JEE (ADVANCED) 2019 PAPER 1
PART-III MATHEMATICS

SECTION 1 (Maximum Marks: 12)

- This section contains **FOUR (04)** questions.
- Each question has **FOUR** options. **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme:
  Full Marks : +3 If ONLY the correct option is chosen;
  Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered);
  Negative Marks : −1 In all other cases.

Q.1 Let S be the set of all complex numbers z satisfying $|z-2+i| \geq \sqrt{5}$. If the complex number z_0 is such that $\frac{1}{|z_0-1|}$ is the maximum of the set $\left\{\frac{1}{|z-1|} : z \in S\right\}$, then the principal argument of $\frac{4-z_0-\overline{z_0}}{z_0-\overline{z_0}+2i}$ is

(A) $-\frac{\pi}{2}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{2}$ (D) $\frac{3\pi}{4}$

Q.2 Let

$$M = \begin{bmatrix} \sin^4\theta & -1-\sin^2\theta \\ 1+\cos^2\theta & \cos^4\theta \end{bmatrix} = \alpha I + \beta M^{-1},$$

where $\alpha = \alpha(\theta)$ and $\beta = \beta(\theta)$ are real numbers, and I is the 2×2 identity matrix. If

α^* is the minimum of the set $\{\alpha(\theta) : \theta \in [0, 2\pi)\}$ and

β^* is the minimum of the set $\{\beta(\theta) : \theta \in [0, 2\pi)\}$,

then the value of $\alpha^* + \beta^*$ is

(A) $-\frac{37}{16}$ (B) $-\frac{31}{16}$ (C) $-\frac{29}{16}$ (D) $-\frac{17}{16}$

Q.3 A line $y = mx + 1$ intersects the circle $(x-3)^2 + (y+2)^2 = 25$ at the points P and Q. If the midpoint of the line segment PQ has x-coordinate $-\frac{3}{5}$, then which one of the following options is correct?

(A) $-3 \leq m < -1$ (B) $2 \leq m < 4$

(C) $4 \leq m < 6$ (D) $6 \leq m < 8$

Q.4 The area of the region $\{(x, y) : \; xy \leq 8, \; 1 \leq y \leq x^2\}$ is

(A) $16 \log_e 2 - \frac{14}{3}$

(B) $8 \log_e 2 - \frac{14}{3}$

(C) $16 \log_e 2 - 6$

(D) $8 \log_e 2 - \frac{7}{3}$

SECTION 2 (Maximum Marks: 32)

- This section contains **EIGHT (08)** questions.
- Each question has **FOUR** options. **ONE OR MORE THAN ONE** of these four option(s) is(are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:

  Full Marks : +4 If only (all) the correct option(s) is(are) chosen;
  Partial Marks : +3 If all the four options are correct but ONLY three options are chosen;
  Partial Marks : +2 If three or more options are correct but ONLY two options are chosen and both of which are correct;
  Partial Marks : +1 If two or more options are correct but ONLY one option is chosen and it is a correct option;
  Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered);
  Negative Marks : −1 In all other cases.
- For example, in a question, if (A), (B) and (D) are the ONLY three options corresponding to correct answers, then
  choosing ONLY (A), (B) and (D) will get +4 marks;
  choosing ONLY (A) and (B) will get +2 marks;
  choosing ONLY (A) and (D) will get +2 marks;
  choosing ONLY (B) and (D) will get +2 marks;
  choosing ONLY (A) will get +1 mark;
  choosing ONLY (B) will get +1 mark;
  choosing ONLY (D) will get +1 mark;
  choosing no option (i.e. the question is unanswered) will get 0 marks; and
  choosing any other combination of options will get −1 mark.

Q.5 Let α and β be the roots of $x^2 - x - 1 = 0$, with $\alpha > \beta$. For all positive integers n, define

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad n \geq 1,$$
$$b_1 = 1 \text{ and } b_n = a_{n-1} + a_{n+1}, \quad n \geq 2.$$

Then which of the following options is/are correct?

(A) $a_1 + a_2 + a_3 + \cdots + a_n = a_{n+2} - 1$ for all $n \geq 1$

(B) $\sum_{n=1}^{\infty} \frac{a_n}{10^n} = \frac{10}{89}$

(C) $b_n = \alpha^n + \beta^n$ for all $n \geq 1$

(D) $\sum_{n=1}^{\infty} \frac{b_n}{10^n} = \frac{8}{89}$

Q.6 Let

$$M = \begin{bmatrix} 0 & 1 & a \\ 1 & 2 & 3 \\ 3 & b & 1 \end{bmatrix} \quad \text{and} \quad \operatorname{adj} M = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$$

where a and b are real numbers. Which of the following options is/are correct?

(A) $a + b = 3$

(B) $(\operatorname{adj} M)^{-1} + \operatorname{adj} M^{-1} = -M$

(C) $\det(\operatorname{adj} M^2) = 81$

(D) If $M \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, then $\alpha - \beta + \gamma = 3$

Q.7 There are three bags B_1, B_2 and B_3. The bag B_1 contains 5 red and 5 green balls, B_2 contains 3 red and 5 green balls, and B_3 contains 5 red and 3 green balls. Bags B_1, B_2 and B_3 have probabilities $\frac{3}{10}$, $\frac{3}{10}$ and $\frac{4}{10}$ respectively of being chosen. A bag is selected at random and a ball is chosen at random from the bag. Then which of the following options is/are correct?

(A) Probability that the chosen ball is green, given that the selected bag is B_3, equals $\frac{3}{8}$

(B) Probability that the chosen ball is green equals $\frac{39}{80}$

(C) Probability that the selected bag is B_3, given that the chosen ball is green, equals $\frac{5}{13}$

(D) Probability that the selected bag is B_3 and the chosen ball is green equals $\frac{3}{10}$

Q.8 In a non-right-angled triangle ΔPQR, let p, q, r denote the lengths of the sides opposite to the angles at P, Q, R respectively. The median from R meets the side PQ at S, the perpendicular from P meets the side QR at E, and RS and PE intersect at O. If $p = \sqrt{3}$, $q = 1$, and the radius of the circumcircle of the ΔPQR equals 1, then which of the following options is/are correct?

(A) Length of $RS = \frac{\sqrt{7}}{2}$

(B) Area of $\Delta SOE = \frac{\sqrt{3}}{12}$

(C) Length of $OE = \frac{1}{6}$

(D) Radius of incircle of $\Delta PQR = \frac{\sqrt{3}}{2}\left(2 - \sqrt{3}\right)$

Q.9 Define the collections $\{E_1, E_2, E_3, \ldots\}$ of ellipses and $\{R_1, R_2, R_3, \ldots\}$ of rectangles as follows:

E_1: $\frac{x^2}{9} + \frac{y^2}{4} = 1$;

R_1: rectangle of largest area, with sides parallel to the axes, inscribed in E_1;

E_n: ellipse $\frac{x^2}{a_n^2} + \frac{y^2}{b_n^2} = 1$ of largest area inscribed in R_{n-1}, $n > 1$;

R_n: rectangle of largest area, with sides parallel to the axes, inscribed in E_n, $n > 1$.

Then which of the following options is/are correct?

(A) The eccentricities of E_{18} and E_{19} are NOT equal

(B) $\sum_{n=1}^{N} (\text{area of } R_n) < 24$, for each positive integer N

(C) The length of latus rectum of E_9 is $\frac{1}{6}$

(D) The distance of a focus from the centre in E_9 is $\frac{\sqrt{5}}{32}$

Q.10 Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be given by

$$f(x) = \begin{cases} x^5 + 5x^4 + 10x^3 + 10x^2 + 3x + 1, & x < 0; \\ x^2 - x + 1, & 0 \leq x < 1; \\ \frac{2}{3}x^3 - 4x^2 + 7x - \frac{8}{3}, & 1 \leq x < 3; \\ (x-2)\log_e(x-2) - x + \frac{10}{3}, & x \geq 3. \end{cases}$$

Then which of the following options is/are correct?

(A) f is increasing on $(-\infty, 0)$

(B) f' has a local maximum at $x = 1$

(C) f is onto

(D) f' is NOT differentiable at $x = 1$

Q.11 Let Γ denote a curve $y = y(x)$ which is in the first quadrant and let the point $(1,0)$ lie on it. Let the tangent to Γ at a point P intersect the y-axis at Y_P. If PY_P has length 1 for each point P on Γ, then which of the following options is/are correct?

(A) $y = \log_e\left(\frac{1+\sqrt{1-x^2}}{x}\right) - \sqrt{1-x^2}$

(B) $xy' + \sqrt{1-x^2} = 0$

(C) $y = -\log_e\left(\frac{1+\sqrt{1-x^2}}{x}\right) + \sqrt{1-x^2}$

(D) $xy' - \sqrt{1-x^2} = 0$

Q.12 Let L_1 and L_2 denote the lines

$$\vec{r} = \hat{\imath} + \lambda\left(-\hat{\imath} + 2\hat{\jmath} + 2\hat{k}\right),\ \lambda \in \mathbb{R} \text{ and}$$
$$\vec{r} = \mu\left(2\hat{\imath} - \hat{\jmath} + 2\hat{k}\right),\ \mu \in \mathbb{R}$$

respectively. If L_3 is a line which is perpendicular to both L_1 and L_2 and cuts both of them, then which of the following options describe(s) L_3?

(A) $\vec{r} = \frac{2}{9}\left(4\hat{\imath} + \hat{\jmath} + \hat{k}\right) + t\left(2\hat{\imath} + 2\hat{\jmath} - \hat{k}\right),\ t \in \mathbb{R}$

(B) $\vec{r} = \frac{2}{9}\left(2\hat{\imath} - \hat{\jmath} + 2\hat{k}\right) + t\left(2\hat{\imath} + 2\hat{\jmath} - \hat{k}\right),\ t \in \mathbb{R}$

(C) $\vec{r} = \frac{1}{3}\left(2\hat{\imath} + \hat{k}\right) + t\left(2\hat{\imath} + 2\hat{\jmath} - \hat{k}\right),\ t \in \mathbb{R}$

(D) $\vec{r} = t\left(2\hat{\imath} + 2\hat{\jmath} - \hat{k}\right),\ t \in \mathbb{R}$

SECTION 3 (Maximum Marks: 18)

- This section contains **SIX (06)** questions. The answer to each question is a **NUMERICAL VALUE.**
- For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer. If the numerical value has more than two decimal places, **truncate/round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme:
  Full Marks : +3 If ONLY the correct numerical value is entered;
  Zero Marks : 0 In all other cases.

Q.13 Let $\omega \neq 1$ be a cube root of unity. Then the minimum of the set

$$\left\{|a + b\omega + c\omega^2|^2 \ : \ a, b, c \text{ distinct non-zero integers}\right\}$$

equals ____

Q.14 Let $AP(a; d)$ denote the set of all the terms of an infinite arithmetic progression with first term a and common difference $d > 0$. If

$$AP(1;3) \cap AP(2;5) \cap AP(3;7) = AP(a;d)$$

then $a + d$ equals___

Q.15 Let S be the sample space of all 3×3 matrices with entries from the set $\{0, 1\}$. Let the events E_1 and E_2 be given by

$$E_1 = \{A \in S \ : \ \det A = 0\} \text{ and}$$
$$E_2 = \{A \in S \ : \text{sum of entries of } A \text{ is } 7\}.$$

If a matrix is chosen at random from S, then the conditional probability $P(E_1|E_2)$ equals ___

Q.16 Let the point B be the reflection of the point $A(2, 3)$ with respect to the line $8x - 6y - 23 = 0$. Let Γ_A and Γ_B be circles of radii 2 and 1 with centres A and B respectively. Let T be a common tangent to the circles Γ_A and Γ_B such that both the circles are on the same side of T. If C is the point of intersection of T and the line passing through A and B, then the length of the line segment AC is___

Q.17 If

$$I = \frac{2}{\pi} \int_{-\pi/4}^{\pi/4} \frac{dx}{(1 + e^{\sin x})(2 - \cos 2x)}$$

then $27\,I^2$ equals___

Q.18 Three lines are given by

$$\vec{r} = \lambda\,\hat{\imath},\ \ \lambda \in \mathbb{R}$$
$$\vec{r} = \mu\,(\hat{\imath} + \hat{\jmath}),\ \ \mu \in \mathbb{R}\ \text{ and}$$
$$\vec{r} = v(\hat{\imath} + \hat{\jmath} + \hat{k}),\ \ v \in \mathbb{R}.$$

Let the lines cut the plane $x + y + z = 1$ at the points A, B and C respectively. If the area of the triangle ABC is Δ then the value of $(6\Delta)^2$ equals___