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MODULE - 3

PROBABILITY DISTRIBUTION

Probability Distribution

- Discrete Random variable
  - Binomial (<25)
  - Poisson (>25)
- Continuous
  - Normal
  - Exponential

Random variable:

In a random experiment, if a real variable is associated with every outcome, then it is called a Random variable / stochastic variable.

If a random variable takes finite or countable infinite number of values then it is called a Discrete Random variable.

If a random variable takes non countable number of values, then it is called a Non-Discrete or Continuous Random variable.

Probability function & Discrete Probability Distribution:

If for each values x_i of a discrete random variable X, we assign a real number $P(x_i)$ such that $P(x_i) >= 0$ and $\sum_i P(x_i) = 1$

Then the function $P(x)$ is called the Probability function.

The set of values x_i, $P(x_i)$ is called a discrete probability distribution of the discrete random variable X.

The function $P(x)$ is called Probability Density Function (PDF) or

~~the p~~ Probability Mass Function (pmf)

$f(x) = P(X \leq x) = \sum_{i=1}^{x} P(x_i)$ is called

the cumulative distribution function (cdf) or distribution function.

Formulae:

1.) Mean $(\mu) = \sum_i x_i P(x_i)$

2.) Variance (V) (σ^2)
$$= \left(\sum_i x_i^2 P(x_i)\right) - \mu^2$$

3.) Standard deviation $(\sigma) = \sqrt{V}$.

Problems:

1. A Random variable X has the following probability function for various values of x.

x	0	1	2	3	4	5	6	7
$P(x)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

i.) Find k

ii.) Evaluate $P(x<6)$, $P(x \geq 6)$ and $P(3 < x \leq 6)$

iii.) Find probability distribution and distribution function.

Ans: we have $P(x_i) \geq 0$.
$$\sum_i P(x_i) = 1.$$

i.) $0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$

$9k + 10k^2 = 1$

$10k^2 + 9k = 1$

$10k^2 + 9k - 1 = 0$

$$k = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$k = \frac{-9 \pm \sqrt{81 - 4(10)(-1)}}{2(10)}$$

$$k = \frac{-9 \pm \sqrt{121}}{20} = \frac{-9 \pm 11}{20}$$

$$k = \frac{-9+11}{20} = \frac{+2}{20} \qquad k = \frac{-9-11}{20} = \frac{-20}{20}$$

$k = \frac{1}{10}$ or $k = -1$ but $k = \frac{1}{10} = 0.1$

ii) $P(x < 6) = P(0) + P(1) + P(2) + P(3) + P(4) + P(5)$

$= 0 + k + k + 2k + 3k + k^2$

$= 8k + k^2$

$= 8(0.1) + (0.1)^2$

$= 0.81$

$P(x \geq 6) = P(6) + P(7)$ (or)

$= 1 - P(x < 6)$

$= 1 - 0.81$

$= 0.19$

$P(3 < x \leq 6) = P(4) + P(5) + P(6)$

$= 3k + k^2 + 2k^2$

$= 3k^2 + 3k$

$= 3(0.1)^2 + 3(0.1)$

$= 0.33$

iii.)

x	0	1	2	3	4	5	6	7
$P(x)$	0	0.1	0.2	0.2	0.3	0.01	0.02	0.17

probability distribution

x	0	1	2	3	4	5	6	7
$P(x)$	0	0.1	0.2	0.2	0.3	0.01	0.02	0.17
$f(x)$	0	0.1	0.3	0.5	0.8	0.81	0.83	1

distribution function)

2. A random variable X takes the values $-3, -2, -1, 0, 1, 2, 3$ such that $P(X = 0) = P(X < 0)$ and $P(X = -3) = P(X = -2) = P(X = -1) = P(X = 1)$
$= P(X = 2) = P(X = 3)$

Find the probability distribution.

Ans:

x	-3	-2	-1	0	1	2	3
P(x)	P_1	P_2	P_3	P_4	P_5	P_6	P_7

Given : $P(X = 0) = P(X < 0)$

$\Rightarrow P_4 = P_1 + P_2 + P_3$ ——— ①

Also given

$P_1 = P_2 = P_3 = P_5 = P_6 = P_7$. ——— ②

also we have

$P_1 + P_2 + P_3 + P_4 + P_5 + P_6 + P_7 = 1$ ——— ③

from ①

$P_1 + P_2 + P_3 + P_1 + P_2 + P_3 + P_5 + P_6 + P_7 = 1$

$2P_1 + 2P_2 + 2P_3 + P_5 + P_6 + P_7 = 1$

from ②

$2P_1 + 2P_1 + 2P_1 + P_1 + P_1 + P_1 = 1$

$9P_1 = 1$

$P_1 = \frac{1}{9}$

$P_1 = 0.1111$ (equal to $P_2, P_3 \dots$ from ②)

$P_4 = P_1 + P_2 + P_3 = 3P_1$

$P_4 = \frac{1}{3}$

$P_4 = 0.333$

∴ Probability Distribution.

x	-3	-2	-1	0	1	2	3
P(x)	0.111	0.111	0.111	0.333	0.1111	0.1111	0.111

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3. If a random variable X takes the values 1, 2, 3, 4, such that $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$

Find the distribution function and cumulative distribution.

Ans:

x	1	2	3	4
$P(x)$	P_1	P_2	P_3	P_4.

also given,

$$2P_1 = 3P_2 = P_3 = 5P_4.$$

Consider, $P(X=3)$ as P_1

so $2(P(X=1)) = P_1$

$$P(X=1) = P_1/2$$

$$P(X=2) = P_1/3$$

$$P(X=4) = P_1/5$$

x	1	2	3	4
$P(x)$	$P_1/2$	$P_1/3$	P_1	$P_1/5$.

$\Rightarrow \frac{P_1}{2} + \frac{P_1}{3} + P_1 + \frac{P_1}{5} = 1$ $\quad \frac{(15+10+30+6)}{30} = \frac{61}{30}$

$$\frac{61 P_1}{30} = 1 \Rightarrow P_1 = \frac{30}{61}$$

$$P_1 = 0.4918.$$

$\therefore$

x	1	2	3	4
$P(x)$	0.2459	0.1639	0.4918	0.09836

For Cumulative distribution function.

x	1	2	3	4
$P(x)$	0.2459	0.1639	0.4918	0.09836
$F(x)$	0.2459	0.4098	0.9016	0.99996 ~ 1

4. The range of a random variable $X = \{1, 2, 3 \dots n\}$ and the probabilities of X are kx. Find the value of k and also compute mean and variance of the probability distribution.

Ans: $x = \{1, 2, 3, \dots\dots n\}$, $P(X) = kx$.

x	1	2	3	.....	n
$P(x)$	k	$2k$	$3k$		nk.

we have,

$$k + 2k + 3k + \dots\dots\dots\dots nk = 1$$

$$k(1 + 2 + 3 + \dots\dots n) = 1$$

$$k \times \frac{n(n+1)}{2} = 1$$

$$\frac{kn(n+1)}{2} = 1 \Rightarrow \boxed{k = \frac{2}{n(n+1)}}$$

Mean, $\mu = \sum_i x_i P(x_i)$

$$\mu = 1.k + 2^2 k + 3^2 k + \dots\dots + n^2 k.$$

$$= k(1^2 + 2^2 + 3^2 + \dots\dots\dots + n^2)$$

$$= k \times \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{\cancel{2}}{\cancel{n(n+1)}} \times \frac{\cancel{n(n+1)}(2n+1)}{\cancel{6}\,3}$$

$$\mu = \frac{2n+1}{3}$$

variance $(V) = \left(\sum_i x_i^2 P(x_i)\right) - \mu^2$

$$= (1^3 k + 2^3 k + 3^3 k + \dots\dots + n^3 k) - \left(\frac{2n+1}{3}\right)^2$$

$$= k(1^3 + 2^3 + 3^3 + \dots\dots\dots + n^3) - \left(\frac{2n+1}{3}\right)^2$$

$$= k \times \frac{n^2(n+1)^2}{4} - \frac{(2n+1)^2}{9}$$

$$= \frac{2}{n(n+1)} \times \frac{n^2(n+1)^2}{4_2} - \frac{(2n+1)^2}{9}$$

$$= \frac{n(n+1)}{2} - \frac{4n^2+1+4n}{9}$$

$$= \frac{n^2+n}{2} - \frac{(4n^2+1+4n)}{9}$$

$$= \frac{9n^2+9n-8n^2-8n-2}{18}$$

$$V = \frac{n^2+n-2}{18}$$

Binomial / Bernoulli's Distribution :

If p is the probability of success and q is the probability of failure, the probability of x success out of n trials, is given by

$$\boxed{P(x) = {}^nC_x\, p^x\, q^{n-x}}$$

where $p+q=1$

* Mean, $\mu = np$
* Variance $(V)(\sigma^2) = npq$
* Standard deviation $(\sigma) = \sqrt{npq}$.

Problems :

5. The probability that a person aged 60 years will live upto 70 years is 0.65. What is the probability that out of 10 persons aged 60; atleast 7 of them will live upto 70.

Ans: Given, $n = 10$,

$p = 0.65$

$q = 1 - 0.65$

$= 0.35$

By Binomial distribution.

$P(x) = {}^{n}C_x \, p^x \, q^{n-x}$

$= {}^{10}C_x \, (0.65)^x \, (0.35)^{10-x}$.

$P(x \geq 7) = P(7) + P(8) + P(9) + P(10)$

$= {}^{10}C_7 \, (0.65)^7 \, (0.35)^{10-7} +$

$= {}^{10}C_8 \, (0.65)^8 \, (0.35)^2 +$

${}^{10}C_9 \, (0.65)^9 \, (0.35)^1 +$

${}^{10}C_{10} \, (0.65)^{10} \, (0.35)^0$

$= 0.2522 + 0.1756 + 0.07249 + 0.01346$

$= 0.51375$

$= \underline{0.5138}$

6. The number of telephone lines busy at an instant of time is a binomial variate with probability 0.1 that a line is busy. If 10 lines are chosen at random, what is the probability that

i) no line is busy

ii) all lines are busy

iii) atleast 1 line is busy

iv) at most 2 lines are busy.

Ans Given, $n = 10$,

$p = 0.1$ (line is busy)

$q = 1 - 0.1$

$= 0.9$ (line is not busy)

By Binomial distribution.

$P(x) = {}^{n}C_x \, p^x \, q^{n-x}$

$= {}^{10}C_x \, (0.1)^x \, q^{10-x}$

i.) no line is busy,

$P(X=0) = {}^{10}C_0 \times (0.1)^0 (0.9)^{10}$

$= \underline{0.3486}$

ii.) all lines busy.

$P(x=10) = {}^{10}C_{10} (0.1)^{10} (0.9)^0$

$= \underline{1 \times 10^{-10}}$

$= \underline{(0.1)^{10}}$

iii.) at least 1 line is busy.

$P(x>=1) = \underbrace{P(1) + P(2) + P(3) + \cdots P(10)}_{\text{success}}$

$= 1 - P(\cancel{x} < 1)$ (failure)

$= 1 - P(0)$

$= 1 - 0.3486$

$= \underline{0.6513}$

iv.) atmost 2 lines busy.

$P(x<=2) = P(0) + P(1) + P(2)$

$= {}^{10}C_0 (0.1)^0 (0.9)^{10} +$

${}^{10}C_1 (0.1)^1 (0.9)^9 +$

${}^{10}C_2 (0.1)^2 (0.9)^8$

$= 0.3486 + 0.3874 + \cancel{0.88046}$ 0.1937

$= \cancel{1.6164}$

$= \underline{0.9298}.$

7. In a sampling, a large no. of parts manufactured by a company, the mean no. of defectives in samples of 20 is 2. Out of 1000 such samples how many would be expected to contain atleast 3 defective parts?

Ans: Given, $n = 20$ $N = 1000$

Mean $= 2$

$np = 2 \Rightarrow 20p = 2$

$\underline{p = 0.1}$

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$q = 1 - 0.1$
$= 0.9$

By binomial distribution,

$P(x) = {}^nC_x \, p^x \, q^{n-x}$
$= {}^{20}C_x \, (0.1)^x \, (0.9)^{20-x}$.

$\therefore P(x \geq 3) = P(3) + P(4) + P(5) + \cdots\cdots + P(20)$
$= 1 - P(x < 3)$
$= 1 - (P(2) + P(1) + P(0))$
$= 1 - ({}^{20}C_2 (0.1)^2 (0.9)^{18} + {}^{20}C_1 (0.1)^1 (0.9)^{19} + {}^{20}C_0 (0.1)^0 (0.9)^{20})$
$= 1 - (0.2851 + 0.2701 + 0.1215)$
$= 0.3233$
~ 0.3231.

$\therefore$ The number of samples expected to contain atleast 3 defectives parts is
$= 1000 \times 0.3231$
$= 323$

8. An airline knows that 5% of the people making reservations on a certain flight will not turn down. Consequently, their policy is to sell 52 tickets for a flight that can only hold 50 people. What is the probability that there will a seat for every passenger who turns up.

Ans

$p = 5\% = \frac{5}{100} = 0.05$ (not turn up)

$q = 1 - p = 1 - 0.05$ (turn up)
$= 0.95$

$n = 52$

For every passenger to have a seat, there should be atleast 2 passengers who will not turn up

$\Rightarrow P(x \geq 2) = P(2) + P(3) \cdots + P(52)$

$= 1 - P(x < 2)$

$= 1 - (P(1) + P(0))$

$= 1 - ({}^{52}C_1 (0.05)^1 (0.95)^{51} + {}^{52}C_0 \times 0.95^{52}$

$= 1 - (0.1900 + 0.06944)$

$= \underline{0.7405}$

The probability that a passenger will not turn up

$p = 5\% = \underline{0.05}$, $q = \underline{0.95}$

& We have

$$P(x) = {}^nC_x \, p^x \, q^{n-x}$$

$$= {}^{52}C_x \, (0.05)^x \, (0.95)^{52-x}$$

We know that, probability of seat available for every passenger who turns up = probability of atleast 2 passengers will not turn up.

(as) written above)

$= \underline{0.7405}$

$\therefore$ Probability that the seat is available for every ~~perp~~ passenger is $0.\underline{7405}$.

Poisson Distribution :

Formula :

$$\boxed{P(x) = \frac{m^x \, e^{-m}}{x!}}$$

$P(x) \rightarrow$ Poisson distribution function.

$x \rightarrow$ Poisson variate.

- Mean, $\mu = m$
- variance, $\sigma^2 = m$
- S.D, $(\sigma) = \sqrt{m}$

* $\boxed{m = np}$

Poisson distribution is applied when,
$n \to \infty$ and $p \to 0$.

9. In a certain factor, turning of the razor blades, there is a small probability of 1/500 for any plate to be defective. The plates are supplied in packets of 10. Use Poisson distribution to calculate the approximate number of packets containing.

i.) no defective
ii.) 1 defective
iii.) 2 defective blades

in a consignment of 10,000 packets.

Ans: $p = \frac{1}{500} = \underline{0.002}$

$n = 10$, $N = 10000$

we have mean, $m = np$
$= 10 \times 0.002$
$= \underline{0.02}$

By Poisson distribution.

we have, $P(x) = \frac{m^x e^{-m}}{x!}$

$$P(x) = \frac{(0.02)^x e^{-0.02}}{x!}$$

i.) $P(x=0) = \frac{(0.02)^0 e^{-0.02}}{0}$

$= 0.9802$

Number of packets containing no defective blades is, $= 10000 \times 0.9802$

$= \underline{9802}$

ii.) $P(x=1) = \frac{(0.02)^1 e^{-0.02}}{1!}$

$= 0.0196$

$\therefore$ No. of packets containing 1 defective blades is, 10000×0.0196

$= \underline{196}$

iii.) $P(x=2) = \frac{(0.02)^2 e^{-0.02}}{2!}$

$= 0.0002$

$\therefore$ No. of packets containing 2 defective blades is $= 10000 \times 0.0002$

$= \underline{2}$

10. If the probability of bad reaction from a certain injection is ~~0.0001~~ 0.001, determine the chance, that out of 2000 individuals, more than 2 will get a bad reaction.

Ans

$n = 2000$

$p = 0.001$

$m = np = 2000 \times 0.001$

$= 2$

By poison distribution, we have

$$p(x) = \frac{m^x e^{-m}}{x!}$$

i.) $p(x>2) = p(x=3) + p(4) + \dots$

$= 1 - [p(x=0) + p(1) + p(2)]$

$= 1 - \left[\frac{m^0 e^{-2}}{0!} + \frac{m^1 e^{-2}}{1!} + \frac{m^2 e^{-2}}{2!}\right]$

$= 1 - \left[e^{-2} + 2e^{-2} + \frac{4e^{-2}}{2}\right]$

$= 0.3223$

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11. 2% of the fuses manufactured by a company are found to be defective. Find the probability that a box containing 200 fuses contains

i.) No defective fuses

ii.) 3 or more of defective fuses.

Ans:

$$p = 2\% = \frac{2}{100} = 0.02$$

$$n = 200$$

$$m = np = 200 \times 0.02$$

$$m = 4$$

By poisson distribution,

$$P(x) = \frac{m^x e^{-m}}{x!}$$

i.) No defective fuses

$$P(x=0) = \frac{m^0 e^{-m}}{0!} = \frac{4^0 (e^{-4})}{1}$$

$$= 0.0183$$

ii.) 3 or more defective fuses

$$P(x \geq 3) = P(3) + P(4) + \dots + P(200)$$

$$= 1 - [P(0) + P(1) + P(2)]$$

$$= 1 - \left[\frac{m^0 e^{-m}}{0!} + \frac{m^1 e^{-m}}{1!} + \frac{m^2 e^{-m}}{2!}\right]$$

$$= 1 - \left[1 e^{-4} + 4 e^{-4} + \frac{\overset{8}{\cancel{16}} e^{-4}}{\cancel{2}}\right]$$

$$= 1 - [13 e^{-4}]$$

$$\cancel{= 0.2381} = 0.7618.$$

∴ Probability of no defective fuses is 0.0183.

& Probability of 3 or more defective fuses is 0.7618

Continuous Probability Distribution :

If for every x, belonging to the range of a continuous random variable X, we assign a real number satisfying the condition,

$$\boxed{\int_{-\infty}^{\infty} f(x)\, dx = 1} \quad ; \quad f(x) \geq 0$$

then $f(x)$ is called a continuation probability function or Probability Density Function (PDF)

If X is a continuous random variable with PDF $f(x)$ then the function $\boxed{F(x) = P(X \leq x) = \int_{-x}^{x} f(x)\, dx}$

is called cumulative distribution function (CDF)

Formulae :

- Mean $\mu = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$
- Variance, $V(\sigma^2) = \left(\int_{-\infty}^{\infty} x^2 \cdot f(x)\, dx\right) - \mu^2$
- ~~Standar~~ $= \left(\int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx\right)$
- Standard Deviation, $\sigma = \sqrt{V}$.

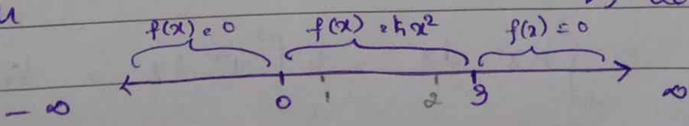
(imp) 12. Find the constant k such that $f(x) = \begin{cases} kx^2, & 0 < x < 3 \\ 0 & \text{otherwise} \end{cases}$

is a PDF. Also find i.) $P(1 < x < 2)$

ii.) $P(x \leq 1)$ iii.) $P(x > 1)$

iv.) Mean v.) Variance

Ans:-



we have $f(x) \geq 0$

$$\int_{-\infty}^{\infty} f(x)\cdot dx = 1$$

$$\Rightarrow \int_{-\infty}^{0} f(x)\cdot dx + \int_{0}^{3} f(x)\cdot dx + \int_{3}^{\infty} f(x)\cdot dx = 1$$

$$\Rightarrow \int_{0}^{3} kx^2\cdot dx = 1$$

$$\Rightarrow k\int_{0}^{3} (x^2)\cdot dx = 1$$

$$\Rightarrow k\left[\frac{x^3}{3}\right]_0^3 = 1 \qquad \Rightarrow \frac{k}{3}\left[3^3 - 0^3\right] = \frac{k}{3}[27] = 1$$

$$\boxed{k = \frac{1}{9}}$$

i.) $P(1 < x < 2) = \int_{1}^{2} f(x)\cdot dx = \int_{1}^{2} kx^2\cdot dx.$

$$= k\int_{1}^{2} x^2\cdot dx = k\left[\frac{x^3}{3}\right]_1^2$$

$$= \frac{k}{3}\left[2^3 - 1^3\right] = \frac{k}{3}[8 - 1]$$

$$= \frac{1}{9\times 3}[7] = \frac{7}{27}$$

ii.) $P(x \le 1) = \int_{-\infty}^{1} f(x)\cdot dx = \int_{-\infty}^{0} f(x)\cdot dx + \int_{0}^{1} f(x)\cdot dx$

$$= \int_{0}^{1} kx^2\cdot dx = k\int_{0}^{1} x^2\cdot dx = k\left[\frac{x^3}{3}\right]_0^1$$

$$= \frac{k}{3}\left[1^3 - 0^3\right] = \frac{1}{9}\times\frac{1}{3}[1] = \frac{1}{27}$$

iii.) $P(x > 1) = \int_{1}^{\infty} f(x)\cdot dx = \int_{1}^{3} f(x)\cdot dx + \int_{3}^{\infty} f(x)\cdot dx$

$$= \int_{1}^{3} kx^2\cdot dx = k\int_{1}^{3} x^2\cdot dx = k\left[\frac{x^3}{3}\right]_1^3$$

$$= \frac{k}{3}\left[3^3 - 1^3\right] = \frac{1}{9}\times\frac{1}{3}[27 - 1] = \frac{26}{27}$$

iv.) Mean, $\mu = \int_{-\infty}^{\infty} x \cdot f(x) \cdot dx = \int_{0}^{3} x \cdot f(x) \cdot dx$

$= \int_0^3 x \cdot kx^2 \cdot dx = k \int_0^3 x^3 \cdot dx$

$= k \left[\frac{x^4}{4}\right]_0^3 = \frac{k}{4}\left[3^4 - 0^4\right] = \frac{k}{4}\left[81\right]$

$= \frac{1 \times \cancel{81}^{9}}{\cancel{9} \times 4} = \underline{\underline{\frac{9}{4}}}$

v.) Variance, $\sigma^2 = \left(\int_{-\infty}^{\infty} x^2 \cdot f(x) \cdot dx\right) - \mu^2$

$= \left(\int_0^3 x^2 \cdot kx^2 \cdot dx\right) - \left(\frac{9}{4}\right)^2$

$= k \int_0^3 x^4 \cdot dx - \frac{81}{16}$

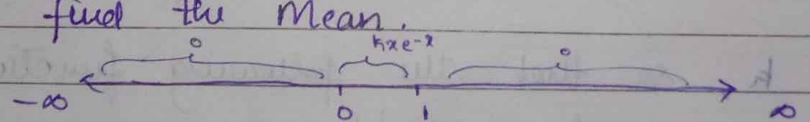
$= k\left[\frac{x^5}{5}\right]_0^3 - \frac{81}{16} = \frac{k}{5}\left[3^5 - 0^5\right] - \frac{81}{16}$

$= \frac{1}{\cancel{9} \times 5}(\cancel{3^5})^{3^3} - \frac{81}{16} = \frac{27}{5} - \frac{81}{16}$

$= \underline{\underline{\frac{27}{80}}}$

13. Find k such that $f(x) = \begin{cases} kxe^{-x} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$ is a PDF and also find the Mean.

Ans:



we have $f(x) \geq 0$. & $\int_{-\infty}^{\infty} f(x) \cdot dx = 1$

$\int_{-\infty}^{0} f(x) \cdot dx + \int_0^1 f(x) \cdot dx + \int_1^{\infty} f(x) \cdot dx = 1$

$\int_0^1 kxe^{-x} \cdot dx = 1$

$\Rightarrow k \int_0^1 x e^{-x} dx = 1 \Rightarrow \cancel{k \int_0^1}$

$\Rightarrow k \left[x \frac{e^{-x}}{-1} - (1)\frac{e^{-x}}{(-1)(-1)} \right]_0^1 = 1$

$\Rightarrow k \left[\frac{-x e^{-x}}{1} - \frac{e^{-x}}{1} \right]_0^1 = 1 \Rightarrow k\left[(-1e^{-1} - e^{-1}) - (0e^{-0} - e^{0}) \right] = 1$

$\Rightarrow k(-2e^{-1} + 1) = 1$

$\boxed{k = \left(\frac{-2}{e} + 1\right) = 1}$

Mean, $\mu = \int_{-\infty}^{\infty} x \cdot f(x) \cdot dx$

$= \int_0^1 x \cdot kx e^{-x} \cdot dx = k \int_0^1 x^2 e^{-x} \cdot dx$

$= k \left[x^2 \left(\frac{e^{-x}}{-1}\right) - (2x)\left(\frac{e^{-x}}{1}\right) + 2\left(\frac{e^{-x}}{-1}\right) \right]_0^1$

$= k\left[-x^2 e^{-x} - 2x e^{-x} - 2e^{-x} \right]_0^1$

$= k\left[(-1e^{-1} + 0) - (2(1)e^{-1} - 0) - (2e^{-1} - 2e^{0}) \right]$

$= k\left[-e^{-1} - 2e^{-1} - 2e^{-1} - 2(1) \right]$

$= k\left[-5e^{-1} - 2 \right]$

$= 3.8393\, k$

$= 3.8393 \left(\left(\frac{-2}{e} + 1\right) \right)$

$= \cancel{1.0105}$ or $= \frac{2e - 5}{e - 2}$

14. Find k, so that the following function can be a PDF of a random variable,

$$f(x) = \begin{cases} 0 & \text{for } x \le 0 \\ kx e^{-4x^2} & \text{for } x > 0 \end{cases}$$

Ans:

$-\infty \quad \overbrace{\qquad}^{0} \quad 0 \quad \overbrace{\qquad}^{kxe^{-4x^2}} \quad \infty$

We have $\int_{-\infty}^{\infty} f(x) \cdot dx = 1 \quad (f(x) \ge 0)$

$$\Rightarrow \int_0^\infty kx \cdot e^{-4x^2} \cdot dx = 1$$

take $-4x^2 = t$

$-8x = \frac{dt}{dx}$

$x\,dx = \frac{-dt}{8}$

$\cancel{k\left[\frac{x \cdot e^{-4x^2}}{-4x}\right]}$

$$\Rightarrow k \int_0^\infty e^t \left(\frac{-dt}{8}\right) = 1$$

$$\Rightarrow -\frac{k}{8}\left[e^t\right]_0^\infty = 1 \Rightarrow -\frac{k}{8}\left[e^{-4x^2}\right]_0^\infty = 1$$

$$\Rightarrow -\frac{k}{8}\left[e^{-\infty} - e^0\right] = 1 \Rightarrow -\frac{k}{8}[0 - 1] = 1$$

$$\Rightarrow \frac{k}{8} = 1 \Rightarrow \underline{k = 8}.$$

15. Find the CDF for the following PDF of a random variable x,

$$f(x) = \begin{cases} 6x - 6x^2, & 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Date / /
Page

Ans: CDF, $F(x) = \int_{-\infty}^{x} f(x)\cdot dx = \int_{0}^{x} f(x)\cdot dx$

$$= \int_0^x (6x - 6x^2)\cdot dx$$

$$= \cancel{6}^{3}\left(\frac{x^2}{\cancel{2}}\right)_0^x - {}^{2}\cancel{6}\,\frac{x^3}{\cancel{3}}\Big|_0^x = 3(x^2 - 0) - 2(x^3 - 0^3)$$

$$= \underline{3x^2 - 2x^3}$$

16. Find the value of 'c' such that $f(x) = \begin{cases} \frac{x}{6} + c & , 0 \leq x \leq 3 \\ 0 & , \text{otherwise} \end{cases}$

is a PDF, also find $P(1 \leq x \leq 2)$

Ans: $f(x) = \int_0^3 \left(\frac{x}{6} + c\right)\cdot dx = 1$

$$= \frac{x^2}{12}\Big|_0^3 + cx\Big|_0^3 = \left(\frac{3^2}{12} - 0\right) + c(3 - 0) = 1$$

$$= \frac{9}{12} + 3c = \frac{3}{4} + 3c = 1 \Rightarrow 3c = 1 - \frac{3}{4} = \frac{4-3}{4} = \frac{1}{4}$$

~~$P(1 \leq x \leq 2) =$~~

$$c = \frac{1}{4 \times 3} = \frac{1}{12}$$

$$P(1 \leq x \leq 2) = \int_1^2 \left(\frac{x}{6} + \frac{1}{12}\right)\cdot dx$$

$$= \frac{x^2}{12}\Big|_1^2 + \frac{1}{12}x\Big|_1^2 = \left(\frac{4}{12} - \frac{1}{12}\right) + \left(\frac{1}{12}(2-1)\right)$$

$$= \left(\frac{3}{12}\right) + \frac{1}{12} = \frac{4}{12} = \underline{\frac{1}{3}}$$

7/7/23

MODULE - 3

Probability Distribution:

Exponential Distribution :

The continuous probability distribution having the PDF $f(x)$ given by,

$$f(x) = \begin{cases} \alpha e^{-\alpha x} & x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

where $\alpha > 0$ is known as the exponential distribution.

- Mean $\mu = \dfrac{1}{\alpha}$
- Variance, $\sigma^2 = \dfrac{1}{\alpha^2}$
- Standard Deviation $(\sigma) = \dfrac{1}{\alpha}$.

17 The length of a telephone ~~communication~~ conversation in booth, has been an exponential distribution and found on an ~~edge~~ average to be 5 minutes. Find the probability that a random call made from the booth,

i.) ends less than 5 mins

ii.) between 5 and 10 mins.

Ans, Average / Mean, $\mu = 5$

we have, $f(x) = \begin{cases} \alpha e^{-\alpha x} & x > 0 \\ 0 & \text{otherwise.} \end{cases}$

Mean, $\mu = \dfrac{1}{\alpha}$

$$5 = \frac{1}{\alpha} \Rightarrow \alpha = \frac{1}{5}$$

Now, $f(x) = \begin{cases} \frac{1}{5} e^{-\frac{1}{5}x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$

i.) $P(x<5) = \int_0^5 f(x) \cdot dx$

$$= \int_0^5 \frac{1}{5} \cdot e^{-1/5 x} \cdot dx$$

$$= \frac{1}{5}\left[\frac{e^{-1/5 x}}{-1/5}\right]_0^5$$

$$= \frac{-5}{5}\left[e^{-1/5(5)} - e^0\right]$$

$$= -1\left[e^{-1} - e^0\right]$$

$$= -(1 - 0.6321)$$

$$= \underline{0.6321}$$

ii.) $P(5 \le x < 10) = \int_5^{10} f(x) \cdot dx$

$$= \int_5^{10} \frac{1}{5} e^{-1/5(x)} \cdot dx$$

$$= \frac{1}{5}\left[\frac{e^{-1/5(x)}}{-1/5}\right]_5^{10}$$

$$= -1\left[e^{-1/5(10)} - e^{-1/5(5)}\right]$$

$$= -1\left[e^{-2} - e^{-1}\right] = -1(0 - 0.2325)$$

$$= \underline{0.2325}$$

18. In a certain town, the duration of a shower is exponentially distributed with mean 5 mins. What is the probability that a shower will last for

i.) 10 mins or more

ii.) less than 10 mins

iii.) between 10 and 12 mins.

Ans: Average / mean $\mu = 5$

$$\frac{1}{\alpha} = 5$$

$$\alpha = \frac{1}{5}$$

we have $f(x) = \begin{cases} \alpha e^{-\alpha x} & , x > 0 \\ 0 & \text{otherwise.} \end{cases}$

$= \begin{cases} 1/5\, e^{-1/5 x} & , x > 0 \\ 0 & \text{otherwise.} \end{cases}$

i) $P(x \geq 10) = \int_{10}^{\infty} f(x) \cdot dx$

$= \int_{10}^{\infty} \frac{1}{5} e^{-1/5 x} \cdot dx$

$= \frac{1}{\cancel{5}} \left[\frac{e^{-1/5 x}}{-1/\cancel{5}} \right]_{10}^{\infty} = -\left[e^{-1/5(\infty)} - e^{-1/5(10)} \right]$

$= -\left[e^{\infty} - e^{-2} \right]$

$= -\left[0 - e^{-2} \right] = +e^{-2}$

$= \underline{0.1353}$

ii) $P(x < 10) = \int_{0}^{10} f(x) \cdot dx$

$= \int_{0}^{10} \frac{1}{5} e^{-1/5 x} \cdot dx$

$= \frac{1}{\cancel{5}} \left[\frac{e^{-1/5 x}}{-1/5} \right]_{0}^{10} = -1 \left[e^{-1/5(10)} - e^{-1/5(0)} \right]$

$= -\left[e^{-2} - e^{0} \right]$

$= \underline{0.8646}$

iii) $P(10 < x < 12) = \int_{10}^{12} f(x) \cdot dx$

$= \int_{10}^{12} \frac{1}{5} e^{-1/5 x} \cdot dx =$

$= \frac{1}{\cancel{5}} \left[\frac{e^{-1/5 x}}{-1/\cancel{5}} \right]_{10}^{12} = -1 \left[e^{-1/5(12)} - e^{-1/5(10)} \right]$

$= -\left[e^{-2.4} - e^{-2} \right] = -\left[-0.044617 \right]$

$= \underline{0.04461}$

$\alpha = +$ (add)

19. If 'x' is an exponential variate with mean 5. Evaluate the following

i.) $P(-\infty < x < 10)$

ii.) $P(x \leq 0 \text{ or } x \geq 1)$

Ans Mean $= 5$ $\mu = 5$

$$\frac{1}{\alpha} = 5 \Rightarrow \alpha = \frac{1}{5}$$

we have $f(x) = \begin{cases} \alpha e^{-\alpha x}, & x > 0 \\ 0 & \text{otherwise} \end{cases}$

i.) $P(-\infty < x < 10) = \int_{-\infty}^{10} f(x)\cdot dx = \int_{-\infty}^{0} f(x)\cdot dx + \int_{0}^{10} f(x)\, dx$

$$= \int_0^{10} \frac{1}{5} e^{-1/5 x}\cdot dx$$

$$= \frac{1}{5}\left[\frac{e^{-1/5 x}}{-1/5}\right]_0^{10} = -1\left[e^{-1/5(10)} - e^{-0}\right]$$

$$= -[e^{-2} - 1] = 1 - e^{-2}$$

$$= \underline{0.8647}$$

ii.) $P(x \leq 0 \text{ or } x \geq 1) = P(x \leq 0) + P(x \geq 1)$

$$= \int_{-\infty}^{0} f(x)\cdot dx + \int_1^{\infty} f(x)\cdot dx$$

$$= 0 + \int_1^{\infty} \frac{1}{5} e^{-1/5 x}\cdot dx$$

$$= \frac{1}{5}\left[\frac{e^{-1/5 x}}{-1/5}\right]_1^{\infty}$$

$$= -1\left[e^{-\infty} - e^{-1/5}\right] = -1(-e^{-1/5})$$

$$= \underline{0.8187}$$

<u>Normal Distribution</u>:

The continuous probability distribution having probability density function (pdf), that is written as

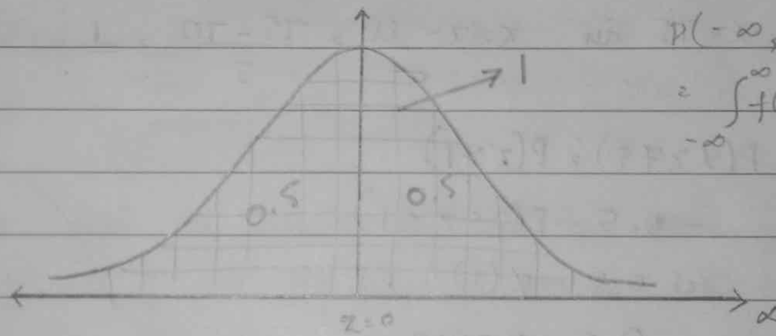
$$\boxed{f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}}$$

where $\sigma > 0$, is known as normal distribution.

Standard Normal distribution:

$$\boxed{z = \frac{x-\mu}{\sigma}}$$

where x is normal variate and
z is standard normal variate &
μ is mean &
σ is standard deviation of standard normal distribution.



$P(-\infty < z < \infty) = \int_{-\infty}^{\infty} f(x)\,dx = 1$

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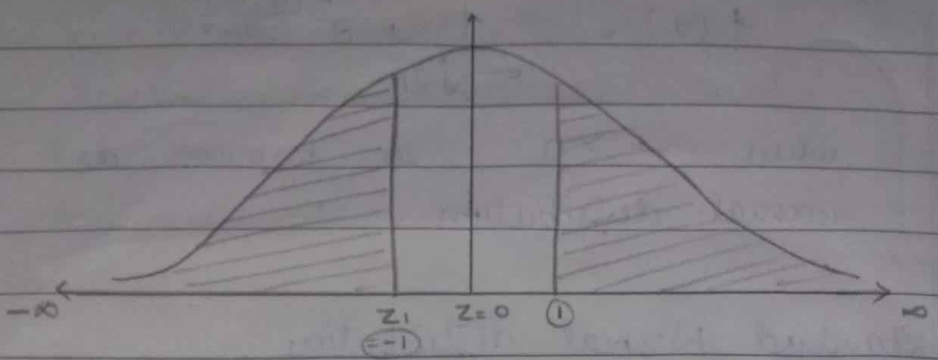
20 The marks of 1000 students in an examination follows a normal distribution with mean 70 and standard deviation 5. Find the number of students whose marks will be i.) less than 65
ii.) more than 75
iii.) 65 to 75.

Ans: Mean, $\mu = 70$, S.D $(\sigma) = 5$, $N = 1000$

i.) $P(x < 65) = ?$

put $x = 65$ in $z = \frac{x-\mu}{\sigma} = \frac{65-70}{5} = \underline{\underline{-1}}$

$P(x<65) = P(z < -1)$



$P(x<65) = P(z<-1) = P(z>1)$

$= 0.5 - P(0 \le z \le 1) = 0.5 - \phi(1)$

$= 0.5 - 0.3413$

$= \underline{0.1587}$.

$\therefore$ No. of students scoring less than 65 marks.

$= 1000 \times 0.1587 = 158.7 \approx \underline{159}$.

ii.) $P(x>75) = ?$

$x = 75$ in $z = \frac{x-\mu}{\sigma} = \frac{75-70}{5} = \underline{1}$.

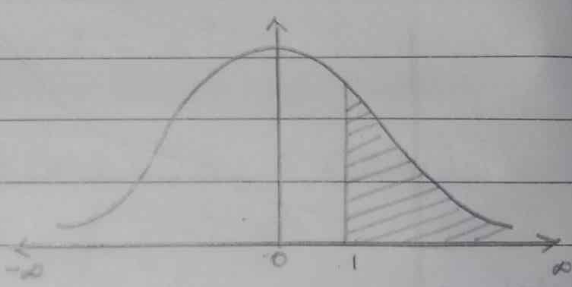
$P(x>75) = P(z>1)$

$= 0.5 - P(0 \le z \le 1)$

$= 0.5 - \phi(1)$

$= 0.5 - 0.3413$

$= \underline{0.1587}$



$\therefore$ No. of students scoring more than 75 marks

$= 1000 \times 0.1587 = 158.7 \approx \underline{159}$.

iii.) $P(65 \le x \le 75) = ?$

when $x = 65$, $z = -1$

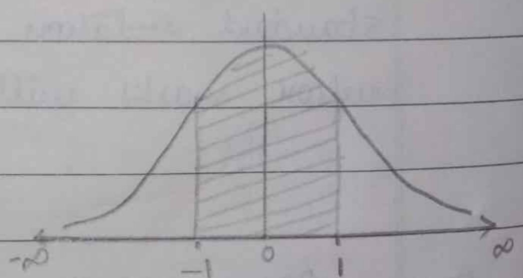
$x = 75$, $z = +1$

$P(65 \le x \le 75) = P(-1 \le z \le 1)$

$= 2 \times P(0 \le z \le 1)$

$= 2\,\phi(1) = 2 \times 0.3413$

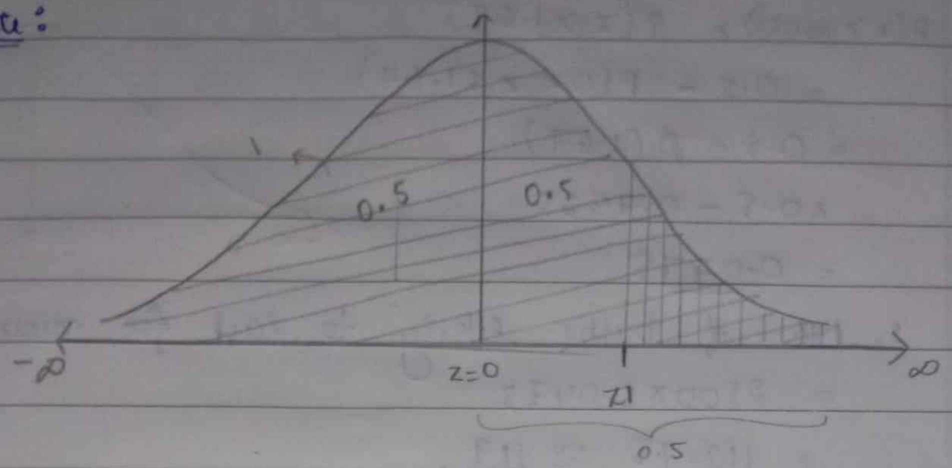
$= \underline{0.6826}$



$\therefore$ No. of students scoring b/w 65 to 75 marks is

$= 1000 \times 0.6826 = 682.6 \approx \underline{683}$.

Note:



- $P(-\infty < z < \infty) = \int_{-\infty}^{\infty} f(x).dx = 1$

- $P(0 \le z < \infty) = P(-\infty < z \le 0) = 0.5$
- $P(z_1 < z < \infty) = 0.5 - P(0 \le z \le z_1)$

$= 0.5 - \phi(z_1)$

$\rightarrow \phi(z_1) = \frac{1}{\sqrt{2\pi}} \int_0^{z_1} e^{-z^2/2} . dz.$

21. In a test on Electric bulbs it was found that the lifetime of a particular brand distributed normally, with an average life of 2000 hrs and standard deviation of 60 ~~6~~ hours. If a firm purchases 2500 bulbs, find the number of bulbs that are likely to last for

i.) More than ~~200~~ 2100 hrs

ii.) less than 1950 hrs

iii.) between 1900 ~~and~~ to 2100 hours.

Ans: Average / Mean = 2000

$SD (\sigma) = 60$

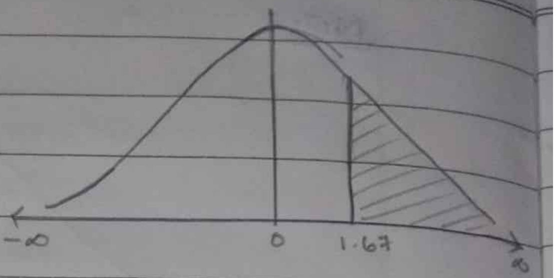
$N = 2500.$

i.) $P(x > 2100) = ?$

$x = 2100, \quad z = \frac{x-\mu}{\sigma} = \frac{2100-2000}{60} = \frac{100}{60} = \underline{1.67}$

$\phi(1.67) = 0.4525$
$\phi(0.83) = 0.2967$

$P(x > 2100) = P(z > 1.67)$
$= 0.5 - P(0 < z < 1.67)$
$= 0.5 - \phi(1.67)$
$= 0.5 - 0.4525$
$= \underline{0.0475}$

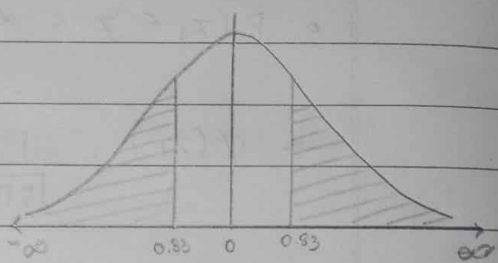


$\therefore$ No. of bulbs likely to last for more than 2100 hrs
$= 2500 \times 0.0475$
$= 118.75 \approx \underline{119}$.

ii) $P(x < 1950) = ?$

$x = 1950, \Rightarrow z = \dfrac{1950 - 2000}{60} = \dfrac{-50}{60} = \underline{-0.833}$

$P(x < 1950) = P(z < -0.83)$
$= P(z > 0.83)$
$= 0.5 - P(0 < z < 0.83)$
$= 0.5 - \phi(0.83)$
$= 0.5 - 0.2967$
$= \underline{0.2033}$

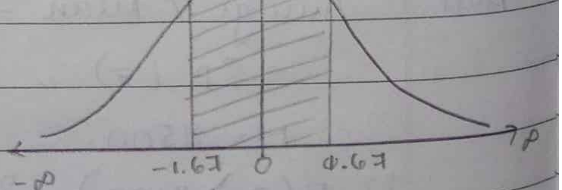


$\therefore$ No. of bulbs likely to last for ~~more~~ less than 1950 hours $= 2500 \times 0.2033$
$= 508.25 \approx \underline{508}$.

iii) $P(1900 < x < 2100)$
when $x = 1900$, $z = \dfrac{1900 - 2000}{60} = \underline{-1.67}$

$x = 2100, \quad z = +\underline{1.67}$

$P(1900 < x < 2100)$
$= P(-1.67 < z < \cancel{2100}\ 1.67)$
$= 2(P(0 < z < 1.67))$
$= 2\,\phi(1.67)$
$= 2(0.4525)$
$= \underline{0.905}$



∴ No. of bulbs likely having lifetime between

1900 - 2100 hrs $= 2500 \times 0.905$

$= 2262.5$

$= \underline{\underline{2262}}$

$\therefore$ No. of bulbs likely having lifetime between 1900 - 2100 hrs $= 2500 \times 0.905$

$= 2262.5$

$= \underline{\underline{2262}}$

11/07/23

22. In a normal distribution, 31% of the items are under 45 and 8% of the items are over 64. Find the mean and standard deviation of the distribution.

Ans Given $P(x < 45) = 31\% = \frac{31}{100} = \underline{\underline{0.31}}$

$P(x > 64) = 8\% = \frac{8}{100} = \underline{\underline{0.08}}$

Put $x = 45$, in $z = \frac{x-\mu}{\sigma}$

$\Rightarrow z = \frac{45-\mu}{\sigma} = z_1$ (say.) —— (1)

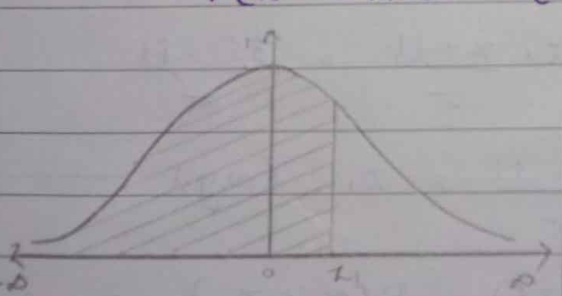
$\Rightarrow P(x < 45) = P(z < z_1)$

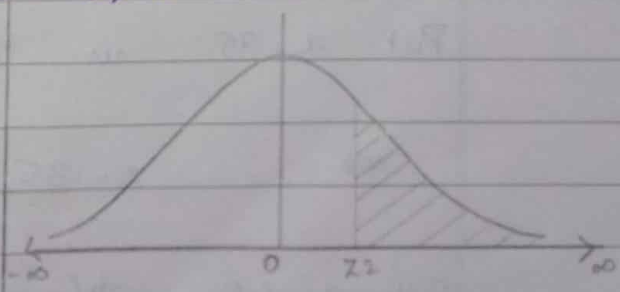
$0.31 = P(z < z_1)$

Put $x = 64$ in $z = \frac{x-\mu}{\sigma}$

$z = \frac{64-\mu}{\sigma} = z_2$ (say) —— (2)

$\Rightarrow P(x > 64) = P(z > z_2) = 0.08$





$P(z < z_1) = P(x < 45) = 0.31$	$P(z > z_2) = P(x > 64) = 0.08$
$\Rightarrow 0.5 + P(0 \leq z \leq z_1) = 0.31$	$\Rightarrow 0.5 - P(0 \leq z \leq z_2) = 0.08$

$0.5 + \phi(z_1) = 0.31$	$0.5 - \phi(z_2) = 0.08$
$\phi(z_1) = 0.31 - 0.5$	$\phi(z_2) = 0.5 - 0.08$
$\phi(z_1) = \underline{-0.19}$	$= \underline{0.42}$

We have $\phi(0.5) = 0.1915 \approx 0.19$ &

$\phi(1.4) = 0.4192 \approx 0.42$

$\therefore z_1 = \underline{-0.5}$ & $z_2 = \underline{1.4}$

Substitute in (1) and (2)

$\frac{45 - \mu}{\sigma} = -0.5$	$\frac{64 - \mu}{\sigma} = 1.4$
$45 - \mu = -0.5\sigma$	$64 - \mu = 1.4\sigma$
$\Rightarrow \mu - 0.5\sigma = 45$	$\mu + 1.4\sigma = 64.$

$\underline{\mu = 50}$

$\underline{\sigma = 10}$

23. In an examination 7% of students scores less than 35% of marks and 89% of students scores less then 60% marks. Find the mean and standard deviation if the marks is normally distributed. Given $P(1.2263) = 0.39$ and $P(1.4757) = 0.43$.

Ans:- Given, $P(x < 35) = 7\% = \frac{7}{100} = \underline{0.07}$

$P(x < 60) = 89\% = \frac{89}{100} = \underline{0.89}$

Put $x = 35$, in $z = \frac{x - \mu}{\sigma} = \frac{35 - \mu}{\sigma}$

$\Rightarrow z = \frac{35 - \mu}{\sigma} = z_1$ (say) —— (1)

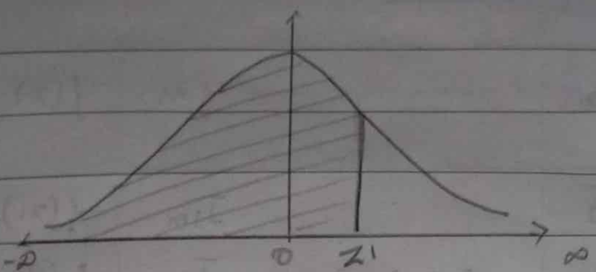
~~Put x = 60~~ $P(x < 35) = P(z < z_1) = 0.07$

Put $x = 60$ in $z = \frac{x - \mu}{\sigma} = \frac{60 - \mu}{\sigma} = z_2$ (say)

—— (2)

48.65
9.28

$\Rightarrow \quad P(x<60) = P(z<z_2) = 0.89.$

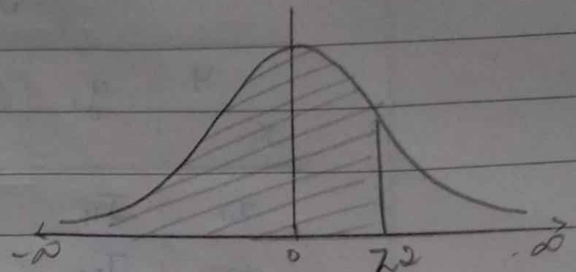


$\cancel{0.5}\, P(z<z_1) = P(x<35) = 0.07$

$0.5 + P(0<z<z_1) = 0.07$

$0.5 + \phi(z_1) = 0.07$

$\phi(z_1) = \underline{-0.43}$



$P(z<z_2) = P(x<60) = 0.89.$

$0.5 + P(0<z<z_2) = 0.89$

$0.5 + \phi(z_2) = 0.89$

$\phi(z_2) = \underline{0.39}$

Given $P(1.2263) = 0.39$ & $P(1.4757) = 0.43$

$\Rightarrow \phi(z_1) = -P(1.4757)$

$z_1 = -1.4757$

$\phi(z_2) = P(1.2263)$

$z_2 = 1.2263$

① & ②

$\dfrac{35-\mu}{\sigma} = -1.4757$

$35 - \mu = -1.4757\sigma$

$\mu - 1.4757\sigma = 35$

$\dfrac{60-\mu}{\sigma} = 1.2263$

$60 - \mu = 1.2263\sigma$

$\mu + 1.2263\sigma = 60$

$\Rightarrow \mu = \underline{48.653}$

$\sigma = \underline{9.252}$

Joint Probability Distribution:

$X \backslash Y$	y_1	y_2	......	y_m	$f(x)$
x_1	J_{11}	J_{12}	....	J_{1m}	$f(x_1)$
x_2	J_{21}	J_{22}	....	J_{2m}	$f(x_2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_n	J_{n1}	J_{n2}	......	J_{nm}	$f(x_n)$
$g(y)$	$g(y_1)$	$g(y_2)$	......	$g(y_m)$	1

* Mean, μ_x = ~~Except~~ Expectation of X

$$= E(X) = \sum_{i=1}^{n} x_i \, f(x_i)$$

μ_y = Expectation of Y

$$= E(Y) = \sum_{j=1}^{m} y_j \, g(y_j)$$

$$\mu_{xy} = E(XY) = \sum_{i=1}^{n} \sum_{j=1}^{m} x_i \, y_j \, J_{ij}$$

* Variance,

$$\sigma_x^2 = \sum_{i=1}^{n} x_i^2 \, f(x_i) - \mu_x^2$$

$$= E(X^2) - \mu_x^2$$

$$\sigma_y^2 = \sum_{j=1}^{m} y_j^2 \, g(y_j) - \mu_y^2$$

$$= E(Y^2) - \mu_y^2.$$

* Covariance = $(\text{cov}(X, Y))$

$$= E(XY) - \mu_x \mu_y.$$

* Correlation of X and Y

$$\rho(X, Y) = \frac{\text{cov}(X, Y)}{\sigma_x \sigma_y}$$

/ /

Note :

- x and y are independent if (all 3 to be true)
  - i.) $E(xy) = E(x) \cdot E(y)$
  - ii.) $Cov(x, y) = 0$
  - iii.) $J_{ij} = f(x_i)\, g(y_j)$.

12/1/23

24 The Joint Probability distribution table for 2 random variable x and y is as follows

x \ y	-2	-1	4	5
1	0.1	0.2	0	0.3
2	0.2	0.1	0.1	0

Determine the marginal probability distribution of X and Y, also compute

i.) Expectation of X, Y, XY.

ii.) Standard Deviation of X, Y

iii.) Covariance of X and Y

iv.) Correlation of X and Y

Further verify that X and Y are dependent random variable, also find $P(X+Y) > 0$ (not independent)

Ans

	x \ y	y_1: -2	y_2: -1	y_3: 4	y_4: 5	f(x).	
x_1	1	0.1 J_{11}	0.2 J_{12}	0 J_{13}	0.3 J_{14}	0.6	$f(x_1)$
x_2	2	0.2 J_{21}	0.1 J_{22}	0.1 J_{23}	0 J_{24}	0. 4	$f(x_2)$
	g(y)	0.3 $g(y_1)$	0.3 $g(y_2)$	0.1 $g(y_3)$	0.3 $g(y_4)$	1	

i.) $E(X) = \mu x = \sum_{i=1}^{n} x_i f(x_i)$

$= (1 \times 0.6) + (2 \times 0.4) = \underline{\underline{1.4}}$.

$$E(y) = \mu_y = \sum_{j=1}^{m} y_j\, g(y_j)$$

$$= (-2 \times 0.3) + (-1 \times 0.3) + (4 \times 0.1) + (5 \times 0.3)$$

$$= -0.6 - 0.3 + 0.4 + 1.5$$

$$= \underline{1}$$

$$E(xy) = \mu_{xy} = \sum_i \sum_j x_i\, y_j\, J_{ij}.$$

$$= (1 \times -2 \times 0.1) + (1 \times -1 \times 0.2) + (1 \times 5 \times 0.3)$$
$$+ (2 \times -2 \times 0.2) + (2 \times -1 \times 0.1) + (2 \times 4 \times 0.1)$$
$$+ (2 \times 5 \times 0)$$

$$= -0.2 - 0.2 + 1.5 - \cancel{0.8} - 0.2 + \cancel{0.8}$$

$$E(xy) = \underline{0.9}$$

ii.) SD of X, Y

variance $\quad \sigma_x^2 = \sum_i x_i^2 f(x_i) - \mu_x^2.$

$$= [(1^2 \times 0.6) + (2^2 \times 0.4)] - (1.4)^2$$

$$= [0.6 + 1.6] - (1.96)$$

$$\sigma_x^2 = \underline{0.24}$$

By $\quad \sigma_y^2 = \sum_j y_j^2\, g(y_j) - \mu_y^2$

$$= [(-2^2 \times 0.3) + (-1^2 \times 0.3) + (4^2 \times 0.1)$$
$$+ (5^2 \times 0.3)] - 1^2$$

$$= \underline{9.6}$$

$\Rightarrow$ Standard Deviation :

$$\sigma_x = \sqrt{V_x} = \sqrt{0.24}$$

$$= \underline{0.489}$$

$$\sigma_y = \underline{3.0983} \quad (\sqrt{9.6})$$

iii.) $\quad Cov(X, Y) = E(X.Y) - \mu_x \mu_y$

$$= 0.9 - (1.4 \times 1)$$

$$= \underline{-0.5}$$

$4-2 = -1$ ✗ $\quad$ $2-2=0$ ✗

$1-(-1) = 0$ ✗ $\quad$ $2-1=1$ ✓

$1+4=5$ ✓ $\quad$ $2+4=6$ ✓

$1+5=6$ ✓ $\quad$ $2+5=7$ ✓

should be greater than 0

iv.) Correlation : $\rho(x,y) = \dfrac{Cov(x,y)}{\sigma_x \sigma_y}$

$$= \frac{-0.5}{0.489 \times 3.0983}$$

$$= \underline{-0.33} \;/\; \underline{-0.329}$$

To check independence:

* we have $J_{ij} \neq f(x_i) \cdot g(y_j)$ and

covariance → $cov(x,y) \neq 0$

and also, $E(xy) \neq E(x)E(y)$

∴ x and y are dependent

→ $P(X+Y>0) = 0 + 0.3 + 0.1 + 0.1 + 0.$

$= \underline{0.5}$

25. X and Y are independent random variables. X takes values 2, 5, 7, with probability 1/2, 1/4, 1/4 respectively. while Y takes values, 3, 4, 5 with probability 1/3, 1/3, 1/3.

i.) Find the joint probability distribution of X and Y

ii.) Show the $cov(X,Y) = 0$

iii.) Find the probability distribution of $Z = X+Y$

Ans:- i.)

X \ Y	3 (y_1)	4 (y_2)	5 (y_3)	$f(x)$
2 (x_1)	1/6 J_{11}	1/6 J_{12}	1/6 J_{13}	1/2
5 (x_2)	1/12 J_{21}	1/12 J_{22}	1/12 J_{23}	1/4
7 (x_3)	1/12 J_{31}	1/12 J_{32}	1/12 J_{33}	1/4
$g(y)$	1/3	1/3	1/3	1

∵ X, Y → independent :

$J_{ij} = f(x_i) \cdot g(y_j)$

ii.) $Cov(x, y) = E(xy) - \mu_x \mu_y$

$\mu_x = \sum_i x_i f(x_i)$

$= (2 \times 1/2) + (5 \times 1/4) + (7 \times 1/4)$

$= 1 + 5/4 + 7/4 = 1 + 3$

$= \underline{4}$.

$\mu_y = \sum_j y_j \, g(y_j)$

$= (3 \times 1/3) + (4 \times 1/3) + (5 \times 1/3)$

$= 1 + 4/3 + 5/3 = 1 + 3$

$= \underline{4}$.

$E(xy) = \sum_i \sum_j x_i y_j J_{ij}$

$= \left(2 \times 3 \times \frac{1}{6}\right) + \left(2 \times 4 \times \frac{1}{6}\right) + \left(2 \times 5 \times \frac{1}{6}\right) +$

$\left(5 \times 3 \times \frac{1}{12}\right) + \left(5 \times 4 \times \frac{1}{12}\right) + \left(5 \times 5 \times \frac{1}{12}\right) +$

$\left(7 \times 3 \times \frac{1}{12}\right) + \left(7 \times 4 \times \frac{1}{12}\right) + \left(7 \times 5 \times \frac{1}{12}\right)$

$= 1 + \frac{4}{3} + \frac{5}{3} + \frac{5}{4} + \frac{5}{3} + \frac{25}{12} + \frac{7}{4} + \frac{7}{3} + \frac{35}{12}$

$= \cancel{16}\ \cancel{16}\ \underline{16}$.

$Cov(x, y) = E(xy) - \mu_x \mu_y$

$= 16 - 4(4)$

$= \underline{0}$.

iii.) $P(z) = P(x + y)$

$\Rightarrow$

z	5	6	7	8	9	10	10	11	12
$P(z)$	1/6	1/6	1/6	1/12	1/12	1/12	1/12	1/12	1/12
						= 1/6			

26. Suppose X and Y are independent Random variables with the following respective distribution.
Find the Joint Distribution of X and Y.
Also verify that Covariance Cov (X, Y) = 0

x_i	1	2
$f(x_i)$	0.7	0.3

y_j	-2	5	8
$g(y_j)$	0.3	0.5	0.2

Ans

X \ Y	-2	5	8	$f(x_i)$
1	0.21	0.35	0.14	0.7
2	0.09	0.15	0.06	0.3
$g(y_j)$	0.3	0.5	0.2	1

$\because$ X and Y are independent

$$J_{ij} = f(x_i) \cdot g(y_j)$$

$$Cov(X, Y) = E(XY) - \mu_x \mu_y$$

$$\mu_x = \sum_i x_i f(x_i)$$

$$= (1 \times 0.7) + (2 \times 0.3)$$

$$= \underline{\underline{1.3}}$$

$$\mu_y = \sum_j y_j g(y_j)$$

$$= (-2 \times 0.3) + (5 \times 0.5) + (8 \times 0.2)$$

$$= \underline{\underline{3.5}}$$

$$E(XY) = \sum_i \sum_j x_i y_j J_{ij}$$

$$= (1 \times -2 \times 0.21) + (1 \times 5 \times 0.35) + (1 \times 8 \times 0.14) + (2 \times -2 \times 0.09) + (2 \times 5 \times 0.15) + (2 \times 8 \times 0.06)$$

$$= \cancel{9}$$

$$= 4.55.$$

$$\operatorname{cov}(x,y) = E(xy) - \mu_x \mu_y$$
$$= 4.55 - (1.3)(3.5)$$
$$= 4.55 - 4.55$$
$$= \underline{\underline{0}}$$

$$cov(x,y) = E(xy) - \mu_x \mu_y$$
$$= 4.55 - \cancel{2}\,(1.3)(3.5)$$
$$= 4.55 - 4.55$$
$$= \underline{0}$$

14/07/23

27 The joint probability distribution of 2 discrete random variables X and Y, is given by $f(x,y) = k(2x+y)$ where x and y are integers such that $0 \le x \le 2$, $0 \le y \le 3$.

i.) Find the value of the constant k.

ii.) Find the marginal probability distribution of X and Y

iii.) Show that the random variables X and Y are dependent.

Ans (discrete probability = integers)

X \ Y	0 (y_1)	1 (y_2)	2 (y_3)	3 (y_4)	f(x)
0 (x_1)	0	k	2k	3k	6k
1 (x_2)	2k	3k	4k	5k	14k
2 (x_3)	4k	5k	6k	7k	22k
g(y)	6k	9k	12k	15k	1

$$J_{ij} = f(x,y) = k(2x+y)$$
~~i.) We have~~ $0 \le x \le 2$, $0 \le y \le 3$

i.) we have $\sum f(x) = 1$ or $\sum g(y) = 1$

$$\Rightarrow 6k + 14k + 22k = 1$$
$$42k = 1$$
$$k = \frac{1}{42}$$

ii.) Probability distribution :

x	0	1	2
$f(x)$	$6k$	$14k$	$22k$
	$=\frac{1}{7}$	$=\frac{1}{3}$	$=\frac{11}{21}$

y	0	1	2	3
$g(y)$	$6k$	$9k$	$12k$	$15k$
	$=\frac{1}{7}$	$=\frac{3}{14}$	$=\frac{2}{7}$	$=\frac{5}{14}$

iii.) We have $J_{ij} \neq f(x_i) \cdot g(y_j)$

for eg:- $f(x_1) g(y_1) = \frac{1}{7} \times \frac{1}{7} = \cancel{\frac{1}{49}} = \frac{1}{49}$

$$J_{11} = 0.$$

$$\therefore J_{11} \neq f(x_1)\, g(y_1).$$

Hence X and Y are dependent.

28. Suppose X and Y are independent random variables with following respective distribution of X and Y. Also verify $cov(x, y) = 0$.

x_i	1	2
$f(x_i)$	0.7	0.3

y_j	-2	5	8
$g(y_j)$	0.3	0.5	0.2

(done - Q 26)

<u>Stochastic Process</u> :

Stochastic process consists of a sequence of experiments in which each experiment has a finite number of outcomes with given probabilities.

<u>Probability vector</u> :

A vector $V = (V_1, V_2 \text{ - - - - } V_n)$ is called the probability vector, if each one of its components are non-negative and their sum equals unity

Eg: a) $v = [1, 0]$

b) $w = \left[\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right]$

Stochastic matrix:

A square matrix P is called a stochastic matrix if all the entries of P are non-negative and the sum of the entries of any row is one.

(OR) A ~~sequence~~ square matrix P is called a stochastic matrix with each row in form of probability vector.

eg:- a) $\begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix}$ b.) $\begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 1/4 & 1/4 \\ 1 & 0 & 0 \end{bmatrix}$

Regular stochastic matrix:

A matrix P is said to be regular stochastic matrix if all the entries of some power P^n are positive.

eg:- $P = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$

$P^2 = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix}$

$\therefore$ P is a regular stochastic matrix.

Note: The regular stochastic matrix P has a unique fixed probability ~~of~~ vector v such that

$vP = v$

(stationary probability vector)

28. Find the unique fixed probability vector of the regular stochastic matrix of $A = \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix}$

Ans Consider $A = \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix}$

We have to find a fixed probability vector i.e. $V = (x, y)$ where $x+y=1$ such that $VA = V$

$$(x, y) \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix} = (x, y)$$

$$\left[\frac{3}{4}x + \frac{1}{2}y, \ \frac{1}{4}x + \frac{1}{2}y\right] = (x, y)$$

$$\frac{3}{4}x + \frac{1}{2}y = x \quad ; \quad \frac{1}{4}x + \frac{1}{2}y = y$$

$$\Rightarrow \frac{1}{4}x = \frac{1}{2}y$$

$$x = 2y.$$

$$x + y = 1 \Rightarrow 3y = 1 \Rightarrow \underline{y = \frac{1}{3}}$$

$$x = 1 - y \Rightarrow x = \frac{2}{3}$$

$$\underline{V = \left(\frac{2}{3}, \frac{1}{3}\right)}$$

29. Find the unique fixed probability vector for the regular stochastic matrix matrix $P = \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix}$

Ans Let $V = (x, y, z)$ (unique fixed prob. vector)

we have $VP = V$

$$\Rightarrow [x, y, z]\begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix} = [x, y, z]$$

$$\Rightarrow \left[0 + \frac{y}{6} + 0, \quad x + \frac{y}{2} + \frac{2z}{3}, \quad \frac{y}{3} + \frac{z}{3}\right] = [x, y, z]$$

$(x(0) + y(1/6) + z(0))$

$$\frac{y}{6} = x \quad (1), \qquad x + \frac{y}{2} + \frac{2z}{3} = y \quad (2), \qquad \frac{y}{3} + \frac{z}{3} = z \quad (3)$$

$y = \underline{6x}$

$$\Rightarrow \frac{6x}{3} + \frac{z}{3} = z$$

$$2x = z - \frac{z}{3} = \frac{2z}{3}$$

$$\underline{6x = 2z}$$

$$\underline{z = 3x}$$

in (2), $x + y + z = 1$ ($\because$ its unique probability vector)

$$x + 6x + 3x = 1$$

$$10x = 1$$

$$\underline{x = 0.1}$$

$\therefore y = 6(0.1) = \underline{0.6}$

$z = 3(0.1) = \underline{0.3}$

$\therefore$ Unique fixed probability vector $= V = \underline{(0.1, 0.6, 0.3)}$

36. S.T $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$ is a Regular stochastic matrix & find the corresponding unique fixed probability vector.

Ans: Consider

$$P^2 = P \cdot P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

/ /

$$P^3 = P^2 P = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$ (calc. till $P^n = P^5$)

Since all entries of P^n for $n = 3$ are strictly +ve

$\therefore$ P is a Regular Stochastic matrix.

Let $V = [x, y, z]$ be a unique fixed probability vector.

we have, $VP = V$

$$\Rightarrow [x, y, z] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = [x, y, z]$$

$$\Rightarrow \left[\frac{1z}{2}, x + \frac{z}{2}, y \right] = [x, y, z]$$

$\Rightarrow \frac{1z}{2} = x$ — (1) ; $x + \frac{z}{2} = y$ — (2) ; $y = z$. — (3)

$z = 2x$

$\Rightarrow x = \frac{z}{2}$

$x + \frac{z}{2} = z$

$2x + z = 2z$

$2x = z$

$\Rightarrow$ W.K.T $x + y + z = 1$

$\cancel{x} \cancel{z} + \cancel{2x}\ 2x$ $\quad \frac{z}{2} + z + z = 1$

$5z = 2 \Rightarrow z = \frac{2}{5}$.

$x = 1/5$,

$y = 2/5$

$\therefore$ Unique fixed probability vector $= \left[\frac{1}{5}, \frac{2}{5}, \frac{2}{5} \right]$