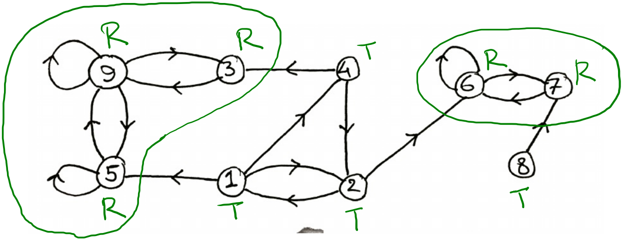
Markov Chain: Recurrent and Transient States

- State s_i is recurrent if "starting from s_i" and from wherever you can go, there is a way of returning to s_i
- If s_i is not recurrent, called transient.



- Recurrence is a class property, i.e. if s_i communicates with s_j and s_i is recurrent, then s_j is recurrent.
- a collection of recurrent states communicating with each other form a **recurrence class**

For a state s_i, we define:

$$f_{ii} = P(X_n = s_i,\ \text{for some } n \geq 1 \mid X_0 = s_i)$$

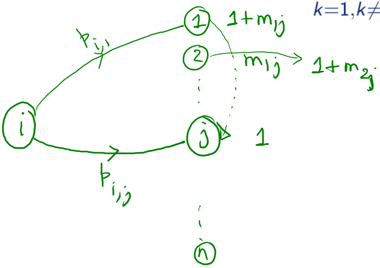
State s_i is *recurrent* if $\boxed{f_{ii} = 1}$ → RW will visit s_i infinitely many times

State s_i is *transient* if $\boxed{f_{ii} < 1}$ → - - - - "finite" no. of times

Random Walk: Mean First Passage Time/ Hitting Time

m_{ij} = Expected number of transitions (time) before a random walker first reaches state s_j, given that walker is currently in state s_i

$$m_{ij} = 1 + \sum_{k=1, k\neq j}^{n} p_{ik} m_{kj}$$



$$m_{ij} = p_{ij} \times 1 + p_{i,1}(1+m_{1j}) + p_{2,j}(1+m_{2j}) + \cdots$$

$$= p_{ij} + \sum_{\substack{k=1 \\ k\neq j}}^{n} p_{ik}(1+m_{kj})$$

$$= \sum_{k=1}^{n} p_{ik} + \sum_{\substack{k=1 \\ k\neq j}}^{n} p_{ik} m_{kj}$$

Random Walk: Mean Recurrence Time

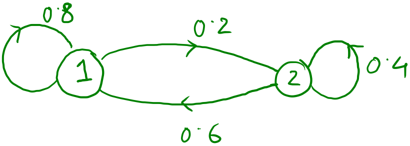
m^*_{ii} = Expected number of transitions (time) before a random walker re-visits state s_i, starting from s_i

$$m^*_{ii} = 1 + \sum_{k=1, k \neq i}^{n} p_{ik} m_{ki}$$

Property: $\boxed{\frac{1}{m^*_{ii}} = \pi_i}$
(discuss the intuitive proof)

Compute: Mean First Passage and Mean Recurrence Time

Example:



Compute: m_{11}^*, m_{22}^*, m_{12}, m_{21}

$$m_{21} = 1 + \sum_{\substack{k=1 \\ k\neq 1}}^{2} p_{2k} m_{k1} = 1 + p_{22} m_{21} = 1 + 0.4\, m_{21}$$

$$\Rightarrow \quad 0.6\, m_{21} = 1 \Rightarrow m_{21} = \frac{10}{6} = \frac{5}{3}.$$

$$m_{12} = 5$$

$$m_{11}^* = 1 + p_{12} m_{21} = 1 + 0.2 \times \frac{5}{3} = \frac{4}{3}$$

$$m_{22}^* = 4$$

Ergodic

- **Positive Recurrence:** If for a recurrent state s_i, m_{ii}^* is finite
- **Ergodic:** Positive recurrent, aperiodic states are called ergodic