JEE (ADVANCED) 2018 PAPER 2
PART-III MATHEMATICS

SECTION 1 (Maximum Marks: 24)

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options for correct answer(s). **ONE OR MORE THAN ONE** of these four option(s) is (are) correct option(s).
- For each question, choose the correct option(s) to answer the question.
- Answer to each question will be evaluated according to the following marking scheme:
  Full Marks : $+4$ If only (all) the correct option(s) is (are) chosen.
  Partial Marks : $+3$ If all the four options are correct but ONLY three options are chosen.
  Partial Marks : $+2$ If three or more options are correct but ONLY two options are chosen, both of which are correct options.
  Partial Marks : $+1$ If two or more options are correct but ONLY one option is chosen and it is a correct option.
  Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered).
  Negative Marks : -2 In all other cases.
- **For Example:** If first, third and fourth are the ONLY three correct options for a question with second option being an incorrect option; selecting only all the three correct options will result in +4 marks. Selecting only two of the three correct options (e.g. the first and fourth options), without selecting any incorrect option (second option in this case), will result in +2 marks. Selecting only one of the three correct options (either first or third or fourth option) , without selecting any incorrect option (second option in this case), will result in +1 marks. Selecting any incorrect option(s) (second option in this case), with or without selection of any correct option(s) will result in -2 marks.

Q.1 For any positive integer n, define $f_n: (0, \infty) \to \mathbb{R}$ as

$$f_n(x) = \sum_{j=1}^{n} \tan^{-1}\left(\frac{1}{1+(x+j)(x+j-1)}\right) \text{ for all } x \in (0, \infty).$$

$\left(\text{Here, the inverse trigonometric function } \tan^{-1}x \text{ assumes values in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).\right)$

Then, which of the following statement(s) is (are) TRUE?

(A) $\sum_{j=1}^{5} \tan^2(f_j(0)) = 55$

(B) $\sum_{j=1}^{10} \left(1 + f_j'(0)\right) \sec^2(f_j(0)) = 10$

(C) For any fixed positive integer n, $\lim_{x\to\infty} \tan(f_n(x)) = \frac{1}{n}$

(D) For any fixed positive integer n, $\lim_{x\to\infty} \sec^2(f_n(x)) = 1$

Q.2 Let T be the line passing through the points $P(-2,7)$ and $Q(2,-5)$. Let F_1 be the set of all pairs of circles (S_1, S_2) such that T is tangent to S_1 at P and tangent to S_2 at Q, and also such that S_1 and S_2 touch each other at a point, say, M. Let E_1 be the set representing the locus of M as the pair (S_1, S_2) varies in F_1. Let the set of all straight line segments joining a pair of distinct points of E_1 and passing through the point $R(1,1)$ be F_2. Let E_2 be the set of the mid-points of the line segments in the set F_2. Then, which of the following statement(s) is (are) TRUE?

(A) The point $(-2,7)$ lies in E_1

(B) The point $\left(\frac{4}{5}, \frac{7}{5}\right)$ does **NOT** lie in E_2

(C) The point $\left(\frac{1}{2}, 1\right)$ lies in E_2

(D) The point $\left(0, \frac{3}{2}\right)$ does **NOT** lie in E_1

Q.3 Let S be the set of all column matrices $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ such that $b_1, b_2, b_3 \in \mathbb{R}$ and the system of equations (in real variables)

$$-x + 2y + 5z = b_1$$
$$2x - 4y + 3z = b_2$$
$$x - 2y + 2z = b_3$$

has at least one solution. Then, which of the following system(s) (in real variables) has (have) at least one solution for each $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in S$?

(A) $x + 2y + 3z = b_1,\ 4y + 5z = b_2$ and $x + 2y + 6z = b_3$

(B) $x + y + 3z = b_1,\ 5x + 2y + 6z = b_2$ and $-2x - y - 3z = b_3$

(C) $-x + 2y - 5z = b_1,\ 2x - 4y + 10z = b_2$ and $x - 2y + 5z = b_3$

(D) $x + 2y + 5z = b_1,\ 2x + 3z = b_2$ and $x + 4y - 5z = b_3$

Q.4 Consider two straight lines, each of which is tangent to both the circle $x^2 + y^2 = \frac{1}{2}$ and the parabola $y^2 = 4x$. Let these lines intersect at the point Q. Consider the ellipse whose center is at the origin $O(0,0)$ and whose semi-major axis is OQ. If the length of the minor axis of this ellipse is $\sqrt{2}$, then which of the following statement(s) is (are) TRUE?

(A) For the ellipse, the eccentricity is $\frac{1}{\sqrt{2}}$ and the length of the latus rectum is 1

(B) For the ellipse, the eccentricity is $\frac{1}{2}$ and the length of the latus rectum is $\frac{1}{2}$

(C) The area of the region bounded by the ellipse between the lines $x = \frac{1}{\sqrt{2}}$ and $x = 1$ is $\frac{1}{4\sqrt{2}}(\pi - 2)$

(D) The area of the region bounded by the ellipse between the lines $x = \frac{1}{\sqrt{2}}$ and $x = 1$ is $\frac{1}{16}(\pi - 2)$

Q.5 Let s, t, r be non-zero complex numbers and L be the set of solutions $z = x + iy$ $\left(x, y \in \mathbb{R},\ i = \sqrt{-1}\right)$ of the equation $sz + t\bar{z} + r = 0$, where $\bar{z} = x - iy$. Then, which of the following statement(s) is (are) TRUE?

(A) If L has exactly one element, then $|s| \neq |t|$

(B) If $|s| = |t|$, then L has infinitely many elements

(C) The number of elements in $L \cap \{z : |z - 1 + i| = 5\}$ is at most 2

(D) If L has more than one element, then L has infinitely many elements

Q.6 Let $f: (0, \pi) \rightarrow \mathbb{R}$ be a twice differentiable function such that

$$\lim_{t \to x} \frac{f(x)\,\sin t - f(t)\sin x}{t - x} = \sin^2 x \text{ for all } x \in (0, \pi).$$

If $f\left(\frac{\pi}{6}\right) = -\frac{\pi}{12}$, then which of the following statement(s) is (are) TRUE?

(A) $f\left(\frac{\pi}{4}\right) = \frac{\pi}{4\sqrt{2}}$

(B) $f(x) < \frac{x^4}{6} - x^2$ for all $x \in (0, \pi)$

(C) There exists $\alpha \in (0, \pi)$ such that $f'(\alpha) = 0$

(D) $f''\left(\frac{\pi}{2}\right) + f\left(\frac{\pi}{2}\right) = 0$

SECTION 2 (Maximum Marks: 24)

- This section contains **EIGHT (08)** questions. The answer to each question is a **NUMERICAL VALUE.**
- For each question, enter the correct numerical value (in decimal notation, truncated/rounded-off to the **second decimal place**; e.g. 6.25, 7.00, -0.33, -.30, 30.27, -127.30) using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme:

  Full Marks : $+3$ If ONLY the correct numerical value is entered as answer.

  Zero Marks : 0 In all other cases.

Q.7 The value of the integral

$$\int_0^{\frac{1}{2}} \frac{1+\sqrt{3}}{\left((x+1)^2(1-x)^6\right)^{\frac{1}{4}}}\, dx$$

is _____ .

Q.8 Let P be a matrix of order 3×3 such that all the entries in P are from the set $\{-1, 0, 1\}$. Then, the maximum possible value of the determinant of P is _____ .

Q.9 Let X be a set with exactly 5 elements and Y be a set with exactly 7 elements. If α is the number of one-one functions from X to Y and β is the number of onto functions from Y to X, then the value of $\frac{1}{5!}(\beta - \alpha)$ is _____ .

Q.10 Let $f: \mathbb{R} \to \mathbb{R}$ be a differentiable function with $f(0) = 0$. If $y = f(x)$ satisfies the differential equation

$$\frac{dy}{dx} = (2+5y)(5y-2),$$

then the value of $\lim_{x \to -\infty} f(x)$ is _____.

Q.11 Let $f: \mathbb{R} \to \mathbb{R}$ be a differentiable function with $f(0) = 1$ and satisfying the equation

$$f(x+y) = f(x)f'(y) + f'(x)f(y) \text{ for all } x, y \in \mathbb{R}.$$

Then, the value of $\log_e(f(4))$ is _____.

Q.12 Let P be a point in the first octant, whose image Q in the plane $x + y = 3$ (that is, the line segment PQ is perpendicular to the plane $x + y = 3$ and the mid-point of PQ lies in the plane $x + y = 3$) lies on the z-axis. Let the distance of P from the x-axis be 5. If R is the image of P in the xy-plane, then the length of PR is _____ .

Q.13 Consider the cube in the first octant with sides OP, OQ and OR of length 1, along the x-axis, y-axis and z-axis, respectively, where $O(0,0,0)$ is the origin. Let $S\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ be the centre of the cube and T be the vertex of the cube opposite to the origin O such that S lies on the diagonal OT. If $\vec{p} = \overrightarrow{SP}$, $\vec{q} = \overrightarrow{SQ}$, $\vec{r} = \overrightarrow{SR}$ and $\vec{t} = \overrightarrow{ST}$, then the value of $\left|(\vec{p} \times \vec{q}) \times (\vec{r} \times \vec{t})\right|$ is _____ .

Q.14 Let

$$X = \left({}^{10}C_1\right)^2 + 2\left({}^{10}C_2\right)^2 + 3\left({}^{10}C_3\right)^2 + \cdots + 10\left({}^{10}C_{10}\right)^2,$$

where ${}^{10}C_r$, $r \in \{1, 2, \cdots, 10\}$ denote binomial coefficients. Then, the value of $\frac{1}{1430} X$ is _____ .

SECTION 3 (Maximum Marks: 12)

- This section contains **FOUR (04)** questions.
- Each question has **TWO (02)** matching lists: **LIST-I** and **LIST-II**.
- **FOUR** options are given representing matching of elements from **LIST-I** and **LIST-II**. **ONLY ONE** of these four options corresponds to a correct matching.
- For each question, choose the option corresponding to the correct matching.
- For each question, marks will be awarded according to the following marking scheme:
  Full Marks : +3 If ONLY the option corresponding to the correct matching is chosen.
  Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered).
  Negative Marks : −1 In all other cases.

Q.15 Let $E_1 = \left\{x \in \mathbb{R} : x \neq 1 \text{ and } \frac{x}{x-1} > 0\right\}$
and $E_2 = \left\{x \in E_1 : \sin^{-1}\left(\log_e\left(\frac{x}{x-1}\right)\right) \text{is a real number}\right\}$.
$\left(\text{Here, the inverse trigonometric function } \sin^{-1}x \text{ assumes values in } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].\right)$
Let $f: E_1 \to \mathbb{R}$ be the function defined by $f(x) = \log_e\left(\frac{x}{x-1}\right)$
and $g: E_2 \to \mathbb{R}$ be the function defined by $g(x) = \sin^{-1}\left(\log_e\left(\frac{x}{x-1}\right)\right)$.

LIST–I	LIST–II
P. The range of f is	**1.** $\left(-\infty, \frac{1}{1-e}\right] \cup \left[\frac{e}{e-1}, \infty\right)$
Q. The range of g contains	**2.** $(0, 1)$
R. The domain of f contains	**3.** $\left[-\frac{1}{2}, \frac{1}{2}\right]$
S. The domain of g is	**4.** $(-\infty, 0) \cup (0, \infty)$
	5. $\left(-\infty, \frac{e}{e-1}\right]$
	6. $(-\infty, 0) \cup \left(\frac{1}{2}, \frac{e}{e-1}\right]$

The correct option is:

(A) **P → 4; Q → 2; R → 1; S → 1**
(B) **P → 3; Q → 3; R → 6; S → 5**
(C) **P → 4; Q → 2; R → 1; S → 6**
(D) **P → 4; Q → 3; R → 6; S → 5**

Q.16 In a high school, a committee has to be formed from a group of 6 boys $M_1, M_2, M_3, M_4, M_5, M_6$ and 5 girls G_1, G_2, G_3, G_4, G_5.

(i) Let α_1 be the total number of ways in which the committee can be formed such that the committee has 5 members, having exactly 3 boys and 2 girls.

(ii) Let α_2 be the total number of ways in which the committee can be formed such that the committee has at least 2 members, and having an equal number of boys and girls.

(iii) Let α_3 be the total number of ways in which the committee can be formed such that the committee has 5 members, at least 2 of them being girls.

(iv) Let α_4 be the total number of ways in which the committee can be formed such that the committee has 4 members, having at least 2 girls and such that both M_1 and G_1 are **NOT** in the committee together.

LIST–I	**LIST–II**
P. The value of α_1 is	**1.** 136
Q. The value of α_2 is	**2.** 189
R. The value of α_3 is	**3.** 192
S. The value of α_4 is	**4.** 200
	5. 381
	6. 461

The correct option is:

(A) **P → 4; Q → 6; R → 2; S → 1**

(B) **P → 1; Q → 4; R → 2; S → 3**

(C) **P → 4; Q → 6; R → 5; S → 2**

(D) **P → 4; Q → 2; R → 3; S → 1**

Q.17 Let $H: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $a > b > 0$, be a hyperbola in the xy-plane whose conjugate axis LM subtends an angle of 60^0 at one of its vertices N. Let the area of the triangle LMN be $4\sqrt{3}$.

LIST–I	LIST–II
P. The length of the conjugate axis of H is	**1.** 8
Q. The eccentricity of H is	**2.** $\frac{4}{\sqrt{3}}$
R. The distance between the foci of H is	**3.** $\frac{2}{\sqrt{3}}$
S. The length of the latus rectum of H is	**4.** 4

The correct option is:

(A) **P → 4; Q → 2; R → 1; S → 3**

(B) **P → 4; Q → 3; R → 1; S → 2**

(C) **P → 4; Q → 1; R → 3; S → 2**

(D) **P → 3; Q → 4; R → 2; S → 1**

Q.18 Let $f_1: \mathbb{R} \rightarrow \mathbb{R}$, $f_2: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow \mathbb{R}$, $f_3: \left(-1, e^{\frac{\pi}{2}} - 2\right) \rightarrow \mathbb{R}$ and $f_4: \mathbb{R} \rightarrow \mathbb{R}$ be functions defined by

(i) $f_1(x) = \sin\left(\sqrt{1 - e^{-x^2}}\right)$,

(ii) $f_2(x) = \begin{cases} \frac{|\sin x|}{\tan^{-1} x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$, where the inverse trigonometric function $\tan^{-1} x$ assumes values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

(iii) $f_3(x) = [\sin(\log_e(x+2))]$, where, for $t \in \mathbb{R}$, $[t]$ denotes the greatest integer less than or equal to t,

(iv) $f_4(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

LIST–I	LIST–II
P. The function f_1 is	**1.** **NOT** continuous at $x = 0$
Q. The function f_2 is	**2.** continuous at $x = 0$ and **NOT** differentiable at $x = 0$
R. The function f_3 is	**3.** differentiable at $x = 0$ and its derivative is **NOT** continuous at $x = 0$
S. The function f_4 is	**4.** differentiable at $x = 0$ and its derivative is continuous at $x = 0$

The correct option is:

(A) **P → 2; Q → 3; R → 1; S → 4**

(B) **P → 4; Q → 1; R → 2; S → 3**

(C) **P → 4; Q → 2; R → 1; S → 3**

(D) **P → 2; Q → 1; R → 4; S → 3**

END OF THE QUESTION PAPER