Handwritten Notes

On

Application of Derivatives

Application of Derivatives.

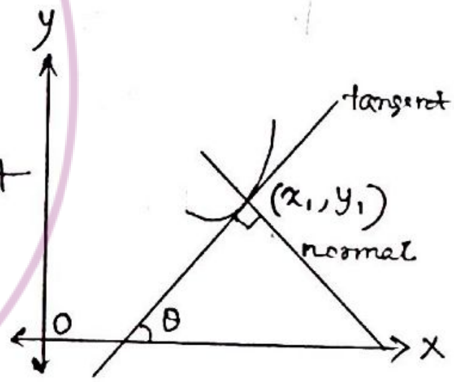
* **Derivative as the Rate of Change:** The rate of change of any variable with respect to some other variable is the derivative of first variable with respect to the other variable.

$$\text{Rate of change} = \frac{d}{dx} f(x).$$

* **Approximation & Differentials:** When Δy & Δx are sufficiently small quantities, then $\frac{\Delta y}{\Delta x} \cong \frac{dy}{dx} = f'(x)$.

i.e. $\Delta y = f'(x) \cdot \Delta x$.

* **Slopes of Tangent & Normal:**



- Tangent: $y = f(x)$ be a continuous curve. Slope of tangent at (x_1, y_1) is $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \tan\theta$.

- Normal: $y = f(x)$ be a continuous curve. Slope of normal at (x_1, y_1) is $\left(-\frac{dx}{dy}\right)_{(x_1, y_1)}$.

* **Equations of Tangent & Normal:**

- Equ^n of tangent at x_1, y_1 : $y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)$

- Equ^n of tangent at (x_1, y_1) that is parallel to x-axis : $y - y_1 = 0$.

• Equⁿ of Normal at (x_1, y_1) :

$$y - y_1 = \left(-\frac{dx}{dy}\right)_{(x_1, y_1)} (x - x_1).$$

• Direct method to find equⁿ of Tangent :

In the standard equⁿ of curve, we may replace x^2 to xx_1, y^2 to yy_1, $2x$ to $x + x_1$, $2y$ to $y + y_1$, xy to $\frac{xy_1 + yx_1}{2}$. [* This method is applied only for any conics of 2nd degree.]

• If a curve passes through the origin, then the equation of the tangent at the origin can be directly written by equating the lowest degree terms of the curve to zero. eg. Equⁿ of tangent of $x^2 + y^2 + 2gx + 2fy = 0$ is $gx + fy = 0$.

• Folium of descartes : In the curve - $x^3 + y^3 - 3xy = 0$ same line is tangent and normal at a given point. The line pair $xy = 0$ is both the tangent as well as normal at $x = 0$.

• Parametric coordinates :

1. $x^{2/3} + y^{2/3} = a^{2/3}$: $x = a\cos^3\theta$, $y = a\sin^3\theta$.
2. $\sqrt{x} + \sqrt{y} = \sqrt{a}$: $x = a\cos^4\theta$, $y = a\sin^4\theta$.
3. $\frac{x^n}{a^n} + \frac{y^n}{b^n} = 1$: $x = a(\sin\theta)^{2/n}$, $y = b(\sin\theta)^{2/n}$
4. $c^2(x^2 + y^2) = x^2y^2$: $x = c\sec\theta$, $y = c\operatorname{cosec}\theta$
5. $y^2 = x^3$: $x = t^2$, $y = t^3$.

Application of Derivatives.

* **Angle of Intersection of two Curves:** The angle is defined as the angle between the tangents to the two curves at their point of intersection.

Let C_1 & C_2 be two curves.

$$m_1 = \tan\theta_1 = \left(\frac{dy}{dx}\right)_{c_1}$$

$$m_2 = \tan\theta_2 = \left(\frac{dy}{dx}\right)_{c_2}$$

Angle of intersection, $\theta = \tan^{-1}\left|\dfrac{\left(\frac{dy}{dx}\right)_{c_1} - \left(\frac{dy}{dx}\right)_{c_2}}{1+\left(\frac{dy}{dx}\right)_{c_1}\left(\frac{dy}{dx}\right)_{c_2}}\right|$

- **Orthogonal Curves:** If the angle of intersection of two curves is a right angle, the two curves are said to be orthogonal.

If the curves are orthogonal,

$$\left(\frac{dy}{dx}\right)_{c_1}\left(\frac{dy}{dx}\right)_{c_2} = -1.$$

* Length of Tangent $= \left| y_1\sqrt{1+\left(\frac{dx}{dy}\right)^2_{x_1,y_1}}\right|$

Length of Normal $= \left| y_1\sqrt{1+\left(\frac{dy}{dx}\right)^2_{x_1,y_1}}\right|$

Length of Subtangent $= \left| y_1\left(\frac{dx}{dy}\right)_{x_1,y_1}\right|$
(Projection of tangent)

Length of Subnormal $= \left| y_1\left(\frac{dy}{dx}\right)_{x_1,y_1}\right|$
(Projection of normal)

* There are two types of monotonic function:
1) Increasing function 2) Decreasing function.

* **Increasing function:**

1. Strictly increasing function: $f(x)$ is known as strictly increasing function in its domain, if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

For strictly increasing function, $f'(x) > 0$.

- Strictly increasing functions can be classified as,

i) Concave up when $f'(x) > 0$ & $f''(x) > 0$, $\forall x \in$ domain

ii) When $f'(x) > 0$ & $f''(x) = 0$ $\forall x \in$ domain

iii) Concave down when $f'(x) > 0$ & $f''(x) < 0$, $\forall x \in$ domain

2. Only increasing or Non-decreasing Function:

$f(x)$ is non-decreasing in its domain, if

$x_1 < x_1 \Rightarrow f(x_1) \leq f(x_2)$.

For non-decreasing function, $f'(x) \geq 0$.

* **Decreasing function:**

1. Strictly decreasing function: $f(x)$ is known as strictly decreasing in its domain if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

For strictly decreasing function, $f'(x) < 0$.

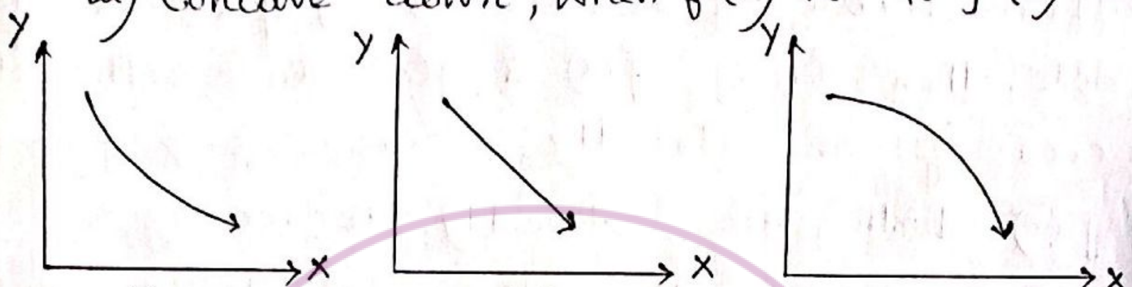
Application of Derivatives.

- Strictly decreasing functions can be classified as,

i) Concave up, when $f'(x) < 0$ & $f''(x) > 0 \ \forall x \in$ dom.

ii) When $f'(x) < 0$ & $f''(x) = 0 \ \forall x \in$ domain

iii) Concave down, when $f'(x) < 0$ & $f''(x) < 0 \ \forall x \in$ dom.



(i) (ii) (iii)

2. Only decreasing or Non-increasing Function:

$f(x)$ is said to be non-increasing, if for,

$x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$

For non-increasing function, $f'(x) \leq 0$.

Non-decreasing Function　　　Non-increasing Function.

* Problem Solving – Leibnitz-rule:

$$\frac{d}{dx}\left[\int_{\phi(x)}^{\psi(x)} f(t)\,dt\right] = f(\psi(x))\left\{\frac{d}{dx}\psi(x)\right\} - f(\phi(x))\left\{\frac{d}{dx}\phi(x)\right\}.$$

* Properties of Monotonic Functions:

1. If $f(x)$ is strictly increasing function on $[a,b]$

$\Rightarrow \begin{cases} f^{-1}(x) \text{ exists.} \\ f^{-1}(x) \text{ is also strictly increasing on } [a,b]. \end{cases}$

*2. If $f(x)$ & $g(x)$ are two continuous & differentiable functions & $fog(x)$ & $gof(x)$ exists, then, (i) if $f(x)$ & $g(x)$ are both strictly increasing or strictly decreasing $\Rightarrow fog(x)$ & $gof(x)$ both are strictly increasing.

(ii) If amongst the two functions one is strictly increasing & other is strictly decreasing $\Rightarrow fog(x)$ & $gof(x)$ both are strictly decreasing.

+ for increasing functions
- for decreasing functions

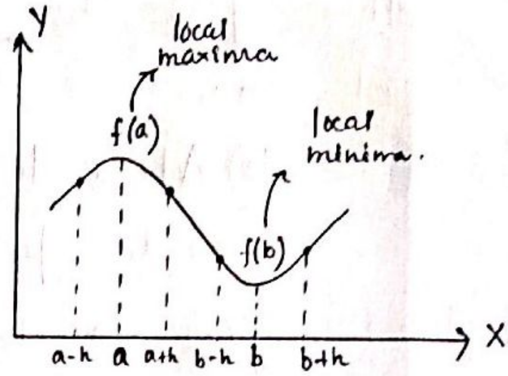
$f'(x)$	$g'(x)$	$(fog)'(x)$ or $(gof)'(x)$
+	+	+
+	−	−
−	+	−
−	−	+

*** Critical Points:** Collection of points for which, i) $f(x)$ does not exist, ii) $f'(x)$ does not exist, iii) $f'(x) = 0$.

*** Comparison of functions:** If we want to compare $f(x)$ & $g(x)$ consider a function $\phi(x) = f(x) - g(x)$ or $\psi(x) = g(x) - f(x)$ & check whether $\phi(x) / \psi(x)$ is increasing or decreasing in given domain of $f(x)$ & $g(x)$.

Application of Derivatives.

* Local Maxima & Minima:



- $f(x)$ is said to have a local maximum at $x=a$, if $f(a)$ is greatest of all values in the suitably small neighbourhood of a, where $x=a$ is an interior point in the domain of $f(x)$.
Analytically, this means $f(a) \geqslant f(a+h)$ & $f(a) \geqslant f(a-h)$, where $h \geqslant 0$.
- $f(x)$ is said to have a local minimum at $x=b$, if $f(b)$ is smallest of all values in the suitably small neighbourhood of b, where $x=b$ is an interior point in the domain of $f(x)$.
Analytically, $f(b) \leq f(b+h)$ & $f(b) \leq f(b-h)$ where $h > 0$ (very small quantity).

* Method of finding Extrema of Continuous Functions:

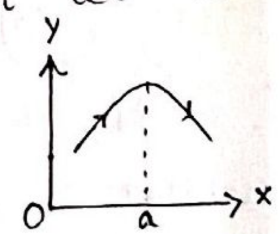
1. First Derivative Test: Applies to continuous fun^n.

a) At a critical point, $x = x_0$

(i) When $f(x)$ attains maximum at $x=a$.

if $f'(x) > 0$ for $x < a$

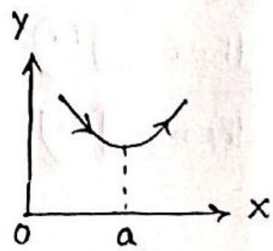
$f'(x) < 0$ for $x > a$



(ii) When $f(x)$ attains minimum at $x=a$.

if $f'(x) < 0$, $x < a$

$f'(x) > 0$, $x > a$.



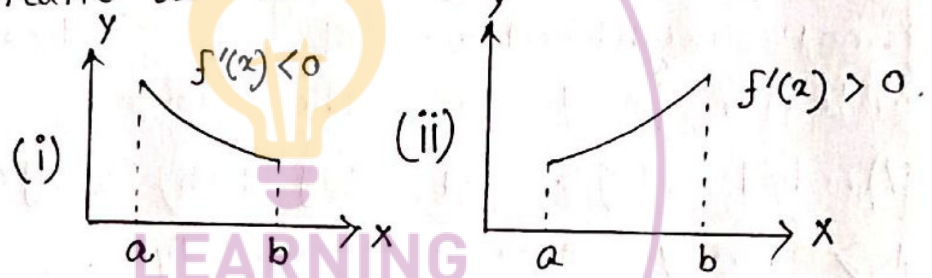
- If sign of $f'(x)$ does not change at x_0, then $f(x)$ has neither a maximum nor a minimum at x_0.

b) <u>At a left end point a & right end point b in [a, b]</u>.

$f(x) \rightarrow$ defined on $[a, b]$.

(i) If $f'(x) < 0$ for $x > a$, then $f(x)$ has local maximum at $x = a$ & local minimum at $x = b$.

(ii) If $f'(x) > 0$ for $x > a$, then $f(x)$ has local minimum at $x = a$ & local maximum at $x = b$.



2. <u>Second Derivative Test</u>: Find the root of $f'(x) = 0$. If $x = a$ is one of the roots, then find $f''(x)$ at $x = a$

i) $f''(a) \rightarrow$ negative, then $f(x)$ is maximum at $x = a$

ii) $f''(a) \rightarrow$ positive, then $f(x)$ is minimum at $x = a$

iii) $f''(a) \rightarrow$ zero.

We find $f'''(a)$. If $f'''(a) \neq 0$ then $f(x)$ has neither maximum nor minimum at $x = a$.

If $f'''(a) = 0$ then find $f^{iv}(a)$. If $f^{iv}(a) \rightarrow$ positive, then $f(x)$ is minimum at $x = a$, $f^{iv}(a) \rightarrow$ negative then $f(x)$ is maximum at $x = a$.

And so on...

* 2nd Derivative Test is not applicable to those critical points where $f'(x)$ remains undefined.

* Global Extrema:

1. **Global Extrema in $[a,b]$**: Find all critical points of $f(x)$ in $[a,b]$ $(c_1, c_2, c_3, \ldots)$.

Now,

Global maxima $= \max\{f(a), f(c_1), f(c_2), \ldots, f(c_n), f(b)\}$

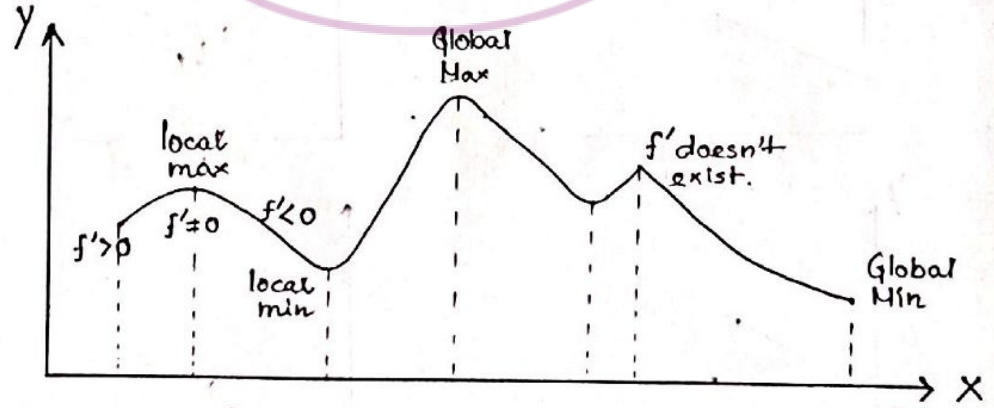
Global minima $= \min\{f(a), f(c_1), f(c_2), \ldots, f(c_n), f(b)\}$.

2. **Global Extrema in (a,b)**: $c_1, c_2, \ldots, c_n$ be the critical points.

Global maxima $= \max\{f(c_1), f(c_2), \ldots, f(c_n)\}$

Global minima $= \min\{f(c_1), f(c_2), \ldots, f(c_n)\}$.

But if $\lim_{x\to a^+} f(x) >$ global maxima or $\lim_{x \to b^-} f(x) <$ global minima then $f(x)$ would not possess global maximum or minimum in (a,b).



Summary Graph.

* Extrema of Discontinuous Functions:

1. Minimum of Discontinuous Functions:

For minimum, at $x = a$

$$f(a) \le f(a+h)$$
$$\& \; f(a) \le f(a-h).$$

2. Maximum of Discontinuous Functions:

For maximum, at $x = a$

$$f(a) \ge f(a+h)$$
$$\& \; f(a) \ge f(a-h).$$

3. Neither Maximum nor minimum exists: