Mathematics for Machine Learning (AI 512)

Amit Chattopadhyay

IIIT-Bangalore

Irreducible and Aperiodic MC

1. Irreducible Markov Chains

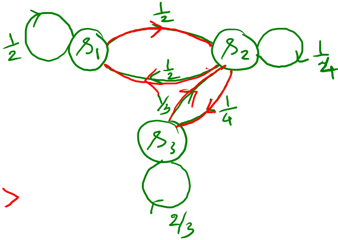
Let $(X_0, X_1, \ldots)$ be a **Markov chain** (MC) with state space $S = \{s_1, s_2, \ldots, s_k\}$ and transition Matrix $\mathbb{P}$

- Irreducibility $\Leftrightarrow$ all states of the Markov Chain can be reached from all others

Example 1: Is the MC with following transition matrix irreducible?

irreducible MC

$$\mathbb{P} = \begin{array}{c c} & \begin{array}{ccc} s_1 & s_2 & s_3 \end{array} \\ \begin{array}{c} s_1 \\ s_2 \\ s_3 \end{array} & \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/4 & 1/4 \\ 0 & 1/3 & 2/3 \end{bmatrix} \end{array}$$



$\mathbb{P}^2 = s_1 \quad s_3$

Let $(X_0, X_1, \dots)$ be a **Markov chain** (MC) with state space $S = \{s_1, s_2, \dots, s_k\}$ and transition Matrix $\mathbb{P}$

Def^n: $p_{ii}^{(0)} = 1 \quad \forall i$

- $s_i \to s_j$: state s_j is **accessible** from a state s_i

  $\Leftrightarrow$ if $\exists$ an integer $n \geq 0$ such that

  $\boxed{p_{ii}^{(n)} > 0 \, ?}$

  $p_{ij}^{(n)} \stackrel{\text{def}}{=} P(X_{m+\boxed{n}} = s_j \mid X_m = s_i) > 0$

  (i,j)-th element $\mathbb{P}^n$

- $s_i \leftrightarrow s_j$: states s_i and s_j **communicate** $\Leftrightarrow$ if $s_i \to s_j$ and $s_j \to s_i$

- An MC is **irreducible** $\Leftrightarrow s_i \leftrightarrow s_j \; \forall s_i, s_j \in S$

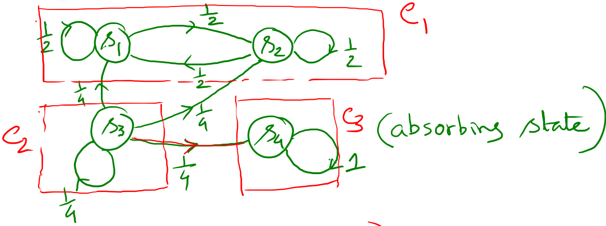
Rreducible Markov Chains: Classes

Property 1: '$\leftrightarrow$' is an **equivalence relation** on S.

Note: Thus '$\leftrightarrow$' partitions the state space S into **equivalence classes**. Communicating states belong to the same class. Any two classes are either disjoint or identical.

Example 2: Is the MC with following transition matrix irreducible?

$$\mathbb{P} = \begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} \begin{array}{c} \begin{array}{cccc} s_1 & s_2 & s_3 & s_4 \end{array} \\ \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$



(not irreducible)

Irreducibility: using Transition (or State Graph)

A MC is **irreducible** $\Leftrightarrow$ its state graph is a **strongly connected graph**

$G = (V, E)$: Directed Graph

- An **ordered pair of vertices** (u, v) is **strongly connected** if $\exists$ a directed path from u to v in G
- **Strongly Connected Graph (SCG):** every 'ordered pair of vertices' is strongly connected
- **Strongly Connected Component (SCC):** a subgraph of G which is strongly connected

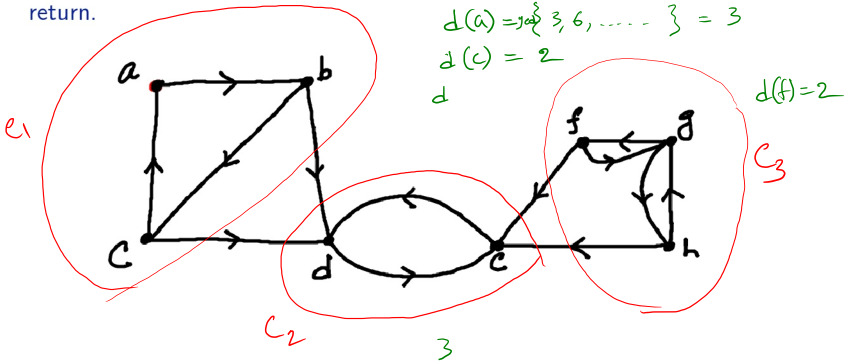
Example 3: Is the MC with following transition matrix irreducible?

$$\mathbb{P} = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.3 & 0.7 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 \\ 0 & 0 & 0.8 & 0.2 \end{bmatrix}$$

(no, check from TG)

1. The strongly connected components of an arbitrary directed graph forms a partition into subgraphs that are themselves strongly connected. (using the same logic as before)

2. Once a Walk leaves a strongly connected component it can never return.



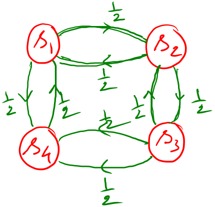
2. Aperiodic Markov Chains

$(X_0, X_1, \ldots)$: a **Markov chain** (MC) with state space $S = \{s_1, s_2, \ldots, s_k\}$ and transition Matrix $\mathbb{P}$

Periodicity $\Leftrightarrow$ how periodically each state of MC can be re-visited

Example 1: What is the period of each state in the MC with the following transition matrix?

$$\mathbb{P} = \begin{array}{cc} & s_1 \\ s_1 & \begin{bmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \end{bmatrix} \end{array}$$



s_i

$\mathbb{P}^2 = \begin{bmatrix} \square & \\ & \end{bmatrix}$

$\mathbb{P}^3 = \begin{bmatrix} 0 & \\ & \end{bmatrix}$

$\mathbb{P}^4 = \begin{bmatrix} \square & \\ & \end{bmatrix}$

$d(s_1) = \gcd\{2, 4, 6, \ldots\} = 2$

$\vdots$

$d(s_4) = 2$

Aperiodicity: Mathematical Definition

$(X_0, X_1, \ldots)$: a **Markov chain** (MC) with state space $S = \{s_1, s_2, \ldots, s_k\}$ and transition Matrix $\mathbb{P}$

$d(s_i) :=$ **period** of a state $s_i \in S = \text{gcd}\ \{n \geq 1 : (\mathbb{P}^n)_{i,i} > 0\}$

$d(s_i) = 1 \Rightarrow$ the state s_i is **aperiodic**

A MC is **aperiodic** $\Leftrightarrow$ all the states of the MC are aperiodic

Example 2: What is the period of each state in the MC with the following transition matrix?

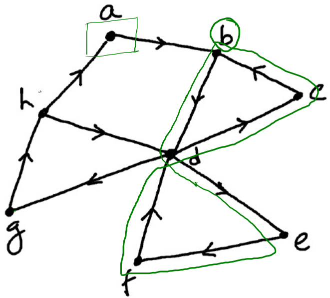
$$\mathbb{P} = \begin{bmatrix} 0 & 1/4 & 0 & 3/4 \\ 1/2 & 0 & 1/3 & 1/6 \\ 0 & 0 & 0 & 1 \\ 0 & 1/2 & 1/4 & 1/4 \end{bmatrix}$$

check!

Aperiodicity: Using Transition Graph (or State Graph)

$d(s_i) = \gcd\ \{n \geq 1 : n = \text{length of } \textbf{cycle} \text{ starting at } s_i\}$

If *gcd of all cycle lengths* $= 1$ in the Transition Graph, then the MC is aperiodic. **Is this true?** No. (It is true only when the MC irreducible)



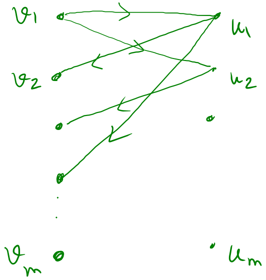
all states are aperiod (Check!)

MC is aperiodic.

Cycle: A **walk** of vertices and edges whose start and end vertices coincide.

Aperiodicity: Properties

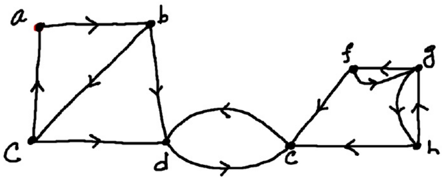
1. **Bipartite graphs** do not contain any odd-length cycles.



(not aperiodic)

$$d(v_1) = \gcd\{2, 4, \ldots\} = 2$$

2. Periodicity of a state is a class property.
3. For a **finite irreducible** MC, all the states have the **same period**.



Property 1: For a finite, aperiodic MC with transition matrix $\mathbb{P}$, there exists a positive integer N such that

$$(\mathbb{P}^m)_{i,i} > 0$$

for all states s_i and all $m > N$.

Property 2: For a finite, irreducible, aperiodic MC with transition matrix $\mathbb{P}$, there exists a positive integer M such that

$$(\mathbb{P}^m)_{i,j} > 0$$

for all states s_i, s_j and all $m \geq M$.

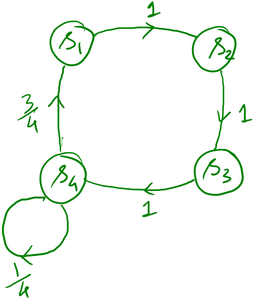
(i) $\mathbb{P}^m$ is a +ve matrix

(ii) sum of elements in each row is 1

Markov matrix.

Example: What is the smallest N so that $(\mathbb{P}^m)_{i,i} > 0$ (for all states s_i and all $m > N$) for the MC with the following transition matrix?

$$\mathbb{P} = \begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} \begin{array}{c} \begin{array}{cccc} s_1 & s_2 & s_3 & s_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3/4 & 0 & 0 & 1/4 \end{bmatrix} \end{array}.$$



$p\,\mathbb{P} = p$