1 (a). If X is a continuous random variable with PDF $f(x)$, then the expected value (or mean) of X is given by

$$\mu = \mu_X = E[X] = \int_{-\infty}^{\infty} x \cdot f(x)\, dx \tag{1}$$

Applying the definition given in eq (1) above to our problem, we compute the expected value of X:

$$E[X] = \int_0^1 x \cdot x\, dx + \int_1^2 x \cdot (2-x)\, dx = \int_0^1 x^2\, dx + \int_1^2 (2x - x^2)\, dx = \frac{1}{3} + \frac{2}{3} = 1 \tag{2}$$

1 (b). Now we calculate the variance and standard deviation of X, by first finding the expected value of X^2.

$$E[X^2] = \int_0^1 x^2 \cdot x\, dx + \int_1^2 x^2 \cdot (2-x)\, dx = \int_0^1 x^3\, dx + \int_1^2 (2x^2 - x^3)\, dx = \frac{1}{4} + \frac{11}{12} = \frac{7}{6} \tag{3}$$

Thus we have

$$Var(X) = E[X^2] - \mu^2 = \frac{7}{6} - 1 = \frac{1}{6} \tag{4}$$

1 (c). $SD(X) = \sqrt{Var(x)} = \frac{1}{\sqrt{6}}$

2.a. Types of random variables

A. Discrete Random Variable

A random variable that takes on a finite or countably infinite number of values (see page 4) is called a discrete random variable while one which takes on a non-countably infinite number of values is called a non-discrete random variable. When the random variable can assume only a countable, sometimes infinite, number of values.

B. Continuous Random Variable

When the random variable can assume an uncountable number of values in a line interval. A non-discrete random variable X is said to be absolutely continuous, or simply continuous, if its distribution function may be represented as:

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)dx \; \{-\infty < x < \infty\} \tag{5}$$

where the function $f(x)$ has the properties

1. $f(x) \geq 0$
2. $\int_{-\infty}^{\infty} f(x)dx = 1$

It follows from the above that if X is a continuous random variable, then the probability that X takes on any one particular value is zero, whereas the interval probability that X lies between two different values, say, a and b, is given by:

$$P(a < X < b) = \int_a^b f(x)dx \tag{6}$$

A Probability Mass Function (PMF) is a function over the sample space of a discrete random variable X which gives the probability that X is equal to a certain value.

Let X be a discrete random variable on a sample space S. Then the probability mass function $f(x)$ is defined as:

$$f(x) = P[X = x] \tag{7}$$

Each probability mass function satisfies the following two conditions:

$$f(x) = \begin{cases} f(x) \geq 0 \ for \ all \ x \in S, \\ \sum_{x \in S} f(x) = 1 \end{cases}$$

2.b. To ensure that first condition in 7.a. holds we need $f(x) \geq 0$, so we see that k cannot be negative. We also need to show that second condition in 7.b. holds true, i.e. $\sum_{x \in S} f(x) = 1$. So we have:

$$f(1) + f(2) + f(3) = 1 \tag{8}$$

Hence

$$k(7+3) + k(14+3) + k(21+3) = 1 \tag{9}$$

$$51k = 1 \tag{10}$$

or

$$k = \frac{1}{51} \tag{11}$$

3.a. Since $f(x)$ satisfies Property 1 as described in 7.b. if $c \geq 0$, it must satisfy Property 2 as described in 7.b. above in order to be a density function. So we have:

$$\int_{-\infty}^{\infty} f(x)dx = \int_0^3 cx^2 dx = \frac{cx^3}{3}\Bigg\|_0^3 = 9c \tag{12}$$

Since this should be equal to 1, we have $c = \frac{1}{9}$

3.b.

$$P(1 < X < 2) = \int_0^2 \frac{1}{9}x^2 dx = \frac{x^3}{27}\Bigg\|_1^2 = \frac{8}{27} - \frac{1}{27} = \frac{7}{27} \tag{13}$$

4.a. The density function $f(x) = \frac{dF(x)}{dx} = \begin{cases} 2e^{-2x} & for\ x > 0 \\ 0 & for\ x < 0 \end{cases}$

4.b. $P(X > 2) = \int_2^\infty 2e^{-2x}dx = -e^{-2x}\Big\|_2^\infty = e^{-4}$

4.c. $P(-3 < X \leq 4) = \int_{-3}^4 2e^{-2x}dx = \int_{-3}^0 0dx + \int_0^4 2e^{-2x}dx = e^{-2x}\Big\|_0^4 = 1 - e^{-8}$

The Poisson Probability Distribution

The Poisson random variable satisfies the following conditions:

1. The number of successes in two disjoint time intervals is independent.

2. The probability of a success during a small time interval is proportional to the entire length of the time interval.

Apart from disjoint time intervals, the Poisson random variable also applies to disjoint regions of space.

Applications

a. birth defects and genetic mutations
b. rare diseases (like Leukemia, but not AIDS because it is infectious and so not independent) - especially in legal cases
c. car accidents
d. traffic flow and ideal gap distance
e. number of typing errors on a page
f. spread of an endangered animal in Africa
g. failure of a machine in one month

The probability distribution of a Poisson random variable X representing the number of successes occurring in a given time interval or a specified region of space is given by the formula:

$$P(X) = \frac{e^{-\mu}\mu^x}{x!}$$

where:

$x = 0, 1, 2, 3 \ldots$
$e = 2.71828$
$\mu =$ mean number of successes in the given time interval or region of space

Mean and Variance of Poisson Distribution

If μ is the average number of successes occurring in a given time interval or region in the Poisson distribution, then the mean and the variance of the Poisson distribution are both equal to μ.

$$E(X) = \mu$$

and

$$V(X) = \sigma^2 = \mu$$

Note: In a Poisson distribution, only one parameter, μ is needed to determine the probability of an event.

5.(a) A life insurance salesman sells on the average 3 life insurance policies per week. We use Poisson's law to calculate the probability that in a given week he will sell some policies

Here, $\mu = 3$

Now "Some policies" means "1 or more policies". We can calculate this out by finding 1 minus the "zero policies" probability:

$$P(X > 0) = 1 - P(x_0)$$

Now

$$P(X) = \frac{e^{-\mu}\mu^x}{x!}$$

so

$$P(x_0) = \frac{e^{-3}3^0}{0!} = 4.9787 \times 10^{-2}$$

Therefore the probability of 1 or more policies is given by Probability

$$= P(X) \geq 0$$

$$= 1 - P(X_0)$$

$$= 1 - 4.9787 \times 10^{-2}$$

$$= 0.95021$$

5.(b) The probability of selling 2 or more, but less than 5 policies is:

$$P(2 \leq X < 5) = P(x_2) + P(x_3) + P(x_4)$$

$$= \frac{e^{-3}3^2}{2!} + \frac{e^{-3}3^3}{3!} + \frac{e^{-3}3^4}{4!} = 0.61611$$

5.(c) Assuming that there are 5 working days per week, the probability that in a given day he will sell one policy is calculates as follows:

Average number of policies sold per day: $\frac{3}{5} = 0.6$

So on a given day ($x = 1$), we have:

$$P(X = 1) = \frac{e^{-0.6}(0.6)^1}{1!}$$

6.(a) Let X be the number of students who passed the exam. Then, X has a binomial distribution with $n = 10$ and $p = 0.2$

The event that at least two students passed the exam is $X \geq 2$.
So,

$$P(X \geq 2) = 1 - P(X < 2) = 1 - p(0) - p(1)$$

$$= 1 - \binom{10}{0}(0.2)^0(0.8)^{10} - \binom{10}{1}(0.2)^1(0.8)^9 = 0.6242$$

6.(b) Expected number of students who passed the exam

$$E(X) = np = 10 \cdot (0.2) = 2$$

6.(c) Suppose that n students are needed to make the probability at least 0.99 that a student will pass the exam. Let A denote the event that a student pass the exam. Then, A^c means that all the students fail the exam. We have, $P(A) = 1 - P(A^c) = 1 - (0.8)^n \geq 0.99$

Solving the inequality, we find that $n \geq \frac{ln(0.01)}{ln(0.8)} = 20.6$. So, the required number of students is 21

7.(a) Vehicles pass through a junction on a busy road at an average rate of 300 per hour. We find the probability that none passes in a given minute as follows:

The average number of cars per minute is: $\mu = \frac{300}{60} = 5$. Since there are no vehicles passing in a given minute means $x = 0$

$$P(x = 0) = \frac{e^{-5}5^0}{0!} = 6.7379 \times 10^{-3}$$

7.(b) Expected number of vehicles passing in two minutes is calculated as follows:

$$E(X) = 5 \times 2 = 10$$

8.(a) Here, we have n = 20 and p = 10% = 0.1

And so Probability that exactly 5 customers will return the items purchased is:

$$P(X = 5) = \binom{n}{p} p^5(1-p)^{n-5} = \binom{20}{5} \times 0.1^5 \times 0.9^{15} = 0.03192$$

8.(b) Probability that a maximum of 5 customers will return the items purchased is:

$$P(X \leq 5) = \sum_{i=0}^{5} \binom{n}{i} p^i(1-p)^{n-i}$$

$$= \binom{20}{0}\times 0.1^0\times 0.9^0+\binom{20}{1}\times 0.1^1\times 0.9^{19}+\binom{20}{2}\times 0.1^2\times 0.9^{18}+\binom{20}{3}\times 0.1^3\times 0.9^{17}+\binom{20}{4}\times 0.1^4\times 0.9^{16}+\binom{20}{5}\times 0.1^5\times 0.9^{15}$$

$$= 0.9887$$

8.(c) Probability that more than 5 customers will return the product is:

$$P(X > 5) = 1 - P(X \leq 5) = 1 - \sum_{i=0}^{5} \binom{20}{i} (0.1)^i \times (0.9)^{20-i} = 0.0113$$

8.(d) The average number of customers who are likely to return the items is:

$$E(X) = n \times p = 20 \times 0.1 = 2$$

8.(e) Variance of a binomial distribution is given by:

$$Var(X) = n \times p \times q = 20 \times 0.1 \times 0.9 = 1.8$$

And the corresponding Standard Deviation $= \sqrt{Var} = \sqrt{1.8} = 1.3416$