

Take Home Assignment 1

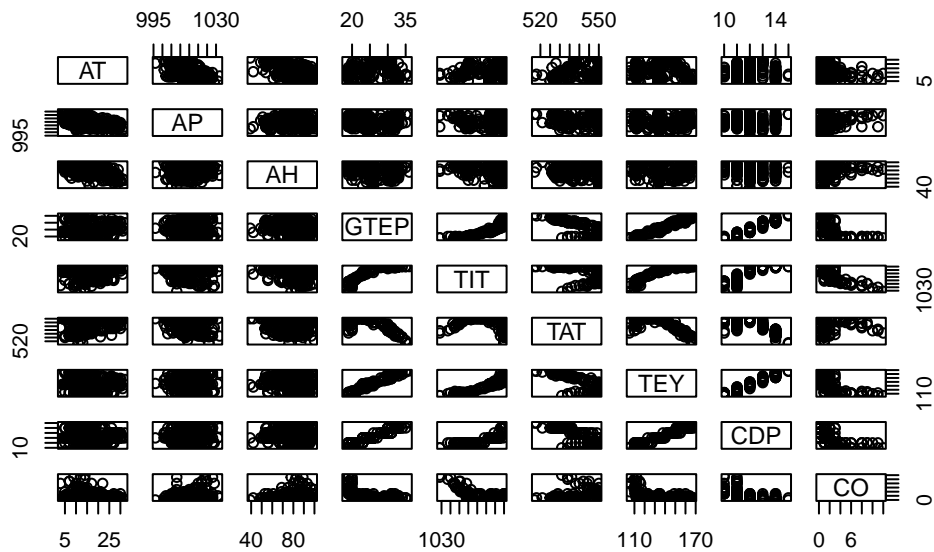
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Task 1

subtask 1

We perform basic inspection:

```
load("Data ST523 813 E2025 Exam.rdata")
df <- Data
pairs(df)
```



```
nrow(df)
```

```
[1] 300
```

```
ncol(df)
```

```
[1] 9
```

```
summary(df)
```

AT		AP		AH		GTEP		TIT	
Min.	: 3.00	Min.	: 996	Min.	: 39.00	Min.	:18.00	Min.	:1028
1st Qu.	:11.00	1st Qu.	:1010	1st Qu.	: 72.00	1st Qu.	:23.00	1st Qu.	:1083
Median	:16.00	Median	:1014	Median	: 82.00	Median	:25.00	Median	:1089
Mean	:16.74	Mean	:1015	Mean	: 79.82	Mean	:25.85	Mean	:1085
3rd Qu.	:22.25	3rd Qu.	:1019	3rd Qu.	: 90.00	3rd Qu.	:30.00	3rd Qu.	:1100
Max.	:32.00	Max.	:1032	Max.	:100.00	Max.	:36.00	Max.	:1100

TAT		TEY		CDP		CO	
Min.	:517.0	Min.	:105.0	Min.	:10.00	Min.	: 0.0
1st Qu.	:536.0	1st Qu.	:130.0	1st Qu.	:12.00	1st Qu.	: 1.0
Median	:550.0	Median	:134.0	Median	:12.00	Median	: 1.0
Mean	:543.8	Mean	:136.5	Mean	:12.29	Mean	: 1.7
3rd Qu.	:550.0	3rd Qu.	:151.0	3rd Qu.	:13.00	3rd Qu.	: 2.0
Max.	:550.0	Max.	:168.0	Max.	:15.00	Max.	:12.0

We fit a default model:

```
model = lm(CO ~ AT + AP + AH + GTEP + TIT + TAT + CDP + TEY, data = df)
```

subtask 2

Display all the estimated parameters:

```
coef(model)
```

(Intercept)	AT	AP	AH	GTEP
124.201440581	-0.021611280	0.009736961	-0.009621796	-0.370650220

TIT	TAT	CDP	TEY
0.029651998	-0.255240883	-0.463761682	-0.068439127

And the one for ambient temperature in particular:

```
coef(model) ["AT"]
```

```
      AT  
-0.02161128
```

From this we know that the estimated change in CO for a 1°C increase in ambient temperature, other predictors constant, is -0.0216 units.

subtask 3

Perform F-Test:

```
summary(model)
```

Call:

```
lm(formula = CO ~ AT + AP + AH + GTEP + TIT + TAT + CDP + TEY,  
    data = df)
```

Residuals:

	Min	1Q	Median	3Q	Max
	-2.8724	-0.5804	-0.0571	0.4270	4.3507

Coefficients:

	Estimate	Std. Error	t value	Pr(> t)
(Intercept)	124.201441	17.933449	6.926	2.78e-11 ***
AT	-0.021611	0.028025	-0.771	0.441242
AP	0.009737	0.012831	0.759	0.448558
AH	-0.009622	0.005824	-1.652	0.099605 .
GTEP	-0.370650	0.158512	-2.338	0.020048 *
TIT	0.029652	0.058304	0.509	0.611437
TAT	-0.255241	0.072992	-3.497	0.000544 ***
CDP	-0.463762	0.221422	-2.094	0.037083 *
TEY	-0.068439	0.071399	-0.959	0.338581

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.9744 on 291 degrees of freedom

Multiple R-squared: 0.7386, Adjusted R-squared: 0.7314

F-statistic: 102.8 on 8 and 291 DF, p-value: < 2.2e-16

As we can see:

F-statistic: 102.8 on 8 and 291 DF, p-value: < 2.2e-16

P-value is very low which indicates the rejection of H_0 - global null hypothesis at significance $\alpha = 0.05$

We have 291 residual degrees of freedom (n - p) and 8 model degrees of freedom (p - 1)

Value of the F-test statistic is 102.8

subtask 4

Let's start by fitting 2 submodels:

```
M_A = lm(CO ~ AT + AP + AH, data = df)
M_B = lm(CO ~ GTEP + TIT + TAT + TEY, data = df)
```

Now try comparing ambient only model to the default:

```
anova(M_A, model)
```

Analysis of Variance Table

Model 1: CO ~ AT + AP + AH

Model 2: CO ~ AT + AP + AH + GTEP + TIT + TAT + CDP + TEY

	Res.Df	RSS	Df	Sum of Sq	F	Pr(>F)
1	296	971.35				
2	291	276.31	5	695.04	146.4	< 2.2e-16 ***

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

As we can see by the corresponding F-statistic (146.4) and p-value (< 2.2e-16) The default model explains the data *much better* i.e.

$$\exists_{i \neq AT, AP, AH} \text{ such that } \beta_i \neq 0$$

So we reject the H_0 that all other predictions aside from AT, AP, AH are = 0.

What about the process only model?:

```
anova(M_B, model)
```

Analysis of Variance Table

Model 1: CO ~ GTEP + TIT + TAT + TEY

Model 2: CO ~ AT + AP + AH + GTEP + TIT + TAT + CDP + TEY

	Res.Df	RSS	Df	Sum of Sq	F	Pr(>F)
1	295	286.13				
2	291	276.31	4	9.8158	2.5844	0.0373 *

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

The rejection of the null hypothesis—that the reduced (process-only) model explains the data equally well—is supported by the F statistic (~ 2.58) and the corresponding p-value (0.037). This suggests that, at a significance level of $\alpha = 0.05$, adding other predictors leads to a statistically significant improvement in model fit. However, for $\alpha = 0.01$, the evidence would not be strong enough to reject the null hypothesis.

Therefore we conclude that we *can reduce* the default model to the Process Only, however only when accepting our significance level to be < 0.04

subtask 5

From the last subtask we can conclude that M_B is better than M_A therefore M_B is chosen as our final model for this subtask.

By inspecting adjusted R^2 between models we can see the difference of total adjusted explained variation between models.

```
R_Ambient = summary(M_A)$adj.r.squared
R_Process = summary(M_B)$adj.r.squared
R_default = summary(model)$adj.r.squared
```

```
R_Ambient
```

```
[1] 0.07171319
```

```
R_Process
```

```
[1] 0.7256326
```

```
R_default
```

```
[1] 0.731403
```

The absolute reduction of unexplained variation for Process Only model M_B:

```
explained_variation_Model_B = summary(M_B)$r.squared
CO = df$CO
TSS = sum((CO - mean(CO))^2)
RSS = sum(residuals(M_B)^2)
absolute_reduction_Model_B = TSS - RSS

explained_variation_Model_B
```

```
[1] 0.7293031
```

```
absolute_reduction_Model_B
```

```
[1] 770.8733
```

Let us introduce another alternative model M_Alt that includes CDP as one of its predictors:

```
M_Alt = lm(CO ~ GTEP + TIT + TAT + CDP + TEY, data = df)

anova(M_Alt, M_B)
```

Analysis of Variance Table

Model 1: CO ~ GTEP + TIT + TAT + CDP + TEY

Model 2: CO ~ GTEP + TIT + TAT + TEY

	Res.Df	RSS	Df	Sum of Sq	F	Pr(>F)
1	294	281.35				
2	295	286.13	-1	-4.7816	4.9967	0.02615 *

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Analyzing this anova table we can conclude that the alternative model is better as the performed partial F test returned low p-value

However we could further reduce the M_Alt:

```
anova(M_Alt)
```

Analysis of Variance Table

Response: CO

	Df	Sum Sq	Mean Sq	F value	Pr(>F)
GTEP	1	341.46	341.46	356.8233	< 2.2e-16 ***
TIT	1	401.59	401.59	419.6583	< 2.2e-16 ***
TAT	1	24.64	24.64	25.7507	6.899e-07 ***
CDP	1	5.96	5.96	6.2230	0.01316 *
TEY	1	2.00	2.00	2.0886	0.14946
Residuals	294	281.35	0.96		

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Here we see that the F test corresponding to the inclusion of TEY predictor shows a relatively high p-value that might indicate corresponding $\beta_{TEY} \approx 0$

It is worth performing R^2 inspection for the alternate model M_Alt that includes CDP predictor and removes TEY as it seemed redundant:

```
M_Alt = lm(CO ~ GTEP + TIT + TAT + CDP, data = df)

explained_variation_ModelAlt = summary(M_Alt)$r.squared
RSS = sum(residuals(M_Alt)^2)
absolute_reduction_ModelAlt = TSS - RSS

explained_variation_ModelAlt
```

```
[1] 0.7319358
```

```
absolute_reduction_ModelAlt
```

```
[1] 773.6562
```

The absolute reduction in the residual sum of squares is only slightly higher for model M_Alt , at the cost of including one additional predictor (CDP), and removing one other (TEY). Depending on our circumstances, we can choose to either stay by M_Alt for marginally better performance or opt for the different M_B , which performs almost equivalently.

Task 2

subtask 1

The model presented in task can also be written as:

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_{45} \end{bmatrix}, \quad X = \begin{bmatrix} 1 & \mathbb{1}_{2,1} & \mathbb{1}_{3,1} & \mathbb{1}_{4,1} & X_1 \\ 1 & \mathbb{1}_{2,2} & \mathbb{1}_{3,2} & \mathbb{1}_{4,2} & X_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \mathbb{1}_{2,45} & \mathbb{1}_{3,45} & \mathbb{1}_{4,45} & X_{45} \end{bmatrix}, \quad \beta = \begin{bmatrix} \mu \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ b \end{bmatrix}, \quad \epsilon = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_{45} \end{bmatrix}.$$

$$\mathbb{1}_{i,j} = \begin{cases} 1 & \text{if } \alpha_j \text{ present in sample } i \\ 0 & \text{otherwise} \end{cases}$$

Where column 1 of X corresponds to always present μ and each column 2, 3, 4 is filled with 0, 1 depending on corresponding $\alpha_{j(i)}$ (indicator function)

Also α_i defined as:

α_2 = Temporary – Research/Academic (relative to Permanent)

α_3 = Temporary – Private Consultant (relative to Permanent)

α_4 = Freelance (relative to Permanent)

We know that:

$$\text{Cov}(\hat{\alpha}_2, \hat{\alpha}_3) = 22,000,000$$

$$\text{Cov}(\hat{\alpha}_2, \hat{\alpha}_4) = 20,000,000$$

$$\text{Cov}(\hat{\alpha}_3, \hat{\alpha}_4) = 21,000,000$$

And:

$$\widehat{\text{SE}}(\hat{\mu}) = 20,000$$

$$\widehat{\text{SE}}(\hat{\alpha}_2) = 24,000$$

$$\widehat{\text{SE}}(\hat{\alpha}_3) = 23,000$$

$$\widehat{\text{SE}}(\hat{\alpha}_4) = 22,000$$

First we want to calculate the CL for $\alpha_2 - \alpha_3$ with confidence 90%

Which is:

$$\begin{aligned}
c &= [0, 1, -1, 0, 0], \\
\hat{\psi} &= c^T \hat{\beta} = \hat{\alpha}_2 - \hat{\alpha}_3, \\
\hat{SE}(\hat{\psi}) &= \sqrt{c Var(\hat{\beta}) c^T}, \\
&= \sqrt{Var(\hat{\alpha}_2 - \hat{\alpha}_3)} \\
&= \sqrt{Var(\hat{\alpha}_2) + Var(\hat{\alpha}_3) - 2Cov(\hat{\alpha}_2, \hat{\alpha}_3)}, \\
CI_{90\%} &= \hat{\psi} \pm t_{45-5, 1-0.10/2} \cdot \hat{SE}(\hat{\psi})
\end{aligned}$$

That then by substitution becomes:

$$\begin{aligned}
\hat{\psi} &= -40,000 - (-10,000) \\
&= -30,000 \\
\hat{SE}(\hat{\psi}) &= \sqrt{(24,000)^2 + (23,000)^2 - 2 \cdot 22,000,000} \\
&\approx 32,573 \\
t_{40, (1-0.10)/2} &\approx 1.68385 \\
CI_{90\%} &= \hat{\psi} \pm t_{45-5, 1-0.10/2} \cdot \hat{SE}(\hat{\psi}) \\
CI_{90\%} &= -30,000 \pm 1.68385 \cdot 32,573 \\
&= -30,000 \pm 54848
\end{aligned}$$

Therefore the Confidence interval is:

$$CI_{90\%} = [-84,848, 24,848]$$

That concludes the first subtask of Task 2.

subtask 2

Next we want to look for statistical evidence for: $\alpha_2 \leq \alpha_3$

Therefore we perform One Sided hypothesis Test on:

$$\begin{aligned}
H_0 &= \alpha_2 - \alpha_3 > 0 \\
H_a &= \alpha_2 - \alpha_3 \leq 0
\end{aligned}$$

H_0 - temporary researchers are earning **more** than temporary private consultants

H_a - the opposite

We are using the same $\hat{\psi}$ from previous subtask:

$$\begin{aligned}\hat{\psi} &= \hat{\alpha}_2 - \hat{\alpha}_3 \\ T &= \frac{(\hat{\alpha}_2 - \hat{\alpha}_3) - 0}{\widehat{\text{SE}}(\hat{\psi})} \\ &= \frac{-30,000}{32,573} \\ &\approx -0.921\end{aligned}$$

Then:

```
qt(0.05, df = 40)
```

```
[1] -1.683851
```

$$\begin{aligned}-0.921 &> -1.683851 \\ T &> t_{40, 0.05}\end{aligned}$$

We cannot reject H_0

Therefore there is not enough statistical evidence that temporary researchers are earning less than temporary private consultants.

Task 3

We have $x_i \in [-2, 2]$ for $i = 1, \dots, n$. The X and β matrices are as follows:

$$X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}.$$

Therefore, the variance of $\hat{\beta}$ is

$$\text{Var}(\hat{\beta}) = \sigma^2 (X^\top X)^{-1} = \sigma^2 \begin{bmatrix} n & \sum_i x_i \\ \sum_i x_i & \sum_i x_i^2 \end{bmatrix}^{-1}.$$

Following the inverse this becomes

$$\text{Var}(\hat{\beta}) = \frac{\sigma^2}{n \sum_i x_i^2 - (\sum_i x_i)^2} \begin{bmatrix} \sum_i x_i^2 & -\sum_i x_i \\ -\sum_i x_i & n \end{bmatrix}.$$

And then, variance of the slope estimate ($\hat{\beta}_1$) is:

$$\text{Var}(\hat{\beta}_1) = \frac{\sigma^2 n}{n \sum_i x_i^2 - (\sum_i x_i)^2}$$

Which further reduces to:

$$\begin{aligned} \text{Var}(\hat{\beta}_1) &= \frac{\sigma^2}{\sum_i x_i^2 - \frac{1}{n}(\sum_i x_i)^2} = \frac{\sigma^2}{\sum_i x_i^2 - \frac{2}{n}(\sum_i x_i)^2 + \frac{1}{n}(\sum_i x_i)^2} = \\ &= \frac{\sigma^2}{\sum_i x_i^2 - 2 \sum_i x_i \cdot \bar{x} + n \bar{x}^2} = \frac{\sigma^2}{\sum_i (x_i - \bar{x})^2} \end{aligned}$$

We want to minimize the $\text{Var}(\hat{\beta}_1)$. Which can be done by maximizing $\sum_i (x_i - \bar{x})^2$ as σ^2 is constant. This can be achieved by spreading x_1, \dots, x_n as much as possible that will ideally set $\bar{x} = \frac{1}{n} \sum_i (x_i) = 0$

We know that:

$$x_i \in [-2, 2], \quad \forall i = 1, \dots, n$$

Then we can choose:

- $x_1, \dots, x_{n/2} = -2$
- $x_{n/2+1}, \dots, x_n = 2$

By that we minimize $\text{Var}(\hat{\beta}_1)$ now equal to:

$$\text{Var}(\hat{\beta}_1) = \frac{\sigma^2}{\sum_i (x_i - \bar{x})^2} = \frac{\sigma^2}{\sum_i (\pm 2 - 0)^2} = \frac{\sigma^2}{4n}$$

And we achieve maximum possible precision

Q.E.D.

Task 4

Only for Master's