DM545 – Linear and Integer Programming

Answers to the Take-home Assignment, Winter 2025

In this assignmnent I will be using Tableau Prinitng function provided on github by Marco always including it as:

```
from util import tableau
```

I will also be using *numpy*, and *fractions* for matrix operations:

```
import numpy as np
from fractions import Fraction
```

Throughout the Excersise a website that was showed during lectures will also be used as solver/visualizer

Task 1

Subtask 1.a

In the following problem:

$$\begin{aligned}
\max\ & 4x_1 + 5x_2 - 7x_3 \\
& -x_1 - x_2 + x_3 \leq 2, \\
& -5x_1 + 10x_3 \leq 10, \\
& x_1 \in [0, 5], \\
& x_2 \in [-1, 1], \\
& x_3 \in [-2, 2].
\end{aligned}$$

By inspection of the objective function we can deduct the potential values of variables:

$$\begin{aligned}
4x_1 &\Rightarrow 4 \text{ is positive } \Rightarrow x_1 \text{ make as large as possible}, \\
5x_2 &\Rightarrow 5 \text{ is positive } \Rightarrow x_2 \text{ make as large as possible}, \\
-7x_3 &\Rightarrow -7 \text{ is negative } \Rightarrow x_3 \text{ make as small as possible}.
\end{aligned}$$

Check potential solution formed by limits of interval bounds $[x_1, x_2, x_3] = [5, 1, -2]$:

$$\begin{aligned}
\text{1st constraint: } & -x_1 - x_2 + x_3 \leq 2 \\
& -5 - 1 + (-2) \leq 2 \\
& -8 \leq 2 \quad \text{is feasible} \\
\text{2nd constraint: } & -5x_1 + 10x_3 \leq 10 \\
& -25 + 10(-2) \leq 10 \\
& -45 \leq 10 \quad \text{is feasible}
\end{aligned}$$

$$\begin{aligned}
&\text{Objecive function value:} \\
&4x_1 + 5x_2 - 7x_3 = \\
&20 + 5 + 14 = \\
&39
\end{aligned}$$

We conclude the optimal solution is $[x_1, x_2, x_3] = [5, 1, -2]$

Subtask 1.b

$$\begin{aligned} \max\ & x_1 + x_2 \\ & sx_1 + tx_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

We can choose (s) and (t) to get different cases:

I) Single Optimal Solution

$$s = 2, \quad t = 1$$

- The slope of the only constraint is different from the slope of the objective function.
- Geometrically, the feasible region intersects the objective function at a single vertex.
- The vertex is our optimal solution

II) Infinite Optimal Solutions

$$s = 1, \quad t = 1$$

- The constraint line is parallel to the objective function.
- Objective function “placed” on the face gives us infinitely many optimal solutions.

III/IV) Infeasible / Unbounded

- If either s or t (or both) are negative, the problem becomes unbounded as each variable balances the other in the constraint allowing the objective function to grow indefinitely.

$$s = -1, \quad t = 1 \text{ is an example of unbounded}$$

- For any $s, t \geq 0$, setting $x_1 = \frac{1}{s}, x_2 = \frac{1}{t}$ gives a feasible solution (except if $s = 0$ or $t = 0$, in which case the value of x_1 or x_2 does not matter as it’s multiplied by 0).

It is **impossible** to set s, t such that the problem becomes *infeasible*, we considered all posssible cases

Task 2

Subtask 2.a

First, we transform the problem into the standard form:

- Change a minimization into a maximization: min$(c^T x)$ into $-$ max $-(c^T x)$.
- Replace equalities with two inequalities.

- Convert all constraints to "$\leq$".

The result is:

$$
\begin{aligned}
-\max -3x_1 - 2x_2 - 7x_3 & \\
-x_1 + x_2 &\leq 10, \\
x_1 - x_2 &\leq -10, \\
-2x_1 + x_2 - x_3 &\leq -10.
\end{aligned}
$$

Then the tableau looks like:

```
T = np.array([[-1,1,0,1,0,0,0,10],
              [1,-1,0,0,1,0,0,-10],
              [-2,1,-1,0,0,1,0,-10],
              [-3,-2,-7,0,0,0,1,0]],dtype=object)

tableau(T)
```

```
|-------+-------+-------+-------+-------+-------+-------+-------+
|    x1 |    x2 |    x3 |    x4 |    x5 |    x6 |    -z |     b |
|-------+-------+-------+-------+-------+-------+-------+-------+
|    -1 |     1 |     0 |     1 |     0 |     0 |     0 |    10 |
|     1 |    -1 |     0 |     0 |     1 |     0 |     0 |   -10 |
|    -2 |     1 |    -1 |     0 |     0 |     1 |     0 |   -10 |
|-------+-------+-------+-------+-------+-------+-------+-------+
|    -3 |    -2 |    -7 |     0 |     0 |     0 |     1 |     0 |
|-------+-------+-------+-------+-------+-------+-------+-------+
```

As we can see the tableau is **optimal** (no positive reduced costs) but **infeasible** (two of the $b_i \leq 0$), we could apply Dual Simplex to work towards feasibility

Subtask 2.b

Tableau is given:

```
T = np.array([[0,0,0,1,1,0,0,0],
              [0,1,1,2,0,-1,0,30],
              [1,0,1,1,0,-1,0,20],
              [0,0,2,-7,0,5,1,-120]],dtype=object)

tableau(T)
```

```
|-------+-------+-------+-------+-------+-------+-------+-------+
|    x1 |    x2 |    x3 |    x4 |    x5 |    x6 |    -z |     b |
|-------+-------+-------+-------+-------+-------+-------+-------+
|     0 |     0 |     0 |     1 |     1 |     0 |     0 |     0 |
|     0 |     1 |     1 |     2 |     0 |    -1 |     0 |    30 |
|     1 |     0 |     1 |     1 |     0 |    -1 |     0 |    20 |
|-------+-------+-------+-------+-------+-------+-------+-------+
|     0 |     0 |     2 |    -7 |     0 |     5 |     1 |  -120 |
|-------+-------+-------+-------+-------+-------+-------+-------+
```

It is worth noting that this tableau is *unbounded* as all coefficients of x_6 in A are negative, but x_6 has a positive reduced cost, however we can still technically perform a change of basis and see the results.

By largest coefficient x_6 would have to enter (and x_3 leave as it's constraint is tighter) and that would make the problem infeasible, and unoptimal

```
# III * -1
T[2] = T[2] * Fraction(-1, 1)

# II + III
T[1] = T[1] + T[2]

# IV - 5*III
T[3] = T[3] - 5*T[2]

tableau(T)
```

```
|-------+-------+-------+-------+-------+-------+-------+-------+
|    x1 |    x2 |    x3 |    x4 |    x5 |    x6 |    -z |     b |
|-------+-------+-------+-------+-------+-------+-------+-------+
|     0 |     0 |     0 |     1 |     1 |     0 |     0 |     0 |
|    -1 |     1 |     0 |     1 |     0 |     0 |     0 |    10 |
|    -1 |     0 |    -1 |    -1 |     0 |     1 |     0 |   -20 |
|-------+-------+-------+-------+-------+-------+-------+-------+
|     5 |     0 |     7 |    -2 |     0 |     0 |     1 |   -20 |
|-------+-------+-------+-------+-------+-------+-------+-------+
```

By ratio test x_3 enters (as x_6 has only negative coefficients in A) and x_1 leaves as $20/1 < 30/1$ (we ignore first line $0/0 = ?$).

Coincidentally by Bland's Rule x_3 enters and x_1 leaves (we take the lowest index)

We can follow with both of them:

```
T = np.array([[0,0,0,1,1,0,0,0],
              [0,1,1,2,0,-1,0,30],
              [1,0,1,1,0,-1,0,20],
              [0,0,2,-7,0,5,1,-120]],dtype=object)


# II - III
T[1] = T[1] - T[2]

# IV - 2*III
T[3] = T[3] - 2*T[2]

tableau(T)
```

```
|-------+-------+-------+-------+-------+-------+-------+-------+
|    x1 |    x2 |    x3 |    x4 |    x5 |    x6 |    -z |     b |
|-------+-------+-------+-------+-------+-------+-------+-------+
|     0 |     0 |     0 |     1 |     1 |     0 |     0 |     0 |
|    -1 |     1 |     0 |     1 |     0 |     0 |     0 |    10 |
|     1 |     0 |     1 |     1 |     0 |    -1 |     0 |    20 |
|-------+-------+-------+-------+-------+-------+-------+-------+
|    -2 |     0 |     0 |    -9 |     0 |     7 |     1 |  -160 |
|-------+-------+-------+-------+-------+-------+-------+-------+
```

As we can see the tableau is still unbounded (by the case of x_6)

Task 3

Subtask 3.a

Tableau given:

```
T = np.array([[1,0,1,-1,0,5],
              [0,1,-2,3,0,15],
              [0,0,-2,-2,1,-110]],dtype=object)

tableau(T)
```

```
|-------+-------+-------+-------+-------+-------+
|    x1 |    x2 |    x3 |    x4 |    -z |     b |
|-------+-------+-------+-------+-------+-------+
|     1 |     0 |     1 |    -1 |     0 |     5 |
|     0 |     1 |    -2 |     3 |     0 |    15 |
|-------+-------+-------+-------+-------+-------+
|     0 |     0 |    -2 |    -2 |     1 |  -110 |
|-------+-------+-------+-------+-------+-------+
```

From the tableau, we observe the following:

- The solution is $[x_1, x_2, x_3, x_4] = [5, 15, 0, 0]$ with objective value 110.
- The reduced costs are $-2, -2$
- The values of dual variables are $2, 2$ (negative reduced costs).
- The shadow prices are the same as the dual variables: $2, 2$
- There is no over-capacity, as all the constraints are tight (no slacks in the basis).

Task 4

Subtask 4.a

We denote original problem as **P** and relaxed original problem as **PR** Let's start by considering the original problem (with marked constraints by potential dual variables α, β, γ)

$$\begin{aligned}
\text{min } & \sum_{i=1}^{n} c_i y_i \\
& \sum_{j=1}^{m} a_{ij} x_{ij} \le b_i y_i, \quad i = 1, \dots, n \quad (\alpha) \\
& \sum_{i=1}^{n} x_{ij} = 1, \quad j = 1, \dots, m \quad (\beta) \\
& y_i \le 1, \quad i = 1, \dots, n \quad (\gamma) \\
& \\
& y_i \ge 0, \quad x_{ij} \ge 0.
\end{aligned}$$

We denote the potential dual variables:

$$\begin{aligned}
\alpha &= [\alpha_1, \dots, \alpha_n] \\
\beta &= [\beta_1, \dots, \beta_m] \\
\gamma &= [\gamma_1, \dots, \gamma_n]
\end{aligned}$$

Then measure the violation of constraints (by putting everything to one side and multiplying by corresponding dual variable):

$$\begin{array}{ccc}
\alpha_1 \, (0 + b_1 y_1 - \sum_{j=1}^{m} a_{1j} x_{1j}) & \beta_1 \, (1 - \sum_{i=1}^{n} x_{i1}) & \gamma_1 \, (1 - y_1) \\
\alpha_2 \, (0 + b_2 y_2 - \sum_{j=1}^{m} a_{2j} x_{2j}) & \beta_2 \, (1 - \sum_{i=1}^{n} x_{i2}) & \gamma_2 \, (1 - y_2) \\
\vdots & \vdots & \vdots \\
\alpha_n \, (0 + b_n y_n - \sum_{j=1}^{m} a_{nj} x_{nj}) & \beta_m \, (1 - \sum_{i=1}^{n} x_{im}) & \gamma_n \, (1 - y_n)
\end{array}$$

Then we denote **PR** relaxed problem:

$$\text{PR}(\alpha, \beta, \gamma) = \min_{\text{by all } y, x \ge 0} \{\vdots\} =$$

$$= \min_{\text{by all } y,x\geq 0} \left\{ \begin{array}{r} c_1 y_1 + \dots + c_n y_n + \\ \alpha_1 \, (b_1 y_1 - \sum_{j=1}^{m} a_{1j} x_{1j}) + \\ \alpha_2 \, (b_2 y_2 - \sum_{j=1}^{m} a_{2j} x_{2j}) + \\ \vdots \\ \alpha_n \, (b_n y_n - \sum_{j=1}^{m} a_{nj} x_{nj}) + \\ \beta_1 \, (1 - \sum_{i=1}^{n} x_{i1}) + \\ \beta_2 \, (1 - \sum_{i=1}^{n} x_{i2}) + \\ \vdots \\ \beta_m \, (1 - \sum_{i=1}^{n} x_{im}) + \\ \gamma_1 \, (1 - y_1) + \\ \gamma_2 \, (1 - y_2) + \\ \vdots \\ \gamma_n \, (1 - y_n) \end{array} \right\} = \min_{\text{by all } y,x\geq 0} \left\{ \begin{array}{r} y_1 \, (c_1 + \alpha_1 b - \gamma_1) + \\ y_2 \, (c_2 + \alpha_2 b - \gamma_2) + \\ \vdots \\ y_1 \, (c_n + \alpha_n b - \gamma_n) + \\ x_{11} \, (0 - \alpha_n a_1 - \beta_1) + \\ x_{21} \, (0 - \alpha_2 a_1 - \beta_2) + \\ \vdots \\ x_{n1} \, (0 - \alpha_n a_1 - \beta_n) + \\ x_{12} \, (0 - \alpha_1 a_2 - \beta_1) + \\ x_{22} \, (0 - \alpha_2 a_2 - \beta_2) + \\ \vdots \\ x_{nm} \, (0 - \alpha_n a_m - \beta_n) + \\ \sum_{j=1}^{m} \beta_j + \\ \sum_{i=1}^{n} \gamma_i + \end{array} \right\}$$

To avoid useless lower bounds we set all parts multiplied by $y, x \geq 0$, (otherwise corresponding variable, for instance $x_{i,j} -> \infty$ and $\min_{\text{by all } y,x\geq 0} = -\infty$)
That's how we get to the dual constraints:

$$\begin{array}{rcr} (c_1 + \alpha_1 b - \gamma_1) \geq 0 & & \alpha_1 b - \gamma_1 \geq -c_1 \\ (c_2 + \alpha_2 b - \gamma_2) \geq 0 & & \alpha_2 b - \gamma_2 \geq -c_2 \\ \vdots & & \vdots \\ (c_n + \alpha_n b - \gamma_n) \geq 0 & & -\alpha_n b - \gamma_n \geq -c_n \\ (0 - \alpha_n a_1 - \beta_1) \geq 0 & & -\alpha_n a_1 - \beta_1 \geq 0 \\ (0 - \alpha_2 a_1 - \beta_2) \geq 0 & \Rightarrow & -\alpha_2 a_1 - \beta_2 \geq 0 \\ \vdots & & \vdots \\ (0 - \alpha_n a_1 - \beta_n) \geq 0 & & -\alpha_n a_1 - \beta_n \geq 0 \\ (0 - \alpha_1 a_2 - \beta_1) \geq 0 & & -\alpha_1 a_2 - \beta_1 \geq 0 \\ (0 - \alpha_2 a_2 - \beta_2) \geq 0 & & -\alpha_2 a_2 - \beta_2 \geq 0 \\ \vdots & & \vdots \\ (0 - \alpha_n a_m - \beta_n) \geq 0 & & -\alpha_n a_m - \beta_n \geq 0 \end{array}$$

To get new Objective function we take the sums uncorrelated with y, x in front of the $\min_{\text{by all } y,x} \{\vdots\}$. which are: $\sum_{j=1}^{m} \beta_j + \sum_{i=1}^{n} \gamma_i$ Now we want to:

$$\max_{\alpha,\beta,\gamma} \left\{ \text{PR}(\alpha,\beta,\gamma) = \sum_{j=1}^{m} \beta_j + \sum_{i=1}^{n} \gamma_i + \min_{\text{by all } y,x} \{:\} \right\}.$$

However to keep the:

$$\text{opt}(\text{PR}(\alpha,\beta,\gamma)) \ \leq \ \text{opt}(P)$$

We need to penalize breaking the constraints.
That will cause the found $\min\limits_{\text{by all } y,x\geq 0}$ to not break any constraint.
We have to make it impossible to achieve minimum when breaking the constraint.
And to do that each dual variable must be chosen so that **violating a original constraint increases the value of the objective**.

For the constraints of type $\alpha \Rightarrow \sum_{j=1}^{m} a_{ij}x_{ij} \leq b_i y_i$

- The violation measure is
  $b_i y_i - \sum_{j=1}^{m} a_{ij}x_{ij}$.
- When **broken**, this becomes **negative**.
- To penalize the objective it requires
  $\alpha_i \leq 0$.

For the constraints of type $\gamma \Rightarrow y_i \leq 1$:

- The violation measure is
  $1 - y_i$.
- When **breaking** this becomes **negative**.
- To ensure the penalty, we need
  $\gamma_i \leq 0$.

For the constraints of type $\beta \Rightarrow \sum_{i=1}^{n} x_{ij} = 1$:

- Violations can happen **in both ways**.
- Therefore we set: $\beta_j \in \mathbb{R}$.

This ensures that always:

$$\text{opt}(\text{PR}(\alpha,\beta,\gamma)) \leq \text{opt}(P)$$

Combinig everything togheter we are left with:

$$
\begin{aligned}
\max\ & \sum_{j=1}^{m} \beta_j + \sum_{i=1}^{n} \gamma_i \\
& \alpha_1 b - \gamma_1 \geq -c_1 \\
& \alpha_2 b - \gamma_2 \geq -c_2 \\
& \vdots \\
& -\alpha_n b - \gamma_n \geq 0 \\
& -\alpha_n a_1 - \beta_1 \geq 0 \\
& -\alpha_2 a_1 - \beta_2 \geq 0 \\
& \vdots \\
& -\alpha_n a_1 - \beta_n \geq 0 \\
& -\alpha_1 a_2 - \beta_1 \geq 0 \\
& -\alpha_2 a_2 - \beta_2 \geq 0 \\
& \vdots \\
& -\alpha_n a_m - \beta_n \geq 0 \\
\\
& \alpha_i \leq 0. \quad i = 1, \dots, n \\
& \beta_j \in \mathbb{R}. \quad j = 1, \dots, m \\
& \gamma_i \leq 0. \quad i = 1, \dots, n
\end{aligned}
$$

$\Rightarrow$
organized as

$$
\begin{aligned}
\max\ & \sum_{j=1}^{m} \beta_j + \sum_{i=1}^{n} \gamma_i \\
\\
& -b\alpha_i + \gamma_i \leq c_i \quad i = 1, \dots, n \\
& \alpha_i a_j + \beta_j \leq 0 \quad i = 1, \dots, n \qquad j = 1, \dots, m \\
\\
& \alpha_i \leq 0. \quad i = 1, \dots, n \\
& \beta_j \in \mathbb{R}. \quad j = 1, \dots, m \\
& \gamma_i \leq 0. \quad i = 1, \dots, n
\end{aligned}
$$

Q.E.D.

Task 5

Subtask 5.a

First write original problem with changed $\min() \Rightarrow -\max -()$ (and with slacks):

$$c = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad A = \begin{bmatrix} -3 & 7 & 1 & 0 \\ 1 & -4 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} -21 \\ 4 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

So far $B = 3, 4$ Let's prepare for Revised Simplex:

$$B = 1, 2$$
$$A_B = \begin{bmatrix} -3 & 7 \\ 1 & -4 \end{bmatrix}$$
$$A_B^{-1} = \frac{1}{det(A_B)} \begin{bmatrix} -4 & -7 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} & -\frac{7}{5} \\ -\frac{1}{5} & -\frac{3}{5} \end{bmatrix}$$

(by using the formula for the inverse of 2x2 matrix, and det() of 2x2 matrix)

$$A_N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$c_B^T = \begin{bmatrix} -1 & -1 \end{bmatrix}$$
$$c_N^T = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

To check whether or not the solution with this basis is optimal we just need to compute $c_n^T - c_B^T A_B^{-1} A_N$ (new reduced costs), which in our case can be easily done by hand:

$$\begin{bmatrix} 0 & 0 \end{bmatrix} - \begin{bmatrix} -1 & -1 \end{bmatrix} \begin{bmatrix} -\frac{4}{5} & -\frac{7}{5} \\ -\frac{1}{5} & -\frac{3}{5} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 0 \end{bmatrix} - \begin{bmatrix} (\frac{4}{5} + \frac{1}{5}) & (\frac{7}{5} + \frac{3}{5}) \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 2 \end{bmatrix} =$$

$$= \begin{bmatrix} -1 & -2 \end{bmatrix}$$

As we can see all reduced costs ≤ 0, therefore solution is optimal

Subtask 5.b

Firstly we need to calculate the values x_1, x_2, x_3, x_4 in the optimal solution this can be done by calculatng $A_B^{-1}b$:

$$A_B^{-1}b = \begin{bmatrix} -\frac{4}{5} & -\frac{7}{5} \\ -\frac{1}{5} & -\frac{3}{5} \end{bmatrix} \begin{bmatrix} -21 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{56}{5} \\ \frac{9}{5} \end{bmatrix}, \quad \text{optimal basis was } B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Therefore $x_1 = 11\frac{1}{5}$ and $x_2 = 1\frac{4}{5}$. Let's calculate new non basic A

$$A_{N_{new}} = A_B^{-1}A_N = \begin{bmatrix} -\frac{4}{5} & -\frac{7}{5} \\ -\frac{1}{5} & -\frac{3}{5} \end{bmatrix}$$

Both of the solution x_1, x_2 fractional, we provide Gomory cuts for both:

$$\text{row } u = 1, 2$$

$$\sum_{j \in N_{new}} (\bar{a}_{uj} - \lfloor \bar{a}_{uj} \rfloor) x_j \ \geq \ \bar{b}_u - \lfloor \bar{b}_u \rfloor$$

for convinience not that $x_3 = s_1$, $x_4 = s_2$ - slacks

Gomory cut for $u = 1$

$$(-\frac{4}{5} - \lfloor -\frac{4}{5} \rfloor) s_1 + (-\frac{9}{5} - \lfloor -\frac{9}{5} \rfloor) s_1 \geq \frac{56}{5} - \lfloor \frac{56}{5} \rfloor$$

$$\frac{1}{5} s_1 + \frac{3}{5} s_2 \geq \frac{1}{5}$$

$$s_1 + 3s_2 \geq 1$$

Gomory cut for $u = 2$

$$(-\frac{1}{5} - \lfloor -\frac{1}{5} \rfloor) s_1 + (-\frac{3}{5} - \lfloor -\frac{3}{5} \rfloor) s_1 \geq \frac{9}{5} - \lfloor \frac{9}{5} \rfloor$$

$$\frac{4}{5} s_1 + \frac{2}{5} s_2 \geq \frac{4}{5}$$

$$4s_1 + 2s_2 \geq 4$$

We can express them in terms of the original variables by substitution, using the equalities from standard form:

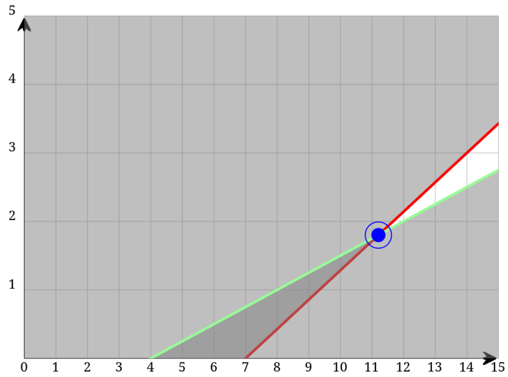
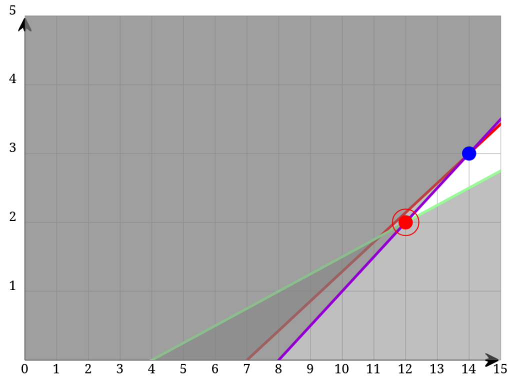
$$-3x_1 + 7x_2 + s_1 = -21$$
$$x_1 - 4x_2 + s_2 = 4$$

$$s_1 = -21 + 3x_1 - 7x_2$$
$$s_2 = 4 - x_1 + 4x_2$$

Gomory cut for $u = 1$	Gomory cut for $u = 2$
$s_1 + 3s_2 \geq 1$	$4s_1 + 2s_2 \geq 4$
$-21 + 3x_1 - 7x_2$ $+12 - 3x_1 + 12x_2 \geq 1$	$-84 + 12x_1 - 28x_2$ $+8 - 2x_1 + 8x_2 \geq 4$
$-9 + 5x_2 \geq 1$	$-76 + 10x_1 - 20x_2 \geq 4$
$x_2 \geq -2$	$x_1 - 2x_2 \geq 8$

Subtask 5.c

Website was used as a visualizer
$x_2 \geq -2$ is not visible as it's weaker than base constraints $x_1, x_2 \geq 0$

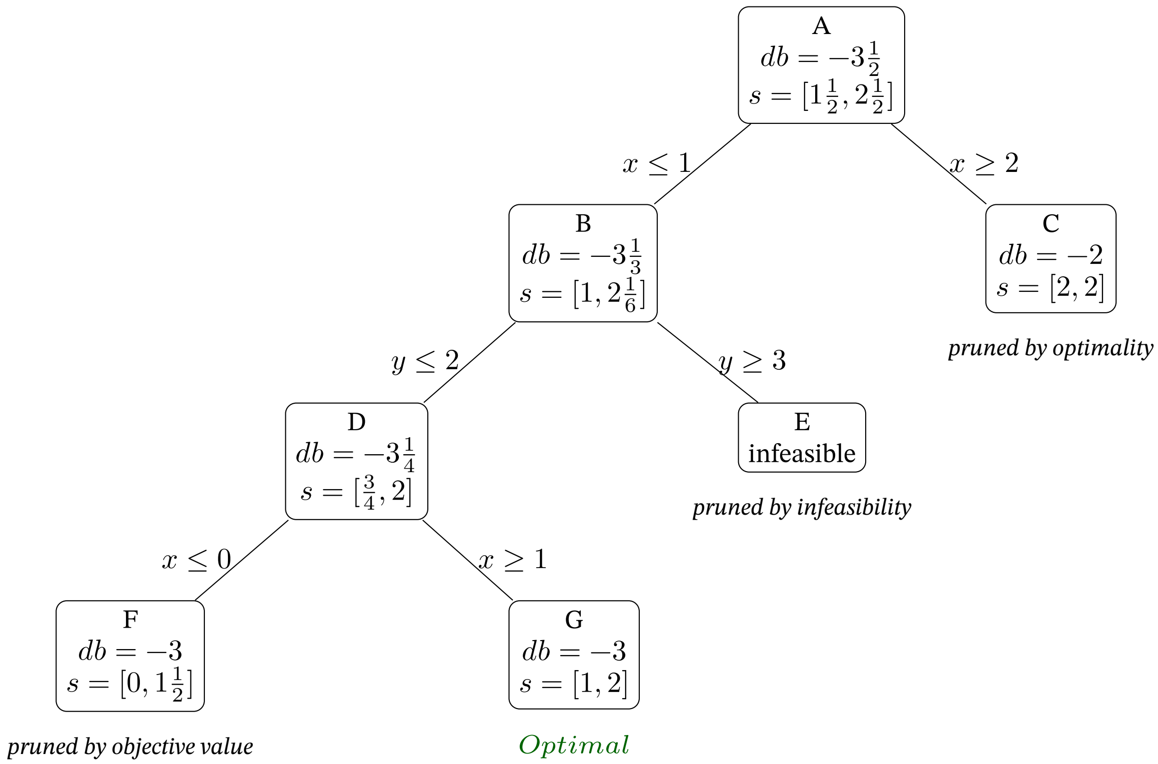
without Gomory cuts	with Gomory cuts
 (56/5, 9/5) Minimum	 (14, 3) (12, 2) Minimum

Task 6

Subtask 6.a

$$
\begin{aligned}
\min\ & x - 2y \\
\text{s.t.}\ & -4x + 6y \leq 9 \\
& x + y \leq 4 \\
& x, y \in \mathbb{N}
\end{aligned}
$$

As the problem has only 2 variables website was used as a visualizer and solver.
Nodes were opened in alphabetical order (i.e. A $\rightarrow$ B $\rightarrow$ C ...)



Task 7

Subtask 7.a

Firstly let us note *Hamming Distance* H_d between two words w_i, w_j of lenght d i.e. $w_i, w_j \in \{\mathbf{0}, \mathbf{1}\}^d$

$$H_d(w_i, w_j) = \sum_{k=1}^{d} \mathbf{1}\big(w_i[k] \neq w_j[k]\big)$$

k being the index of a word being checked and **1** indicator function

Our problem can then be expressed as:

$$\max \left\{\min \left\{H_d(w_i, w_j)\right\}\right\} \text{ for } i, i \in \{1 \cdots N\}$$

across all $w_i \neq w_j$ of $|w_i| = |w_j|$

We transform it into *Mathematical Programming form* by first denoting:

$$M_d = \min \left\{H_d(w_i, w_j)\right\} \text{ for } i, i \in \{1 \cdots N\}$$

And subsequently transforming the formulation into:

$$\begin{aligned} &\max M_d \\ &\sum_{k=1}^{d} \mathbf{1}\big(w_i[k] \neq w_j[k]\big) \geq M_d \\ &\text{for } 1 \leq i < j \leq N \end{aligned}$$

To make the constraints more workable:

$$\begin{aligned} &\max M_d \\ &\sum_{k=1}^{d} |w_i[k] - w_j[k]| \geq M_d \\ &\text{for } 1 \leq i < j \leq N \\ &\underset{j,k,i}{\forall} \; w_i[k], w_j[k] \in \{\mathbf{0}, \mathbf{1}\} \end{aligned}$$

This formulation has exactly $\frac{N(N-1)}{2}$ constraints (all unique pairs) and $N \cdot d$ variables (N words of lenght d)
We would like however to get rid of not explicitily linear absolute value in the constraints. We can do that by introductiong another variable that will replace it, let's call it γ:

$$\gamma_{ijk} \Leftarrow |w_i[k] - w_j[k]|$$

We have to bound it so that in all possible situations it is forced to hold a correct value:

$$\begin{aligned}
&\gamma_{ijk} \leq 2 - (w_i[k] + w_j[k]) \\
&\gamma_{ijk} \leq w_i[k] + w_j[k] \\
&\gamma_{ijk} \geq w_i[k] - w_j[k] \\
&\gamma_{ijk} \geq w_j[k] - w_i[k]
\end{aligned}$$

We see that using previously mentioned bounds we can replace absolute value in all cases:

$$\begin{array}{ll}
w_i[k], w_j[k] = (0,0) & w_i[k], w_j[k] = (1,1) \\
& \\
\gamma_{ijk} \leq 2 & \gamma_{ijk} \leq 0 \\
\gamma_{ijk} \leq 0 & \gamma_{ijk} \leq 2 \\
\gamma_{ijk} \geq 0 & \gamma_{ijk} \geq 0 \\
\gamma_{ijk} \geq 0 & \gamma_{ijk} \geq 0 \\
\Rightarrow \gamma_{ijk} = 0 & \Rightarrow \gamma_{ijk} = 0 \\
w_i[k], w_j[k] = (0,1) & w_i[k], w_j[k] = (1,0) \\
& \\
\gamma_{ijk} \leq 1 & \gamma_{ijk} \leq 1 \\
\gamma_{ijk} \leq 1 & \gamma_{ijk} \leq 1 \\
\gamma_{ijk} \geq -1 & \gamma_{ijk} \geq 1 \\
\gamma_{ijk} \geq 1 & \gamma_{ijk} \geq -1 \\
\Rightarrow \gamma_{ijk} = 1 & \Rightarrow \gamma_{ijk} = 1
\end{array}$$

Combinig that our new formulation looks as follows:

$$\begin{aligned}
&\max M_d \\
&\left\{ \sum_{k=1}^{d} \gamma_{ijk} \geq M_d \right\} \quad 1 \leq i < j \leq N \\
&\left.\begin{cases}
\gamma_{ijk} \leq 2 - (w_i[k] + w_j[k]) \\
\gamma_{ijk} \leq w_i[k] + w_j[k] \\
\gamma_{ijk} \geq w_i[k] - w_j[k] \\
\gamma_{ijk} \geq w_j[k] - w_i[k]
\end{cases}\right\} \begin{array}{l} 1 \leq i < j \leq N \\ \\ 1 \leq k \leq d \end{array} \\
&\forall_{j,k,i} \ \gamma_{ijk} \in \mathbb{R} \\
&\forall_{j,k,i} \ w_i[k], w_j[k] \in \{\mathbf{0}, \mathbf{1}\}
\end{aligned}$$

That yields a total of $(\frac{N(N-1)}{2} + 4d\frac{N(N-1)}{2})$ constraints and $(N \cdot d + d\frac{N(N-1)}{2})$ variables $(w_i[k] + \gamma_{ijk})$

Subtask 7.b

$$\max M_d$$

$$\sum_{k=1}^{d} \gamma_{ijk} \geq M_d \qquad 1 \leq i < j \leq N$$

$$\left.\begin{aligned} \gamma_{ijk} &\leq 2 - (w_i[k] + w_j[k]) \\ \gamma_{ijk} &\leq w_i[k] + w_j[k] \\ \gamma_{ijk} &\geq w_i[k] - w_j[k] \\ \gamma_{ijk} &\geq w_j[k] - w_i[k] \end{aligned}\right. \quad \begin{aligned} &1 \leq i < j \leq N \\ &1 \leq k \leq d \end{aligned}$$

$$\forall_{j,k,i} \; \gamma_{ijk} \in \mathbb{R}$$
$$\forall_{j,k,i} \; w_i[k], w_j[k] \in \{\mathbf{0}, \mathbf{1}\}$$

In the final formulation the problem belongs to the **Mixed Integer Linear Programming** family as its constraints are all linear and some of the variables (γ) can be real, the other ($w_i[k]$) integer. It's worth noting however that all integer variables here are **binary** therefore it's more of a *Mixed Binary Linear Programming*, which *might* be a bit easier to solve than regular *Integer* programming.

Task 8

Subtask 8.a

Let's first denote:

- $j \in N$ = set of availible assets to invest in
- $t \in T$ = set of all months where return data is given
- r_{jt} = value of historical return on investment j from month t to month $t+1$
- ϵ = parameter determining minimum reward
- x_j = variable representing fraction of total resources put into asset j
- τ_t = variable that will replace the absolute value in objective function

Then our variables used in model will be:

$$\hat{R}_j = \frac{1}{T}\sum_{t=1}^{T} r_{jt}$$

$$\alpha = \max\left(0, \min_j \hat{R}_j\right)$$

$$\beta = \max_j \hat{R}_j$$

$$\begin{aligned}\hat{R} &= \sum_{j=1}^{N} x_j R_j = \\ &= \text{reward}\end{aligned}$$

$$\begin{aligned}\widehat{\text{MAD}} &= \frac{1}{T}\sum_{t=1}^{T}\left[\left|\sum_{j=1}^{N} x_j(r_j t - \hat{R}_j)\right|\right] \\ &= \text{risk}\end{aligned}$$

We want to replace the absolute value with something more linear, let's introduce a set of additional variables responisble for that, τ_t Assuming that all the fractions have to sum up to 1, i.e. we *have to* invest all of our resources, the model is as follows:

$$\min \quad \frac{1}{T}\sum_{t=1}^{T}\tau_t$$

$$\sum_{j=1}^{N} x_j \hat{R}_j \geq \alpha + \epsilon(\beta - \alpha),$$
$$\sum_{j=1}^{N} x_j = 1,$$

$$\tau_t \geq \sum_{j=1}^{N} x_j(r_{jt} - \hat{R}_j) \quad t = 1, \dots T$$
$$\tau_t \geq -\sum_{j=1}^{N} x_j(r_{jt} - \hat{R}_j) \quad t = 1, \dots T$$

$$\underset{j}{\forall}\, x_j \geq 0$$

Subtask 8.b

Model:

$$\min \quad \frac{1}{T}\sum_{t=1}^{T}\tau_t$$

$$\sum_{j=1}^{N} x_j \hat{R}_j \geq \alpha + \epsilon(\beta - \alpha),$$
$$\sum_{j=1}^{N} x_j = 1,$$

$$\tau_t \geq \sum_{j=1}^{N} x_j (r_{jt} - \hat{R}_j) \quad t = 1, \dots T$$
$$\tau_t \geq -\sum_{j=1}^{N} x_j (r_{jt} - \hat{R}_j) \quad t = 1, \dots T$$

$$\underset{j}{\forall}\, x_j \geq 0$$
$$\underset{t}{\forall}\, \tau_t \in \mathbb{R}$$

Has exactly $2N$ variables and $2N + 2$ (or $3N + 2$ if we include $x_j \geq 0$) constraints

Subtask 8.c

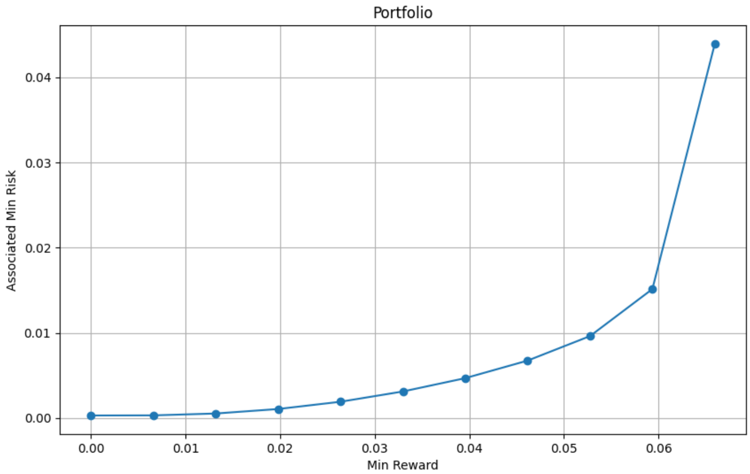
As gurobi was available only with license I used **mip** solver that is free. The soluton is based on provided template, adjusted to use **mip** and my model:

```
from mip import Model, xsum, CONTINUOUS, MINIMIZE
import matplotlib.pyplot as plt
import sys
import numpy as np

SEE_DETAILED_ALLOCATIONS = False

class Data:
    def __init__(self, filename):
        with open(filename, "r") as filehandle:
            lines=filehandle.readlines()

        self.N = int(lines[0].strip("\r\n"))
        self.T = int(lines[1].strip("\r\n"))
        self.r = np.zeros([self.N, self.T])

        for line in lines[2:]:
            line=line.strip("\r\n");
            parts=line.split(" ");
            j=int(parts[0])-1
            t=int(parts[1])-1
            val=float(parts[2])
            self.r[j,t]=val

        self.mean_r = np.mean(self.r, axis=1)
        self.min_r = np.max([0, np.min(self.mean_r)])
        self.max_r = np.max(self.mean_r)

        print("min_r:", self.min_r, "max_r:", self.max_r)


def solve(data, epsilon):
    m = Model("portfolio", sense=MINIMIZE)

    N = data.N
    T = data.T
    R_hat = data.mean_r
    r = data.r

    # The Minimum Return
    B = data.min_r + (epsilon / 100.0) * (data.max_r - data.min_r)

    # Decision variables
    x = [m.add_var(var_type=CONTINUOUS, lb=0) for _ in range(N)]
    tau = [m.add_var(var_type=CONTINUOUS) for _ in range(T)]
```

```
    # Objective function
    m.objective = (1 / T) * xsum(tau[t] for t in range(T))

    # Minimum Return
    m += xsum(R_hat[j] * x[j] for j in range(N)) >= B

    # All money
    m += xsum(x[j] for j in range(N)) == 1

    # Tau constraints
    for t in range(T):
        expr = xsum(x[j] * (r[j, t] - R_hat[j]) for j in range(N))
        m += (tau[t] >= expr)
        m += (tau[t] >= -expr)

    status = m.optimize()

    if status is None:
        return B, 0, None

    x_values = [x[j].x for j in range(N)]
    return B, m.objective_value, x_values



def main(argv):
    if len(argv) != 1:
        usage()
    instance = Data(argv[0])
    portfolio_info = []

    epsilons = np.arange(0, 101, 10)
    rewards = np.zeros_like(epsilons, dtype=np.float64)
    risks = np.zeros_like(epsilons, dtype=np.float64)

    for i, epsilon in enumerate(epsilons):
        print(f"=== Solve for epsilon = {epsilon}")
        B_val, obj_val, x_values = solve(instance, epsilon)

        rewards[i] = B_val
        risks[i] = obj_val
        positive_x = [(j+1, x) for j, x in enumerate(x_values) if x > 0]
        if not SEE_DETAILED_ALLOCATIONS:
            positive_x = len(positive_x)

        portfolio_info.append({
            "epsilon": epsilon,
            "reward": B_val,
            "risk": obj_val,
            "allocations": positive_x
```

```
        })

    plt.figure(figsize=(10,6))
    plt.plot(rewards, risks, '-o')
    plt.xlabel("Min Reward")
    plt.ylabel("Associated Min Risk")
    plt.title("Portfolio")
    plt.grid(True)

    plt.figure(figsize=(6, 10))
    plt.axis("off")
    text_str = ""
    for info in portfolio_info:
        epsilon = info['epsilon']
        positive_x = info['allocations']
        if(SEE_DETAILED_ALLOCATIONS):
            alloc_str = ", ".join([f"x_{j} = {x:.4f}" for j, x in positive_x])
        else:
            alloc_str = f"different assets: {positive_x}"
        text_str += f"\n  = {epsilon}: \n{alloc_str}\n"

    plt.text(0, 1, text_str, fontsize=12, va='top', ha='left', wrap=True)

    plt.tight_layout()
    plt.show()

def usage():
    print("Reads data from datafilename")
    print("Usage: [\"datafilename\"]\n")
    raise SystemExit


if __name__ == "__main__":
    main(sys.argv[1:])
```

Risk/Reward plot



ε = 0:
different assets: 244

ε = 10:
different assets: 237

ε = 20:
different assets: 228

ε = 30:
different assets: 188

ε = 40:
different assets: 143

ε = 50:
different assets: 111

ε = 60:
different assets: 72

ε = 70:
different assets: 50

ε = 80:
different assets: 27

ε = 90:
different assets: 14

ε = 100:
different assets: 1

Subtask 8.d

Worst-period return objective:

$$\max \min_{t=1,\dots,T} \sum_{j=1}^{n} x_j r_{jt}$$

We can try to transform it:

$$\min \gamma$$
$$\min_{t\in T} \left\{ \sum_{j=1}^{n} x_j r_{jt} \right\} \leq \gamma$$

This formulation should work (because we need to make γ only bigger than the smallest one)
We can try to remove the min part again by introducing another variable ω. We make ω negative in the objective to penalize it being less than minimum across all T

$$\begin{aligned} \min \ & \gamma - \omega \\ & \gamma \geq \omega \\ & \omega \leq \sum_{j=1}^{n} x_j r_{jt} \quad t = 1, \dots, T \end{aligned}$$

Therefore *Worst-period return* could be used, but the objective function would have to be corrected by the value of ω and in general this formulation is harder to work with.

Mean-variance:

$$\hat{\sigma}^2 = \sum_i \sum_j x_i x_j \hat{\sigma}_i \hat{\sigma}_j \rho_{ij}$$

Cannot be used in a linear programming problem, as it involves multiplication of decision variables $x_i x_j$. This would make it a quadratic problem.

Subtask 8.e

We can model this feature by adding aditional variable $\theta_j \in \{\mathbf{0}, \mathbf{1}\}$ indicator-type for x_j. We know that upper bound of each x_j is 1 (you cannot spend more than 100% of your resources), θ_j can be used by introducing a new constraint for each x_j:

$$\min \quad \frac{1}{T}\sum_{t=1}^{T} \tau_t$$

$$\sum_{j=1}^{N} x_j \hat{R}_j \geq \alpha + \epsilon(\beta - \alpha),$$
$$\sum_{j=1}^{N} x_j = 1,$$
$$\sum_{j=1}^{N} \theta_j \leq M,$$

$$\tau_t \geq \sum_{j=1}^{N} x_j (r_{jt} - \hat{R}_j) \quad t = 1, \dots T$$
$$\tau_t \geq -\sum_{j=1}^{N} x_j (r_{jt} - \hat{R}_j) \quad t = 1, \dots T$$

$$x_j \leq \theta_j \quad j = 1, \dots, N$$

$$\underset{j}{\forall}\, x_j \geq 0$$
$$\underset{j}{\forall}\, \theta_j \in \{\mathbf{0}, \mathbf{1}\}$$
$$\underset{t}{\forall}\, \tau_t \in \mathbb{R}$$

Here for each x_j:

- *Not Chosen* : $\theta_j = 0$ makes corresponding $x_j = 0$
- *Chosen* : $\theta_j = 1$ makes corresponding $x_j \in [0, 1]$

Sum of all θ_j ensures that amount of different assets bought is $\leq M$

Subtask 8.f

We can reuse model from previous task:

$$\min \quad \frac{1}{T}\sum_{t=1}^{T}\tau_t$$

$$\sum_{j=1}^{N} x_j \hat{R}_j \geq \alpha + \epsilon(\beta - \alpha),$$
$$\sum_{j=1}^{N} x_j = 1,$$
$$\sum_{j=1}^{N} \theta_j \leq M,$$

$$\tau_t \geq \sum_{j=1}^{N} x_j(r_{jt} - \hat{R}_j) \quad t = 1, \dots T$$
$$\tau_t \geq -\sum_{j=1}^{N} x_j(r_{jt} - \hat{R}_j) \quad t = 1, \dots T$$

$$x_j \leq \theta_j \quad j = 1, \dots, N$$

$$\forall_j \, x_j \geq 0$$
$$\forall_j \, \theta_j \in \{\mathbf{0}, \mathbf{1}\}$$
$$\forall_t \, \tau_t \in \mathbb{R}$$

By removing maximum different assets constraint and adding a new one with corresponding variable v that will be responsible for minimum x value. We can do that by adding another constraint that, when corresponding $\theta_j = 1$ will enforce $x_j \in [v, 1]$

$$\min \quad \frac{1}{T}\sum_{t=1}^{T}\tau_t$$

$$\sum_{j=1}^{N} x_j\,\hat{R}_j \geq \alpha + \epsilon(\beta - \alpha),$$
$$\sum_{j=1}^{N} x_j = 1,$$
$$\tau_t \geq \sum_{j=1}^{N} x_j(r_{jt} - \hat{R}_j) \quad t = 1, \dots T$$
$$\tau_t \geq -\sum_{j=1}^{N} x_j(r_{jt} - \hat{R}_j) \quad t = 1, \dots T$$

$$x_j \leq \theta_j \quad j = 1, \dots, N$$

$$\theta_j v \leq x_j \quad j = 1, \dots, N$$

$$\underset{j}{\forall}\, x_j \geq 0$$
$$\underset{j}{\forall}\, \theta_j \in \{\mathbf{0}, \mathbf{1}\}$$
$$\underset{t}{\forall}\, \tau_t \in \mathbb{R}$$

This ensures that all x stay within set bounds $0 \cup [v, 1]$.