Convex Optimisation and KKT Conditions

Vijay Jaisankar

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Agenda

1. Convex Sets and Functions
2. Lagrangian Definition
3. Dual Problems
4. Strong Duality and KKT Conditions

Convexity

Convex Set

- $\forall x, y \in S, t \in [0, 1],\ tx + (1 - t)y \in S.$
- "Potato good, Potato with hole bad."

Convex Function

- $\forall x, y \in S, t \in [0,1],\ f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$
- "Straight line joining two points on the function is *above* the function itself."
- Hessian Matrix Positive Semi Definite.
- For convex functions g, *sub-level sets* of the foem $\{x|g(x) \leq 0\}$ are also convex - useful going forward

Lagrangian

Also, some simple examples of Convex Optimisation.

Why Convex Optimisation is Overpowered

Problem 1

Equality Constraint

Minimise $f(x)$ subject to $g(x) = 0$,where f and g are convex.

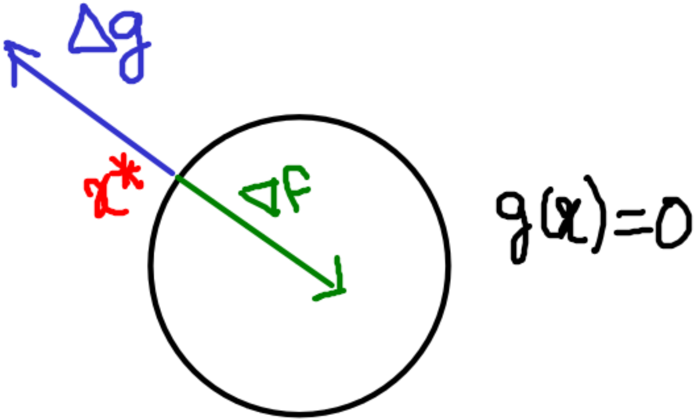


Figure: Problem 1 - Pic 1

Problem 1

- $\exists v \in R, \nabla f(x) + v \cdot \nabla g(x) = 0$
- What if we had a function that could combine these into a simpler format?

Problem 1 - Lagrangian

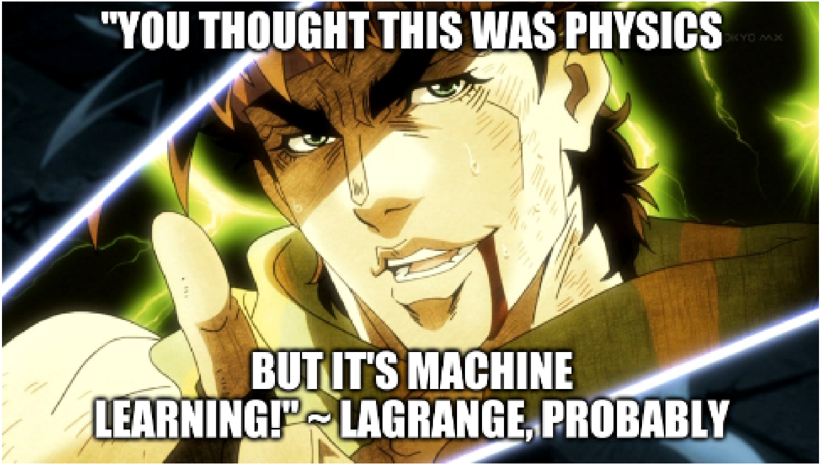


Figure: The Lagrangian - also, a Jojo reference.

Problem 1 - Lagrangian

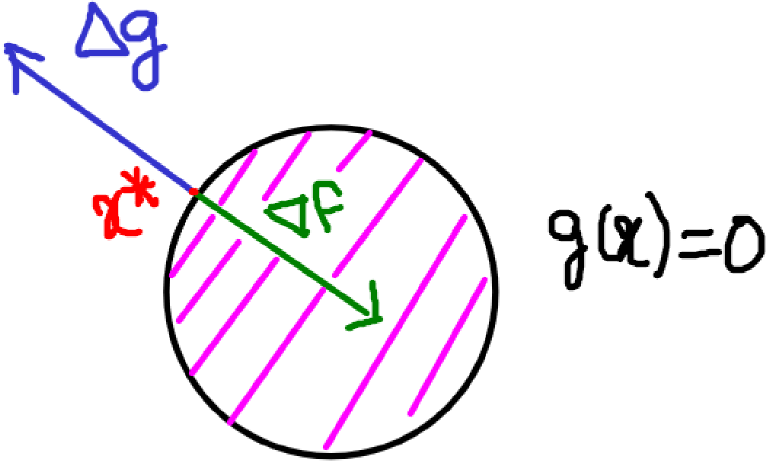
$L(x, v) = f(x) + v \cdot g(x)$
To find the optimal point x^*, we need to find the **saddle point** of L.

Problem 2

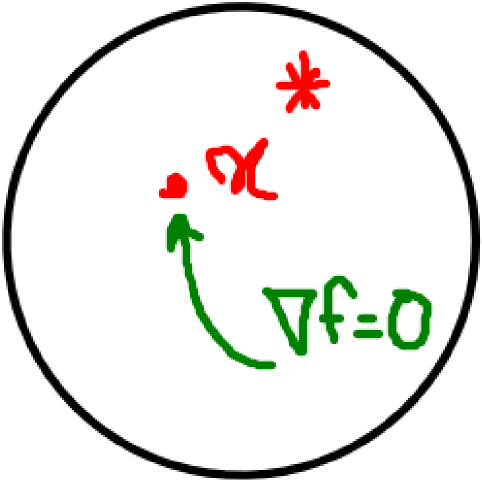
Inequality Constraint Minimise $f(x)$ subject to $g(x) \leq 0$,where f and g are convex. We distinguish two cases:

- Constraint is *active*: Optimal point is on $g(x) = 0$
- Constraint is *inactive*: Optimal point is not on $g(x) = 0$, but somewhere inside.

Active Constraint



Inactive Constraint



Problem 2 - Lagrangian

Much like in *Problem 1*, we can abstract the cases into a *Lagrangian*. $L(x, \lambda) = f(x) + \lambda \cdot g(x), \lambda \geq 0$.

Also, to make our lives easier, we will consider an extra condition :
$\lambda \cdot g(x^*) = 0$
~~KKT Condition Foreshadowing!~~

Dual Problems

Formal Definition of Convex Optimisation

- Minimise $f_0(x)$ subject to $f_i(x) \leq 0$ and $h_j(x) = 0$.
- $p^* := f_0(x^*)$
- The essence of the Lagrangian Approach: **For each equality constraint j and inequality constraint i, we introduce $v_j \in R$, and $\lambda_i \geq 0$, respectively.**
- $L(x, \lambda, v) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{k} v_j h_j(x)$

Dual Problem

$g(\lambda, v) = inf_x \; L(x, \lambda, v)$ i.e "Fix x and do the same process"
g is always concave(!)

Interesting Fact : $\forall \lambda_i \geq 0, v_j \in R, \; g(\lambda, v) \leq p^*$
We can use this fact to get the **highest lower bound** possible for p^*.

Dual Optimisation Problem

To get the best lower bound possible for p^*, we consider the following problem: $max_{\lambda,v} \quad g(\lambda, v)$ where $\lambda_i \geq 0$, $v_j \in R$. Let the

optimal solution be $d^* := g(\lambda^*, v^*)$.

Strong Duality and KKT Conditions

Strong Duality

- $p^* = d^*$
- If strong Duality holds, if x^* is the solution of the formal problem, and λ^*, v^* are the solutions of the Dual Optimisation problem, $[x^*, \lambda^*, v^*]$ is the saddle point of the Lagrangian.
- Won't it be good even if the other side holds? It'll make things a whole lot easier. Spoiler: They are.

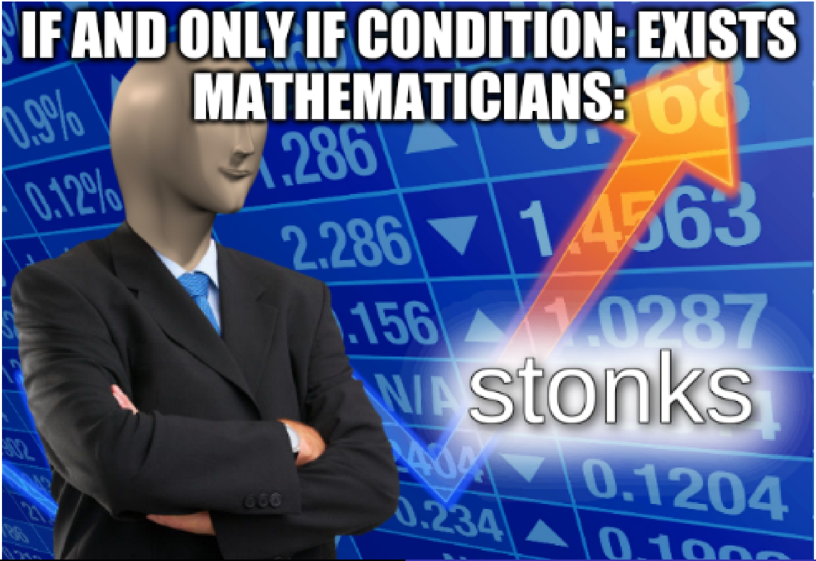
KKT Conditions

Figure: The joy of a double implication.

KKT Conditions

1. $\forall i, j \quad f_i(x) \leq 0$ and $h_j(x) = 0$
2. $\forall \lambda_i \geq 0$
3. $\forall i \quad \lambda_i \cdot g_i(x^*) = 0$
4. Second Order Hessian should be Positive (much like any minimum solution.)

KKT Conditions are equivalent to the Strong Duality Condition.



Summary

We learnt about:

1. Convex Sets and Functions
2. Lagrangian Definition
3. Dual Problems
4. Strong Duality and KKT Conditions

References

- https://www.youtube.com/watch?v=wtpHmTSLZ4c&list=PL05umP7R6ij1a6KdEy8PVE9zoCv6SlHRS&index=89
- https://www.stat.cmu.edu/~ryantibs/convexopt-S15/scribes/12-kkt-scribed.pdf

Thank you :)