Final Project

Shrihari Pande

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1 Introduction

This projects attempts to solve the interface dynamics of a two component imiscible flow with surface tension. In particular, Level set method is adopted to to track the location of the interface.
Level-set method introduces a smooth function, typically denoted by ϕ whose zero level denotes the position of interface at any instant.This project aims to solve the level set equation in addition to modified Navier-Stokes equations with surface tension in 2 dimensions using finite difference scheme in MATLAB.
Initially, an overview of governing equation and their discretization schemes are discussed. Description of the MATLAB code is provided and the results from level set method without the surface tension are compared against the Front tracking method of the similar problem. In particular, the interface shapes and locations at 3 different times and two different length scales are compared.The results show a good agreement between both the methods, with both of them becoming closer to each other for smaller vales of length scale in level set.The result of the code considering the surface is also provided but not compared with the Front tracking method since the results obtained were not favourable.
Lastly,the possible explanations for the inaccurate results of the surface tension level set code are discussed.

2 Problem Description/Methodology

The equations governing the fluid flows an are as follows:

$$\nabla.\mathbf{u} = 0 \tag{1}$$

$$\frac{\partial \phi}{\partial t} + \mathbf{u}.\nabla\phi = 0 \tag{2}$$

where ϕ is the level set function and $\mathbf{u}$ is the velocity field.

This equation was discretized using first order upwind scheme,

$$\phi_{i,j}^{n+1} = \phi_{i,j}^{n} - \Delta t\left(max(u_{i,j},0)D_x^-\phi_{i,j}^n + max(u_{i,j},0)D_x^+\phi_{i,j}^n + max(u_{i,j},0)D_y^-\phi_{i,j}^n + max(u_{i,j},0)D_y^+\phi_{i,j}^n\right) \tag{3}$$

Where

$$D_x^+ = \frac{\phi_{i+1,j}^n - \phi_{i,j}^n}{\Delta x} \tag{4}$$

$$D_x^- = \frac{\phi_{i,j}^n - \phi_{i-1,j}^n}{\Delta x} \tag{5}$$

Level set function was then converted into a signed distance function by rewriting equation (2)with source term as

$$\frac{\partial \varphi}{\partial t} + F|\Delta\varphi| = F \tag{6}$$

Equation (6) was discretized as follows:

$$\varphi_{i,j}^{n+1} = \varphi_{i,j}^{n} - \Delta t\left(max(F_{i,j},0)\nabla^+ min(F_{i,j},0)\nabla^-\right) \tag{7}$$

Where

$$\nabla^+ = \sqrt{max(D_x^-\varphi_{i,j},0)^2 + min(D_x^+\varphi_{i,j},0)^2 + max(D_x^-\varphi_{i,j},0)^2 + min(D_x^+\varphi_{i,j},0)^2} \tag{8}$$

$$\nabla^- = \sqrt{max(D_x^+\varphi_{i,j},0)^2 + min(D_x^-\varphi_{i,j},0)^2 + max(D_x^+\varphi_{i,j},0)^2 + min(D_x^-\varphi_{i,j},0)^2} \tag{9}$$

with $\varphi(\mathbf{x},0) = \tilde{\phi}^{n+1}, \varphi(\mathbf{x},\infty) = \phi^{n+1}$

Here ,$\tilde{\phi}^{n+1}$ is the solution obtained from equation(3).

F in equation (6) is the normal velocity of the interface, and is defined as

$$F = sign_{\epsilon}(\tilde{\phi}^{n+1}) = \frac{\tilde{\phi}^{n+1}}{\sqrt{(\tilde{\phi}^{n+1})^2 + \epsilon^2}} \tag{10}$$

Where ϵ is the length scale signifying the thickness of the interface and was taken $\epsilon = \frac{3}{2}\Delta x$ After ϕ was made into a signed distance, the momentum equation from Navier -stokes equation with additional Surface tension force was solved. The momentum equation accounting for surface tension is as follows:

$$\rho\frac{\partial \mathbf{u}}{\partial t} + \rho\nabla.(\mathbf{u}\otimes\mathbf{u}) = -\nabla p + \mu\Delta\mathbf{u} + \gamma\kappa\delta(\mathbf{n}_\Gamma)n_\Gamma \tag{11}$$

With ϕ being signed distance, characteristic function(χ) signifying the density of the fluid can be made continuous as is defined as

$$\chi_s(\phi) = 1 \quad \phi < -\epsilon \tag{12}$$

$$\chi_s(\phi) = 0.5 + 0.5\frac{\phi}{\epsilon} + \frac{1}{2\pi}sin\left(\frac{\pi\phi}{\epsilon}\right) \quad -\epsilon \leq \phi \leq \epsilon \tag{13}$$

$$\chi_s(\phi) = 1 \quad \phi > \epsilon \tag{14}$$

Similarly dirac delta in surface tension can be defined as

$$\delta_s(\phi) = \frac{d\chi_s(\phi)}{d\phi} \tag{15}$$

Using the fact $\kappa = \nabla.n_\Gamma, \quad n_\Gamma = \frac{\nabla.\phi}{|\nabla\phi|}, \quad |\nabla\phi| = 1$(From equation 6), surface tension force can be written in equation(11)

$$\gamma\kappa\delta(\mathbf{n}_\Gamma)n_\Gamma = \gamma\Delta\phi\delta_s(\phi)\nabla\phi \tag{16}$$

The surface tension force was discretized as

$$F_{gx} = \gamma\delta_s\left(\frac{\phi^n_{i-1,j} - 2\phi^n_{i,j} + \phi^n_{i-1,j}}{\Delta x^2} + \frac{\phi^n_{i,j-1} - 2\phi^n_{i,j} + \phi^n_{i,j-1}}{\Delta y^2}\right)\frac{\phi^n_{i+1,j} - \phi^n_{i,j}}{\Delta x} \tag{17}$$

$$F_{gy} = \gamma\delta_s\left(\frac{\phi^n_{i-1,j} - 2\phi^n_{i,j} + \phi^n_{i-1,j}}{\Delta x^2} + \frac{\phi^n_{i,j-1} - 2\phi^n_{i,j} + \phi^n_{i,j-1}}{\Delta y^2}\right)\frac{\phi^n_{i,j+1} - \phi^n_{i,j}}{\Delta y} \tag{18}$$

Finally, momentum equation(11) was discretized on a staggered grid with projection method and is discretized as follows:

$$u^* = u^n + \Delta t\left(-\frac{\partial(u^n u^n)}{\partial x} - \frac{\partial(u^n v^n)}{\partial y} + \frac{\mu}{\rho^{n+1}}\left(\frac{\partial^2(u^n)}{\partial x^2} + \frac{\partial^2(u^n)}{\partial y^2}\right)\right) + \Delta t g_x + \Delta t\frac{F_{gx}}{\rho^{n+1}} \tag{19}$$

$$v^* = v^n + \Delta t\left(-\frac{\partial(u^n v^n)}{\partial x} - \frac{\partial(v^n v^n)}{\partial y} + \frac{\mu}{\rho^{n+1}}\left(\frac{\partial^2(v^n)}{\partial x^2} + \frac{\partial^2(v^n)}{\partial y^2}\right)\right) + \Delta t g_x + \Delta t\frac{F_{gy}}{\rho^{n+1}} \tag{20}$$

$$\frac{\partial}{\partial x}\left(\frac{1}{\rho^{n+1}}\frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\rho^{n+1}}\frac{\partial P}{\partial y}\right) = \frac{1}{\Delta t}\left(\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y}\right) \tag{21}$$

$$u^{n+1} = u^* - \frac{\Delta t}{\rho^{n+1}}\frac{\partial P}{\partial x} \tag{22}$$

$$v^{n+1} = v^* - \frac{\Delta t}{\rho^{n+1}}\frac{\partial P}{\partial y} \tag{23}$$

$$\rho^{(n+1)} = \frac{\rho_{i+1,j} - \rho_{i,j}}{2} \tag{24}$$

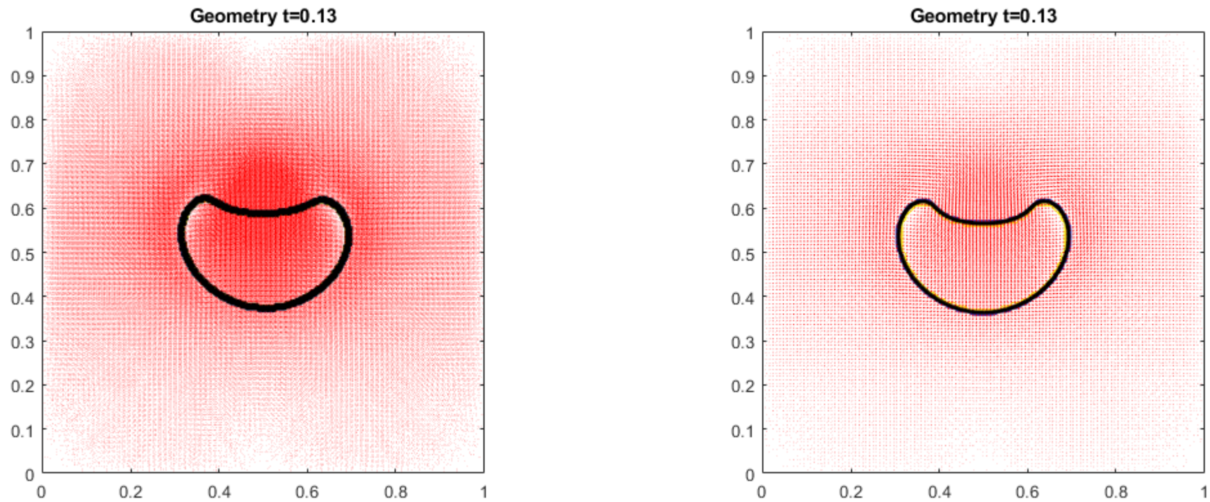
Equation(21) is discretized using finite central difference and solved using Successive Over Relaxation(SOR)

3 Code Description

A staggered grid of pressure and velocity was chosen and level set was initialized at the points pf pressure. The level set was then initialized as a signed distance function by setting a circle as an interface and taking the minimum distance from all the level set points to the circle with setting the values to negative of their current if the points are inside the circle.Similarly characteristic function (χ) is set according to its smeared out functions. The computation domain is chosen as a 1 by 1 square box. The level set is then evolved as written in Equation (3). It is then converted to a signed distance function by solving equation(6). After the characteristic function and density are evolve by the smeared out function.Dirac delta is also calculated. Then x and y component of the surface tension forces are calculate and added to temporary velocities. Pressure equation is then solved using SOR. No flux boundary conditions are imposed for pressure and level set throughout the code. True velcoties are then calculate an the solution of is plotted by finding the points that are close to the length scale which is initialized in the beginning of the code.

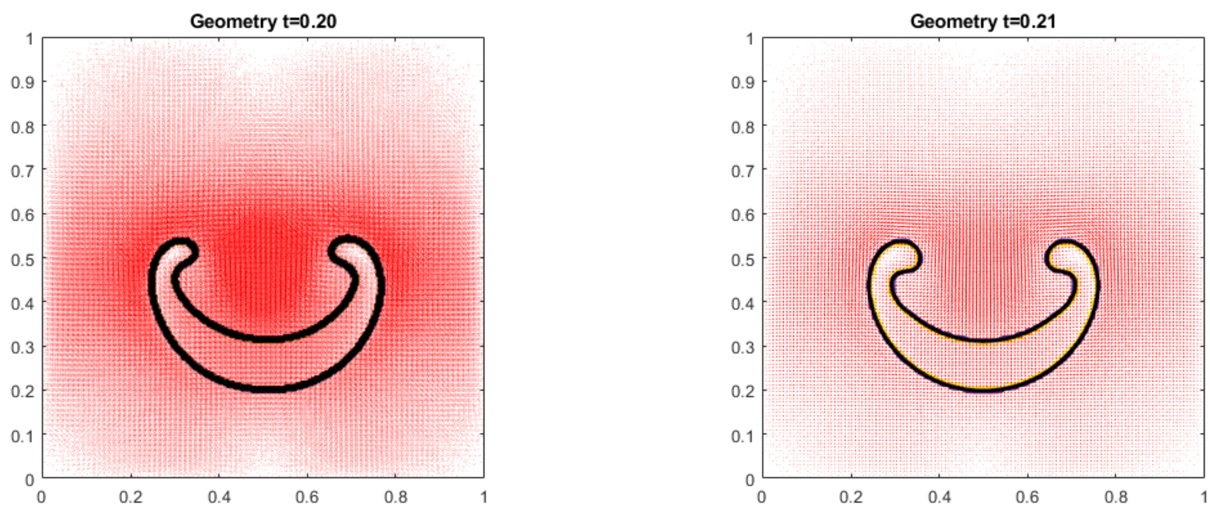
4 Results and Discussion

After running the code the following results were obtained:



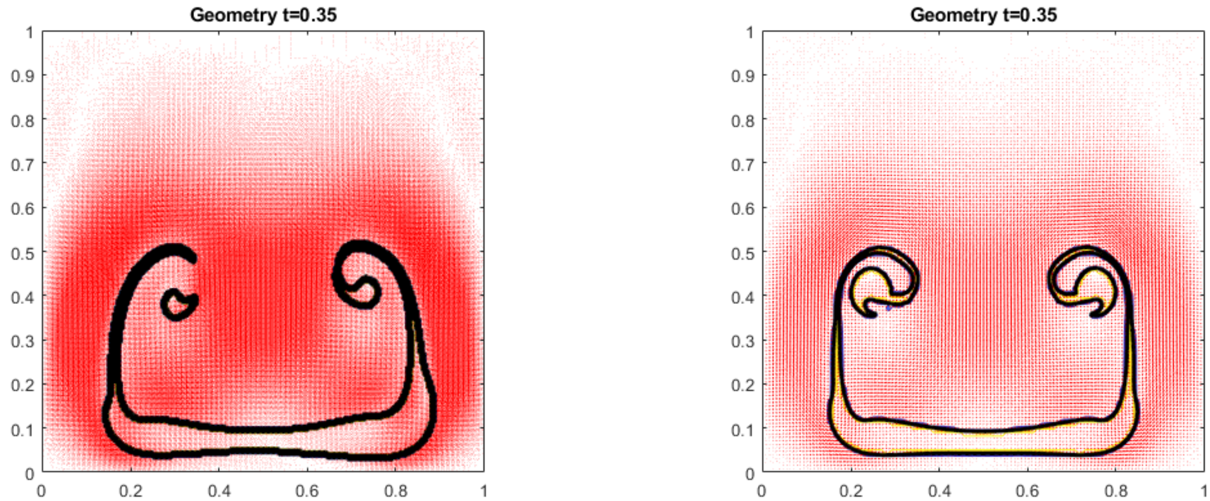
(a) (b)

Figure 1: Shape of the Interface Without Surface Tension at t=0.13s by Level Set Method(1a) and Front Tracking Method(1b)



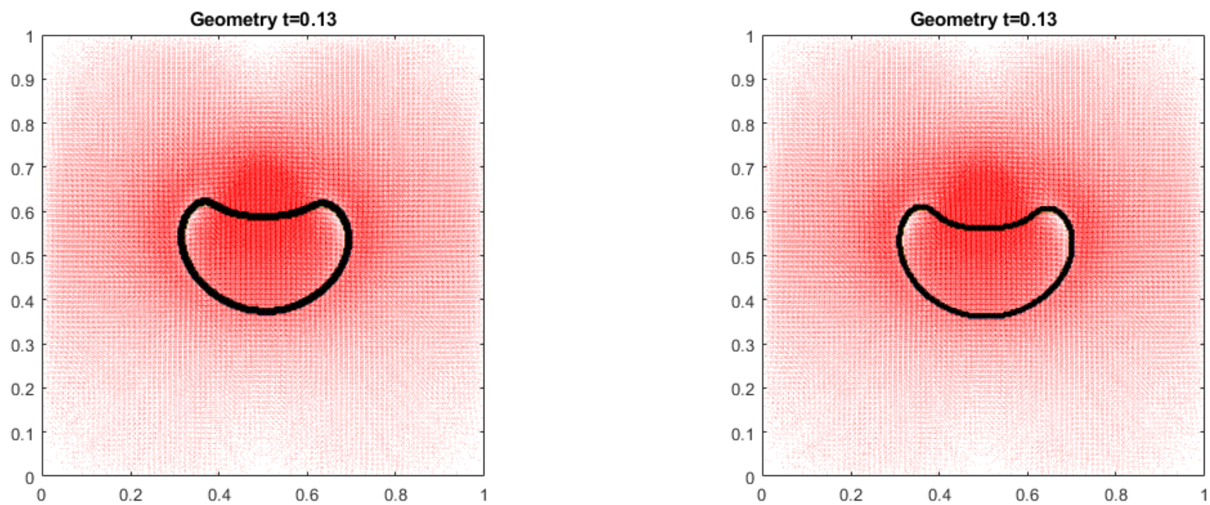
(a) (b)

Figure 2: Shape of the Interface without Surface Tension at t=0.2s by Level Set Method(2a) and Front Tracking Method(2b)



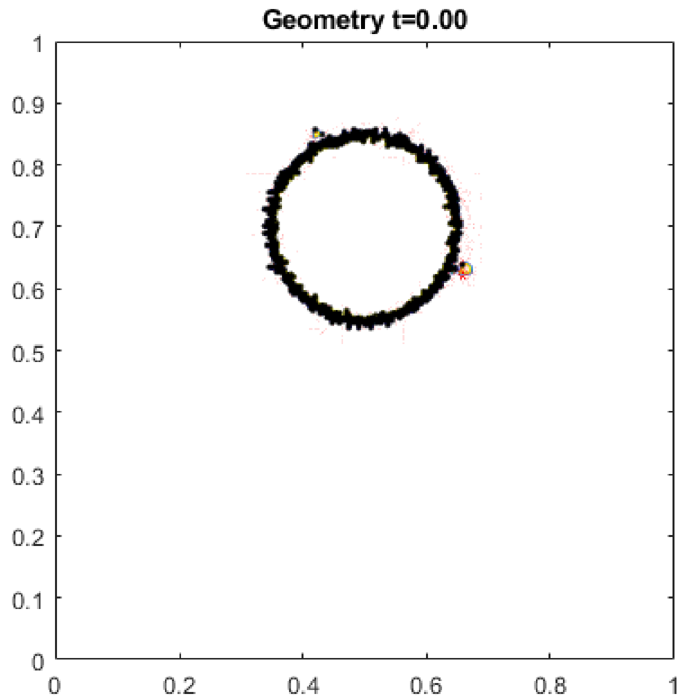
(a) (b)

Figure 3: Shape of the Interface without Surface Tension at t=0.2s by Level Set Method(3a) and Front Tracking Method(3b)



(a) (b)

Figure 4: Shape of the Interface without Surface Tension at t=0.35s with$\epsilon = 1.5\Delta x$(4a) and $\epsilon = \Delta x$(4b)



(a)

Figure 5: Shape of the Interface with Surface Tension

Level set methods agree with the Front tracking method for the initial time steps when surface tension is not taken into account. A slight difference in shape of the interface which is expected since in Level set the interface is extracted from the underlying grid of level on basis if the points is has ϕ value close to zero , set while in Front tracking the front has a separate grid and all the points can be accounted for.This leads to,in Level Set loss of some key points of the interface , since numerically it is difficult to extract level set points corresponding to zero level set. It can also be observed that as the front gets closer to the walls, the interface breaks in Level set while it remains intact in case of Front tracking(Figure3)

Similarly, the interface resembles more with the interface of the Front tracking method with smaller length scale(Figure 4), which is expected since at lower thickness interface, the equations become more accurate.

5 Reasons for the Inaccurate results and in Surface tension

A couple of reasons that may have lead to inaccuracy when implementing the code with surface tension could be the presence of higher order derivatives in calculation of surface tension. Since the force inclues calculation of the Laplacian of the level set, errors could have been propagated. In addition to that, the the curvature of the interface is not not smooth since it is interpolated from underlying grid. A finer grid was also tried to rectify that, but the solution was still unstable.