (1{\cdot}01)^2 = 1{\cdot}0301$$

At $x_2 = 0{\cdot}02$, $\quad y_2 = y_1 + h\, f(x_1, y_1)$

$$= 1{\cdot}01 + 0{\cdot}01\,(1{\cdot}0301) = 1{\cdot}020301$$

Example 9 : Using Euler's method find the approximate value of y at $x = 0{\cdot}6$, taking $h = 0{\cdot}2$, given $\dfrac{dy}{dx} = 1 - 2xy$ and $y(0) = 0$.

Sol. : $\quad f(x_0, y_0) = 1 - 0 = 1$

At $x_1 = 0{\cdot}2$, $\quad y_1 = y_0 + h\, f(x_0, y_0) = 0 + 0{\cdot}2\,(1)$

$$= 0{\cdot}2$$

Now, $\quad f(x_1, y_1) = 1 - 2x_1\, y_1 = 1 - 2\,(0{\cdot}2)\,(0{\cdot}2) = 0{\cdot}92$

At $x_2 = 0{\cdot}4$, $\quad y_2 = y_1 + h\, f(x_1, y_1) = 0{\cdot}2 + 0{\cdot}2\,(0.92)$

$$= 0{\cdot}384$$

Now, $\quad f(x_2, y_2) = 1 - 2x_2\, y_2 = 1 - 2\,(0{\cdot}4)\,(0{\cdot}384) = 0{\cdot}6928$

At $x_3 = 0{\cdot}6$, $\quad y_3 = y_2 + h\, f(x_2, y_2) = 0{\cdot}384 + 0{\cdot}2\,(0{\cdot}6928)$

$$= 0{\cdot}52256$$

Example 10 : Solve $\dfrac{dy}{dx} = \log(x + y)$, $y(1) = 2$ for $x = 1{\cdot}2$ taking only one approximation by Euler's modified method.

Sol. : $x_0 = 1$, $y_0 = 2$, $h = 0{\cdot}2$, $x_1 = 1{\cdot}2$

$$f(x_0, y_0) = \log(x_0 + y_0) = \log(1 + 2) = 1{\cdot}0986$$

$$y_1 = y_0 + h\, f(x_0, y_0) = 2 + 0{\cdot}2\,(1{\cdot}0986) = 2{\cdot}2197$$

First approximation gives

$$y_1^{(1)} = y_0 + \frac{h}{2}[\, f(x_0, y_0) + f(x_1, y_1)]$$

$$f(x_0, y_0) + f(x_1, y_1) = \log(1 + 2) + \log(1{\cdot}2 + 2{\cdot}2197)$$

$$= 1{\cdot}0986 + 1{\cdot}2295 = 2{\cdot}3281$$

$$\therefore \quad y_1^{(1)} = 2 + \frac{0{\cdot}2}{2}(2{\cdot}3281) = 2{\cdot}2328$$

Example 11 : Solve $\dfrac{dy}{dx} = x^2 + y$, $y(0) = 0{\cdot}94$ for $x = 0{\cdot}1$ by Euler's modified method taking only one approximation.

Sol. : $\quad x_0 = 0$, $y_0 = 0{\cdot}94$, $h = 0{\cdot}1$, $x_1 = 0{\cdot}1$

$$f(x_0, y_0) = 0^2 + 0{\cdot}94 = 0{\cdot}94$$

$$y_1 = y_0 + h\, f(x_0, y_0) = 0{\cdot}94 + 0{\cdot}1\,(0{\cdot}94) = 1{\cdot}034$$

$$f(x_0, y_0) + f(x_1, y_1) = [\,0{\cdot}94\,] + [\,(0{\cdot}1)^2 + 1{\cdot}034\,] = 1{\cdot}984$$

First approximation gives

$$y_1^{(1)} = y_0 + \frac{h}{2}[\, f(x_0, y_0) + f(x_1, y_1)]$$

$$= 0{\cdot}94 + \frac{0{\cdot}1}{2}(1{\cdot}984) = 1{\cdot}0392$$

Example 12 : Solve $\dfrac{dy}{dx} = x + y$, $y(0) = 1$ at $x = 0{\cdot}5$ by Euler's modified method taking only one approximation.

Sol. : $\quad x_0 = 0$, $y_0 = 1$, $h = 0{\cdot}5$, $x_1 = 0{\cdot}5$.

$$f(x_0, y_0) = 0 + 1 = 1$$

$$y_1 = y_0 + h\, f(x_0, y_0) = 1 + 0{\cdot}5\,(1) = 1{\cdot}5$$

$$f(x_0, y_0) + f(x_1, y_1) = 1 + [\,0{\cdot}5 + 1{\cdot}5\,] = 3$$

First approximation gives

$$y_1^{(1)} = y_0 + \frac{h}{2}[\, f(x_0, y_0) + f(x_1, y_1)] = 1 + \frac{0{\cdot}5}{2}(3) = 1{\cdot}75.$$

Example 13 : Solve $\dfrac{dy}{dx} = \dfrac{xy}{1 + x^2}$ with initial conditions $x_0 = 0$, $y_0 = 1$ for $x = 0{\cdot}1$ using Runge-Kutta method of fourth order, taking $h = 0{\cdot}1$.

Sol. : $x_0 = 0$, $y_0 = 1$, $h = 0{\cdot}1$

$$k_1 = h \cdot f(x_0, y_0) = 0{\cdot}1\left[\frac{(0)(1)}{1 + 0^2}\right] = 0$$

$$k_2 = h \cdot f\left[x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right] = 0{\cdot}1\left[\frac{\left(0+\frac{0{\cdot}1}{2}\right)\left(1+\frac{0}{2}\right)}{1+\left(\frac{0{\cdot}1}{2}\right)^2}\right] = 0{\cdot}005$$

$$k_3 = h \cdot f\left[x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right] = 0{\cdot}1\left[\frac{\left(0+\frac{0{\cdot}1}{2}\right)\left(\frac{1+0{\cdot}005}{2}\right)}{1+\left(\frac{0{\cdot}1}{2}\right)^2}\right] = 0{\cdot}005$$

$$k_4 = h \cdot f(x_0 + h,\ y_0 + k_3) = 0{\cdot}1\left[\frac{(0+0{\cdot}1)(1+0{\cdot}005)}{1+(0{\cdot}1)^2}\right] = 0{\cdot}00995$$

$$k = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] = 0{\cdot}004992$$

$\therefore\ y_1 = 1 + 0{\cdot}004992 = 1{\cdot}004992.$

Example 14 : Using Runge-Kutta method of fourth order solve $\frac{dy}{dx} = x + y$ given $x = 0$, $y = 1$ for $x = 0{\cdot}1$.

Sol. : $x_0 = 0,\ y_0 = 1,\ h = 0{\cdot}1$

$$k_1 = h \cdot f(x_0, y_0) = 0{\cdot}1\ (0 + 1) = 0{\cdot}1$$

$$k_2 = h \cdot f\left[x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right]$$

$$= 0{\cdot}1\left[\left(0+\frac{0{\cdot}1}{2}\right)+\left(1+\frac{0{\cdot}1}{2}\right)\right] = 0{\cdot}11$$

$$k_3 = h \cdot f\left[x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right]$$

$$= 0{\cdot}1\left[\left(0+\frac{0{\cdot}1}{2}\right)+\left(1+\frac{0{\cdot}11}{2}\right)\right] = 0{\cdot}1105$$

$$k_4 = h \cdot f(x_0 + h,\ y_0 + k_3)$$

$$= 0{\cdot}1\ [(0 + 0{\cdot}1) + (1 + 0{\cdot}1105)] = 0{\cdot}12105$$

$$k = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}[0{\cdot}1 + 0{\cdot}22 + 0{\cdot}2210 + 0{\cdot}2421] = 0{\cdot}1305$$

$\therefore\ y = y_0 + k = 1{\cdot}1305.$

Example 15 : Solve $\frac{dy}{dx} = -y$, with initial conditions $x_0 = 0$, $y_0 = 1$ for $x = 0{\cdot}1$ using Runge-Kutta method of fourth order.

Sol. : $x_0 = 0,\ y_0 = 1,\ h = 0{\cdot}1$

$$k_1 = h \cdot f(x_0, y_0) = 0{\cdot}1\ (-1) = -0{\cdot}1$$

$$k_2 = h f\left[x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right] = 0{\cdot}1\left[-\left(1-\frac{0{\cdot}1}{2}\right)\right] = -0{\cdot}095$$

$$k_3 = h f\left[x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right] = 0{\cdot}1\left[-\left(1-\frac{0{\cdot}095}{2}\right)\right] = -0{\cdot}09525$$

$$k_4 = h f(x_0 + h,\ y_0 + k_3) = 0{\cdot}1\ [-(1 - 0{\cdot}09525)] = -0{\cdot}090475$$

$$k = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] = -0{\cdot}09516$$

$\therefore\ y = y_0 + k = 0{\cdot}90484.$

***

CHAPTER 6

Beta, Gamma Functions

Exercise I (Page 6-5)

Evaluate the following integrals : Class (a) : 3 Marks

Example 1 : $\int_0^\infty e^{-x^3}\,dx$

Sol. : Put $x^3 = t \quad \therefore\ I = \int_0^\infty e^{-t}\cdot\frac{1}{3}t^{-2/3}dt = \frac{1}{3}\Gamma\left(\frac{1}{3}\right)$.

Example 2 : $\int_0^\infty e^{-x^2/4}\,dx$

Sol. : Put $\frac{x^2}{4} = t \quad \therefore\ I = \int_0^\infty e^{-t}\cdot t^{-1/2}dt = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Example 3 : $\int_0^\infty e^{-x^5}\,dx$

Sol. : Put $x^5 = t \quad \therefore\ I = \int_0^\infty e^{-t}\cdot\frac{1}{5}t^{-1/5}\,dt = \frac{1}{5}\Gamma\left(\frac{4}{5}\right)$.

Exercise II (Page 6-8)

Evaluate the following integrals : Class (a) : 3 Marks

Example 1 : $\int_0^\infty (x^2+4)\,e^{-2x^2}dx$

Sol. : Put $2x^2 = t \quad \therefore\ I = \frac{1}{2\sqrt{2}}\left[\int_0^\infty e^{-t}\cdot t^{-1/2} + 4\int_0^\infty e^{-t}\cdot t^{-1/2}\,dt\right]$

$$= \frac{1}{2\sqrt{2}}\left[\frac{1}{2}\Gamma\left(\frac{1}{2}\right) + 4\Gamma\left(\frac{1}{2}\right)\right] = \frac{9}{4\sqrt{2}}\sqrt{\pi}.$$

Example 2 : $\int_0^\infty x^2 e^{-x^4}dx \cdot \int_0^\infty e^{-x^4}dx$

Sol. : Put $x^4 = t \quad \therefore\ I = \frac{1}{16}\int_0^\infty e^{-t}\cdot t^{-1/4}\,dt \cdot \int_0^\infty e^{-t}\cdot t^{-3/4}\,dt$

$$\therefore\ I = \frac{1}{16}\cdot\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right) = \frac{\pi}{8\sqrt{2}}.$$

Example 3 : $\int_0^\infty \sqrt{x}\,e^{-x^2}\,dx \cdot \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\,dx$

Sol. : Put $x^2 = t$,

$$\therefore\ I = \frac{1}{4}\int_0^\infty e^{-t}\cdot t^{-1/4}\,dt \cdot \int_0^\infty e^{-t}\cdot t^{-3/4}\,dt$$

$$= \frac{1}{4}\cdot\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right) = \frac{\sqrt{2}\cdot\pi}{4}.$$

Example 4 : $\int_0^\infty x\,e^{-x^8}dx \cdot \int_0^\infty x^2 e^{-x^8}dx$

Sol. : Put $x^8 = t$,

$$\therefore\ I = \frac{1}{64}\int_0^\infty e^{-t}\cdot t^{-3/4}\,dt \cdot \int_0^\infty e^{-t}\cdot t^{-1/4}\,dt$$

$$= \frac{1}{64}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{\sqrt{2}\cdot\pi}{32}.$$

Exercise III (Page 6-11)

Evaluate the following integrals : Class (a) : 3 Marks

Example 1 : $\int_0^1 (\log x)^5\,dx$

Sol. : Put $\log x = -t$, $\quad \therefore\ I = \int_0^\infty e^{-t}\cdot t^5\,dt = \Gamma 6 = 5! = 120$.

Example 2 : $\int_0^1 x^3\left[\log\left(\frac{1}{x}\right)\right]^4 dx$

Sol. : Put $\log x = -t$, $\quad \therefore\ I = \int_\infty^0 e^{-3t}(t)^4(-e^{-t})\,dt = \int_0^\infty e^{-4t}t^4\,dt$

Now put $4t = u$, $\quad \therefore\ I = \int_0^\infty e^{-u}\left(\frac{u}{4}\right)^4\cdot\frac{du}{4} = \frac{1}{4^5}\Gamma 5$.

Example 3 : $\int_0^1 \frac{dx}{\sqrt{x\cdot\log(1/x)}}$

Sol. : Put $\log x = -t$, $\therefore\ I = \int_0^1 \frac{dx}{\sqrt{x(\log 1 - \log x)}} = \int_0^1 \frac{dx}{\sqrt{x(-\log x)}}$

$$= \int_\infty^0 (e^{-t})^{-1/2}\cdot(t)^{-1/2}(-e^{-t})\,dt$$

$$= \int_0^\infty e^{-t/2}\cdot t^{-1/2}\,dt$$

Now put $\frac{t}{2} = u$, $\quad \therefore\ I = \int_0^\infty e^{-u}(2u)^{-1/2}\cdot 2\,du = \sqrt{2}\cdot\Gamma\left(\frac{1}{2}\right) = \sqrt{2\pi}$.

Example 4 : $\int_0^1 \sqrt{x \cdot \log(1/x)} \cdot dx$

Sol. : Put $\log x = -t$, $\therefore\ I = \int_0^\infty e^{-(3/2)t} \cdot t^{1/2}\, dt$

Now, put $\frac{3t}{2} = u$, $\therefore\ I = \int_0^\infty e^{-u} \cdot \left(\frac{2}{3}u\right)^{1/2} \cdot \left(\frac{2}{3}\right) du = \frac{2}{3}\sqrt{\frac{2}{3}}\,\Gamma\left(\frac{3}{2}\right)$.

Example 5 : $\int_0^1 \sqrt[3]{\log(1/x)}\, dx$

Sol. : Put $\log x = -t$, $\therefore\ I = \int_0^1 (-\log x)^{1/3}\, dx = \int_0^\infty t^{1/3} e^{-t}\, dt = \Gamma\left(\frac{4}{3}\right)$.

Example 6u : $\int_0^1 \sqrt{\log(1/x)}\, dx$

Sol. : Put $\log x = -t$, $\therefore\ I = \int_0^1 (-\log x)^{1/2}\, dx = \int_0^1 t^{1/2} e^{-t}\, dt = \Gamma\left(\frac{3}{2}\right)$.

Exercise IV (Page 6-13)

Evaluate the following integrals : Class (a) : 3 Marks

Example 1 : $\int_0^\infty \frac{x^{m-1}}{(m-1)^x}\, dx$

Sol. : Put $(m-1)^x = e^t$ i.e. $x = \frac{t}{\log(m-1)}$

$$\therefore\ I = \frac{1}{[\log(m-1)]^m} \cdot \int_0^\infty e^{-t} \cdot t^{m-1}\, dt = \frac{\Gamma m}{[\log(m-1)]^m}.$$

Example 2 : $\int_0^\infty 5^{-4x^2}\, dx$

Sol. : Put $5^{-4x^2} = e^{-t}$; $x = \frac{1}{2\sqrt{\log 5}} \cdot t^{1/2}$

$$\therefore\ I = \int_0^\infty e^{-t} \cdot \frac{1}{4\sqrt{\log 5}} \cdot t^{-1/2}\, dt = \frac{1}{4\sqrt{\log 5}} \cdot \Gamma\left(\frac{1}{2}\right).$$

Example 3 : $\int_0^\infty 3^{-4x^2}\, dx$

Sol. : Put $3^{-4x^2} = e^{-t}$ $\therefore\ x = \frac{1}{2\sqrt{\log 3}} \cdot t^{1/2}$

$$\therefore\ I = \int_0^\infty e^{-t} \cdot \frac{1}{4\sqrt{\log 3}} \cdot t^{-1/2}\, dt = \frac{1}{4\sqrt{\log 3}} \cdot \Gamma\left(\frac{1}{2}\right).$$

Exercise V (Page 6-22)

Evaluate the following : Class (a) : 3 Marks

Example 1 : $\int_0^9 x^{3/2}(9-x)^{1/2}\, dx$

Sol. : Put $x = 9t$, $\therefore\ I = \int_0^1 9^3 t^{3/2}(1-t)^{1/2}\, dt = 729\, B\left(\frac{5}{2}, \frac{3}{2}\right)$.

Example 2 : $\int_0^1 (1-\sqrt[5]{x})^{3/2}\, dx$

Sol. : Put $x^{1/5} = t$, $\therefore\ I = \int_0^1 (1-t)^{3/2} \cdot 5t^4\, dt = 5\, B\left(\frac{5}{2}, 5\right)$.

Example 3 : $\int_0^2 \frac{x^2}{\sqrt{2-x}}\, dx$

Sol. : Put $x = 2t$, $\therefore\ I = \int_0^1 2^2 t^2 \cdot 2^{-1/2}(1-t)^{1/2} \cdot 2\, dt = 2^{5/2}\, B\left(3, \frac{1}{2}\right)$.

Example 4 : $\int_0^2 x \cdot \sqrt[3]{8-x^3} \cdot dx$

Sol. : Put $x^3 = 8t$, $\therefore\ I = \frac{8}{3}\int_0^1 t^{-1/3}(1-t)^{1/3}\, dt = \frac{8}{3} B\left(\frac{2}{3}, \frac{4}{3}\right)$.

Example 5 : $\int_0^1 x^6(1-x)^{1/2}\, dx$

Sol. : By definition $I = B\left(7, \frac{3}{2}\right)$.

Example 6 : $\int_0^1 \frac{x^2(4-x^2)}{\sqrt{1-x^2}}\, dx$

Sol. : $I = \int_0^1 \frac{4x^2}{\sqrt{1-x^2}}\, dx - \int_0^1 \frac{x^4}{\sqrt{1-x^2}}\, dx$. Put $x^2 = t$.

$$\therefore\ I_1 = 2\int_0^1 t^{1/2}(1-t)^{1/2}\, dt = 2B\left(\frac{3}{2}, \frac{1}{2}\right)$$

$$\therefore\ I_2 = \frac{1}{2}\int_0^1 t^{3/2}(1-t)^{-1/2}\, dt = \frac{1}{2} B\left(\frac{5}{2}, \frac{1}{2}\right).$$

Exercise VI (Page 6-26)

Evaluate the following : Class (a) : 3 Marks

Example 1 : $\int_0^1 x^2(1-x^2)^4\, dx$

Sol. : Put $x^2 = t$, $\therefore\ I = \frac{1}{2}\int_0^1 t^{1/2}(1-t)^4\, dt = \frac{1}{2} B\left(\frac{3}{2}, 5\right)$.

Example 2 : $\int_0^2 x^7 (16 - x^4)^{10}\, dx$

Sol. : Put $x^4 = 16t$, $\therefore\ I = 2^6 \cdot 16^{10} \int_0^1 t(1-t)^{10}\, dt = 2^6 \cdot 16^{10} \cdot B(2, 11)$.

Example 3 : $\int_0^1 \frac{x^n}{\sqrt{1-x^2}}\, dx$

Sol. : $I = \int_0^1 x^n (1-x^2)^{-1/2}\, dx$. Put $x^2 = t$.

$\therefore\ I = \frac{1}{2}\int_0^1 t^{(n/2)-(1/2)} \cdot (1-t)^{-1/2}\, dt = \frac{1}{2} B\left(\frac{n}{2} + \frac{1}{2}, \frac{1}{2}\right)$.

Example 4 : $\int_0^1 \frac{dx}{\sqrt[n]{1-x^n}}$

Sol. : Put $x^n = t$, $\therefore\ I = \int_0^1 (1-t)^{-1/n} \cdot \frac{1}{n} \cdot t^{(1/n)-1}\, dt = \frac{1}{n} B\left(-\frac{1}{n}+1, \frac{1}{n}\right)$.

Example 5 : $\int_0^1 \frac{x\, dx}{\sqrt{1-x^5}}$

Sol. : Put $x^5 = t$, $\therefore\ I = \frac{1}{5}\int_0^1 t^{-3/5}(1-t)^{-1/2}\, dt = \frac{1}{5} B\left(\frac{2}{5}, \frac{1}{2}\right)$.

Example 6 : $\int_0^1 \sqrt{x}\,(1-x^2)^{1/3}\, dx$

Sol. : Put $x^2 = t$, $\therefore\ I = \frac{1}{2}\int_0^1 t^{-1/4}(1-t)^{1/3}\, dt = \frac{1}{2} B\left(\frac{3}{4}, \frac{4}{3}\right)$.

Example 7 : $\int_0^{2a} \frac{x^{9/2}}{\sqrt{2a-x}}\, dx$

Sol. : Put $x = 2at$, $\therefore\ I = (2a)^5 \int_0^1 t^{9/2}(1-t)^{-1/2}\, dt = (2a)^5 B\left(\frac{11}{2}, \frac{1}{2}\right)$.

Example 8 : $\int_0^2 x \cdot \sqrt{2x - x^2}\, dx$

Sol. : Put $x = 2t$, $\therefore\ I = 2^3 \int_0^1 t^{3/2}(1-t)^{1/2}\, dt = 8\, B\left(\frac{5}{2}, \frac{3}{2}\right) = \frac{\pi}{2}$.

Example 9 : $\int_0^4 x^2 \sqrt{4x - x^2}\, dx$

Sol. : Put $x = 4t$, $\therefore\ I = 2^8 \int_0^1 t^{5/2}(1-t)^{1/2}\, dt = 2^8 B\left(\frac{7}{2}, \frac{3}{2}\right) = 10\pi$.

Prove the following results : Class (a) : 3 Marks

Example 10 : $\int_0^1 \frac{x}{\sqrt{1-x^4}}\, dx = \frac{\pi}{4}$

Sol. : Put $x^4 = t$, $\therefore\ I = \frac{1}{4}\int_0^1 t^{-1/2}(1-t)^{-1/2}\, dt = \frac{1}{4} B\left(\frac{1}{2}, \frac{1}{2}\right)$.

Example 11 : $\int_0^a x^4 (a^2 - x^2)^{1/2} dx = \frac{a^6 \pi}{32}$

Sol. : Put $x^2 = a^2 t$, $\therefore\ I = \frac{a^6}{2}\int_0^1 t^{3/2}(1-t)^{1/2}\, dt = \frac{a^6}{2} B\left(\frac{5}{2}, \frac{3}{2}\right)$. ■

Example 12 : $\int_0^a x^2 (a^2 - x^2)^{3/2} dx = \frac{a^6 \pi}{32}$

Sol. : Put $x^2 = a^2 t$, $x = a t^{1/2}$,

$\therefore\ I = \frac{a^6}{2}\int_0^1 t^{1/2}(1-t)^{3/2}\, dt = \frac{a^6}{2} B\left(\frac{3}{2}, \frac{5}{2}\right)$. ■

Example 13 : $\int_0^a x^6 (a^4 - x^4)^{1/4} dx = \frac{\sqrt{2}}{128} a^8 \pi$

Sol. : Put $x^4 = a^4 t$, $x = a t^{1/4}$

$\therefore\ I = \frac{a^8}{4}\int_0^1 t^{3/4}(1-t)^{1/4}\, dt = \frac{a^8}{4} B\left(\frac{7}{4}, \frac{5}{4}\right) = \frac{\sqrt{2}}{128} a^6 \pi$. ■

Example 14 : $\int_0^a x^{10} (a^6 - x^6)^{1/6} dx = \frac{5}{6^3} a^{12} \pi$

Sol. : Put $x^6 = a^6 t$, $x = a t^{1/6}$

$\therefore\ I = \frac{a^{12}}{6}\int_0^1 t^{5/6}(1-t)^{1/6}\, dt = \frac{a^{12}}{6} B\left(\frac{11}{6}, \frac{7}{6}\right)$. ■

Evaluate the following : Class (c) : 8 Marks

Example 15 : $\int_0^1 \frac{x^2\, dx}{\sqrt{1-x^4}} \int_0^1 \frac{dx}{\sqrt{1-x^4}}$

Sol. : Put $x^4 = t$, $\therefore\ I = \frac{1}{4}\int_0^1 t^{-1/4}(1-t)^{-1/2}\, dt \cdot \int_0^1 \frac{1}{4}(1-t)^{-1/2} t^{-3/4}\, dt$

$= \frac{1}{4} B\left(\frac{3}{4}, \frac{1}{2}\right) \cdot \frac{1}{4} B\left(\frac{1}{4}, \frac{1}{2}\right) = \frac{\pi}{4}$. ■

Exercise VII (Page 6-28)

Evaluate the following : Class (a) : 3 Marks

Example 1 : $\int_0^1 \frac{dx}{\sqrt{1-x^8}}$

Sol. : Put $x^8 = t$, $\therefore\ I = \int_0^1 (1-t)^{-1/2} \cdot \frac{1}{8} \cdot t^{-7/8}\, dt = \frac{1}{8} B\left(\frac{1}{2}, \frac{1}{8}\right)$. ■

Example 2 : $\int_0^1 \sqrt{1-x^m}\, dx$

Sol. : Put $x^m = t$, $\therefore\ I = \int_0^1 (1-t)^{1/2} \cdot \frac{1}{m} \cdot t^{(1/m)-1}\, dt = \frac{1}{m} B\left(\frac{1}{m}, \frac{3}{2}\right)$. ■

Example 3 : $\int_0^1 \sqrt{1-x^8}\, dx$

Sol. : Put $x^8 = t$, $\therefore\ I = \int_0^1 (1-t)^{1/2} \cdot \frac{1}{8} \cdot t^{-7/8}\, dt = \frac{1}{8} B\left(\frac{3}{2}, \frac{1}{8}\right)$.

Example 4 : $\int_0^1 \sqrt{1-x^6}\, dx$

Sol. : Put $x^6 = t$, $\therefore\ I = \int_0^1 (1-t)^{1/2} \cdot \frac{1}{6} \cdot t^{-5/6}\, dt = \frac{1}{6} B\left(\frac{3}{2}, \frac{1}{6}\right)$.

Exercise VIII (Page 6-31)

Evaluate the following integrals. (1 to 10) : Class (a) : 3 Marks

Example 1 : $\int_0^1 \frac{x^7}{\sqrt{1-x^2}}\, dx$

Sol. : Put $x = \sin\theta$, $\therefore\ I = \int_0^{\pi/2} \sin^7\theta\, d\theta = \frac{16}{35}$.

Example 2 : $\int_0^1 \frac{x^6}{\sqrt{1-x^2}}\, dx$

Sol. : Put $x = \sin\theta$, $\therefore\ I = \int_0^{\pi/2} \sin^6\theta\, d\theta = \frac{5\pi}{32}$.

Example 3 : $\int_0^1 \frac{x^2(4-x^4)}{\sqrt{1-x^2}}\, dx$

Sol. : Put $x = \sin\theta$,

$$\therefore\ I = 4\int_0^{\pi/2} \sin^2\theta\, d\theta - \int_0^{\pi/2} \sin^6\theta\, d\theta = \pi - \frac{5}{32}\pi = \frac{27}{32}\pi.$$

Example 4 : $\int_0^{\pi/6} \sin^6 3\theta \cdot d\theta$

Sol. : Put $3\theta = \sin\Phi$, $\therefore\ I = \int_0^{\pi/2} \sin^6\Phi \cdot \frac{1}{3}\, d\Phi = \frac{5}{96}\pi$.

Example 5 : $\int_0^{\pi} (1-\cos\theta)^5 \cdot d\theta$

Sol. : $I = \int_0^{\pi/2} (2\sin^2(\theta/2))^5 d\theta$. Put $\frac{\theta}{2} = \Phi$

$$\therefore\ I = 2^6 \int_0^{\pi/2} \sin^{10}\Phi\, d\Phi = \frac{63}{8}\pi.$$

Example 6 : $\int_0^1 \frac{x^9}{\sqrt{1-x^4}}\, dx$

Sol. : Put $x^2 = \sin\theta$, $\therefore\ I = \frac{1}{2}\int_0^{\pi/2} \sin^4\theta\, d\theta = \frac{3\pi}{32}$.

Example 7 : $\int_0^{\pi} \frac{\sin^4\theta}{(1+\cos\theta)^2} \cdot d\theta$

Sol. : $I = \int_0^{\pi} \frac{\sin^4\theta}{[2\cos^2(\theta/2)]^2}\, d\theta$. Put $\frac{\theta}{2} = \Phi$

$$\therefore\ I = \frac{1}{2}\int_0^{\pi/2} \frac{(2\sin\Phi\cos\Phi)^4}{\cos^4\Phi}\, d\Phi = 2^3 \int_0^{\pi/2} \sin^4\Phi\, d\Phi = \frac{3\pi}{2}.$$

Example 8 : $\int_0^1 x^4 \cos^{-1} x\, dx$

Sol. : Integrating by parts,

$$\therefore\ I = \left[\cos^{-1}x \cdot \frac{x^5}{5}\right]_0^1 - \int_0^1 \frac{x^5}{5} \cdot \frac{(-1)}{\sqrt{1-x^2}}\, dx = \frac{1}{5}\int_0^1 \frac{x^5}{\sqrt{1-x^2}}\, dx$$

Put $x = \sin\theta$, $\therefore\ I = \frac{1}{5}\int_0^{\pi/2} \sin^5\theta\, d\theta = \frac{8}{75}$.

Example 9 : $\int_0^{\infty} \frac{t^6}{(1+t^2)^4} \cdot dt$

Sol. : Put $t = \tan\theta$, $\therefore\ I = \int_0^{\pi/2} \sin^6\theta\, d\theta = \frac{5\pi}{32}$.

Example 10 : $\int_0^1 x^5 \cdot \sqrt{\left\{\frac{1-x^2}{1+x^2}\right\}}\, dx$

Sol. : $\int_0^1 x^5 \sqrt{\left(\frac{1-x^2}{1+x^2}\right)}\, dx = \int_0^1 x^4 \cdot \frac{(1-x^2)}{\sqrt{1-x^4}} \cdot x\, dx$

Put $x^2 = \sin\theta$, $\therefore\ I = \frac{1}{2}\int_0^{\pi/2} (\sin^2\theta - \sin^3\theta)\, d\theta = \frac{3\pi - 8}{24}$.

Example 11 : Prove that $\int_0^{\pi/2} \sin^{2n} x\, dx = \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2^n \cdot n!} \cdot \frac{\pi}{2}$

Sol. :

$$\int_0^{\pi/2} \sin^{2n} x\, dx = \frac{(2n-1)\cdot(2n-3)\cdot(2n-5)\ldots 3\cdot 1}{2n\cdot(2n-2)(2n-4)\ldots 4\cdot 2} \cdot \frac{\pi}{2}$$

$$= \frac{1\cdot 3\cdot 5\ldots(2n-1)}{2(n)\cdot 2(n-1)\cdot 2(n-2)\ldots 2(2)\cdot 2\cdot 1} \cdot \frac{\pi}{2}$$

$$= \frac{1\cdot 3\cdot 5\ldots(2n-1)}{2^n \cdot n!} \cdot \frac{\pi}{2}.$$

Example 12 : Prove that $\int_0^{\pi/2} \sin^p x\, dx \int_0^{\pi/2} \sin^{p+1} x\, dx = \frac{1}{(p+1)} \cdot \frac{\pi}{2}$.

Sol. : $\int_0^{\pi/2} \sin^p x\, dx \int_0^{\pi/2} \sin^{p+1} x\, dx$

$$= \frac{(p-1)\cdot(p-3)\ldots 3\cdot 1}{p\cdot(p-2)\ldots 4\cdot 2} \cdot \frac{\pi}{2} \cdot \frac{p\cdot(p-2)\ldots 2}{(p+1)(p-1)\ldots 3\cdot 1}$$

Note that if p is odd then $p+1$ is even and if p is even then $p+1$ is odd.

$\therefore\ I = \dfrac{1}{(p+1)} \cdot \dfrac{\pi}{2}$

Exercise IX (Page 6-34)

Evaluate the following integrals : Class (a) : 3 Marks

Example 1 : $\displaystyle\int_0^\infty \frac{dx}{(1+x^2)^{3/2}}$

Sol. : Put $x = \tan\theta$. $\quad \therefore\ I = \displaystyle\int_0^{\pi/2} \cos^3\theta\, d\theta = \frac{4 \cdot 2}{5 \cdot 3 \cdot 1} = \frac{8}{15}$.

Example 2 : $\displaystyle\int_0^\infty \frac{dx}{(a^2+x^2)^5}$

Sol. : Put $x = a\tan\theta$. $\quad \therefore\ I = \dfrac{1}{a^9}\displaystyle\int_0^{\pi/2} \cos^8\theta\, d\theta = \frac{35}{a^9 \cdot 256} \cdot \pi$.

Example 3 : $\displaystyle\int_0^\infty \frac{x^2}{(1+x^6)^3}\, dx$

Sol. : Put $x^3 = t$. $\quad \therefore\ I = \displaystyle\int_0^{\pi/2} \frac{1}{(1+t^2)^3} \cdot \frac{dt}{3}$

Put $t = \tan\theta$. $\quad \therefore\ I = \dfrac{1}{3}\displaystyle\int_0^{\pi/2} \cos^4\theta\, d\theta = \frac{\pi}{16}$.

Example 4 : $\displaystyle\int_0^\infty \frac{x^2}{(a^2+x^6)^{5/2}}\, dx$

Sol. : Put $x^3 = at$. $\quad \therefore\ I = \displaystyle\int_0^\infty \frac{1}{3} \cdot \frac{a\, dt}{a^2(1+t^2)^{5/2}}$.

Put $t = \tan\theta$. $\quad \therefore\ I = \dfrac{1}{3a^4}\displaystyle\int_0^{\pi/2} \cos^3\theta\, d\theta = \frac{2}{9a^4}$.

Example 5 : $\displaystyle\int_0^\pi (1+\cos\theta)^3\, d\theta$

Sol. : $\quad I = \displaystyle\int_0^\pi [2\cos^2(\theta/2)]^3\, d\theta$. Put $\dfrac{\theta}{2} = t$

$\therefore\ I = \displaystyle\int_0^{\pi/2} 8 \cdot \cos^6 t \cdot 2\, dt = \frac{5}{2}\pi$.

Example 6 : $\displaystyle\int_0^{\pi/4} \cos^7 2\theta\, d\theta$

Sol. : Put $2\theta = t$. $\quad \therefore\ I = \displaystyle\int_0^{\pi/2} \cos^7 t \cdot \frac{1}{2}\, dt = \frac{6}{35}$. ■

Example 7 : $\displaystyle\int_0^{\pi/4} (1+\cos 4\theta)^5\, d\theta$

Sol. : $\quad I = \displaystyle\int_0^{\pi/4} (2\cos^2 2\theta)^5\, d\theta$. Put $2\theta = t$, $2\, d\theta = dt$

$\therefore\ I = \displaystyle\int_0^{\pi/2} 2^5 \cdot \cos^{10} t \cdot \frac{dt}{2} = \frac{63}{32}\pi$. ■

Example 8 : $\displaystyle\int_0^\infty \frac{dx}{(x^2+1)^5}$

Sol. : Put $x = \tan\theta$. $\quad \therefore\ I = \displaystyle\int_0^{\pi/2} \cos^8\theta\, d\theta = \frac{35}{256}\pi$. ■

Example 9 : $\displaystyle\int_0^\infty (1+x^2)^{-n-(1/2)}\, dx$

Sol. : Put $x = \tan\theta$.

$\therefore\ I = \displaystyle\int_0^{\pi/2} (\sec^2\theta)^{-n-(1/2)} \cdot \sec^2\theta\, d\theta$

$= \displaystyle\int_0^{\pi/2} \cos^{2n-1}\theta\, d\theta = \frac{(2n-2)(2n-4)(2n-6)}{(2n-1)(2n-2)(2n-5)} \cdots \frac{4 \cdot 2}{3 \cdot 1}$

$= \dfrac{(2n-2)^2(2n-4)^2(2n-6)^2 \ldots 4^2 \cdot 2^2}{(2n-1)(2n-2)(2n-3)(2n-4) \ldots 4 \cdot 3 \cdot 2 \cdot 1}$

$= \dfrac{2^{2(n-1)}[(n-1)!]^2}{(2n-1)!}$ ■

Exercise X (Page 6-40)

(A) Evaluate the following integrals : Class (a) : 3 Marks

Example 1 : $\displaystyle\int_0^{\pi/4} \cos^3 2\theta \cdot \sin^2 4\theta\, d\theta$

Sol. : Put $2\theta = t$.

$\therefore\ I = \displaystyle\int_0^{\pi/2} \cos^3 t (2\sin t\cos t)^2 \frac{dt}{2} = 2\int_0^{\pi/2} \sin^2 t \cos^5 t\, dt = \frac{16}{105}$. ■

Example 2 : $\displaystyle\int_0^{\pi/6} \cos^6 3\theta \cdot \sin^2 6\theta\, d\theta$

Sol. : Put $3\theta = t$.

$\therefore\ I = \displaystyle\int_0^{\pi/2} \cos^6 t \cdot 4\sin^2 t\cos^2 t \frac{dt}{3} = \frac{4}{3}\int_0^{\pi/2} \cos^8 t \sin^2 t\, dt = \frac{7}{384}\pi$. ■

Example 3 : $\displaystyle\int_0^\pi \sin^2\theta \cdot (1+\cos\theta)^4\, d\theta$

Sol. : $\quad I = \displaystyle\int_0^\pi 2^2 \sin^2(\theta/2)\cos^2(\theta/2) \cdot 2^4 \cdot \cos^8(\theta/2)\, d\theta$.

Put $\dfrac{\theta}{2} = t$. $\quad \therefore\ I = 2^8\displaystyle\int_0^{\pi/2} \sin^2 t \cos^{10} t \cdot 2 \cdot dt$

$= 2^7 \cdot \dfrac{1 \cdot 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1}{12 \cdot 10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} \cdot \dfrac{\pi}{2} = \dfrac{21}{16}\pi$. ■

Example 4 : $\int_{-\pi}^{\pi} \sin^4 x \cos^2 x \, dx$

Sol. : $I = 2\int_0^{\pi} \sin^4 x \cos^2 x \, dx$

$$= 2\left[\int_0^{\pi/2} \sin^4 x \cos^2 x \, dx + \int_0^{\pi/2} \sin^4(\pi - x) \cdot \cos^2(\pi - x) \, dx\right]$$

$$= 2\left[\int_0^{\pi/2} \sin^4 x \cos^2 x + \int_0^{\pi/2} \sin^4 x \cos^2 x \, dx\right]$$

$$\therefore \; I = 4\int_0^{\pi/2} \sin^4 x \cos^2 x \, dx = \frac{\pi}{8}.$$

Example 5 : $\int_0^{\pi} x \sin^7 x \cos^4 x \, dx$

Sol. : $I = \int_0^{\pi} (\pi - x) \sin^7(\pi - x) \cos^4(\pi - x) \, dx = \pi \int_0^{\pi} \sin^7 x \cdot \cos^4 x \, dx - I$

$$2I = \pi \int_0^{\pi} \sin^7 x \cos^4 x \, dx$$

$$= \pi\left[\int_0^{\pi/2} \sin^7 x \cos^4 x \, dx + \int_0^{\pi/2} \sin^7(\pi - x) \cos^4(\pi - x) \, dx\right]$$

$$= 2\pi \int_0^{\pi/2} \sin^7 x \cos^4 x \, dx$$

$$\therefore \; I = \int_0^{\pi/2} \sin^7 x \cos^4 x \, dx = \frac{16}{1155}.$$

Example 6 : $\int_{-\pi/2}^{\pi/2} \cos^3 \theta \cdot (1 + \sin\theta)^2 \, d\theta$

Sol. : $I = \int_{-\pi/2}^{\pi/2} \cos^3 \theta (1 + 2\sin\theta + \sin^2\theta) \, d\theta \qquad [\sin^2\theta = 1 - \cos^2\theta]$

$$= \int_{-\pi/2}^{\pi/2} 2\cos^3\theta \, d\theta + 2\int_{-\pi/2}^{\pi/2} \cos^3\theta \sin\theta \, d\theta - \int_{-\pi/2}^{\pi/2} \cos^5\theta \, d\theta$$

$$= 4\int_0^{\pi/2} \cos^3\theta \, d\theta + 2\int_{-\pi/2}^{\pi/2} \cos^3\theta \sin\theta \, d\theta - 2\int_0^{\pi/2} \cos^5\theta \, d\theta$$

$$= 4 \cdot \frac{2}{3} \cdot 1 + 0 - 2 \cdot \frac{4 \cdot 2 \cdot 1}{5 \cdot 3} = \frac{8}{5}.$$

Example 7 : $\int_0^{2\pi} \sin^2\theta \cdot (1 + \cos\theta)^4 \, d\theta$

Sol. : $I = \int_0^{2\pi} 4\sin^2(\theta/2) \cdot \cos^2(\theta/2) \cdot 2^4 \cos^8(\theta/2) \, d\theta.$

Put $\frac{\theta}{2} = t.$

$$\therefore \; I = 2^6 \int_0^{\pi} \sin^2 t \cos^{10} t \cdot 2t \cdot dt$$

$$= 2^7 \int_0^{\pi/2} \sin^2 t \cos^{10} t \cdot 2 \, dt = \frac{21\pi}{8}.$$

Example 8 : $\int_{-\pi/2}^{\pi/2} \sin^4 x \cos^2 x \, dx$

Sol. : $I = 2\int_0^{\pi/2} \sin^4 x \cos^2 x \, dx = 2 \cdot \frac{3 \cdot 1 \cdot 1}{6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2} = \frac{\pi}{16}.$

Example 9 : $\int_0^1 x^6 \sqrt{1 - x^2} \, dx$

Sol. : Put $x = \sin\theta$, $\therefore \; I = \int_0^{\pi/2} \sin^6\theta \cdot \cos\theta \cos\theta \, d\theta$

$$= \int_0^{\pi/2} \sin^6\theta \cos^2\theta \, d\theta = \frac{5\pi}{256}.$$

[OR put $x = \sqrt{t}$, $\therefore \; I = \frac{1}{2} B\left(\frac{7}{2}, \frac{3}{2}\right)$.]

Example 10 : $\int_0^1 x^2 \cdot (1 - x^2)^{3/2} \, dx$

Sol. : Put $x = \sin\theta$,

$$\therefore \; I = \int_0^{\pi/2} \sin^2\theta \cos^3\theta \cdot \cos\theta \, d\theta = \frac{1 \cdot 3 \cdot 1}{6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2} = \frac{\pi}{32}.$$

[OR put $x = \sqrt{t}$, $\therefore \; I = \frac{1}{2} B\left(\frac{9}{2}, \frac{5}{2}\right)$.]

Example 11 : $\int_0^1 x^2 \cdot (1 - \sqrt{x})^5 \, dx$

Sol. : Put $\sqrt{x} = t$, $\therefore \; I = \int_0^1 t^4 (1 - t)^5 \cdot 2t \, dt = 2\int_0^1 t^5 (1 - t)^5 \, dt$

$$= 2 \cdot B(6, 6) = \frac{2 \cdot 5! \cdot 5!}{11!} = \frac{1}{1386}$$

Example 12 : $\int_0^a x^4 \cdot \sqrt{a^2 - x^2} \, dx$

Sol. : Put $x = a\sin\theta$, $\therefore \; I = a^6 \int_0^{\pi/2} \sin^4\theta \cdot \cos^2\theta \, d\theta = \frac{a^6\pi}{32}.$

[OR put $x = \sqrt{t}$, $\therefore \; I = \frac{a^6}{2} B\left(\frac{5}{2}, \frac{3}{2}\right)$.]

Example 13 : $\int_0^a x^4 \cdot (1 - x^2)^{3/2} \, dx$

Sol. : Put $x = \sin\theta$, $\therefore \; I = \int_0^{\pi/2} \sin^4\theta \cos^4\theta \, d\theta = \frac{3\pi}{256}.$

[OR put $x = \sqrt{t}$, $\therefore \; I = \frac{1}{2} B\left(\frac{5}{2}, \frac{5}{2}\right)$.]

Example 14 : $\int_0^a x^2 \cdot \sqrt{(ax - x^2)} \, dx$

Sol. : Put $x = a\sin^2\theta$, $\therefore \; I = 2a^4 \int_0^{\pi/2} \sin^6\theta \cos^2\theta \, d\theta = \frac{5\pi a^4}{128}.$

[OR put $x = at$, $\therefore\ I = \int_0^a x^{5/2}(a-x)^{1/2}dx = a^4 B\left(\frac{7}{2}, \frac{3}{2}\right)$.]

Example 15 : $\int_0^1 x^{3/2} \cdot (1-x)^{3/2}\, dx$

Sol. : Put $x = \sin^2\theta$, $\therefore\ I = 2\int_0^{\pi/2} \sin^4\theta \cos^4\theta\, d\theta = \frac{3\pi}{128}$.

[OR By definition $\therefore\ I = B\left(\frac{5}{2}, \frac{5}{2}\right)$.]

Example 16 : $\int_0^{\pi} x \sin^5 x \cos^6 x\, dx$

Sol. : $I = \int_0^{\pi}(\pi - x)\sin^5(\pi - x)\cdot\cos^6(\pi - x)\, dx$

$2I = \pi\int_0^{\pi}\sin^5 x\cos^6 x\, dx = 2\pi\int_0^{\pi/2}\sin^5 x\cos^6 x\, dx = \frac{8\pi}{693}$.

Example 17 : $\int_0^{\pi} x \sin^4 x \cos^6 x\, dx$

Sol. : $I = \int_0^{\pi}(\pi - x)\sin^4(\pi - x)\cos^6(\pi - x)\, dx = \pi\int_0^{\pi}\sin^4 x\cos^6 x\, dx - I$

$\therefore\ 2I = 2\pi\int_0^{\pi/2}\sin^4 x\cos^6 x\, dx \quad \therefore\ I = \frac{3}{512}\pi^2$.

(B) Evaluate the following (1 to 8) : Class (a) : 3 Marks

Example 1 : $\int_0^{\pi/2}\frac{d\theta}{\sqrt{\sin\theta}} \cdot \int_0^{\pi/2}\sqrt{\sin\theta}\, d\theta$

Sol. : $I = \int_0^{\pi/2}\sin^{-1/2}\theta\, d\theta \cdot \int_0^{\pi/2}\sin^{1/2}\theta\, d\theta$

$I = \frac{1}{2}B\left(\frac{1}{4}, \frac{1}{2}\right)\cdot\frac{1}{2}B\left(\frac{3}{4}, \frac{1}{2}\right) = \pi$.

Example 2 : $\int_0^{\pi/2}(\sin 2x)^{2t-1}\, dx$

Sol. : $I = \int_0^{\pi/2} 2^{2t-1}\cdot\sin^{2t-1}x\cdot\cos^{2t-1}x\cdot dt = 2^{2t-1}\cdot\frac{1}{2}B(t, t)$

$= 2^{2t-2}\cdot\frac{\Gamma t\ \Gamma t}{\Gamma 2t} = 2^{2t-2}\cdot\frac{(\Gamma t)^2}{\Gamma 2t}$.

Example 3 : $\int_0^{\pi}\sin^2\theta\cdot(1+\cos\theta)^4 d\theta$

Sol. : $I = \int_0^{\pi} 2^2\sin^2(\theta/2)\cdot\cos^2(\theta/2)\cdot[2\cos^2(\theta/2)]^4\cdot d\theta$

Put $\frac{\theta}{2} = \Phi$, $\therefore\ I = 2^6\int_0^{\pi/2}\sin^2\Phi\cos^{10}\Phi\cdot 2\, d\Phi$

$= 2^7\cdot\frac{1}{2}\cdot\frac{\Gamma 3/2\ \Gamma 11/2}{\Gamma 7} = \frac{21\pi}{16}$.

Example 4 : $\int_0^{\pi/4}\cos^3 2x\cdot\sin^4 4x\, dx$

Sol. : $I = \int_0^{\pi/4}\cos^3 2x\cdot(2\sin 2x\cos 2x)^4\, dx$

Put $2x = \Phi$, $\therefore\ I = 2^3\int_0^{\pi/2}\sin^4\Phi\cos^7\Phi\, d\Phi = 4B\left(\frac{5}{2}, 4\right)$. ■

Example 5 : $\int_0^{\pi/2}\sqrt{\cot\theta}\, d\theta$

Sol. : $I = \int_0^{\pi/2}\sin^{-1/2}\theta\cos^{1/2}\theta\, d\theta = \frac{1}{2}\cdot\frac{\Gamma 3/4\ \Gamma 1/4}{\Gamma 1} = \frac{\pi}{\sqrt{2}}$. ■

Example 6 : $\int_0^{\pi/8}\sin^4 8\theta\cos^6 4\theta\, d\theta$

Sol. : $I = 2^4\int_0^{\pi/8}\sin^{10}4\theta\cdot\cos^4 4\theta\cdot d\theta$

Put $4\theta = \Phi$, $\therefore\ I = 2^4\int_0^{\pi/2}\sin^{10}\Phi\cos^4\Phi\cdot\frac{1}{4}d\Phi = 4\cdot\frac{1}{2}B\left(\frac{11}{2}, \frac{5}{2}\right)$. ■

Example 7 : $\int_0^{\infty}\left(\frac{t}{1+t^2}\right)^3 dt$

Sol. : Put $t = \tan\theta$, $\therefore\ I = \int_0^{\pi/2}\frac{\tan^3\theta}{\sec^6\theta}\cdot\sec^2\theta\, d\theta$

$= \int_0^{\pi/2}\sin^3\theta\cos\theta\, d\theta = \frac{1}{2}B(2, 1) = \frac{1}{4}$. ■

Example 8 : $\int_0^{\infty}\left(\frac{x}{1+x^2}\right)^6 dx$

Sol. : Put $x = \tan\theta$, $\therefore\ I = \int_0^{\pi/2}\frac{\tan^6\theta}{\sec^{12}\theta}\cdot\sec^2\theta\, d\theta$

$\therefore\ I = \int_0^{\pi/2}\sin^6\theta\cdot\cos^4\theta\, d\theta = \frac{3\pi}{512}$. ■

Example 9 : Prove that

$$\int_{-\pi/6}^{\pi/3}(\sqrt{3}\sin\theta + \cos\theta)^{1/4} d\theta = 2^{-3/4}B\left(\frac{1}{2}, \frac{5}{8}\right)$$

Sol. : $I = \int_{-\pi/6}^{\pi/3}(2)^{1/4}\left(\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta\right)^{1/4} d\theta$

$= 2^{1/4}\int_{-\pi/6}^{\pi/3}\left(\sin\theta\cos\frac{\pi}{6} + \cos\theta\sin\frac{\pi}{6}\right)^{1/4} d\theta$

$= 2^{1/4}\int_{-\pi/6}^{\pi/3}\left[\sin\left(\theta + \frac{\pi}{6}\right)\right]^{1/4} d\theta$

Now put $\theta + \frac{\pi}{6} = \Phi$, $\therefore\ I = 2^{1/4}\int_0^{\pi/2} \sin^{1/4}\Phi \cdot \cos^0\Phi\, d\Phi$

Example 10 : Prove that

$$\int_{-\pi/3}^{\pi/6} (\sin\theta + \sqrt{3}\cos\theta)^{1/6}\, d\theta = 2^{-5/6} B\left(\frac{1}{2}, \frac{7}{12}\right)$$

Sol. : $I = \int_{-\pi/3}^{\pi/6} 2^{1/6}\left(\sin\theta \cdot \frac{1}{2} + \cos\theta \cdot \frac{\sqrt{3}}{2}\right)^{1/6} d\theta$

$$= \int_{-\pi/3}^{\pi/6} 2^{1/6}\left(\sin\theta\cos\frac{\pi}{3} + \cos\theta\sin\frac{\pi}{3}\right)^{1/6} d\theta$$

$$= \int_{-\pi/3}^{\pi/6} 2^{1/6}\left[\sin\left(\theta + \frac{\pi}{3}\right)\right]^{1/6} d\theta$$

Now put $\theta + \frac{\pi}{3} = \Phi$, $\therefore\ I = 2^{1/6}\int_0^{\pi/2} \sin^{1/6}\Phi \cdot \cos^0\Phi\, d\Phi$.

Example 11 : Express the integral $\int_{-\pi/4}^{\pi/4} (\sin\theta + \cos\theta)^{1/5}\, d\theta$ as a Gamma function and show that its value is $2^{-9/10}\, B\,(3/5, 11/2)$.

Sol. : $I = \int_{-\pi/4}^{\pi/4} (\sin\theta + \cos\theta)^{1/5}\, d\theta$

$$= \int_{-\pi/4}^{\pi/4} (\sqrt{2})^{1/5}\left(\sin\theta\frac{1}{\sqrt{2}} + \cos\theta\frac{1}{\sqrt{2}}\right)^{1/5} d\theta$$

$$= \int_{-\pi/4}^{\pi/4} 2^{1/10}\left[\sin\left(\theta + \frac{\pi}{4}\right)\right]^{1/5} d\theta$$

Now put $\theta + \frac{\pi}{4} = \Phi$, $\therefore\ I = \int_0^{\pi/2} 2^{1/10}[\sin\Phi]^{1/5}\cos^0\Phi\, d\theta$.

$$I = 2^{10} \cdot \frac{1}{2} \cdot B\left(\frac{3}{5}, \frac{1}{2}\right).$$

Exercise XI (Page 6-42)

Evaluate the following integrals : Class (a) : 3 Marks

Example 1 : $\int_0^\infty \frac{dx}{(1+x^2)^{5/2}}$

Sol. : Put $x = \tan\theta$

$$\therefore\ I = \int_0^{\pi/2} \frac{\sec^2\theta}{(\sec^2\theta)^{5/2}}\, d\theta = \int_0^{\pi/2} \sec^{-3}\theta\, d\theta = \int_0^{\pi/2} \cos^3\theta\, d\theta$$

Example 2 : $\int_0^\infty \left(\frac{x}{1+x^2}\right)^3 dx$

Sol. : Put $x = \tan\theta$ $\therefore\ \frac{x}{1+x^2} = \frac{\tan\theta}{\sec^2\theta} = \sin\theta\cos\theta$

$$\therefore\ I = \int_0^{\pi/2} \sin^3\theta\cos^3\theta \cdot \sec^2\theta\, d\theta = \int_0^{\pi/2} \sin^3\theta\cos\theta\, d\theta$$

$$= \frac{2 \cdot 1}{4 \cdot 2} = \frac{1}{4}.$$

Example 3 : $\int_0^\infty \left(\frac{x}{1+x^2}\right)^6 dx$

Sol. : Put $x = \tan\theta$

$$\therefore\ I = \int_0^{\pi/2} \sin^6\theta\cos^6\theta\sec^2\theta\, d\theta = \int_0^{\pi/2} \sin^6\theta\cos^4\theta\, d\theta$$

$$= \frac{5 \cdot 3 \cdot 1 \cdot 3 \cdot 1}{10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} = \frac{\pi}{2}.$$

Example 4 : $\int_0^\infty \frac{x^2}{(1+x^6)^{5/2}}\, dx$

Sol. : Put $x^3 = t$ $\therefore\ 3x^2\, dx = dt$ $\therefore\ x^2\, dx = \frac{dt}{3}$.

$$\therefore\ I = \int_0^\infty \frac{1}{(1+t^2)^{5/2}} \cdot \frac{dt}{3} = \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}. \qquad \text{[By Ex. 1]}$$

Example 5 : $\int_0^\infty \frac{x^3}{(1+x^8)^4}\, dx$

Sol. : Put $x^4 = t$ $\therefore\ 4x^3\, dx = dt$ $\therefore\ I = \int_0^\infty \frac{1}{(1+t^2)^4} \cdot \frac{dt}{4}$

Put $t = \tan\theta$

$$\therefore\ I = \int_0^{\pi/2} \frac{1}{\sec^8\theta} \cdot \sec^2\theta \cdot \frac{d\theta}{4} = \int_0^{\pi/2} \cos^6\theta \cdot \frac{d\theta}{4}$$

Exercise XII (Page 6-45)

Prove that : Class (a) : 3 Marks

Example 1 : $\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx = B(m, n)$

Sol. : In solved Ex. 1 above put $a = 1$, $b = 1$.

Example 2 : $\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx = B(n, m)$

Sol. : In solved Ex. 1 above put $a = 1$, $b = 1$.

Example 3 : $\int_0^\infty \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}} = 2B(m, n)$

Sol. : In solved Ex. 1 above put $a = 1$, $b = 1$.

Example 4 : $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\,dx = 0$

Sol. : $I = \int_0^\infty \frac{(x^8 - x^{14})}{(1+x)^{24}}\,dx$; $I_1 = \int_0^\infty \frac{x^{9-1}}{(1+x)^{15+9}}\,dx = B(9, 15)$

$I_2 = \int_0^\infty \frac{x^{15-1}}{(1+x)^{15+9}}\,dx = B(15, 9)$

Example 5 : $\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}\,dx = \frac{1}{5005}$

Sol. : $I = \int_0^\infty \frac{x^4+x^9}{(1+x)^{15}}\,dx$

$I_1 = \int_0^\infty \frac{x^4}{(1+x)^{15}}\,dx = \int_0^\infty \frac{x^{5-1}}{(1+x)^{5+10}}\,dx = B(5, 10)$

$\therefore\ I = 2B(5, 10)$.

Example 6 : $\int_0^\infty \frac{x^6(1-x^{10})}{(1+x)^{24}}\,dx = 0$

Sol. : $I = \int_0^\infty \frac{x^6 - x^{16}}{(1+x)^{24}}\,dx = 0$

$I_1 = \int_0^\infty \frac{x^6}{(1+x)^{24}}\,dx = B(7, 17)$

$I_2 = \int_0^\infty \frac{x^{16}}{(1+x)^{24}}\,dx = B(17, 7)$

Example 7 : $\int_0^\infty \frac{x^3(1+x^8)}{(1+x)^{13}}\,dx = \frac{83}{990}$

Sol. : $I = \int_0^\infty \frac{x^3}{(1+x)^{13}}\,dx + \int_0^\infty \frac{x^{11}}{(1+x)^{13}}\,dx$

$I = B(4, 9) + B(12, 1)$ $\therefore\ I = \frac{3!\,8!}{12!} + \frac{11!\,0!}{12!}$

Example 8 : $\int_0^\infty \frac{x^6 - x^3}{(1+x^3)^5}\,x^2\,dx = 0$

Sol. : $I = \int_0^\infty \frac{x^6}{(1+x^3)^5}\,x^2\,dx - \int_0^\infty \frac{x^3}{(1+x^3)^5}\,x^2\,dx$

Put $x^3 = t$, $\therefore\ I_1 = \frac{1}{3}B(3, 2)$; $I_2 = \frac{1}{3}B(2, 3)$

Exercise XIII (Page 6-49)

Prove that : Class (a) : 3 Marks

Example 1 : $\int_{-1}^1 (1+x)^m (1-x)^n\,dx = 2^{m+n+1} B(m+1, n+1)$.

Hence, evaluate $\int_{-1}^1 \sqrt{\left(\frac{1+x}{1-x}\right)} \cdot dx$.

Sol. : Put $x + 1 = 2t$,

$\therefore\ I = \int_0^1 (2t)^m [1-(2t-1)]^n \cdot 2\,dt$

$= 2^{m+n+1} \int_0^1 t^m (1-t)^n \cdot dt = 2^{m+n+1} B(m+n, n+1)$

$\therefore\ \int_{-1}^1 \sqrt{\left(\frac{1+x}{1-x}\right)} \cdot dx = \int_{-1}^1 (1+x)^{1/2}(1-x)^{-1/2}\,dx$

Now put $m = \frac{1}{2}$, $n = \frac{1}{2}$ in the above result. ■

Example 2 : $\int_a^b \sqrt[n]{(x-a)(b-x)}\,dx = (b-a)^{(2/n)+1} \cdot B\left(\frac{1}{n}+1, \frac{1}{n}+1\right)$

Sol. : Put $x - a = (b-a)\,t$, $dx = (b-a)\,dt$

and $b - x = (b-a)(1-t)$

$\therefore\ I = \int_0^1 (b-a)^{1/n} \cdot t^{1/n} \cdot (b-a)^{1/n} \cdot (1-t)^{1/n} \cdot (b-a)\,dt$

$= (b-a)^{(2/n)+1} \int_0^1 t^{1/n}(1-t)^{1/n}\,dt$

$= (b-a)^{(2/n)+1} B\left(\frac{1}{n}+1, \frac{1}{n}+1\right)$. ■

Example 3 : $\int_3^7 \sqrt[4]{(x-3)(7-x)} \cdot dx = \frac{2(\Gamma 1/4)^2}{3\sqrt{\pi}}$

Sol. : Put $x - 3 = (7-3)\,t = 4t$ *i.e.* $x = 3 + 4t$ $\therefore\ dx = 4\,dt$

and $7 - x = 7 - (3 + 4t) = 4 - 4t = 4(1-t)$

$\therefore\ I = \int_0^1 [4t \cdot 4(1-t)]^{1/4} \cdot 4 \cdot dt$

$= 2 \cdot 4 \int_0^1 t^{1/4}(1-t)^{1/4}\,dt = 8B\left(\frac{5}{4}, \frac{5}{4}\right)$. ■

Example 4 : $\int_5^9 \sqrt[4]{(9-x)(x-5)} \cdot dx = \frac{2(\Gamma 1/4)^2}{3\sqrt{\pi}}$

Sol. : Put $x - 5 = (9-5)\,t = 4t$ *i.e.* $x = 5 + 4t$ $\therefore\ dx = 4\,dt$

and $9 - x = 9 - (5 + 4t) = 4(1-t)$

$\therefore\ I = \int_0^1 [4t \cdot 4(1-t)]^{1/4} \cdot 4\,dt = \frac{2(\Gamma 1/4)^2}{3\sqrt{\pi}}$ as above. ■

Example 5 : $\int_5^6 (x-5)^5 (6-x)^6\, dx = B(6, 7)$

Sol. : Put $x - 5 = (6-5)\, t = t$ i.e. $x = 5 + t$, $dx = dt$

and $6 - x = 6 - (5 + t) = 1 - t$.

When $x = 5$, $t = 0$; when $x = 6$, $t = 1$

$\therefore\ I = \int_0^1 t^5 (1-t)^6\, dt = B(6, 7)$.

Miscellaneous Exercise (Page 6-61)

(A) Class (a) : 3 Marks

Example 1 : Find the value of

(i) $B\left(\frac{4}{3}, \frac{5}{3}\right)$ (ii) $B\left(\frac{3}{2}, \frac{1}{2}\right)$ (iii) $\overline{\left|\frac{1}{4}\right|}\ \overline{\left|\frac{2}{4}\right|}\ \overline{\left|\frac{3}{4}\right|}\ \overline{\left|\frac{4}{4}\right|}$

Sol. : (i) $B\left(\frac{4}{3}, \frac{5}{3}\right) = \frac{\overline{|4/3}\ \overline{|5/3}}{\overline{|3}} = \frac{(1/3)\overline{|1/3} \cdot (2/3)\overline{|2/3}}{2!}$

$$= \frac{2}{9} \cdot \frac{1}{2} \cdot \overline{\left|\frac{1}{3}\right|}\ \overline{\left|\frac{2}{3}\right|} = \frac{1}{9} \cdot \frac{\pi}{\sin(\pi/3)}.$$

(ii) $B\left(\frac{3}{2}, \frac{1}{2}\right) = \frac{\overline{|3/2}\ \overline{|1/2}}{\overline{|2}} = \frac{(1/2)\overline{|1/2}\ \overline{|1/2}}{2!} = \frac{1}{4}\pi$

(iii) $\left(\overline{\left|\frac{1}{4}\right|}\ \overline{\left|\frac{3}{4}\right|}\right) \cdot \overline{\left|\frac{1}{2}\right|} = \sqrt{2} \cdot \pi \cdot \sqrt{\pi}$

Example 2 : (i) Given $\overline{|1{\cdot}6} = 0{\cdot}8935$, find the value of $\overline{|-2{\cdot}4}$.

(ii) Given $\sqrt{\pi} = 1{\cdot}772$, find the value $\overline{|-4{\cdot}5}$.

Sol. : (i) $\overline{|-2{\cdot}4} = \frac{\overline{|-1{\cdot}4}}{(-2{\cdot}4)} = \frac{1}{(-2{\cdot}4)} \cdot \frac{1}{(-1{\cdot}4)} \cdot \frac{1}{(-0{\cdot}4)} \cdot \frac{\overline{|0{\cdot}6}}{\overline{|0} \cdot \overline{|6}}$

(ii) $\overline{|-4{\cdot}5} = \frac{\overline{|-3{\cdot}5}}{(-4{\cdot}5)} = \frac{1}{(-4{\cdot}5)} \cdot \frac{1}{(-3{\cdot}5)} \cdot \frac{1}{(-2{\cdot}5)} \cdot \frac{1}{(-1{\cdot}5)} \cdot \frac{\overline{|0{\cdot}5}}{(-0{\cdot}5)}$

Example 3 : Show that

(i) $\overline{|x}\ \overline{|-x} = -\frac{\pi}{x \sin x\pi}$ (ii) $\overline{\left|\frac{1}{2} + x\right|}\ \overline{\left|\frac{1}{2} - x\right|} = \frac{\pi}{\cos \pi x}$

Sol. : (i) $\overline{|1-x} = \overline{|1+(-x)} = (-x)\overline{|-x}$ $\therefore\ \overline{|x}\ \overline{|1-x} = (-x)\ \overline{|x}\ \overline{|-x}$

$\therefore\ \frac{\pi}{\sin \pi x} = (-x)\ \overline{|x}\ \overline{|-x}$ $\therefore\ \overline{|x}\ \overline{|-x} = -\frac{\pi}{x \sin(\pi x)}$.

(ii) $\overline{\left|\frac{1}{2} + x\right|} = \overline{\left|1 - \left(\frac{1}{2} - x\right)\right|}$

$$\therefore\ \overline{\left|\frac{1}{2} + x\right|}\ \overline{\left|\frac{1}{2} - x\right|} = \overline{\left|1 - \left(\frac{1}{2} - x\right)\right|}\ \overline{\left|\left(\frac{1}{2} - x\right)\right|}$$

$$= \overline{|y}\ \overline{|1-y} \text{ where } y = \frac{1}{2} - x$$

But $\overline{|y}\ \overline{|1-y} = \frac{\pi}{\sin \pi y} = \frac{\pi}{\sin \pi[(1/2) - x]}$

$$= \frac{\pi}{\sin[(\pi/2) - \pi x]} = \frac{\pi}{\cos \pi x}.$$

Example 4 : Show that $\overline{\left|\frac{3}{2} - n\right|}\ \overline{\left|\frac{3}{2} + n\right|} = \left(\frac{1}{4} - n^2\right) \pi \sec n\pi$.

Sol. : l.h.s. $= \left\{\left(\frac{1}{2} - n\right)\overline{\left|\frac{1}{2} - n\right|}\right\}\left\{\left(\frac{1}{2} + n\right)\overline{\left|\frac{1}{2} + n\right|}\right\}$

$$= \left(\frac{1}{4} - n^2\right)\overline{\left|n + \frac{1}{2}\right|}\ \overline{\left|1 - \left(n + \frac{1}{2}\right)\right|}$$

$$= \left(\frac{1}{4} - n^2\right)\overline{|y}\ \overline{|1-y} \text{ where } y = n + \frac{1}{2}$$

$$= \left(\frac{1}{4} - n^2\right) \cdot \frac{\pi}{\sin \pi y} = \left(\frac{1}{4} - n^2\right) \cdot \frac{\pi}{\sin \pi[n + (1/2)]}$$

Example 5 : Using Beta function, prove that $\int_0^\infty \frac{dx}{1 + x^2} = \frac{\pi}{2}$.

Sol. : Put $x = \tan \theta$, $dx = \sec^2 \theta\, d\theta$.

$$\therefore\ I = \int_0^{\pi/2} \frac{1}{\sec^2 \theta} \cdot \sec^2 \theta\, d\theta = \int_0^{\pi/2} d\theta$$

$$= \int_0^{\pi/2} \sin^0 \theta \cos^0 \theta\, d\theta = \frac{1}{2} B\left(\frac{1}{2}, \frac{1}{2}\right)$$

$$= \frac{1}{2} \cdot \frac{\overline{|1/2}\ \overline{|1/2}}{\overline{|1}} = \frac{1}{2} \cdot \sqrt{\pi} \cdot \sqrt{\pi} = \frac{\pi}{2}.$$

[OR $\int_0^{\pi/2} \sin^0 \theta\, d\theta = \frac{\pi}{2}$]

Example 6 : Show that $\int_0^1 \left(\frac{1}{x} - 1\right)^{1/4} dx = \frac{\pi}{2\sqrt{2}}$.

Sol. : $I = \int_0^1 (1-x)^{1/4} \cdot x^{-1/4}\, dx = B\left(\frac{5}{4}, \frac{3}{4}\right)$

$$= \frac{\overline{|5/4}\ \overline{|3/4}}{\overline{|2}} = \frac{1}{4} \cdot \frac{\overline{|1/4}\ \overline{|3/4}}{1!} = \frac{\sqrt{2} \cdot \pi}{4} = \frac{\pi}{2\sqrt{2}}.$$

Example 7 : Prove that $\int_0^1 \frac{x^2}{(1-x^4)^{1/2}}\,dx \cdot \int_0^\infty \frac{1}{(1+x^4)^{1/2}}\,dx = \frac{\pi}{4\sqrt{2}}$.

Sol. : In I_1, put $x^2 = \sin\theta$, $x = \sqrt{\sin\theta}$; $dx = \frac{1}{2\sqrt{\sin\theta}}\cos\theta\,d\theta$

In I_2, put $x^2 = \tan\theta$, $x = \sqrt{\tan\theta}$;

$$dx = \frac{1}{2\sqrt{\tan\theta}}\sec^2\theta\,d\theta = \frac{1}{2}(\sin\theta)^{-1/2}\cos^{-3/2}\theta\,d\theta$$

$$\therefore\ I = \int_0^{\pi/2} \sin\theta\cdot\cos^{-1}\theta\cdot\frac{1}{2}\cdot\sin^{-1/2}\theta\cdot\cos\theta\,d\theta$$

$$\times\int_0^{\pi/2}\cos\theta\cdot\frac{1}{2}\cdot\sin^{-1/2}\theta\cdot\cos^{-3/2}\theta\,d\theta$$

$$I = \int_0^{\pi/2}\frac{1}{2}\sin^{1/2}\theta\,d\theta\cdot\frac{1}{2}\int_0^{\pi/2}\sin^{-1/2}\theta\cdot\cos^{-1/2}\theta\,d\theta$$

$$= \frac{1}{4}\cdot\frac{1}{2}\cdot\frac{\overline{|1/2}\ \overline{|3/4}}{\overline{|5/4}}\cdot\frac{1}{2}\cdot\frac{\overline{|1/4}\ \overline{|1/4}}{\overline{|1/2}}.$$

Example 8 : Prove that $\iint_D x^{l-1}y^{m-1}\,dx\,dy = a^{l+m}\frac{\overline{|l}\ \overline{|m}}{\overline{|l+m+1}}$

where D is the domain $x \ge 0$, $y \ge 0$, $x + y \le a$.

Sol. : Put $x = au$, $y = av$. $\therefore\ I = \int_0^1\int_0^{1-v}(au)^{l-1}(av)^{m-1}a^2\,du\,dv$

$$= \frac{a^{l+m}}{l}\int_0^1 v^{m-1}(1-v)^l\,du = \frac{a^{l+m}}{l}B(m, l+1).$$

Example 9 : If $B(n, 2) = \frac{1}{42}$ and n is a positive integer find n.

Sol. : $\frac{\overline{|n}\ \overline{|2}}{\overline{|n+2}} = \frac{1}{42}$ $\quad\therefore\ \frac{\overline{|n}\cdot 1!}{(n+1)\,n\,\overline{|n}} = \frac{1}{42}$

$\therefore\ n^2 + n - 42 = 0 \quad \therefore\ (n+7)(n-6) = 0.$

Example 10 : If $B(n, 3) = \frac{1}{60}$ and n is a positive integer, find n.

Sol. : $\frac{\overline{|n}\ \overline{|3}}{\overline{|n+3}} = \frac{1}{60}$ $\quad\therefore\ \frac{2!}{(n+2)(n+1)\cdot n\,\overline{|n}} = \frac{1}{60}$

$\therefore\ n^3 + 3n^2 + 2n - 120 = 0$

$\therefore\ (n-4)(n^2 + 7n - 28) = 0 \quad \therefore\ n = 4.$

(B) Class (c) : 8 Marks

Example 1 : Prove that $\int_0^\infty \frac{e^{2mx} + e^{-2mx}}{(e^x + e^{-x})^{2n}}\,dx = \frac{1}{2}B(m+n, n-m).$

Sol. : Multiply by e^{2nx} in denominator and numerator.

$$\therefore\ I = \int_0^\infty \frac{e^{2(m+n)x} + e^{-2(m-n)x}}{(e^{2x}+1)^{2n}}\,dx$$

Now put $e^{2x} = t$, $\quad\therefore\ I = \int_1^\infty \frac{t^{m+n} + t^{-(m-n)}}{(t+1)^{2n}}\cdot\frac{1}{2t}\,dt$

Now put $t = \frac{1}{y}$, $\quad\therefore\ I = \frac{1}{2}\int_0^1 \frac{y^{m+n-1} + y^{-(m-n+1)}}{(1+y)^{2n}}\,dy$

In I_1 put $y = \frac{u}{1-u}$,

$$\therefore\ I_1 = \int_0^{1/2} u^{m+n-1}\cdot(1-u)^{-(m-n+1)}\,du$$

Similarly, (changing the sign of m)

$$I_2 = \int_0^{1/2} u^{-m+n-1}\cdot(1-u)^{m+n-1}\,du$$

Now put $u = 1 - z$ in I_2,

$$\therefore\ I_2 = \int_{1/2}^1 (1-u)^{-(m-n+1)}\cdot u^{m+n-1}\,du$$

$$\therefore\ I_1 + I_2 = \int_0^1 u^{m+n-1}\cdot u^{n-m-1}\,du = B(m+n, n-m).$$

Example 2 : Prove that $\int_0^\infty \frac{x^2\,dx}{(1+x^4)^{3/2}}\cdot\int_0^\infty \frac{dx}{(1+x^4)^{1/2}} = \frac{\pi}{4}$.

Sol. : Put $x^4 = t$, $\quad\therefore\ I_1 = \frac{1}{4}\int_0^1 \frac{t^{-1/4}}{(1+t)^{3/2}}\,dt = \frac{1}{4}B\left(\frac{3}{4}, \frac{3}{4}\right)$

$$I_2 = \frac{1}{4}\int_0^1 \frac{t^{-3/4}}{(1+t)^{1/2}}\,dt = \frac{1}{4}B\left(\frac{1}{4}, \frac{1}{4}\right)$$

$$\therefore\ I = \frac{1}{16}\cdot\frac{\overline{|3/4}\ \overline{|3/4}}{(1/2)\overline{|1/2}}\cdot\frac{\overline{|1/4}\ \overline{|1/4}}{\overline{|1/2}} = \frac{1}{8}\cdot\frac{2\pi^2}{\pi} = \frac{\pi}{4}.$$

Example 3 : Express $\int_{-1}^1 (1+x)^m(1-x)^n\,dx$ **as a Beta Function.**

Sol. : Put $1 + x = 2t$,

$$\therefore\ I = \int_0^1 (2t)^m(1+1-2t)^n\cdot 2\,dt = 2^{m+n-1}B(m+1, n+1).$$

Example 4 : Prove that $\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{2^{(2-n)/n}\cdot(\overline{|1/n})^2}{n\cdot\overline{|2/n}}$.

Sol. : Put $x^n = t$,

$$\therefore\ I = \int_0^1 \frac{1}{n}\cdot t^{(1/n)-1}\cdot(1-t)^{-1/2}\,dt = \frac{1}{n}\cdot\frac{\overline{|1/n}\ \overline{|1/2}}{\overline{|(1/n)+(1/2)}}.$$

By duplication formula,

$$2^{2(1/n)-1} \cdot \Gamma\left(\frac{1}{n}\right) \Gamma\left(\frac{1}{n}+\frac{1}{2}\right) = \sqrt{\pi} \cdot \Gamma\left(2 \cdot \frac{1}{n}\right)$$

$$\therefore \quad I = \frac{1}{n} \cdot \frac{\Gamma(1/n)\,\Gamma(1/2)}{1} \cdot \frac{2^{2(1/n)-1}}{\sqrt{\pi}\,\Gamma(2/n)} \cdot \Gamma\left(\frac{1}{n}\right)$$

Example 5 : Prove that $\int_0^\infty \sec h^6 x\, dx = \frac{8}{15}$.

Sol. : Put $e^x = \tan\theta \quad \therefore \; dx = \dfrac{d\theta}{\sec\theta\cos\theta}$

$$\cos hx = \frac{e^x + e^{-x}}{2} = \frac{\tan\theta + \cot\theta}{2}$$

$$= \frac{1}{2}\left[\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right] = \frac{1}{2} \cdot \frac{1}{\sin\theta\cos\theta}$$

$$\therefore \quad I = \frac{1}{2}\int_{-\infty}^{\infty} \sec h^6 x\, dx$$

$$= \frac{1}{2}\int_0^{\pi/2} 2^6 \sin^6\theta \cos^6\theta \cdot \frac{1}{\sin\theta\cos\theta}\, d\theta$$

$$= \frac{1}{2} \cdot 2^6 \int_0^{\pi/2} \sin^5\theta \cos^5\theta\, d\theta = 2^5 \cdot \frac{1}{2} B(3, 3) = \frac{8}{15}.$$

Example 6 : Prove that $\int_0^2 x \cdot \sqrt[3]{8 - x^3}\, dx = \frac{16\pi}{9\sqrt{3}}$.

Sol. : Put $x^3 = 8t$,

$$\therefore \quad I = \int_0^1 2t^{1/3} \cdot 2(1-t)^{1/3} \cdot \frac{2}{3} t^{-2/3}\, dt$$

$$\therefore \quad I = \frac{8}{3} B\left(\frac{2}{3}, \frac{4}{3}\right) = \frac{16\pi}{9\sqrt{3}}.$$

Example 7 : Prove that $B(n, n) = 2\int_0^{1/2} (t - t^2)^{n-1} dt$.

Sol. : $B(n,n) = \int_0^{1/2} t^{n-1} \cdot (1-t)^{n-1} dt + \int_{1/2}^1 t^{n-1} \cdot (1-t)^{n-1} dt$

In the second integral, put $1 - t = x \quad \therefore \; -dt = dx$

$$\therefore \quad I_2 = \int_{1/2}^0 (1-x)^{n-1} \cdot x^{n-1} (-dx)$$

Example 8 : Find $\int_0^3 \frac{x^{3/2}}{\sqrt{3-x}}\, dx \int_0^1 \frac{dx}{\sqrt{1-x}}$.

Sol. : In I_1 put $x = 3t$,

$$\therefore \quad I_1 = \int_0^1 \frac{3^{3/2} \cdot t^{3/2}}{\sqrt{3} \cdot (1-t)^{1/2}}\, dt = 9B\left(\frac{5}{2}, \frac{1}{2}\right)$$

In I_2 put $x^4 = t$,

$$\therefore \quad I_2 = \int_0^1 (1-t)^{-1/2} \cdot \frac{1}{4} t^{-3/4}\, dt$$

$$\therefore \quad I = \frac{9}{4} B\left(\frac{5}{2}, \frac{1}{2}\right) B\left(\frac{1}{4}, \frac{1}{2}\right).$$

Example 9 : Prove that $\int_0^1 \frac{(1-x^4)^{3/4}}{(1+x^4)^2}\, dx = \frac{3\pi}{2^{15/4}}$.

Sol. : Putting $x^4 = t$,

$$\therefore \quad I = \frac{1}{4}\int_0^1 \frac{(1-t)^{3/4}}{(1+t)^2} \cdot t^{-3/4}\, dt$$

Now, put $t = \dfrac{u}{2-u}$

$$\therefore \quad I = \frac{1}{2^{9/4}}\int_0^1 u^{-3/4} (1-u)^{3/4}\, du = \frac{1}{2^2 \cdot 2^{1/4}} B\left(\frac{1}{4}, \frac{7}{4}\right).$$

Example 10 : Prove that

$$\int_a^b (x-a)^m (b-x)^n\, dx = (b-a)^{m+n+1} B(m+1, n+1)$$

Hence deduce that

(i) $\int_{-1}^1 (1+x)^m (1-x)^n\, dx = 2^{n+n+1} B(m+1, n+1)$

(ii) $\int_0^n x^n (1-x)^p dx = n^{p+n+1} B(n+1, p+1)$

Sol. : See Type IX, page (6-48) of the Text Book.

(i) Put $a = -1$, $b = 1$. (ii) Put $a = 0$, $b = n$, $m = n$.

Exercise XIV (Page 6-63)

Class (a) : 3 marks

Example 1 : Evaluate $\int_0^\infty x^2 e^{-x^2} dx$.

Sol. : Put $x^2 = t$, $x = \sqrt{t}$

$$\therefore \quad I = \int_0^\infty t \cdot e^t \cdot \frac{1}{2\sqrt{t}}\, dt = \frac{1}{2}\int_0^\infty e^{-t} \cdot \sqrt{t}\, dt$$

$$= \frac{1}{2}\Gamma\left(\frac{3}{2}\right) = \frac{1}{2} \cdot \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{4}.$$

Example 2 : Evaluate $\int_0^\infty \sqrt{x} \cdot e^{-x^2} dx$.

Sol. : Put $x^2 = t$, $x = \sqrt{t}$

$$\therefore \quad I = \int_0^\infty t^{1/4} \cdot e^{-t} \frac{1}{2\sqrt{t}}\, dt = \frac{1}{2}\int_0^\infty e^{-t} \cdot t^{-1/4}\, dt = \frac{1}{2}\Gamma\left(\frac{3}{4}\right)$$

Example 3 : Evaluate $\int_0^\infty x^n e^{-\sqrt{ax}}\,dx.$

Sol. : Put $\sqrt{ax} = t,\ x = \frac{t^2}{a}$

$$\therefore\ I = \int_0^\infty \frac{t^{2n}}{a^n} e^{-t} \cdot 2t \frac{dt}{a} = \frac{2}{a^{n+1}} \int_0^\infty e^{-t} \cdot t^{2n+1}\,dt$$

$$= \frac{2}{a^{n+1}} \cdot \overline{|2n+2}$$

Example 4 : Evaluate $\int_0^1 (-\log x)^5\,dx.$

Sol. : Put $\log x = -t,\ x = e^{-t},\ dx = -e^{-t}\,dt$

$$\therefore\ I = \int_\infty^0 (t)^5 \cdot e^{-t}(-dt) = \int_0^\infty e^{-t} \cdot t^5\,dt = \overline{|6}$$

Example 5 : Evaluate $\int_0^1 x^3 \left[\log\left(\frac{1}{x}\right)\right]^4 dx.$

Sol. : Put $\log x = -t,\ x = e^{-t},\ dx = -e^{-t}\,dt.$

$$\therefore\ I = \int_\infty^0 e^{-3t} \cdot (t)^4 (-e^{-t})\,dt = \int_0^\infty e^{-4t} \cdot t^4\,dt$$

Put $4t = u.$

$$\therefore\ I = \int_0^\infty e^{-u} \left(\frac{u}{4}\right)^4 \cdot \frac{du}{4} = \frac{1}{4^5}\overline{|5}$$

Example 6 : Evaluate $\int_0^1 \sqrt[3]{\log(1/x)} \cdot dx.$

Sol. : Put $\log x = -t,\ x = e^{-t},\ dx = -e^{-t}\,dt.$

$$\therefore\ I = \int_\infty^0 (t)^{1/3} \cdot (-e^{t})\,dt = \int_0^\infty e^{-t} \cdot t^{1/3}\,dt = \overline{\left|\frac{4}{3}\right.}$$

Example 7 : Evaluate $\int_0^\infty 3^{-4x^2}\,dx.$

Sol. : Put $3^{-4x^2} = e^{-t}$

$$\therefore\ -t = \log 3^{-4x^2} = -4x^2 \log 3 \qquad \therefore\ \frac{t}{4\log 3} = x^2$$

$$\therefore\ x = \frac{1}{2\sqrt{\log 3}} \cdot t^{1/2} \qquad \therefore\ dx = \frac{t^{-1/2}}{4\sqrt{\log 3}}\,dt$$

$$\therefore\ I = \int_0^\infty e^{-t} \cdot \frac{1}{4\sqrt{\log 3}} \cdot t^{-1/2}\,dt = \frac{1}{4\sqrt{\log 3}}\overline{\left|\frac{1}{2}\right.}$$

Example 8 : Evaluate $\int_0^4 \sqrt{x}\,(4-x)^{3/2}\,dx.$

Sol. : Put $x = 4t,\ dx = 4\,dt.$

$$\therefore\ I = \int_0^1 2\sqrt{t} \cdot 2^3 (1-t)^{3/2} \cdot 4\,dt$$

$$= 64 \int_0^1 t^{1/2} (1-t)^{3/2}\,dt = 64\,B\left(\frac{3}{2}, \frac{5}{2}\right) = 64\,\frac{\overline{|3/2}\ \overline{|5/2}}{\overline{|4}}$$

$$= 64 \cdot \frac{(1/2)\,\overline{|1/2} \cdot (3/2)(1/2)\overline{|1/2}}{3!} = 4\pi$$

Example 9 : Evaluate $\int_0^2 x^3 \sqrt{2-x} \cdot dx.$

Sol. : Put $x = 2t \qquad \therefore\ dx = 2\,dt$

$$\therefore\ I = \int_0^1 8t^3 \sqrt{2} \cdot (1-t)^{1/2} \cdot 2\,dt = 16\sqrt{2} \int_0^1 t^3 (1-t)^{1/2}\,dt$$

$$= 16\sqrt{2} \cdot B\left(4, \frac{3}{2}\right) = 16\sqrt{2} \cdot \frac{\overline{|4}\ \overline{|3/2}}{\overline{|11/2}} = \frac{512}{315}\sqrt{2}$$

Example 10 : Evaluate $\int_0^2 x^4 (8-x^3)^{-1/3}\,dx.$

Sol. : Put $x^3 = 8t,\ x = 2t^{1/3},\ dx = \frac{2}{3} t^{-2/3}\,dt$

$$\therefore\ I = \int_0^1 2^4 \cdot t^{4/3} \cdot 8^{-1/3} (1-t)^{-1/3} \cdot \frac{2}{3} t^{-2/3}\,dt$$

$$= \int_0^1 16 \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot t^{2/3} (1-t)^{-1/3}\,dt = \frac{16}{3} B\left(\frac{5}{3}, \frac{2}{3}\right)$$

Example 11 : Evaluate $\int_0^{2a} x\sqrt{2ax - x^2} \cdot dx.$

Sol. : Put $x = 2\,at,\ dx = 2a\,dt.$

$$\therefore\ I = \int_0^{2a} x \cdot x^{1/2} \sqrt{2a - x} \cdot dx$$

$$= \int_0^1 (2at)^{3/2} \cdot \sqrt{2a} \cdot (1-t)^{1/2} \cdot 2a\,dt$$

$$= (2a)^3 \int_0^1 t^{3/2} (1-t)^{1/2}\,dt = 8a^3 \cdot B\left(\frac{5}{2}, \frac{3}{2}\right) = \frac{\pi}{2} a^3$$

Example 12 : Evaluate $\int_0^1 \frac{dx}{\sqrt{1-x^4}}.$

Sol. : Put $x^4 = t,\ x = t^{1/4},\ dx = \frac{1}{4} t^{-3/4}\,dt$

$$\therefore\ I = \int_0^1 \frac{1}{4} t^{-3/4} (1-t)^{-1/2} = \frac{1}{4} B\left(\frac{1}{4}, \frac{1}{2}\right)$$

Example 13 : Evaluate $\int_0^3 \frac{x^{3/2}}{\sqrt{3-x}}\,dx$.

Sol. : Put $x = 3\sin^2\theta$, $dx = 6\sin\theta\cos\theta\,d\theta$.

$$\therefore\ I = \int_0^{\pi/2} \frac{3\sqrt{3}\cdot\sin^3\theta}{\sqrt{3}\cdot\cos\theta}\,6\sin\theta\cos\theta\,d\theta$$

$$= \int_0^{\pi/2} 18\cdot\sin^4\theta\,d\theta = 18\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{27}{8}\pi$$

Example 14 : Find $B\left(\frac{5}{2}, \frac{3}{2}\right)$.

Sol. :
$$B\left(\frac{5}{2}, \frac{3}{2}\right) = \frac{\overline{|5/2}\ \overline{|3/2}}{\overline{|4}}$$

$$= \frac{(3/2)\ (1/2)\ \overline{|1/2}\ (1/2)\ \overline{|1/2}}{3!} = \frac{\pi}{16}$$

Example 15 : If $B(n, 3) = \frac{1}{3}$, **find** n.

Sol. :
$$\frac{\overline{|n}\ \overline{|3}}{\overline{|n+3}} = \frac{1}{3} \qquad \therefore\ \frac{\overline{|n}\cdot 2!}{(n+2)(n+1)\ n\overline{|n}} = \frac{1}{3}$$

$$\therefore\ (n+1)(n+2)\,n = 6 \qquad \therefore\ n^3 + 3n^2 + 2n - 6 = 0$$

$$\therefore\ (n-1)(n^2 + 4n + 6) = 0 \quad \therefore\ n = 1$$

Example 16 : If $\overline{|1\cdot 8} = 0\cdot 9314$, **find** $\overline{|4\cdot 8}$.

Sol. : We know that $\overline{|n} = (n-1)\overline{|n-1}$

$$\therefore\ \overline{|4\cdot 8} = (3\cdot 8)\overline{|3\cdot 8} = (3\cdot 8)\cdot(2\cdot 8)\cdot\overline{|2\cdot 8}$$

$$= (3\cdot 8)\cdot(2\cdot 8)\cdot(1\cdot 8)\cdot\overline{|1\cdot 8} = 17\cdot 8382$$

Example 17 : If $\overline{\left|\frac{3}{5}\right.} = 1.4891$, **find** $\overline{\left|\frac{18}{5}\right.}$.

Sol. : We know that $\overline{|n} = (n-1)\overline{|n-1}$

$$\therefore\ \overline{\left|\frac{18}{5}\right.} = \left(\frac{13}{5}\right)\overline{\left|\frac{13}{5}\right.} = \left(\frac{13}{5}\right)\left(\frac{8}{5}\right)\overline{\left|\frac{8}{5}\right.}$$

$$= \left(\frac{13}{5}\right)\left(\frac{8}{5}\right)\left(\frac{3}{5}\right)\overline{\left|\frac{3}{5}\right.} = 3\cdot 7168$$

Example 18 : Evaluate $\int_0^\infty e^{-x^2}\,dx$.

Sol. : Putting $x^2 = t$, $x = \sqrt{t}$, $dx = \frac{1}{2\sqrt{t}}\,dt$, we get

$$I = \int_0^\infty \frac{1}{2\sqrt{t}}\cdot e^{-t}\,dt = \frac{1}{2}\int_0^\infty t^{-1/2}\cdot e^{-t}\,dt = \frac{1}{2}\overline{\left|\frac{1}{2}\right.} = \frac{\sqrt{\pi}}{2}.$$

Example 19 : Evaluate $B\left(\frac{3}{2}, \frac{3}{2}\right)$.

Sol. :
$$B\left(\frac{3}{2}, \frac{3}{2}\right) = \frac{\overline{|3/2}\ \overline{|3/2}}{\overline{|3}} = \frac{\left[(1/2)\overline{|1/2}\right]^2}{2!}$$

$$= \frac{1}{2}\cdot\left[\left(\frac{1}{2}\right)\sqrt{\pi}\right]^2 = \frac{\pi}{8}.$$

Example 20 : Evaluate $\int_0^{\pi/4} \sin^5 2\theta\,d\theta$.

Sol. : Put $2\theta = t$, $\theta = \frac{t}{2}$, $d\theta = \frac{1}{2}\,dt$.

When $\theta = 0$, $t = 0$; when $\theta = \frac{\pi}{4}$, $t = \frac{\pi}{2}$.

$$\therefore\ I = \int_0^{\pi/2} \sin^5 t\cdot\frac{dt}{2} = \frac{1}{2}\cdot\frac{4\cdot 2}{5\cdot 3\cdot 1} = \frac{4}{15}.$$

Example 21 : Evaluate $\int_0^{\pi/6} \cos^5 3\theta\,d\theta$.

Sol. : Put $3\theta = t$, $\theta = \frac{t}{3}$, $d\theta = \frac{1}{3}\,dt$.

When $\theta = 0$, $t = 0$; when $\theta = \frac{\pi}{6}$, $t = \frac{\pi}{2}$.

$$\therefore\ I = \int_0^{\pi/2} \frac{1}{3}\cos^6 t\,dt = \frac{1}{3}\cdot\frac{5\cdot 3\cdot 1}{6\cdot 4\cdot 2}\cdot\frac{\pi}{2} = \frac{5}{96}.$$

Example 22 : Show that $B\left(\frac{1}{4}, \frac{3}{4}\right) = \sqrt{2}\cdot\pi$.

Sol. : We have
$$B\left(\frac{1}{4}, \frac{3}{4}\right) = \frac{\overline{|1/4}\ \overline{|3/4}}{\overline{|1}} = \frac{\sqrt{2}\cdot\pi}{1} = \sqrt{2}\,\pi. \quad \text{[By Ex. 24 below]}$$

Example 23. : State the duplication formula and evaluate $\overline{|5/2}$.

Sol. : The duplication formula states that

$$2^{2m-1}\overline{|m}\ \overline{|m + (1/2)} = \sqrt{\pi}\cdot\overline{|2m}$$

$$\therefore\ \overline{\left|m + \frac{1}{2}\right.} = \frac{\sqrt{\pi}\cdot\overline{|2m}}{2^{2m-1}\ \overline{|m}}$$

Putting $m = 2$, we get
$$\overline{\left|\frac{5}{2}\right.} = \frac{\sqrt{\pi}\cdot\overline{|4}}{2^5\ \overline{|2}} = \frac{\sqrt{\pi}\cdot 3!}{2^3\cdot 1!} = \frac{3}{4}\sqrt{\pi}.$$

Example 24 : State the duplication formula and prove that

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \sqrt{2}\cdot\pi.$$

Sol. : The duplication formula states that

$$2^{2m-1}\Gamma(m)\,\Gamma(m+(1/2)) = \sqrt{\pi}\cdot\Gamma(2m)$$

Putting $m = \frac{1}{4}$, we get $\quad 2^{-1/2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \sqrt{\pi}\cdot\Gamma\left(\frac{1}{2}\right)$

$$\therefore\ \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{\sqrt{\pi}\cdot\sqrt{\pi}}{2^{-1/2}} = \sqrt{2}\cdot\pi.$$

Example 25 : Prove that $\int_0^{\pi/2}\sqrt{\cot\theta}\,d\theta = \frac{\pi}{\sqrt{2}}$.

Sol. : We have

$$I = \int_0^{\pi/2}(\cos\theta)^{1/2}(\sin\theta)^{-1/2}\,d\theta = \frac{1}{2}B\left(\frac{3}{4},\frac{1}{4}\right)$$

$$= \frac{1}{2}\cdot\frac{\Gamma(3/4)\,\Gamma(1/4)}{\Gamma(1)} = \frac{1}{2}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right) = \frac{1}{2}\sqrt{2}\cdot\pi = \frac{\pi}{\sqrt{2}}.$$

Example 26 : Prove that $B\left(\frac{1}{6},\frac{5}{6}\right) = 2\pi$.

Sol. : We have

$$B\left(\frac{1}{6},\frac{5}{6}\right) = \frac{\Gamma(1/6)\,\Gamma(5/6)}{\Gamma(1)} = 2\pi \quad \text{[By (2), page 6-53 of the text book]}$$

***

CHAPTER 7

Differentiation Under Integral Sign

Exercise I (Page 7-14)

Assuming the validity of differentiation under the integral sign prove that : Class (c) : 8 Marks

Example 1 : $\int_0^\infty \frac{e^{-\alpha x}\sin x}{x}\,dx = \cot^{-1}\alpha$.

Deduce that $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$.

Sol. : $\quad I(\alpha) = \int_0^\infty \frac{e^{-\alpha x}\sin x}{x}\,dx$

$$\therefore\ \frac{dI}{d\alpha} = -\int_0^\infty e^{-\alpha x}\sin x\,dx$$

$$= -\frac{1}{\alpha^2+1}\left[e^{-\alpha x}(-\alpha\sin x - 1\cdot\cos x)\right]_0^\infty = -\frac{1}{\alpha^2+1}$$

Integrating, we get $\quad \therefore\ I = \frac{\pi}{2} - \tan^{-1}\alpha = \cot^{-1}\alpha.$

Now, put $\alpha = 0$, $\quad \therefore\ \int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$

Example 2 : $\int_0^\infty e^{-x^2}\cos 2ax\ dx = \frac{\sqrt{\pi}}{2}e^{-a^2}$.

Sol. : $\quad I(a) = \int_0^\infty e^{-x^2}\cos 2ax\,dx$

$$\therefore\ \frac{dI}{da} = \int_0^\infty [e^{-x^2}(-2x)]\cdot\sin 2ax\,dx$$

$$= \left[\sin 2ax\cdot e^{-x^2}\right]_0^\infty - \int_0^\infty e^{-x^2}\cdot\cos 2ax\cdot 2a\,dx$$

$$= 0 - 2aI \quad \therefore\ \frac{dI}{I} = -2a\,da$$

Integrating, $\quad \log I = -a^2 + \log c. \quad \therefore\ I = \frac{\sqrt{\pi}}{2}e^{-a^2}.$

Example 3 : $\int_0^\infty \frac{1-\cos ax}{x^2}\,dx = \frac{\pi a}{2}$.

Sol. : $\frac{dI}{da} = \int_0^\infty \frac{\sin ax}{x}\,dx = \int_0^\infty \frac{\sin t}{t/a}\cdot\frac{dt}{a}$ where $ax = t$.

$\therefore \frac{dI}{da} = \int_0^\infty \left(\frac{\sin t}{t}\right)dt = \frac{\pi}{2}$ [By Ex. (1)] $\therefore I = \frac{\pi}{2}a + c$

$I(0) = \int_0^\infty \frac{1-1}{x^2}\,dx = 0 \quad \therefore c = 0 \quad \therefore I = \frac{\pi}{2}a.$

Example 4 : $\int_0^\pi \frac{dx}{a-\cos x} = \frac{\pi}{\sqrt{a^2-1}}$, $(a > 0)$. **Deduce that**

(i) $\int_0^\pi \frac{dx}{(a-\cos x)^2} = \frac{\pi a}{(a^2-1)^{3/2}}$ (ii) $\int_0^\pi \frac{dx}{(2-\cos x)^2} = \frac{2\pi}{3\sqrt{3}}$

Sol. : Put $t = \tan\left(\frac{x}{2}\right)$,

$$\therefore I = \int_0^\infty \frac{2}{(a+1)}\cdot\frac{1}{[(a-1)/(a+1)]+t^2}\,dt$$

$$= \frac{2}{a+1}\cdot\sqrt{\frac{a+1}{a-1}}\cdot\left[\tan^{-1} t\sqrt{\frac{a+1}{a-1}}\right]_0^\infty$$

$$= \frac{2}{\sqrt{a^2-1}}\cdot\frac{\pi}{2} = \frac{\pi}{\sqrt{a^2-1}}$$

D.U.I.S. w.r.t. a,

$$\therefore \int_0^\infty \frac{dx}{(a-\cos x)^2} = \frac{\pi a}{(a^2-1)^{3/2}}.$$

Put $a = 2$. $\therefore \int_0^\infty \frac{dx}{(2-\cos x)^2} = \frac{2\pi}{3\sqrt{3}}$.

Example 5 : $\int_0^\infty \frac{\log(1+ax^2)}{x^2}\,dx = \pi\sqrt{a}$, $(a > 0)$.

Also deduce that $\int_0^\infty \frac{\log(1+x^2)}{x^2}\,dx = \pi$.

Sol. : $I(a) = \int_0^\infty \frac{\log(1+ax^2)}{x^2}\,dx$

$$\therefore \frac{dI}{da} = \frac{1}{a}\int_0^\infty \frac{1}{(1/a)+x^2}\,dx = \frac{1}{a}\cdot\left(\frac{\sqrt{a}}{1}\right)\left[\tan^{-1} x\cdot\sqrt{\frac{a}{1}}\right]_0^\infty = \frac{1}{\sqrt{a}}\cdot\frac{\pi}{2}$$

By integration, $\therefore I = \pi\sqrt{a}$

For deduction put $a = 1$, $\therefore \int_0^\infty \frac{\log(1+x)^2}{x^2}\,dx = \pi$.

Example 6 : $\int_0^\pi \frac{\log(1+\alpha\cos x)}{\cos x}\,dx = \pi\sin^{-1}\alpha$, $|\alpha| < 1$.

Sol. : $\frac{dI}{d\alpha} = \int_0^\pi \frac{1}{1+\alpha\cos x}\,dx$. Put $t = \tan\left(\frac{x}{2}\right)$.

$$\therefore \frac{dI}{d\alpha} = \frac{2}{(1-\alpha)}\int_0^\infty \frac{1}{[(1+\alpha)/(1-\alpha)]+t^2}\,dt$$

$$= \frac{2}{1-\alpha}\cdot\sqrt{\frac{1-\alpha}{1+\alpha}}\cdot\left[\tan^{-1} t\cdot\sqrt{\frac{1-\alpha}{1+\alpha}}\right]_0^\infty$$

$$= \frac{2}{\sqrt{1-\alpha^2}}\cdot\frac{\pi}{2} = \frac{\pi}{\sqrt{1-\alpha^2}}.$$

Integrating, we get, $\therefore I = \pi\sin^{-1}\alpha + c$. And $c = 0$.

Example 7 : $\int_0^x \frac{dx}{(x^2+a^2)^2} = \frac{1}{2a^3}\tan^{-1}\left(\frac{x}{a}\right) + \frac{x}{2a^2(x^2+a^2)}$

Sol. : Consider $\int_0^x \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$.

D.U.I.S. w.r.t. a,

$$\int_0^x -\frac{2a}{(x^2+a^2)^2}\,dx = -\frac{1}{a^2}\tan^{-1}\frac{x}{a} + \frac{1}{a}\cdot\frac{1}{1+(x^2/a^2)}\cdot\left(-\frac{x}{a^2}\right)$$

$$\therefore \int_0^x \frac{dx}{x^2+a^2} = \frac{1}{2a^3}\tan^{-1}\frac{x}{a} + \frac{x}{2a^2(x^2+a^2)}.$$

Example 8 : $\int_0^\infty x e^{-ax}\sin bx\,dx = \frac{2ab}{(a^2+b^2)^2}$.

Sol. : Consider $\int_0^\infty e^{-ax}\sin bx\,dx = \frac{e^{-ax}}{a^2+b^2}[-a\sin bx - b\cos bx]_0^\infty = \frac{b}{a^2+b^2}$

D.U.I.S. w.r.t. a.

$$\int_0^\infty e^{-ax}\sin bx(-x)\,dx = -\frac{b}{(a^2+b^2)^2}\cdot 2a$$

Example 9 : Verify the rule of differentiation under the integral sign for $\int_0^\infty e^{-at}\cos bt\,dt$.

Sol. : We have to show that $\frac{dI}{da} = \int_0^\infty \frac{\partial f}{\partial a}\,dt$

(a) Now, $I = \int_0^\infty e^{-at}\cos bt\,dt = \frac{a}{a^2+b^2}$

$$\therefore \frac{dI}{da} = \frac{b^2-a^2}{(a^2+b^2)^2}$$

(b) Now, $\int_0^\infty \frac{\partial f}{\partial a}\,dt = -\int_0^\infty t \cdot e^{-at} \cos bt\, dt$

By integrating by parts, we get

$$= -\left[\left\{t \cdot \frac{e^{-at}}{a^2+b^2}(-a\cos bt + b\sin bt)\right\}_0^\infty - \int_0^\infty \frac{e^{-at}}{a^2+b^2}(-a\cos bt + b\sin bt)\cdot 1 \cdot dt\right]$$

$$= 0 + \int_0^\infty -\frac{e^{-at}}{a^2+b^2}\, a\cos bt + \int_0^\infty \frac{e^{-at}}{a^2+b^2}\, b\sin bt\, dt$$

$$= \frac{1}{a^2+b^2}\left\{\left[\frac{-a e^{-at}}{a^2+b^2}(-a\cos bt + b\sin bt)\right]_0^\infty + \left[\frac{e^{-at} b}{a^2+b^2}(-a\sin bt - b\cos bt)\right]_0^\infty\right\}$$

$$= \frac{b^2-a^2}{(a^2+b^2)^2}.$$

Example 10 : Evaluate $\int_0^{\pi/2} \frac{dx}{1+a\sin^2 x}$ and deduce that

$$\int_0^{\pi/2} \frac{\sin^2 x}{(3+\sin^2 x)^2}\, dx = \frac{\pi\sqrt{3}}{96}.$$

Sol. : We have $\int_0^{\pi/2} \frac{dx}{1+a\sin^2 x} = \int_0^{\pi/2} \frac{dx}{(a+1)\sin^2 x + \cos^2 x}$

$$= \int_0^{\pi/2} \frac{\sec^2 x\, dx}{(a+1)\tan^2 x + 1}$$

Now, put $t = \tan x$,

$$\therefore\ I = \frac{1}{a+1}\int_0^\infty \frac{dt}{(1/\sqrt{a+1})^2 + t^2} = \frac{\pi}{2\sqrt{a+1}}$$

D.U.I.S. w.r.t. a,

$$\int_0^\pi -\frac{\sin^2 x}{(1+a\sin^2 x)^2}\, dx = \frac{\pi}{2}\left(-\frac{1}{2}\right)(a+1)^{-3/2}$$

$$\therefore\ \int_0^{\pi/2} \frac{\sin^2 x}{(1+a\sin^2 x)}\, dx = \frac{\pi}{4}(a+1)^{-3/2}$$

Now, put $a = \frac{1}{3}$, $\therefore \int_0^{\pi/2} \frac{\sin^2 x\, dx}{(3+\sin^2 x)} = \frac{\pi\sqrt{3}}{96}$.

Exercise II (Page 7-22)

Assuming the validity of differentiation under the integral sign, prove the following : Class (c) : 8 Marks

Example 1 : $\int_0^1 \frac{x^a - x^b}{\log x}\, dx = \log\left(\frac{a+1}{b+1}\right)$. ..... (1)

Sol. : $I(a) = \int_0^1 \frac{x^a - x^b}{\log x}\, dx$

$$\therefore\ \frac{dI}{da} = \int_0^1 x^a\, dx = \left[\frac{x^{a+1}}{a+1}\right]_0^1 = \frac{1}{a+1}$$

Integrating, $I = \log(a+1) + c$ $\quad\therefore\ I(b) = \log(b+1) + c.$

From (1), $I(b) = \int_0^1 \frac{x^b - x^b}{\log x}\, dx = 0$ $\quad\therefore\ c = -\log(b+1)$

$$\therefore\ I = \log(a+1) - \log(b+1) = \log\left(\frac{a+1}{b+1}\right).$$ ■

Example 2 : $\int_0^\infty \frac{\tan^{-1} ax - \tan^{-1} bx}{x}\, dx = \frac{\pi}{2}\log\left(\frac{a}{b}\right)$.

Sol. : $I = \int_0^\infty \frac{\tan^{-1} ax - \tan^{-1} bx}{x}\, dx$ ..... (1)

$$\therefore\ \frac{dI}{da} = \int_0^\infty \frac{dx}{1+a^2x^2} = \frac{1}{a^2}\left[a\tan(ax)\right]_0^\infty = \frac{\pi}{2a}.$$

Integrating, $I = \frac{\pi}{2}\log a + c$ $\quad\therefore\ I(b) = \frac{\pi}{2}\log b + c$

From (1), $I(b) = \int_0^\infty \frac{\tan^{-1} ax - \tan^{-1} ax}{x}\, dx = 0$ $\quad\therefore\ c = -\frac{\pi}{2}\log b$

$$\therefore\ I = \frac{\pi}{2}\log a - \frac{\pi}{2}\log b = \frac{\pi}{2}\log\left(\frac{a}{b}\right).$$ ■

Example 3 : $\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\sin mx\, dx = \tan^{-1}\left(\frac{b}{m}\right) - \tan^{-1}\left(\frac{a}{m}\right)$.

Sol. : $I = \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\sin mx\, dx$ ..... (1)

$$\therefore\ \frac{dI}{da} = -\int_0^\infty e^{-ax}\sin mx\, dx$$

$$= -\left[\frac{e^{-ax}}{a^2+m^2}(-a\sin mx - m\cos mx)\right]_0^\infty = -\frac{m}{a^2+m^2}$$

Integrating, $I = -\tan^{-1}\left(\frac{a}{m}\right) + c \quad \therefore \; I(b) = -\tan^{-1}\left(\frac{b}{m}\right) + c$

From (1),

$$I(b) = \int_0^\infty \frac{e^{-bx} - e^{-bx}}{x} \sin x \, dx = 0 \qquad \therefore \; c = \tan^{-1}\left(\frac{b}{x}\right)$$

$$\therefore \; I = -\tan^{-1}\left(\frac{a}{m}\right) + \tan^{-1}\left(\frac{b}{m}\right).$$

Example 4 : Evaluate $\int_0^\pi \frac{dx}{a + b\cos x}$, $a > 0$, $|b| < a$. Hence, prove that

$$\int_0^\pi \frac{\cos x}{(a + b\cos x)^2} dx = -\frac{\pi b}{(a^2 - b^2)^{3/2}}$$

and $\int_0^\pi \frac{\cos x}{(5 + 4\cos x)^2} dx = -\frac{4\pi}{27}$.

Sol. : See solved Ex. 8, page 7-20 of the Text Book.

For deduction put $a = 5$, $b = 4$

$$\therefore \; \int_0^\pi \frac{\cos x}{(5 + 4\cos x)^2} dx = -\frac{\pi \cdot 4}{(25 - 16)^{3/2}} = -\frac{4\pi}{27}.$$

Example 5 : Evaluate $\int_0^\pi \frac{dx}{a + b\cos x}$, $a > 0$, $|b| < a$. Hence, deduce that $\int_0^\pi \frac{dx}{(5 + 4\cos x)^3} = \frac{11\pi}{81}$.

Sol. : See solved Ex. 8, page 7-20 of the Text Book.

$$\int_0^\pi \frac{dx}{(a + b\cos x)^2} = \frac{\pi a}{(a^2 - b^2)^{3/2}}$$

D.U.I.S. w.r.t. a, we get

$$\int_0^\pi \frac{dx}{(a + b\cos x)^3} = \frac{\pi}{2} \cdot \frac{(2a^2 + b^2)}{(a^2 - b^2)^{5/2}}$$

Put $a = 5$, $b = 4$.

Example 6 : By differentiating $\int_0^\infty \frac{dx}{x^2 + a^2} = \frac{\pi}{2a}$ w.r.t. a under the integral sign successively, prove that

$$\int_0^\infty \frac{dx}{(x^2 + a^2)^{n+1}} = \frac{(2n)!\,\pi}{2^{2n+1} \cdot (n!)^2 \, a^{2n+1}}.$$

Sol. : $\int_0^\infty \frac{dx}{(x^2 + a^2)} = \frac{\pi}{2a}$. D.U.I.S. $\int_0^\infty \frac{dx}{(x^2 + a^2)^2} = \frac{\pi}{2^2 a^3}$

Again D.U.I.S., $\int_0^\infty \frac{-2 \cdot 2a}{(x^2 + a^2)^3} dx = \frac{\pi}{2^2} \cdot \frac{(-3)}{a^4}$

$$\int_0^\infty \frac{dx}{(x^2 + a^2)^3} = \frac{\pi \cdot 1 \cdot 3}{2^4 \cdot a^5}$$

D.U.I.S. $\therefore \; \int_0^\infty \frac{-3 \cdot 2a}{(x^2 + a^2)^4} dx = \frac{\pi \cdot 1 \cdot 3 \cdot (-5)}{2^4 \cdot a^6}$

$$\therefore \; \int_0^\infty \frac{dx}{(x^2 + a^2)^4} dx = \frac{\pi \cdot 1 \cdot 3 \cdot 5}{2^4 \cdot 2 \cdot 3 \cdot a^7}$$

$$= \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}{2^4 \cdot (1 \cdot 2 \cdot 3) \cdot (2 \cdot 4 \cdot 6)} \cdot \frac{\pi}{a^7}$$

$$= \frac{6!}{2^7 (1 \cdot 2 \cdot 3)^2 a^7}$$ [Now generalise]

***

CHAPTER 8

Rectification

Exercise I (Page 8-4)

Solve the following examples : Class (b) : 6 Marks

Example 1 : Find the lengths of the following curves as stated.

(1) $y = \frac{1}{3}(x^2+2)^{3/2}$ from $x = 0$ to $x = 3$.

(2 $y = 2x^{3/2}$ from $x = 0$ to 7.

(3) $y = \frac{3}{4}x^{4/3} - \frac{3}{8}x^{2/3} + 5$ from $x = 1$ to $x = 8$.

Sol. : (1) $\frac{dy}{dx} = \frac{1}{3}\cdot\frac{3}{2}(x^2+2)^{1/2}\cdot 2x = x\sqrt{x^2+2}$

$$\therefore\ S = \int_0^3 \sqrt{1+x^2(x^2+2)}\,dx = \int_0^3 \sqrt{x^4+2x^2+1}\,dx$$

$$= \int_0^3 (x^2+1)\,dx = \left[\frac{x^3}{3}+x\right]_0^3 = 12$$

■

(2) $\frac{dy}{dx} = 2\cdot\frac{3}{2}\sqrt{x}$

$$\therefore\ S = \int_0^7 \sqrt{1+9x}\,dx = \frac{2}{3}\left[\frac{(1+9x)^{3/2}}{9}\right]_0^7 = \frac{1022}{7}$$

■

(3) $\frac{dy}{dx} = \frac{3}{4}\cdot\frac{4}{3}\cdot x^{1/3} - \frac{3}{8}\cdot\frac{2}{3}\cdot x^{-1/3} = x^{1/3} - \frac{1}{4}\cdot x^{-1/3}$

$$\therefore\ S = \int_1^8 \sqrt{1+\left(x^{1/3}-\frac{1}{4}x^{-1/3}\right)^2}\,dx$$

$$= \int_1^8 \sqrt{\left(1+x^{2/3}-\frac{1}{2}+\frac{1}{16}x^{-2/3}\right)}\,dx = \int_1^8 \sqrt{\left(x^{1/3}+\frac{1}{4}x^{-1/3}\right)^2}\,dx$$

$$= \int_1^8 \left(x^{1/3}+\frac{1}{4}x^{-1/3}\right)dx = \left[\frac{3}{4}x^{4/3}+\frac{1}{4}\cdot\frac{3}{2}x^{2/3}\right]_1^8$$

$$\therefore\ S = \left[\left(\frac{3}{4}\cdot 16+\frac{3}{8}\cdot 4\right)-\left(\frac{3}{4}+\frac{3}{8}\right)\right] = \frac{99}{8}$$

■

Example 2 : Find the length of the arc of the parabola $y^2 = 8x$ cut off by the latus rectum.

Sol. : In the solved Ex. 1, page 8-5 of the Text Book, put $a = 2$. ■

Example 3 : Find the length of the parabola $x^2 = 4by$ cut-off by its latus rectum.

Sol. : $y = \frac{x^2}{4b}\ \therefore\ \frac{dy}{dx} = \frac{x}{2b}$

$$\therefore\ S = 2\int_0^{2b}\sqrt{1+\frac{x^2}{4b^2}}\cdot dx = \frac{2}{2b}\int_0^{2b}\sqrt{4b^2+x^2}\cdot dx$$

$$= \frac{1}{b}\left[\frac{x}{2}\sqrt{x^2+4b^2}+\frac{4b^2}{2}\log\left(x+\sqrt{x^2+4b^2}\right)\right]_0^{2b}$$

■

Example 4 : Prove that the length of the arc of the curve

$y = \log\left(\frac{e^x-1}{e^x+1}\right)$ **from $x = 1$ and $x = 2$ is $\log\left(e+\frac{1}{e}\right)$.**

Sol. : $\frac{dy}{dx} = \frac{e^x}{e^x-1}-\frac{e^x}{e^x+1} = \frac{2e^x}{e^{2x}-1}$

$$\therefore\ S = \int_1^2\sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot dx = \int_1^2\frac{\sqrt{(e^{4x}-2e^{2x}+1+4e^{2x})}}{(e^{2x}-1)}\,dx$$

$$= \int_1^2\left(\frac{e^{2x}+1}{e^{2x}-1}\right)dx = \int_1^2\left(\frac{e^x+e^x}{e^x-e^{-x}}\right)dx = \left[\log(e^x-e^{-x})\right]_1^2$$

■

Example 5 : Find the length of the arc of the curve

$y = \log\left(\tan h\frac{x}{2}\right)$ **from $x = 1$ to $x = 2$.**

Sol. : $y = \log\tanh\left(\frac{x}{2}\right) = \log\left(\frac{e^{x/2}-e^{x/2}}{e^{x/2}+e^{x/2}}\right) = \log\left(\frac{e^x-1}{e^x+1}\right)$.

Thus, this Example is same as Example 4 above. ■

Example 6 : Find the length of the arc of the curve $y = \log\sec x$ from $x = 0$ to $x = \pi/3$.

Sol. : $y = \log\sec x\quad\therefore\ \frac{dy}{dx} = \frac{1}{\sec x}\cdot\sec x\tan x = \tan x$

$$\therefore\ S = \int_0^{\pi/3}\sqrt{1+\tan^2 x}\cdot dx = \int_0^{\pi/3}\sec x\cdot dx$$

$$= \left[\log(\sec x+\tan x)\right]_0^{\pi/3} = \log(2+\sqrt{3}).$$

■

Exercise II (Page 8-7)

Solve the following examples : Class (b) : 6 Marks

Example 1 : Find the length of the curve $x = \frac{y^3}{3} + \frac{1}{4y}$ from $y = 1$ to $y = 2$.

Sol. : $\frac{dx}{dy} = y^2 - \frac{1}{4y^2}$

$$\therefore \; S = \int_1^2 \sqrt{1 + y^4 - \frac{1}{2} + \frac{1}{16}y^4} \cdot dy = \int_1^2 \left(y^2 + \frac{1}{4y^2}\right) dy$$

$$= \left[\frac{y^3}{3} - \frac{1}{4y}\right]_1^2 = \frac{59}{24}.$$

Example 2 : Show that the length of the arc of the curve $4ax = y^2 - 2a^2 \log \cdot \frac{y}{a} - a^2$ from $(0, a)$ to any point $P(x, y)$ is $\frac{y^2}{2a} - \frac{a}{2} - x$.

Sol. : $4a\frac{dx}{dy} = 2y - \frac{2a^2}{y} = 2\frac{(y^2 - a^2)}{y}$

Now, $S = \int_a^y \sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \int_a^y \frac{(y^2 + a^2)}{2ay} dy$

$$= \frac{1}{2a}\int_a^y \left(y + \frac{a^2}{y}\right) dy = \frac{1}{2a}\left[\frac{y^2}{2} + a^2 \log y\right].$$

Exercise III (Page 8-12)

Find the lengths of the following curves : Class (b) : 6 Marks

Example 1 : $x = a(2\cos\theta + \cos 2\theta)$, $y = a(2\sin\theta + \sin 2\theta)$ from $\theta = 0$ to any point θ.

Sol. : $\frac{dx}{d\theta} = a(-2\sin\theta - 2\sin 2\theta) = -4a\sin\frac{3\theta}{2}\cos\frac{\theta}{2}$

$\frac{dy}{d\theta} = a(2\cos\theta + 2\cos 2\theta) = 4a\cos\frac{3\theta}{2}\cos\frac{\theta}{2}$

Now, $S = \int_0^\theta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \cdot d\theta$

$$= \int_0^\theta \sqrt{16a^2\sin^2\frac{3\theta}{2}\cos^2\frac{\theta}{2} + 16\cos^2\frac{3\theta}{2}\cos^2\frac{\theta}{2}} \cdot d\theta$$

$$\therefore \; S = \int_0^\theta 4a\cos\frac{\theta}{2} \cdot d\theta$$

Example 2 : $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ from $\theta = 0$ to $\theta = 2\pi$.

Sol. : $\frac{dx}{d\theta} = a(1 - \cos\theta)$; $\frac{dy}{d\theta} = a\sin\theta$

$$\therefore \; S = \int_0^{2\pi} \sqrt{a^2(1 - \cos\theta)^2 + a^2\sin^2\theta} \cdot d\theta$$

$$= a\int_0^{2\pi} \sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta} \cdot d\theta = 2a\int_0^{2\pi} \sin\frac{\theta}{2} d\theta$$

Example 3 : $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ from $\theta = 0$ to $\theta = 2\pi$.

Sol. : $\frac{dx}{d\theta} = a(-\sin\theta + \sin\theta + \theta\cos\theta) = a\theta\cos\theta$

$\frac{dy}{d\theta} = a(\cos\theta - \cos\theta + a\sin\theta) = a\theta\sin\theta$

$$\therefore \; S = \int_0^{2\pi} \sqrt{a^2\theta^2\cos^2\theta + a^2\theta^2\sin^2\theta} \cdot d\theta.$$

Example 4 : $x = a e^\theta \sin\theta$, $y = a e^\theta \cos\theta$ from $\theta = 0$ to $\theta = \pi / 2$.

Sol. : $\frac{dx}{d\theta} = a[e^\theta \sin\theta + e^\theta \cos\theta] = a e^\theta(\sin\theta + \cos\theta)$

$\frac{dy}{d\theta} = a e^\theta(\cos\theta - \sin\theta)$

$$\therefore \; S = \int_0^{\pi/2} \sqrt{[a^2e^{2\theta}(\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta) + a^2 e^{2\theta}(\sin^2\theta + \cos^2\theta - 2\sin\theta\cos\theta)]}\, d\theta$$

$$\therefore \; S = \int_0^{\pi/2} \sqrt{2} \cdot a e^\theta \, d\theta.$$

Example 5 : $x = a(3\cos\theta - \cos 3\theta)$, $y = a(3\sin\theta - \sin 3\theta)$ from $\theta = \pi / 2$ to any point θ.

Sol. : $\frac{dx}{d\theta} = a(-3\sin\theta + 3\sin 3\theta) = 6a\cos 2\theta\sin\theta$

$\frac{dy}{d\theta} = a(3\cos\theta - 3\cos 3\theta) = 6a\sin 2\theta\sin\theta$

$$\therefore \; S = \int_{\pi/2}^{\theta} 6a\sqrt{\cos^2 2\theta\sin^2\theta + \sin^2 2\theta\sin^2\theta} \cdot d\theta = -6a\cos\theta.$$

Example 6 : $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$ between two consecutive cusps.

Sol. : $\frac{dx}{d\theta} = a(1 + \cos\theta)$; $\frac{dy}{d\theta} = -a\sin\theta$

$$\therefore\ S = 2\int_0^{\pi} a\sqrt{1+\cos^2\theta + 2\cos\theta + \sin^2\theta}\cdot d\theta$$

$$= 2a\int_0^{\pi}[2\cos(\theta/2)]\,d\theta$$

Example 7 : $x = e^{\theta}[\sin(\theta/2) + 2\cos(\theta/2)]$, $y = e^{\theta}[\cos(\theta/2) - 2\sin(\theta/2)]$ from $\theta = 0$ to $\theta = \pi$.

Sol. :

$$\frac{dx}{d\theta} = e^{\theta}\left[\frac{1}{2}\cos\frac{\theta}{2} - \sin\frac{\theta}{2} + \sin\frac{\theta}{2} + 2\cos\frac{\theta}{2}\right] = e^{\theta}\left(\frac{5}{2}\cos\frac{\theta}{2}\right)$$

$$\frac{dy}{d\theta} = e^{\theta}\left[\cos\frac{\theta}{2} - 2\sin\frac{\theta}{2} - \frac{1}{2}\sin\frac{\theta}{2} - \cos\frac{\theta}{2}\right] = e^{\theta}\left(-\frac{5}{2}\right)\sin\frac{\theta}{2}$$

$$\therefore\ S = \int_0^{\pi}\sqrt{e^{2\theta}\cdot\left(\frac{5}{2}\right)^2\left[\sin^2\left(\frac{\theta}{2}\right) + \cos^2\left(\frac{\theta}{2}\right)\right]}\cdot d\theta$$

Example 8 : $x = \log(\sec\theta + \tan\theta) - \sin\theta$, $y = \cos\theta$ from $\theta = 0$ to any point θ.

Sol. :

$$\frac{dx}{d\theta} = \frac{\sec\theta\tan\theta + \sec^2\theta}{\sec\theta + \tan\theta} - \cos\theta = \sin\theta\tan\theta$$

$$\frac{dy}{d\theta} = -\sin\theta$$

$$\therefore\ S = \int_0^{\theta}\sin\theta\sec\theta\,d\theta = \int_0^{\theta}\tan\theta\,d\theta = \log\sec\theta.$$

Example 9 : $x = a(t - \tan h\, t)$, $y = a\sec h\, t$ from $t = 0$ to any point t.

Sol. :

$$\frac{dx}{dt} = a(1 - \sec h^2 t)\ ;\ \frac{dy}{dt} = a\sec h\, t\tan h\, t$$

$$\therefore\ S = \int_0^{t} a\sqrt{1 + \sec h^4 t - 2\sec h^2 t + \sec h^2 t\tan^2 h\, t}\cdot dt$$

$$= a\int_0^{t}\tan h\, t\,dt = a[\log\cos h\, t]_0^{t}$$

Example 10 : $x = 1 - \cos t + (3/5)\,t$, $y = (4/5)\sin t$ from $t = 0$ to $t = \pi$.

Sol. :

$$\frac{dx}{dt} = \sin t + \frac{3}{5}\ ;\ \frac{dy}{dt} = \frac{4}{5}\cos t$$

$$\therefore\ S = \int_0^{\pi}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\cdot dt$$

$$= \int_0^{\pi}\sqrt{\left[\sin^2 t + 2\cdot\frac{3}{5}\sin t + \frac{9}{25} + \frac{16}{25}\cos^2 t\right]}\cdot dt$$

$$\therefore\ S = \int_0^{\pi}\left(1 + \frac{3}{5}\sin t\right)dt$$

Example 11 : $x = a\cos t + a\,t\sin t$, $y = a\sin t - at\cos t$ from $t = 0$ to $t = \pi/2$.

Sol. :

$$\frac{dx}{dt} = -a\sin t + a\sin t + at\cos t = at\cos t$$

$$\frac{dy}{dt} = a\cos t - a\cos t + at\sin t = at\sin t$$

$$\therefore\ S = \int_0^{\pi/2} at\,dt.$$

Exercise IV (Page 8-16)

Show that the length of the loop : Class (b) : 6 Marks

Example 1 : $3ay^2 = x^2(a - x)$ is $4a/\sqrt{3}$.

Sol. :

$$6ay\frac{dy}{dx} = x^2(-1) + 2x(a - x)\ ;\ \frac{dy}{dx} = \frac{2ax - 3x^2}{6ay}$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{3x - 4a}{2\sqrt{3a}\cdot\sqrt{a - x}}$$

$$\therefore\ S = 2\int_0^{a}\frac{3x - 4a}{2\sqrt{3a}\cdot\sqrt{a - x}}\,dx = \frac{2}{2\sqrt{3a}}\int_0^{a}\frac{(3x - 3a) - a}{\sqrt{a - x}}\,dx$$

$$= -\frac{1}{\sqrt{3a}}\cdot\int_0^{a}\left[\frac{a}{\sqrt{a - x}} + 3\sqrt{a - x}\right]dx$$

$$= -\frac{1}{\sqrt{3a}}\cdot\left[-2a\sqrt{a - x} - \frac{3\times 2}{3}(a - x)^{3/2}\right]_0^{a}$$

Example 2 : $9ay^2 = x(x - 3a)^2$ is $4\sqrt{3}\cdot a$.

Sol. :

$$18a\frac{dy}{dx} = (x - 3a)^2 + 2x(x - 3a)$$

$$= (x - 3a)[x - 3a + 2x] = 3(x - 3a)(x - a)$$

$$\therefore\ S = 2\int_0^{3a}\left(\frac{x + a}{2\sqrt{ax}}\right)dx = \frac{1}{2\sqrt{a}}\int_0^{3a}\left(\sqrt{x} + \frac{a}{\sqrt{x}}\right)dx$$

Exercise V (Page 8-24)

Solve the following examples : Class (b) : 6 Marks

Example 1 : Find the circumferences of a circle of radius a by using the polar equation of the circle $r = a$.

Sol. :

$$S = 4\int_0^{\pi/2}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\cdot d\theta \qquad r = a,\ \frac{dr}{d\theta} = 0$$

$$= 4\int_0^{\pi/2} a \cdot d\theta = 4 \cdot a\left[\frac{\pi}{2}\right] = 2\pi a.$$

Example 2 : Find the length of the curve $r = \cos^3(\theta/3)$ from $\theta = 0$ to $\theta = \pi/4$.

Sol. : $$S = \int_0^{\pi/4} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \cdot d\theta$$

$$r = \cos^3\frac{\theta}{3}, \quad \frac{dr}{d\theta} = -\cos^2\frac{\theta}{3} \cdot \sin\frac{\theta}{3}$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \cos^6\frac{\theta}{3} + \cos^4\frac{\theta}{2}\sin^2\frac{\theta}{2} = \cos^4\frac{\theta}{3}$$

$$\therefore \quad S = \int_0^{\pi/4} \cos^2\frac{\theta}{3}\, d\theta = \frac{1}{2}\int_0^{\pi/4}\left(1 + \cos\frac{2\theta}{3}\right) d\theta$$

$$= \frac{1}{2}\left[\theta + \sin\frac{2\theta}{3} \cdot \frac{3}{2}\right]_0^{\pi/4} = \frac{1}{2}\left(\frac{\pi}{4} + \frac{3}{2} \cdot \frac{1}{2}\right) = \frac{\pi + 3}{8}.$$

Example 3 : Find the length of the curve $r = \sqrt{1 + \cos 2\theta}$ from $\theta = 0$ to $\theta = 2\pi\sqrt{2}$.

Sol. : $$r = \sqrt{1 + \cos 2\theta} = \sqrt{2}\cos\theta, \quad \frac{dr}{d\theta} = -\sqrt{2}\sin\theta$$

$$\therefore \quad S = \int_0^{2\pi\sqrt{2}} \sqrt{2\cos^2\theta + 2\sin^2\theta} \cdot d\theta$$

$$= \sqrt{2}\int_0^{2\pi\sqrt{2}} d\theta = \sqrt{2}\,(2\pi\sqrt{2}) = 4\pi.$$

Example 4 : Find the length of the curve $r = \theta^2$ from $\theta = 0$ to $\theta = 2\sqrt{3}$.

Sol. : $$r = \theta^2, \quad \frac{dr}{d\theta} = 2\theta$$

$$\therefore \quad S = \int_0^{2\sqrt{3}} \sqrt{\theta^4 + 4\theta^2} \cdot d\theta = \int_0^{2\sqrt{3}} \theta\sqrt{\theta^2 + 4} \cdot d\theta$$

Put $\theta^2 + 4 = t \qquad \therefore \quad 2\theta\, d\theta = dt$

$$\therefore \quad S = \int_4^{16} \sqrt{t} \cdot \frac{dt}{2} = \frac{1}{2}\left[\frac{t^{3/2}}{3/2}\right]_4^{16} = \frac{56}{3}.$$

Example 5 : Find the length of the cardioide $r = a(1 - \cos\theta)$ lying inside the circle $r = a\cos\theta$.

Sol. : $a(1 - \cos\theta) = a\cos\theta \qquad \therefore \quad \theta = \pi/3.$

$$OA = 2\int_0^{\pi/3} \sqrt{r^2 + (dr/d\theta)^2} \cdot d\theta$$

$$OA = 2\int_0^{\pi/3} \sqrt{a^2(1 - \cos\theta)^2 + a^2\sin^2\theta} \cdot d\theta$$

$$= 2\int_0^{\pi/2} 2a\sin\frac{\theta}{2}\, d\theta.$$

Example 6 : Find the length of the spiral $r = a^{m\theta}$ lying inside the circle $r = a$.

Sol. : $$S = \int_{-\infty}^{0} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \cdot d\theta$$

$$= \int_{-\infty}^{0} \sqrt{a^{2m\theta} + m^2 a^{2m\theta}} \cdot d\theta$$

$$= \int_{-\infty}^{0} \sqrt{1 + m^2} \cdot a^{m\theta}\, d\theta$$

Example 7 : Taking $s = 0$ at $\theta = 0$, find the length of the arc OP of the spiral $r = a e^{\theta\cot\alpha}$ from O to $P(\theta)$.

Sol. : $$S = \int_0^{\theta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \cdot d\theta = \int_0^{\theta} a\sqrt{1 + \cot^2\alpha} \cdot e^{\theta\cot\alpha} \cdot d\theta$$

$$= a\,\mathrm{cosec}\,\alpha\left[\frac{e^{\theta\cot\alpha}}{\cot\alpha}\right]_0^{\theta}$$

Example 8 : Find the length of the cardiode $r = a(1 + \sin\theta)$.

Sol. : $$S = 2\int_{-\pi/2}^{\pi/2} \sqrt{a^2(1 + \sin\theta)^2 + a^2\cos^2\theta} \cdot d\theta$$

$$= 2\int_{-\pi/2}^{\pi/2} a \cdot \sqrt{2}\left[\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right] \cdot d\theta$$

Exercise - VI (Page 8-24)

Solve the following examples : Class (a) : 3 or 4 Marks

Example 1 : Find the length of the following curve using integration $y = 2x + 5$ from $x = 1$ to $x = 3$.

Sol. : $$\frac{dy}{dx} = 2; \qquad s = \int_1^3 \sqrt{1 + 4} \cdot dx = \sqrt{5}\,[x]_1^3 = 2\sqrt{5}$$

Example 2 : Find the length of the parabola $2y = x^2$ from $x = 0$ to $x = 1$.

Sol. : $$s = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx = \int_0^1 \sqrt{1 + x^2} \cdot dx$$

$$\therefore \quad s = \left[\frac{x}{2}\sqrt{x+1} + \frac{1}{2}\log\left(x + \sqrt{x^2+1}\right)\right] = \sqrt{2} + \frac{1}{2}\log(1+\sqrt{2})$$

Example 3 : Find the length of the curve $y = x^{3/2}$ from $x = 0$ to $x = 5$.

Sol. :

$$s = \int_0^5 \sqrt{1 + \frac{9}{4}x} \cdot dx = \frac{1}{2}\int_0^5 \sqrt{4+9x}\, dx$$

$$= \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{9}\left[(4+9x)^{3/2}\right]_0^5 = \frac{1}{27}(7^3 - 8) = \frac{335}{7}$$

Example 4 : Find the length of the curve $y = \log \cos x$ from $x = 0$ to $x = \pi/3$.

Sol. :

$$\frac{dy}{dx} = -\tan x$$

$$s = \int_0^{\pi/3} \sqrt{1+\tan^2 x}\, dx = \int_0^{\pi/3} \sec x\, dx$$

$$= \left[\log \sec x + \tan x\right]_0^{\pi/3} = \log(2+\sqrt{3})$$

Example 5 : Find the length of the curve $y = \sqrt{4-x^2}$ from $x = 0$ to $x = 2$.

Sol. :

$$\frac{dy}{dx} = \frac{-2x}{2\sqrt{4-x^2}} = -\frac{x}{\sqrt{4-x^2}} \quad \therefore \quad 1 + \left(\frac{dy}{dx}\right)^2 = \frac{x^2}{4-x^2}$$

$$\therefore \quad s = \int_0^2 \sqrt{1 + \frac{x^2}{4-x^2}} \cdot dx = \int_0^2 \frac{2}{\sqrt{4-x^2}}\, dx$$

$$= 2\left[\sin^{-1}\frac{x}{2}\right]_0^2 = 2 \cdot \frac{\pi}{2} = \pi.$$

Example 6 : Find the length of the curve $y = x^2 - \frac{1}{8} \cdot \log x$ from $x = 1$ to $x = 2$.

Sol. :

$$\frac{dy}{dx} = 2x - \frac{1}{8x}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + 4x^2 - \frac{1}{2} + \frac{1}{64x^2}$$

$$= 4x^2 + \frac{1}{2} + \frac{1}{64x^2} = \left(2x + \frac{1}{8x}\right)^2$$

$$\therefore \quad s = \int_1^2 \left(2x + \frac{1}{8x}\right) dx = \left[x^2 + \frac{\log x}{8}\right]_1^2$$

$$\therefore \quad s = 4 + \frac{1}{8}\log 2 - 1 = 3 + \log \sqrt[8]{2}.$$

Example 7 : Find the length of the curve $x = \frac{y^3}{3} + \frac{1}{4y}$ from $y = 1$ to $y = 3$.

Sol. :

$$\frac{dx}{dy} = y^2 - \frac{1}{4y^2}$$

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + y^4 - \frac{1}{2} + \frac{1}{16y^4}} = y^2 + \frac{1}{4y^2}$$

$$\therefore \quad s = \int_1^3 \left(y^2 + \frac{1}{4y^2}\right) dy = \left[\frac{y^3}{3} - \frac{1}{4y}\right]_1^3 = \frac{53}{6}.$$

Example 8 : Find the length of the curve $x = \frac{y^{3/2}}{3} - y^{1/2}$ from $y = 1$ to $y = 9$.

Sol. :

$$\frac{dx}{dy} = \frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2}$$

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{1}{4}y - \frac{1}{2} + \frac{1}{4y} = \left(\frac{1}{2}\sqrt{y} + \frac{1}{2\sqrt{y}}\right)^2$$

$$\therefore \quad s = \int_1^9 \left(\frac{1}{2}\sqrt{y} + \frac{1}{2}\sqrt{y}\right) dy = \left[\frac{1}{3}y^{3/2} + \sqrt{y}\right]_1^9 = \frac{32}{3}.$$

Example 9 : Find the length of the curve $x = t^3$, $y = \frac{3t^2}{2}$ from $t = 0$ to $t = \sqrt{3}$.

Sol. :

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9t^4 + 9t^2$$

$$s = \int_0^{\sqrt{3}} 3t\sqrt{1+t^2} \cdot dt = \left[(1+t^2)^{3/2}\right]_0^{\sqrt{3}} = 8 - 1 = 7$$

Example 10 : Find the circumference of the circle $x^2 + y^2 = a^2$.

Sol. : $y = \sqrt{a^2 - x^2}$, $\frac{dy}{dx} = -\frac{x}{\sqrt{a^2 - x^2}}$

$$s = 4\int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = 4\int_0^a \sqrt{1 + \frac{x^2}{a^2 - x^2}}$$

$$= 4a \cdot \int_0^a \frac{dx}{\sqrt{a^2 - x^2}} = 4a\left[\sin\frac{x}{a}\right]_0^a = 2\pi.$$

CHAPTER 9

Double Integrals

Exercise I (Page 9-7)

Evaluate the following integrals : Class (a) : 3 Marks

Example 1 : $\int_0^{\pi/2}\int_0^{3(1-\cos t)} x^2 \sin t \, dx\, dt$

Sol. : $I = \int_0^{\pi/2} \sin t \left[\frac{x^3}{3}\right]_0^{3(1-\cos t)} dt$

$= 9\cdot 8\int_0^{\pi/2} 2\sin\frac{t}{2}\cos\frac{t}{2}\cdot\sin^6\frac{t}{2}\,dt$ $\left[\text{Put } y = \sin\frac{t}{2}\right]$

$= 9\cdot 8\int_0^1 2\cdot 2y^7\,dy = 9\cdot 8\cdot 4\left[\frac{y^8}{8}\right]_0^1$

Example 2 : $\int_0^1\int_0^x xy(x^2y + xy^2)\,dy\,dx$

Sol. : $I = \int_0^1\int_0^x (x^3y^2 + x^2y^3)\,dy\,dx$

$= \int_0^1\left[\frac{x^3y^3}{3} + \frac{x^2y^4}{4}\right]_0^x dx = \int_0^1\left(\frac{x^6}{3} + \frac{x^6}{4}\right)dx$

$= \frac{7}{12}\int_0^1 x^6\,dx$

Example 3 : $\int_0^{\pi/2}\int_{\pi/2}^{\pi} \cos(x+y)\,dy\,dx$

Sol. : $I = \int_0^{\pi/2}[\sin(x+y)\,dx]_{\pi/2}^{\pi}dx = \int_0^{\pi/2}(-\sin x - \cos x)\,dx$

Example 4 : $\int_0^4\int_0^{\sqrt{\frac{1}{2}(1-y^2)}} \frac{dx\,dy}{\sqrt{1-x^2-y^2}}$

Sol. : $I = \int_0^4\left[\sin^{-1}\left(\frac{x}{\sqrt{1-y^2}}\right)\right]_0^{\sqrt{(1/2)(1-y^2)}} dy = \int_0^4\frac{\pi}{4}\,dy$

Example 5 : $\int_1^2\int_0^x \frac{1}{x^2+y^2}\,dy\,dx$

Sol. : $I = \int_1^2\frac{1}{x}\left[\tan^{-1}\frac{y}{x}\right]_0^x dx = \frac{\pi}{4}\int_1^2\frac{dx}{x}$

Example 6 : $\int_0^2\int_0^{\sqrt{2x-x^2}} xy\,dy\,dx$

Sol. : $I = \int_0^2\left[x\cdot\frac{y^2}{2}\right]_0^{\sqrt{2x-x^2}} dx = \frac{1}{2}\int_0^2(2x^2 - x^3)\,dx$

Example 7 : $\int_0^1\int_0^x (x^2+y^2)\,x\,dy\,dx$

Sol. : $I = \int_0^1\left[x^3y + \frac{xy^3}{3}\right]_0^x dx = \frac{4}{3}\int_0^1 x^4\,dx$

Example 8 : $\int_0^2\int_0^{x^2} x(x^2+y^2)\,dy\,dx$

Sol. : $I = \int_0^2\left[x^3y + x\frac{y^3}{3}\right]_0^{x^2} dx = \int_0^2\left[x^5 + \frac{x^7}{3}\right]dx$

Example 9 : $\int_0^a\int_0^{\sqrt{a^2-x^2}} x^2y\,dy\,dx$

Sol. : $I = \int_0^a x^2\cdot\left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx = \frac{1}{2}\int_0^a(a^2x^2 - x^4)\,dx$

Example 10 : $\int_0^{a\sqrt{3}}\int_0^{\sqrt{x^2+a^2}} \frac{x\,dy\,dx}{x^2+y^2+a^2}$

Sol. : $I = \int_0^{a\sqrt{3}}\int_0^{\sqrt{x^2+a^2}} x\cdot\frac{dy}{y^2+(\sqrt{x^2+a^2})^2}\,dx$

$= \int_0^{a\sqrt{3}}\frac{x}{\sqrt{x^2+a^2}}\left[\tan^{-1}\frac{y}{\sqrt{x^2+a^2}}\right]_0^{\sqrt{x^2+a^2}} dx$

$= \frac{\pi}{4}\left[\sqrt{x^2+a^2}\right]_0^{a\sqrt{3}}$

Example 11 : $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} xy\,dy\,dx$

Sol. : $I = \int_0^{2a} x\cdot\left[\frac{y^2}{2}\right]_0^{\sqrt{2ax-x^2}} dx = \frac{1}{2}\int_0^{2a}(2ax^2 - x^3)\,dx$

Example 12 : $\int_0^5 \int_{2-x}^{2+x} dy\, dx$

Sol. : $I = \int_0^5 [y]_{2-x}^{2+x} dx = \int_0^5 2x\, dx.$

Example 13 : $\int_1^2 \int_{-(2-y)}^{(2-y)} 2x^2y^2\, dx\, dy$

Sol. : $I = \int_1^2 2y^2 \left[\frac{x^3}{3}\right]_{-(2-y)}^{(2-y)} dy$

$= \frac{4}{3}\int_1^2 y^2 (8 - 12y + 6y^2 - y^3)\, dy$

$= \frac{4}{3}\int_1^2 (8y^2 - 12y^3 + 6y^4 - y^5)\, dy$

Example 14 : $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\, dx}{1 + x^2 + y^2}$

Sol. : $I = \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1}\left(\frac{y}{\sqrt{1+x^2}}\right)\right]_0^{\sqrt{1+x^2}} dx$

$= \frac{\pi}{4}\int_0^1 \frac{dx}{\sqrt{1+x^2}}.$

Example 15 : $\int_0^\infty \int_0^\infty \frac{e^{-x^3}}{\sqrt{x}} \cdot e^{-y^6} \cdot y^4\, dx\, dy$

Sol. : Put $x^3 = p$, $3x^2\, dx = dp$ and $y^6 = q$, $6y^5\, dy = dq$.

$\therefore\ I = \int_0^{-\infty} e^{-p} \cdot p^{-1/6} \cdot \frac{1}{3} p^{-2/3}\, dp \cdot \int_0^\infty e^{-q} \cdot q^{4/6} \cdot \frac{1}{6} q^{-5/6}\, dq$

$= \frac{1}{18}\int_0^\infty e^{-p} p^{-5/6}\, dp \cdot \int_0^\infty e^{-q} q^{-1/6}\, dq$

$= \frac{1}{18}\Gamma\left(-\frac{5}{6}+1\right) \cdot \Gamma\left(-\frac{1}{6}+1\right)$

$= \frac{1}{18}\Gamma\left(\frac{1}{6}\right) \cdot \Gamma\left(\frac{5}{6}\right) = \frac{1}{18} \cdot 2\pi = \frac{\pi}{9}.$ [By (2), page 6-53 of the text] ■

Exercise II (Page 9-9)

Sketch the region of integration and then evaluate each of the following integrals : Class (a) : 3 Marks

Example 1 : $\int_0^1 \int_0^x (x^2 + y^2)\, x\, dy\, dx$

Sol. : $I = \int_0^1 \int_0^x (x^3 + xy^2)\, dy\, dx$

$= \int_0^1 \left[x^3y + \frac{xy^3}{3}\right]_0^x dx$

$= \int_0^1 \left(x^4 + \frac{x^4}{3}\right) dx$

$= \frac{4}{3}\int_0^1 x^4\, dx = \frac{4}{3}\left[\frac{x^5}{5}\right]_0^1$ [See Fig. 1 below]

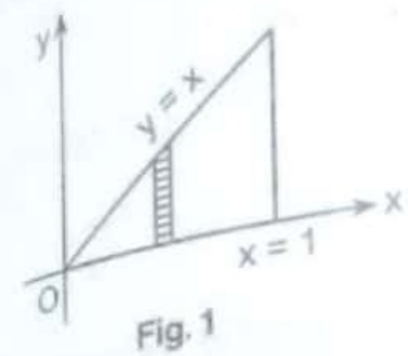


Fig. 1

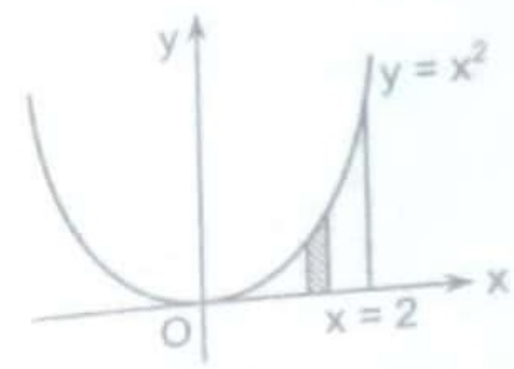


Fig. 2

Example 2 : $\int_0^2 \int_0^{x^2} x(x^2 + y^2)\, dy\, dx$

Sol. : $I = \int_0^2 \int_0^{x^2} (x^3 + xy^2)\, dy\, dx$

$= \int_0^2 \left[x^3y + \frac{xy^3}{3}\right]_0^{x^2} dx$

$= \int_0^2 \frac{4}{3}x^5\, dx = \frac{4}{3}\left[\frac{x^6}{6}\right]_0^2$ [See Fig. 2 above] ■

Example 3 : $\int_0^1 \int_0^x e^{x+y}\, dy\, dx$

Sol. : $I = \int_0^1 e^x \cdot \left[e^y\right]_0^x dx$

$= \int_0^1 e^x \cdot (e^x - 1)\, dx$

$= \int_0^1 (e^{2x} - e^x)\, dx = \left[\frac{e^{2x}}{2} - e^x\right]_0^1$ [See Fig. 1 above] ■

Example 4 : $\int_0^a \int_0^{\sqrt{a^2-x^2}} x^2y\, dy\, dx$

Sol. : $I = \int_0^a \left[\frac{x^2y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$

$= \frac{1}{2}\int_0^a x^2(a^2-x^2)\,dx$

$= \frac{1}{2}\left[a^2\frac{x^3}{3} - \frac{x^5}{5}\right]_0^a$ ■

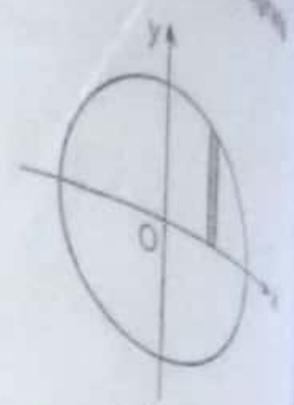


Fig. 3

Example 5 : $\int_0^1\int_0^x (x^2+y^2)\,dy\,dx$ [See Fig. 1 on previous page]

Sol. : $I = \int_0^1\left[x^2y + \frac{y^3}{3}\right]_0^x dx = \int_0^1\left(x^3 + \frac{x^3}{3}\right)dx$

$= \frac{4}{3}\int_0^1 x^3\,dx = \frac{1}{3}\left[x^4\right]_0^1$ ■

Example 6 : $\int_0^a\int_0^{\sqrt{a^2-x^2}} xy\,dy\,dx$ [See Fig. 3 above]

Sol. : $I = \int_0^a\left[x\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx = \int_0^a \frac{x}{2}(a^2-x^2)\,dx$

$= \frac{1}{2}\left[a^2\frac{x^2}{2} - \frac{x^4}{4}\right]_0^a = \frac{a^4}{8}.$ ■

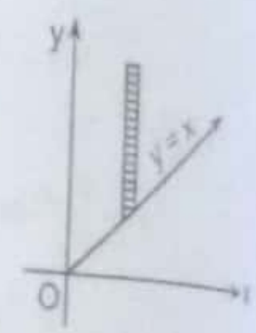


Fig. 4

Example 7 : $\int_0^\infty\int_x^\infty e^{-y}\,dy\,dx$

Sol. : $I = \int_0^\infty\left[-e^{-y}\right]_x^\infty dx = -\int_0^\infty\left[0-e^{-x}\right]dx$

$= \int_0^\infty e^{-x}\,dx = \left[-e^{-x}\right]_0^\infty$ [See Fig. 4] ■

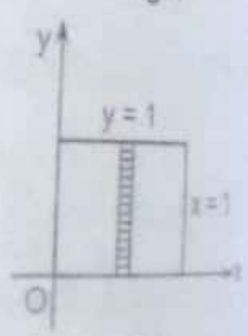


Fig. 5

Example 8 : $\int_0^1\int_0^1 (x+2)\,dy\,dx$

Sol. : $I = \int_0^1\left[xy+2y\right]_0^1 dx = \int_0^1 (x+2)\,dx$

$= \left[\frac{x^2}{2} + 2x\right]_0^1$ [See Fig. 5] ■

Exercise III (Page 9-11)

Evaluate the following polar integrals : Class (a) : 3 Marks

Example 1 : $\int_0^\pi\int_0^{a\sin\theta} r\,dr\,d\theta$

Sol. : $I = \int_0^{\pi/2}\left[\frac{r^2}{2}\right]_0^{a\sin\theta} d\theta = \frac{a^2}{2}\cdot\int_0^\pi \frac{1-\cos 2\theta}{2}\,d\theta$ ■

Example 2 : $\int_0^{\pi/2}\int_0^{a\cos\theta} r\sin\theta\,dr\,d\theta$

Sol. : $I = \int_0^{\pi/2}\sin\theta\left[\frac{r^2}{2}\right]_0^{a\cos\theta} d\theta = \frac{a^2}{2}\left[-\frac{\cos^3\theta}{3}\right]_0^{\pi/2}$ ■

Example 3 : $\int_0^{\pi/2}\int_0^{a(1+\sin\theta)} r^2\cos\theta\,dr\,d\theta$

Sol. : $I = \int_0^{\pi/2}\cos\theta\cdot\left[\frac{r^3}{3}\right]_0^{a(1+\sin\theta)} d\theta$

$I = \frac{a^3}{3}\int_0^{\pi/2}(1+\sin\theta)^3\cos\theta\,d\theta = \frac{a^3}{3}\left[\frac{(1+\sin\theta)^4}{4}\right]_0^{\pi/2}$ ■

Example 4 : $\int_0^{\pi/2}\int_0^{a\cos\theta} r^2\,dr\,d\theta$

Sol. : $I = \int_0^{\pi/2}\left[\frac{r^3}{3}\right]_0^{a\cos\theta} d\theta = \frac{1}{3}\int_0^{\pi/2} a^3\cos^3\theta\,d\theta = \frac{2a^3}{9}.$ ■

Example 5 : $\int_0^{\pi/4}\int_0^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2}\,dr\,d\theta$

Sol. : $I = \int_0^{\pi/2}\frac{1}{2}\left[-\frac{1}{1+r^2}\right]_0^{\sqrt{\cos 2\theta}} d\theta = -\frac{1}{2}\int_0^{\pi/2}\left(\frac{1}{2}\sec^2\theta - 1\right)d\theta$

$= -\frac{1}{2}\left[\frac{1}{2}\tan\theta - \theta\right]_0^{\pi/4}$ ■

Example 6 : $\int_0^{\pi/2}\int_0^{a\sin\theta} r\sqrt{a^2-r^2}\,dr\,d\theta$

Sol. : $I = -\int_0^{\pi/2}\left[\frac{1}{3}(a^2-r^2)^{3/2}\right]_0^{a\cos\theta} d\theta = -\frac{1}{3}\int_0^{\pi/2} a^3\cdot\sin^3\theta\,d\theta$ ■

Example 7 : $\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\,dr\,d\theta$

Sol. : $I = \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta = \frac{a^2}{2}\int_0^{\pi}\left(1+2\cos\theta+\frac{1+\cos 2\theta}{2}\right)d\theta$

Example 8 : $\int_0^{\pi/2}\int_0^{2a\cos\theta} r^2\sin\theta\,dr\,d\theta$

Sol. : $I = \int_0^{\pi/2}\left[\frac{r^3}{3}\sin\theta\right]_0^{2a\cos\theta} d\theta = \frac{8a^3}{3\times 4}\left[-\cos^4\theta\right]_0^{\pi/2}$

Exercise IV (Page 9-28)

Change the order of integration and evaluate : Class (b) : 6 Marks

Example 1 : $\int_0^b\int_{x^2/b}^{x} \frac{x\,dy\,dx}{(b-y)\sqrt{by-y^2}}$

Sol. : $y = \frac{x^2}{b}$ $\therefore x^2 = by,\ y = x$

$\therefore I = \int_0^b\int_0^{\sqrt{by}} \frac{x\,dy\,dx}{(b-y)\sqrt{by-x^2}} = \int_0^b\left[\frac{1}{b-y}(-1)\cdot\sqrt{by-x^2}\right]_y^{\sqrt{by}} dy$

Put $y = bt^2$, $dy = 2b\,dt$

$\therefore I = 2b\int_0^1 \frac{t^2}{\sqrt{1-t^2}}dt = -2b\int_0^1 \frac{1-t^2-1}{\sqrt{1-t^2}}dt$

$= -2b\left[\int_0^1\sqrt{1-t^2}\,dt - \int_0^1\frac{dt}{\sqrt{1-t^2}}\right]$

Example 2 : $\int_1^2\int_1^{x^2}\frac{x^2}{y}\,dy\,dx$

Sol. : $y = 1,\ y = x^2,\ x = 1,\ x = 2.$

$\therefore I = \int_1^4\int_{\sqrt{y}}^2 \frac{x^2}{y}dx\,dy = \int_1^4\left[\frac{x^3}{3y}\right]_{\sqrt{y}}^2 dy$

$= \frac{1}{3}\int_1^4\left(\frac{8}{y}-y^{1/2}\right)dy = \frac{1}{3}\left[8\log y - \frac{2}{3}y^{3/2}\right]_1^4$

Example 3 : $\int_3^5\int_0^{4/x} xy\,dy\,dx$

Sol. : $y = 0,\ y = \frac{4}{x}$ *i.e.* $xy = 4,\ x = 3,\ x = 5.$

$\therefore I = \int_0^{4/5}\int_3^5 xy\,dx\,dy + \int_{4/5}^{4/3}\int_3^{4/y} xy\,dx\,dy$

$= \int_0^{4/5} y\left[\frac{x^2}{2}\right]_3^5 dy + \int_{4/5}^{4/3} y\cdot\left[\frac{x^2}{2}\right]_3^{4/y} dy$

$= 8\int_0^{4/5} y\,dy + \int_{4/5}^{4/3}\left(\frac{8}{y}-\frac{9}{2}y\right)dy$

Example 4 : $\int_0^1\int_0^{\sqrt{1-x^2}} \frac{y\,dy\,dx}{(1+y^2)\sqrt{1-x^2-y^2}}$

Sol. : $y = 0,\ y = \sqrt{1-x^2}$ *i.e.* $x^2+y^2 = 1,\ x = 0,\ x = 1.$

$\therefore I = \int_0^1\int_0^{\sqrt{1-y^2}} \frac{y}{1+y^2}\cdot\frac{1}{\sqrt{(1-y^2)-x^2}}dx$

$= \int_0^1 \frac{y}{1+y^2}\left[\sin^{-1}\left\{\frac{x}{\sqrt{1-y^2}}\right\}\right]_0^{\sqrt{1-y^2}} = \frac{\pi}{2}\int_0^1\frac{y}{1+y^2}dy$

Example 5 : $\int_0^{a/\sqrt{2}}\int_x^{\sqrt{a^2-x^2}} y^2\,dy\,dx$

Sol. : $I = \int_0^{a/\sqrt{2}}\int_0^y y^2\,dy\,dx + \int_{a/\sqrt{2}}^a\int_0^{\sqrt{a^2-y^2}} y^2\,dy\,dx$

$I_1 = \int_0^{a/\sqrt{2}}\left[y^2x\right]_0^y dy = \int_0^{a/\sqrt{2}} y^3 dy = \left[\frac{y^4}{4}\right]_0^{a/\sqrt{2}} = \frac{a^4}{16}$

$I_2 = \int_{a/\sqrt{2}}^a\left[y^2x\right]_0^{\sqrt{a^2-y^2}} dy = \int_{a/\sqrt{2}}^a y^2\sqrt{a^2-y^2}\cdot dy$

Now, put $y = a\sin\theta$

$\therefore I_2 = \int_{\pi/4}^{\pi/2} a^2\sin^2\theta\cdot a\cos\theta\cdot a\cos\theta\,d\theta = \frac{a^4}{4}\int_{\pi/4}^{\pi/2}\sin^2 2\theta\,d\theta$

$\therefore I = I_1 + I_2 = \frac{a^4}{16}+\frac{a^4\pi}{32} = \frac{a^4}{32}[2+\pi].$

Example 6 : $\int_0^1\int_{1-\sqrt{1-y}}^{1+\sqrt{1-y}} dx\,dy$

Sol. : $x = 1-\sqrt{1-y}$; $x = 1+\sqrt{1-y}$

$(x-1)^2 = 1-y$ $\therefore x^2-2x+1 = 1-y$ $\therefore y = 2x-x^2$

$\therefore I = \int_0^2\int_0^{2x-x^2} dx\,dy = \int_0^2[y]_0^{2x-x^2}dx = \int_0^2[2x-x^2]dx$

Example 7 : $\int_{-1}^{2}\int_{x^2}^{x+2} dy\,dx$

Sol. : $y = x^2,\ y = x+2\ ;\ x^2 - x - 2 = 0,\ (x-2)(x+1) = 0$

$$\therefore\ I = \int_0^1\int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy + \int_1^4\int_{y-2}^{\sqrt{y}} dx\,dy$$

$$\therefore\ I_1 = \int_0^1 [x]_{-\sqrt{y}}^{\sqrt{y}} dy\ ;\ I_2 = \int_1^4\int_{y-2}^{\sqrt{y}} dx\,dy = \int_1^4 [x]_{y-2}^{\sqrt{y}} dy$$

Example 8 : $\int_0^1\int_1^{\sqrt{2-y^2}} \dfrac{y\,dx\,dy}{\sqrt{(2-x^2)(1-x^2y^2)}}$

Sol. : $x = 1,\ x = \sqrt{2-y^2}\ ;\ x^2 + y^2 = 2$. When $y = 0,\ x^2 = 2$.

$$\therefore\ I = \int_1^{\sqrt{2}}\int_0^{\sqrt{2-x^2}} \frac{y\,dy\,dx}{\sqrt{(2-x^2)(1-x^2y^2)}}$$

$$= \int_1^{\sqrt{2}} \frac{1}{\sqrt{2-x^2}}\left[-\frac{\sqrt{1-x^2y^2}}{x^2}\right]_0^{\sqrt{2-x^2}} dx = \int_1^{\sqrt{2}} \frac{\sqrt{2-x^2}}{x^2}\,dx$$

Put $x = \sqrt{2}\cdot\cos\theta$

$$\therefore\ I = \int_0^{\pi/4} \tan^2\theta\,d\theta = \int_0^{\pi/4} (\sec^2\theta - 1)\,d\theta$$

Example 9 : $\int_0^2\int_{\sqrt{2y}}^{2} \dfrac{x^2\,dx\,dy}{\sqrt{x^4 - 4y^2}}$

Sol. : $$I = \int_0^2\int_{y=0}^{x^2/2} \frac{x^2\,dy\,dx}{\sqrt{x^4-4y^2}} = \int_0^2 x^2\int_0^{x^2/2} \frac{dy}{2\sqrt{(x^2/2)-y^2}}$$

$$= \int_0^2 \frac{x^2}{2}\left[\sin^{-1}\frac{y\cdot 2}{x^2}\right]_0^{x^2/2} dx$$

Example 10 : $\int_0^a\int_0^y \dfrac{dx\,dy}{\sqrt{(a^2-x^2)(a-y)(y-x)}}$

Sol. : Compare this with the solved Ex. No. 13, page 9-20 of the Text Book.

$$\therefore\ I = \int_0^a \frac{dx}{\sqrt{a^2-x^2}}\cdot\pi = \pi\left[\log\left(x+\sqrt{x^2+a^2}\right)\right]_0^a$$

Example 11 : $\int_0^1\int_{\sqrt{x}}^{1} e^{x/y}\,dy\,dx$

Sol. : $$I = \int_0^1\int_0^{y^2} e^{x/y}\,dx\,dy = \int_0^1 \left[y\,e^{x/y}\right]_0^{y^2} dy$$

$$= \int_0^1 \left[y\,e^y - y\right] dx = \left[y\,e^y - e^y - \frac{y^2}{2}\right]_0^1$$

Example 12 : $\int_0^{\pi/2}\int_0^y \cos 2y\cdot\sqrt{1-a^2\sin^2 x}\cdot dx\,dy$

Sol. : $$I = \int_0^{\pi/2}\int_x^{\pi/2} \cos 2y\cdot\sqrt{1-a^2\sin^2 x}\cdot dy\,dx$$

$$= \int_0^{\pi/2} \sqrt{1-a^2\sin^2 x}\cdot\left[\frac{1}{2}\sin 2y\right]_x^{\pi/2} dx.$$

$$= \int_0^{\pi/2} \sqrt{1-a^2\sin^2 x}\cdot\left[0 - \frac{1}{2}\sin 2x\right] dx$$

Put $1 - a^2\sin^2 x = t^2$, $\quad\therefore\ I = \dfrac{1}{a^2}\int_1^{\sqrt{1-a^2}} t\cdot t\,dt.$

Example 13 : $\int_0^a\int_{x/a}^{\sqrt{x/a}} (x^2+y^2)\,dy\,dx$

Sol. : $$I = \int_0^1\int_{ay^2}^{ay} (x^2+y^2)\,dx\,dy = \int_0^1 \left[\frac{x^3}{3} + y^2x\right]_{ay^2}^{ay} dy$$

$$= \int_0^1 \left[\frac{a^3y^3}{3} + ay^3 - \frac{a^3y^6}{3} - ay^4\right] dy$$

Example 14 : $\int_0^a\int_y^a \dfrac{x^2}{\sqrt{x^2+y^2}}\,dx\,dy$

Sol. : $$I = \int_0^a\int_0^x \frac{x^2}{\sqrt{x^2+y^2}}\,dy\,dx = \int_0^a \left\{x^2\left[\log\left(x+\sqrt{x^2+y^2}\right)\right]_0^x\right\} dx$$

$$= \int_0^a x^2\left[\log\left(x+\sqrt{2}\cdot x\right) - \log x\right] dx = \log\left(1+\sqrt{2}\right)\int_0^a x^2\,dx$$

Example 15 : $\int_0^1\int_{x^2}^{2-x} xy\,dy\,dx$

Sol. : $$I = \int_0^1\int_0^{\sqrt{y}} xy\,dx\,dy + \int_1^2\int_0^{2-y} xy\,dx\,dy$$

$$= \int_0^1 y\left[\frac{x^2}{2}\right]_0^{\sqrt{y}} dy + \int_1^2 y\left[\frac{x^2}{2}\right]_0^{2-y} dy$$

$$= \int_0^1 \frac{y^2}{2}\,dy + \int_1^2 \frac{y}{2}(2-y)^2\,dy$$

Example 16 : $\int_0^a\int_0^x x\sqrt{(a^2-y^2)(x^2-y^2)}\,dy\,dx$

Sol. : $$I = \int_0^a\int_y^a \sqrt{a^2-y^2}\cdot\sqrt{x^2-y^2}\cdot x\,dx\,dy$$

$$= \int_0^a \sqrt{a^2-y^2}\left[\frac{1}{3}(x^2-y^2)^{3/2}\right]_y^a dy$$

$$= \frac{1}{3}\int_0^a \sqrt{a^2 - y^2}\,(a^2 - y^2)^{3/2}\,dy.$$

Example 17 : $\int_0^\infty \int_0^\infty e^{-x^2(1+t^2)}\, x\, dx\, dt$

Sol. : $$I = \int_0^\infty \left[\frac{e^{-x^2(1+t^2)}}{-2(1+t^2)}\right]_0^\infty dt = \frac{1}{2}\int_0^\infty \frac{dt}{1+t^2} = \frac{1}{2}\left[\tan^{-1} t\right]_0^\infty$$

Example 18 : $\int_0^a \int_0^{2\sqrt{ax}} x^2\, dy\, dx$

Sol. : $$I = \int_0^{2a}\int_{y^2/4a}^{a} x^2\, dx\, dy = \int_0^{2a}\left[\frac{x^3}{3}\right]_{y^2/4a}^{a} dy$$

$$\therefore\ I = \frac{1}{3}\int_0^{2a}\left[a^3 - \frac{y^6}{64a^3}\right] dy$$

Example 19 : $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$

Sol. : $$I = \int_0^\infty \int_0^y \frac{e^{-y}}{y}\, dx\, dy = \int_0^\infty \left[\frac{e^{-y}}{y}\cdot x\right]_0^y dy = \int_0^\infty e^{-y}\, dy$$

Example 20 : Express as a single integral and then evaluate

$$\int_0^1\int_0^y (x^2+y^2)\, dx\, dy + \int_1^2\int_0^{2-y} (x^2+y^2)\, dx\, dy.$$

Sol. : $$I = \int_0^1\int_x^{2-x} (x^2+y^2)\, dx\, dy = \int_0^1\left[x^2 y + \frac{y^3}{3}\right]_x^{2-x} dx$$

Exercise V (Page 9-39)

Change to polar coordinates and evaluate : Class (a) : 4 Marks

Example 1 : $\int_0^a \int_0^{\sqrt{a^2-x^2}} y^2\sqrt{(x^2+y^2)}\, dy\, dx.$

Sol. : $y = 0,\ y = \sqrt{a^2 - x^2}$ *i.e.* $x^2 + y^2 = a^2$

$x = 0,\ x = a \quad \therefore r = 0,\ r = a,\ \theta = 0,\ \theta = \pi/2.$

$$\therefore\ I = \int_0^{\pi/2}\int_0^a \sin^2\theta \cdot r^4\, d\theta\, dr = \int_0^{\pi/2}\sin^2\theta\left[\frac{r^5}{5}\right]_0^a d\theta$$

$$= \frac{a^5}{5}\int_0^{\pi/2}\frac{(1-\cos 2\theta)}{2}\, d\theta = \frac{a^5}{10}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

Example 2 : $\int_0^4 \int_y^{4+\sqrt{16-y^2}} \frac{dx\, dy}{(64 + x^2 + y^2)^2}.$

Sol. : $(x-4)^2 + y^2 = 16,\ y = 0,\ y = 4$

$\therefore\ x = y \quad \therefore\ r\cos\theta = r\sin\theta \quad \therefore\ \theta = \pi/4$

$x^2 + y^2 = 8x, \quad \therefore\ r = 8\cos\theta.$

$$\therefore\ I = \int_0^{\pi/4}\int_0^{8\cos\theta}\frac{r\, d\theta\, dr}{(64+r^2)^2} = \int_0^{\pi/4} -\frac{1}{2}\left[\frac{1}{(64+r^2)}\right]_0^{8\cos\theta} d\theta$$

$$= \frac{1}{128}\int_0^{\pi/4}\left[1 - \frac{1}{1+\cos^2\theta}\right] d\theta$$

$$= \frac{1}{128}\int_0^{\pi/4}\left[1 - \frac{\sec^2\theta}{2+\tan^2\theta}\right] d\theta \qquad \text{Put } \tan\theta = t.$$

Example 3 : $\int_0^a \int_y^a \frac{x\, dx\, dy}{(x^2+y^2)}.$

Sol. : $x = y \quad \therefore\ \theta = \pi/4, \quad x = r\cos\theta = a.$

$$\therefore\ I = \int_0^{\pi/4}\int_0^{a/\cos\theta}\cos\theta\, dr$$

$$= \int_0^{\pi/4}\cos\theta\left[r\right]_0^{a/\cos\theta} d\theta = \int_0^{\pi/4} a\, d\theta = a\frac{\pi}{4}.$$

Example 4 : $\int_0^1 \int_x^{\sqrt{(2x-x^2)}} \frac{1}{\sqrt{x^2+y^2}}\, dy\, dx.$

Sol. : $y^2 = 2x - x^2$ *i.e.* $x^2 + y^2 = 2x$ *i.e.* $r = 2\cos\theta.$

$y = x$ *i.e.* $\theta = \pi/4.$

$$\therefore\ I = \int_{\theta=\pi/4}^{\pi/2}\int_{r=0}^{2\cos\theta} dr\, d\theta = \int_{\theta=\pi/4}^{\pi/2}\left[r\right]_0^{2\cos\theta} d\theta$$

$$= \int_{\theta=\pi/4}^{\pi/2}\left[2\cos\theta\right] d\theta$$

Example 5 : $\int_0^a \int_0^x \frac{x^3\, dy\, dx}{\sqrt{(x^2+y^2)}}.$

Sol. : $y = x, \quad \theta = \pi/4, \quad x = a, \quad r\cos\theta = a.$

$$I = \int_0^{\pi/4}\int_0^{a/\cos\theta}\cos^3\theta \cdot r^3\, d\theta\, dr = \int_0^{\pi/4}\cos^3\theta\left[\frac{r^4}{4}\right]_0^{a/\cos\theta} d\theta$$

$$= \frac{a^4}{4}\int_0^{\pi/4}\sec\theta\, d\theta = \frac{a^4}{4}\left[\log(\sec\theta + \tan\theta)\right]_0^{\pi/4}$$

Example 6 : $\int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}}\, dx\, dy.$

Sol. : $x = y,\quad x = a,\quad y = 0,\quad y = a,\quad \theta = \pi/4,\quad r\cos\theta = a.$

$$I = \int_0^{\pi/4} \int_0^{a/\cos\theta} r\cos^2\theta \cdot r\, d\theta\, dr = \int_0^{\pi/4} \cos^2\theta \left[\frac{r^3}{3}\right]_0^{a/\cos\theta} d\theta$$

$$= \frac{a^3}{3}\int_0^{\pi/2} \sec\theta\, d\theta = \frac{a^3}{3}\log[\sec\theta + \tan\theta]_0^{\pi/4}$$

Example 7 : $\int_0^a \int_y^a \frac{x^2\, dx\, dy}{(x^2+y^2)^{3/2}}.$

Sol. : See solved Ex. 2, page 9-30 of the Text Book.

$$\therefore\ I = \int_0^{\pi/4} \int_0^{a/\cos\theta} \cos^2\theta\, dr\, d\theta$$

$$\therefore\ I = \int_0^{\pi/4} \cos^2\theta [\, r\,]_0^{a/\cos\theta} d\theta = a\int_0^{\pi/4} \cos\theta\, d\theta$$

Example 8 : $\int_0^1 \int_0^{\sqrt{x-x^2}} \frac{4xy}{x^2+y^2} e^{-(x^2+y^2)}\, dy\, dx.$

Sol. : $y^2 = x - x^2,\ x^2 + y^2 = x,\ r = \cos\theta.$

$y = 0,\quad \theta = 0,\quad y = 1,\quad \theta = \pi/2.$

$$I = \int_0^{\pi/2} \int_0^{\cos\theta} 4e^{-r^2} r\sin\theta\cos\theta\, d\theta\, dr$$

$$= \int_0^{\pi/2} 4\left[-\frac{1}{2}e^{-r^2}\right]_0^{\cos\theta} \sin\theta\cos\theta\, d\theta$$

$$= -2\int_0^{\pi/2} (e^{-\cos^2\theta} - 1)\sin\theta\cos\theta\, d\theta$$

$$= -2\left[+\frac{e^{-\cos^2\theta}}{2} + \frac{\cos 2\theta}{2}\right]_0^{\pi/2}$$

Example 9 : $\int_0^4 \int_{\sqrt{4x-x^2}}^{\sqrt{16-x^2}} \frac{dy\, dx}{\sqrt{16-x^2-y^2}}.$

Sol. : See solved Ex. 8 page 9-33 of the Text Book and put $a = 4$.

Example 10 : $\int_0^2 \int_{\sqrt{2x-x^2}}^{\sqrt{4-x^2}} \frac{dy\, dx}{\sqrt{4-x^2-y^2}}.$

Sol. : See solved Ex. 8 page 9-33 of the Text Book and put $a = 2$.

Example 11 : $\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{xy\, e^{-(x^2+y^2)}}{x^2+y^2}\, dy\, dx.$

Sol. : See solved Ex. 8, page 9-33 of the Text Book.

$$\therefore\ I = \int_0^{\pi/2} \int_{a\cos\theta}^{a} \sin\theta\cos\theta \cdot e^{-r^2} \cdot r\, d\theta\, dr$$

$$= \int_0^{\pi/2} \sin\theta\cos\theta \left[-\frac{e^{-r^2}}{2}\right]_{a\cos\theta}^{a} d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2} \sin\theta\cos\theta\left[e^{-a^2} - e^{-a^2\cos^2\theta}\right] d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} -e^{-a^2}\sin\theta\cos\theta\, d\theta + \frac{1}{2}\int_0^{\pi/2} e^{-a^2\cos^2\theta}\sin\theta\cos\theta\, d\theta$$

$$= -\frac{e^{-a^2}}{2}\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} + \frac{1}{2}\left[\frac{-e^{-a^2\cos^2\theta}}{2a^2}\right]_0^{\pi/2}.$$

Example 12 : $\int_0^a \int_0^{\sqrt{a^2-x^2}} y\sqrt{(x^2+y^2)}\, dy\, dx.$

Sol. : $y^2 = a^2 - x^2,\quad r = a,\quad x = 0,\quad x = a,\quad \theta = 0,\quad \theta = \pi/2.$

$$I = \int_0^{\pi/2} \int_0^a r^3 \sin\theta\, dr\, d\theta = \int_0^{\pi/2} \sin\theta\, d\theta \int_0^a r^3\, dr.$$

Example 13 : $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dx\, dy}{(a^2+x^2+y^2)^{3/2}}.$

Sol. :

$$I = 4\int_0^{\pi/2} \int_0^{\infty} \frac{r\, dr\, d\theta}{(a^2+r^2)^{3/2}} = 4\int_0^{\pi/2} d\theta \int_0^{\infty} \frac{r}{(a^2+r^2)^{3/2}}\, dr$$

$$= 4[\theta]_0^{\pi/2}\left[-\frac{1}{\sqrt{a^2+r^2}}\right]_0^{\infty}$$

Example 14 : $\int_0^{2a} \int_0^{\sqrt{2ax-x^2}} (x^2+y^2)\, dy\, dx.$

Sol. :

$$I = \int_0^{\pi/2} \int_0^{2a\cos\theta} r^2 \cdot r\, dr\, d\theta = \int_0^{\pi/2} \left[\frac{r^4}{4}\right]_0^{2a\cos\theta} d\theta$$

$$= \frac{16a^4}{4}\int_0^{\pi/2} \cos^4\theta\, d\theta = 4a^4\left[\frac{3\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2}\right] = \frac{3\pi a^4}{4}.$$

Exercise VI (Page 9-50)

Solve the following examples : Class (b) : 6 Marks

Example 1 : Evaluate $\iint (x^2 - y^2)\, dx\, dy$ over the area of the triangle whose vertices are at the points (0, 1) (1, 1) (1, 2).

Sol. : $$I = \int_1^2 \int_{y-1}^1 (x^2 - y^2)\, dx\, dy = \int_1^2 \left[\frac{x^3}{3} - y^2 x\right]_{y-1}^1 dy$$

$$I = \int_1^2 \left[\frac{1}{3} - 2y^2 - \frac{(y-1)^3}{3} + y^3\right] dy$$

Example 2 : Evaluate $\iint x^{n-1} y^{m-1}\, dx\, dy$ over the triangle given by $x \geq 0,\ y \geq 0,\ x + y \leq 1$.

Sol. : $$I = \int_0^1 \int_0^{1-y} x^{n-1} y^{m-1}\, dx\, dy = \int_0^1 \left[\frac{x^n}{n}\right]_0^{1-y} \cdot y^{m-1}\, dy$$

$$= \frac{1}{n}\int_0^1 (1-y)^n \cdot y^{m-1}\, dy = \frac{1}{n} B(n+1, m)$$

Example 3 : Evaluate $\iint dx\, dy$ throughout the area bounded by $y = x^2$ and $x + y = 2$.

Sol. : $$I = \int_{-2}^1 \int_{x^2}^{2-x} dx\, dy = \int_{-2}^1 [\, y\,]_{x^2}^{2-x}\, dx$$

$$= \int_{-2}^1 (2 - x - x^2)\, dx$$

Example 4 : Evaluate $\iint (x^2 + y^2)\, dx\, dy$ over the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Sol. : $$I = 4\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2 - x^2}} (x^2 + y^2)\, dx\, dy = 4\int_0^a \left[x^2 y + \frac{y^3}{3}\right]_0^{\frac{b}{a}\sqrt{a^2 - x^2}} dx$$

$$= 4\int_0^a \left[\frac{b}{a} \cdot x^2 \sqrt{a^2 - x^2} + \frac{b^3}{3a^3}(a^2 - x^2)^{3/2}\right] dx$$

Put $x = a \sin\theta,\ dx = a\cos\theta\ d\theta$

$$\therefore\ I = 4\int_0^{\pi/2} \left(a^3 b \sin^2\theta \cos^2\theta + \frac{ab^3}{3}\cos^4\theta\right) d\theta$$

$$= 4\left[a^3 b \cdot \frac{1 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2} + \frac{ab^3}{3} \cdot \frac{3 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2}\right]$$

Example 5 : Evaluate $\iint x(x^2 + y^2)\, dx\, dy$ over the positive quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Sol. : $$I = \int_0^a \int_0^{\frac{b}{a}\sqrt{a^2 - x^2}} x(x^2 + y^2)\, dx\, dy = \int_0^a \left[x^3 y + \frac{xy^3}{3}\right]_0^{\frac{b}{a}\sqrt{a^2 - x^2}} dx$$

$$I = \int_0^a \left[\frac{b}{a} \cdot x^3 \sqrt{a^2 - x^2} + \frac{x}{3} \cdot \frac{b^3}{a^3}(a^2 - x^2)^{3/2}\right] dx$$

Put $x = a\sin\theta,\ dx = a\cos\theta\ d\theta$.

$$\therefore\ I = \int_0^{\pi/2} \left[a^4 b \cdot (\sin^3\theta \cos^2\theta) + \frac{a^2 b^3}{3}\sin\theta\cos^4\theta\right] d\theta$$

$$= a^4 b \cdot \frac{2 \cdot 1}{5 \cdot 3 \cdot 1} + \frac{a^2 b^3}{3} \cdot \frac{1 \cdot 3}{5 \cdot 3 \cdot 1}$$

Example 6 : Evaluate $\iint x^{m-1} y^{n-1}\, dx\, dy$ over the positive quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Sol. : $$I = \int_0^a \int_0^{\frac{b}{a}\sqrt{a^2 - x^2}} x^{m-1} y^{n-1}\, dx\, dy$$

$$= \int_0^a x^{m-1} \left[\frac{y^n}{n}\right]_0^{(b/a)\sqrt{a^2 - x^2}} dx = \frac{1}{n}\int_0^a x^{m-1} \cdot \frac{b^n}{a^n}(a^2 - x^2)^{n/2}\, dx$$

Put $x = a\sin\theta,\ dx = a\cos\theta\ d\theta$.

$$\therefore\ I = \frac{1}{n} \cdot a^m b^n \int_0^{\pi/2} \sin^{m-1}\theta \cdot \cos^{n+1}\, d\theta$$

$$= \frac{1}{n} \cdot a^m b^n \cdot \frac{1}{2} B\left(\frac{m}{2}, \frac{n+2}{2}\right).$$

Example 7 : Evaluate $\iint_R \frac{y\, e^{2y}}{\sqrt{(1-x)(x-y)}}\, dx\, dy$ where R is the region of the triangle whose vertices are (0, 0), (1, 0) and (1, 1).

Sol. : $$I = \int_0^1 \int_{x=y}^1 \frac{y\, e^{2y}}{\sqrt{(1-x)(1-y)}}\, dx\, dy = \int_0^1 \frac{y\, e^{2y}}{\sqrt{1-y}}\left[-2\sqrt{1-x}\right]_y^1 dy$$

$$= 2\int_0^1 y\, e^{2y}\, dy = 2\left[y\frac{e^{2y}}{2} - \int \frac{e^{2y}}{2} \cdot 1 dy\right]_0^1$$

Example 8 : Evaluate $\iint (x^2 + y^2)\, dx\, dy$ over the area of the triangle whose vertices are (0, 0) (1, 0), (1, 2).

Sol. : $$I = \int_0^1 \int_0^{2x} (x^2 + y^2)\, dy\, dx = \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_1^{2x} dx$$

$$\therefore \quad I = \int_0^1 \left(2x^3 + \frac{8x^3}{3}\right) dx$$

Example 9 : Evaluate $\iint xy\, dx\, dy$ over the area bounded by $y^2 = 4x$ and $y = 2x - 4$.

Sol. :
$$I = \int_{-2}^{4} \int_{x=y^2/4}^{(y+4)/2} xy\, dx\, dy = \int_{-2}^{4} y \left[\frac{x^2}{2}\right]_{y^2/4}^{(y+4)/2} dy$$
$$= \frac{1}{32}\int_{-2}^{4} (4y^3 + 32y^2 + 64y - y^5)\, dy = \frac{45}{2}$$

Example 10 : Evaluate $\iint_R (x^2 + y^2)\, dx\, dy$ over the area enclosed by the curves $y = 4x$, $x + y = 3$, $y = 0$, $y = 2$.

Sol. :
$$I = \int_{y=0}^{2} \int_{x=y/4}^{3-y} (x^2 + y^2)\, dx\, dy = \int_0^2 \left[\frac{x^3}{3} + y^2 x\right]_{y/4}^{(3-y)} dy$$
$$= \int_0^2 \left[\frac{(3-y)^3}{3} + y^2(3-y) - \frac{y^3}{3 \cdot 64} - \frac{y^3}{4}\right] dy$$

Example 11 : Evaluate $\iint x^2 y^2\, dx\, dy$ over the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Sol. : See solved Ex. 19, page 9-48 of the Text Book and put $m = 3$, $n = 3$.

Example 12 : Evaluate $\iint y\, dx\, dy$ through out the area bounded by $y = x^2$ and $x + y = 2$.

Sol. :
$$I = \int_0^1 \int_{x^2}^{2-x} y\, dx\, dy = \frac{1}{2}\int_0^1 [(2-x)^2 - x^4]\, dx$$

Example 13 : Evaluate $\iint e^{3x+4y}\, dx\, dy$ over the triangle $x = 0$, $y = 0$, $x + y = 1$.

Sol. :
$$I = \int_{x=0}^{1} \int_{y=0}^{y=1-x} e^{3x+4y}\, dx\, dy = \int_0^1 e^{3x} \cdot \left[\frac{e^{4y}}{4}\right]_0^{1-x} dx$$
$$= \frac{1}{4}\int_0^1 e^{3x}\left[e^{4(1-x)} - 1\right] dx = \frac{1}{4}\int_0^1 (e^{4-x} - e^{3x})\, dx$$

Example 14 : Evaluate $\iint \frac{xy}{\sqrt{1-y^2}}\, dx\, dy$ over the positive quadrant of the circle $x^2 + y^2 = 1$.

Sol. :
$$I = \int_0^1 \int_0^{\sqrt{1-y^2}} \frac{y}{\sqrt{1-y^2}}\, x\, dx\, dy = \int_0^1 \left[\frac{y}{\sqrt{1-y^2}} \cdot \frac{x^2}{2}\right]_0^{\sqrt{1-y^2}} dy$$

$$\therefore \quad I = \frac{1}{2}\int_0^1 y \cdot \sqrt{1-y^2}\, dy = \frac{1}{2}\left[\frac{1}{3}(1-y^2)^{3/2}\right]_0^1$$

Example 15 : Evaluate $\iint \frac{dy\, dx}{\sqrt{1-2x^2-y^2}}$ over the first quadrant of the ellipse $2x^2 + y^2 = 1$.

Sol. :
$$I = \int_{y=0}^{1} \int_{x=0}^{\sqrt{(1-y^2)}/\sqrt{2}} \frac{dx}{\sqrt{(1-y^2) - 2x^2}}$$
$$= \int_0^1 \int_0^{\sqrt{(1-y^2)}/\sqrt{2}} \frac{dx}{\sqrt{2} \cdot \sqrt{\frac{(1-y^2)}{2} - x^2}}\, dy$$
$$= \frac{1}{\sqrt{2}} \cdot \int_0^1 \left[\sin^{-1}\left(x \cdot \frac{\sqrt{2}}{\sqrt{1-y^2}}\right)\right]_0^{\sqrt{(1-y^2)}/\sqrt{2}} dx = \frac{1}{\sqrt{2}}\int_0^1 \frac{\pi}{2}\, dx.$$

Example 16 : Evaluate $\iint_R xy(x-1)\, dx\, dy$ where R is the region bounded by $xy = 4$, $y = 0$, $x = 1$, $x = 4$.

Sol. :
$$I = \int_{x=1}^{4} \int_{y=0}^{4/x} (x^2 y - xy)\, dy\, dx = \int_1^4 \left[\frac{x^2 y^2}{2} - \frac{xy^2}{2}\right]_0^{4/x} dx$$
$$= \int_1^4 \left(8 - \frac{8}{x}\right) dx.$$

Example 17 : Evaluate $\iint_R xy(x+y)\, dx\, dy$ where R is the region bounded by $x^2 = y$ and $x = y$.

Sol. :
$$I = \int_{x=0}^{1} \int_{y=x^2}^{x} (x^2 y + xy^2)\, dy\, dx = \int_0^1 \left[x^2 \frac{y^2}{2} + \frac{xy^3}{3}\right]_{x^2}^{x} dx$$
$$= \int_0^1 \left[\left(\frac{x^4}{2} + \frac{x^4}{3}\right) - \left(\frac{x^6}{2} + \frac{x^7}{3}\right)\right] dx$$

Exercise VII (Page 9-54)

Solve the following examples : Class (b) : 6 Marks

Example 1 : Evaluate $\iint r\sqrt{a^2 - r^2} \cdot dr\, d\theta$ over upper half of the circle $r = a\cos\theta$.

Sol. : $I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{a\cos\theta} r\sqrt{a^2 - r^2}\, dr\, d\theta = -\frac{1}{3}\int_0^{\pi/2} \left[(a^2 - r^2)^{3/2}\right]_0^{a\cos\theta} d\theta$

$= -\frac{1}{3}\int_0^{\pi/2} (a^3\cos^3\theta - a^3)\, d\theta$

Example 2 : Evaluate $\iint r^2\, dr\, d\theta$ over the area between the circles $r = a\cos\theta$, $r = 2a\cos\theta$. ■

Sol. : $I = \int_{-\pi/2}^{\pi/2} \int_{a\cos\theta}^{2a\cos\theta} r^2\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} \left[\frac{r^3}{3}\right]_{a\cos\theta}^{2a\cos\theta} d\theta$

$= \frac{1}{3}\cdot 2a^3 \int_0^{\pi/2} 7a^3\cos^3\theta\, d\theta$

Example 3 : Evaluate $\iint \frac{r\, dr\, d\theta}{\sqrt{r^2 + 4}}$ over one loop of $r^2 = 4\cos 2\theta$. ■

Sol. : See solved Ex. 3, page 9-53 of the Text Book. Put $a = 2$.

Example 4 : Evaluate $\iint \frac{r\, dr\, d\theta}{(a^2 + r^2)^2}$ over one loop of lemniscate $r^2 = a^2\cos 2\theta$. ■

Sol. : $I = \int_{-\pi/4}^{\pi/4} \int_0^{a\sqrt{\cos 2\theta}} \frac{r\, dr\, d\theta}{(a^2 + r^2)^2} = 2\int_0^{\pi/4} \left[-\frac{1}{2}\cdot\frac{1}{(a^2 + r^2)}\right]_0^{a\sqrt{\cos 2\theta}} d\theta$

$= 2\int_0^{\pi/4} \left[\frac{1}{a^2} - \frac{1}{a^2\cdot 2\cos^2\theta}\right] d\theta = \frac{1}{a^2}[\theta - \tan\theta]$

Example 5 : Evaluate $\iint r\sin\theta\, dA$ over the cardioide $r = a(1 + \cos\theta)$ above the initial line. ■

Sol. : $I = \int_0^{\pi} \int_0^{a(1+\cos\theta)} r\sin\theta\, r\, dr\, d\theta = \int_0^{\pi/2} \sin\theta \left[\frac{r^3}{3}\right]_0^{a(1+\cos\theta)} d\theta$

$= \frac{a^3}{3}\int_0^{\pi} (1 + \cos\theta)^3 \sin\theta\, d\theta = -\frac{a^3}{4\cdot 3}\left[(1 + \cos\theta)^4\right]_0^{\pi}$ ■

Exercise VIII (Page 9-60)

Evaluate the following integrals over the region stated, by changing to polar coordinates : Class (a) : 6 Marks

Example 1 : $\iint x^2y^2\, dx\, dy$ over the circle $x^2 + y^2 = a^2$.

Sol. : $I = 4\int_0^{\pi/2} \int_0^a r^4 \sin^2\theta\cos^2\theta\, r\, d\theta\, dr$

$= 4\int_0^{\pi/2} \left[\frac{r^8}{6}\right]_0^a \sin^2\theta\cos^2\theta\, d\theta = \frac{2}{3}a^8 \int_0^{\pi/2} \sin^2\theta\cos^2\theta\, d\theta$ ■

Example 2 : $\iint \sin(x^2 + y^2)\, dx\, dy$ over the circle $x^2 + y^2 = a^2$.

Sol. : $I = \int_0^{2\pi} \int_0^a \sin(r^2)\, r\, d\theta\, dr = \int_0^{2\pi} \frac{1}{2}\left[-\cos r^2\right]_0^a d\theta = \frac{1}{2}\int_0^{2\pi} (1 - \cos a^2)\, d\theta$ ■

Example 3 : $\iint \frac{dx\, dy}{(1 + x^2 + y^2)^{3/2}}$ over the region bounded by $y = 0$, $x = y$, $x = 1$.

Sol. : $I = \int_0^{\pi/4} \int_0^{1/\cos\theta} \frac{r\, d\theta\, dr}{(1 + r^2)^{3/2}} = \int_0^{\pi/4} \left[-(1 + r^2)^{-1/2}\right]_0^{1/\cos\theta} d\theta$

$= \int_0^{\pi/4} \left[1 - \left(1 + \frac{1}{\cos^2\theta}\right)^{-1/2}\right] d\theta = \int_0^{\pi/4} \left[1 - \frac{\cos\theta}{\sqrt{1 + \cos^2\theta}}\right] d\theta$

$= \int_0^{\pi/4} \left[1 - \frac{\cos\theta}{\sqrt{2 - \sin^2\theta}}\right] d\theta = \left[\theta - \sin^{-1}\left(\frac{\sin\theta}{\sqrt{2}}\right)\right]_0^{\pi/4}$ ■

Example 4 : $\iint \frac{(x^2 + y^2)^2}{x^2y^2}$ over the area bounded by

(i) $x^2 + y^2 = ax$, $ax = by$, (ii) $x^2 + y^2 = by$, $ax = by$.

Sol. : $\because x^2 + y^2 = ax \quad \therefore r = a\cos\theta$

$\because ax = by \quad \therefore ar\cos\theta = br\sin\theta \quad \therefore \theta = \tan^{-1}(a/b)$

(i) $I = \int_{\tan^{-1}(a/b)}^{\pi/2} \int_0^{a\cos\theta} \frac{r^2}{r^4\sin^2\theta\cos^2\theta}\, r\, d\theta\, dr$

$= \int_{\tan^{-1}(a/b)}^{\pi/2} \frac{1}{\sin^2\theta\cos^2\theta}\left[\frac{r^2}{2}\right]_0^{a\cos\theta} d\theta$

$= \frac{a^2}{2}\int_{\tan^{-1}(a/b)}^{\pi/2} \operatorname{cosec}^2\theta\, d\theta$

(ii) $I = \int_0^{\tan^{-1}(a/b)} \int_0^{b\sin\theta} \frac{r^2}{r^2\sin^2\theta\cos^2\theta}\, r\, d\theta\, dr$

$= \int_0^{\tan^{-1}(a/b)} \frac{1}{\sin^2\theta\cos^2\theta}\cdot\left[\frac{r^2}{2}\right]_0^{b\sin\theta} d\theta$

$= \frac{b^2}{2}\int_0^{\tan^{-1}(a/b)} \sec^2\theta\, d\theta$ ■

Example 5 : $\iint xy\,(x^2 + y^2)^{3/2}\, dx\, dy$ over the first quadrant of the ellipse $x^2 + y^2 = a^2$.

Sol. : $I = \int_0^{\pi/2} \int_0^a r^2\sin\theta\cos\theta\cdot r^3\cdot r\, dr\, d\theta$

$= \int_0^{\pi/2} \sin\theta\cos\theta\, d\theta \int_0^a r^6\, dr.$ ■

Example 6 : $\iint (x^2 + y^2)\, x\, dx\, dy$ **over the positive quadrant of the circle** $x^2 + y^2 = a^2$.

Sol. : As above.

$$I = \int_0^{\pi/2} \int_0^a r^2 \cdot r \cos\theta\, r\, dr\, d\theta = \int_0^{\pi/2} \cos\theta\, d\theta \int_0^a r^4\, dr$$

Example 7 : $\iint \dfrac{dx\, dy}{(a^2 + x^2 + y^2)^{3/2}}$ **over the entire** xy **plane.**

Sol. : See Ex. 13, Exercise - V, page 9-39 of the Text Book.

Example 8 : $\iint e^{-(x^2+y^2)}\, dx\, dy$ **over the positive quadrant of the** xy **plane.**

Sol. : See solved Ex. 12, page 9-35 of the Text Book.

Exercise IX (Page 9-61)

(A) Find the following integrals : Class (a) : 3 Marks

Example 1 : $\int_0^1 \int_0^1 x^2 y^2\, dx\, dy$

Sol. : $$I = \int_0^1 x^2\, dx \int_0^1 y^2\, dy = \left[\frac{x^3}{3}\right]_0^1 \left[\frac{y^3}{3}\right]_0^1 = \frac{1}{9}$$

Example 2 : $\int_0^{\pi/2} \int_0^1 r^3 \sin\theta \cos\theta\, d\theta$

Sol. : $$I = \int_0^1 r^3\, d\theta \int_0^{\pi/2} \sin\theta \cos\theta = \left[\frac{r^4}{4}\right]_0^1 \left[\frac{\sin^2\theta}{2}\right]_0^1 = \frac{1}{8}$$

Example 3 : $\int_0^\infty \int_0^\infty \dfrac{dx\, dy}{(1+x^2)(1+y^2)}$

Sol. : $$I = \int_0^\infty \frac{dx}{1+x^2} \int_0^\infty \frac{dy}{1+y^2} = \left[\tan^{-1} x\right]_0^\infty \left[\tan^{-1} y\right]_0^\infty = \frac{\pi^2}{4}.$$

Example 4 : $\int_1^2 \int_0^{3y} y\, dx\, dy$

Sol. : $$I = \int_1^2 y\left[x\right]_0^{3y} dy = \int_1^2 3y^2\, dy = \left[y^3\right]_1^2 = 7$$

Example 5 : $\int_0^1 \int_0^{\pi/2} r \sin\theta\, dr\, d\theta$

Sol. : $$I = \int_0^1 r\, dr \int_0^{\pi/2} \sin\theta\, d\theta = \left[\frac{r^2}{2}\right]_0^1 \left[-\cos\theta\right]_0^{\pi/2} = \frac{1}{2}$$

(B) Evaluate the following integrals by changing to polar coordinates : Class (a) : 4 Marks

Example 1 : $\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2 + y^2)\, dx\, dy$

Sol. : See Fig. 9.76 on page 9-61 in the text book.

$$I = \int_0^{\pi/2} \int_0^1 r^2 \cdot r\, dr\, d\theta = \int_0^{\pi/2} d\theta \int_0^1 r^3\, dr = \frac{\pi}{8}$$

Example 2 : $\int_0^a \int_y^a x\, dx\, dy$

Sol. : See Fig. 9.77 on page 9-61 in the text book.

$$I = \int_0^{\pi/4} \int_0^{a/\cos\theta} r^2 \cos\theta\, dr\, d\theta$$

$$= \int_0^{\pi/4} \cos\theta \left[\frac{r^3}{3}\right]_0^{a/\cos\theta} d\theta = \frac{a^3}{3} \int_0^{\pi/4} \sec^2\theta\, d\theta$$

$$= \frac{a^3}{3}\left[\tan\theta\right]_0^{\pi/4} = \frac{a^3}{3}.$$

***

CHAPTER 10

Area

Exercise I (Page 10-12)

Find the areas by double integration : Class (b) : 6 Marks

Example 1 : The area bounded by $xy = 2$, $4y = x^2$ and $y = 4$.

Sol. : $y = \frac{2}{x}$ $\quad \therefore\ 8 = x^3 \quad \therefore\ x = 2,\ y = 1.$

$$A = \int_1^4 \int_{2/y}^{2\sqrt{y}} dx\, dy = \int_1^4 \left[2\sqrt{y} - \frac{2}{y}\right] dy$$

$$= \left[2 \cdot \frac{y^{3/2}}{3/2} - 2\log y\right]_1^4 = \frac{28}{3} - 4\log 2.$$

■

Example 2 : The area bounded by the lines $y = x + 2$, $y = -x + 2$, $x = 5$.

Sol. : The line $x = 5$ intersects the two lines in $(5, -3)$, $(5, 7)$.

$$A = \int_0^5 \int_{2-x}^{2+x} dy\, dx = \int_0^5 [2 + x - 2 + x]\, dx$$

$$= \int_0^5 2x\, dx = \left[x^2\right]_0^5 = 25.$$

■

Example 3 : The area between the circle $x^2 + y^2 - 2ax = 0$ and the parabola $y^2 = ax$.

Sol. : $(x - a)^2 + y^2 = a^2.$

Points of intersection are $x^2 - ax = 0$, $x(x - a) = 0$. $(0, 0)$, (a, a), $(a, -a)$.

$$\therefore\ A = 2\int_0^a \int_{\sqrt{ax}}^{\sqrt{2ax - x^2}} dx\, dy = 2\int_0^a \left[\sqrt{2ax - x^2} - \sqrt{ax}\right] dx$$

$$= 2\int_0^a \left[\sqrt{a^2 - (x - a)^2} - \sqrt{ax}\right] dx$$

$$= 2\left[\frac{x - a}{2} \cdot \sqrt{a^2 - (x - a)^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x - a}{a}\right) - \sqrt{a} \cdot \frac{2}{3}x^{3/2}\right]_0^a$$

■

Example 4 : The area bounded by the parabolas $y^2 = 4ax$ and $x^2 = 4ay$.

Sol. : Points of intersection are $y^4 = 16a^2x^2 = 16 \cdot a^2\, 4ay \quad \therefore\ y = 4a$

$x^2 = 16a^2 \quad \therefore\ x = 4a \qquad (4a, 4a),\ (0, 0).$

$$\therefore\ A = \int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dx\, dy = \int_0^{4a} \left[2\sqrt{ax} - \frac{x^2}{4a}\right] dx$$

■

Example 5 : The area between the parabola $y = 4x - x^2$ and the line $y = x$.

Sol. : $A = \int_0^3 \int_x^{4x - x^2} dy\, dx = \int_0^3 [\,y\,]_x^{4x - x^2}\, dx = \int_0^3 [4x - x^2 - x]\, dx$

■

Example 6 : The larger of the two areas into which the circle $r^2 = 16^2$ is divided by the parabola $y^2 = 24x$.

Sol. : See solved Ex. 21, page 10-10 of the Text Book and put $a = 4$.

The larger area $= \pi \cdot 16^2 - \frac{64}{3}(49 + \sqrt{3}).$

■

Example 7 : The total area of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$.

Sol. : $A = 4\int_0^a \int_0^{(a^{2/3} - x^{2/3})^{3/2}} dy\, dx = 4\int_0^a [a^{2/3} - x^{2/3}]\, dx$

Put $x = a\cos^3\theta$, $dx = -3a\cos^2\theta \sin\theta\, d\theta$

$$\therefore\ A = 4\int_{\pi/2}^0 a[1 - (\cos^3\theta)^{2/3}]^{3/2}(-3a)\cos^2\theta \sin\theta\, d\theta.$$

■

Example 8 : The total area included between the two branches of the curve $y^2(4 - x)(x - 2) = x^2$ and its two asymptotes.

Sol. : See solved Ex. 12, page 10-6 of the Text Book.

Put $a = 4$ and $b = 2$.

■

Example 9 : The area between the parabola $y^2 = 4ax$ and the line $2x - 3y + 4a = 0$.

Sol. : The point of intersection $\left(\frac{2x + 4a}{3}\right)^2 = 4ax$

$\therefore\ x = a,\ x = 4a \quad ; \quad \therefore\ y = 2a,\ y = 4a. \qquad (a, 2a), (4a, 4a)$

$$\therefore\ A = \int_a^{4a} \int_{(2x + 4a)/3}^{2\sqrt{ax}} dy\, dx = \int_a^{4a} [\,y\,]_{(2x + 4a)/3}^{2\sqrt{ax}}\, dy$$

■

Example 10 : The area of the circle $x^2 + y^2 = 2ax$ which lies outside the circle $x^2 + y^2 = a^2$.

Sol. : Putting $x = r\cos\theta$, $y = r\sin\theta$, $x^2 + y^2 - 2ax = 0$ becomes $r = 2a\cos\theta$ and $x^2 + y^2 = a^2$ becomes $r = a$.

$$\therefore\ A = 2\int_0^{\pi/3}\int_0^{2a\cos\theta} r\,dr\,d\theta = 2\int_0^{\pi/3}\left[\frac{r^2}{2}\right]_a^{2a\cos\theta} d\theta$$

$$= a^2\int_0^{\pi/3}\left[\frac{4(1+\cos 2\theta)}{2} - 1\right]d\theta$$

■

Example 11 : The area between the circles $x^2 + y^2 - 4ax = 0$ and $x^2 + y^2 - 2ax = 0$.

Sol. : Putting $x = r\cos\theta$, $y = r\sin\theta$. $x^2 + y^2 = 4ax$ becomes $r = 4a\cos\theta$ and $x^2 + y^2 = 2ax$ becomes $r = 2a\cos\theta$.

$$\therefore\ A = 2\int_0^{\pi/2}\int_{2a\cos\theta}^{4a\cos\theta} r\,dr\,d\theta = \int_0^{\pi/2}\left[r^2\right]_{2a\cos\theta}^{4a\cos\theta} d\theta$$

$$= \int_0^{\pi/2}(16a^2\cos^2\theta - 4a^2\cos^2\theta)\,d\theta$$

$$= 12a^2\int_0^{\pi/2}\cos^2\theta\,d\theta = 3\pi a^2.$$

■

Exercise II (Page 10-20)

Find the areas by double integration : Class (b) : 6 Marks

Example 1 : Find the area of the cardioid $r = a(1 - \cos\theta)$.

Sol. : $$A = 2\int_0^{\pi}\int_0^{a(1-\cos\theta)} r\,dr\,d\theta = 2\int_0^{\pi}\frac{a^2(1-\cos\theta)^2}{2}\,d\theta$$

$$= a^2\int_0^{\pi} 4\sin^4(\theta/2)\,d\theta = 4a^2\int_0^{\pi/2}\sin^4\Phi\cdot 2\,d\Phi \quad \left[\text{Put } \frac{\theta}{2} = \Phi\right]$$ ■

Example 2 : Find the area between the circles

$$r = 2\sin\theta,\quad r = 4\sin\theta.$$

Sol. : See solved Ex. 8, page 10-16 of the Text Book. Put $a = 1$ and $b = 2$. ■

Example 3 : Find the area of the crescent bounded by the circles

$$r = \sqrt{2} \text{ and } r = 2\cos\theta.$$

Sol. : See solved Ex. 9, page 10-17 of the Text Book. Put $a = 1$. ■

Example 4 : Find the area inside the circle $r = 2\sin\theta$ and outside the cardioid $r = 2(1 - \cos\theta)$.

Sol. : See solved Ex. 5, page 10-15 of the Text Book. Put $a = 2$. ■

Exercise - III (Page 10-20)

Find the areas by double integration : Class (a) : 3 or 4 Marks

Example 1 : The area bounded by $y^2 = 4x$ and the ordinate at $x = 1$.

Sol. : See Fig. 10.38 on page 10-20 in the text book.

$$A = 2\int_0^1\int_0^{2\sqrt{x}} dy\,dx = 2\int_0^1\left[y\right]_0^{2\sqrt{x}} dx$$

$$= 2\int_0^1 2\sqrt{x} = 4\left[\frac{x^{3/2}}{3/2}\right]_0^1 = \frac{8}{3}.$$

■

Example 2 : The area bounded by $y = e^x$ and the line $y = x$ from $x = 0$ to $x = 2$.

Sol. : See Fig. 10.39 on page 10-20 in the text book.

$$A = \int_0^2\int_x^{e^x} dy\,dx = \int_0^2\left[y\right]_x^{e^x} dx$$

$$= \int_0^2(e^x - x)\,dx = \left[e^x - \frac{x^2}{2}\right]_0^2 = e^2 - 3.$$

■

Example 3 : The area bounded by $y = x$ and $y = x^{1/3}$ in the first quadrant.

Sol. : See Fig. 10.40 on page 10-20 in the text book.

$$A = \int_0^1\int_x^{x^{1/3}} dy\,dx = \int_0^1\left[x^{1/3} - x\right]dx$$

$$= \frac{3}{4} - \frac{1}{2} = \frac{1}{4}.$$

■

Example 4 : The area bounded by $y = x^3$, $y = x^5$ in the first quadrant.

Sol. : See Fig. 10.41 on page 10-21 in the text book.

$$A = \int_0^1\int_{x^5}^{x^3} dy\,dx = \int_0^1\left[x^5 - x^3\right]dx$$

$$= \left[\frac{x^6}{6} - \frac{x^4}{4}\right]_0^1 = \frac{1}{12}.$$

■

Example 5 : The area bounded by the line $y = a + x$ and $y = a - x$ and $x = a$.

Sol. : See Fig. 10.42 on page 10-22 in the text book.

$$A = \int_0^a\int_{a-x}^{a+x} dy\,dx = \int_0^a[a + x - a + x]\,dx$$

$$= \int_0^a 2x\,dx = \left[x^2\right]_0^a = a^2.$$

■

Example 6 : The area bounded by $y^2 = 4x$ and the line $y = 2x$.

Sol. : See Fig. 10.43 on page 10-22 in the text book.

$$A = \int_0^1 \int_{2x}^{2\sqrt{x}} dy\, dx = \int_0^1 \left[2\sqrt{x} - 2x\right] dx$$

$$= 2\left[\frac{x^{3/2}}{3/2} - \frac{x^2}{2}\right]_0^1 = 2\left[\frac{2}{3} - \frac{1}{2}\right] = \frac{1}{3}.$$

■

Example 7 : The area between the curves $y^2 = x^3$ and $y = x^2$.

Sol. : See Fig. 10.44 on page 10-21 in the text book.

$$A = \int_0^1 \int_{x^{3/2}}^{x^2} dy\, dx = \int_0^1 \left[x^2 - x^{3/2}\right] dx$$

$$= \left[\frac{x^3}{3} - \frac{x^{5/2}}{5/2}\right]_0^1 = \frac{1}{3} - \frac{2}{5} = \frac{1}{15}.$$

■

Example 8 : The area between the curve $y = 2x + x^2$, the x-axis and the ordinates at $x = 1$, $x = 3$.

Sol. : See Fig. 10.45 on page 10-21 in the text book.

$$A = \int_1^3 \int_0^{2x + x^2} dy\, dx = \int_1^3 (x^2 + 2x)\, dx$$

$$= \left[\frac{x^3}{3} + x^2\right]_1^3 = (9 + 9) - \left(\frac{1}{3} + 1\right) = \frac{50}{3}.$$

■

Example 9 : The area bounded by $y = x^2$ and $y^2 = x$.

Sol. : See Fig. 10.46 on page 10-21 in the text book.

$$A = \int_0^1 \int_{x^2}^{\sqrt{x}} dy\, dx = \int_0^1 \left[\sqrt{x} - x^2\right] dx$$

$$A = \left[\frac{x^{3/2}}{3/2} - \frac{x^3}{3}\right]_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$$

■

Example 10 : The area of a circle of radius a (use polar form).

Sol. : See Fig. 10.47 on page 10-22 in the text book.

$$A = 4\int_0^{\pi/2} \int_0^a r\, dr\, d\theta$$

$$= 4\left[\theta\right]_0^{\pi/2} \left[\frac{r^2}{2}\right]_0^a = 4 \cdot \frac{\pi a^2}{4} = \pi a^2$$

■

Example 11 : The area between the circles $r = \sin\theta$ and $r = 2\sin\theta$.

Sol. : See Fig. 10.48 on page 10-22 in the text book.

$$A = 2\int_0^{\pi/2} \int_{\sin\theta}^{2\sin\theta} r\, dr\, d\theta = 2\int_0^{\pi/2} \left[\frac{r^2}{2}\right]_{\sin\theta}^{2\sin\theta}$$

$$= \int_0^{\pi/2} (4\sin^2\theta - \sin^2\theta)\, d\theta = 3\int_0^{\pi/2} \sin^2\theta\, d\theta$$

$$= 3\int_0^{\pi/2} \left(\frac{1 - \cos 2\theta}{2}\right) d\theta = \frac{3}{2} \cdot \left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{3}{2} \cdot \frac{\pi}{2} = \frac{3\pi}{4}.$$

■

***

CHAPTER

11 Mass of a Lamina

Exercise I (Page 11-8)

Solve the following examples :-Class (b) : 6 Marks

Type - I

Example 1 : Find the mass of the lamina bounded by the curve $y^2 = x^3$ and the line $y = x$, if the density at a point varies as the distance of the point from the x-axis.

Sol. : See solved Ex. 1 page 11-1 of the Text Book. Put $a = 1$, $b = 1$. ■

Example 2 : Find the mass of the lamina bounded by $4y^2 = x^3$ and $y = x$, if the density at any point varies as the distance of the point from the x-axis.

Sol. : See solved Ex. 1 page 11-1 of the Text Book. Put $a = 4$, $b = 1$. ■

Example 3 : Find the mass of the lamina bounded by $ay^2 = x^3$ and the line $y = x$ if the density at any point varies as the distance of the point from the x-axis.

Sol. : See solved Ex. 1 page 11-1 of the Text Book. Put $b = 1$.

Example 4 : Find the mass of the lamina bounded by the curve $16y^2 = x^3$ and the line $2y = x$, if the density at a point varies as the square of the distance of the point from the x-axis.

Sol. : $16y^2 = x^3$, $16\dfrac{x^2}{4} = x^3$ $\therefore$ $x = 4$.

Point of intersection is (4, 2).

$$\text{Mass} = \int_0^4 \int_{x^{3/2}/4}^{x/2} ky^2\, dx\, dy = k\int_0^4 \left[\frac{y^3}{3}\right]_{x^{3/2}/4}^{x/2} dx$$

$$= \frac{k}{3}\int_0^4 \left(\frac{x^3}{8} - \frac{x^{9/2}}{64}\right) dx.$$ ■

Example 5 : The boundaries of a plate can be given by $x = 0$, $y = 0$, $x = 1$ and $y = e^x$. If the density at any point varies as the square of its distance from the origin, find the mass of the plate.

Sol. : Points of intersection are (0, 1), (1, e).

$$\text{Mass} = \int_0^1 \int_0^{e^x} k(x^2 + y^2)\, dy\, dx$$

$$= k\int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_0^{e^x} dx$$

$$= k\int_0^1 \left(x^2 e^x + \frac{e^{3x}}{3}\right) dx$$ ■

Example 6 : A lamina is bounded by the curves $y = x^2 - 3x$ and $y = 2x$. If the density at any point $P(x, y)$ is $k\,xy$, find the mass of the lamina.

Sol. : Points of intersection $x^2 - 3x = 2x$ *i.e.* $x = 0$ or 5

$y = 2x$ $\therefore$ $y = 0$ or 10

$$\text{Mass} = \int_0^5 \int_{x^2-3x}^{2x} k\,xy\, dy\, dx = k\int_0^5 \left[\frac{xy^2}{2}\right]_{x^2-3x}^{2x} dx$$

$$= \frac{k}{2}\int_0^5 [x(2x)^2 - x(x^2 - 3x)^2]\, dx$$

$$= \frac{k}{2}\int_0^5 (4x^3 - x^5 + 6x^4 - 9x^3)\, dx.$$ ■

Example 7 : Find the mass of the lamina bounded by $x^2 + 2y - 4 = 0$ and the x-axis, if the density at any point is k times its distance from the x-axis.

Sol. : $$\text{Mass} = 2\int_0^2 \int_0^{(4-x^2)/2} ky\, dy\, dx$$

$$= 2k\int_0^2 \left[\frac{y^2}{2}\right]_0^{(4-x^2)/2} dx = \int_0^2 \frac{(4 - x^2)^2}{4}\, dx$$ ■

Example 8 : Find the mass of the lamina of the region included between the curves $y = \log x$, $y = 0$, $x = 2$, having uniform density.

Sol. : $$\text{Mass} = \int_1^2 \int_0^{\log x} dy\, dx = \int_1^2 (\log x)\, dx$$

$$= \left[\log x \cdot x - \int dx\right]_1^2 = [x \log x - x]_1^2$$

$$= (2\log 2 - 2) - (0 - 1) = 2\log 2 - 1.$$ ■

Example 9 : Find the mass of the lamina bounded by the curves $y^2 = x$ and $x^2 = y$, if the density of the lamina at any point varies as the square of its distance from the origin.

Sol. : See Ex. 4, page 11-3 of the text. Put $a = 1$. ■

Type - II

Example 1 : The density at any point of a uniform circular lamina of radius a varies as its distance from a fixed point on the circumference of the circle. Find the mass of the lamina.

Sol. : The circle is $r = 2a\cos\theta$.

$$\text{Mass} = 2\int_0^{\pi/2}\int_0^{2a\cos\theta} kr \cdot r\, d\theta\, dr$$

$$= 2k\int_0^{\pi/2}\left[\frac{r^3}{3}\right]_0^{2a\cos\theta} d\theta$$

$$= 2k\int_0^{\pi/2}\frac{8a^3\cos^3\theta}{3}\, d\theta$$

■

Example 2 : If the density at any point of the curve $x = a(\theta + \sin\theta)$ and $y = a(1 + \cos\theta)$ varies as the distance from the x-axis, find the mass of the lamina bounded by the curve, ordinates at the two cusps and the tangent at the vertex.

Sol. : $\text{Mass} = 2\iint ky\, dx\, dy$

Now y varies from y to $2a$.

$$\therefore\ \text{Mass} = 2\int k\left[\frac{y^2}{2}\right]_y^{2a} dx = k\int(4a^2 - y^2)\, dx$$

If we put $y = a(1 + \cos\theta)$ and $dx = a(1 + \cos\theta)\, d\theta$ then θ varies from π to 0 in the first quadrant.

$$\therefore\ \text{Mass} = k\int_\pi^0 [4a^2 - a^2(1+\cos\theta)^2]\, a(1+\cos\theta)\, d\theta$$

$$= 4ka^3\int_\pi^0(1+\cos\theta)\, d\theta - ka^3\int_\pi^0(1+\cos\theta)^3\, d\theta$$

$$= -4ka^3\int_0^\pi[2\cos^2(\theta/2)]\, d\theta + ka^3\int_0^\pi[2\cos^2(\theta/2)]^3\, d\theta$$

$$= -8ka^3\int_0^\pi\cos^2(\theta/2)\, d\theta + 8ka^3\int_0^\pi\cos^6(\theta/2)\, d\theta$$

Put $\theta/2 = \Phi$ $\quad\therefore\ d\theta = 2\, d\Phi$

$$\therefore\ \text{Mass} = -8ka^3\int_0^{\pi/2}\cos^2\Phi \cdot 2\, d\Phi + 8ka^3\int_0^{\pi/2}\cos^6\Phi \cdot 2\, d\Phi$$

$$= -16ka^3\int_0^{\pi/2}\cos^2\Phi\, d\theta + 16ka^3\int_0^{\pi/2}\cos^6\Phi\, d\Phi$$

$$= 16ka^3\left[\int_0^{\pi/2}\cos^6\Phi\, d\Phi - \int_0^{\pi/2}\cos^2\Phi\, d\Phi\right] = \frac{3\pi ka^3}{2}$$

■

Example 3 : Find the mass of a plate in the form of a cardioid $r = a(1 - \cos\theta)$ if the density at any point of the plate varies as its distance from the pole.

Sol. :

$$\text{Mass} = 2\int_0^\pi\int_0^{a(1-\cos\theta)} k \cdot r \cdot r\, d\theta\, dr$$

$$= 2k\int_0^\pi\left[\frac{r^3}{3}\right]_0^{a(1-\cos\theta)} d\theta = \frac{2k}{3}\int_0^\pi a^3(1-\cos\theta)^3\, d\theta$$

$$= \frac{2ka^3}{3}\cdot 8\int_0^\pi\sin^6(\theta/2)\, d\theta \qquad \left[\text{Put } \frac{\theta}{2} = t\right]$$

$$= \frac{16ka^3}{3}\int_0^{\pi/2}\sin^6 t \cdot 2\, dt.$$

■

Example 4 : The density at any point of a non-uniform circular lamina of radius a varies as (i) the distance and (ii) the cube of its distance from a fixed point on the circumference of the circle. Find the mass of the lamina in each case.

Sol. : (i) $M = 2\int_0^{\pi/2}\int_0^{2a\cos\theta}(kr)\, r\, dr\, d\theta$

$$= 2k\int_0^{\pi/2}\left[\frac{r^3}{3}\right]_0^{2a\cos\theta} d\theta = \frac{2k}{3}\int_0^{\pi/2} 8a^3\cos^3\theta\, d\theta$$

(ii) $M = 2\int_0^{\pi/2}\int_0^{2a\cos\theta}(kr^3)\, r\, dr\, d\theta$

$$= 2k\int_0^{\pi/2}\left[\frac{r^5}{5}\right]_0^{2a\cos\theta} d\theta = \frac{2k}{5}\int_0^{\pi/2} 32a^5\cos^5\theta\, d\theta$$

■

Example 5 : Find the mass of a plate in the form of Lemniscate $r^2 = a^2\sin 2\theta$, whose density varies as the square of its distance from the pole.

Sol. : $M = \int_0^{\pi/2}\int_0^{a\sqrt{\sin 2\theta}} kr^2 \cdot r\, dr\, d\theta = k\int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^{a\sqrt{\sin 2\theta}} d\theta$

$$= k\int_0^{\pi/2}\frac{a^4}{4}\sin^2 2\theta\, d\theta = k\int_0^{\pi/2}\frac{a^4}{4}\, 4\sin^2\theta\cos^2\theta\, d\theta$$

■

Example 6 : Find the mass of the plate which is inside the circle $r = 2a\cos\theta$ and outside the circle $r = a$ if the density varies as the distance from the pole.

Sol. : $M = 2\int_0^{\pi/3}\int_a^{2a\cos\theta} kr \cdot r\, dr\, d\theta$

At the point of intersection $a = 2a\cos\theta$

i.e. $\cos\theta = \frac{1}{2}$ *i.e.* $\theta = \frac{\pi}{3}$.

$$\therefore \quad M = 2k\int_0^{\pi/3}\left[\frac{r^3}{3}\right]_a^{2a\cos\theta} d\theta$$

$$= \frac{2k}{3}\int_0^{\pi/3}(8a^3\cos^3\theta - a^3)\,d\theta$$

$$= \frac{2ka^3}{3}\int_0^{\pi/3}(8\cos^3\theta - 1)\,d\theta$$

$$= \frac{2ka^3}{3}\int_0^{\pi}\left[\frac{8(\cos 3\theta + 3\cos\theta)}{4} - 1\right]d\theta$$

Exercise - II (Page 11-9)

Solve the following examples : Class (a) : 3 or 4 Marks

Example 1 : Find the mass of the lamina bounded by the line $y = ax$, the x-axis and the line $x = a$, if the density at a point varies as the distance of the point from the x-axis.

Sol. : See Fig. 11.13 on page 11-9 in the text book.

$$M = \int_0^a\int_0^{ax} ky\,dy\,dx = \int_0^a k\left[\frac{y^2}{2}\right]_0^{ax} dx$$

$$= \frac{k}{2}\int_0^a a^2x^2\,dx = \frac{ka^2}{2}\left[\frac{x^3}{3}\right]_0^a = \frac{ka^5}{6}.$$

Example 2 : Find the mass of the lamina bounded by the lines $x = 0$, $y = 0$, $x = a$, $y = b$, if the density varies as the product of the distances from the axes.

Sol. : See Fig. 11.14 on page 11-9 in the text book.

$$M = \int_0^a\int_0^b kxy\,dy\,dx = k\int_0^a x\left[\frac{y^2}{2}\right]_0^b dx$$

$$= k\frac{b^2}{2}\int_0^a x\,dx = k\frac{b^2}{2}\cdot\frac{a^2}{2} = \frac{ka^2b^2}{4}.$$

Example 3 : Find the mass of the lamina bounded by the parabola $y^2 = ax$, the x-axis and the line $x = a$, if the density varies as the product of the distances of the point from the axes.

Sol. : See Fig. 11.4 on page 11-3 in the text book.

$$M = \int_0^a\int_0^{\sqrt{ax}} kxy\,dy\,dx = k\int_0^a x\left[\frac{y^2}{2}\right]_0^{\sqrt{ax}} dx$$

$$\therefore \quad M = \frac{k}{2}\int_0^a a\cdot x^2\,dx = \frac{ka}{2}\left[\frac{x^3}{3}\right]_0^a = \frac{ka^4}{6}.$$

Example 4 : Find the mass of a circular lamina of radius a, if the density at a point varies as its distance from a fixed point on its circumference.

Sol. : See Fig. 11.9 on page 11-6 in the text book.

$$M = 2\int_0^{\pi/2}\int_0^{2a\cos\theta} k\,r\cdot r\,d\theta\,dr = 2\int_0^{\pi/2} k\left[\frac{r^3}{3}\right]_0^{2a\cos\theta} d\theta$$

$$= \frac{2k}{3}\int_0^{\pi/2} 8a^3\cos^3\theta\,d\theta = \frac{16ka^3}{3}\cdot\frac{2}{3}\cdot 1 = \frac{32ka^3}{9}.$$

***

CHAPTER 12

Triple Integral

Exercise I (Page 12-8)

Evaluate the following integrals : Class (b) : 6 Marks

Example 1 : $\int_0^1 dx \int_0^2 dy \int_1^2 x^2 yz\, dz$.

Sol. : $I = \int_0^1 dx \int_0^2 dy \left[x^2 y \frac{z^2}{2}\right]_1^2$

$= \int_0^1 dx \int_0^2 \left(2x^2 y - \frac{x^2 y}{2}\right) dy$

$= \int_0^1 dx \int_0^2 \frac{3}{2} x^2 y\, dy = \int_0^1 \frac{3}{2}\left[x^2 \frac{y^2}{2}\right]_0^2 dx$

Example 2 : $\int_0^2 \int_0^z \int_0^{yz} xyz\, dx\, dy\, dz$.

Sol. : $I = \int_0^2 \int_0^z \left[\frac{x^2}{2}\right]_0^{yz} yz\, dy\, dz = \int_0^2 \int_0^z \frac{1}{2}[y^3 z^3]\, dy\, dz$

$= \int_0^2 \frac{1}{2}\left[\frac{y^4}{4}\right]_0^z z^3\, dz = \int_0^2 \frac{1}{8} \cdot z^7\, dz = \frac{1}{64}\left[z^8\right]_0^2 = 4$

Example 3 : $\int_0^2 \int_0^x \int_0^{x+y} e^{x+y+z}\, dx\, dy\, dz$.

Sol. : $I = \int_0^2 \int_0^x \left[e^{x+y+z}\right]_0^{x+y} dy\, dx = \int_0^2 \int_0^x \left[e^{2(x+y)} - e^{x+y}\right] dy\, dx$

$= \int_0^2 \left[\frac{e^{2(x+y)}}{2} - e^{x+y}\right]_0^x dx$

$= \int_0^2 \left[\frac{e^{4x}}{2} - e^{2x} - \frac{e^{2x}}{2} + e^x\right] dx$

Example 4 : $\int_0^{\log 2} \int_0^x \int_0^{x+\log y} e^{x+y+z}\, dx\, dy\, dz$.

Sol. : $I = \int_0^{\log 2} \int_0^x \left[e^{x+y+z}\right]_0^{x+\log y} dx\, dy$

$= \int_0^{\log 2} \int_0^x \left[e^{x+y+x+\log y} - e^{x+y}\right] dx\, dy$

$I = \int_0^{\log 2} \int_0^x \left[e^{2x+y} y - e^{x+y}\right] dy\, dx$

$= \int_0^{\log 2} \left[e^{2x}(ye^y - e^y) - e^{x+y}\right]_0^x dx$

Example 5 : $\int_0^a \int_0^a \int_0^a (y^2 z^2 + z^2 x^2 + x^2 y^2)\, dx\, dy\, dz$.

Sol. : $I = \int_0^a \int_0^a \left[y^2 z^2 x + z^2 \frac{x^3}{3} + y^2 \frac{x^3}{3}\right]_0^a dy\, dz$

$= \int_0^a \int_0^a \left(ay^2 z^2 + \frac{a^3}{3} z^2 + \frac{a^3}{3} y^2\right) dy\, dz$

$= \int_0^a \left[az^2 \frac{y^3}{3} + \frac{a^3}{3} z^2 y + \frac{a^3}{3} \cdot \frac{y^3}{3}\right]_0^a dz$

$= \int_0^a \left(\frac{a^4}{3} z^2 + \frac{a^4}{3} z^2 + \frac{a^6}{9}\right) dz$

Example 6 : $\int_0^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_0^x z^2\, dx\, dy\, dz$.

Sol. : $I = \int_0^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_0^y z^2\, dx\, dy\, dz = \int_0^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left[\frac{z^3}{3}\right]_0^y dy\, dx$

$= \int_0^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left[\frac{y^3}{3}\right] dy\, dx = \int_0^a \left[\frac{y^4}{12}\right]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} = \frac{a^5}{5}$

Example 7 : $\int_0^a \int_0^{a-x} \int_0^{a-x-y} (x^2 + y^2 + z^2)\, dz\, dy\, dx$.

Sol. : $I = \int_0^a \int_0^{a-x} \left[x^2 z + y^2 z + \frac{z^3}{3}\right]_0^{a-x-y} dy\, dx$

$= \int_0^a \int_0^{a-x} \left[x^2(a-x) - x^2 y + (a-x) y^2 - y^3 + \frac{(a-x-y)^3}{3}\right] dy\, dx$

$= \int_0^a \left[x^2(a-x) y - x^2 \frac{y^2}{2} + (a-x)\frac{y^3}{3} - \frac{y^4}{4} - \frac{(a-x-y)^4}{12}\right]_0^{a-x} dx$

$$= \int_0^a \left[\frac{x^2(a-x)^2}{2} + \frac{(a-x)^4}{6} \right] dx$$

$$= \int_0^a \left\{ \frac{1}{2}[a^2x^2 - 2ax^3 + x^4] + \frac{(a-x)^4}{6} \right\} dx$$

Example 8 : $\int_0^1 \int_0^{1-x} \int_0^{x+y} e^z \, dx \, dy \, dz$.

Sol. : $I = \int_0^1 \int_0^{1-x} \left[e^z\right]_0^{x+y} dx \, dy = \int_0^1 \int_0^{1-x} (e^{x+y} - 1) \, dy \, dx$

$$= \int_0^1 \left[e^{x+y} - y\right]_0^{1-x} dx = \int_0^1 \left[e - (1-x) - e^x\right] dx$$

■

Example 9 : $\int_0^4 \int_0^{2\sqrt{z}} \int_0^{\sqrt{4z-x^2}} dz \, dx \, dy$.

Sol. : $I = \int_0^4 \int_0^{2\sqrt{z}} [\, y \,]_0^{\sqrt{4z-x^2}} dz \, dx = \int_0^4 \int_0^{2\sqrt{z}} \sqrt{4z-x^2} \, dz \, dx$

$$= \int_0^4 \left[\frac{x}{2}\sqrt{4z-x^2} + 2z \sin^{-1}\left(\frac{x}{2\sqrt{z}}\right) \right]_0^{2\sqrt{z}} dz$$

■

Example 10 : $\int_0^a \int_0^x \int_0^{\sqrt{x+y}} z \, dz \, dy \, dx$.

Sol. : $I = \int_0^a \int_0^x \left[\frac{z^2}{2}\right]_0^{\sqrt{x+y}} dy \, dx = \frac{1}{2}\int_0^a \int_0^x (x+y) \, dy \, dx$

$$= \frac{1}{2}\int_0^a \left[xy + \frac{y^2}{2} \right]_0^x dx$$

■

Example 11 : $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xyz \, dx \, dy \, dz$.

Sol. : $I = \frac{1}{2}\int_0^a \int_0^{\sqrt{a^2-x^2}} [xy(a^2 - x^2 - y^2)] \, dy \, dx$

$$= \frac{1}{2}\int_0^a \int_0^{\sqrt{a^2-x^2}} (a^2xy - x^3y - xy^3) \, dy \, dx$$

$$= \frac{1}{2}\int_0^a \left[a^2x\frac{y^2}{2} - x^3\frac{y^2}{2} - x\frac{y^4}{4} \right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_0^a \left[\frac{a^4x}{2} - \frac{a^2x^3}{2} - \frac{a^2x^3}{2} + \frac{x^5}{2} - \frac{a^4x}{4} + \frac{a^2x^3}{2} - \frac{x^5}{4} \right] dx$$

$$= \frac{1}{2}\int_0^a \left[\frac{a^4x}{4} - \frac{a^2x^3}{2} + \frac{x^5}{4} \right] dx$$

■

Exercise II (Page 12-12)

Solve the following examples : Class (b) : 6 Marks

Example 1 : Evaluate $\iiint x^2yz \, dx \, dy \, dz$ throughout the volume bounded by $x = 0, y = 0, z = 0, x + y + z = 1$.

Sol. : See solved Ex. 4, page 12-10 of the Text Book. Put $a = b = c = 1$.

$$\therefore \; I = \frac{a^3b^2c}{2520} = \frac{1}{2520}.$$

■

Example 2 : Evaluate $\iiint xyz \, dx \, dy \, dz$ throughout the volume bounded $x = 0, y = 0, z = 0, x + y + z = 1$.

Sol. : See solved Ex. 5, page 12-11 of the Text Book. Put $l = 2, m = 2, n = 2$.

$$\therefore \; I = \frac{1}{(l+m+n)} \cdot \frac{\lfloor l \; \lfloor m \; \lfloor n}{\lfloor l+m+n}$$

$$= \frac{1}{6} \cdot \frac{\lfloor 2 \; \lfloor 2 \; \lfloor 2}{\lfloor 6} = \frac{1}{6} \cdot \frac{1 \cdot 1 \cdot 1}{5!} = \frac{1}{720}.$$

■

Example 3 : Evaluate $\iiint x^2 y^2 z^2 \, dx \, dy \, dz$ throughout the volume bounded $x = 0, y = 0, z = 0, x + y + z = 1$.

Sol. : See solved Ex. 5, page 12-11 of the Text Book. Put $l = 3, m = 3, n = 3$.

$$\therefore \; I = \frac{1}{9} \cdot \frac{\lfloor 3 \; \lfloor 3 \; \lfloor 3}{\lfloor 9} = \frac{2! \; 2! \; 2!}{9(8!)} = \frac{1}{45360}.$$

■

Example 4 : Evaluate $\iiint x^2 \, dx \, dy \, dz$ throughout the volume of the tetrahedron $x \geq 0, y \geq 0, z \geq 0$ and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} \leq 1$.

Sol. : See solved Ex. 5, page 12-11 of the Text Book. Put $x = au, \; y = bv, z = cv$ and then $l = 3, m = 1, n = 1$.

$$\therefore \; I = a^3bc \cdot \frac{1}{5} \cdot \frac{\lfloor 3 \; \lfloor 1 \; \lfloor 1}{\lfloor 5} = a^3bc \cdot \frac{2! \; 2! \; 2!}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{a^3bc}{60}.$$

■

Example 5 : Evaluate $\iiint dx \, dy \, dz$ over the volume of the tetrahedron bounded by $x = 0, y = 0, z = 0$ and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

Sol. : See solved Ex. 2, page 10-9 and Ex. 5, page 10-11 of the Text Book. Put $x = au, \; y = bv, z = cw$ and then $l = 1, m = 1, n = 1$.

$$\therefore \; I = \frac{abc}{3} \cdot \frac{\lfloor 1 \; \lfloor 1 \; \lfloor 1}{\lfloor 3} = \frac{abc}{6}.$$

■

Exercise III (Page 12-19)

Solve the following examples : Class (b) : 6 Marks

Example 1 : Evaluate $\iiint \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{3/2}}$ **over the volume bounded by the spheres** $x^2+y^2+z^2=16$ **and** $x^2+y^2+z^2=25$.

Sol. : See solved Ex. 8, page 12-17 of the Text Book. Put $a=4$, $b=5$. ■

Example 2 : Evaluate $\iiint \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{1/2}}$ **over the volume bounded by the spheres** $x^2+y^2+z^2=a^2$ **and** $x^2+y^2+z^2=b^2$, $a>b>0$.

Sol. : Using spherical coordinates,

$$I=\iiint \frac{1}{r}\cdot r^2\sin\theta\,dr\,d\theta\,d\Phi = 4\int_0^{\pi/2} d\Phi \int_0^{\pi/2}\sin\theta\,d\theta\int_a^b r\,dr$$

$$=4\left[\Phi\right]_0^{\pi/2}\left[-\cos\theta\right]_0^{\pi/2}\left[\frac{r^2}{2}\right]_a^b = 4\cdot\frac{\pi}{2}\cdot[1]\cdot(b^2-a^2)$$

$$=2\pi(b^2-a^2).$$ ■

Example 3 : Evaluate $\iiint x^2\,dx\,dy\,dz$ **over the volume of the ellipsoid**

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Sol. : Put $x=au$, $y=bv$, $z=cw$

$$\therefore\ I=8\iiint a^2u^2\cdot a\,du\,b\,dv\,c\,dw = 8a^3bc\int u^2\,du\,dv\,dw$$

Now, put $w=r\cos\theta$, $v=r\sin\theta\sin\Phi$, $u=r\sin\theta\cos\Phi$.

$$\therefore\ du\,dv\,dw=r^2\sin\theta\,dr\,d\theta\,d\Phi$$

$$\therefore\ I=8a^3bc\iiint r^2\sin^2\theta\cos^2\Phi\,r^2\sin\theta\,dr\,d\theta\,d\Phi$$

$$=8a^3bc\int_0^{\pi/2}\cos^2\Phi\,d\Phi\int_0^{\pi/2}\sin^3\theta\,d\theta\int_0^1 r^4\,dr$$

$$=8a^3bc\left[\frac{1}{2}\cdot\frac{\pi}{2}\right]\cdot\left[\frac{2}{3}\right]\cdot\left[\frac{1}{5}\right]=\frac{4\pi a^3bc}{15}.$$ ■

Example 4 : Evaluate $\iiint \sqrt{\frac{x^2}{1^2}+\frac{y^2}{2^2}+\frac{z^2}{3^2}}\,dx\,dy\,dz$ **over the volume of the ellipsoid** $\frac{x^2}{1^2}+\frac{y^2}{2^2}+\frac{z^2}{3^2}=1$.

Sol. : See solved Ex. 13, page 12-19 of the Text Book. Put $a=1$, $b=2$, $c=3$.

Exercise IV (Page 12-22)

Solve the following examples : Class (b) : 6 Marks

Example 1 : Evaluate the integral $\iiint \sqrt{x^2+y^2}\,dV$ **throughout the volume bounded by the cylinder** $x^2+y^2=1$ **over the paraboloid** $z=1-x^2-y^2$ **below the plane** $z=4$.

Sol. : Put $x=r\cos\theta$, $y=r\sin\theta$, $z=z$. $\quad dx\,dy\,dz=r\,dr\,d\theta\,dz$

$$I=\int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=1-r^2}^{4} r\cdot r\,dr\,d\theta\,dz=\int_{\theta=0}^{2\pi}\int_{r=0}^{1} r^2\left[z\right]_{1-r^2}^{4}dr\,d\theta$$

$$=\int_{\theta=0}^{2\pi}\int_{r=0}^{1}(3r^2+r^4)\,dr\,d\theta=\int_{\theta=0}^{2\pi}\frac{6}{5}\,d\theta=\frac{12\pi}{5}.$$ ■

Example 2 : Evaluate $\iiint z^2\,dx\,dy\,dz$ **over the volume common to the sphere** $x^2+y^2+z^2=1$ **and the cylinder** $x^2+y^2=x$.

Sol. : See solved Ex. 4, page 12-25 of the Text Book. Put $a=1$. ■

Exercise - V (Page 12-22)

Evaluate the following triple integrals : Class (a) : 3 or 4 Marks

Example 1 : $\int_0^1\int_0^1\int_0^1 xyz\,dx\,dy\,dz$ ■

Sol. : $I=\int_0^1 x\,dx\int_0^1 y\,dy\int_0^1 z\,dz=\frac{1}{8}.$

Example 2 : $\int_0^1\int_0^x\int_0^{xy} dx\,dy\,dz$

Sol. : $I=\int_0^1\int_0^x xy\,dy\,dx=\int_0^1 x\left[\frac{y^2}{2}\right]_0^x dx$

$$=\int_0^1\frac{x^3}{3}\,dx=\frac{1}{2}\left[\frac{x^4}{4}\right]_0^1=\frac{1}{8}.$$ ■

Example 3 : $\int_0^\infty\int_0^\infty\int_0^\infty e^{-(x+y+z)}\,dx\,dy\,dz$

Sol. : $I=\int_0^\infty e^{-x}\,dx\int_0^\infty e^{-y}\,dy\int_0^\infty e^{-z}\,dz$

$$=\left[-e^{-x}\right]_0^\infty\left[-e^{-y}\right]_0^\infty\left[-e^{-z}\right]_0^\infty$$

$$=[-(0-1)][-(0-1)][-(0-1)]=1.$$ ■

Example 4 : $\int_0^1 \int_0^1 \int_0^1 e^{(x+y+z)}\, dx\, dy\, dz$

Sol. : $I = \left[e^x\right]_0^1 \left[e^y\right]_0^1 \left[e^z\right]_0^1 = (e-1)^3.$

Example 5 : $\int_{-1}^1 \int_{-2}^2 \int_{-3}^3 dx\, dy\, dz$

Sol. :
$$I = 2[x]_0^1\, 2[y]_0^2\, 2[z]_0^3$$
$$= 2(1) \cdot 2(2) \cdot 2(3) = 48$$

Example 6 : $\int_0^1 dx \int_0^2 dy \int_1^2 x^2\, yz\, dz$

Sol. :
$$I = \int_0^1 dx \int_0^2 dy\, (x^2\, y) \left[\frac{z^2}{2}\right]_1^2$$
$$= \frac{1}{2}\int_0^1 dx \int_0^2 3\, x^2\, y\, dy = \frac{3}{2}\int_0^1 dx\, (x^2) \left[\frac{y^2}{2}\right]_0^2$$
$$= 3\int_0^1 x^2\, dx = 3\left[\frac{x^3}{3}\right]_0^1 = 1.$$

Example 7 : $\int_0^1 \int_1^z \int_0^{yz} xyz\, dx\, dy\, dz$

Sol. :
$$I = \int_0^1 \int_1^z y^2 \left[\frac{x^2}{2}\right]_0^{yz} dy\, dz = \frac{1}{2}\int_0^1 \int_1^z y^3\, z^3\, dy\, dz$$
$$= \frac{1}{2}\int_0^1 z^3 \left[\frac{y^4}{4}\right]_1^z dz = \frac{1}{8}\int_0^1 (z^7 - z^3)\, dz$$
$$= \frac{1}{8}\left[\frac{z^8}{8} - \frac{z^4}{4}\right]_0^1 = \frac{1}{8}\left[\frac{1}{8} - \frac{1}{4}\right] = -\frac{1}{64}.$$

Example 8 : $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x\, dz\, dx\, dy$

Sol. :
$$I = \int_0^1 \int_{y^2}^1 x\left[z\right]_0^{1-x} dy\, dx = \int_0^1 \int_{y^2}^1 (x - x^2)\, dy\, dx$$
$$= \int_0^1 \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{y^2}^1 dy = \int_0^1 \left(\frac{1}{6} - \frac{y^4}{2} + \frac{y^6}{3}\right) dy$$
$$= \left[\frac{y}{6} - \frac{y^5}{10} + \frac{y^7}{21}\right]_0^1 = \frac{4}{35}.$$

Example 9 : $\int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 r\, dr\, d\theta\, d\phi$

Sol. :
$$I = \left[\theta\right]_0^{\pi/2} \left[\phi\right]_0^{\pi/2} \left[\frac{r^2}{2}\right]_0^1 = \frac{\pi^2}{8}$$

Example 10 : $\int_0^a \int_0^{\pi/2} \int_0^{\pi/2} r \cos\theta \cos\phi\, dr\, d\theta\, d\phi$

Sol. :
$$I = \left[\frac{r^2}{2}\right]_0^a \left[\sin\theta\right]_0^{\pi/2} \left[\sin\phi\right]_0^{\pi/2} = \frac{a^2}{2}.$$

Example 11 : $\int_0^{\pi} \int_0^{\pi/3} \int_0^1 r^2 \sin\theta\, dr\, d\theta\, d\phi$

Sol. :
$$I = \int_0^{\pi} \int_0^{\pi/3} \sin\theta \left[\frac{r^3}{3}\right]_0^1 d\theta\, d\phi = \frac{1}{3}\int_0^{\pi} \left[-\cos\theta\right]_0^{\pi/3} d\phi$$
$$= \frac{1}{3}\int_0^{\pi} \left[1 - \frac{1}{2}\right] d\phi = \frac{\pi}{6}.$$

Example 12 : $\int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 r^2 \sin\theta\, dr\, d\theta\, d\phi$

Sol. :
$$I = \int_0^{\pi/2} \int_0^{\pi/2} \sin\theta \left[\frac{r^3}{3}\right]_0^1 d\theta\, d\phi$$
$$= \int_0^{\pi/2} \frac{1}{3}\left[-\cos\theta\right]_0^{\pi/2} d\phi = \int_0^{\pi/2} \frac{1}{3}\, d\phi = \frac{\pi}{6}.$$

Example 13 : $\int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 r^4 \sin\theta\, dr\, d\theta\, d\phi$

Sol. :
$$I = \int_0^{\pi/2} d\phi \int_0^{\pi/2} \sin\theta\, d\theta \int_0^1 r^4\, dr$$
$$= \left[\phi\right]_0^{\pi/2} \left[-\cos\theta\right]_0^{\pi/2} \left[\frac{r^5}{5}\right] = \frac{\pi}{10}.$$

Example 14 : $\int_0^{\pi} \int_0^{\pi} \int_a^b \frac{\sin\theta}{r}\, dr\, d\theta\, d\phi$

Sol. :
$$I = \int_0^{\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \int_a^b \frac{dr}{r}$$
$$= \left[\phi\right]_0^{\pi} \left[-\cos\theta\right]_0^{\pi} \left[\log r\right]_a^b = 2\pi \log\left(\frac{b}{a}\right).$$

***

CHAPTER

13 Volumes of Solids

Exercise I (Page 13-17)

Solve the following examples : Class (c) : 8 Marks

Example 1 : Find the volume of the triangular prism formed by the plane $ay - bx = 0$, $y = 0$, $x = 0$, $x = a$, $z = 0$, $z = c$.

Sol. : $V = \int_{x=0}^{a} \int_{y=0}^{bx/a} \int_{z=0}^{z=c} dx\, dy\, dz = \int_{x=0}^{a} \int_{y=0}^{bx/a} [z]_0^c\, dx\, dy$

$= c \int_0^a \int_{y=0}^{bx/a} dy\, dx = c \int_0^a [y]_0^{bx/a}\, dx$

Example 2 : Find the volume bounded by the coordinate planes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

Sol. : Put $x = au$, $y = bv$, $z = cw$, then $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ becomes $u + v + w = 1$ and $dx\, dy\, dz = abc\, du\, dv\, dw$.

$$\therefore V = \int_0^1 \int_0^{1-u} \int_0^{1-u-v} abc\, du\, dv\, dw$$

$$= abc \int_0^1 \int_0^{1-u} [1 - u - v]\, du\, dv$$

$$= abc \int_0^1 \left[v - uv - \frac{v^2}{2} \right]_0^{1-u} du$$

$$= abc \int_0^1 \left[(1-u) - u(1-u) - \frac{(1-u)^2}{2} \right] du$$

Example 3 : Find the volume bounded by the coordinate planes and the plane $x + y + z = 1$.

Sol. : Put $a = b = c = 1$ in the above Ex. No. 2.

Example 4 : Find the volume bounded by the coordinates planes and the plane $lx + my + nz = 1$.

Sol. : Put $a = \frac{1}{l}$, $b = \frac{1}{m}$, $c = \frac{1}{n}$ in the above Ex. No. 2.

Example 5 : Find the volume bounded by $y^2 = x$, $x^2 = y$ and the planes $z = 0$ to $z = a$.

Sol. : $V = \iint_R z\, dx\, dy = \iint a\, dx\, dy$

$= \int_{x=0}^{1} \int_{y=\sqrt{x^2}}^{\sqrt{x}} a\, dy\, dx = \int_0^1 [ay]_{x^2}^{\sqrt{x}}\, dx$

$= a \int_0^1 \left[\sqrt{x} - x^2 \right] dx = a \left[\frac{x^{3/2}}{3/2} - \frac{x^3}{3} \right]_0^1 = a \left[\frac{2}{3} - \frac{1}{3} \right] = \frac{a}{3}$

Example 6 : Find the volume bounded by $y^2 = x$, $x^2 = y$ and the planes $z = 0$ and $z = 3$.

Sol. : $V = \iint_R z\, dx\, dy = \iint_R 3\, dx\, dy$

$= \int_0^1 \int_{x^2}^{\sqrt{x}} 3\, dy\, dx = 3 \int_0^1 [y]_{x^2}^{\sqrt{x}}\, dx$

Example 7 : Find the volume bounded by the cylinders $y^2 = x$ and $x^2 = y$ between the planes $z = 0$ and $x + y + z = 2$.

Sol. : $V = \iint_R z\, dx\, dy = \iint_R (2 - x - y)\, dx\, dy$

$= \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} (2 - x - y)\, dx\, dy = \int_0^1 \left[2y - xy - \frac{y^2}{2} \right]_{x^2}^{\sqrt{x}} dx$

$= \int_0^1 \left[\left(2\sqrt{x} - x^{3/2} - \frac{x}{2} \right) - \left(2x^2 - x^3 - \frac{x^4}{2} \right) \right] dx$

Example 8 : Find the volume bounded by the cylinder $x^2 + y^2 = a^2$ and the planes $y + z = 2a$ and $z = 0$.

Sol. : $V = \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_0^{2a-y} dz\, dy\, dx$

$= \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} [2a - y]\, dy\, dx = \int_{-a}^{a} \left[2ay - y^2 \right]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx$

$= \int_{-a}^{a} \left\{ \left[2a\sqrt{a^2 - x^2} - (a^2 - x^2) \right] - \left[-2a\sqrt{a^2 - x^2} - (a^2 - x^2) \right] \right\} dx$

$= 2 \int_0^a 4a\sqrt{a^2 - x^2}\, dx.$ Put $x = a \sin\theta$.

Example 9 : Find the volume bounded by $x^2 + y^2 = bz$ and $x^2 + y^2 = a^2$.

Sol. : $V = 4 \int_0^{\pi/2} \int_0^a \int_0^{r^2/b} r\, dr\, d\theta\, dz = 4 \int_0^{\pi/2} \int_0^a [z]_0^{r^2/b}\, r\, dr\, d\theta$

$= 4 \int_0^{\pi/2} \int_0^a \left(\frac{r^2}{b} \right) r\, dr\, d\theta = 4 \int_0^{\pi/2} \int_0^a \left(\frac{r^3}{b} \right) dr\, d\theta$

Example 10 : By using triple integration find the volume cut off from the sphere $x^2 + y^2 + z^2 = 16$ by the plane $z = 0$ and the cylinder $x^2 + y^2 = 4x$.

Sol. :

$$V = 2\int_0^{\pi/2}\int_0^{4\cos\theta}\int_{z=0}^{\sqrt{16-r^2}} r\,dr\,d\theta\,dz$$

$$= 2\int_0^{\pi/2}\int_0^{4\cos\theta} [z]_0^{\sqrt{16-r^2}}\, r\,dr\,d\theta$$

$$= 2\int_0^{\pi/2}\int_0^{4\cos\theta} (\sqrt{16-r^2})\, r\,dr\,d\theta$$

$$= 2\int_0^{\pi/2}\left[\frac{(16-r^2)^{3/2}}{-3/2}\cdot\frac{1}{2}\right]_0^{4\cos\theta} d\theta$$

$$= \frac{2}{3}\int_0^{\pi/2}(-64\cos^3\theta + 64)\,d\theta$$

$$= \frac{2}{3}\cdot 64\left[\int_0^{\pi/2} d\theta - \int_0^{\pi/2}\cos^3\theta\,d\theta\right]$$

■

Example 11 : Find the volume enclosed by the cylinders $x^2+y^2=4x$ and $z^2 = 4x$ by triple integration.

Sol. : See solved Ex. 8, page 13-10 of the Text Book. Put $a = 2$.

■

Example 12 : Find the volume enclosed by the cylinder $x^2+y^2=9$ and the planes $x + z = 5$ and $z = 0$.

Sol. :

$$V = 4\int_0^{\pi/2}\int_0^{3}\int_0^{5-r\cos\theta} dz\,r\,dr\,d\theta = 4\int_0^{\pi/2}\int_0^{3}[5 - r\cos\theta]\,r\,dr\,d\theta$$

$$= 4\int_0^{\pi/2}\left[\frac{5r^2}{2} - \frac{r^3}{3}\cos\theta\right]_0^3 d\theta = 4\int_0^{\pi/2}\left(5\cdot\frac{9}{2} - 9\cos\theta\right)d\theta$$

■

Example 13 : Find the volume of the sphere $x^2+y^2+z^2=a^2$ by triple integration.

Sol. :

$$V = 8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{a} r^2\sin\theta\,dr\,d\theta\,d\Phi$$

$$= 8\int_0^{\pi/2} d\Phi\int_0^{\pi/2}\sin\theta\,d\theta\int_0^{a} r^2\,dr$$

$$= 8[\Phi]_0^{\pi/2}[-\cos\theta]_0^{\pi/2}\left[\frac{r^3}{3}\right]_0^a$$

$$= 8\cdot\frac{\pi}{2}(0+1)\left(\frac{a^3}{3}\right) = \frac{4\pi a^3}{3}.$$

■

Example 14 : Find the volume enclosed by the sphere $x^2 + y^2 + z^2 = a^2$ and the cone $x^2 + y^2 = z^2 \tan\alpha$.

Sol. :

$$V = \int_0^{2\pi}\int_0^{\alpha}\int_0^{a} r^2\sin\theta\,dr\,d\theta\,d\Phi$$

$$= \int_0^{2\pi} d\Phi\int_0^{\alpha}\sin\theta\,d\theta\int_0^{a} r^2\,dr$$

■

Example 15 : Find the volume bounded in the first octant by the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$ and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1$.

Sol. :

$$V_1 = abc\int_0^{\pi/2} d\Phi\int_0^{\pi/2}\sin\theta\,d\theta\int_0^1 r^2\,dr$$

$$= abc[\Phi]_0^{\pi/2}[-\cos\theta]_0^{\pi/2}\left[\frac{r^3}{3}\right]_0^1 = abc\cdot\frac{\pi}{2}(1)\cdot\frac{1}{3} = \frac{\pi abc}{6}$$

$$V_2 = \int_0^1\int_0^{1-u}\int_0^{1-u-v} abc\,du\,dv\,dw$$

$$= abc\int_0^1\int_0^{1-u}[w]_0^{1-u-v}\,du\,dv = abc\int_0^1\int_0^{1-u}(1-u-v)\,du\,dv$$

$$= abc\int_0^1\left[(1-u)v - \frac{v^2}{2}\right]_0^{1-u} du = abc\int_0^1\left[(1-u)^2 - \frac{(1-u)^2}{2}\right]du$$

$$= \frac{abc}{2}\int_0^1(1-u)^2\,du = \frac{abc}{6}.$$

■

Example 16 : Find the volume of region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 2x$.

Sol. : The circle of intersection is $x^2 + y^2 = 2x$

i.e. $r^2 = 2r\cos\theta$ *i.e.* $r = 2\cos\theta$.

$$\therefore\ V = \iiint dx\,dy\,dz = \iint_R\int_{x^2+y^2}^{2x} dx\,dy\,dz$$

$$= \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta}[2x - (x^2+y^2)]\,r\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta}[2r\cos\theta - r^2]\,r\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\left[2\frac{r^3}{3}\cos\theta - \frac{r^4}{4}\right]_0^{2\cos\theta} d\theta = \int_{-\pi/2}^{\pi/2}\frac{4}{3}\cos^4\theta\,d\theta$$

$$= 2\int_0^{\pi/2}\frac{4}{3}\cos^4\theta\,d\theta = 2\cdot\frac{4}{3}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{2}.$$

■

Example 17 : Find the volume of region above the xy - plane enclosed by the paraboloid $az = x^2 + y^2$ and the cylinder $x^2 + y^2 = 2ax$.

Sol. : $$V = \int_{-\pi/2}^{\pi/2} \int_0^{2a\cos\theta} \int_0^{r^2/a} r\,dr\,d\theta\,dz$$

$$= 2\int_0^{\pi/2} \int_0^{2a\cos\theta} [z]_0^{r^2/a}\, r\,dr\,d\theta$$

$$= 2\int_0^{\pi/2} \int_0^{2a\cos\theta} \left[\frac{r^2}{a}\right] r\,dr\,d\theta = 2\int_0^{\pi/2} \left[\frac{r^4}{4a}\right]_0^{2a\cos\theta} d\theta$$

■

Example 18 : Find the volume of the paraboloid $x^2 + y^2 = 4z$ cut off by the plane $z = 4$.

Sol. : $$V = 4\int_0^{\pi/2} \int_0^4 \int_{r^2/4}^4 r\,dr\,d\theta\,dz$$

$$= 4\int_0^{\pi/2} \int_0^4 [z]_{r^2/4}^4\, r\,dr\,d\theta = 4\int_0^{\pi/2} \int_0^4 \left(4r - \frac{r^3}{4}\right) dr$$

■

Example 19 : Find the volume of the solid bounded by the plane $z = 0$, the paraboloid $z = x^2 + y^2 + 2$ and the cylinder $x^2 + y^2 = 4$.

Sol. : $$V = 4\int_0^{\pi/2} \int_0^2 \int_0^{r^2+2} r\,dr\,d\theta\,dz = 4\int_0^{\pi/2} \int_0^2 [z]_0^{r^2+2}\, r\,dr\,d\theta$$

$$= 4\int_0^{\pi/2} \int_0^2 (r^2 + 2)\, r\,dr\,d\theta = 4\int_0^{\pi/2} \int_0^2 (r^3 + 2r)\,dr\,d\theta$$

■

Example 20 : Find the volume of the paraboloid $x^2 + y^2 = az$ cut off by the plane $z = a$.

Sol. : The integersection is $x^2 + y^2 = a^2$ *i.e.* $r = a$.

$$\therefore\ V = 4\int_0^{\pi/2} \int_0^a \int_{z=r^2/a}^{z=a} r\,dr\,d\theta\,dz = 4\int_0^{\pi/2} \int_0^a \left[a - \frac{r^2}{a}\right] r\,dr\,d\theta$$

$$= 4\int_0^{\pi/2} \left[\frac{ar^2}{2} - \frac{r^4}{4a}\right]_0^a d\theta = 4 \cdot \frac{a^3}{4}\int_0^{\pi/2} d\theta = \frac{\pi a^3}{2}.$$

Example 21 : Find the volume cut off from the paraboloid $x^2 + \dfrac{y^2}{9} + z = 1$ by the plane $z = 0$.

Sol. : $$V = \iint_R z\,dx\,dy = \iint \left(1 - x^2 - \frac{y^2}{9}\right) dx\,dy$$

$$= \int_{-1}^1 \int_{-3\sqrt{1-x^2}}^{3\sqrt{1-x^2}} \left(1 - x^2 - \frac{y^2}{9}\right) dy\,dx$$

$$= 2\int_0^1 2\int_0^{3\sqrt{1-x^2}} \left(1 - x^2 - \frac{y^2}{9}\right) dy\,dx$$

$$= 4\int_0^1 \left[(1 - x^2)\,y - \frac{y^3}{27}\right]_0^{3\sqrt{1-x^2}} dx$$

$$= 4\int_0^1 \left[3(1 - x^2)^{3/2} - \frac{27}{27}(1 - x^2)^{3/2}\right] dx$$

$$= 8\int_0^1 (1 - x^2)^{3/2}\,dx \qquad \text{Put } x = \sin\theta.$$

■

Example 22 : Find the volume bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy - plane.

Sol. : $$V = \iint z\,dx\,dy = \iint (4 - x^2 - y^2)\,dx\,dy$$

$$= 2\int_0^2 2\int_0^{\sqrt{4-x^2}} (4 - x^2 - y^2)\,dy\,dx$$

$$= 4\int_0^2 \left[(4 - x^2)y - \frac{y^3}{3}\right]_0^{\sqrt{4-x^2}} dx$$

$$= 4\int_0^2 \left[(4 - x^2)\cdot\sqrt{4 - x^2} - \frac{1}{3}(4 - x^2)^{3/2}\right] dx$$

$$= 4\int_0^2 \left[(4 - x^2)^{3/2} - \frac{1}{3}(4 - x^2)^{3/2}\right] dx$$

$$= \frac{8}{3}\int_0^2 (4 - x^2)^{3/2}\,dx. \qquad \text{Put } x = 2\sin\theta,\ dx = 2\cos\theta\,d\theta.$$

■

Example 23 : Find the volume bounded by the paraboloid $z = 4 - x^2 - \dfrac{y^2}{4}$ and the plane $z = 0$.

Sol. : $$V = \iint z\,dx\,dy = \iint \left[(4 - x^2) - \frac{y^2}{4}\right] dx\,dy$$

$$V = 2\int_0^2 2\int_0^{2\sqrt{4-x^2}} \left[(4 - x^2) - \frac{y^2}{4}\right] dy\,dx$$

$$= 4\int_0^2 \left[(4 - x^2)\,y - \frac{y^3}{3 \cdot 4}\right]_0^{2\sqrt{4-x^2}} dx$$

$$= 4\int_0^2 \left[2(4 - x^2)^{3/2} - \frac{8}{12}(4 - x^2)^{3/2}\right] dx$$

$$= 4 \cdot \left(\frac{4}{3}\right)\int_0^2 (4 - x^2)^{3/2}\,dx. \quad \text{Put } x = 2\sin\theta.$$

$$= \frac{16}{3}\int_0^{\pi/2} 16\cos^4\theta\,d\theta = \frac{16 \cdot 16}{3} \cdot \frac{3 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2} = 16\pi.$$

■

Exercise - II (Page 13-20)

Solve the following examples : Class (a) : 3 or 4 Marks

Example 1 : Find the volume of the cuboid of sides a, b, c by triple integration.

Sol. : $V = \int_0^a \int_0^b \int_0^c dx\, dy\, dz$

$= [x]_0^a [y]_0^b [z]_0^c = abc$

Example 2 : Find the volume enclosed by the planes $y + z = 1$, $z = 0$, $y = 0$, $x = 0$, $x = 1$.

Sol. : Refer Fig. 13.48 on page 13-20 in the text book.

Sol. : $V = \int_{x=0}^1 \int_{y=0}^1 \int_{z=0}^{1-y} dz\, dy\, dx = \int_{x=0}^1 \int_{y=0}^1 [z]_0^{1-y}\, dy\, dx$

$= \int_{x=0}^1 \int_{y=0}^1 (1 - y)\, dy\, dx = \int_{x=0}^1 \left[y - \frac{y^2}{2}\right]_0^1 dx$

$= \int_0^1 \frac{1}{2}\, dx = \frac{1}{2}$

Example 3 : Find the volume of the sphere of radius a using triple integration.

Sol. : Refer Fig. 13.49 on page 13-20 in the text book.

Sol. : $V = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^a r^2 \sin\theta\, dr\, d\theta\, d\phi$

$= 8 \int_0^{\pi/2} d\phi \int_0^{\pi/2} \sin\theta\, d\theta \int_0^a r^2\, dr$

$= 8\left[\frac{\pi}{2}\right][-\cos\theta]_0^{\pi/2}\left[\frac{a^3}{3}\right] = \frac{4}{3}\pi a^3.$

***

CHAPTER 14 Numerical Integration

Exercise I (Page 14-7)

Solve the following examples : Class (a) : 4 Marks

Example 1 : Given the following values of e^x.

x :	0·0	0·5	1·0	1·5	2·0	2·5
e^x :	1·0	1·65	2·72	4·48	7·39	12·18

Evaluate $\int_0^{2\cdot5} e^x\, dx$ using Trapezoidal rule.

Sol. : $I = \frac{h}{2}[X + 2R]$

$X = y_0 + y_5 = 1\cdot0 + 12\cdot18 = 13\cdot18$

$R = y_1 + y_2 + y_3 + y_4 = 16\cdot24, \qquad h = 0\cdot5$

$\therefore\ I = \frac{0\cdot5}{2}[13\cdot18 - 32\cdot48] = 11\cdot415$

Example 2 : Find using the Trapezoidal rule from the following table the area bounded by the curve and the x-axis from $x = 7\cdot47$ to $x = 7\cdot52$.

x :	7·47	7·48	7·49	7·50	7·51	7·52
$f(x)$:	1·93	1·95	1·98	2·01	2·03	2·06

Sol. : $A = \frac{h}{2}[X + 2R]$

$X = y_0 + y_5 = 1\cdot93 + 2\cdot06 = 3\cdot99$

$R = y_1 + y_2 + y_3 + y_4 = 7\cdot97, \qquad h = 0\cdot01$

$\therefore\ A = \frac{0\cdot01}{2}[3\cdot99 + 2(7\cdot97)] = 0\cdot0996$

Example 3 : A curve is drawn to pass through the points given by

x :	1·0	1·5	2·0	2·5	3·0	3·5	4·0
y :	2·0	2·4	2·7	2·8	3·0	2·6	2·1

Estimate the area bounded by the curve between the x-axis and the lines $x = 1$ and $x = 4$, by Simpson's rule.

Sol. : $A = \frac{h}{3}[X + 2E + 4O]$

$X = 2 + 2{\cdot}1 = 4{\cdot}1,$ $\quad E = y_2 + y_4 = 2{\cdot}7 + 3 = 5{\cdot}7,$

$O = 2{\cdot}4 + 2{\cdot}8 + 2{\cdot}6 = 7{\cdot}8,$ $\quad h = 0{\cdot}5.$

$\therefore \quad A = \frac{0{\cdot}5}{3}[4{\cdot}1 + 2(5{\cdot}7) + 4(4{\cdot}8)] = 7{\cdot}783.$ ■

Example 4 : A river is 90 meter wide. The depth d meters from one bank is given by the following table. Find approximately the area of the cross-section.

a :	0	10	20	30	40	50	60	70	80	90
d :	0	4	7	9	12	15	14	8	3	1

Sol. : Since the last ordinate is odd, we cannot use Simpson's rule. We shall use Trapezoidal rule.

$$\therefore \quad A = \frac{h}{2}[X + 2R]$$

$$= \frac{10}{2}[(0 + 1) + 2(4 + 7 + 9 + 12 + 15 + 14 + 8 + 3)]$$

$$= 725$$ ■

Example 5 : Calculate upto 5 decimal places $\int_4^{5{\cdot}2} \log_e x \cdot dx$ by using Trapezoidal rule from the following table.

x :	4·0	4·2	4·4	4·6	4·8	5·0	5·2
$\log_e x$:	1·3863	1·4351	1·4816	1·5260	1·5886	1·6094	1·6486

Sol. : $I = \frac{h}{2}[X + 2R]$

$X = y_0 + y_6 = 3{\cdot}0349$

$R = y_1 + y_3 + y_4 + y_5 = 7{\cdot}6407, \quad h = 0{\cdot}2$

$\therefore \quad I = \frac{0{\cdot}2}{2}[3{\cdot}0349 + 2(7{\cdot}6407)] = 1{\cdot}83163.$ ■

Example 6 : Given the following values of sin x

x :	0	$2\pi/10$	$4\pi/10$	$6\pi/10$	$8\pi/10$	π
$\sin x$:	0·000	0·5878	0·9511	0·9511	0·5878	0·000

Evaluate $\int_0^{\pi} \sin x\, dx$ and compare it with its exact value.

Sol. : Since the last ordinate is odd we cannot use Simpson's rule. Hence, we use Trapezoidal rule.

$$I = \frac{h}{2}[X + 2R]$$

$$= \frac{\pi/5}{2}[(0 + 0) + 2(0{\cdot}5878 + 0{\cdot}9511 + 0{\cdot}9511 + 0{\cdot}5878)]$$

$$= \frac{\pi}{10}[6{\cdot}1556] = 1{\cdot}9338 \quad \therefore \quad \frac{1}{2}I = 0{\cdot}9669$$

$$\frac{1}{2}\int_0^{\pi} \sin x\, dx = \frac{1}{2}[-\cos x]_0^{\pi} = -\frac{1}{2}[-1 - 1] = 1.$$ ■

Example 7 : Apply Simpson's 1/3rd rule to find $\int_0^6 \frac{1}{1+x}\, dx$.

Sol. :

x :	0	1	2	3	4	5	6
$y = \frac{1}{1+x}$:	1	0·5	0·3333	0·25	0·20	0·1667	0·1429

$$I = \frac{h}{3}[X + 2E + 4O]$$

$$= \frac{1}{3}[1{\cdot}1429 + 2(0{\cdot}5333) + 4(0{\cdot}91167)]$$

$$= 1{\cdot}9588$$ ■

Example 8 : Use any rule to evaluate approximately $\int_0^1 \frac{dx}{1+x^2}$ taking five equidistant intervals and find the value of π.

Sol. :

x :	0	0·2	0·4	0·6	0·8	1
$y = \frac{1}{1+x^2}$:	1	0·9615	0·8621	0·7353	0·6097	0·5

We shall use Trapezoidal rule

$$I = \frac{h}{2}[X + 2R]$$

$X = 1 + 0{\cdot}5 + 1{\cdot}5$

$R = 0{\cdot}9615 + 0{\cdot}8621 + 0{\cdot}7353 + 0{\cdot}6097 = 3{\cdot}1679$

$h = 0{\cdot}2$

$$\therefore \quad I = \frac{0{\cdot}2}{2}[1{\cdot}5 + 2(3{\cdot}1686)] = 0{\cdot}7837$$

$$I = \int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_0^1 = \tan^{-1} 1 = \frac{\pi}{4}.$$

$$\therefore \quad \pi = 4I = 4\,(0{\cdot}7837) = 3{\cdot}1348.$$ ■

Note : Since the last ordinate is odd numbered, we cannot use Simpson's rule.

Example 9 : Calculate by trapezoidal rule an approximate value of $\int_0^1 e^x\, dx$ in steps of 0·20.

Sol. :

x :	0	0·20	0·40	0·60	0·80	1·00
e^x :	1	1·2214	1·4918	1·8221	2·2255	2·7183

$$I = \frac{h}{2}[X + 2R] = \frac{0{\cdot}20}{2}[3{\cdot}7183 + 2(6{\cdot}7608)] = 1{\cdot}72399$$ ■

Example 10 : Evaluate in two ways $\int_4^{5\cdot2} \log_e x\,dx$ by dividing the interval [4, 5·2] into six equal parts.

Sol. :

x	:	4	4·2	4·4	4·6	4·8	5	5·2
$y = \log e^x$	:	1·3863	1·4351	1·4816	1·5261	1·5686	1·6094	1·6487

(i) Trapezoidal rule

$$I = \frac{h}{2}[X + 2R] = \frac{0\cdot2}{2}[3\cdot035 + 2(7\cdot6208)] = 1\cdot8277$$

(ii) Simpson's rule

$$I = \frac{h}{3}[X + 2E + 4O]$$

$$= \frac{0\cdot2}{3}[3\cdot035 + 2(3\cdot0502) + 4(4\cdot5706)] = 1\cdot8278$$

■

Example 11 : Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by taking suitable sub-intervals and hence, find an approximate value of π by using (i) Trapezoidal rule, (ii) Simpson's 1/3rd rule.

Sol. :

x	:	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
$y = \frac{1}{1+x^2}$	:	1	$\frac{36}{37}$	$\frac{9}{10}$	$\frac{4}{5}$	$\frac{9}{13}$	$\frac{36}{61}$	$\frac{1}{2}$

(i) Trapezoidal rule

$$I = \frac{h}{2}[X + 2R]$$

$$= \left(\frac{1/6}{2}\right)\left[\left(1+\frac{1}{2}\right) + 2\left(\frac{36}{37}+\frac{9}{10}+\frac{4}{5}+\frac{9}{13}+\frac{36}{61}\right)\right]$$

$$= \frac{1}{12}[1\cdot5 + 2(3\cdot9554)] = 0\cdot7842$$

$$I = \int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_0^1 = \tan^{-1} 1 = \frac{\pi}{4}.$$

$$\therefore\ \pi = 4\,(0.7842) = 3.1368.$$

(ii) Simpson's rule

$$I = \frac{h}{3}[X + 2E + 4O]$$

$$= \left(\frac{1/6}{3}\right)\left[1\cdot5 + 2\left(\frac{9}{10}+\frac{9}{13}\right) + 4\left(\frac{36}{37}+\frac{4}{5}+\frac{36}{61}\right)\right]$$

$$= \frac{1}{18}[1\cdot5 + 2(1\cdot5923) + 4(2\cdot3631)] = 0\cdot7854$$

$$\therefore\ \pi = 4\,(0.7854) = 3.1416.$$

■

Example 12 : Calculate $\int_2^{10} \frac{dx}{1+x}$ upto 4 decimal places by dividing the range into eight equal parts by Simpson's 1/3rd rule.

Sol. :

x	:	2	3	4	5	6
$y = \frac{1}{1+x}$	:	0.3333	0.2500	0.2000	0.1667	0.1429

7	8	9	10
0·1250	0·1111	0·1000	0·0909

$$I = \frac{h}{3}[X + 2E + 4O]$$

$$= \frac{1}{3}[(0\cdot4242) + 2(0\cdot4540) + 4(0\cdot6417)]$$

$$= 1\cdot2996$$

■

Example 13 : Apply Simpson's 1/3rd rule to find the value of $\int_0^6 \frac{dx}{1+x^2}$ upto two places of decimals.

Sol. :

x	:	0	1	2	3	4	5	6
$y = \frac{1}{1+x^2}$	:	1	0·50	0·20	0·10	0·06	0·04	0·03

$$I = \frac{h}{3}[X + 2E + 4O]$$

$X = 1 + 0\cdot03 = 1\cdot03,$ $\quad E = 0\cdot2 + 0\cdot06 = 0\cdot26,$

$O = 0\cdot5 + 0\cdot1 + 0\cdot04 = 0\cdot64,$ $\quad h = 1.$

$$\therefore\ I = \frac{1}{3}[1\cdot03 + 2(0\cdot26) + 4(0\cdot64)] = 1\cdot37.$$

■

Example 14 : Evaluate $\int_{-3}^{3} x^4 dx$ by Trapezoidal rule and compare it with the exact value.

Sol. :

x	:	−3	−2	−1	0	1	2	3
$y = x^4$	:	81	16	1	0	1	16	81

$$I = \frac{h}{2}[X + 2R] = \frac{1}{2}[162 + 2(34)] = 115$$

Exact value, $I = \int_{-3}^{3} x^4\,dx = \left[\frac{x^5}{5}\right]_{-3}^{3} = \frac{1}{5}[243 + 243] = 97\cdot2$

■

Example 15 : Using Simpson's one third rule obtain $\int_1^2 \frac{dx}{x}$ taking 10 equal intervals. Hence, obtain approximate value of log 2 upto five places of decimals.

Sol. :

x :	1	1·1	1·2	1·3	1·4	1·5	1·6	1·7	1·8	1·9	2
$y = \frac{1}{x}$:	1	0.90909	0.83333	0.76923	0.71429	0·66667	0·62500	0·58824	0·55555	0·52632	0.50000

$I = \frac{h}{3}[X + 2E + 4O]$

$X = 1·5,$

$E = 0·83333 + 0·71429 + 0·62500 + 0·55555 = 2·7282$

$O = 0·90909 + 0·76923 + 0·66667 + 0·58824 + 0·52632 = 3·45955$

$\therefore \quad I = \frac{0·1}{3}[1·5 + 2(2·72821 + 4(3·45955)] = 0·69315$

$\int_1^2 \frac{1}{x}\,dx = [\log x]_1^2 = \log 2 \qquad \therefore \quad \log 2 = 0·69315$ ■

Example 16 : Evaluate $\int_{0·5}^{0·7} \sqrt{x}\, e^{-x}\, dx$ using Simpson's (1/8)th rule.

Sol.

x :	0.5	0.55	0.60	0.65	0.70
y :	0.4289	0.4279	0.4251	0.4209	0.4155

$I = \frac{1(0·05)}{3}[0·8444 + 2(0·4251) + 4(0·8488)] = 0·08483$ ■

Exercise II (Page 14-20)

Solve the following examples : Class (a) : 4 Marks

Example 1 : A curve is drawn to pass through the points given by

x :	1.0	1.5	2.0	2.5	3.0	3.5	4.0
y :	2.0	2.4	2.7	2.8	3.0	2.6	2.1

Estimate the area bounded by the curve between the x-axis, and the lines $x = 1$ and $x = 4$ by Simpson's (3/8)th rule.

Sol. : $X = y_0 + y_6 = 2 + 2·1 = 4·1$

$T = y_3 = 2·8$

$R = y_1 + y_2 + y_4 + y_5 = 2·4 + 2·7 + 3·0 + 2·6 = 10·7$

$h = 0·5$

$I = \frac{3h}{8}[X + 2T + 3R]$

$= \frac{3(0·5)}{8}[4·1 + 2(2·8) + 3(10·7)]$

$= 7·8375$ ■

Example 2 : A river is 90 meter wide. The depth d meters from one bank is given by the following table. Find approximately the area of the cross-section by Simpson's (3/8)th rule.

x :	0	10	20	30	40	50	60	70	80	90
d :	0	4	7	9	12	15	14	8	3	1

Sol. : $X = y_0 + y_9 = 0 + 1 = 1$

$T = y_3 + y_6 = 9 + 14 = 23$

$R = y_1 + y_2 + y_4 + y_5 + y_7 + y_8$

$= 4 + 7 + 12 + 15 + 8 + 3 = 49, \quad h = 10$

$A = \frac{3h}{8}[X + 2T + 3R] = \frac{3(10)}{8}[1 + 2(23) + 3(49)] = 727·5$ ■

Example 3 : Evaluate $\int_4^{5·2} \log x\, dx$ from the following data using Simpson's (3/8)th rule.

x :	4	4·2	4·4	4·6	4·8	5	5·2
$\log x$:	1·3863	1·4351	1·4816	1·5261	1·5686	1·6094	1·6487

Also find the error.

Sol. : $X = y_0 + y_6 = 1·3863 + 1·6487 = 3·0350$

$T = y_3 = 1·5261$

$R = y_1 + y_2 + y_4 + y_5 = 1·4351 + 1·4816 + 1·5686 + 1·6094$

$= 6·1147, \qquad h = 0·2$

$I = \frac{3h}{8}[X + 2T + 3R]$

$= \frac{3(0·2)}{8}[3·0350 + 2(1·5261) + 3(6·1147)]$

$= 1·8323475$

Now, $\int_4^{5·2} \log x\, dx = \left[\log x \cdot x - \int x \cdot \frac{dx}{x}\right]_4^{5·2}$

$= [x \log x - x]_4^{5·2} = [x(\log x - 1)]_4^{5·2}$

$= 3·3730248 - 1·5451771 = 1·8278474$

Error = 0·0045. ■

Example 4 : Evaluate $\int \frac{dx}{1 + x^2}$ between suitable intervals and hence, find an approximate value of π by Simpson's (3/8)th rule.

Sol. : Since for applying Simpson's $3/8^{th}$ rule the last ordinate should be a multiple of 3, we divide the interval into sub-intervals of length 1/6.

Sol. :

x	:	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
$y = \frac{1}{1+x^2}$	:	1	$\frac{36}{37}$	$\frac{9}{10}$	$\frac{4}{5}$	$\frac{9}{13}$	$\frac{36}{61}$	$\frac{1}{2}$

$$I = \frac{3h}{8}[X + 2T + 3R]$$

$$= \frac{3(1/6)}{8}\left[1.5 + 2\left(\frac{4}{5}\right) + 3\left(\frac{36}{37} + \frac{9}{10} + \frac{9}{13} + \frac{36}{61}\right)\right]$$

$$= \frac{1}{16}[1.5 + 1.6 + 9.4663] = 0.7854$$

Now, $I = \int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_0^1 = \frac{\pi}{4}$

$\therefore \pi = 4I = 4(0.7854) = 3.1416.$ ■

Example 5 : A body is in the form of a solid of revolution. The diameter d in cms. of its sections at various distances x cms. from one end are as given below.

x	:	0	2.5	5.0	7.5	10.0	12.5	15.0
d	:	5	5.5	6.0	6.75	6.25	5.5	4.0

Estimate the volume of the solid.

Sol. : Since $V = \int \frac{\pi}{4} d^2 \cdot dx$, we prepare the following table.

x	:	0.0	2.5	5.0	7.5	10.0	12.5	15.0
d^2	:	25.0	30.25	36.0	45.56	39.06	30.25	16

$$I = \int \frac{d^2}{4} dx$$

$X = y_0 + y_6 = 25 + 16 = 41$

$T = y_3 = 45.56$

$R = y_1 + y_2 + y_4 + y_5 = 30.25 + 36.0 + 39.06 + 30.25$

$= 135.56$

$$\therefore I = \frac{3h}{8}[X + 2T + 3R]$$

$$= \frac{3(2.5)}{8}[41 + 2(45.56) + 3(135.56)]$$

$= 505.12$

$V = \frac{\pi}{4} \cdot I = \frac{\pi}{4}(505.12) = 396.58$ cub. cms. ■

Example 6 : Using Simpson's (3/8)th rule, find $\int_0^6 f(x)\,dx$ where

x	:	0	1	2	3	4	5	6
$f(x)$	:	6.9897	7.4036	7.7815	8.1291	8.4510	8.7506	9.0309

Sol. : $I = \frac{3h}{8}[X + 2T + 3R]$

$X = 6.9897 + 9.0309 = 16.0206$

$T = 8.1291$

$R = 7.4036 + 7.7815 + 8.4510 + 8.7506 = 32.3867$

$$\therefore I = \frac{3(1)}{8}[16.0206 + 2(8.1291) + 3(32.3867)]$$

$= 48.5396$ ■

Example 7 : Use Simpson's (3/8)th rule to find $\int_{20}^{26} y\,dx$. Use the tabulated values of y given below.

x	:	20	21	22	23	24	25	26
y	:	95.20	96.85	97.77	98.68	96.56	100.41	101.24

Sol. : $I = \frac{3h}{8}[X + 2T + 3R]$

$X = 95.20 + 101.24 = 196.44$

$T = 98.68$

$R = 96.85 + 97.77 + 96.56 + 100.41 = 391.59$

$$\therefore I = \frac{3(1)}{8}[196.44 + 2(98.68) + 3(391.59)] = 588.2137$$ ■

Example 8 : Using Simpson's (3/8)th rule, find $\int_0^{0.3} \sqrt{1 - 8x^3} \cdot dx$.

Sol. :

x	:	0.0	0.1	0.2	0.3
$y = \sqrt{1-8x^3}$	:	1	0.9960	0.9675	0.8854

$$I = \frac{3h}{8}[X + 2T + 3R]$$

$$= \frac{3(0.1)}{8}[(1.8854) + 2(0) + 3(0.9960) + 0.9675]$$

$= 0.2916$ ■

Example 9 : Evaluate $\int_{-3}^{3} x^4 dx$ by Simpson's (3/8)th rule and compare it with the exact value.

Sol. :

x	:	−3	−2	−1	0	1	2	3
$y = x^4$	:	81	16	1	0	1	16	81

$$I = \frac{3h}{8}[X + 2T + 3R]$$

$$= \frac{3(1)}{8}[(81 + 81) + 2(0) + 3(34)]$$

$$= 99$$

$$\int_{-3}^{3} x^4\, dx = \left[\frac{x^5}{5}\right]_{-3}^{3} = \frac{2(3^5)}{5} = 97{\cdot}2.$$

■

Example 10 : Evaluate $\int_0^1 \frac{dx}{1+x}$ using Simpson's (3/8)th rule.

Sol. :

x	:	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
$y = \frac{1}{1+x}$	:	1	$\frac{6}{7}$	$\frac{6}{8}$	$\frac{6}{9}$	$\frac{6}{10}$	$\frac{6}{11}$	$\frac{6}{12}$

$$I = \frac{3h}{8}[X + 2T + 3R]$$

$$= \frac{3(1/6)}{8}[1{\cdot}5 + 2(0{\cdot}6667) + 3(2{\cdot}7526)]$$

$$= 0{\cdot}6932$$

■

Example 11 : Evaluate $\int_0^6 \frac{dx}{1+x}$ using Simpson's (3/8)th rule. Compare it with the exact value.

Sol. :

x	:	0	1	2	3	4	5	6
$y = \frac{1}{1+x}$	:	1	0·5	0·3333	0·25	0·20	0·1667	0·1429

$$I = \frac{3h}{8}[X + 2T + 3R]$$

$$= \frac{3(1)}{8}[1{\cdot}1429 + 2(0{\cdot}25) + 3(1{\cdot}2)]$$

$$= 1{\cdot}9661$$

■

$$\int_0^6 \frac{dx}{1+x} = [\log(1+x)]_0^6 = \log 7 = 1{\cdot}9459$$

***

Scanned by CamSca