IIIT-Bangalore
Course: AI 512 Mathematics for ML
Tutorial 1

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1. Consider the first 6 examples 4.1 to 4.6 in Ref. Book by S.M. Ross (page: $186 - 187$). For each of the examples:
   (a) compute the transition graph
   (b) check irreducibility of the MC and find the equivalence classes
   (c) check periodicity of the MC.

2. Consider Ex 4.1 (in Ref. Book by S.M. Ross), in which the weather is considered as a two-state Markov Chain. Take $\alpha = 0.7$, $\beta = 0.4$. Calculate the probability that it will rain on the fourth day given that it is raining today.

3. Consider Ex 4.4 (in Ref. Book by S.M. Ross); what is the probability that it will rain on Thursday given that it has rained on Monday and Tuesday.

4. Solve the first 5 problems 2.1 to 2.5 in Ref. Book by Olle Häggström (page: 15).

Example 4.1 (Forecasting the Weather) Suppose that the chance of rain tomorrow depends on previous weather conditions only through whether or not it is raining today and not on past weather conditions. Suppose also that if it rains today, then it will rain tomorrow with probability α; and if it does not rain today, then it will rain tomorrow with probability β.

If we say that the process is in state 0 when it rains and state 1 when it does not rain, then the preceding is a two-state Markov chain whose transition probabilities are given by

$$\mathbf{P} = \begin{Vmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{Vmatrix} \quad \blacksquare$$

Example 4.2 (A Communications System) Consider a communications system which transmits the digits 0 and 1. Each digit transmitted must pass through several stages, at each of which there is a probability p that the digit entered will be unchanged when it leaves. Letting X_n denote the digit entering the nth stage, then $\{X_n, n = 0, 1, \dots\}$ is a two-state Markov chain having a transition probability matrix

$$\mathbf{P} = \begin{Vmatrix} p & 1-p \\ 1-p & p \end{Vmatrix} \quad \blacksquare$$

Example 4.3 On any given day Gary is either cheerful (C), so-so (S), or glum (G). If he is cheerful today, then he will be C, S, or G tomorrow with respective probabilities 0.5, 0.4, 0.1. If he is feeling so-so today, then he will be C, S, or G tomorrow with probabilities 0.3, 0.4, 0.3. If he is glum today, then he will be C, S, or G tomorrow with probabilities 0.2, 0.3, 0.5.

Letting X_n denote Gary's mood on the nth day, then $\{X_n, n \geqslant 0\}$ is a three-state Markov chain (state $0 = C$, state $1 = S$, state $2 = G$) with transition probability matrix

$$\mathbf{P} = \begin{Vmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{Vmatrix} \quad \blacksquare$$

Example 4.4 (Transforming a Process into a Markov Chain) Suppose that whether or not it rains today depends on previous weather conditions through the last two days. Specifically, suppose that if it has rained for the past two days, then it will rain tomorrow with probability 0.7; if it rained today but not yesterday, then it will rain tomorrow with probability 0.5; if it rained yesterday but not today, then it will rain tomorrow with probability 0.4; if it has not rained in the past two days, then it will rain tomorrow with probability 0.2.

If we let the state at time n depend only on whether or not it is raining at time n, then the preceding model is not a Markov chain (why not?). However, we can transform this model into a Markov chain by saying that the state at any time is determined by the weather conditions during both that day and the previous day. In other words, we can say that the process is in

state 0	if it rained both today and yesterday,
state 1	if it rained today but not yesterday,
state 2	if it rained yesterday but not today,
state 3	if it did not rain either yesterday or today.

The preceding would then represent a four-state Markov chain having a transition probability matrix

$$\mathbf{P} = \begin{Vmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{Vmatrix}$$

You should carefully check the matrix $\mathbf{P}$, and make sure you understand how it was obtained. ■

Example 4.5 (A Random Walk Model) A Markov chain whose state space is given by the integers $i = 0, \pm 1, \pm 2, \ldots$ is said to be a random walk if, for some number $0 < p < 1$,

$$P_{i,i+1} = p = 1 - P_{i,i-1}, \qquad i = 0, \pm 1, \ldots$$

The preceding Markov chain is called a *random walk* for we may think of it as being a model for an individual walking on a straight line who at each point of time either takes one step to the right with probability p or one step to the left with probability $1 - p$. ■

Example 4.6 (A Gambling Model) Consider a gambler who, at each play of the game, either wins \$1 with probability p or loses \$1 with probability $1 - p$. If we suppose that our gambler quits playing either when he goes broke or he attains a fortune of \$$N$, then the gambler's fortune is a Markov chain having transition probabilities

$$P_{i,i+1} = p = 1 - P_{i,i-1}, \qquad i = 1, 2, \ldots, N - 1,$$
$$P_{00} = P_{NN} = 1$$

States 0 and N are called *absorbing* states since once entered they are never left. Note that the preceding is a finite state random walk with absorbing barriers (states 0 and N). ■

Problems

2.1 **(5)** Consider the Markov chain corresponding to the random walker in Figure 1, with transition matrix P and initial distribution $\mu^{(0)}$ given by (11) and (14).

(a) Compute the square P^2 of the transition matrix P. How can we interpret P^2? (See Theorem 2.1, or glance ahead at Problem 2.5.)

(b) Prove by induction that

$$\mu^{(n)} = \begin{cases} (0, \frac{1}{2}, 0, \frac{1}{2}) & \text{for } n = 1, 3, 5, \ldots \\ (\frac{1}{2}, 0, \frac{1}{2}, 0) & \text{for } n = 2, 4, 6, \ldots . \end{cases}$$

2.2 **(2)** Suppose that we modify the random walk example in Figure 1 as follows. At each integer time, the random walker tosses *two* coins. The first coin is to decide whether to stay or go. If it comes up heads, he stays where he is, whereas if it comes up tails, he lets the second coin decide whether he should move one step clockwise, or one step counterclockwise. Write down the transition matrix, and draw the transition graph, for this new Markov chain.

2.3 **(5)** Consider Example 2.1 (the Gothenburg weather), and suppose that the Markov chain starts on a rainy day, so that $\mu^{(0)} = (1, 0)$.

(a) Prove by induction that

$$\mu^{(n)} = (\tfrac{1}{2}(1 + 2^{-n}), \tfrac{1}{2}(1 - 2^{-n}))$$

for every n.

(b) What happens to $\mu^{(n)}$ in the limit as n tends to infinity?

2.4 **(6)**

(a) Consider Example 2.2 (the Los Angeles weather), and suppose that the Markov chain starts with initial distribution $(\frac{1}{6}, \frac{5}{6})$. Show that $\mu^{(n)} = \mu^{(0)}$ for any n, so that in other words the distribution remains the same at all times.[7]

(b) Can you find an initial distribution for the Markov chain in Example 2.1 for which we get similar behavior as in (a)? Compare this result to the one in Problem 2.3 (b).

2.5 **(6)** Let $(X_0, X_1, \ldots)$ be a Markov chain with state space $\{s_1, \ldots, s_k\}$ and transition matrix P. Show, by arguing as in the proof of Theorem 2.1, that for any $m, n \geq 0$ we have

$$\mathbf{P}(X_{m+n} = s_j \mid X_m = s_i) = (P^n)_{i,j} \, .$$

[7] Such a Markov chain is said to be in **equilibrium**, and its distribution is said to be **stationary**. This is a very important topic, which will be treated carefully in Chapter 5.