[illegible]:

Stochastic Process:

Stochastic process consists of a sequence of experiments in which each experiment has a finite number of outcomes with given probabilities.

Probability Vector:

A vector $V = (V_1, V_2 - - - - V_n)$ is called the probability vector, if each one of its components are non-negative and their sum equals unity.

Eg:- (a) $v = [1, 0]$ (b) $w = [\frac{1}{2}, \frac{1}{4}, \frac{1}{4}]$.

Stochastic matrix:

A square matrix P is called a stochastic matrix if all the entries of P are non-negative and the sum of the entries of any row is one.

(OR) A square matrix P is called a stochastic matrix with each row in the form of probability vector.

Eg:- (a) $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$ (b) $\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 1 & 0 & 0 \end{bmatrix}$

Regular stochastic matrix:

A matrix P is said to be regular stochastic matrix if all the entries of some power P^n are positive.

Eg:- $P = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$$P^2 = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

$\therefore$ P is a regular stochastic matrix.

** The regular stochastic matrix P has a unique fixed probability vector V such that $\underline{\underline{VP = V}}$.

Problems:

1) Find the unique fixed probability vector of the regular stochastic matrix of $A = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

Soln:- Consider $A = \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix}$

We have to find a fixed probability vector i.e $V = (x, y)$ where $x + y = 1$ such that $VA = V$.

$$(x, y) \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix} = (x, y)$$

$$\left[\frac{3}{4}x + \frac{1}{2}y, \ \frac{1}{4}x + \frac{1}{2}y \right] = (x, y)$$

$$\frac{3}{4}x + \frac{1}{2}y = x \quad ; \quad \frac{1}{4}x + \frac{1}{2}y = y$$

$$\Rightarrow \frac{1}{4}x = \frac{1}{2}y$$

$$x = 2y$$

$$\because x + y = 1 \Rightarrow 3y = 1 \Rightarrow y = 1/3.$$

$$\therefore x = 1 - y \Rightarrow x = 2/3.$$

$$\therefore \boxed{V = (2/3, 1/3)}.$$

2) Find the unique fixed probability vector for the regular stochastic matrix $P = \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix}$

Soln: - Let $V = (x, y, z)$ be the unique fixed probability vector associated with P

$\Rightarrow x + y + z = 1 \; (1)$ such that $VP = V$.

$$(x, y, z)\begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix} = (x, y, z)$$

$$\Rightarrow \left(\frac{y}{6}, \; x + \frac{y}{2} + \frac{2z}{3}, \; \frac{y}{3} + \frac{z}{3} \right) = (x, y, z)$$

$$\Rightarrow \frac{y}{6} = x \; ; \quad x + \frac{y}{2} + \frac{2z}{3} = y \; ; \quad \frac{y}{3} + \frac{z}{3} = z.$$

$$\Rightarrow y = 6x. \qquad \frac{y}{3} = z - \frac{z}{3}$$

$$\Rightarrow \frac{6x}{3} = \frac{2z}{3}$$

$$\Rightarrow 6x = 2z \Rightarrow z = 3x.$$

$\therefore$

$\therefore$ From (1), $x + 6x + 3x = 1 \Rightarrow 10x = 1 \Rightarrow x = \frac{1}{10} = 0.1$

$\therefore \; y = 6x = \frac{6}{10} = 0.6$

$z = 3x = \frac{3}{10} = 0.3.$

$\therefore$ The required unique fixed probability vector associated with P is

$$\boxed{V = (x, y, z) = \left(\frac{1}{10}, \frac{6}{10}, \frac{3}{10} \right) \text{ or } (0.1, 0.6, 0.3)}$$

3) Show that $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$ is an Regular stochastic matrix and find the corresponding unique fixed probability vector.

Soln :- Consider,

$$P^2 = P \cdot P = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

$$P^3 = P^2 \cdot P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

Since all the entries of P^n for $n=3$ are strictly positive.

$\therefore$ P is a regular stochastic matrix.

Let $V = (x, y, z)$ be a unique fixed probability vector associated with P. $\Rightarrow x+y+z=1 \; \text{--}(1) \Rightarrow VP = V$.

$$(x, y, z) \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = (x, y, z)$$

$$\left(\frac{z}{2}, \; x + \frac{z}{2}, \; y \right) = (x, y, z)$$

$$\Rightarrow \frac{z}{2} = x, \quad x + \frac{z}{2} = y, \quad y = z$$

$$\Rightarrow z = 2x.$$

Also $y = 2x$.

$\therefore (1) \Rightarrow x + 2x + 2x = 1 \Rightarrow 5x = 1 \Rightarrow x = 1/5$.

$y = 2/5, \; z = 2/5$

$\therefore$ The required unique fixed probability vector associated with P is $\boxed{V = (x, y, z) = \left(1/5, 2/5, 2/5 \right)}$

4) Find unique fixed probability vector for regular stochastic matrix $P = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$

Sol^n: Let $v = (x, y, z)$ be the unique fixed probability vector associated with P.

$\Rightarrow \underbrace{x + y + z = 1}_{(1)} \Rightarrow vP = v$.

$$(x, y, z)\begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} = (x, y, z)$$

$$\Rightarrow \left(\frac{x}{2} + \frac{y}{2}, \frac{x}{4} + z, \frac{x}{4} + \frac{y}{2}\right) = (x, y, z)$$

$$\Rightarrow \frac{x}{2} + \frac{y}{2} = x, \quad \frac{x}{4} + z = y, \quad \frac{x}{4} + \frac{y}{2} = z.$$

$$\Rightarrow \frac{y}{2} = x - \frac{x}{2} = \frac{x}{2} \Rightarrow y = x.$$

Also, $\frac{x}{4} + z = y \Rightarrow z = y - \frac{x}{4} \Rightarrow z = x - \frac{x}{4} \Rightarrow z = \frac{3x}{4}$.

$(1) \Rightarrow \quad x + x + \frac{3x}{4} = 1 \Rightarrow \frac{11x}{4} = 1 \Rightarrow x = \frac{4}{11}$.

$\Rightarrow \quad = \frac{3x}{4} = \frac{3}{4}\cdot\frac{4}{11} = \frac{3}{11}$.

$\Rightarrow y = x \Rightarrow y = \frac{4}{11}$.

$\therefore$ The required unique fixed probability vector associated with P is

$$\boxed{v = (x, y, z) = \left(\frac{4}{11}, \frac{4}{11}, \frac{3}{11}\right)}$$

5) Find the unique fixed probability vector for the regular stochastic matrix $P = \begin{bmatrix} 0 & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Sol^n:- $V = (x, y, z) = (4/13, 8/13, 1/13)$

Markov chains :-

A stochastic process which is such that the generation of the probability distribution depends only on the present state is called markov process, if the state space of the process is discrete then it is called markov chain.

Absorbing state of a Markov chain :-

In a Markov chain, the process reaches to a certain state after which it continuous to remain in the same state. Such a state is called absorbing state of the Markov chain.

The transition probabilities P_{ij} are such that

$$P_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise.} \end{cases}$$

If $P^{(0)}$ is initial probability distribution then the Probability distribution in the n^{th} step is given by $P^{(n)} = P^{(0)} P^n$.

Problems :-

1) The transition matrix P of Markov chain is given by $P = \begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix}$ with a initial probability distribution $P^{(0)} = (1/4, 3/4)$. Determine (i) $P_{21}^{(2)}$ (ii) $P_{12}^{(2)}$ (iii) $P^{(2)}$ (iv) $P_1^{(2)}$ (v) the vector $P^{(0)} P^{(n)}$ approaches.

Solⁿ:- Given $P = \begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix}$

$$P^2 = \begin{bmatrix} 5/8 & 3/8 \\ 9/16 & 7/16 \end{bmatrix}$$

$P_{21}^{(2)} = 9/16$; $P_{12}^{(2)} = 3/8$

$$p^{(2)} = p^{(0)} P^2 = \begin{bmatrix} 1/4 & 3/4 \end{bmatrix} \begin{bmatrix} 5/8 & 3/8 \\ 9/16 & 7/16 \end{bmatrix} = \begin{bmatrix} 37/64 & 27/64 \end{bmatrix}$$

$P_1^{(2)} = 37/64$.

Let $V = (x, y)$ where $x + y = 1$; $VA = V$.

$$(x, y) \begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix} = (x, y).$$

$$\frac{1}{2}x + \frac{3}{4}y = x \; ; \; \frac{1}{2}x + \frac{1}{4}y = y$$

$$x = \frac{3}{2}y \; ; \; \frac{5}{2}y = 1$$

$$\therefore x = \frac{3}{5}, \; y = \frac{2}{5}.$$

$$\therefore \boxed{V = (3/5, 2/5)}$$

2) The transition probability matrix of a markov chain is given by $P = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$ and the initial probability distribution is $p^{(0)} = [1/2, 1/2, 0]$. Find i) $P_{13}^{(2)}$, ii) $P_{23}^{(2)}$, (iii) P^2, (iv) $P_1^{(2)}$.

p^2

$p^{(0)} p^2$

Soln :- Let $P = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$

$$P^2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 3/8 & 1/4 & 3/8 \\ 1/2 & 0 & 1/2 \\ 11/16 & 1/8 & 3/16 \end{bmatrix} \end{array}$$

i) $\underline{P_{13}^{(2)} = 3/8}$; ii) $\underline{P_{23}^{(2)} = 1/2}$;

iii) $P_1^{(2)} = p^{(2)} = p^{(0)} P^2$

$$= \left[\frac{1}{2}, \frac{1}{2}, 0\right] \begin{bmatrix} 3/8 & 1/4 & 3/8 \\ 1/2 & 0 & 1/2 \\ 11/16 & 8 & 3/16 \end{bmatrix}$$

$$= \left[135/256, \ 21/128 \ \ 79/256\right]$$

$\therefore \ \underline{P_1^{(2)} = 135/256}.$

Irreducible Markov chain :-

A markov chain is said to be irreducible if its transition probability matrix (TPM) is the regular stochastic matrix. (Entries are positive.)

Problems :-

1) Prove that the Markov chain where TPM $P = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$ is irreducible. Also find the corresponding stationary probability vector.

Solⁿ:- Let $P = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} \Rightarrow P^2 = \begin{bmatrix} 1/2 & 1/6 & 1/3 \\ 1/4 & 7/12 & 1/6 \\ 1/4 & 1/3 & 5/12 \end{bmatrix}$

Since all the entries in P^2 are positive.

We conclude that the TPM is regular stochastic matrix.

$\therefore$ P is irreducible.

To find a unique fixed Probability vector.

Let $v = (x, y, z) \Rightarrow x + y + z = 1 \Rightarrow VA = V$.

$$(x, y, z) \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} = (x, y, z).$$

$$\left(\frac{y}{2} + \frac{z}{2}, \frac{2}{3}x + \frac{z}{2}, \frac{x}{3} + \frac{y}{2}\right) = (x, y, z)$$

$$\frac{y}{2} + \frac{z}{2} = x, \quad \frac{2}{3}x + \frac{z}{2} = y, \quad \frac{x}{3} + \frac{y}{2} = z.$$

On solving we get

$$x = 1/3, \quad y = 10/27, \quad z = 8/27$$

$$\therefore \boxed{V = [1/3 \quad 10/27 \quad 8/27]}$$

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2) A students study habits are as follows. If he studied 1 night he is 70% sure not to study the next night. On the otherhand if he does not study one night, he is 60% sure not to study the next night. In the long run how often does he study?

Solⁿ:- Let A : Studying B : Not studying.

P_{11} : Probability from A to A = 30% = 0.3

P_{12} : Probability from A to B : 70% = 0.7

P_{21} : Probability from B to A = 40% : 0.4

P_{22} : Probability from B to B = 60% : 0.6

Hence, TPM for the given problem is

$$P = \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix}$$

In order to find the happening in long run, we have to find the unique probability vector.

i.e we have to find $V = (x, y)$ when $x + y = 1$ such that $VP = V$.

$$(x, y)\begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix} = (x, y)$$

$$0.3x + 0.4y = x.$$
$$0.7x + 0.6y = y$$
$$x = \frac{4}{7}y.$$

$\therefore \quad x + y = 1 \Rightarrow x = \frac{4}{11}, \; y = \frac{7}{11}$

$\therefore \quad \boxed{V = [4/11 \;, 7/11]}$ $\therefore$ We conclude that in the long run the student will study $4/11 = 36.36\%$ of time.

3) A software engineer goes to his work place every day by motor bike or by car. He never goes by bike on two consecutive days but if he goes by car on a day he is equally likely to go by car or by bike on the next day. Find the transition matrix for the chain of the mode of transport he uses. If car is used on the first day of the week find the probability that i) bike is used after 4 days (or on the fifth day).

Soln:- Let $S = \{$Motor Bike, Car$\}$

The transition probability matrix for the chain of mode of transport

$$P = \begin{array}{c c} & \begin{array}{cc} B & C \end{array} \\ \begin{array}{c} B \\ C \end{array} & \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} \end{array}$$

Given car is used on the first day of the week.

$\therefore$ The initial probability distribution of the mode of transport is given by

$$p^{(0)} = (0, 1).$$

Since the car is used, the probability that bike is used is 0 and the probability that car is used is 1.

$\therefore$ The probability distribution of the mode of transport after 4 days is given by

$$p^{(4)} = p^{(0)} P^4 = (p_1^{(4)}, p_2^{(4)})$$

$$P^4 = \begin{bmatrix} 3/8 & 5/8 \\ 5/16 & 11/16 \end{bmatrix}$$

Consider, $p^{(4)} = p^{(0)} P^4 = (0,1)\begin{bmatrix} 3/8 & 5/8 \\ 5/16 & 11/16 \end{bmatrix} = [5/16, 11/16]$

$\therefore$ After 4 days, probability of using a bike $= p_1^{(4)} = 5/16$

probability of using a car $= p_2^{(4)} = 11/16$.

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4) Three boys A, B and C are throwing ball to each other. A always throws the ball to B and B always throws the ball to C, but C is just likely to throw the ball to B as to A. If C was the first person to throw the ball find the probabilities that i) A has the ball, ii) B has the ball, iii) C has the ball. After three throws (or for the fourth throw)

Soln:- The TPM is

$$P = \begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \end{array}$$

Initially C has the ball.

$\therefore$ the initial probability distribution is given by

$p^{(0)} = (0, 0, 1)$

Since the probabilities are desired after three throws we have to find $p^{(3)} = p^{(0)} \cdot p^3$.

$$p^3 = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

$$p^3 = p^{(0)} \cdot p^{3} = [0\ 0\ 1] \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

$$p^3 = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$$

Thus after three throws the probability that the ball with A is $1/4$, B is $1/4$ & C is $1/2$.

5) A sales man S sells in three cities A, B, C. He never sells in the same city on successive days. If he sells in city A on a day, then the next day he sells in city B. However if he sells in either B or C, then the next day he is twice as likely to sell in city A as in the other city. In a long run how often does he sell in each of the cities?

Soln:- The tPM of the process is

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 2/3 & 0 & 1/3 \\ 2/3 & 1/3 & 0 \end{bmatrix} \end{array}$$

In order to find the happening in long run, we have to find the unique probability vector

Let $V = (x, y, z) \Rightarrow x+y+z=1 \ni: VP = V$.

$$(x, y, z)\begin{bmatrix} 0 & 1 & 0 \\ 2/3 & 0 & 1/3 \\ 2/3 & 1/3 & 0 \end{bmatrix} = (x, y, z)$$

$$\left(\frac{2}{3}y + \frac{2}{3}z,\ x + \frac{z}{3},\ \frac{y}{3}\right) = (x, y, z)$$

$$\frac{2}{3}y + \frac{2}{3}z = x,\quad x + \frac{z}{3} = y,\quad \frac{y}{3} = z$$

$$x = \frac{8}{3}z,\quad \Rightarrow y = 3z.$$

$$\therefore \frac{8}{3}z + 3z + z = 1 \Rightarrow \frac{20}{3}z = 1 \Rightarrow z = \frac{3}{20}.$$

$$\Rightarrow y = \frac{9}{20}$$

$$\Rightarrow x = \frac{8}{20}$$

$$\therefore V = \left[\frac{8}{20}, \frac{9}{20}, \frac{3}{20}\right]$$

In long run,

S Sells in city A with probability $= \frac{8}{20}$

S Sells in city B with Probability $= \frac{9}{20}$

S Sells in city C with probability $= \frac{3}{20}$.

6) Every year, a man trades his car for a new car. If he has Maruti, he trades it for an Ambassador. If he has an Ambassador, he trades it for a Santro. However if he has Santro he is just likely to trade it for a new Maruti or an Ambassador. Santro as to trade it for a Santro as to trade it for a Santro. In 2000, he bought his first car which was Santro. Find the probability that he has

i) 2002 Santro ii) 2002 Maruti.

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Solⁿ:- The transition probability matrix of the process is

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \end{array}$$

A : Maruti
B : Ambassador.
C : Santro.

With 2000 as the first year, 2002 is to be regarded as 2 years later.

$$P^2 = P.P = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{9} & \frac{4}{9} & \underline{\frac{4}{9}} \end{bmatrix} \begin{bmatrix} AA & AB & AC \\ BA & BB & BC \\ CA & CB & CC \end{bmatrix}$$

i) In 2002 Santro $= \dfrac{4}{9}$ (CC)

ii) In 2002 Maruti $= \dfrac{1}{9}$ (CA)