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MODULE - 1

CALCULUS OF COMPLEX FUNCTIONS

1. $z = x + iy$ (cartesian)
2. $z = re^{i\theta}$ (polar form)
3. $e^{i\theta} = \cos\theta + i\sin\theta$
4. $r = \sqrt{x^2 + y^2}$

   $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$|z| = r = \sqrt{x^2 + y^2}$

3. $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

   $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

4. $\cos(i\theta) = \cosh\theta$

   $\sin(i\theta) = i\sinh\theta$.

5. $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

Function of a Complex variable:

$f(z) = u(x, y) + i\, v(x, y)$ → Cartesian form

$f(z) = u(r, \theta) + i\, v(r, \theta)$ → Polar form.

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Analytic function:

A complex valued function $w = f(z)$ (w or ω) is said to be analytic at a point $z = z_0$ if

$$\frac{dw}{dz} = f'(z) = \lim_{\delta z \to 0} \frac{f(z + \delta z) - f(z)}{\delta z}$$

exists and is unique at z_0 and in the neighbourhood of z_0, further $f(z)$ is to be analytic in a region if it's analytic at every point of the region

$\delta z \to$ small variation in z ⇒ there will be variation in x & y as well

- Analytic function is also called Regular function or Holomorphic function.

* **Theorem 1:**

Cauchy - Riemann equations in cartesian form:
(C-R equation)

The necessary condition that the function, $w = f(z)$; $w = f(z) = u(x,y) + iv(x,y)$, be analytic at any point $z = x + iy$ is that, there exists four continuous first order partial derivatives,
$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ & satisfies the equation

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \quad \text{and} \quad \boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

These are known as C-R Equation.

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Proof: Let $f(z)$ be analytic at a point $z = x + iy$
By definition,
$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$ exists and is unique

where $f(z) = u(x,y) + iv(x,y)$ and
$$f(z+\delta z) = u(x+\delta x, y+\delta y) + i\,v(x+\delta x, y+\delta y)$$
where δz is an increment in z corresponding to increments $\delta x, \delta y$ in x, y.

Now, $$f'(z) = \lim_{\delta z \to 0} \frac{u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y) - (u(x,y) + iv(x,y))}{\delta z}$$

$$f'(z) = \lim_{\delta z \to 0} \frac{u(x+\delta x, y+\delta y) - u(x,y)}{\delta z}$$

$$+ i \lim_{\delta z \to 0} \frac{v(x+\delta x, y+\delta y) - v(x,y)}{\delta z} \quad \text{—— (1)}$$

* $\delta z = (z + \delta z) - z$ where $z = x + iy$

$z + \delta z = (x + \delta x) + i(y + \delta y)$

$= \cancel{x+iy}\,(x + \delta x) + i(y + \delta y) - (x + iy)$

$\delta z = \delta x + i\,\delta y$

<u>case</u> 1: put $\delta x = 0$,

$\Rightarrow \delta z = i\,\delta y$

when $\delta z \to 0 \Rightarrow \delta y \to 0$

eqn (1):

$$f'(z) = \lim_{\delta y \to 0} \frac{u(x, y+\delta y) - u(x,y)}{i\delta y} + i \lim_{\delta y \to 0} \frac{v(x, y+\delta y) - v(x,y)}{i\delta y}$$

(definition of partial derivative)

$$= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$[1/i = -i]$

$$\boxed{f'(z) = -i u_y + v_y} \quad \text{——— (2)}$$

<u>case 2</u>: put $\delta y = 0$

$\Rightarrow \delta z = \delta x$

when $\delta z \to 0 \Rightarrow \delta x \to 0$

eqn (1):

$$f'(z) = \lim_{\delta x \to 0} \frac{u(x+\delta x, y) - u(x,y)}{\delta x} + i \lim_{\delta x \to 0} \frac{v(x+\delta x, y) - v(x,y)}{\delta x}$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\boxed{f'(z) = u_x + i v_x} \quad \text{——— (3)}$$

from (2) and (3) $f'(z) = -i u_y + v_y = u_x + i v_x$.

$\Rightarrow$ $\boxed{u_x = v_y}$ & $\boxed{v_x = -u_y}$

Hence the proof

* Theorem 2:

Cauchy-Riemann Equation in Polar form (C-R equation)

If $f(z) = u(r,\theta) + i v(r,\theta)$ is analytic at a point $z = re^{i\theta}$ then there exists four continuous 1st order partial derivatives $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta}$

and satisfies the equation.

$$\boxed{u_r = \frac{1}{r} v_\theta} \quad \text{and} \quad \boxed{v_r = -\frac{1}{r} u_\theta}$$

(or) $\quad r u_r = v_\theta$ and $r v_r = -u_\theta$

Proof: Let $f(z)$ be analytic at $z = re^{i\theta}$

By definition,

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$ exists and is unique.

where $f(z) = u(r,\theta) + i v(r,\theta)$ and

$$f(z+\delta z) = u(r+\delta r, \theta+\delta\theta) + i(r+\delta r, \theta+\delta\theta)$$

where δz is increment in z corresponding to increments $\delta r, \delta\theta$ in r, θ,

Now,

$$f'(z) = \lim_{\delta z \to 0} \frac{u(r+\delta r, \theta+\delta\theta) + i v(r+\delta r, \theta+\delta\theta) - (u(r,\theta) + i v(r,\theta))}{\delta z}$$

$$f'(z) = \lim_{\delta z \to 0} \frac{u(r+\delta r, \theta+\delta\theta) - u(r,\theta)}{\delta z} +$$

$$i \lim_{\delta z \to 0} \frac{v(r+\delta r, \theta+\delta\theta) - v(r,\theta)}{\delta z} \quad \text{——— (1)}$$

$\delta z = (z+\delta z) - \cancel{\delta} z$ where

* $$\delta z = \frac{\partial z}{\partial r}\delta r + \frac{\partial z}{\partial \theta}\delta\theta$$

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$$= \frac{\partial (re^{i\theta})}{\partial r}\delta r + \frac{\partial (re^{i\theta})}{\partial \theta}\delta\theta.$$

$$= e^{i\theta}\delta r + r.ie^{i\theta}\,\delta\theta.$$

$$\delta z = e^{i\theta}\delta r + ire^{i\theta}\,\delta\theta.$$

case 1: put $\delta r = 0$

$$\Rightarrow \delta z = ire^{i\theta}\,\delta\theta.$$

when $\delta z \to 0 \Rightarrow \delta\theta \to 0$.

eq (1) becomes.

$$f'(z) = \lim_{\delta\theta\to 0} \frac{u(r, \theta+\delta\theta) - u(r,\theta)}{ire^{i\theta}\,\delta\theta}$$

$$+ \cancel{i} \lim_{\delta\theta\to 0} \frac{v(r, \theta+\delta\theta) - v(r,\theta)}{\cancel{i}re^{i\theta}\,\delta\theta}.$$

$$= \frac{1}{ire^{i\theta}} \cancel{\lim} \frac{\partial u}{\partial\theta} + \frac{1}{re^{i\theta}}\frac{\partial v}{\partial\theta}.$$

$$\boxed{f'(z) = \cancel{\lim_{\delta\theta\to 0}} \frac{-i}{re^{i\theta}} u_\theta + \frac{1}{re^{i\theta}} v_\theta} \quad \text{—— (2)}$$

case 2: put $\delta\theta = 0$

$$\Rightarrow \delta z = e^{i\theta}\delta r$$

when $\delta z \to 0 \Rightarrow \delta r \to 0$

eq (1)

$$\Rightarrow f'(z) = \lim_{\delta r\to 0} \frac{u(r+\delta r, \theta) - u(r,\theta)}{e^{i\theta}\,\delta r}$$

$$+ i \lim_{\delta r\to 0} \frac{v(r+\delta r, \theta) - v(r,\theta)}{e^{i\theta}.\,\delta r}.$$

$$f'(z) = \frac{1}{e^{i\theta}}\frac{\partial u}{\partial r} + \frac{i}{e^{i\theta}}\frac{\partial v}{\partial r}$$

$$\boxed{f'(z) = e^{-i\theta}\left[u_r + i v_r\right]} \quad \text{—— (3)}$$

From eq (2) and (3), we have,

$$f'(z) = \frac{-i}{re^{i\theta}} u_\theta + \frac{1}{re^{i\theta}} v_\theta = e^{-i\theta}(u_r + i v_r)$$

$$\Rightarrow e^{-i\theta}\left[\frac{-i}{r} u_\theta + \frac{1}{r} v_\theta\right] = e^{-i\theta}\left[u_r + i v_r\right]$$

$$u_r = \frac{1}{r} v_\theta \quad \& \quad v_r = -\frac{1}{r} u_\theta$$

$$\Rightarrow \boxed{r u_r = v_\theta} \quad \& \quad \boxed{r v_r = -u_\theta}$$

Note:

* Harmonic function:

1.) The real part of an analytic function is harmonic if $u_{xx} + u_{yy} = 0$

2.) The imaginary part of an analytic function is harmonic if $v_{xx} + v_{yy} = 0$

(Cartesian)

3.) The real part is harmonic if

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$$

4.) Imaginary part is harmonic if

$$v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} v_{\theta\theta} = 0$$

} polar form.

Problems:

1. Show that $w = z + e^z$ is analytic and also find $\frac{dw}{dz}$.

Ans: Given $w = z + e^z$

put $z = x+iy$ (complex no.)

$$f(z) = w = x + iy + e^{(x+iy)}$$

$$= (x+iy) + e^x (e^{iy})$$

$$= (x+iy) + e^x(\cos y + i \sin y)$$

$$= (x+iy) + e^x \cos y + i e^x \sin y.$$

$$= \underbrace{(x + e^x \cos y)}_{u} + i \underbrace{(y + e^x \sin y)}_{v}$$

Here, $u = x + e^x \cos y \quad v = y + e^x \sin y.$

$u_x = \frac{\partial u}{\partial x} = \cancel{x + e}$

$= 1 + \cos y . e^x$

$u_y = \frac{\partial u}{\partial y}$

$= 0 + e^x(-\sin y)$

$= - e^x \sin y$

$v_x = \frac{\partial v}{\partial x}$

$= 0 + \sin y \, e^x = e^x \sin y$

$v_y = \frac{\partial v}{\partial y}$

$= 1 + e^x \cos y$

$u_x = v_y$ & $v_x = - u_y$.

$\therefore f(z)$ is analytic

$\frac{dw}{dz} = f'(z) = -i u_y + v_y = u_x + i v_x$

(from derivation, eq. (2) and (2))

$= u_x + i v_x$

$= \cancel{1 + e^x \cos} (1 + e^x \cos y) + i (e^x \sin y)$

$= 1 + e^x \cos y + i \, \cancel{\sin} \, e^x \sin y$

$= 1 + e^x (\cos y + i \sin y)$

$= 1 + e^x e^{iy}$

$= 1 + e^{x+iy}$

$\frac{dw}{dz} = 1 + e^z$

(OR) Milne Thompson Method:

$z = x + iy$

put $y = 0 \Rightarrow z = x$

in eq:-

$f'(z) = 1 + e^z \cos 0 + \cancel{i e^z \sin 0}$

$f'(z) = 1 + e^z$.

2. Show that $f(z) = \sin z$ is analytic and also find $f'(z)$

Ans) Given $f(z) = \sin z$

put $z = x + iy$ (for splitting easily)

$\frac{d(\cosh x)}{dx} = \sinh x$ $\quad \frac{d(\sinh x)}{dx} = \cosh x$

$\cosh 0 = 1$, $\sinh 0 = 0$

$$f(z) = \sin(x+iy)$$
$$= \sin x \cos iy + \cos x \sin iy.$$
$$= \underbrace{\sin x \cosh y}_{u} + \underbrace{\cos x \sinh y}_{v}.$$

thus, $u = \sin x . \cosh y$ $\quad v = \cos x \sinh y.$

$u_x = \frac{\partial u}{\partial x}$ $\quad V_x = \frac{\partial v}{\partial x} =$

$= \cos x \cosh y$ $\quad = \sinh y(-\sin x)$

$u_y = \frac{\partial u}{\partial y}$ $\quad V_y = \frac{\partial v}{\partial y}$

$= \sin x \sinh y$ $\quad = \cos x \cosh y$

$u_x = V_y$ & $V_x = -u_y$.

$\frac{dw}{dz} = f'(z) = u_x + iV_x$

$= \cos x \cosh y + i(\sinh y(-\sin x))$

$=$

From Milne's theorem,

$y = 0$, $\Rightarrow z = x$

$f'(z) = \cos z \cosh(0) - i \sin 0 \sin z$

$= \cos z - 0$

$f'(z) = \underline{\cos z}$

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3. Show that $f(z) = \cos z$ is analytic also find $f'(z)$.

Ans, Given $f(z) = \cos z$

put $z = x+iy$

$$f(z) = \cos(x+iy)$$
$$= \cos x . \cos iy - \sin x . \sin iy$$
$$= \underbrace{\cos x . \cosh y}_{u} - \underbrace{\sin x . \sinh y}_{v}$$

$u = \cos x \cosh y$ $\quad v = -\sin x . \sinh y$

$\log(ab) = \log a + \log b$

$u_x = \frac{\partial u}{\partial x} = \cosh y . (-\sin x)$ $\quad$ $v_x = \frac{\partial v}{\partial x} = -\sinh y \cos x$.

$u_y = \frac{\partial u}{\partial y} = \cos x . \sinh y$ $\quad$ $v_y = \frac{\partial v}{\partial y} = -\sin x \cosh y$.

$u_x = v_y$

$v_x = -u_y$.

$\therefore f(z)$ is analytic function.

$\frac{dw}{dz} = f'(z) =$

$= u_x + i v_x$

$= -\sin x \cosh y \; \cancel{+} - i \cos x \sinh y$.

By Milne Thompson's ~~theor~~ method,

$y = 0 \implies z = x$.

$f'(z) = -\sin z . \cos 0 - \underline{i \cos 0 \cancel{\sin 0}}$.

$= -\sin z \, (1)$

$f'(z) = \underline{\underline{-\sin z}}$

4. Show that $w = \log z$, $z \neq 0$ is analytic, also find dw/dz

Ans: Given $w = \log z$

put $z = re^{i\theta}$

$w = \log(\cancel{e}\, re^{i\theta})$

$= \log r + \log e^{i\theta}$

$= \log r + i\theta . \log e$ $\quad$ $(\because \log e = 1)$

$w = \underbrace{\log r}_{u} + \underbrace{i\theta}_{v}$

Here $u = \log r$, $v = \theta$.

$U_r = \frac{\partial u}{\partial r}$ $\quad$ $V_r = \frac{\partial v}{\partial r}$

$= \frac{1}{r}$ $\quad$ $= 0$

$U_\theta = 0$ $\quad$ $V_\theta = 1$

$r U_r = \frac{1}{r} \times r = 1 = V_\theta$

$r U_r = V_\theta$

$r V_r = r(0) = 0 = U_\theta$.

$r V_r = -U_\theta$.

$\therefore$ w is analytic.

$\frac{dw}{dz} = e^{i\theta} (u_r + i v_r)$

$= e^{-i\theta} \left(\frac{1}{r} + i(0) \right)$

$= \frac{e^{-i\theta}}{r}$

$= \frac{1}{r e^{i\theta}}$

$= \frac{1}{z}$

(a) By Milne Thomson's method.

$= \cancel{e^{-i\theta}}$ $\quad \theta = 0 \Rightarrow z = r$.

$\Rightarrow \frac{dw}{dz} = e^{-i(0)} \left(\frac{1}{z} + i(0) \right)$

$= \frac{1}{z}$

5. Given, $f(z) = z^n$, find $f'(z)$ and prove it's analytic.

$e^{i\theta} = \cos\theta + i\sin\theta$
$e^{-i\theta} = \quad -$

Ans $f(z) = z^n$

take, $z = re^{i\theta}$.

$f(z) = (re^{i\theta})^n$

$= r^n e^{ni\theta}$.

$= r^n(\cos n\theta + i\sin n\theta)$

$f(z) = \underbrace{r^n\cos n\theta}_{u} + \underbrace{r^n i\sin n\theta}_{v}$.

$u = r^n\cos n\theta$, $v = r^n\sin n\theta$.

$u_r = \cos n\theta\, r^{n-1}.n$	$v_r = \sin n\theta . r^{n-1}.n$
$u_\theta = r^n.(-\sin n\theta).n$	$v_\theta = r^n\cos n\theta . n$.

$r u_r = r . r^{n-1}.n\cos n\theta = r^{n-1+1} n\cos n\theta$

$= r^n n\cos n\theta = v_\theta$.

$r v_r = r . r^{n-1}. n\sin n\theta = r^{n-1+1} n.\sin n\theta$

$= r^n n\sin n\theta = -u_\theta$.

$\therefore f(z)$ is analytic.

$f'(z) = e^{-i\theta}(u_r + i v_r)$

$= e^{-i\theta}(n r^{n-1}\cos n\theta + i n r^{n-1}\sin n\theta)$

$= e^{-i\theta}(n.r^{n-1}(\cos n\theta + i\sin n\theta))$

By $= e^{-i\theta}$. Milne's Thompson's method,

$r = z$, $\theta = 0$

$f'(z) = e^{i(0)}(n r^{n-1}\cos 0 + i(\sin 0))$

$= e^0(n r^{n-1})$

$f'(z) = e(\underline{\underline{n z^{n-1}}}$

6. Show that $f(z) = \left(r + \frac{k^2}{r}\right)\cos\theta + i\left(r - \frac{k^2}{r}\right)\sin\theta$

$r \neq 0$, is a regular function, of $z = re^{i\theta}$, Also find $f'(z)$

Ans Given $f(z) = \left(r + \frac{k^2}{r}\right)\cos\theta + i\left(r - \frac{k^2}{r}\right)\sin\theta$.

$$z = re^{i\theta}$$

$$\cancel{f(z) = \left(r + \frac{k^2}{r}\right)}$$

$$u = \left(r + \frac{k^2}{r}\right)\cos\theta \quad , \quad v = \left(r \mp \frac{k^2}{r}\right)\sin\theta$$

$$u_r = \left(1 - \frac{k^2}{r^2}\right)\cos\theta \quad \Big| \quad v_r = \left(1 + \frac{k^2}{r^2}\right)\sin\theta.$$

$$u_\theta = \left(r + \frac{k^2}{r}\right)(-\sin\theta) \quad \Big| \quad v_\theta = \left(r - \frac{k^2}{r}\right)\cdot\cos\theta.$$

$$r\,u_r = r\left(1 - \frac{k^2}{r^2}\right)\cos\theta = \left(r - \frac{k^2}{r}\right)\cos\theta$$

$$= v_\theta$$

$$r\,v_r = r\left(1 + \frac{k^2}{r^2}\right)\sin\theta = \left(r + \frac{k^2}{r}\right)\sin\theta$$

$$= -u_\theta.$$

$\therefore$ $f(z)$ is a regular function (analytic)

$$f'(z) = e^{-i\theta}(u_r + i v_r)$$

$$= e^{-i\theta}\left(\left(1 - \frac{k^2}{r^2}\right)\cos\theta + i\left(1 + \frac{k^2}{r^2}\right)\sin\theta\right)$$

by milne thompsons method

$$\theta = 0 \implies z = r$$

$$= e^0\left(\left(1 - \frac{k^2}{z^2}\right)\cos 0 + i\left(1 + \frac{k^2}{z^2}\right)\sin 0\right)$$

$$f'(z) = 1 - \frac{k^2}{z^2}$$

II Type (2):

1. Construct the Analytic function whose real part is $\log\sqrt{x^2+y^2}$.

Ans:

$$u = \log\sqrt{x^2+y^2}$$

$$u = \frac{1}{2}\log(x^2+y^2)^{1/2}$$

$$u = \frac{1}{2}\log(x^2+y^2)$$

$$u_x = \frac{1}{2}\cdot\frac{1}{x^2+y^2}(2x) = \frac{x}{x^2+y^2}$$

$$u_y = \frac{1}{2}\cdot\frac{1}{x^2+y^2}(2y) = \frac{y}{x^2+y^2}$$

we have $f'(z) = u_x + i v_x$

also have $v_x = -u_y$.

$$f'(z) = u_x - i u_y.$$

$$= \frac{x}{x^2+y^2} - i\left(\frac{-y}{x^2+y^2}\right)$$

Apply Milne Thompson method,

$$y = 0, \quad x = z$$

$$f'(z) = \frac{z}{z^2}$$

$$f'(z) = \frac{1}{z}$$

$$f(z) = \int f'(z) = \int \frac{1}{z}\,dz$$

$$= \underline{\log z + C}$$

8. Construct / Find the Analytic function, $f(z)$ whose imaginary part is $e^x(x\sin y + y\cos y)$

Ans: Given, $v = e^x(x\sin y + y\cos y)$

$$v = e^x x\sin y + e^x y\cos y.$$

$$v_x = \sin y(x\cdot e^x + e^x) + y\cos y\, e^x$$

$$v_y = xe^x\cdot\cos y + e^x[y(-\sin y) + \cos y]$$

$$= xe^x\cos y - e^x y\sin y + e^x\cos y.$$

~~$U_x = V_y$ & $V_y = -U_x$~~

$U_x = V_y$ & $V_x = -U_y$

we have $f'(z) = U_x + i V_x$

$= V_y + i V_x$

$= i(e^x x \sin y + e^x \sin y + e^x y \cos y) +$
$(e^x x \cos y - e^x y \sin y + e^x \cos y)$

by Milne-Thompson's method.

$y = 0 \Rightarrow z = x$

$f'(z) = (e^z z \cos 0 - \cancel{e^z (0) \sin 0} + e^z \cos 0) +$
$i(\cancel{e^z z \sin 0} + \cancel{e^z \sin 0} + \cancel{e^z (0) \cos 0})$

$f'(z) = e^z z + e^z$

& $f(z) = \int f'(z).\, dz$ ILATE

$= \int e^z . z + e^z \,.\, dz$

$= z . e^z - (1) e^z + c + e^z$

$= z . e^z - e^z + e^z + c$

$f(z) = \underline{z e^z} + c$

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Q. Find analytic function $f(z)$, when
$u = e^{-x} \{ (x^2 - y^2) \cos y + 2xy \sin y \}$

Ans. $u = e^{-x} \{ (x^2 - y^2) \cos y + 2xy \sin y \}$
$= e^{-x} (x^2 - y^2) \cos y + e^{-x} 2xy \sin y$.

~~$U_x = \cos y . (e^{-x} (2x) - (x^2 - y^2) e^{-x}$~~

$U_x = e^{-x} \{ \cos y (2x) + 2y \sin y (1) \} - e^{-x} \{ (x^2 - y^2) \cos y + 2xy \sin y \}$

$U_y = e^{-x} \{ -\sin y (x^2 - y^2) + \cos y (-2y) + 2x (y \cos y + \sin y) \}$

$f'(z) = U_x + i V_x$

By C-R equation, we have $U_x = V_y$ & $V_x = -U_y$

$f'(z) = U_x - i U_y$

$= e^{-x}\{\cos y(2x) + 2y\sin y\} - e^{-x}\{(x^2-y^2)\cos y + 2xy\sin y\}$

$+ i\{e^{-x}\{-\sin y(x^2-y^2) + \cos y(-2y) + 2x(y\cos y + \sin y)\}$

By Milne Thompson's method,

put $y=0$, $x=z$.

$f'(z) = e^{-z}\{\cos 0(2z) + 2(0)\} - e^{-z}\{(z^2)\cos 0 + 2(0)\}$

$\cancel{+ i\{e^{-z}\{-\sin 0(z^2-0^2) + \cos 0(-2(0)) + 2x(0+0)\}}$

$= e^{-z}(2z) - e^{-z}(z^2)$

$= e^{-z}[2z - z^2]$

$f'(z) = e^{-z}[2z-z^2]$

$\int f'(z) = \int [2z-z^2]\, e^{-z}.dz$

$= (2z-z^2)\dfrac{e^{-z}}{-1} - (2-2z)\dfrac{e^{-z}}{-1} + (-2)\dfrac{e^{-z}}{-1} + C$

$= \cancel{(-2z+z^2)e^{-z}} -$

$= e^{z}[-2z+z^2+2-2z+2]$

$= e^{z}.z^2 + c$

H.W

~~10.~~ $u = e^{2x}(x\cos 2y - y\sin 2y)$

Ans $u_x = e^{2x}(\cos 2y) + (x\cos 2y - y\sin 2y)2e^{2x}$

$u_y = e^{2x}(-2x\sin 2y - (2y\cos 2y + \sin 2y))$

$= e^{2x}(-2x\sin 2y - 2y\cos 2y - \sin 2y)$

$\because f'(z) = u_x + iv_x$

by C-R equation, we have $u_x = v_y$ & $v_x = -u_y$.

$\Rightarrow f'(z) = u_x - iu_y$

$= e^{2x}(\cos 2y) + 2e^{2x}(x\cos 2y - y\sin 2y) -$

$i(e^{2x}(-2x\sin 2y - 2y\cos 2y - \sin 2y))$

By Milne Thompson's method,

put $x = z$

& $y = 0$

$$f'(z) = e^{2z}(\cos 0) + 2e^{2z}(z\cos 0 - 0(0)) - i(e^{2z}(-\cancel{2z}\sin 0 - \cancel{2(0)}\cos 0 - \cancel{\sin 0}))$$

$$= e^{2z} + 2e^{2z}(z)$$

$$= e^{2z} + 2z e^{2z}$$

$$f'(z) = e^{2z}(1+2z)$$

ILATE

$$\Rightarrow f(z) = \int e^{2z}(1+2z)\,dz$$

$$= \int (1+2z)e^{2z}\,dz$$

$$= (1+2z)\frac{e^{2z}}{2} - (2)\frac{e^{2z}}{4} + C$$

$$= e^{2z}\left(\frac{\cancel{1}+2z}{2} - \frac{\cancel{2}\cancel{1}}{2}\right) + C$$

$$= e^{2z}\left(\frac{2z}{2}\right) + C$$

$$f(z) = e^{2z}z + C$$

4. Find analytic function whose real part is $\frac{\sin 2x}{\cosh 2y - \cos 2x}$

Ans Given $u = \frac{\sin 2x}{\cosh 2y - \cos 2x}$

$$u_x = \frac{(\cosh 2y - \cos 2x)2\cos 2x - \sin 2x(0 - (2(-\sin 2x)))}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{(\cosh 2y - \cos 2x)\, 2\cos 2x - \sin 2x\,(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$u_y = \sin 2x \left[\frac{-1}{(\cosh 2y - \cos 2x)^2} \times 2\sinh 2y \right]$$

$$= \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

$$f'(z) = u_x + i v_x$$

By C-R equation, $u_x = v_y$ and $v_x = -u_y$.

$$f'(z) = u_x - i u_y$$

By Milne-Thomson method, $x = z \rightarrow y = 0$.

$$f'(z) = (u_x)_{(z,0)} - i (u_y)_{(z,0)}$$

$$f'(z) = \frac{(\cosh 2y - \cos 2z)\, 2\cos 2z - 2\sin 2z \cdot \sin 2z}{(\cosh 2y - \cos 2z)^2} - i \left[\frac{-2\sin 2z \sinh 2(0)}{(\cosh 2y - \cos 2z)^2} \right]$$

$$= \frac{(1 - \cos 2z) \cdot 2\cos 2z - 2\sin^2 2z}{(1 - \cos 2z)^2}$$

$$= \frac{2\cos 2z - 2\cos^2 2z - 2\sin^2 2z}{(1 - \cos 2z)^2}$$

$$= \frac{2\cos 2z - 2(\cos^2 2z + \sin^2 2z)}{(1 - \cos 2z)^2}$$

$$= \frac{2\cos 2z - 2(1)}{(1 - \cos 2z)^2}$$

$$f'(z) = \frac{2(\cos 2x - 1)}{(1 - \cos 2z)^2}$$

$$= \frac{-2\,(1 - \cos 2z)}{(1 - \cos 2z)^2}$$

$$f'(z) = \frac{-2}{(1 - \cos 2z)}$$

$$= \frac{-2}{2\sin^2 z} = -\operatorname{cosec}^2 z$$

$f(x) = \int \csc^2 x \, dx$
$f(x) = \cot x + C$

12. If $\phi + i\psi$ represents the complex potential of an electrostatic field where $\psi = (x^2 - y^2) + \frac{x}{x^2+y^2}$ find the complex potential of the function of complex variable z and hence determine ϕ.

Ans: $\psi \to \psi_x, \psi_y$

$\psi_x = (2x) + \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2}$

$= 2x + \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$

$\psi_x = 2x + \frac{(y^2-x^2)}{(x^2+y^2)^2}$

$\psi_z = -2y + x\left(\frac{1}{(x^2+y^2)^2}(2y)\right)$

$= -2y \mp \frac{2xy}{(x^2+xy^2)^2}$

$f'(z) = \phi_x + i\psi_x$

By C-R equation,

$\phi_x = \psi_y$ & $\psi_x = -\phi_y$

$\Rightarrow f'(z) = \psi_y + i\psi_x$

Milne-thompson method.

$z = x$ & $y = 0$

$f'(z) = i\,2x + \frac{(y^2-x^2)}{(x^2+y^2)^2} + i\left(-2y - \frac{2xy}{(x^2+y^2)^2}\right)$

$= i\,2z + \frac{(0-z^2)}{(-z^2)^2} + i\left(-2(0) - \frac{2z(0)}{(\quad)^2}\right)$

$= i\left(\frac{2z + (-z^2)}{z^4}\right) = i\left(2z - \frac{z^2}{z^4}\right)$

any problem remove i from denominator

/ /

$f'(z) = i\left(2z - \frac{1}{z^2}\right)$

$\int f'(z) = i\int 2z - \frac{1}{z^2} \cdot dz$ $\qquad z^{-2} = \frac{z^{-2+1}}{-2+1}$

$= i\left[2\left(\frac{z^2}{2}\right) - \left(-\frac{1}{z}\right)\right]$ $\qquad = \frac{z^{-1}}{-1} = -\frac{1}{z}$

$f(z) = i\left[z^2 + \frac{1}{z}\right] + c$

put $z = x + iy$

$f(z) = i\left[(x+iy)^2 + \frac{1}{(x+iy)}\right] + c$

$= i\left[x^2 + (iy)^2 + 2xiy + \frac{1}{x+iy}\right] + c$

$= i\left[(x^2 - y^2 + 2xyi + \frac{1}{(x+iy)} \times \frac{(x-iy)}{(x-iy)}\right] + c$

$= i\left[x^2 + 2i\,xy - y^2 + \frac{(x-iy)}{(x^2-(iy)^2)}\right] + c$

$= i\left[(x^2 + 2ixy - y^2) + \frac{(x-iy)}{(x^2+y^2)}\right] + c$

$= i\left[x^2 - y^2 + 2ixy + \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2}\right] + c$

$= ix^2 - iy^2 + 2(-1)xy + \frac{ix}{x^2+y^2} - (-1)\frac{y}{x^2+y^2} + c$

$= \left(ix^2 - iy^2 - 2xy + \frac{ix}{x^2+y^2} + \frac{y}{x^2+y^2}\right) + c$

$= \underbrace{\left(-2xy + \frac{y}{x^2+y^2}\right)}_{\phi} + i\underbrace{\left((x^2 - y^2) + \frac{x}{x^2+y^2}\right)}_{\psi} + c$

$\Rightarrow \phi = -2xy + \frac{y}{x^2+y^2}$

conjugate ; find v

III Type (3) :

13. Show that $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic & find its harmonic conjugate. Also find the corresponding analytic function $f(z)$.

Ans $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$u_x = 3x^2 - 3y^2 + 6x$

$u_{xx} = 6x + 6.$

$u_y = -6xy - 6y$

$u_{yy} = -6x - 6$

$\Rightarrow \quad u_{xx} + u_{yy} = \cancel{6x} + \cancel{6} - \cancel{6x} - \cancel{6}$

$= \underline{0}$

$\therefore$ u is harmonic.

$v = ?$

By C-R equation, we have

$u_x = v_y$ & $v_x = -u_y$.

$\therefore u_x = v_y$	$v_x = -u_y$
$v_y = 3x^2 - 3y^2 + 6x$	$= -(-6xy - 6y)$
	$= +6xy + 6y.$
$\Rightarrow v = \int (3x^2 - 3y^2 + 6x)\, dy$	$v = \int (6xy + 6y)\, dx$
$= 3x^2 y - \frac{3y^3}{3} + 6xy + f(x)$	$= \cancel{6}^{3} xy \frac{x^2}{\cancel{2}} + 6yx + g(y)$
$= 3x^2 y - y^3 + 6xy + f(x)$	$= 3x^2 y + 6yx + g(y)$

we have $g(y) = -y^3$, $f(x) = 0$

$v = \underbrace{3x^2 y + 6xy}_{\text{common term}} + f(x) + g(y)$

$v = \underline{\underline{3x^2 y + 6xy}} - y^3$

$f(z) = u + iv$

$$f(z) = (x^3 - 3xy^2 + 3x^2 - 3y^2 + 1) + i(3x^2y + 6xy - y^3)$$

By Milne Thompson's method

$x = z$ & $y = 0$.

$$f(z) = z^3 + 3z^2 + 1$$

14. Determine which of the following function is harmonic. Find the conjugate harmonic function & express $u + iv$ as an analytic function of z

i) $v = \log\sqrt{x^2+y^2}$

ii) $v = \cos x \sinh y$

Ans/ $v = \log\sqrt{x^2+y^2}$.

$\cancel{v_x} = v = \log(x^2+y^2)^{1/2}$

$$= \frac{1}{2}\log(x^2+y^2)$$

$$v_x = \frac{1}{2}\frac{1}{(x^2+y^2)}(2x) = \frac{x}{x^2+y^2}$$

$$v_{xx} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$v_y = \frac{1}{2}\frac{1}{(x^2+y^2)}(2y) = \frac{y}{x^2+y^2}$$

$$v_{yy} = \frac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2} = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$v_{xx} + v_{yy} = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} = 0$$

$\therefore \quad = 0$

$\therefore$ v is harmonic.

$u = ?$

By C-R equation,

$u_x = v_y$ & $v_x = -u_y$

$\Rightarrow u_y = -v_x$

ii.) $v = \cos x \sinh y$

$v_x = \sinh y(-\sin x) = -\sin x \sinh y$

$v_{xx} = -\cos x \sinh y$

$v_y = \cos x \cosh y$

$v_{yy} = \cos x \sinh y$

$\therefore v_{xx} + v_{yy} = -\cos x \sinh y + \cos x \sinh y$

$= \underline{0}$

$\therefore$ v is harmonic.

$u = ?$

By C-R equation,

$u_x = v_y$ & $v_x = -u_y$

$\Rightarrow u_x = \cos x \cosh y$

$u_y = -v_x$

$= +\sin x \sinh y$

$u = \int \cos x \cosh y \, dx$

$= \sin x \cdot \cosh y + f(y)$

$=$

$u = \int (\sin x \cdot \sinh y) \, dy$

$= \sin x \cdot \cosh y + g(x)$

$g(x) = 0$ & $f(y) = 0$

$\Rightarrow u = \sin x \cosh y + f(y) + g(x)$.

$u = \underline{\sin x} \cosh y$

$f(z) = u + iv = (\sin x \cosh y) + i(\cos x \sinh y)$

By milne thompson's method,

$x = z$ & $y = 0$

$f(z) = \sin z \cosh 0 + i(\cos z \sinh 0)$

$= \sin z (1) + i(0)$

$= \underline{\sin z}$.

i) $v = \log \sqrt{x^2+y} = \frac{1}{2} \log(x^2+y)$

$v_x = \frac{1}{2} \frac{1}{(x^2+y)} (2x) = \frac{x}{x^2+y}$

$$V_{xx} = \frac{1}{2}\,\frac{(x^2+y)(1) - x(2x)}{(x^2+y)^2} = \frac{x^2+y-2x^2}{(x^2+y)^2}$$

$$V_{xx} = \frac{y-x^2}{(x^2+y)^2}$$

$$V_y = \frac{1}{2}\,\frac{1}{(x^2+y)}(1) = \frac{1}{2(x^2+y)}$$

$$V_{yy} = \frac{-1}{2(x^2+y)^2}(1) = \frac{-1}{2(x^2+y)^2}$$

$$V_{xx} + V_{yy} = \frac{y^2-x^2}{(x^2+y)^2} - \frac{1}{2(x^2+y)^2}$$

$$= \frac{2y^2-2x^2-1}{2(x^2+y)^2}$$

$$\neq 0$$

$\therefore$ v is not harmonic.

15. Show that $u = e^x(x\cos y - y\sin y)$ is harmonic & find its harmonic conjugate. Also determine corr. analytic function.

Ans: $u = e^x(x\cos y - y\sin y)$

$$u_x = e^x(\cos y) + (x\cos y - y\sin y)e^x$$

$$u_{xx} = e^x\cos y + e^x(\cos y) + (x\cos y - y\sin y)e^x$$

$$= 2e^x\cos y + e^x(x\cos y - y\sin y)$$

$$u_y = e^x(x(-\sin y) - (y\cos y + \sin y))$$

$$= e^x(-x\sin y - y\cos y - \sin y)$$

$$u_{yy} = e^x(-x\cos y - (y(-\sin y) + \cos y) - \cos y)$$

$$= e^x(-x\cos y + y\sin y - \cos y - \cos y)$$

$$= e^x(-x\cos y + y\sin y - 2\cos y)$$

$$= -e^x x\cos y + e^x y\sin y - 2e^x\cos y$$

$$u_{xx} = 2e^x\cos y + e^x x\cos y - e^x y\sin y$$

$$u_{xx} + u_{yy} = 2e^x \cos y + e^x x \cos y - e^x y \sin y$$
$$- e^x x \cos y + e^x y \sin y - 2e^x \cos y$$
$$= 0$$

$\therefore$ u is harmonic.

By C-R eqn,

$u_x = v_y$ E_1 $v_x = -u_y$.

$v_y = e^x (\cos y + x \cos y - y \sin y)$

$v_x = -e^x(-x \sin y - y\cos y - \sin y)$.

$v = \int e^x (\cos y + x \cos y - y \sin y) \, dy$

$$v = \int (e^x x \cos y + e^x \cos y - e^x y \sin y) \, dy.$$
$$= e^x [x \sin y + \sin y - (-y \cos y - \sin y)] + f(x)$$
$$= e^x [x \sin y + \sin y + y \cos y + \sin y] + f(x)$$
$$= e^x [2 \sin y + x \sin y + y \cos y] + f(x)$$
$$= e^x [\sin y (2+x) + y \cos y] + f(x)$$

$$v_x = -e^x (-x \sin y - y \cos y - \sin y) . dx$$
$$= \int -e^x (-x \sin y - y \cos y - \sin y) . dx$$
$$= \int (e^x x \sin y + e^x y \cos y + e^x \sin y) \, dx$$
$$= (x e^x + e^x) \sin y + e^x y \cos y + e^x \sin y + g(y)$$
$$= e^x (x+1+1) \sin y + e^x y \cos y + g(y)$$
$$= e^x [(x+2) \sin y + y \cos y] + g(y).$$

$g(y) = 0, \quad f(x) = 0$

$v = e^x [\sin y (x+2) + y \cos y]$

$\therefore f(z) = u + iv$

By Milne Thompson method, $x = z, y = 0$

$f(z) = e^z . z$

30/05/23

16. Show that $u = \left(r + \frac{1}{r}\right)\cos\theta$ is harmonic. Find its harmonic conjugate, also find $f(z)$

Ans Given $u = \left(r + \frac{1}{r}\right)\cos\theta$

$u_r = \cos\theta\left(1 - \frac{1}{r^2}\right)$ $\quad \left(r^{-2} \Rightarrow -2r^{-2-1} = -2r^{-3}\right)$

$u_{rr} = \cos\theta\left(0 - \left(\frac{-2}{r^3}\right)\right)$

$= \cos\theta \cdot \frac{2}{r^3}$

$u_\theta = \left(r + \frac{1}{r}\right)(-\sin\theta) = -\left(r + \frac{1}{r}\right)\sin\theta$

$u_{\theta\theta} = \left(r + \frac{1}{r}\right)(-\cos\theta) = -\left(r + \frac{1}{r}\right)\cos\theta$

$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$

$= \cos\theta \cdot \frac{2}{r^3} + \frac{1}{r}\left(1 - \frac{1}{r^2}\right)\cos\theta + \frac{1}{r^2}\left(-\left(r + \frac{1}{r}\right)\cos\theta\right)$

$= \cos\theta \cdot \frac{2}{r^3} + \left(\frac{1}{r} - \frac{1}{r^3}\right)\cos\theta - \left(\left(\frac{1}{r} + \frac{1}{r^3}\right)\cos\theta\right)$

$= \cos\theta\frac{2}{r^3} + \cancel{\frac{1}{r}\cos\theta} - \frac{1}{r^3}\cos\theta - \cancel{\frac{1}{r}\cos\theta} - \frac{1}{r^3}\cos\theta$

$= \cancel{\cos\theta\frac{2}{r^3}} - \cancel{\frac{2\cos\theta}{r^3}}$

$= \underline{0}$

$\therefore$ u is harmonic.

Using C-R equation.

$r u_r = v_\theta$ & $r v_r = -u_\theta$

$r u_r = v_\theta$ $\quad v_r = -\frac{1}{r}u_\theta$

$v_\theta = r u_r$

$$V_\theta = r\left(\cos\theta\left(1 - \frac{1}{r^2}\right)\right)$$

$$V_\theta = \cos\theta\left(r - \frac{1}{r}\right)$$

$$v = \int \cos\theta\left(r - \frac{1}{r}\right) . d\theta$$

$$= \left(r - \frac{1}{r}\right)\sin\theta + f(r)$$

$$V_r = -\frac{1}{r}\left(-\left(r + \frac{1}{r}\right)\sin\theta\right)$$

$$V_r = \left(1 + \frac{1}{r^2}\right)\sin\theta$$

$$v = \int\left(1 + \frac{1}{r^2}\right)\sin\theta . dr$$

$$r^{-2} = \frac{r^{-2+1}}{-2+1} = \frac{r^{-1}}{-1} = -\frac{1}{r}$$

$$= \sin\theta . \left(r - \frac{1}{r}\right) + g(\theta)$$

$$v = \left(r - \frac{1}{r}\right)\sin\theta + g(\theta)$$

Comparing v, $f(r) = 0$, $g(\theta) = 0$

$$\Rightarrow v = \sin\theta\left(r - \frac{1}{r}\right)$$

$$f(z) = u + iv$$

$$f(z) = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta.$$

By Milne Thompson method, $r = z$ and $\theta = 0$

$$f(z) = \left(z + \frac{1}{z}\right)\cos 0 + i\cancel{\left(z - \frac{1}{z}\right)\sin 0}$$

$$f(z) = z + \frac{1}{z}$$

imp 17 If $f(z)$ is a regular function of z, show that
$$\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = |f'(z)|^2$$

Ans We have $f(z) = u + iv$

$$|f(z)| = \sqrt{u^2+v^2} = \phi \text{ (say)} \quad \text{——— (1)}$$

substitute in LHS

$$\left\{\frac{\partial}{\partial x}(\phi)\right\}^2 + \left\{\frac{\partial}{\partial y}(\phi)\right\}^2 =$$

$$= \phi_x^2 + \phi_y^2$$

We have to show

$$\phi_x^2 + \phi_y^2 = |f'(z)|^2$$

from (1) $\phi = \sqrt{u^2+v^2}$

$$\phi^2 = u^2 + v^2$$

→ diff wrt x partially

$$2\phi\phi_x = 2uu_x + 2vv_x.$$

$$\Rightarrow \phi\phi_x = uu_x + vv_x \quad \text{——— (2)}$$

→ diff wrt y partially

$$2\phi\phi_y = 2uu_y + 2vv_y$$

$$\phi\phi_y = uu_y + vv_y. \quad \text{——— (3)}$$

$(2)^2 + (3)^2$

$$\phi^2\phi_x^2 + \phi^2\phi_y^2 = (uu_x + vv_x)^2 + (uu_y + vv_y)^2$$

$$\phi^2(\phi_x^2 + \phi_y^2) = u^2u_x^2 + v^2v_x^2 + 2uu_xvv_x + u^2u_y^2 + v^2v_y^2 + 2uu_yvv_y$$

By C-R equation,

$u_x = v_y$ & $v_x = -u_y$

write in terms of x,

so put $v_y = u_x$ & $u_y = -v_x$ in the equation,

$$= u^2u_x^2 + v^2v_x^2 + 2uu_xvv_x + u^2u_x^2 + v^2v_x^2 + 2u(-v_x)v(u_x)$$

that
$|^2$.

$$\phi^2(\phi_x^2+\phi_y^2) = 2u^2u_x^2 + 2v^2v_x^2.$$

$$\phi^2(\phi_x^2+\phi_y^2) = 2\left[u^2u_x^2 + 2v^2v_x^2\right]$$

$$= u^2(u_x^2+v_x^2) + v^2(v_x^2+u_x^2)$$

$$\phi^2(\phi_x^2+\phi_y^2) = \underline{(u^2+v^2)}(u_x^2+v_x^2)$$

— (7)

$$\phi_x^2+\phi_y^2 = \frac{(u^2+v^2)(u_x^2+v_x^2)}{\phi^2}$$

$$\phi_x^2+\phi_y^2 = \frac{\cancel{(u^2+v^2)}(u_x^2+v_x^2)}{\cancel{(u^2+v^2)}}$$

$$= u_x^2+v_x^2. \quad \text{—— (4)}$$

We have $f'(z) = u_x + iv_x$

$$|f'(z)| = \sqrt{u_x^2+v_x^2}$$

$$\underline{|f'(z)|^2} = (u_x^2+v_x^2) \quad \text{—— (5)}$$

From (4) and (5),

we have

$$\underline{\phi_x^2+\phi_y^2} = |f'(z)|^2$$

18.

18. If $f(z)$ is Analytic, show that

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]|f(z)|^2 = 4|f'(z)|^2$$

Ans: we have $f(z) = u+iv$

$$|f(z)| = \sqrt{u^2+v^2}$$

$$|f(z)|^2 = u^2+v^2 = \phi \quad \text{(say)}$$

$$\text{LHS} = \left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]\phi$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}$$

$$= \phi_{xx} + \phi_{yy}.$$

we have show $\phi_{xx} + \phi_{yy} = 4 |f'(z)|^2$

diff (1) wrt to x partially

$$\phi_x = 2u u_x + 2v v_x$$

$$\phi_{xx} = 2[u u_{xx} + u_x u_x] + 2[v v_{xx} + v_x v_x]$$

$$\phi_{xx} = 2[u u_{xx} + (u_x)^2 + v v_{xx} + (v_x)^2] \quad (2)$$

diff (1) wrt y partially.

$$\phi_y = 2u u_y + 2v v_y$$

$$\phi_{yy} = 2[u u_{yy} + u_y u_y] + 2[v v_{yy} + v_y v_y]$$

$$\phi_{yy} = 2[u u_{yy} + u_y^2 + v v_{yy} + v_y^2] \quad (3)$$

(2) + (3).

$\phi_{xx} + \phi_{yy}$

$$= 2[u u_{xx} + u_x^2 + v v_{xx} + v_x^2] + 2[u u_{yy} + u_y^2 + v v_{yy} + v_y^2]$$

$$= 2[u(u_{xx} + u_{yy}) + v(v_{xx} + v_{yy}) + u_x^2 + v_x^2 + u_y^2 + v_y^2]$$

Since $f(z)$ is Analytic, u and v are harmonic.

$$\Rightarrow u_{xx} + u_{yy} = 0,$$

$$v_{xx} + v_{yy} = 0$$

$$\therefore \phi_{xx} + \phi_{yy} = 2[u_x^2 + u_y^2 + v_x^2 + v_y^2]$$

By C-R equation, $u_x = v_y$, and $v_x = -u_y$.

$$\Rightarrow \phi_{xx} + \phi_{yy} = 2[u_x^2 + (-v_x)^2 + v_x^2 + (u_x)^2]$$

$$= 2[u_x^2 + v_x^2 + v_x^2 + u_x^2]$$

$$= 2\left(2u_x^2 + 2v_x^2\right)$$

$$= 4\left(u_x^2 + v_x^2\right)$$

we have RHS $= f'(z) = u_x + iv_x$

$$|f'(z)| = \sqrt{u_x^2 + v_x^2}.$$

$$|f'(z)| = u_x^2 + v_x^2$$

$$\therefore \phi_{xx} + \phi_{yy} = 4\,|f'(z)|^2.$$