Tutorial 2

1. Find the stationary Distribution for $\mathbb{P} = \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}$.

HW

② Consider the MC with TM : $P = \begin{bmatrix} 5/12 & 5/12 & 1/6 \\ 1/4 & 1/4 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$

(i) Is there a unique stationary distribution?

(ii) Can we converge via Power Iteration?

Sol. Note eigenvalues are : $1, \underbrace{0, 0}$

$$\left.\begin{array}{l} AM(0) = 2 \\ GM(0) = 1 \end{array}\right\} \Rightarrow \text{not diagonalizable}$$

Power Iteration (HW) works!

P is irreducible & aperiodic

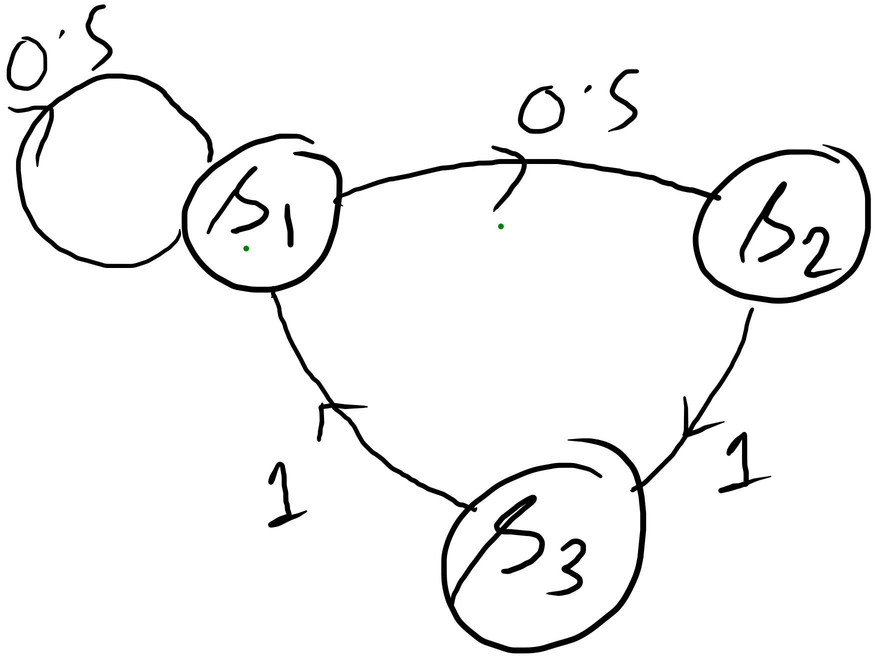
$\Rightarrow$ has unique stationary distribution.

③ Consider the MC with TM : $\mathbb{P} = \begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \end{array} \begin{array}{c} \begin{array}{ccc} s_1 & s_2 & s_3 \end{array} \\ \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$.

(i) Is there a unique stationary distribution?

(ii) Can we converge via Power Iteration?

Sol.



Irreducible & aperiodic

$\Rightarrow$ has unique stationary distribution

(x) Matrix $\mathbb{P}$ has two complex eigenvalues $\Rightarrow$ Pow Iteration works!

"Convergence" using Power Iteration: (Ref. Wikipedia Power Iteration)

Assumptions:

(1) The eigenvalues of the matrix $\mathbb{P}$ satisfies:

$$\lambda_1 = 1 > \underbrace{|\lambda_2| \geq |\lambda_2| \geq \cdots \geq |\lambda_N|}$$

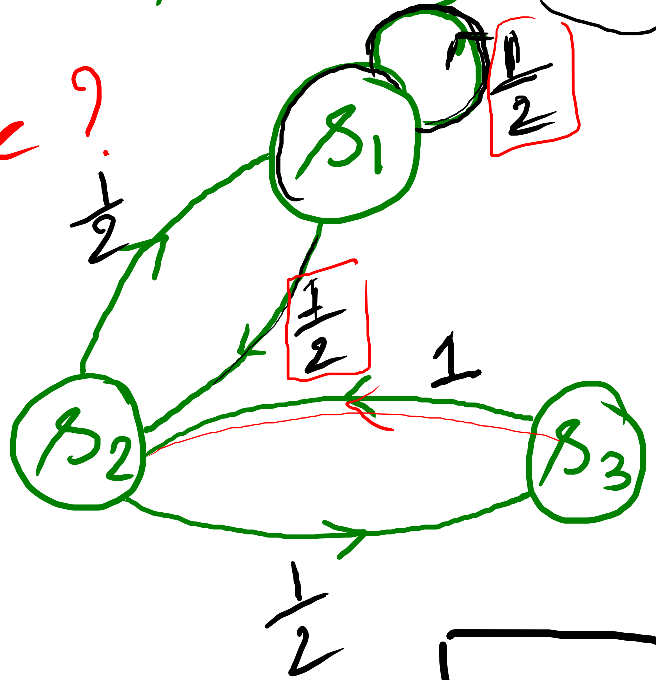
(2) Starting distribution $\underline{p}^{(0)}$ has a $\underbrace{\text{non-zero component}}$ in the direction of an eigenvector associated with the $\underbrace{\text{dominant eigenvalue}}$.

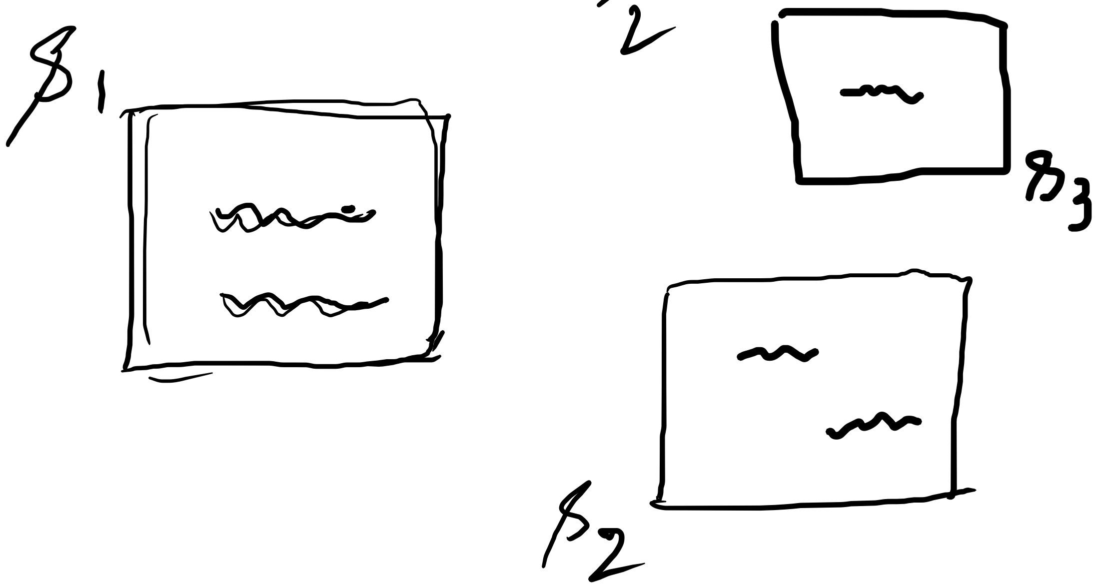
i.e. $\underline{p}^{(0)} = \theta_1 \underline{v}_1 + \theta_2 \underline{v}_2 + \cdots + \theta_N \underline{v}_N$ with $\boxed{\theta_1 \neq 0}$ ⊛

④ Compute the stationary distribution for Random Walk on the following undirected graph:

irreducible? Yes

aperiodic? Yes





$$
\mathbb{P} = \begin{array}{cc|ccc}
 & t \backslash t+1 & p_1^{(t+1)}\ s_1 & p_2^{(t+1)}\ \boxed{s_2} & p_3^{(t+1)}\ s_3 \\
\hline
p_1^{(t)} & \boxed{s_1} & \frac{1}{2} & \times\ \boxed{\frac{1}{2}} & 0 \\
p_2^{(t)} & \boxed{s_2} & \frac{1}{2} & +\ \times\ \boxed{0} & \frac{1}{2} \\
p_3^{(t)} & \boxed{s_3} & 0 & +\ \times\ \boxed{1} & 0
\end{array}
$$

$$\underline{p}^{(t)} = \left[p_1^{(t)} \quad p_2^{(t)} \quad p_3^{(t)} \right]$$

$$\underline{p}^{(t+1)} = \left[p_1^{(t+1)} \quad p_2^{(t+1)} \quad p_3^{(t+1)} \right]$$

$$\boxed{p_j^{(t+1)} = \sum_{i=1}^{3} p_i^{(t)} p_{ij}}$$

$$\boxed{\underline{p}^{(t+1)} = \underline{p}^{(t)} \mathbb{P}}$$

The RW has unique stationary distribution

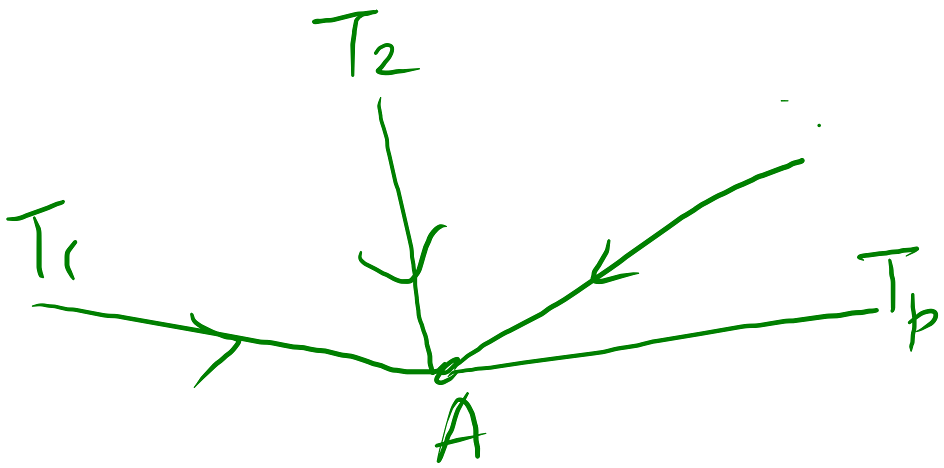
Apply Power Iteration Method to converge

HW (Try yourself!)

<u>Page Rank : of a Page</u> frequency ratio of a RW will be visiting that page over long time

RW of length M

$$PR(A) = \lim_{M \to \infty} \frac{n(A)}{M}$$



$$= P(T_1)\,P(A|T_1) + P(T_2)\,P(A|T_2)$$
$$+ \cdots + P(T_p)\,P(A|T_p)$$