Arranging Beetles

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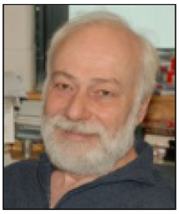

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Picture an insect swarm, perhaps thousands of ladybug beetles on a juicy plant stalk, or a giant ball of bees in busy hive. The entire cluster is in constant, tangled motion, and insects are crawling under and over each other in every direction as they go about their work. One cannot help but wonder how they ever find their way from one side to the other.

The following Olympiad problem [1], and especially a three-dimensional variation, evoke an image similar:

> 65 beetles are placed in 65 cells of a 9×9 table. In each move, every beetle creeps to an adjacent cell. No beetle makes two horizontal or two vertical moves in succession. Prove that after several moves at least two beetles will be in the same cell.

The beetle problem in a square lattice

How should one begin? In the words [2] of the King to the White Rabbit: "Begin at the beginning and go on till you come to the end; then stop." We take this to mean we are free to investigate the problem as we see fit, letting curiosity and experimentation guide us as to what seems most relevant.

Experimenting with how 65 beetles in a 9×9 grid might move around, though a diligent and direct approach, would result in an enormous number of possibilities to explore and it might be hard to discern any patterns in the chaos. We rephrase the problem so as to make it easier to explore smaller versions of it: What is the maximum number of beetles that can be arranged to walk in a $n \times n$ grid subject to the same movement constraints and so that, after any movement, no two beetles share the same cell?

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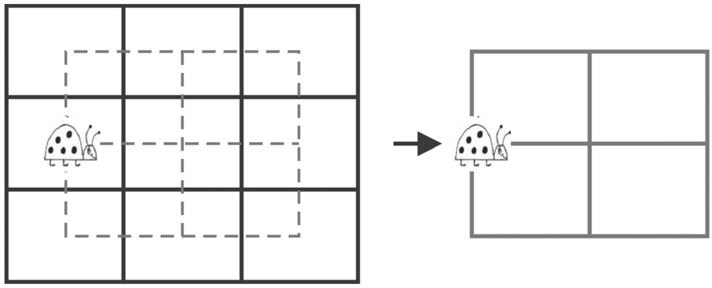

Figure 1. Cells of 3 × 3 table as vertices in 3 × 3 grid graph.

For the small case of a 2 × 2 grid the maximum is four beetles, since we can place a beetle in each cell and have them follow each other in either a clockwise or counterclockwise fashion. And from this it follows that in the case of a $2k \times 2k$ grid - by partitioning it into 2 × 2 grids each with 4 beetleswe can arrange $(2k)^2$ beetles to walk as required and this is clearly the maximum. Furthermore, since an 8 × 8 grid exists inside a 9 × 9 grid, we can arrange 64 beetles to walk in the 9 × 9 grid of the original problem, with two sides of the larger grid completely unused by the beetles. So close! And there are quite a few (81 − 64=17) cells that are never visited by a beetle at all. So why, as the problem claims, does trying to squeeze one extra beetle in always result in some beetle eventually having a cellmate?

We replace the configuration of cells in the table by an associated adjacency graph: replace each cell by a single vertex (think of this vertex as the center location of the cell), and join two such vertices by an edge if and only if the corresponding cells share a common side. (see Figure 1.) Each cell becomes a vertex and adjacent cells become connected by graph edges, and the beetles all move simultaneously between vertices along graph edges. There are four "corner" vertices of degree two, four "side" vertices of degree three, and one "central" vertex of degree four.

We experiment with another small case. After trying to arrange beetles in a 3 × 3 grid, one finds several ways to arrange four beetles but not more. But how can we be *sure* that it is impossible to arrange five or more beetles without two beetles eventually ending up in the same cell? Through continued experimentation one discovers that, because of the restriction about not making two horizontal or vertical moves in succession, the corners are special: a beetle in any of the four corner vertices must, after two moves, end up at the center vertex. Consequently, there can be at most one beetle in any of the four corner vertices, because otherwise after two moves there would be least two beetles in the central vertex. So any arrangement of beetles in the 3 × 3 grid can have at most one beetle amongst the four corner vertices, and one beetle in each of the other five other vertices, and hence at most 6 beetles in total.

Is such an arrangement of 6 beetles possible? If we try placing beetles with just one in a corner, one in the center, and four in the remaining side vertices, then we find a problem. The four beetles in side vertices cannot all move to the center, so at least three must go to corner vertices. But, building on the previous insight, in two more moves those corner beetles will again result in multiple beetles in the center. In fact we can say a little bit more because we still get the collision in the center even if there are three beetles on the side vertices. We conclude that we can have at most two beetles in the side vertices before any movement.

Thus in an arrangement of beetles in the 3 × 3 grid, there can be at most two beetles in the side vertices, one beetle in the center vertex, and at most one beetle in the corner

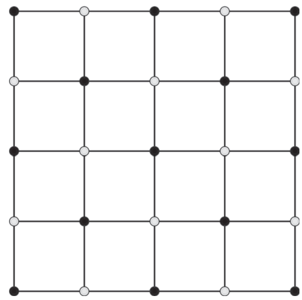

Figure 2. Two types of vertices in the 5x5 grid.

vertices, and thus there are at most four beetles in any arrangement of beetles in the 3×3 grid, as suspected.

After a beetle makes two consecutive moves, each of its x and y coordinates must change by exactly one. So if we visualize where a beetle can end up after an even number of moves we find an alternating pattern dividing the vertices into two types. For example in the 5×5 grid, a beetle starting at the center vertex and moving an even number of times must be on one of the dark vertices illustrated in Figure 2. This invariance provides some order in the apparent chaos of beetle movements. With this in mind, we will consider grouping the vertices with an eye to parity.

We define some notation for the general problem in a square grid. Placing the bottom left corner vertex at coordinates (0, 0), we see that an $n \times n$ square board can be replaced by the integer lattice

$$S(n) = \{(x, y) | x, y \in \mathbb{Z} \text{ and } 0 \leq x, y < n\}.$$

The problem we now consider is as follows.

Problem. Choose any $(2k)^2 + 1$ vertices of the lattice $S(2k + 1)$ and place a beetle on each. Per move, each beetle simultaneously creeps to an adjacent vertex. No beetle makes either two horizontal or two vertical moves in succession. Show that after some number of moves, at least two beetles occupy the same vertex.

We define a *valid arrangement* of beetles in this lattice as some number of beetles located on the vertices of the lattice, forever moving (simultaneously) from vertex to vertex so that no beetle makes two horizontal or two vertical moves consecutively, and always, after any movement, with at most one beetle at any lattice vertex. Our problem, rephrased, is to show that the number of beetles in a valid arrangement is at most $4k^2$.

We partition the vertices of $S(2k + 1)$ into $EE \cup OO \cup EO \cup OE$, where EE consists of those vertices for which both coordinates are even:

$$EE = \{(even, even)\} = \{(x, y) \in S(2k + 1) | x \equiv 0 \bmod 2, y \equiv 0 \bmod 2\},$$

and similarly $OE = \{(odd, even)\}$, and so on.

Because of the movement constraint, after any 2 consecutive moves, beetles in the set EE end up in the set OO, beetles in OO end up in EE, beetles in EO end up in OE, and the beetles in OE end up in EO. We denote this as the '2-move rule':

$$EE \longleftrightarrow_{2\,moves} OO,$$
$$EO \longleftrightarrow_{2\,moves} OE.$$

In the 3×3 example, we saw that the sizes of these partition classes helped upper-bound the number of beetles we could arrange. What are their sizes in the case of $S(2k+1)$? An easy analysis shows that

$$
\begin{aligned}
|EE| &= (k+1)^2, \\
|OO| &= k^2, \\
|EO| &= k(k+1), \\
|OE| &= k(k+1).
\end{aligned}
$$

For a set A of vertices within $S(2k+1)$, let $\mathbf{mb}(\mathbf{A})$ denote the maximum number of beetles in set A after any movement of the beetles, with the maximum taken over all possible valid arrangements of beetles in $S(2k+1)$.

The "2-move rule" tells us subsets of the lattice within which the beetles exchange themselves after 2 moves. Because these sets are different sizes they constrain the amount of beetles involved. Since beetles in sets OO and EE exchange, we must have $\mathrm{mb}(EE) = \mathrm{mb}(OO)$. But obviously $\mathrm{mb}(OO) \leq |OO| = k^2$. Thus, $\mathrm{mb}(EE) = \mathrm{mb}(OO) \leq k^2$ also.

We also see that $\mathrm{mb}(EE \cup OO) \leq \mathrm{mb}(EE) + \mathrm{mb}(OO) \leq 2k^2$. Here, we are using the property

$$
\mathrm{mb}(A \cup B) \leq \mathrm{mb}(A) + \mathrm{mb}(B), \text{ for any subsets } A \text{ and } B, \tag{1}
$$

which follows immediately from the definition. (This type of property is sometimes called "subadditivity" and we will need it again later.) Obviously $\mathrm{mb}(EO \cup OE) \leq |EO \cup OE| = 2(k^2 + k)$.

So it also follows that

$$
\begin{aligned}
\mathrm{mb}(S(2k+1)) &= \mathrm{mb}(EE \cup OO \cup EO \cup OE) \\
&\leq \mathrm{mb}(EE \cup OO) + \mathrm{mb}(EO \cup OE) \\
&\leq 2k^2 + 2(k^2 + k).
\end{aligned}
$$

In the case of the 9×9 grid, this gives an upper bound of 64+8 = 72 beetles in any arrangement of beetles. Progress, but we want to upper-bound the number of beetles by 64 to solve the problem.

In the 3×3 example, it was also helpful to notice that after *one* move beetles on the side vertices have to go to either the center vertex or the corner vertices. Similarly, a beetle on an OE or EO vertex has to move to a vertex in OO or EE, and indeed vice-versa. So, after any 1 move, we have a similar observation, which might be condensed as :

$$
EE \cup OO \longleftrightarrow_{1\,move} EO \cup OE.
$$

From this rule, it follows that $\mathrm{mb}(EE \cup OO) = \mathrm{mb}(EO \cup OE)$, and so we can improve the above argument to

$$
\mathrm{mb}(S(2k+1)) \leq 2\,\mathrm{mb}(EE \cup OO) \leq 2(2k^2) = 4k^2.
$$

This is the result we wanted to prove.

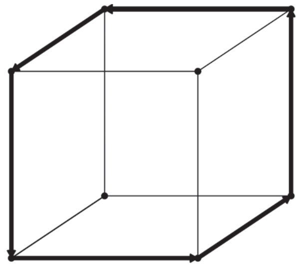

Figure 3. A skew hexagon in $C(2)$.

Interlude and generalization

We have done what we set out to do; we arrived at a solution. It was not particularly difficult, and there are shorter ways to prove the result. An interested reader might wish to consult [3] where this problem is one of several solved by a variety of clever coloring arguments.

It is our firm belief that even the most elementary areas of mathematics are never more than a simple creative leap away from challenging research projects. Mathematics is as much a creative process as are literature and music. Hardy [4] says "A mathematician, like a painter or a poet, is a maker of patterns." So we may consider this solution not as a destination, but instead as a crossroads that, depending on our interests, can lead us in any number of new directions.

The original problem suggests several natural variations and related questions. For example, consider beetles moving in the 3-dimensional lattice

$$C(n) = \{(x, y, z) | x, y, z \in \mathbb{Z} \text{ and } 0 \leq x, y, z < n\},$$

under the movement constraint "In any three successive moves, a beetle must must crawl once in each of the x, y and z directions." In this case, it is natural to ask: what is the maximum number of beetles that can be crawling in the lattice $C(n)$ subject to this movement constraint and without beetles ever being forced to occupy the same vertex?

As before, it is instructive to consider a small case such as beetles moving on the vertices of the $2 \times 2 \times 2$ cube $C(2)$. Figure 3 shows the path traced by 6 beetles arranged to walk along a skew hexagon in $C(2)$. While it is easy to think that we cannot improve on this, a bit of reflection soon shows that the problem statement allows for an arrangement of 8 beetles walking around, which is clearly a maximum.

To see this, it is worth remembering the legal maxim that "what is not explicitly forbidden is permitted." In particular, while beetles must not end up in the same vertex (cell) after a move, there is nothing in the original problem statement that forbids them from passing each other on the same edge during movement itself! We choose to proceed with this investigation by assuming that such moves are permitted.

With beetles upon every vertex of the cube $C(2)$, imagine their simultaneous walks as a rotation of the cube through 90 degrees about the axis that passes through the midpoints of the top and bottom faces. After performing two rotations like this, each of the 8 beetles has completed a move in the x and y direction. Now all beetles move in the z direction, which we can imagine as a reflection of the cube (that is, reflect in the plane $z = 1/2$). Now repeat the two rotations described above, followed once again by the reflection. At the conclusion of this composition of symmetries, all the beetles are back in their original positions, and are free to continue moving again the same way.

This is a valid arrangement of 8 beetles in $C(2)$—each beetle crawls around a skew hexagon but they never collide at a vertex.

It follows at once that, by partitioning the $C(2k)$ into k^2 smaller $C(2)$'s, we can completely saturate the vertices of $C(2k)$ with a valid arrangement of beetles.

As we move on to analyzing the $C(2k+1)$-case, we might want to assume that at most $8k^3$ vertices can be placed but more honestly we have no data to support this. Perhaps we can see if we can first try to find an upper bound based on parity like we did before as a way to get more familiar with this variant of the problem. A beetle moving from position (x, y, z) adds or subtracts the vectors $(1, 0, 0)$, $(0, 1, 0)$ or $(0, 0, 1)$ from its position vector. After three moves, the parity of each coordinate has been flipped, and so three consecutive moves are seen to be equivalent (modulo 2) to adding the vector $(1, 1, 1)$ to the initial vector (x, y, z).

As in the two-dimensional case, partition the vertices of $C(2k+1)$ into 8 classes based on the residues modulo 2 of each coordinate. This equivalence relation partitions the $(2k+1)^3$ vertices into eight equivalence classes; we will name these $[0, 0, 0], [0, 0, 1], \ldots[1, 1, 1]$.

It is not hard to confirm that the cardinality of these classes is as follows.

$$\begin{aligned}
|[0,0,0]| &= (k+1)^3\\
|[0,0,1]| = |[0,1,0]| = |[1,0,0]| &= k(k+1)^2\\
|[1,1,0]| = |[1,0,1]| = |[0,1,1]| &= k^2(k+1)\\
|[1,1,1]| &= k^3
\end{aligned}$$

After three moves, a beetle must change each of its x,y,and z coordinates by exactly 1. This gives rise to the '3-move rule' which interchanges the eight parity classes in the following manner:

$$\begin{aligned}
[0,0,0] &\longleftrightarrow_{3\,moves} [1,1,1]\\
[1,0,0] &\longleftrightarrow_{3\,moves} [0,1,1]\\
[0,1,0] &\longleftrightarrow_{3\,moves} [1,0,1]\\
[0,0,1] &\longleftrightarrow_{3\,moves} [1,1,0]
\end{aligned}$$

In a valid arrangement, the beetles will continually exchange between the pairs above. So in each exchange the number of beetles cannot be more than twice the size of the smaller equivalence class. So, with the function mb now defined in the context of the lattice $C(2k+1)$, the 3-move rule implies that

$$\begin{aligned}
&\mathrm{mb}([0,0,0] \cup [1,1,1]) \leq 2k^3\\
&\mathrm{mb}([1,0,0] \cup [0,1,1]) \leq 2k^2(k+1)\\
&\mathrm{mb}([0,1,0] \cup [1,0,1]) \leq 2k^2(k+1)\\
&\mathrm{mb}([0,0,1] \cup [1,1,0]) \leq 2k^2(k+1),
\end{aligned}$$

and so, using the subadditive property (1) from earlier and adding up gives the upper bound $\mathrm{mb}(C(2k+1)) \leq 8k^3 + 6k^2$.

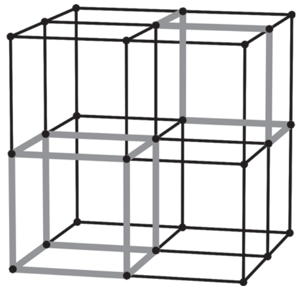

Figure 4. 14 beetles in a $3 \times 3 \times 3$ cube.

Does there exist an arrangement of beetles which realizes this upper bound? There really is no reason to think that this bound is tight—the 3-dimensional case is surely more complex than the 2-dimensional case. How might we proceed?

As before, let us investigate in more detail the smallest non-trivial case of this problem—when k equals 1. Here the formula above gives us an upper bound of 14 for the number of beetles we can arrange in the cube $C(3)$. Can we find a valid arrangement of 14 beetles on the 27 vertices of the cube $C(3)$? If not, can we see any way to improve our upper bound? Either way we will learn more about the nature of the generalized problem.

If we notice that 14 = 6+8, then we might recall our two arrangements of beetles in the cube $C(2)$ discussed earlier; these two arrangements can be combined to get a valid arrangement of 14 beetles in $C(3)$! This arrangement is shown in Figure 4; the two overlapping $C(2)$'s naturally containing the smaller beetle arrangements share the central vertex of the $C(3)$, but note that this central shared vertex is only visited by beetles in the $C(2)$ in the lower left corner of the lattice shown.

Surely though the fact that this bound is met with equality when $k = 1$ is a fluke, a coincidence, some consequence of the special case $k = 1$. But in fact in the next section, we will see that in this generalization to three dimensions things are not so much more complicated after all.

The beetle problem in a cubical lattice

In this section we answer the following question.

Problem. What is the maximum number of beetles that can be arranged to walk in the cube $C(2k + 1)$ such that in any three successive moves, a beetle must must crawl once in each of the x, y and z directions (in some order), and after any movement no beetles are located at the same lattice vertex.

Solution.

As before, denote the desired maximum by $\mathrm{mb}(C(2k + 1))$. Let $\mathbf{U(k)} = 8k^3 + 6k^2$. In the last section we saw that $\mathrm{mb}(C(2k + 1)) \le U(k)$. We also saw that this bound holds with equality when $k = 1$, so that $\mathrm{mb}(C(3)) = U(1)$. In this section we show that this upper bound can always be met with equality:

$$\text{For each } k \in \mathbb{N}, \quad \mathrm{mb}(C(2k + 1)) = U(k) = 8k^3 + 6k^2.$$

We will describe an arrangement of $U(k)$ beetles in the general cubical lattice $C(2k + 1)$ recursively; we decompose $C(2k + 1)$ into smaller rectangular and cubical lattices and combine arrangements of beetles in these smaller lattices to construct the desired arrangement in $C(2k+1)$.

Figure 5 illustrates a configuration of 88 beetles in a $5 \times 5 \times 5$ lattice. Furthermore, since $U(2) = 8(2)^3 + 6(2)^2 = 88$ we see the bound can also be met with equality for the value $k = 2$, and so

$$\mathrm{mb}(C(5)) = U(2).$$

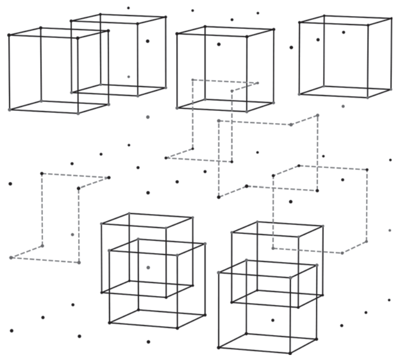

Figure 5. 88 beetles arranged in a $5 \times 5 \times 5$ lattice.

We decompose the $C(2k+1)$ lattice into a $2 \times (2k+1) \times (2k+1)$ lattice, four $2 \times (2k-1) \times (2k-1)$ lattices, a $2 \times (2k-3) \times (2k-3)$ lattice, and a $C(2k-3)$ lattice as indicated in Figure 6.

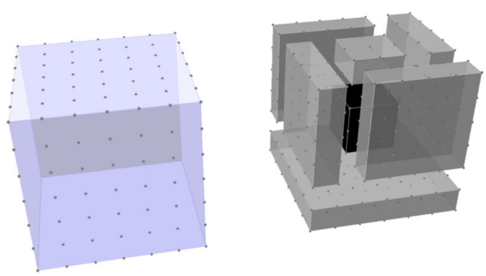

Figure 6. Decomposing a $7 \times 7 \times 7$ lattice into smaller lattices.

The arrangement of beetles we will place in the $2 \times (2k+1) \times (2k+1)$ lattice

$$\{(x, y, z) | 0 \le x < 2k+1, 0 \le y < 2k+1, 0 \le z < 2\}$$

is most easily described by referencing Figure 7, which illustrates the case $k = 3$. Arrange 8 beetles to walk in each of $2(1 + 2 + \cdots + (k-1)) = k(k-1)$ disjoint $2 \times 2 \times 2$ cubes on either side of the main diagonal, and arrange 6 beetles in each of $2k$ skewed hexagons along the diagonal.

This results in a valid arrangement of $8k(k-1) + 6(2k) = 8k^2 + 4k$ beetles walking within the $2 \times (2k+1) \times (2k+1)$ lattice.

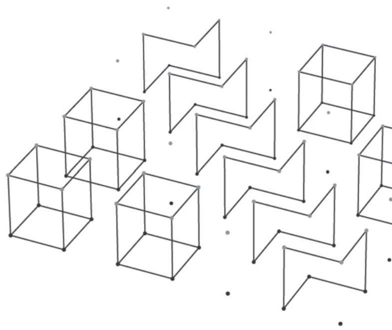

Figure 7. An arrangement of beetles in a $2 \times 7 \times 7$ lattice.

Similarly in a $2 \times (2k-1) \times (2k-1)$ lattice, there is a valid arrangement of $8(k-1)^2 + 4(k-1)$ beetles, and there is a valid arrangement of $8(k-2)^2 + 4(k-2)$ beetles in the $2 \times (2k-3) \times (2k-3)$ lattice.

Inductively assume we can arrange $U(k-2) = 8(k-2)^3 + 6(k-2)^2$ beetles in $C(2k-3)$. We have seen this is true when $k-2$ equals 1 or 2.

Then the number of beetles we have in the above decomposition of $(C(2k+1)$ is

$$\begin{aligned} (8k^2 + 4k)+ \\ 4(8(k-1)^2 + 4(k-1))+ \\ 8(k-2)^2 + 4(k-2)+ \\ 8(k-2)^3 + 6(k-2)^2 \\ & = 8k^3 + 6k^2. \end{aligned}$$

This configuration of beetles is a valid arrangement of beetles in the $C(2k+1)$ lattice, and inductively we conclude that

$$\mathrm{mb}(C(2k+1)) = 8k^3 + 6k^2 \text{ for each natural number } k.$$

Conclusion

The authors' consideration of the original Olympiad problem, and the investigation of the generalization, occurred in the context of a larger investigation about problem solving generally. We have tried to capture some of the lessons learned in that process in this narrative.

When solving a problem from a mathematics competition, or even from a mathematics textbook, you are typically working towards a *known* and well-articulated goal. You also know that the result you have been asked to prove is actually true and all you need to do is prove it! In fact a well-known trick in problem-solving circles is to exploit the fact that competition problems *must have* at least one relatively short and elegant solution. Simply knowing this can guide your attempt to solve the problem.

The generalized problem has no such guarantees. It is unclear if anyone else has ever considered the problem. It is unclear whether the problem will submit to an analysis like its predecessor. In considering the generalization, we somehow leave the realm of problem solving and enter the realm of research. This undermines the sense of confidence with which one approaches the task when compared to the comfort of looking for a solution that you already know exists.

Insight is well fed by curiosity and experimentation, and clearly, we join in emphasizing the value of exploring small cases in the pursuit of a problem. Invariably, doing so provides insights and suggest new approaches. Edison [5] noted that "Genius is one percent inspiration and ninety-nine percent perspiration." The mind has an uncanny ability of "connecting the dots," a point made lucidly by Gauss [6]: "Finally, two days ago, I succeeded. Like a sudden flash of lightening, the riddle was solved. I am unable to say what was the conducting thread that connected what I previously knew with what made my success possible." It is also a ubiquitous experience in mathematics to see a clever proof of some result, and think "I follow this argument well enough, but I don't see how I could have *ever* come up with this." Students should remember that a proof, necessarily prioritizing clarity, rigour and economy, can flow nothing like the perhaps messy sequence of explorations and ideas that led to the proofs creation.

Of course, this is not the end of the tale, for no problem ever has a true end. For example, it is natural to consider the problem on graphs and surfaces more complicated than the lattices here. What happens if the beetles must move differently, perhaps like knights in a chess game? Why does a "1-move" bound not appear to be needed in the case of the cube? And most obviously, it is impossible not to ask what happens in the cubical lattice when two beetles also cannot share an edge during movement. The maximum number of beetles cannot be more than the case we consider here. There is such an arrangement of 14 beetles in $C(3)$, but it is unknown what happens for larger lattices.

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Summary. This article, directed at undergraduate problem solvers and their mentors, discusses and solves an Olympiad problem and a generalization. We interweave subtle commentary about problem solving generally.

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A Proof of the Reflective Property of the Parabola

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It is well known that a parabola has an important property having to do with reflection. Specifically, a ray of light originating at the focus of the parabola is reflected by the parabola along a line parallel to the parabola's axis of symmetry. Because of this, the parabolic shape is used in the construction of flashlight and headlight reflectors, telescope mirrors, radar dishes, and satellite dishes. There are various ways of proving this reflection property, some of which are simple [1, 2]. We wish to present a slight variation. Its novelty is its use of the properties of a rhombus.

We may state the theorem as follows. See Figure 1. We draw the parabola having focus F and directrix d. We call its vertex V. We choose any point P on the parabola other than V. We draw the tangent line t to the parabola at P. We define the angle $\angle 1$ to be the acute angle formed by $\overline{FP}$ and t, and with V in the interior. Then we draw the ray r with endpoint P pointing away from d. We define $\angle 2$ to be the acute angle formed by r and t. We wish to prove that $\angle 1 \cong \angle 2$.

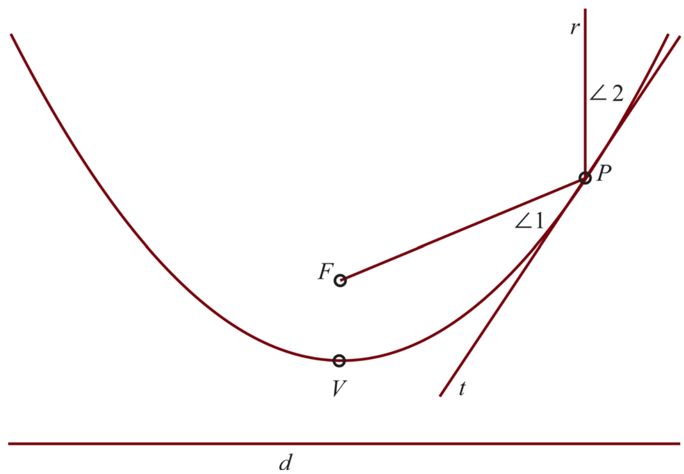

Figure 1. A parabola with focus F and directrix d, showing a tangent t to the parabola at P and the angles $\angle 1$ and $\angle 2$.

Now, remove the tangent line t for the moment. We let Q be the foot of the perpendicular from P to d. See Figure 2. We let M be the midpoint of $\overline{FQ}$, and let S be the reflection of P in M. Therefore, MF = MQ and MP = MS. So, by side-angle-side, $\Delta FMP \cong \Delta QMS$, and therefore FP = QS. By a similar argument, PQ = SF. Now, since the distance from a point on a parabola to the focus is equal to the distance from the point to the directrix, we have FP = PQ. Therefore, quadrilateral FPQS is a rhombus.

We now need to show that line $\overleftrightarrow{MP}$ is indeed the tangent line t. Let X be any point on that line other than P. Since the diagonals of a rhombus are perpendicular, line $\overleftrightarrow{MP}$ is the perpendicular bisector of $\overline{FQ}$. If X is not the point M, then by side-angle-side, $\Delta FMX \cong \Delta QMX$, and therefore, FX = XQ. If X is the point M, then obviously we still have FX = XQ. Now drop a perpendicular from X to the directrix d. Call the foot

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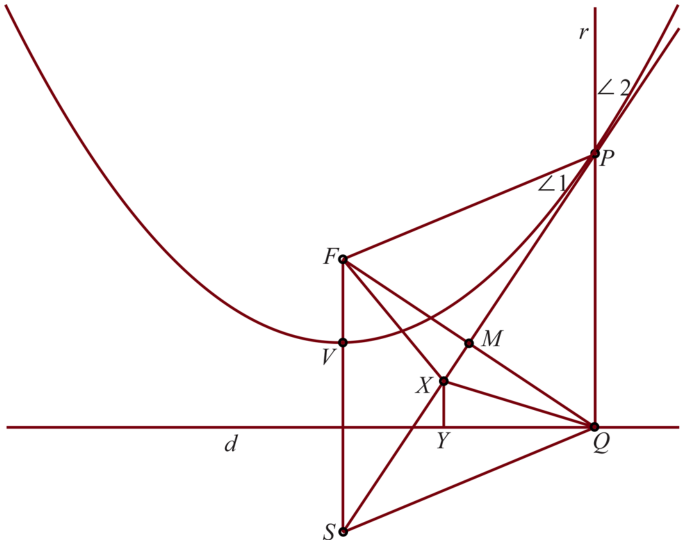

Figure 2. The parabola with the constructed rhombus.

of that perpendicular Y. If X is not on d, ΔXYQ is a right triangle with hypotenuse QX. Therefore, $XY < QX$, and so $XY \neq FX$. Therefore, X cannot lie on the parabola. If, on the other hand, X lies on the directrix d, then X and Y are the same point, and so $XY = 0 \neq FX$, and again X cannot lie on the parabola.

P is the only point on $\overleftrightarrow{MP}$ that lies on the parabola, and so $\overleftrightarrow{MP}$ is the tangent line t.

We now define $\angle 1$ and $\angle 2$ as before. The line segment $\overline{PS}$ is a diagonal of the rhombus. Therefore, $\angle 1 \cong \angle MPQ$. The angles $\angle MPQ$ and $\angle 2$ are vertical angles and thus congruent. Therefore $\angle 1 \cong \angle 2$, which completes the proof.

Summary. We prove the reflection property of a parabola using the properties of rhombus.

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Using Linear Interpolation to Implement the Change of Variables in Double Integrals

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A double integral, $\iint_D f(x, y)\, dA$, often can be solved by a change of variables. Typically, what makes an integral difficult to evaluate is either the form of the integrand, f, or the shape of the region D. Therefore, we can change the variables to either make the integrand f simpler (and hope for a nice new parameterization of D to work with), or make the region D simpler (and cross fingers for a nice new form of f). This note focuses on the latter approach, using linear interpolation to introduce new variables that transform a non-rectangular integration region into a rectangular one.

Linear Interpolation. A linear interpolation that constructs new data points to fill the space between two given data points A and B enjoys the simple form of

$$A + t(B - A) \quad (0 \le t \le 1). \tag{1}$$

When A and B are two points, Equation (1) draws the line segment from A to B. If one of A and B (or both of them) is a curve, and they are parametrized on the same interval $a \le s \le b$, then Equation (1) becomes

$$A(s) + t\big[B(s) - A(s)\big], \tag{2}$$

which represents an area over a rectangular parametric domain $(s, t) \in [a, b] \times [0, 1]$ (provided that the interpolation line segments between $A(s)$ and $B(s)$ are disjoint, with the possible exceptions of the endpoints).

Example 1. Use linear interpolation to parameterize the region enclosed by the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ (Figure 1(a)).

Solution. In Equation (2), let $A(s) = \langle 0, 0\rangle$ (the origin) and $B(s) = \langle 2\cos s, 3\sin s\rangle$ (the ellipse), the elliptical region is represented by $\langle 2t\cos s, 3t\sin s\rangle$, where $(s, t) \in [0, 2\pi] \times [0, 1]$. ■

Example 2. Use linear interpolation to parameterize the region enclosed by the trapezoid with the four vertices $(1, 0)$, $(2, 0)$, $(0, 1)$ and $(0, 2)$ (Figure 1(b)).

Solution. In Equation (2), let $A(s) = \langle 0, s\rangle$ (the edge on the y-axis) and $B(s) = \langle s, 0\rangle$ (the edge on the x-axis), the trapezoid is represented by $\langle st, s - st\rangle$, where $(s, t) \in [1, 2] \times [0, 1]$. ■

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(a) (b)

Figure 1. (a) Linear interpolation between an ellipse and its center; (b) Linear interpolation between two nonparallel edges of a trapezoid. In each panel, a red line segment is a linear interpolation line for a fixed s value. Multiple s values are sampled. If the s values were continuously sampled in their domains, then the regions would be completely filled.

Solving Double Integrals by a Change of Variables. The parametric form of an integration region, obtained from a linear interpolation, provides an obvious choice for a change of variables: we can replace the original variables in the integrand by their counterparts in the newly derived parametric form.

Example 3. Evaluate the integral $\iint_D y^2\, dA$, where D is the elliptical region enclosed by $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

Solution. Based on the parametric form of the elliptical region in Example 1, let $x = 2t\cos s$ and $y = 3t\sin s$. The Jacobian of the transform is $\frac{\partial(x,y)}{\partial(s,t)} = -6t$. Therefore,

$$\iint_D y^2\, dA = \int_0^1 \int_0^{2\pi} (9t^2\sin^2 s)(6t)\, ds\, dt = 54\int_0^1 t^3\, dt \int_0^{2\pi} \frac{1-\cos 2s}{2}\, ds = \frac{27}{2}\pi.$$

■

Example 4. Evaluate the integral $\iint_D \frac{x-y}{x+y}\, dA$, where D is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,1)$ and $(0,2)$.

Solution. Based on the parametric form of the trapezoidal region in Example 2, let $x = st$ and $y = s - st$. The Jacobian of the transform is $\frac{\partial(x,y)}{\partial(s,t)} = -s$. Thus,

$$\iint_D \frac{x-y}{x+y}\, dA = \int_0^1 \int_1^2 s(2t-1)ds\, dt = \int_0^1 (2t-1)\, dt \int_1^2 s\, ds = 0.$$

■

Remark 1. It is worthy noting to students that the choice of linear interpolation is not unique. For instance, in Example 4, we can also use the two parallel edges $\langle s, 1-s\rangle$ and $\langle 2s, 2-2s\rangle$ to reconstruct the trapezoid: $\langle s+st, (1-s)+t(1-s)\rangle$, $(s,t) \in [0,1]\times[0,1]$. Thus, the change of variables will be $x = s + st$ and $y = (1-s) + t(1-s)$, which will be nice to leave as an exercise for students to try.

Remark 2. When the change of variables is primarily concerned with making the integrand simpler (the first case mentioned in the opening paragraph), the new variables are typically introduced in terms of the original variables. One then has to rewrite the original variables in terms of the new variables in order to calculate the Jacobian and redraw the integration region in the transformed parametric space in order to set up the updated integral. These two steps are skipped in the linear interpolation induced change of variables method. As shown in Examples 3 and 4, the Jacobians are ready to

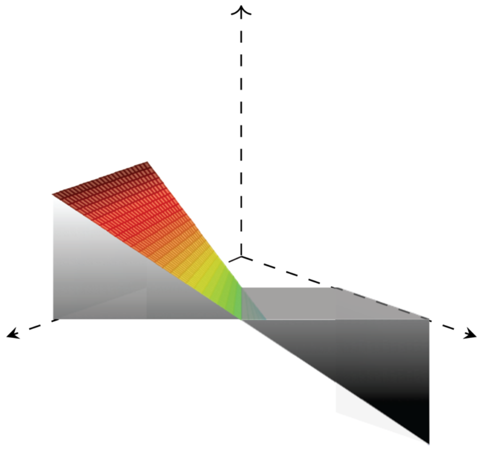

Figure 2. Visualize Example 4 as a signed volume.

be calculated and the transformed regions are always rectangles, so that the integrals can be easily set up without any additional sketching.

Remark 3. This method of change of variables provides a standardized procedure to solve a double integral on a non-rectangular region. The unification does take away the joy of searching for insightful observations when a change of variable is necessary (e.g., $u = x/2$ and $v = y/2$ in Example 3, as presented in most calculus books), and can be inefficient when a double integral on a non-rectangular region does not need a change of variables. On the other hand, it does give students a clear path to a solution.

Remark 4. It is a pleasure for students to make sense of double integrals visually. For example, the integral in Example 4 is zero due to the combined symmetry of the integrand and the trapezoidal domain (Figure 2). Because all 3D surface graphic functions (e.g., "parametric_plot3d" in SageMath and "Surface" in GeoGebra) require a rectangular domain, the parameterization given by Equation (2) is particularly useful in 3D graphing and thus helps students to visualize double integrals using software. (In fact, the idea for this note emerged when the author was drawing the 3D solids associated with the double integrals on SageMath in a multivariable calculus class.)

Acknowledgments. The author is grateful to the referee and the Classroom Capsules editors for their questions and suggestions that lead to the better version of the article. The author would also like to thank to his colleagues Carolyn Yackel and Margaret Symington for reading the earlier drafts and providing valuable suggestions.

Summary. The change of variables method can be applied to simplify either the integrand or the region of integration. This note addresses the second task, using linear interpolation to transform a non-rectangular region into a rectangular one. The approach provides a standardized way to solve a double integral when a change of variables is necessary and can be useful when using software to do 3D graphing.

PROBLEMS AND SOLUTIONS

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This section contains problems intended to challenge students and teachers of college mathematics. We urge you to participate actively *both* by submitting solutions and by proposing problems that are new and interesting. To promote variety, the editors welcome problem proposals that span the entire undergraduate curriculum.

Proposed problems should be sent to **Greg Oman**, either by email (preferred) as a pdf, TEX, or Word attachment or by mail to the address provided above. Whenever possible, a proposed problem should be accompanied by a solution, appropriate references, and any other material that would be helpful to the editors. Proposers should submit problems only if the proposed problem is not under consideration by another journal.

Solutions to the problems in this issue should be sent to **Chip Curtis**, either by email as a pdf, TEX, or Word attachment (preferred) or by mail to the address provided above, no later than July 15, 2022. Sending both pdf and TEX files is ideal.

PROBLEMS

1216. *Proposed by Oluwatobi Alabi, Government Science Secondary School Pyakasa Abuja, Abuja, Nigeria.*

For an integer $n \geq 3$, find a closed form expression for the number of ways to tile an $n \times n$ square with 1×1 squares and $(n-1) \times 1$ rectangles (each of which may be placed horizontally or vertically).

1217. *Proposed by Eugen Ionascu, Columbus State University, Columbus, GA.*

Prove the following:

1. There exists a unique function $f : \mathbb{R} \to \mathbb{R}$ which satisfies the following equation for every $x \in \mathbb{R}$:

$$f(-x) = 1 + \int_0^x \cos(t) f(x-t) dt.$$

   Moreover, express f explicitly in terms of elementary functions.

2. For every non-negative integer k, $f^k(0) = (-1)^{\lfloor \frac{k+1}{2} \rfloor} F_k$, where $F_0 = 0$, $F_1 = 1$, $F_{k+2} = F_k + F_{k+1}$, and $\lfloor x \rfloor$ denote the greatest integer less than or equal to a real number x.

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1218. *Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Las Palmas de Gran Canaria, Spain.*

The *Pell* and *Pell-Lucas numbers*, $\{P_n : n \in \mathbb{N}\}$ and $\{Q_n : n \in \mathbb{N}\}$, respectively, are defined recursively as follows: $P_0 = 0$, $P_1 = 1$, $Q_0 = Q_1 = 2$, and (for each sequence) $u_{n+1} = 2u_n + u_{n-1}$ for $n \geq 1$. Next, let $n \in \mathbb{N}$, and let $A_n(x)$ and $B_n(x)$ be polynomials of degree n with real coefficients such that for $0 \leq i \leq n$, we have $A_n(i) = P_i$ and $B_n(i) = Q_i$. Find $A_n(n+1)$ and $B_n(n+1)$ in terms of P_{n+1} and Q_{n+1}, respectively.

1219. *Proposed by Greg Oman, University of Colorado at Colorado Springs, Colorado Springs, CO.*

Let R be a commutative ring with identity $1 \neq 0$. Recall that if I and J are ideals of R, then the *product* of I and J is defined as follows:

$$IJ := \{i_1 j_1 + \cdots + i_n j_n : i_k \in I,\ j_k \in J, n \in \mathbb{Z}^+\}.$$

Prove that R is a field if and only if for every ideal I and J of R, we have $IJ \in \{I, J\}$.

1220. *Proposed by Jeff Stuart, Pacific Lutheran University, Tacoma, WA.*

Let A be an $n \times n$ real or complex matrix with $n \geq 2$. Let $\mathrm{co}(A)$ denote the matrix of cofactors of A, that is, for each i and j, $(\mathrm{co}(A))_{ij}$ is the product of $(-1)^{i+j}$ and the determinant of the matrix obtained by deleting the ith row and jth column of A. Prove the following:

1. For $n = 2$, show that $\mathrm{co}(\mathrm{co}(A)) =!$.
2. For $n > 2$, show that there is a unique singular A such that $\mathrm{co}(\mathrm{co}(A)) = A$.
3. For $n > 2$, find a condition on $\det(A)$ that is satisfied exactly when A is invertible and $\mathrm{co}(\mathrm{co}(A)) = A$.

It was brought to our attention that CMJ problem 1208 has already appeared as problem 12256 in the May 2021 issue of the Monthly (by a different proposer). Accordingly, we will not be featuring a solution to this problem. We apologize for the error.

SOLUTIONS

An equilateral triangle in an isosceles triangle

1191. *Proposed by Herb Bailey, Rose-Hulman Institute of Technology, Terre Haute, IN.*

An isosceles triangle has incenter I, circumcenter O, side length S, and base length W. Show that there is a unique value of $\frac{S}{W}$ so that there exists a point P on one of the two sides of length S such that triangle IOP is equilateral. Find this value. *Solution by the*

Eagle Problem Solvers, Georgia Southern University, Statesboro, GA and Savannah, GA.

The unique value is

$$\frac{S}{W} = \frac{1 + \sqrt{3 + 2\sqrt{7/3}}}{2} \approx 1.7303506.$$

Position the isosceles triangle ABC with $A = (0, a)$, $B = (W/2, 0)$ and $C = (-W/2, 0)$. Then $a^2 = S^2 - \frac{W^2}{4}$ and $a = \frac{\sqrt{4S^2 - W^2}}{2}$. The circumcenter O lies at the intersection of the perpendicular bisectors of AB and BC. The midpoint of AB is $(W/4, a/2)$ and the slope of AB is $\frac{-2a}{W}$, so the perpendicular bisector of AB has slope $\frac{W}{2a}$ and equation

$$y = \frac{a}{2} + \frac{W}{2a}\left(x - \frac{W}{4}\right).$$

Since the perpendicular bisector of BC is the y-axis, then the circumcenter O has y-coordinate

$$y_O = \frac{a}{2} - \frac{W^2}{8a} = \frac{4a^2 - W^2}{8a} = \frac{2S^2 - W^2}{2\sqrt{4S^2 - W^2}}.$$

Since the y-axis bisects $\angle A$, then the incenter I also lies on the y-axis; its y-coordinate is given by

$$y_I = \frac{Wa}{2S + W} = \frac{W\sqrt{4S^2 - W^2}}{2(2S + W)}.$$

Thus, the distance between I and O is given by

$$IO = y_O - y_I = \frac{2S^2 - W^2}{2\sqrt{4S^2 - W^2}} - \frac{W\sqrt{4S^2 - W^2}}{2(2S + W)} = \frac{S(S - W)}{\sqrt{4S^2 - W^2}}.$$

If a point P is equidistant from O and I, then its y-coordinate must be given by

$$\begin{aligned} y_P &= \frac{y_O + y_I}{2} \\ &= \frac{2S^2 - W^2}{4\sqrt{4S^2 - W^2}} + \frac{W\sqrt{4S^2 - W^2}}{4(2S + W)} \\ &= \frac{2S^2 - W^2 + W(2S - W)}{4\sqrt{4S^2 - W^2}} \\ &= \frac{S^2 + SW - W^2}{2\sqrt{4S^2 - W^2}}. \end{aligned}$$

If P also lies on AB, then its x-coordinate must be given by

$$\begin{aligned} x_P &= \frac{W}{2}\left(1 - \frac{y_P}{a}\right) \\ &= \frac{W}{2}\left(1 - \frac{S^2 + SW - W^2}{4S^2 - W^2}\right) \\ &= \frac{SW(3S - W)}{2(4S^2 - W^2)}. \end{aligned}$$

Thus, the square of the distance between O and P is

$$OP^2 = \left(\frac{IO}{2}\right)^2 + x_P^2$$

$$= \frac{S^2(S-W)^2}{4(4S^2-W^2)} + \frac{S^2W^2(3S-W)^2}{4(4S^2-W^2)^2}$$
$$= \frac{S^3\left(S^3-2S^2W+3SW^2-W^3\right)}{\left(4S^2-W^2\right)^2}.$$

If triangle IOP is equilateral, then $IO^2 = OP^2$; that is,

$$\frac{S^2(S-W)^2}{4S^2-W^2} = \frac{S^3\left(S^3-2S^2W+3SW^2-W^3\right)}{\left(4S^2-W^2\right)^2}$$
$$(S-W)^2\left(4S^2-W^2\right) = S^4-2S^3W+3S^2W^2-SW^3$$
$$3S^4-6S^3W+3SW^3-W^4=0.$$

Dividing by $W^4 \neq 0$ and letting $x = S/W$ gives the equation

$$3x^4-6x^3+3x-1=0.$$

Substituting $x = z+1/2$ and multiplying by 16, we get

$$48z^4-72z^2-1=0,$$

so that

$$z^2 = \frac{72 \pm 16\sqrt{21}}{96} = \frac{3}{4} \pm \frac{\sqrt{21}}{6} = \frac{3 \pm 2\sqrt{7/3}}{4},$$

$$z = \frac{\pm\sqrt{3 \pm 2\sqrt{7/3}}}{2},$$

and

$$x = \frac{1 \pm \sqrt{3 \pm 2\sqrt{7/3}}}{2}.$$

Since $x = \frac{S}{W}$ is a positive real number, there is a unique solution:

$$\frac{S}{W} = \frac{1+\sqrt{3+2\sqrt{7/3}}}{2} \approx 1.7303506.$$

Also solved by Michel Bataille, Rouen, France; James Duemmel, Bellingham, WA; Jeffrey Groah, Lone Star C. - Montgomery; Eugene Herman, Grinnell C.; Elias Lampakis, Kiparissia, Greece; Volkhard Schindler, Berlin, Germany; Randy Schwartz, Schoolcraft C. (retired); Albert Stadler, Herrliberg, Switzerland; Enrique Treviño, Lakeforest C.; Michael Vowe, Therwil, Switzerland; and the proposer.

Ubiquitous zero divisors without nontrivial nilpotent elements implies infinite

1192. *Proposed by Greg Oman, University of Colorado at Colorado Springs, Colorado Springs, CO.*

Let R be a commutative ring (not assumed to have an identity). Recall that an element $x \in R$ is a *zero divisor* if there is some nonzero $y \in R$ such that $xy = 0$; x is nilpotent if $x^n = 0$ for some positive integer n (note that we do *not* require a zero divisor to be nonzero).

(a) Prove or disprove: there exists a finite commutative ring R for which

1. every element of R is a zero divisor, and
2. the only nilpotent element of R is 0.

(b) Does your answer change if "finite" is replaced with "infinite"?

Solution by Northwestern University Math Problem Solving Group.

1. The answer is *negative*, i.e., there is no finite commutative ring satisfying 1 and 2. If $R = \{0\}$ (the trivial ring), then 0 is not a zero divisor, sit fails to satisfy 1. Hence, we may assume that R is non-trivial, and the proof proceeds as follows.

   Let $S = \{x_1, x_2, ..., x_n\}$ be a maximal set (n maximum) of distinct non-zero elements of R with the property $x_i x_j = 0$ for every $i \neq j$. Denote $s = x_1 + x_2 + \cdots + x_n$ its sum. Then

   (a) We h ave $s \neq 0$ because otherwise $x_1 = -x_2 - \cdots - x_n$, hence $x_1^2 = -x_2x_1 - \cdots - x_nx_1 = 0$, contradicting the assumption that 0 is the only nilpotent element.

   (b) Since all elements of R are zero divisors, there must be a non-zero r such that $0 = rs = rx_1 + rx_2 + \cdots + rx_n$. Hence, for each $i = 1, ..., n$, we have

   $$rx_i = -\sum_{\substack{j=1 \\ j\neq i}}^{n} \rightarrow (rx_i)^2 = r\,(rx_i)\,x_i = -r\sum_{\substack{j=1 \\ j\neq i}}^{n} rx_jx_i = 0 \rightarrow rx_i = 0.$$

   This implies that the set $S' = \{r, x_1, x_2, ..., x_n\}$ also has the property that every pair of distinct elements in it has product zero, but S' has $n + 1$ elements, contradicting the maximality of S.

2. For infinite rings, the answer is *affirmative*. An example is the ring R of infinite sequences of integers with finitely many non-zero elements (and term-wise addition and multiplication). This ring satisfies the required properties, as shown below.

   - Property 1: If $\{a_n\}_{n\in\mathbb{N}}$ is in R, then there will be some (in fact infinitely many) $m \in \mathbb{N}$ such that $a_m = 0$. Given a fixed m such that $a_m = 0$, let $b_m = 1$ and $b_n = 0$ for $n \neq m$. Then we have that $\{b_n\}_{n\in\mathbb{N}}$ is not zero, but $a_nb_n = 0$ for every n, so that $\{a_n\}_{n\in\mathbb{N}}$ is a zero divisor.
   - If $k \geq 1$, then, for each n, $a_n^k = 0$ if and only if $a_n = 0$. Hence the zero element of R, consisting of the sequence with all terms zero, is the only nilpotent element in R.

Also solved by Egle Bettio and Liceo Benedetti-Tommaseo, Venezia, Italy; Anthony Bevelacqua, U. of N. Dakota; Paul Budney, Sunderland, MA; Elias Lampakis, Kiparissia, Greece; and the proposer.

A function that is a polynomial over the rationals in each slot separately need not be a polynomial over $\mathbb{Q}^2$

1193. *Proposed by George Stoica, Saint John, New Brunswick, Canada.*

Let $f : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ be a function such that $y \to f(a, y)$ is a polynomial over $\mathbb{Q}$ for every $a \in \mathbb{Q}$ and $x \to f(x, b)$ is a polynomial over $\mathbb{Q}$ for every $b \in \mathbb{Q}$. Is it true that $f(x, y)$ is a polynomial in $(x, y) \in \mathbb{Q}^2$?

Solution by Paul Budney, Sunderland, Massachusetts.

Such functions exist which cannot be defined by a polynomial in $\mathbb{Q}[x, y]$. Let $r_1, r_2, \ldots$ be a faithfully-indexed sequence of the rationals. Define $f : \mathbb{Q}^2 \to \mathbb{Q}$ by

$$f(x, y) = \sum_{k=1}^{\infty} \prod_{i=1}^{k} (x - r_i)(y - r_i).$$

For each $(x, y) = (r_m, r_n) \in \mathbb{Q}^2$, this series has only finitely many non-zero terms, so it converes on $\mathbb{Q}^2$. For any rational $x = r_n$, if $n > 1$,

$$f(r_n, y) = \sum_{k=1}^{n-1} \prod_{i=1}^{k} (r_n - r_i)(y - r_i) \in \mathbb{Q}[y],$$

a polynomial of degree $n - 1$. If $n = 1$, $f(r_1, y)$. Similarly, for $n > 1$, $f(x, r_n) \in \mathbb{Q}[x]$, a polynomial of degree $n - 1$. Also, $f(x, r_1) = 0$. Now, if f is defined by a polynomial $f(x, y) \in \mathbb{Q}[x, y]$, we can choose a positive integer $n > \deg[f(x, y)] = d > 0$. But then $f(r_{n+1}, y)$ is a polynomial of degree n and also a polynomial of degree at most $d < n$. This is impossible since non-constant polynomials have only finitely many zeros. Thus $f(x, y)$ can't be defined by a polynomial in $\mathbb{Q}[x, y]$.

Also solved by Gerald Edgar, Denver, CO; Albert Natian, Los Angeles Valley C.; Kenneth Schilling, U. of Michigan - Flint; and the proposer. One incomplete solution and one incorrect solution were received.

A two-variable inequality over the integers

1194. *Proposed by Andrew Simoson, King University, Bristol, TN.*

Let a and b be positive integers with $a \geq b$. Prove the following:
(a) $\frac{b}{a+b} + \frac{a+b}{b} > \sqrt{5}$, and
(b) either $\frac{a}{a+b} + \frac{a+b}{a} > \sqrt{5}$ or $\frac{a}{b} + \frac{b}{a} > \sqrt{5}$.

Solution by Charlie Mumma, Seattle, Washington.

For convenience, set $c = (\sqrt{5} - 1)/2$, $d = (\sqrt{5} + 1)/2$, and $f(x) = x + 1/x$. Observe that f is strictly decreasing on $(0, 1)$, strictly increasing on $(1, \infty)$, and $f(c) = f(d) = \sqrt{5}$. Since $a \geq b$, $(a + b)/b \geq 2 > d$, which proves (a) $[f((a + b)/b) > f(d)]$. Next notice that when $a/b + b/a \leq \sqrt{5}$, $c \leq b/a \leq 1$. Hence $(a + b)/a = 1 + b/a \geq 1 + c = d$. If $a = b$, $a/(a + b) + (a + b)/a = 5/2 > \sqrt{5}$. However, for $b = ca$, $a/(a + b) + (a + b)/a = a/b + b/a = \sqrt{5}$. Thus (b) is true so long as a/b is not the golden ratio (a condition less stringent than the requirement that both a and b be integers).

Also solved by Ulrich Abel, Technische Hochschule Mittelhessen, Germany and Georg Arends, Eschweiler, Germany (jointly); Farrukh Rakhimjanovich Ataev, Westminster International U., Tashkent, Uzbekistan;

Michel Bataille, Rouen, France; Brian Beasley, Presybterian C.; Brian Bradie, Christopher Newport U.; Kyle Calderhead, Malone U.; John Christopher, California St. U., Sacramento; Christopher Newport U. Problem Solving Seminar; Matthew Creek, Assumption U.; Richard Daquila, Muskingum U.; Eagle Problem Solvers, Georgia Southern U.; Habin Far, Lone Star C. - Montgomery; Dmitry Fleischman, Santa Monica, CA; Davide Fusi, U. of South Florida Beaufort; Russ Gordon, Whitman C.; Lixing Han, U. of Michigan - Flint and Xinjia Tang, Chang Zhou U.; Eugene Herman, Grinnell C.; Donald Hooley, Bluffton, OH; Tom Jager, Calvin U.; A. Bathi Kasturiarachi, Kent St. U. at Stark; Elias Lampakis, Kiparissia, Greece; Kee-Wai Lau, Hong Kong, China; Seungheon Lee, Yonsei U.; Graham Lord, Princeton, NJ; Rhea Malik; Northwestern U. Math Problem Solving Group; Ángel Plaza and Francisco Perdomo, Universidad de Las Palmas de Gran Canaria, Las Palmas, Spain; Mark Sand; Randy Schwartz, Schoolcraft C. (retired); Albert Stadler, Herrliberg, Switzerland; Enrique Treviño, Lake Forest C.; Michael Vowe, Therwil, Switzerland; Roy Willits; Lienhard Wimmer; and the proposer.

A sum of harmonic sums

1195. *Proposed by Marián Štofka, Slovak University of Technology, Bratislava, Slovak Republic.*

Prove the following:

$$\sum_{k=1}^{\infty} \frac{H_k}{k+1}\left(\frac{\pi^2}{6} - H_{k+1,2}\right) = \frac{\pi^4}{90},$$

where $H_k = \sum_{i=1}^{k} \frac{1}{i}$ is the kth harmonic number and $H_{k,2} = \sum_{i=1}^{k} \frac{1}{i^2}$ is the kth generalized harmonic number.

Solution by Russ Gordon, Whitman College, Walla Walla, WA.

Since $\frac{1}{6}\pi^2 = \sum_{k=1}^{\infty} \left(1/k^2\right)$, we can express the given sum as

$$\sum_{\infty}^{n=1} \sum_{k=n+2}^{\infty} \frac{}{(n+1)k^2} = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{}{(n+1)(n+k+1)^2}.$$

Using integration by parts, it is not difficult to verify that

$$\int_0^1 -x^{n-1} \ln x \, dx = \frac{1}{n^2} \text{ and } \int_0^1 x^{n-1} (\ln x)^2 \, dx = \frac{2}{n^3}$$

for each positive integer n. We also make note of the following Macluarin series:

$$-\ln(1-x) = \sum_{n=1}^{\infty} \frac{1}{n} x^n \text{ and } \frac{(\ln(1-x))^2}{2x} = \sum_{n=1}^{\infty} \frac{h_n}{n+1} x^n.$$

Using this information, we find that

$$\begin{aligned}
\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{h_n}{(n+1)(n+k+1)^2} &= \sum_{n=1}^{\infty} \frac{h_n}{n+1} \sum_{k=1}^{\infty} \int_0^1 -x^{n+k} \ln x \, dx \\
&= \sum_{n=1}^{\infty} \frac{h_n}{n+1} \int_0^1 \frac{-x^{n+1}}{1-x} \ln x \, dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 \frac{-\ln x}{1-x} \sum_{n=1}^{\infty} \frac{h_n}{n+1} x^{n+1}\, dx \\
&= \int_0^1 \frac{-\ln x}{1-x} \cdot \frac{(\ln(1-x))^2}{2}\, dx \\
&= \frac{1}{2} \int_0^1 \frac{-\ln(1-x)(\ln x)^2}{x}\, dx \\
&= \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 x^{n-1} (\ln x)^2\, dx \\
&= \frac{1}{2} \sum_{n=1}^{\infty} \frac{2}{n^4} \\
&= \frac{\pi^4}{90},
\end{aligned}
$$

the desired result.

Also solved by Michel Bataille, Rouen, France; Gerald Bilodeau, Boston Latin School; Khristo Boyadzhiev, Ohio Northern U.; Paul Bracken, U. of Texas, Edinburg; Brian Bradie, Christopher Newport U.; Bruce Burdick, Roger Williams U.; Hongwei Chen, Christopher Newport U.; Lixing Han, U. of Michigan-Flint and Xinjia Tang, Chang Zhou U.; Eugene Herman, Grinnell C.; Omran Kouba, Higher Inst. for Applied Sci. and Tech., Damascus, Syria. Elias Lampakis, Kiparissia, Greece; Albert Stadler, Herrliberg, Switzerland; Seán Stewart, Bomaderry, NSW, Australia; Michael Vowe, Therwil, Switzerland; and the proposer.

Editor's note: The name of James Brenneis was omitted from the list of solvers of problem 1183 in the November 2021 issue. We apologize for the omission.