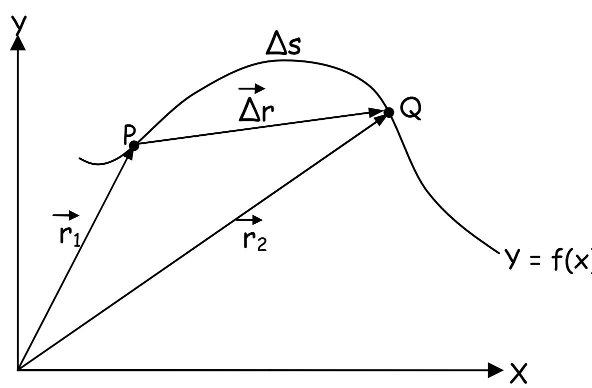
KINEMATICS OF PARTICLES

INTRODUCTION :

Kinematics is a branch of mechanics that describes the motion of particles or bodies without reference to the forces which actually cause the motion. Kinematics is often described as "geometry of motion". Some engineering applications of Kinematics are – cams, gears, linkages & other machine elements to control 'or' produce certain desired motion (as in case of robots) and calculation of flight trajectories of aircrafts, rockets & spacecrafts.

BASIC DEFINITIONS:

Consider a particle moving from position P to Q, along the path **y = f(x)**, in Δt seconds.



Position :

It is the location of the particle, with respect to a given reference frame, at any given instant of time and is represented as **r**. As time changes the position of the particle also changes. Hence,

$$\mathbf{r} = \mathbf{f(t)}$$

Displacement :

It is the change in position of the particle in an interval of time Δt seconds.

$$\Delta \mathbf{r} = \mathbf{r_2} - \mathbf{r_1}$$

Note : The displacement of the particle is independent of path covered by the particle and depends only on initial & final positions of the particle.

Distance :

It is the actual length of the path covered by the particle in an interval of time Δt seconds.

Note : When a particle moves along a curve, the magnitude of its displacement is always less than the distance covered by it.

Velocity :

Average Velocity $\mathbf{v_{avg}} = \dfrac{\text{Displacement}}{\text{time}} = \dfrac{\Delta r}{\Delta t}$

Instantaneous Velocity $\mathbf{v_{inst}} = \lim\limits_{\Delta t \to 0} \dfrac{\Delta r}{\Delta t} = \dfrac{dr}{dt}$

Speed :

Average speed $\mathbf{v_{avg}} = \dfrac{\text{Distance}}{\text{time}} = \dfrac{\Delta s}{\Delta t}$

Instantaneous speed $\mathbf{v_{inst}} = \lim\limits_{\Delta t \to 0} \dfrac{\Delta s}{\Delta t} = \dfrac{ds}{dt}$

Note:

- The average speed of a particle is always more than the magnitude of average velocity.
- The magnitude of instantaneous speed is always equal to the magnitude of instantaneous velocity.

Acceleration :

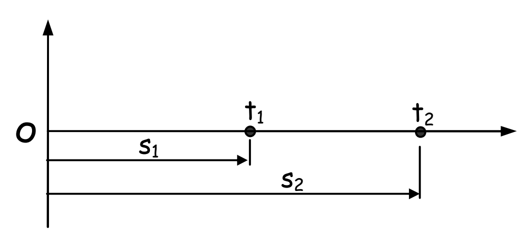
Average acceleration $\mathbf{a_{avg}} = \dfrac{\text{Change in velocity}}{\text{time}} = \dfrac{\Delta v}{\Delta t}$

Instantaneous acceleration $\mathbf{a_{inst}} = \lim\limits_{\Delta t \to 0} \dfrac{\Delta v}{\Delta t} = \dfrac{dv}{dt}$

VARIABLE ACCELERATION MOTION ALONG A STRAIGHT LINE:

Case (i) : Consider a particle moving along a straight line such that the distance / position of the particle from the origin is a function of time ie., **s = f(t).**

Then - velocity '**v**' $= \dfrac{ds}{dt}$; acceleration '**a**' $= \dfrac{dv}{dt}$



Case (ii) : Let acceleration $a = f(t)$.

Now $a = \frac{dv}{dt}$

Hence $\int dv = \int a.dt$

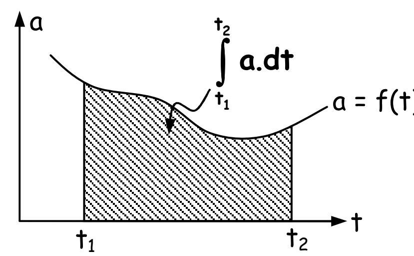
Therefore velocity **'v'** = $\int a.dt = g(t)$

But, $v = \frac{ds}{dt}$.

Then - $\int ds = \int v.dt$

Therefore, position / distance of the particle from 'O' - **'s'** = $\int v.dt = h(t)$

Case (iii) : Consider a particle whose acceleration is varying with respect to time as shown.

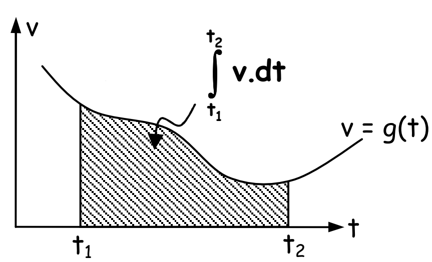


Now - $a = \frac{dv}{dt}$.

Hence - $\int_{v_1}^{v_2} dv = \int_{t_1}^{t_2} a.dt$

ie. **Change in velocity** 'Δv' = $\int_{t_1}^{t_2} a.dt$ = **Area under a -t plot**

Let the velocity of the particle varies with respect to time as shown.

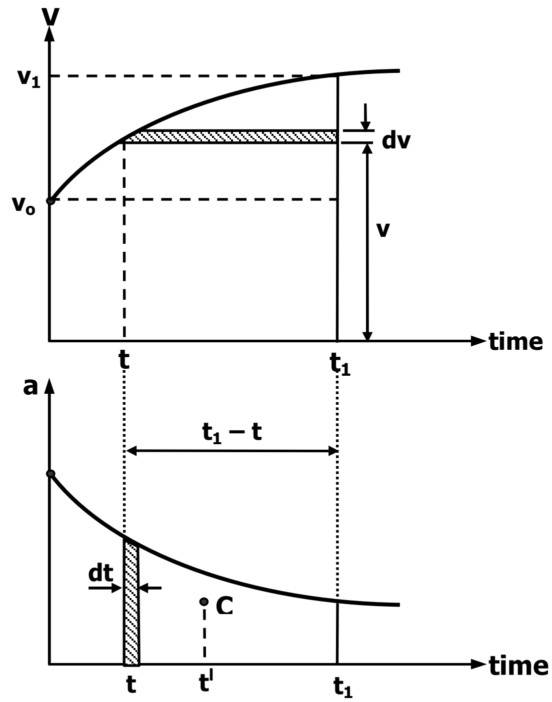


Then - $v = \frac{ds}{dt}$.

Hence $\int_{s_1}^{s_2} ds = \int_{t_1}^{t_2} v.dt$

ie. **Displacement** 'Δs' = $\int_{t_1}^{t_2} v.dt$ = **Area under v -t plot**

We know that area under Velocity – time plot between any two time limits gives the change in position or displacement of the particle.



$$x_1 - x_0 = v_0 t_1 + \int_{v_0}^{v_1} (t_1 - t)\, dv$$

But dv = a.dt

$$x_1 - x_0 = v_0 t_1 + \int_{0}^{t_1} (t_1 - t)\, a.dt$$

$$= v_0 t_1 + (t_1 - t) \int_{0}^{t_1} a.dt$$

Note that the integral component represents the moment of area under a-t plot, between t_0 to t_1, with respect to 't_1'. This method of solution, therefore, is called the "Moment – Area" method.

If the abscissa t^l of centroid of the area is known, then the position co-ordinate may be obtained by:

$$\boxed{x_1 = x_o + v_o.t + (\text{area under a-t plot})\,(t_1 - t^l)}$$

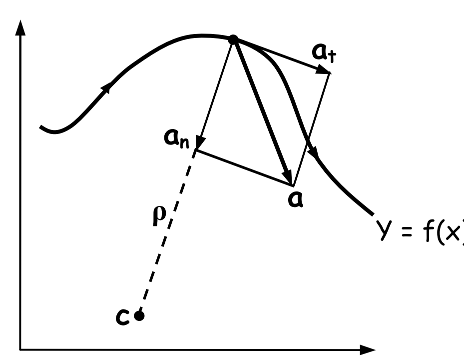
Case (iv) : Consider a particle whose acceleration is a function of position ie. **a = f(s).**

$$a = \frac{dv}{dt} = \frac{dv}{ds} \cdot \frac{ds}{dt} = v.\ \frac{dv}{ds}$$

Hence, $\int a.ds = \int v.dv$

PLANE CURVILINEAR MOTION :

Consider a particle moving along a path **y = f(x)** such that its distance covered along the path is a function of time, ie, **s = f(t)**.



The particle, at any instant, is subjected to two components of acceleration –

(i) a normal component 'a_n' directed towards the centre of rotation (also called centripetal acceleration) &

(ii) a tangential component 'a_t'.

Then the resultant linear acceleration of the particle is given by –

$$\mathbf{a = \sqrt{(a_n{}^2) + (a_t{}^2)}}$$

where - $\mathbf{a_n = \dfrac{v^2}{\rho}}$; $\mathbf{v = \dfrac{ds}{dt}}$ & $\rho =$ radius of curvature of the path.

& $\mathbf{a_t = \dfrac{dv}{dt}}$

Note: The radius of curvature, at any point P(x,y) on a plane curve, is given by –

$$\boxed{\frac{1}{\rho} = \frac{\dfrac{d^2y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}}$$

An alternate formula for the radius of curvature is derived as follows:

Now, $\dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{dt}{dx} = \dfrac{v_y}{v_x}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \frac{1}{v_x}\frac{d}{dt}\left(\frac{v_y}{v_x}\right) = \frac{1}{v_x}.\frac{v_x.a_y - v_y.a_x}{v_x{}^2}$$

On substituting these in the above expression for radius of curvature, we get -

$$\boxed{\frac{1}{\rho} = \frac{v_x a_y - v_y a_x}{\left[v_x{}^2 + v_y{}^2\right]^{3/2}}}$$

Note :

- **Normal acceleration is due to change in direction of velocity vector.**
- **Tangential acceleration is due to the change in magnitude of the velocity vector.**
- **If the speed of the particle is constant along the path, then the tangential acceleration of the particle is zero and the particle is subjected to only normal acceleration.**

PROJECTILE MOTION :

Consider a particle projected with an initial velocity 'u' at an angle 'α' with the horizontal. Let P(x,y) be the position of the particle after 't' seconds.
For small / short range motion the acceleration due to gravity 'g' acts vertically down and hence there is no acceleration component in x-direction. Therefore, the horizontal component of velocity remains constant.

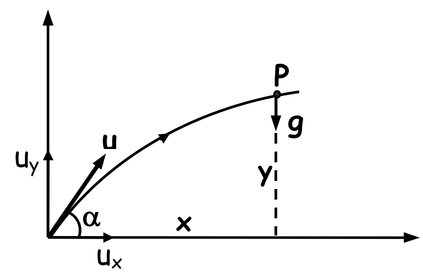
Hence - $x = u_x.t = u\cos\alpha.t \rightarrow$ (i)

$$y = u_y.t - \tfrac{1}{2}gt^2$$
$$= u\sin\alpha.t - \tfrac{1}{2}gt^2 \rightarrow \text{(ii)}$$

Replacing $t = \dfrac{x}{u\cos\alpha}$ in eq.(ii), we get –

$$y = u\sin\alpha.\frac{x}{u\cos\alpha} - \frac{g}{2}\left(\frac{x}{u\cos\alpha}\right)^2$$

Therefore - $\boxed{y = x.\tan\alpha - \dfrac{gx^2}{2u^2\cos^2\alpha}}$



Now, for a given angle & velocity of projection, the above equation can be written in the form - $\mathbf{y = ax - bx^2}$ which is parabolic.

Max. Height '**H**' = $\dfrac{u^2\sin^2\alpha}{2g}$

Range '**R**' = $\dfrac{u^2\sin 2\alpha}{g}$

For the range to be maximum →
$\alpha = 45^o$
and $R_{max} = \dfrac{u^2}{g}$