MODULE – 2

Introduction to Graph Theory :

Graphs :-

Definition :- A graph is a pair (V, E), where V is a nonempty set and E is a set of unordered pairs of elements taken from the set V.

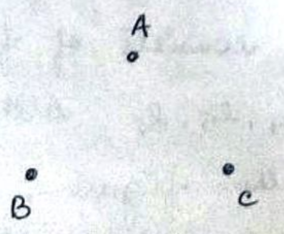
For a graph (V, E), the elements of V are called vertices and the elements of E are called undirected edges or edges. The set V is called the vertex set and the set E is called the edge set.

The graph (V, E) is also denoted by G = (V, E) or G = G(V, E) where there is no ambiguity.

NOTE: The vertex set in a graph|digraph has to be non-empty. Thus, a graph|digraph must contain at least one vertex. But, the edge set can be empty. This means that a graph|digraph need not contain any edge.

A graph|digraph containing no edges is called a null graph. A null graph with only one vertex is called a trivial graph

Example of a null graph with 3 vertices



A graph|digraph with only a finite number of vertices as well as only a finite number of edges is called finite graph|digraph; otherwise it is called an infinit graph / digraph.

Order and Size :

The number of vertices in a (finite) graph is called the order of the graph. ~~and the number of edc~~

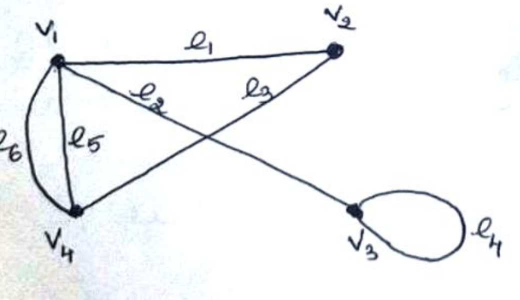
The number of edges in a graph is called the Size of the graph.

For a graph $G = (V, E)$, the cardinality of the set V, namely $|V|$ is called the order of G and the cardinality of the set E, namely $|E|$ is called the size of G.

A graph of order n and size m is called a (n, m) graph.
A null graph with n vertices is a $(n, 0)$ graph.

End vertices, loop, multiple edges :

If v_i and v_j denote two vertices of a graph and if e_k denotes ~~and~~ edge joining v_i and v_j, then v_i and v_j are called the end vertices of e_K.



We note that this graph consists of four vertices v_1, v_2, v_3, v_4 and six edges $e_1, e_2, e_3, e_4, e_5, e_6$. Although the edges e_2 and e_3 seem to intersect in the figure, their point of intersection is not a vertex of the graph. We observe that the edges e_1, e_2, e_3 have distinct end vertices, but the edge e_4 has the same vertex v_3 as both of its end vertices; that is $e_4 = \{v_3, v_3\}$. An edge such as e_4 is called a loop.

We also observe that both of the edges e_5 and e_6 have the same end vertices v_1, v_4; that is, $e_5 = \{v_1, v_4\}$ and $e_6 = \{v_1, v_4\}$. Edges such as these are called parallel edges. If in a graph there are two or more edges with the same end vertices, the edges are called multiple edges.

Simple graph, Multigraph, General graph:

A graph which does not contain loops and multiple edges is called simple graph.

A graph which does not contain a loop is called a loop-free graph.

A graph which contains multiple edges but no loops is called a multigraph.

A graph which contains multiple edges or loops is called a general graph.

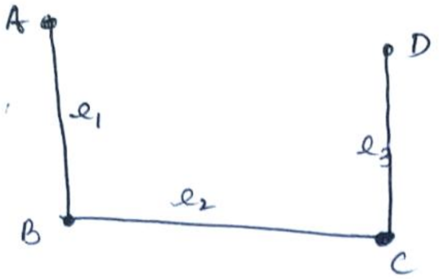
If names are assigned to vertices of a graph then the graph is called a labeled graph; otherwise unlabeled graph.

Incidence:

When a vertex v of a graph G is an end vertex of an edge e of the graph G, we say that the edge e is incident on (or to) the vertex v.

Two non-parallel edges are said to be adjacent edges if they are incident on a common vertex.

Two vertices are said to be adjacent vertices if there is an edge joining them.



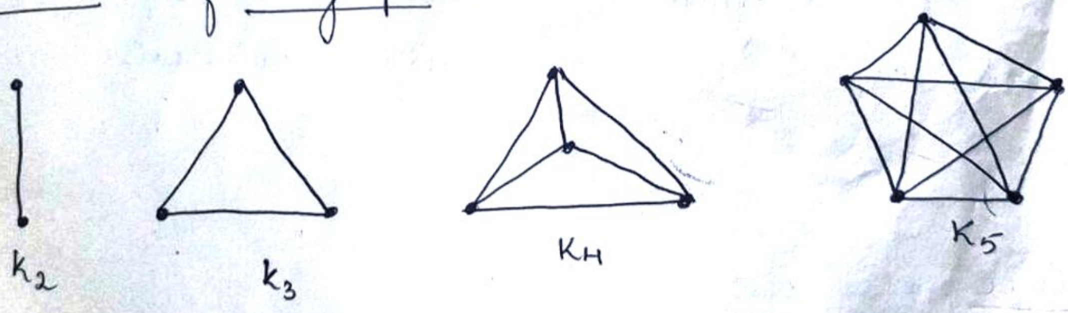
In the graph shown in figure, A and B are adjacent vertices and e_1 & e_2 are adjacent edges. But, A and C are not adjacent vertices, e_1 and e_3 are not adjacent edges.

<u>Complete graph</u> :-

A simple graph of order ≥ 2 in which there is an edge between every pair of vertices is called a <u>complete graph</u>. In other words, a complete graph is a simple graph of order ≥ 2 in which every pair of distinct vertices are adjacent.

A complete graph with $n (\geq 2)$ vertices is denoted by K_n. Complete graph with five vertices, namely K_5 is called the <u>Kuratowski's first graph</u>.

Eg:



<u>Bipartite graph</u> :

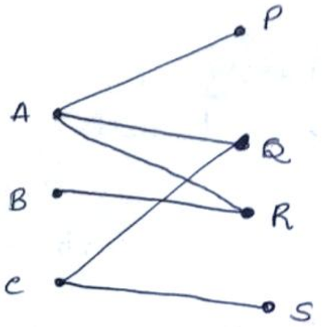
Suppose a simple graph G is such that its vertex set V is the union of two of its mutually

Bipartite graph :-

Suppose a simple graph G is such that its vertex set V is the union of two of its mutually disjoint nonempty subsets V_1 and V_2 which are such that each edge in G joins a vertex in V_1 and a vertex in V_2. Then G is called a bipartite graph. If E is the edge set of this graph, the graph is denoted by $G = (V_1, V_2; E)$ or $G = G(V_1, V_2; E)$.

The sets V_1 and V_2 are called bipartites of the vertex set V.

Eg :-



Consider the graph G as shown in figure above for which the vertex set is $V = \{A, B, C, P, Q, R, S\}$ and the edge set is $E = \{AP, AQ, AR, BR, CQ, CS\}$. Note that the set V is the union of two of its subsets $V_1 = \{A, B, C\}$ and $V_2 = \{P, Q, R, S\}$ which are such that (i) V_1 and V_2 are disjoint (ii) every edge in G joins a vertex in V_1 and a vertex in V_2, (iii) G contains no edge that joins two vertices both of which are in V_1 or V_2. This graph is a bipartic graph with $V_1 = \{A, B, C\}$ and $V_2 = \{P, Q, R, S\}$ as bipartites.

Complete Bipartite graph :

A bipartite graph $G = \{V_1, V_2; E\}$ is called a complete bipartite graph if there is an edge between every vertex in V_1 and every vertex in V_2.

① Draw a diagram of the graph $G = (V, E)$ in each of the following cases :

(i) $V = \{A, B, C, D\}$ $E = \{AB, AC, AD, CD\}$

(ii) $V = \{v_1, v_2, v_3, v_4, v_5\}$ $E = \{v_1v_2, v_1v_3, v_2v_3, v_4v_5\}$

(iii) $V = \{P, Q, R, S, T\}$ $E = \{PS, QR, QS\}$

(iv) $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. $E = \{v_1v_4, v_1v_6, v_4v_6, v_3v_2, v_3v_5, v_2v_5\}$

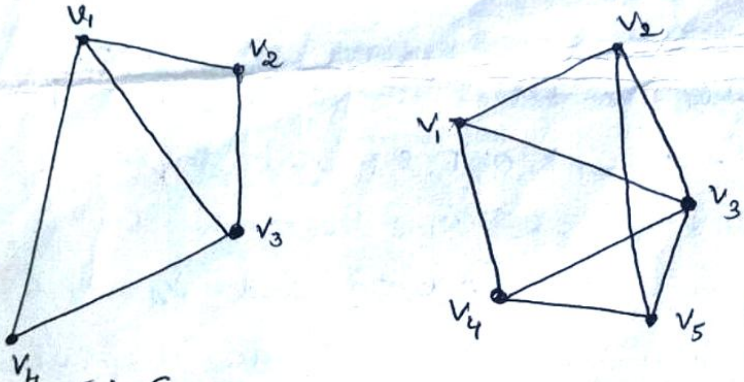
<u>Subgraphs</u> :

Given two graphs G and G_1, we say that G_1 is a subgraph of G if the following conditions hold :

(i) All the vertices and all the edges of G_1 are in G.

(ii) Each edge of G_1 has the same end vertices in G as in G_1.

Note: A subgraph is a graph which is a part of another graph.



(a) G_1 (b) G

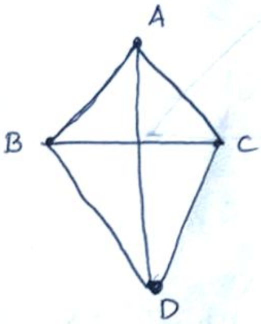
Consider the two graphs G_1 and G as shown in figures (a) & (b) respectively. We observe that all vertices and edges of the graph G_1 are in the graph G and that every edge in G_1 has the same end vertices in G as in G_1. Therefore, G_1 is a subgraph of G. ~~In the diagram of G, the~~

~~(i) Every graph~~

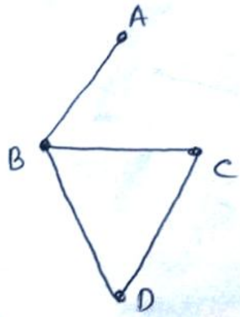
Spanning Subgraph :-

Given a graph $G = (V, E)$, if there is a subgraph $G_1 = (V_1, E_1)$ of G such that $V_1 = V$, then G_1 is called a Spanning Subgraph of G.

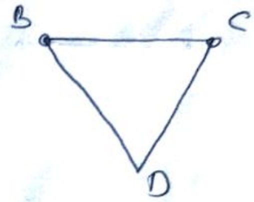
In other words, a subgraph G_1 of a graph G is a spanning subgraph of G whenever G_1 contains all vertices of G.



(a)

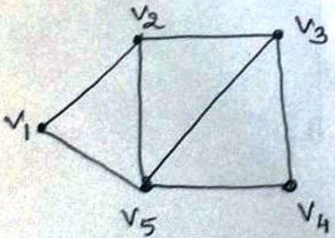


(b)

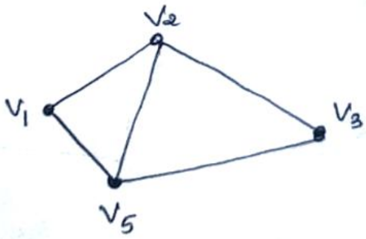


Induced Subgraph :-

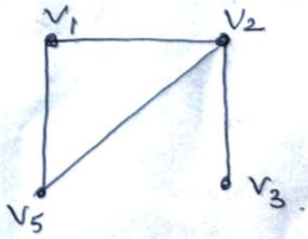
Given a graph $G = (V, E)$, suppose there is a subgraph $G_1 = (V_1, E_1)$ of G such that every edge $\{A, B\}$ of G, where $A, B \in V_1$ is an edge of G_1 also. Then G_1 is called an induced subgraph of G and is denoted by $\langle V_1 \rangle$.



Ex (a) G



(b) G_1



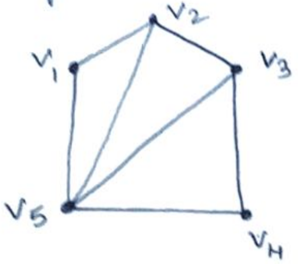
(c) G_2

For the graph G shown in figure (a), the graph G_1 (b) is an induced subgraph - induced by the set of vertices $V_1 = \{v_1, v_2, v_3, v_5\}$ where as graph G_2 (c) is not an induced subgraph.

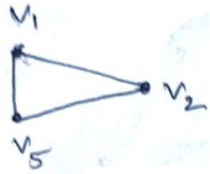
Edge - disjoint and Vertex - disjoint Subgraphs :

Let G be a graph and G_1 and G_2 be two subgraphs of G. Then (1) G_1 and G_2 are said to be edge - disjoint if they do not have any edge in common.

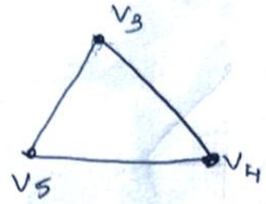
(2) G_1 and G_2 are said to be vertex - disjoint if they do not have any common edge and any common vertex.



(a) G



(b) G_1



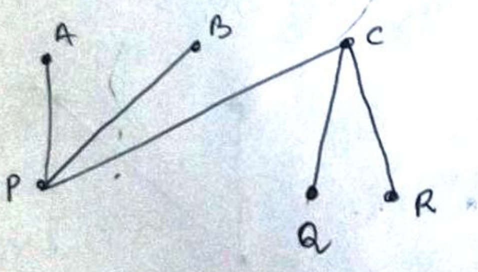
(c) G_2

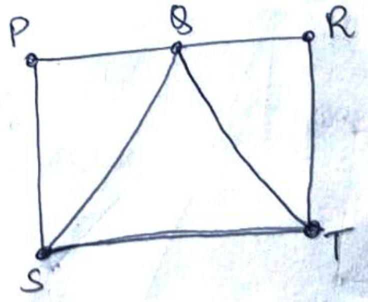
For the graph G shown in (a), the graphs G_1 and G_2 [shown in fig (b) & (c)] are edge - disjoint subgraphs.

~~Problem~~

Problems :-

1) Indicate the order and size of each of the graphs shown below





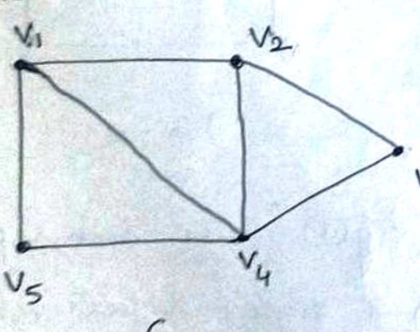
Operations on Graphs :-

Consider two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Then the graph whose vertex set is $V_1 \cup V_2$ and the edge set is $E_1 \cup E_2$ is called the Union of G_1 and G_2, it is denoted by $G_1 \cup G_2$. Thus $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$.

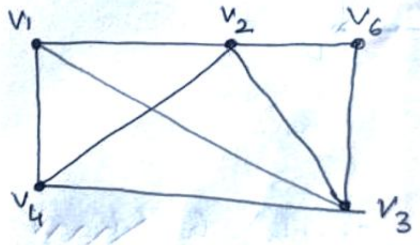
Similarly, if $V_1 \cap V_2 \neq \phi$, then the graph whose vertex set is $V_1 \cap V_2$ and the edge set is $E_1 \cap E_2$ is called the intersection of G_1 and G_2; it is denoted by $G_1 \cap G_2$.

Thus $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$, if $V_1 \cap V_2 \neq \phi$

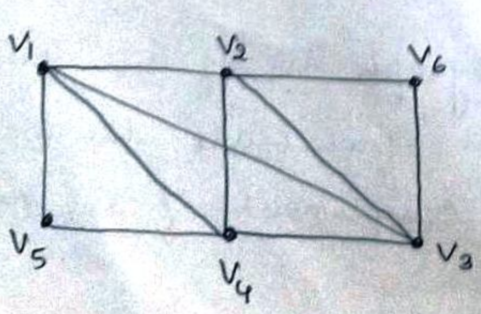
Suppose we consider the graph whose vertex set is $V_1 \cup V_2$ and the edge set is $E_1 \Delta E_2$ where $E_1 \Delta E_2$ is the symmetric difference of E_1 and E_2 i.e. $E_1 \Delta E_2 = (E_1 \cup E_2) - (E_1 \cap E_2)$. This graph is called the ring sum of G_1 and G_2. It is denoted by $G_1 \Delta G_2$.



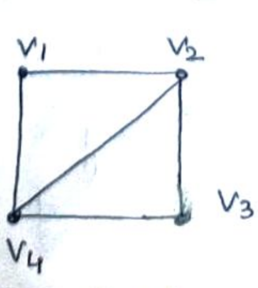
G_1



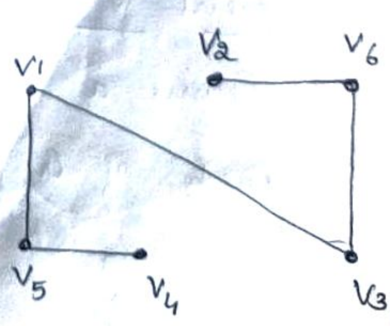
G_2



(a) $G_1 \cup G_2$



(b) $G_1 \cap G_2$



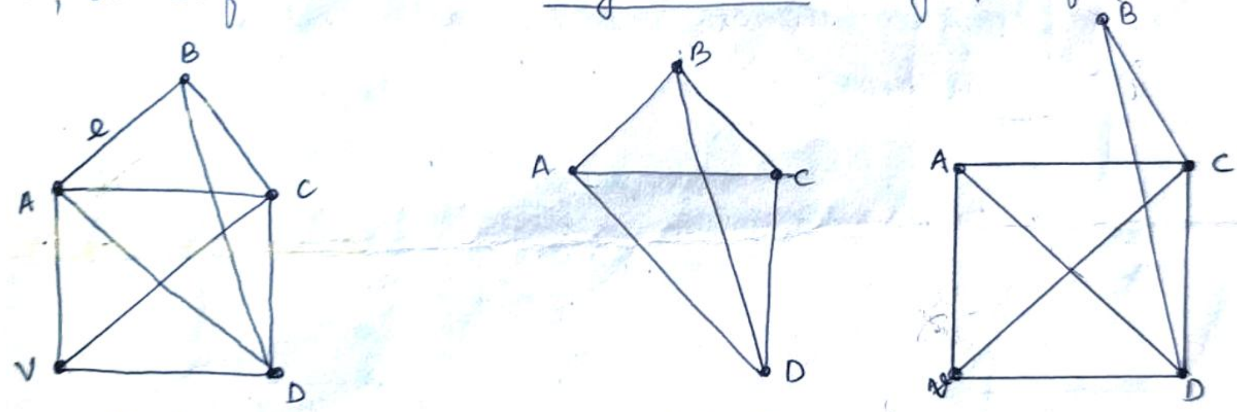
(c) $G_1 \Delta G_2$

Decomposition :-

We say that a graph G is decomposed into two subgraphs G_1 and G_2 if $G_1 \cup G_2 = G$ and $G_1 \cap G_2 =$ Null graph.

Deletion : If v is a vertex in a graph G, then $G-v$ denotes the subgraph of G obtained by deleting v and all edges incident on v, from G. This subgraph, $G-v$, is referred to as vertex - deleted subgraph of G.

If 'e' is an edge in a graph G, then $G-e$ denotes the subgraph of G obtained by deleting e from G. This subgraph $G-e$, is referred to as edge - deleted subgraph of G.



a) G.	b) $G-v$.	c) $G-e$

For the graph G ~~(fig a)~~ shown in fig(a), the subgraph $G-v$ and $G-e$ are shown in fig(b) & fig(c) respectively.

Complement of a Subgraph :-

Given a graph G and a subgraph G_1 of G, the subgraph of G obtained by deleting from G all the edges that belong to G_1 is called the complement of G_1 in G. It is denoted by $\overline{G_1}$.

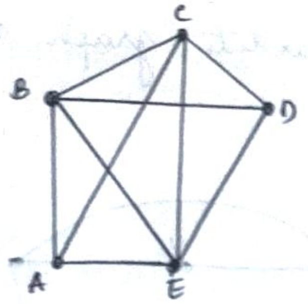


Fig a) G

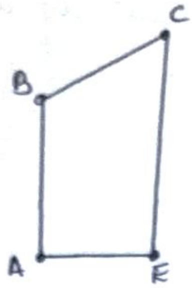


Fig b) G_1

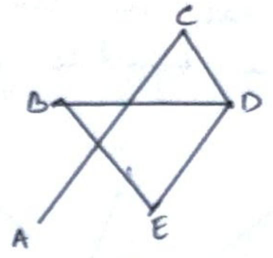


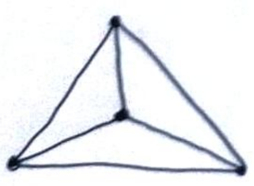
Fig (c) $\overline{G_1}$

Example: Consider the graph G as shown in Fig(a). Let G_1 be the subgraph of G as shown in Fig(b). The complement of G_1 in G, namely $\overline{G_1}$, is as shown in Fig(c)

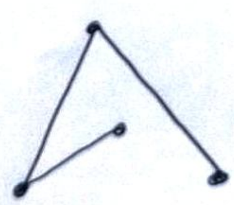
Complement of a simple graph :-

The complement $\overline{G}$ of a simple graph G with n vertices is that graph which is obtained by deleting those edges in K_n which belong to G. Thus $\overline{G} = K_n - G = K_n \Delta G$

Example: Consider a complete graph K_4 as shown in the figure (i) and a simple graph G of order 4 is shown in figure (ii). The complement, $\overline{G}$ of G is shown in figure (iii)



(i) K_4



(ii) G

(iii) $\overline{G}$

① Show that the compliment of a bipartite graph need not be a bipartite graph.

Soln]:

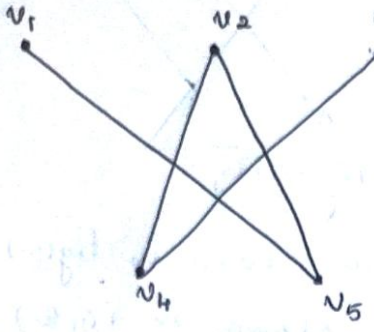


Fig (a)

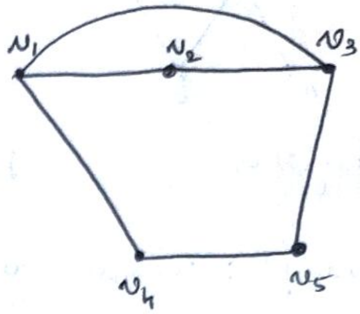


Fig (b)

Figure (a) shows a bipartite graph which is of order 5. The compliment of this graph is shown in Fig (b), this is not a bipartite graph.