Example 1

Given optimisation problem say

Minimize $-x-3y$

subject to $x+y=6$

$-x+y \leq 4$

Lagrangian function can be written as,

$$L(x,y,\lambda,v) = -x-3y+\lambda(-x+y-4)+v(x+y-6)$$

WHERE $\lambda \geq 0$

NOTE: Notice hear that there is condition on λ, but v is a free variable

The dual problem is equal to following saddle point problem

$$\max_{\lambda \geq 0, v} \left(\min_{x,y} L(x,y,\lambda,v) \right)$$

To make lagrange dual free of x and y primal parameters let us take

$$\begin{pmatrix} \frac{\partial L}{\partial x} \\ \frac{\partial L}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\frac{\partial L}{\partial x} = \frac{d}{dx}\left(-x-3y+\lambda(-x+y-4)+v(x+y-6)\right)$$

$$= -1+0+\lambda(-1+0+0)+ v(0+1-0)$$

$$\Rightarrow -1+v-\lambda = 0 \rightarrow (1)$$

$$\frac{\partial L}{\partial y} = -0 - 3 + \lambda(1) + v(1) = 0$$

$$\Rightarrow \quad -3 + \lambda + v = 0 \qquad - (2)$$

when λ and v satisfy these two conditions we have

$$\min_{x,y} L(x,y,\lambda,v) = -x - 3y + \lambda(-x+y-4) + v(x+y-6)$$

$$= x(\underline{-1-\lambda+v}) + y(\underline{-3+\lambda+v}) - 4\lambda - 6v$$

from (1) & (2) these are 0

$$= -4\lambda - 6v.$$

The dual problem becomes

$$\boxed{\begin{array}{ll} \text{Maximise} & g = -4\lambda - 6v \\ \text{Subject to} & \left(\begin{array}{c} -1-\lambda+v=0 \\ -3+\lambda+v=0 \\ \lambda \geq 0 \end{array}\right) \end{array}}$$

Note: Notice that $\lambda \geq 0$ is required by lagrangian function & v is free variable.

Solving Primal problem

$$\min \quad -x - 3y$$

$$\text{subject to} \quad x + y = 6$$

$$-x + y \leq 4$$

$$L(x,y,\lambda,v) = -x - 3y + \lambda(-x+y-4) + v(x+y-6)$$

Say x^*, y^* are optimal primal solutions, then they should follow stationary condition of KKT conditions. & also complementary slackness

Complementary Slackness says

$$\lambda_i^* f_i(x^*) = 0 \quad \forall\ i \in \{1, \cdots n\}$$

which are basically for

INEQUALITY CONSTRAINTS.

Here we have 1 inequality constraint, allowing us for 2 cases

which is a) $-x + y - 4 = 0$

b) $-x + y - 4 < 0$

Case a

$$-x + y - 4 = 0$$

It means $\lambda \neq 0$

(So this Inequality has now became Equality Constraint in this case)

$$L(x, y, \lambda, v) = -x - 3y + \lambda(-x + y - 4) + v(x + y - 6)$$

So we can proceede and now solve as Larangian method for Equality Constraints.

do $\nabla_x L$, $\nabla_y L$, $\nabla_\lambda L$, $\nabla_v L$ & make it $\Rightarrow 0$

we have.

$$\nabla_\lambda L \Rightarrow -0 + 1(-x + y - 4) + 0 = 0$$

$$\Rightarrow -x + y - 4 = 0 \quad - (3)$$

$$\nabla_v L \Rightarrow 0 + 0 + (x + y - 6) = 0$$

$$\Rightarrow x + y - 6 = 0 \quad - (4)$$

Solve (3) & (4) we have

$$-x + y - 4 = 0$$

$$x + y - 6 = 0$$

Add them

$$2y - 10 = 0 \quad \therefore\ y = 5$$

$$\& \ -x + y - 4 = 0 \Rightarrow -x + 5 - 4 = 0 \Rightarrow x = 1$$

$\therefore \quad x^* = 1, \quad y^* = 5$

And minimal value of function under given constraints is

$$f(x,y) = -x - 3y$$
$$= -1 - 3(5)$$

$$\boxed{\begin{aligned} f^*(x,y) &= -16 \\ \text{Min value} &= -16 \end{aligned}}$$

Case b $\quad -x + y - 4 \neq 0 \quad$ in this case $\lambda = 0$

$$L(x,y,v) = -x - 3y + v(x+y-6)$$

make $\nabla_x L$, $\nabla_y L$, $\nabla_v L$ & make them 0

& solve the equations.

$$\left.\begin{aligned} \nabla_x L &\Rightarrow -1 + v = 0 \\ \nabla_y L &\Rightarrow -3 + v = 0 \end{aligned}\right\} \Rightarrow v = 2$$

$$\nabla_v L \Rightarrow x + y - 6 = 0 \rightarrow \underline{x + y = 6}$$
$$\downarrow$$

It just says $x+y=6$ as condition, & there can be many such values / pairs of (x,y) whose sum is 6, the above case a answer $(1,5)$ also satisfy this so can now take $(1,5)$ as our primal solution

$\therefore$ from both case $(x^*, y^*) = (1,5)$ satisfy

$$\therefore \quad \boxed{p^* = -16, \quad x^* = 1, \quad y^* = 5}$$

Solving the dual problem

$$\max \quad g = -4\lambda - 6v$$

subject to
$$\begin{pmatrix} -1-\lambda+v=0 \\ -3+\lambda+v=0 \\ \lambda \geq 0 \end{pmatrix}$$

if we see the constraints they are just linear equations $\Rightarrow$ we can solve them directly

$$-1-\lambda+v=0$$
$$-3+\lambda+v=0$$

adding both equations $\Rightarrow$ $-4+2v=0$

$$2v=4$$
$$\underline{v=2}$$

substituting in $-1-\lambda+v=0$

$$-1-\lambda+2=0$$
$$-\lambda+1=0$$
$$\Rightarrow \underline{\lambda=1}$$

This also satisfies the $\lambda \geq 0 \Rightarrow$ condition

so $(\lambda^*, v^*) = (1, 2)$

$\therefore$ Maximal value of g is $\Rightarrow -4\lambda - 6v$

$$\Rightarrow -4(1) - 6(2)$$
$$\Rightarrow -4-12 \Rightarrow \underline{\underline{-16}}.$$

$$\therefore \quad d^* = -16$$

$\therefore$ Duality gap $= p^* - d^*$

$$= -16 - (-16)$$
$$= 0$$

$\therefore$ This follows Strong duality

Also now let us verify whether KKT conditions are being satisfied by obtained or not by x^*, y^*, λ^*, v^*

Say $f_0(x,y) = -x-3y$
$f_1(x,y) = -x+y-4$
$h_1(x,y) = x+y-6$

(i) Primal feasibility

$$f_i(x^*,y^*) \le 0 \quad \forall i \in \{1,2 \ldots m\}$$

means all inequalities constraints must be satisfied

Here only $f_1(x,y)$ is an inequality constraint which is

$$-x+y-4$$

at (x^*, y^*) it will be $-1+5-4$
$= 4-4$
$= 0 \le 0$

$\therefore$ satisfied

Also,

$$h_1(x^*, y^*) = 0$$

$\Rightarrow x^*+y^*-6 \Rightarrow 1+5-6$
$\Rightarrow 0$

$\therefore$ satisfied.

(ii) dual feasibility

$\lambda^* \ge 0$ here λ^* is 1 which is ≥ 0

so satisfied

(do not take $v \rightarrow$ as it comes from equality constraints)

(iii) Complementary slackness

$$\lambda^*(f_1(x^*, y^*)) = 0$$

λ^* is 1, $f_1(x^*, y^*) = -x^* + y^* - 4$
$= -1+5-4$
$= 0.$

$\therefore \lambda^* f_1(x^*, y^*) = 0 \rightarrow$ satisfied

(iv) Stationary,

It is straight forward, as just derivate & substitute values in them.

It is also satisfied.

we have seen,

1) KKT conditions are satisfied

2) It is convex optimisation problem

3) Slater's condition is being hold

we can see that all functions for optimising is convex functions and also Equality constraint is affine, Objective function & inequality constraint are convex in nature

∴ So it falls under convex optimisation branch under constraint optimisation.

Let us see whether Slaters conditions are being hold

min $\quad -x - 3y$

subject to $\quad -x + y \le 4$

$\quad x + y = 6$

we need strictly feasible set

if we see (x, y) as $(3,3)$, $(2,4)$

∴ multiple points satisfy slater conditions

Example at $(3,3)$

$-3 + 3 - 4 < 0$ AND $3 + 3 - 6 = 0$

So there exists atleast one such point

∴ Slater's condition is being hold.

Example-2

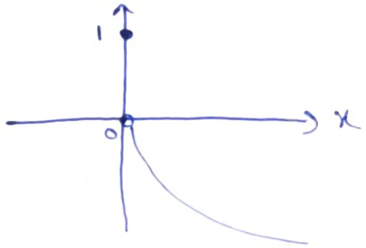
$$\min_{x \in R} \quad f(x) = \begin{cases} -\sqrt{x} & x > 0 \\ 1 & x = 0 \\ \infty & x < 0 \end{cases}$$

Subject to $\quad x \leq 0.$

Solution for primal problem:

this is straight forward problem as it was only subjected to $x \leq 0$, $f(x)$ can now be 1 at $x=0$ and ∞ if $x<0$, out of them 1 is minimal value and will be obtained at $x=0$

$$\boxed{\therefore \quad x^* = 0, \quad p^* = 1}$$



dual of this function,

let us write the Lagrangian of this problem,

$$L(x, \lambda) = \begin{cases} -\sqrt{x} + \lambda(x) & \text{if } x > 0 \\ 1 + \lambda(x) & x = 0 \\ \infty & x < 0 \end{cases}$$

To obtain dual function let us make $L(x, \lambda)$ free from x, make $\nabla_x(x, \lambda)$ at different step-sub functions and make them 0, as we did earlier.

for $x > 0$, $\nabla_x L(x, \lambda) = \frac{\partial}{\partial x}\left(-\sqrt{x} + \lambda(x)\right)$

$$\Rightarrow -\frac{1}{2} x^{\frac{1}{2}-1} + \lambda = 0$$

$$\Rightarrow \lambda = \frac{1}{2\sqrt{x}}$$

so $x = \frac{1}{4\lambda^2}$

$$-\sqrt{x} + \lambda x \Rightarrow -\frac{1}{2\lambda} + \lambda\left(\frac{1}{4\lambda^2}\right)$$

$$\Rightarrow -\frac{1}{2\lambda} + \frac{1}{4\lambda}$$

$$\Rightarrow \frac{-1}{4\lambda}$$

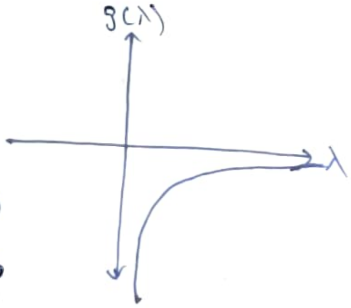
for case when $x = 0$, $x < 0$, we can easily observe $\lambda \leq 0$ goes to $-$infinity

$\therefore$ dual function is,

$$\boxed{g(\lambda) = \begin{cases} \frac{-1}{4\lambda}, & \lambda > 0 \\ -\infty & \lambda \leq 0 \end{cases}}$$

Solving for dual problem:

This is also a straight forward problem as we can see $-\frac{1}{4\lambda}$ looks as



The dual optimum value is when λ is very very large, then $g(\lambda) \to 0$

& that is the maximum value which will occur for very large λ

ie $\lambda \to +\infty$

(But it is not feasible / attainable practically)

so $\boxed{d^* = 0}$ which $\lambda \to +\infty$

Duality gap: $p^* - d^* = 1 - 0$

$= 1 \neq 0.$

This is a convex problem with non-zero duality gap $\rightarrow$ a non-typical case.

Let us verify KKT Conditions are being verified

(i) Primal feasability $\because$ our $x^* = 0,\ p^* = 1,\ \lambda^* = +\infty,\ d^* = 0$

$$x \leq 0 \Rightarrow 0 \leq 0$$

it doesn't mean some number just a symbol of non-attainability

(ii) dual feasibility $\rightarrow \lambda^* \geq 0 \rightarrow$ Satisfied

(iii) Complementary Slackness

$$\lambda^* f_1(x^*) = 0 \Rightarrow \infty(0)$$

which is undefined

$\therefore$ KKT condition failed.

Is Slater Condition holding ?? : No

But if now Question/ optimisation problem becomes

$$\min_{x \in \mathbb{R}} \quad f(x) = \begin{cases} -\sqrt{x} & x > 0 \\ 1 & x = 0 \\ +\infty & x < 0 \end{cases}$$

Subject to $\quad x \leq a \quad$ where $a > 0$

Then Primal optimal value $\boxed{p^* = -\sqrt{a}}$

dual function is

$$g(\lambda) = \begin{cases} -\dfrac{1}{4\lambda} - a\lambda, & \lambda > 0 \\ -\infty & \lambda \leq 0 \end{cases}$$

dual optimal value is $d^* = g\left(\dfrac{1}{2\sqrt{a}}\right) = -\sqrt{a}$

$$\therefore \boxed{\begin{aligned} d^* &= -\sqrt{a} \\ \lambda^* &= \frac{1}{2\sqrt{a}} \end{aligned}}$$

Now Strong duality holds in this case

Exercise

what has happened / Changed in two questions ??

& how that little change has effected our duality gap

Hint: Slater Condition, KKT Condition.

Example 3

$$\underset{x \in R^2}{\text{minimize}} \quad x_1^2 + x_2$$

Subject to $\quad 2x_1 + x_2 \geq 4$

$x_2 \geq 1$

find the lagrangian duel of the above optimisation problem.

Rewriting above optimisation problem as

$$\min_{x \in R^2} \quad x_1^2 + x_2$$

sub. to $\quad 4 - 2x_1 - x_2 \leq 0$

$1 - x_2 \leq 0$

lagrangian is then,

$$L(x, \lambda) = x_1^2 + x_2 + \lambda_1 (4 - 2x_1 - x_2) + \lambda_2 (1 - x_2)$$

objective of dual problem to be defined is

$$\theta_D(\lambda) = \min_x L(x, \lambda)$$

To make it free from x and only be depend on λ, take differentiation with respect to primal attributes x_1, x_2 and replace them

$$\frac{\partial}{\partial x_1} L(x,\lambda) = 2x_1 - 2\lambda_1 = 0$$

$$x_1 = \lambda_1$$

$$\frac{\partial}{\partial x_2}\left(L(x,\lambda)\right) \Rightarrow 1 - \lambda_1 - \lambda_2 = 0$$

$$\begin{aligned}
\theta_D(\lambda) &= \min_{x} L(x,\lambda) \\
&= \min_{x_2} \left[\lambda_1^2 + x_2 + \lambda_1(4 - 2\lambda_1 - x_2) + \lambda_2(1 - x_2) \right] \\
&= \min_{x_2} \left[-\lambda_1^2 + 4\lambda_1 + \lambda_2 + x_2(1 - \lambda_1 - \lambda_2) \right] \\
&= \begin{cases} -\lambda_1^2 + 4\lambda_1 + \lambda_2 & \text{if } 1 - \lambda_1 - \lambda_2 = 0 \\ -\infty & \text{otherwise} \end{cases}
\end{aligned}$$

so dual problem is given by

$$\underset{\lambda \in \mathbb{R}^2}{\text{maximise}} \quad -\lambda_1^2 + 4\lambda_1 + \lambda_2$$

$$\text{subject to} \quad \begin{aligned} \lambda_1 &\geq 0 \\ \lambda_2 &\geq 0 \\ 1 - \lambda_1 - \lambda_2 &= 0 \end{aligned}$$

NOTE: Important to observe that dual problem is Concave quadratic program in Variables of λ

Example 4:

consider convex optimisation problem

$$\text{minimize} \quad e^{-x}$$
$$\text{subject to} \quad \frac{x^2}{y} \leq 0$$

with variables x, y, and domain

$$D = \{(x,y) \mid y > 0\} \quad (x,y) \in \mathbb{R}^2$$

Solution for primal problem:

we have given in Domain that $y > 0$,

and in constraint we have $\frac{x^2}{y} \leq 0$ in $\mathbb{R}^2$

there doesnot exist any x^2, for which value is < 0 → so only way $\frac{x^2}{y} \leq 0$ holds is when $x = 0$ and $y > 0$

So at that such points minimum value of primal objective function is 1 $(e^{-0} = 1)$

$$p^* = 1$$

Lagrangian is,

$$L(x,y,\lambda) = e^{-x} + \lambda\left(\frac{x^2}{y}\right)$$

dual function is

$$g(\lambda) = \inf_{(x,y) \in D} L(x,y,\lambda)$$

$$= \begin{cases} 0 & \text{if } \lambda \geq 0 \\ -\infty & \text{otherwise} \end{cases}$$

dual problem is,

$$\text{maximise} \quad 0$$
$$\text{subject to} \quad \lambda \geq 0$$

optimisation variable in dual problem is λ

& we can see that for any $\lambda \geq 0$ it is dual optimal

So any (x, y) such that $x = 0$ & $y > 0$ is Prime optimal

& any $\lambda \geq 0$ is dual optimal

Although primal and dual optimal values are both attained, Strong duality doesnot hold.

<u>Exercise</u>:

why do you think this problem arises?

(or) changing which part of question can now make Strong duality possible?