Probability & MC Simulation

End Sem

Unit - 4

Joint Probability Distribution Function →

JPDF for the joint event $\{X \le x, Y \le y\}$ is defined as

$$F_{X,Y}(x,y) = P\{X \le x, Y \le y\} = P(A \cap B)$$

JPDF is

$$F_{X,Y}(x,y) = \sum_{n=1}^{N}\sum_{m=1}^{M} P(x_n, y_m)\, u(x-x_n)\, u(y-y_m)$$

Q.

(X,Y)	$(0,0)$	$(1,2)$	$(2,3)$	$(3,2)$
$P(x,y)$	0.2	0.3	0.4	0.1

Find dist. function $F_{X,Y}(x,y)$ & Marginal density functions.

Solution →

$$F_{X,Y}(x,y) = \sum_{n=1}^{N}\sum_{m=1}^{M} P(x_n, y_m)\, u(x-x_n)\, u(y-y_m)$$

$$F_{X,Y}(x,y) = 0.2u(x)u(y) + 0.3u(x-1)u(y-2) + 0.4u(x-2)u(y-3) + 0.1u(x-3)u(y-2)$$

Marginal Density fn →

$$F_X(x) = F_{X,Y}(x, \infty)$$

$$F_X(x) = 0.2u(x) + 0.3u(x-1) + 0.4u(x-2) + 0.1u(x-3)$$

&

$$F_Y(y) = F_{X,Y}(\infty, y)$$

$$F_Y(y) = 0.2u(y) + 0.3u(y-2) + 0.4u(y-3) + 0.1u(y-2)$$

Q.

(X,Y)	-1	0	1
0	$\frac{3}{18}$	$\frac{2}{18}$	$\frac{3}{18}$
1	$\frac{1}{18}$	$\frac{K}{18}$	$\frac{1}{18}$
2	$\frac{2}{18}$	$\frac{1}{18}$	$\frac{2}{18}$

For k →

$$\frac{3}{18}+\frac{1}{18}+\frac{2}{18}+\frac{2}{18}+\frac{k}{18}+\frac{1}{18}+\frac{3}{18}+\frac{1}{18}+\frac{2}{18}=1$$

$$\frac{8}{18}+\frac{k}{18}+\frac{7}{18}=1 \Rightarrow \frac{15}{18}+\frac{k}{18}=1$$

$$\frac{k}{18}=1-\frac{15}{18}$$

$$\frac{k}{18}=\frac{3}{18}$$

$$\boxed{k=3}$$

JDPF →

$$F_{x,y}(x,y)=\sum_{n=1}^{N}\sum_{m=1}^{M} P(x_n,y_m)\,u(x-x_n)\,u(y-y_m)$$

$$F_{x,y}(x,y)=\frac{1}{18}\Big[3u(x)u(y+1)+2u(x)u(y)+3u(x)u(y-1)$$
$$+u(x-1)u(y+1)+3u(x-1)u(y)+u(x-1)u(y-1)$$
$$+2u(x-2)u(y+1)+u(x-2)u(y)+2u(x-2)u(y-1)\Big]$$

Marginal →

$$F_X(x) = F_{X,Y}(x, \infty)$$

$$F_X(x) = \frac{1}{18}\Big[3u(x) + 2u(x) + 3u(x-1) + u(x-1) + 3u(x-1) + u(x-1) + 2u(x-2) + u(x-2) + 2u(x-2)\Big]$$

Same for $F_Y(y)$

Joint Probability Density Function →

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x\, \partial y}$$

Joint Probability Density function is →

$$f_{X,Y}(x,y) = \sum_{n=1}^{N} \sum_{m=1}^{M} P(x_n, y_m)\, \delta(x - x_n)\, \delta(y - y_m)$$

$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(x,y)\, dx\, dy$$

Q:

$$F_{X,Y}(x,y) = Ae^{-(2x+y)}, \quad x \geq 0, y \geq 0$$

Find value of A

Marginal density functions

$$\int_{x\geq 0}\int_{y \geq 0} f_{X,Y}(x,y)\,dx\,dy = 1$$

$$\int_0^{\infty}\int_0^{\infty} Ae^{-(2x+y)}\,dx\,dy = 1$$

$$A\left(\int_0^{\infty} e^{-2x}\,dx\right)\left(\int_0^{\infty} e^{-y}\,dy\right) = 1$$

$$\int_0^{\infty} e^{-2x}\,dx = \frac{1}{2}, \quad \int_0^{\infty} e^{-y}\,dy = 1$$

$$A \times \frac{1}{2} \times 1 = 2, \quad \boxed{A = 2}$$

JPDF = $f_{X,Y}(x,y) = 2e^{-(2x+y)}, \quad x \geq 0, y \geq 0$

Marginal Density Fn →

$$F_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$$

$$F_X(x) = \int_{0}^{\infty} 2e^{-(2x+y)}\, dy$$

$$f_X(x) = 2e^{-2x} \int_{0}^{\infty} e^{-y}\, dy$$

$$= 2e^{-2x} \left[-e^{-y}\right]_0^{\infty}$$

$$= 2e^{-2x}\left[1-0\right] = 2e^{-2x}$$

$$\boxed{f_x(x) = 2e^{-2x}} \quad , \quad x \geq 0$$

Same for $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$

$$f_Y(y) = \int_{0}^{\infty} 2e^{-(2x+y)}\, dx$$

$$f_Y(y) = 2e^{-y} \int_{0}^{\infty} e^{-2x}\, dx$$

$$= 2e^{-y}\left[-\frac{1}{2}e^{-2x}\right]_0^{\infty}$$

$$= 2e^{-y}\left(\frac{1}{2}\right) = e^{-y}$$

$$\boxed{f_Y(y) = e^{-y}} \quad y \geq 0$$

Marginal Density fns →

$$\boxed{f_X(x) = 2e^{-2x}, \quad x \geq 0 \quad \& \quad f_Y(y) = e^{-y}, \; y \geq 0}$$

Q. $f_{X,Y}(x,y) = \begin{cases} be^{-(x+y)}, & 0<x<a,\ 0<y<\infty, \\ 0, & \text{elsewhere} \end{cases}$

find constant b (in terms of a) given it's a valid JDF.

Soln →

$$\iint f_{X,Y}(x,y)\,dx\,dy = 1$$

$$\int_0^\infty \int_0^a be^{-(x+y)}\,dx\,dy = 1$$

$$b\left(\int_0^a e^{-x}\,dx\right)\left(\int_0^\infty e^{-y}\,dy\right) = 1$$

$$b\left(\left[-e^{-x}\right]_0^a\right)\left(\left[-e^{-y}\right]_0^\infty\right) = 1$$

$$b\left(1-e^{-a}\right)(1) = 1$$

$$\boxed{b = \frac{1}{1-e^{-a}}}$$

Valid joint pdf $\rightarrow$

$$\boxed{f_{X,Y}(x,y) = \begin{cases} \dfrac{e^{-(x+y)}}{1-e^{-a}}, & 0<x<a,\ 0<y<\infty, \\ 0 & \text{elsewhere} \end{cases}}$$

Q: Roll a pair of unbiased dice. If X denotes the smaller outcome & Y denotes the larger outcome. What is the JPDF of X & Y?

soln →

outcomes = 36

$X = 2 \; , \; Y = 3$

$$f(2,3) = P(X=2, Y=3) = \frac{2}{36} = \frac{1}{18}$$

6	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$
5	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	0
4	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	0	0
3	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	0	0	0
2	$\frac{2}{36}$	$\frac{2}{36}$	0	0	0	0
1	$\frac{2}{36}$	0	0	0	0	0
	1	2	3	4	5	6

$$f(x,y) = \begin{cases} 1/36 & \text{if } 1 \le x = y \le 6, \\ 2/36, & \text{if } 1 \le x < y \le 6, \\ 0, & \text{otherwise.} \end{cases}$$

Q. $$f(x,y) = \begin{cases} kxy^2, & \text{if } 0 < x < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

value of constant k?

soln →

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx\, dy$$

$$1 = \int_0^1 \int_0^y kxy^2\, dx\, dy$$

$$1 = \int_0^1 ky^2 \left(\int_0^y x \, dx \right) dy$$

$$1 = \int_0^1 ky^2 \left(\left[x^2 \right]_0^y \right) dy$$

$$1 = \int_0^1 ky^2 \times y^2 \, dy$$

$$1 = \frac{k}{2} \int_0^1 y^4 \, dy$$

$$1 = \frac{k}{10} \left[y^5 \right]_0^1$$

$$1 = \frac{k}{10} \Rightarrow \boxed{k = 10}$$

→ Conditional Distribution & Density fns

The distribution of random variable X when the distribution function of a random variable Y is known at some value of y is defined as the Conditional Distribution function of X.

$$F_X(x \mid Y = y) = \frac{\int_{-\infty}^{x} f_{X,Y}(x,y)\,dx}{f_Y(y)}$$

Conditional Density fn of X →

$$= \frac{\int_{-\infty}^{x} \frac{d}{dx} f_{X,Y}(x,y)\,dx}{f_Y(y)}$$

Q. Find the conditional density functions for the joint density function.

$$f_{X,Y}(x,y) = 4xy e^{-(x^2+y^2)} u(x) u(y)$$

Soln →

Conditional density functions →

$$f_X(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

and

$$f_Y(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$f_X(x) = \int_0^{\infty} f_{X,Y}(x,y)\, dy$$

$$= \int_0^\infty 4xy e^{-x^2} e^{-y^2} u(x)\, dy$$

$$= 4xe^{-x^2} u(x) \int_0^\infty y e^{-y^2}\, dy$$

$$= 4xe^{-x^2} u(x) \left[\frac{1}{2} e^{-y^2} \right]_0^\infty$$

$$= 4xe^{-x^2} u(x) \left[\frac{1}{2} \right]$$

$$\boxed{\begin{aligned} f_X(x) &= 2xe^{-x^2} u(x) \\ f_Y(y) &= 2ye^{-y^2} u(y) \end{aligned}}$$

→ Conditional density functions.

→ Statistical Independence of random variables

The two random variables are said to be statistically independent if and only if the joint probability is equal to the product of individual probab.

→ Independence from joind PDF →

$$f_{X,Y}(x,y) = \frac{1}{18} e^{(-x/6 - y/3)}(u(x)\,u(y))$$

Soln ⇒

Computing the marginal of X.

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

$$= \int_0^{\infty} \frac{1}{18} e^{-x/6} e^{-y/3}\,dx$$

$$= \frac{1}{18} e^{-y/3} \int_0^\infty e^{-x/6} \, dx$$

$$= \frac{1}{18} e^{-y/3} \left[-6 e^{-x/6} \right]_0^\infty$$

$$= \frac{1}{18}^{-y/3} (-6)$$

$$= + \frac{6}{18}^{-y/3} = \boxed{+ \frac{1}{3}^{-y/3}} \quad y \geq 0$$

<u>Marginal of X</u> $\rightarrow$

$$f_X(x) = \int_0^\infty \frac{1}{18} e^{-x/6} e^{-y/3}$$

$$= \frac{1}{18} e^{-x/6} \int_0^\infty e^{-y/3} \, dy$$

$$= \frac{1}{18} e^{-x/6} \left[+3 e^{-y/3} \right]_0^\infty$$

$$= \frac{1}{18} e^{-x/6} \left(\cancel{+}3 \right)$$

$$\boxed{= + \frac{1}{6} e^{-x/6}} \quad x \geq 0.$$

<u>Multiplying the marginals</u> →

$$f_X(x)\, f_Y(y) = \left(\frac{1}{6} e^{-x/6} \right) \left(\frac{1}{3} e^{-y/3} \right)$$

$$= \boxed{\frac{1}{18} e^{-x/6 - y/3} = f_{X,Y}(x, y)}$$

∴ X & Y are independent.

→ Covariance

Let X and Y be any 2 random variables, Then.

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

Proof →

$$Cov(X,Y) = E((X-\mu_x)(Y-\mu_y))$$

$$= E(XY - \mu_y X - \mu_x Y + \mu_x\mu_y)$$

$$= E(XY) - \mu_x E(Y) - \mu_y E(X) + \mu_x\mu_y$$

$$= E(XY) - \mu_x\mu_y - \mu_y\mu_x + \mu_x\mu_y$$

$$= E(XY) - \mu_x\mu_y$$

$$= \boxed{E(XY) - E(X)E(Y)}$$

Corollary →

$$Cov(X, X) = \sigma_X^2$$

$$Cov(X, X) = E(XX) - E(X)E(X)$$

$$= E(X^2) - \mu_x^2$$

$$= var(X)$$

$$= \sigma^2_X$$

Q:

$$f(x, y) = \begin{cases} \dfrac{x+2y}{18}, & \text{for } x = 1, 2;\ y = 1, 2. \\ 0, & \text{elsewhere} \end{cases}$$

Covariance σ_{XY} between X & Y?

soln →

Marginal of X →

$$f_1(x) = \sum_{y=1}^{2} \frac{x+2y}{18}$$

$$= \frac{x+2}{18} + \frac{x+2(2)}{18}$$

$$= \frac{2x+6}{18}$$

Expected value of x →

$$E(X) = \sum_{x=1}^{2} x f_1(x) = 1 f_1(1) + 2 f_1(2)$$

$$= \frac{2(1)+6}{18} + \frac{2(2)+6}{18}$$

$$= \frac{8}{18} + 2\left(\frac{10}{18}\right) = \frac{28}{18}$$

Marginal of $Y \rightarrow$

$$f_2(y) = \sum_{1}^{2} \frac{x+2y}{18} = \frac{1}{18}(3+4y)$$

Expected value of $y \rightarrow$

$$E(y) = \sum_{y=1}^{2} x f_1(y) = 1f_1(1) + 2f_1(2)$$

$$= \frac{3+4}{18} + 2\frac{(3+8)}{18}$$

$$= \frac{7}{18} + \frac{22}{18} = \frac{29}{18}$$

Product moment of X & Y is

$$E(XY) = \sum_{x=1}^{2} \sum_{y=1}^{2} xy f(x,y)$$

$$= f(1,1) + 2f(1,2) + 2f(2,1) + 4f(2,2)$$

$$= \frac{3}{18} + 2 \times \frac{5}{18} + 2\frac{4}{18} + 4\frac{6}{18}$$

$$= \frac{45}{18}$$

Covariance b/w X & Y →

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

$$= \frac{45}{18} - \left(\frac{28}{18}\right)\left(\frac{29}{18}\right)$$

$$= \frac{45}{18} - \frac{28 \times 29}{18 \times 18}$$

$$= \frac{45}{18} - \frac{812}{324}$$

$$= \frac{45(18) - 812}{324}$$

$$= \frac{810 - 812}{324} = \frac{-2}{324} = \frac{-1}{162}$$

$$= \boxed{-0.00617}$$

Q. $f(x,y) = \begin{cases} x+y, & 0 < x, y < 1, \\ 0, & \text{elsewhere} \end{cases}$

covariance b/w X & Y?

Soln →

Marginal function of X →

$$f_1(x) = \int_0^1 (x+y)\, dy$$

$$= \left[xy + \frac{y^2}{2} \right]_{y=0}^{1}$$

$$= x + \frac{1}{2}$$

$$E(X) = \int_0^1 x f_1(x)\, dx$$

$$= \int_0^1 x \left(x + \frac{1}{2} \right) dx$$

$$= \int_0^1 \left(x^2 + \frac{x}{2} \right) dx$$

$$= \left[\frac{x^3}{3} + \frac{x^2}{4} \right]_0^1$$

$$= \frac{1}{3} + \frac{1}{4}$$

$$= \frac{4+3}{12} = \boxed{\frac{7}{12}}$$

<u>Marginal fn of y</u> →

$$f_2(y) = \int_0^1 (x+y)\, dx$$

$$= \int_0^1 \frac{x^2}{2} + xy$$

$$= \left[\frac{x^2}{2} + xy \right]_0^1 = \left[\frac{1}{2} + y \right]$$

$$E(y) = \int_0^1 y f_1(y)\, dx$$

$$= \int_0^1 y\left(\frac{1}{2}+y\right) dx$$

$$= \int_0^1 \left(y/2 + y^2\right) dx$$

$$= \left[\frac{y^2}{4} + \frac{y^3}{3}\right]_0^1$$

$$= \left[\frac{1}{4} + \frac{1}{3}\right]$$

$$= \boxed{\frac{7}{12}}$$

$$\boxed{E(x) = E(y)}$$

Product moment of X & Y →

$$E(XY) = \int_0^1 \int_0^1 xy(x+y)\,dx\,dy$$

$$= \int_0^1 \int_0^1 (x^2 y + xy^2)\,dx\,dy$$

$$= \int_0^1 \left[\frac{x^3 y}{3} + \frac{x^2 y^2}{2} \right]_0^1 dy$$

$$= \int_0^1 \left[\frac{y}{3} + \frac{y^2}{3} \right] dy$$

$$= \left[\frac{y^2}{6} + \frac{y^3}{6} \right]_0^1$$

$$= \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$$

Covariance b/w X & Y →

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

$$= \frac{1}{3} - \frac{7}{12} \times \frac{7}{12}$$

$$= \frac{1}{3} - \frac{49}{144}$$

$$= \frac{48 - 49}{144}$$

$$= \boxed{\frac{-1}{144}}$$ Ans.

→ Central Limit Theorem

It states that probability density function of a sum of N independent random variables approaches a Gaussian distribution as $N \to \infty$

Q. $f(x,y) = \begin{cases} k(x+y), & 0 \le x \le 2,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$

$$\int_0^1 \int_0^2 k(x+y)\, dx\, dy = 1$$

$$k \int_0^1 \int_0^2 \frac{x^2}{2} + xy = 1$$

$$k \int_0^1 \left[2 + 2y \right] = 1$$

$$k \int_0^1 2y + \frac{2y^2}{2} = 1$$

$$k \left[2y + y^2 \right]_0^1 = 1$$

$$k [3] = 1$$

$$\boxed{k = \frac{1}{3}}$$

$$f_x(x) = \int_0^1 \frac{1}{3}(x+y)\,dy$$

$$= \frac{1}{3}\int_0^1 xy + \frac{y^2}{2}$$

$$= \frac{1}{3}\left[x + \frac{1}{2} \right]$$

$$= \boxed{\frac{1}{3}x + \frac{1}{6}}$$

Marginal of y

$$f_Y(y) = \int_0^2 \frac{1}{3}(x+y)\,dx$$

$$= \frac{1}{3}\int_0^2 \frac{x^2}{2} + xy$$

$$= \frac{1}{3}\left[\frac{x^2}{2} + xy\right]_0^2$$

$$= \frac{1}{3}\left[\frac{4}{2} + 2y\right]$$

$$= \boxed{\frac{2}{3} + \frac{2y}{3}}$$

$S = (1,2), (1,3), (1,4), (1,5), (1,6)$
$(2,3), (2,4), (2,5), 2,6)$
$(3,4), (3,5), (3,6)$
$(4,5), (4,6)$
$(5,6)$

Marginal of x

$\frac{1}{36}($

kxy, $x = 1, 2, 3$

$y = 1, 2, 3$

$$\sum \sum p(x, y)$$

$$k \left[\sum_{x=1}^{3} (x + 2x + 3x) \right] = 1$$

$$k \left[6 \sum_{x=1}^{3} x \right] = 1$$

$$6k (1 + 2 + 3) = 1$$

$$\boxed{k = \frac{1}{36}}$$

$$f(x,y) = \begin{cases} kxy^2 , & 0 < x < y < 1 \\ 0 , & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, dx\, dy$$

$$\int_{y=0}^{y=1} \int_{x=0}^{x=y} kxy^2 \, dx\, dy = 1$$

$$k \int_{y=0}^{y=1} \left(\frac{x^2}{2} \right)_{x=0}^{x=y} y^2 \, dy = 1$$

$$\frac{k}{2} \int_{y=0}^{y=1} y^4 \, dy = 1$$

$$\frac{k}{2} \left(\frac{y^5}{5} \right)_{y=0}^{y=1} = 1$$

$$\frac{k}{2} \left(\frac{1}{5} \right) = 1 \Rightarrow \boxed{k = 10}$$

$$f(x,y) = \begin{cases} 6/5\,(x^2 + 2xy) & 0 \le x \le 1, 0 \le y \le 1 \\ 0 \; ; & \text{otherwise} \end{cases}$$

$$P(A) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy$$

$$= \int_{0}^{y=1}\int_{0}^{x=y} \frac{6}{5}(x^2 + 2xy)\,dx\,dy$$

$$= \frac{6}{5}\int_{0}^{1} \left(\frac{x^3}{3} + 2 \times \frac{x^2}{2}y\right)_{0}^{x=y} dy$$

$$= \frac{6}{5}\int_{0}^{y=1} \left(\frac{x^3 + 3x^2 y}{3}\right)_{0}^{x=y} dy$$

$$= \frac{6}{5}\int_{0}^{y=1} \frac{4y^3}{3}\,dy$$

$$= \frac{6}{5}\left[\frac{4y^4}{12}\right]_0^{y=1}$$

$$= \frac{6}{5}\left[\frac{4}{12}\right]$$

$$= \frac{\cancel{6}}{5} \times \frac{4}{\cancel{12}^{2}} = \frac{4}{10} = \boxed{\frac{2}{5}}$$ Ans

Marginal

$$f_x(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$f_y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$f_x(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_{-\sqrt{x}}^{\sqrt{x}} \frac{3}{4}\,dy$$

$$= \frac{3}{4}\left[\sqrt{x} - (-\sqrt{x})\right]$$

$$= \frac{3}{\cancel{4}}\,\cancel{2}\sqrt{x}$$

$$= \frac{3\sqrt{x}}{2}$$

$$f_X(x) = \int_{y=x}^{y=\infty} 2e^{-x-y}\, dy$$

$$= 2\left[e^{-x-y} \right]_{y=x}^{y=\infty}$$

$$= 2e^{-2x}$$

$$f_Y(y) = \int_{x=0}^{x=y} 2e^{-x-y}\, dx$$

$$= 2\left[e^{-y-y} - e^{0-y} \right]$$

$$= 2\left[e^{-2y} - e^{-y} \right]$$

$$= 2e^{-2y} - 2e^{-y}$$

If CDF is given then,
PDF is,

$$\boxed{f(x,y) = \frac{\partial^2 F}{\partial x \partial y}}$$

$$f(x,y) = \frac{\partial^2 F}{\partial x \partial y}$$

$$= \frac{1}{5} \frac{\partial}{\partial x} \frac{\partial}{\partial y} (2x^3 y + 3x^2 y^2)$$

$$= \frac{1}{5} \frac{\partial}{\partial x} (2x^3 + 6x^2 y)$$

$$= \frac{1}{5} (6x^2 + 12xy)$$

$$= \frac{6}{5} (x^2 + 2xy)$$

$$f_X(x) = \int_0^\infty 6 e^{-2x} e^{-3y} \, dy$$

$$= 6 \int_0^\infty e^{-2x} e^{-3y} \, dy$$

$$= 6 e^{2x} \left(\frac{e^{-3y}}{-3} \right)_0^\infty$$

$$= 6 e^{2x} \times \left(\frac{e^{-\infty} - e^{0}}{-3} \right)$$

$$= \frac{\cancel{6}^{2} e^{2x}}{-\cancel{3}_{1}} = -2e^{2x}$$

$$f_Y(y) = 6 \int_0^\infty e^{-2x} e^{-3y} \, dx$$

$$= 6 e^{-3y} \left[\frac{e^{-2x}}{-2} \right]_0^\infty =$$

$$= -3 e^{-3y} \left[e^{2\infty} - e^{0} \right]$$

$$= -3 e^{-3y}$$

JDPF $f(x,y) = kxy \quad 0<x<1$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ " y "

mf, cdf, dep. ind. value of k

$$F_X(x) = \int_0^1 kxy \, dy$$

$$= k\int_0^1 xy \, dy$$

$$= kx\left[\frac{y^2}{2}\right]_0^1$$

$$= kx \times \frac{1}{2} = \frac{kx}{2}$$

$$F_y(y) = \int_0^1 kxy \, dx$$

$$= ky\left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{k}{2}y$$

$$cdf = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x) \, dx\,dy = 1$$

$$= \int_0^1\int_0^{x=y} kxy \, dx\,dy$$

$$= k\int_0^1 \left[\frac{x^2}{2}\right]_0^{x=1} y = k\int_0^1 \frac{1}{2}y$$

$$= \frac{k}{2}\left[\frac{y^2}{2}\right] = \frac{k}{2} \times \frac{1}{2} = k = 4$$

Cond. Dist. Density

$$F_X(x \mid y = y) = \frac{\int_{-\infty}^{x} f_{x,y}(x, y)\, dx}{f_Y(y)}$$

Density →

$$= \frac{\int_{-\infty}^{x} \frac{d}{dx} f_{x,y}(x, y)\, dx}{f_Y(y)}$$

$$f_{x,y}(x,y) = 4xy e^{-(x^2+y^2)} u(x)\, u(y)$$

$$f_X(x|y) = \frac{f_{x,y}(x,y)}{f_Y(y)}$$

$$f_Y(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$f_X(x) = \int_0^\infty 4xy e^{-x^2} e^{-y^2} u(x)\, dy$$

$$= 4x e^{-x^2} u(x) \int_0^\infty y e^{-y^2}\, dy$$

$$= \text{〃} \left[\frac{e^{-y^2}}{2} \right]_0^\infty$$

$$= 4x e^{-x^2} u(x) \times \frac{1}{2}$$

$$f_Y(y) = 2ye^{-y^2} u(y)$$

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y)$$

Proof->

$$\text{Cov}(X, Y) = E((X - \mu_x)(Y - \mu_y))$$

$$= E(XY - \mu_y X - \mu_x Y + \mu_x \mu_y)$$

$$= E(XY) - \mu_x \mu_y - \mu_y \mu_x + \mu_x \mu_y$$

$$= E(XY) - \mu_x \mu_y$$

$$= \boxed{E(XY) - E(X)E(Y)}$$