

ST523 – Statistical Modelling

Answers to the Take-home 2 Assignment, Winter 2025, Jan Ryszkiewicz

Task 1

subtask 1

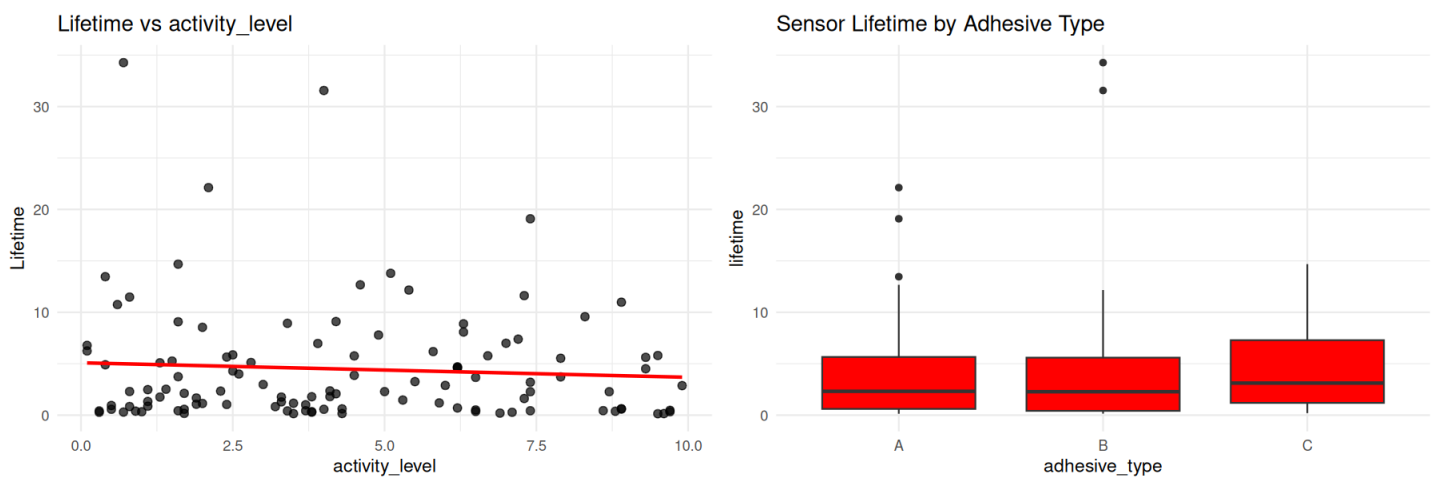
The dataset contains 113 observations across 9 variables. Summary statistics are presented below:

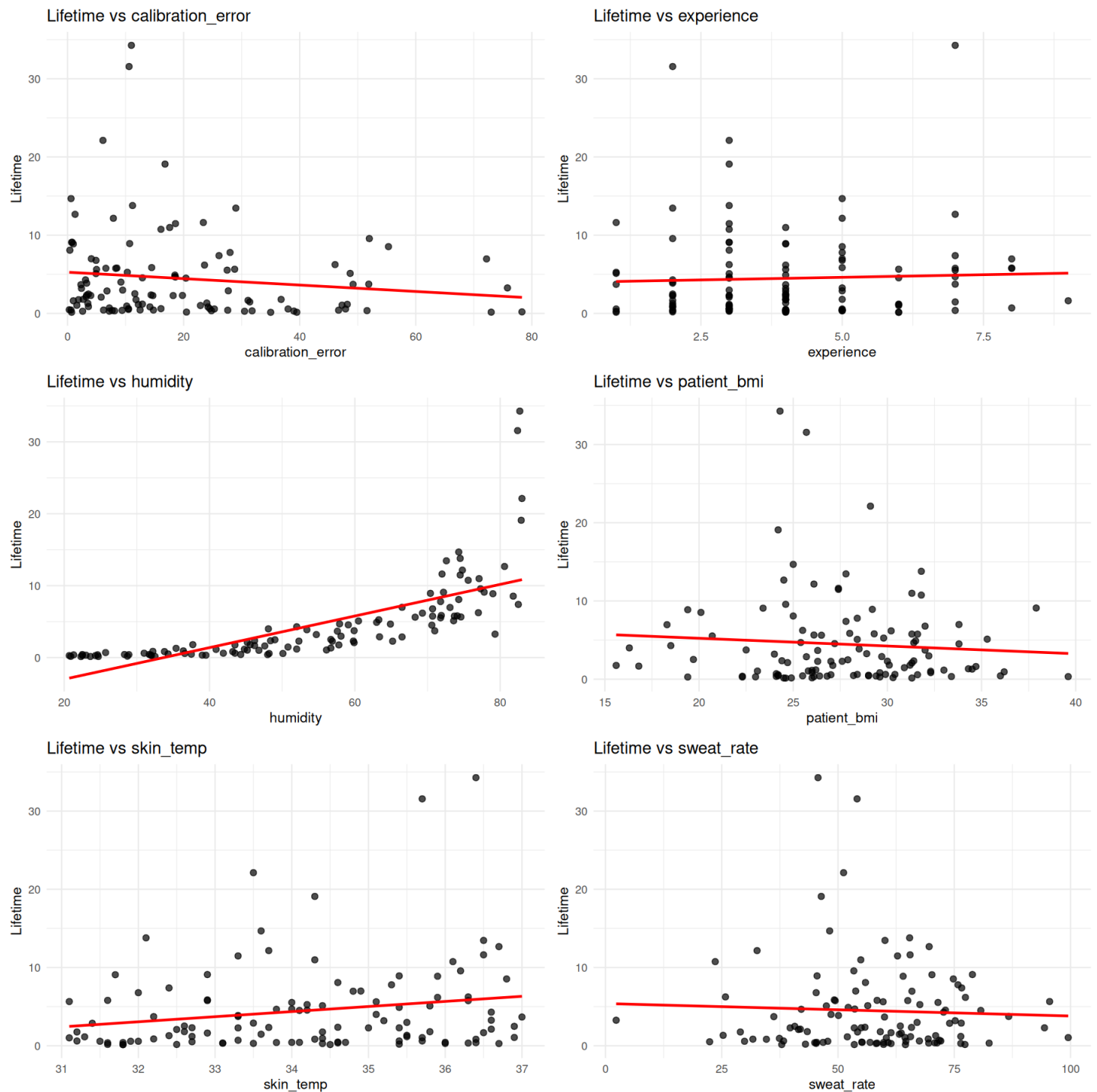
lifetime	skin_temp	humidity	activity_level
Min. : 0.140	Min. :31.10	Min. :20.70	Min. :0.100
1st Qu.: 0.610	1st Qu.:32.70	1st Qu.:39.00	1st Qu.:1.700
Median : 2.340	Median :34.30	Median :56.10	Median :4.000
Mean : 4.478	Mean :34.18	Mean :54.06	Mean :4.345
3rd Qu.: 5.800	3rd Qu.:35.70	3rd Qu.:71.80	3rd Qu.:6.500
Max. :34.270	Max. :37.00	Max. :83.00	Max. :9.900
sweat_rate	calibration_error	patient_bmi	experience
Min. : 2.3	Min. : 0.30	Min. :15.60	Min. :1.000
1st Qu.:46.8	1st Qu.: 5.80	1st Qu.:24.70	1st Qu.:3.000
Median :58.4	Median :12.90	Median :27.60	Median :4.000
Mean :57.5	Mean :19.27	Mean :27.61	Mean :3.965
3rd Qu.:69.4	3rd Qu.:27.60	3rd Qu.:31.20	3rd Qu.:5.000
Max. :99.5	Max. :78.30	Max. :39.60	Max. :9.000

For the adhesive, as it is a factor variable, we only show the overall amount of data entries per type:

A	B	C
52	35	26

Relationship with Lifetime plots:





It is worth noting that on the plots we can see 4 points which significantly higher lifetime stands out from the rest

subtask 2

Since lifetime is strictly positive and continuous, I considered two GLM families:

- Gamma distribution with log and canonical (inverse) links
- Inverse Gaussian distribution with log link

Log link function is chosen as the relationship between lifetime and humidity seems exponential - other continuous predictors appear linear on the plots. The canonical link for Inverse Gaussian caused convergence issues, so it was excluded.

I do not consider Poisson distribution as it is designed for count data, not continuous responses. Binomial on the other hand is mainly used for proportion data which is not present in our example.

We compare candidate models based on AIC:

Gamma_Log	Inv_Gaussian_Log	Gamma_Canonical
296.7400	376.2993	423.0153

and residual deviance:

Inv_Gaussian_Log	Gamma_Log	Gamma_Canonical
16.71100	16.83515	49.09622

The Gamma distribution with log linking was selected based on lowest AIC. Even though Inverse Gaussian had slightly lower residual deviance, the substantial AIC difference ($\Delta \approx 80$) and the excessive cubic mean-variance relationship (μ^3) were not necessary given the visually observed variance structure on the plots. It is worth bearing in mind the existence of categorical predictor in the model - adhesive_type - during the model selection the p-value is low enough to consider it statistically significant.

Full fit model can be tested for interaction using $(...)^2$ syntax, I performed sensible pairwise tests for the predictors including their interaction terms, ie. relating activity_level to, for instance, patient_bmi, sweat_rate and humidity, relating experience:humidity or experience:calibration_error etc, yielded no statistically significant interaction terms between the predictors (their p-values were > 0.05). **However when testing individual interaction between each predictor and adhesive_type I noticed the existence of an interaction between patient_bmi and the aforementioned adhesive_type, what motivates the inclusion of this term in my model.**

We consider 2 different predictor selection methods:

- Stepwise, based on AIC using step(), on default model with included patient_bmi interaction:

```
Model_1 <- step(glm(lifetime ~ skin_temp + humidity + activity_level +
  sweat_rate + calibration_error +
  experience + patient_bmi * adhesive_type,
  data = df,
  family = Gamma(link = "log")), trace=0)
```

- P-value based selection (threshold: $p < 0.05$):

	Estimate	Std. Error	t value	Pr(> t)
(Intercept)	-3.96381	0.96503	-4.10746	0.00008
skin_temp	0.06108	0.02342	2.60771	0.01050
humidity	0.06645	0.00208	32.00586	0.00000
activity_level	-0.04027	0.01343	-2.99882	0.00341
sweat_rate	0.00112	0.00236	0.47682	0.63452
calibration_error	-0.01492	0.00212	-7.04694	0.00000
experience	0.01808	0.02260	0.79968	0.42577
patient_bmi	-0.02418	0.01216	-1.98889	0.04942
adhesive_typeB	-0.21996	0.55061	-0.39948	0.69038
adhesive_typeC	-1.00510	0.59047	-1.70220	0.09179
patient_bmi:adhesive_typeB	0.01197	0.01996	0.59984	0.54996
patient_bmi:adhesive_typeC	0.04333	0.02073	2.09055	0.03908

Initial full model coefficients were deemed:

- Significant ($p < 0.05$): humidity, calibration_error, skin_temp, activity_level, patient_bmi
- Non-significant : sweat_rate ($p=0.635$), experience ($p=0.426$),

- Adhesive type C showed significance by interaction with bmi during testing, therefore all categorical predictors of `adhesive_type` retained.

Comparison between two models:

- Model 1 - `step()` best on default model with included `patient_bmi` interaction.
- Model 2 - p-value based choice on default model with included `patient_bmi` interaction.

Is not needed as after the stepwise model selection based on Akaike Information Criterion and selection based on p values we end up with the same model, different in one predictor. After that I decided to try running `step()` command on a model with all pairwise interactions ($(\dots)^2$ for all predictors). It returned a model with lower AIC (~19 lower), but it included all predictors and a lot of additional interactions (additional 18 (!) terms) that were less interpretable. In the following tasks I will be using the parsimonious Model 2 (equal to Model 1) for clarity and meaningful interpretation, balancing it with the quality of fit.

subtask 3

Final model is chosen to be Gamma GLM with log link with following predictors:

	Estimate	Std. Error	t value	Pr(> t)
(Intercept)	-3.5105	0.8666	-4.0509	0.0001
skin_temp	0.0528	0.0222	2.3737	0.0195
humidity	0.0666	0.0020	32.6798	0.0000
activity_level	-0.0373	0.0130	-2.8742	0.0049
calibration_error	-0.0147	0.0021	-7.0385	0.0000
patient_bmi	-0.0263	0.0119	-2.2086	0.0294
adhesive_typeB	-0.2300	0.5471	-0.4204	0.6751
adhesive_typeC	-0.9801	0.5860	-1.6727	0.0974
patient_bmi:adhesive_typeB	0.0126	0.0198	0.6360	0.5262
patient_bmi:adhesive_typeC	0.0421	0.0205	2.0481	0.0431

Since experience is not in the final model, it does not affect the prediction. Using `patient_bmi = 30`, `activity_level=4` and sample means for remaining predictors (`adhesive_type = A` as reference):

(Intercept)
1.961085

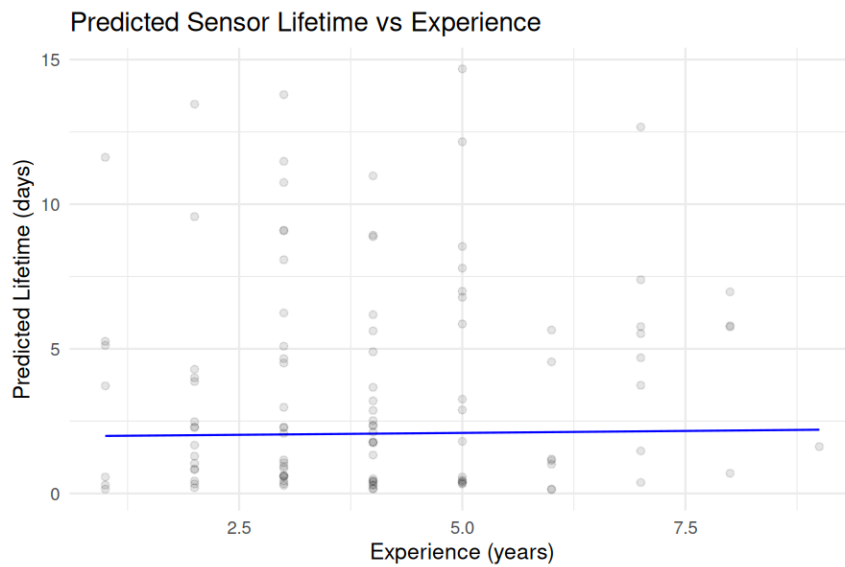
calculated by:

$$\hat{\mu} = \exp(\beta_0 + \beta_1 \cdot x_{\text{skin temp}} + \beta_2 \cdot x_{\text{humidity}} + \beta_3 \cdot x_{\text{activity level}} + \beta_4 \cdot x_{\text{calibration error}} + \beta_5 \cdot x_{\text{patient bmi}} + \dots)$$

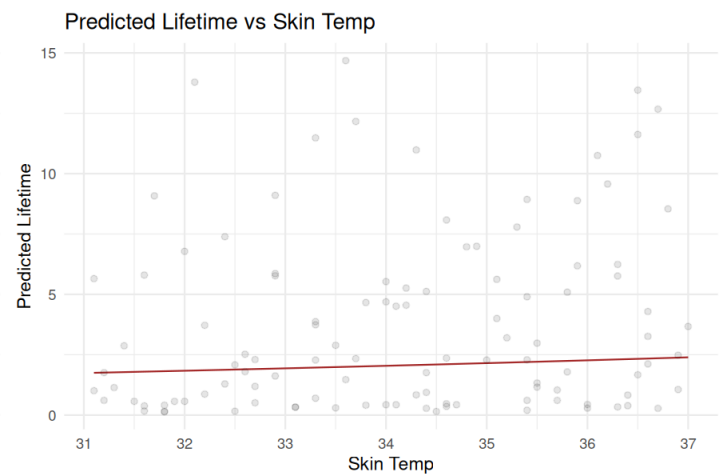
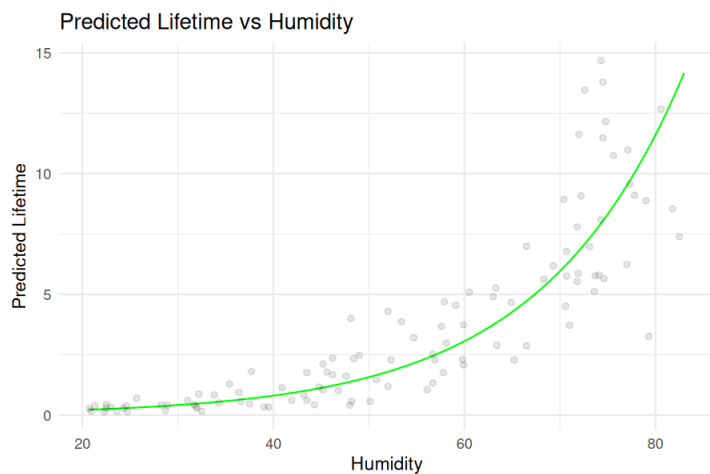
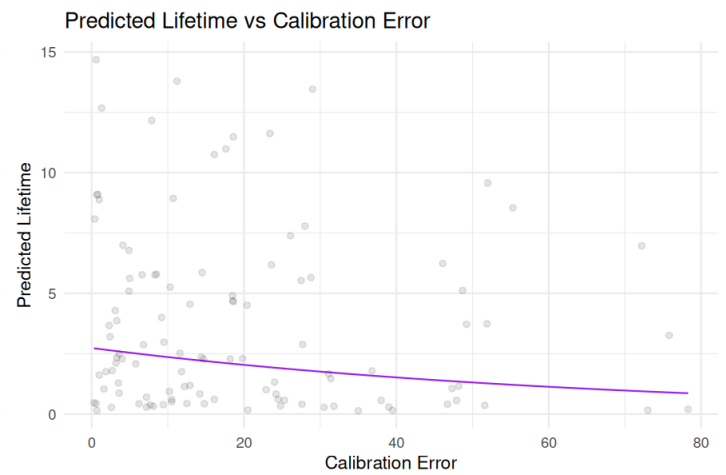
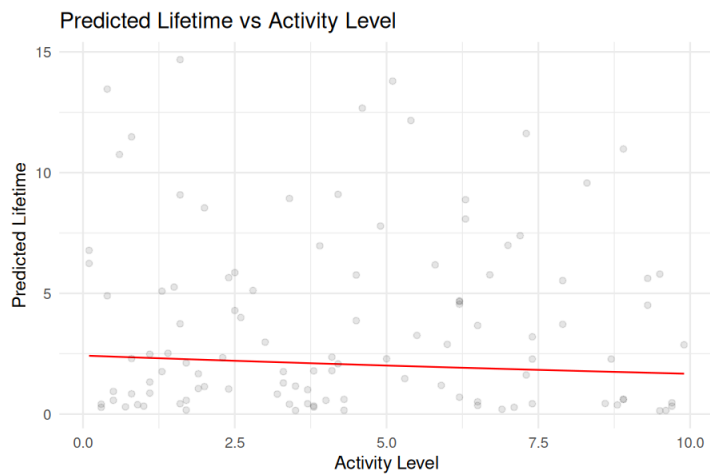
Where “...” normally represent the `adhesive_type` and interaction terms between `patient_bmi` and `adhesive_type`, however in our case they are all set to 0, because we use adhesive type A, which is the reference category in R (and also the most common type in the data).

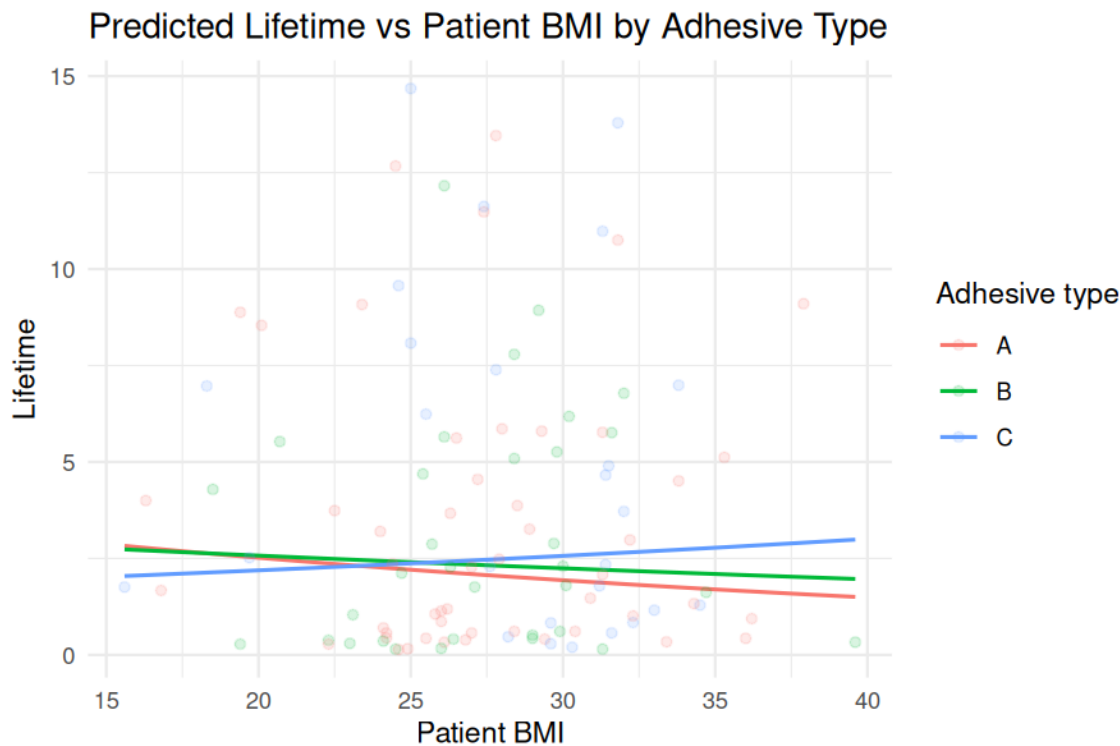
Since experience was removed from the final model, changing experience by 1 year produces no significant change in expected lifetime.

Plots presented here deliberately cut off the lifetime >15 outlier points (only 4 of them) to increase visibility. As experience is not included as predictor in our final model following plot is made using default, full-fit model including all predictors.



As we can see, user experience does not seem to impact the lifetime in a meaningful way compared to our chosen predictors, which effect is clearly visible on the following plots:





Here we can clearly see why the inclusion of the interaction term is necessary. The effect of adhesive type on sensor lifetime varies with patient BMI, as indicated by the differing slopes of the predicted curves.

Final Model Deviance:

```
[1] 16.29032
```

Final Model AIC:

```
[1] 292.9319
```

Final Model's Dispersion parameter is:

```
summary(final_model)$dispersion
```

```
[1] 0.151927
```

This indicates underdispersion. While less problematic than overdispersion, this suggests the variance is smaller than expected under the Gamma model. The model fit quality seems enough, though a potential quasi-Gamma approach could be considered if precision is critical.

Sensor lifetime is significantly influenced by environmental and technical factors: higher skin temperature and higher humidity increases lifetime, while higher calibration_error and activity_level decrease it. Humidity shows the strongest effect, followed by calibration_error, while skin temperature and activity level have more modest impacts. The data points strongly towards an interaction between `patien_bmi` and `adhesive_type` meaning that their specific combination influences the results in a different way. User experience, along with their sweat rate, do not significantly impact sensor lifetime and were excluded from the final model.

Task 2

Because our transformation g_α is monotonically increasing (trivial checks for $y \geq 0$ and $y < 0$)

$$P(Y_i < y_i) = P(g_\alpha(Y_i) < g_\alpha(y_i))$$

And since $g_\alpha(Y_i)$ is normally distributed, we can use the standard normal CDF Φ with normalized Y_i with its mean and standard error:

$$P(Y_i < y_i) = \Phi\left(\frac{g_\alpha(y_i) - \mathbf{x}_i^\top \beta}{\sigma}\right)$$

Taking the derivative with respect to y_i gives us:

$$\frac{d}{dy_i} P(Y_i < y_i) = \phi\left(\frac{g_\alpha(y_i) - \mathbf{x}_i^\top \beta}{\sigma}\right) \cdot \frac{\dot{g}_\alpha(y_i)}{\sigma}$$

where ϕ is the standard normal density and $\dot{g}_\alpha(y_i)$ being the derivative of $g_\alpha(y_i)$ defined as:

$$\dot{g}_\alpha(y_i) = \begin{cases} (y_i + 1)^{\alpha-1}, & y \geq 0 \\ (1 - y_i)^{1-\alpha}, & y < 0 \end{cases}$$

The final density of Y_i is:

$$\begin{aligned} f_{Y_i}(y_i) &= \phi\left(\frac{g_\alpha(y_i) - \mathbf{x}_i^\top \beta}{\sigma}\right) \cdot \frac{\dot{g}_\alpha(y_i)}{\sigma} = \\ &= \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2} \left(\frac{g_\alpha(y_i) - \mathbf{x}_i^\top \beta}{\sigma}\right)^2\right\} \cdot \frac{\dot{g}_\alpha(y_i)}{\sigma} \end{aligned}$$

$$\ell(\alpha, \beta, \sigma^2; y_1, \dots, y_n) = \sum_{i=1}^n \log f_{Y_i}(y_i)$$

For independent Y_i with density f_{Y_i} , this becomes:

$$\ell(\alpha, \beta, \sigma^2; y_1, \dots, y_n) = \log \left[\prod_{i=1}^n \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2} \left(\frac{g_\alpha(y_i) - \mathbf{x}_i^\top \beta}{\sigma}\right)^2\right\} \cdot \frac{\dot{g}_\alpha(y_i)}{\sigma} \right]$$

The estimates derived from equating the derivative of loglikelihood with respect to β and σ to 0 (as we did on the lecture) are:

$$\hat{\beta} = (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top g_\alpha(\mathbf{Y})$$

$$\hat{\sigma}^2 = \frac{1}{n} RSS_g$$

Where

$$RSS_g = \sum_{i=1}^n \left(g_\alpha(y_i) - \mathbf{x}_i^\top \hat{\beta} \right)^2$$

$$l(\alpha; y_1, \dots, y_n) = \log \left[\prod_{i=1}^n \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{1}{2} \left(\frac{g_\alpha(y_i) - \mathbf{x}_i^\top \hat{\beta}}{\hat{\sigma}} \right)^2 \right\} \cdot \frac{\dot{g}_\alpha(y_i)}{\hat{\sigma}} \right]$$

Which equals to:

$$= -\frac{n}{2} \log(2\pi) - \frac{1}{2\hat{\sigma}^2} \sum_{i=1}^n \left(g_\alpha(y_i) - \mathbf{x}_i^\top \hat{\beta} \right)^2 + \sum_{i=1}^n \log \dot{g}_\alpha(y_i) - n \log(\hat{\sigma})$$

Reduced by substituting $RSS_g = \sum_{i=1}^n \left(g_\alpha(y_i) - \mathbf{x}_i^\top \hat{\beta} \right)^2$ and $\hat{\sigma}^2 = \frac{1}{n} RSS_g$:

$$-\frac{n}{2} \log(2\pi) - \frac{n}{2} + \sum_{i=1}^n \log \dot{g}_\alpha(y_i) - n \log \left(\sqrt{\frac{RSS_g}{n}} \right)$$

And further into:

$$l(\alpha; y_1, \dots, y_n) = \sum_{i=1}^n \log \dot{g}_\alpha(y_i) - \frac{n}{2} \log \left(\frac{RSS_g}{n} \right) + const.$$

I decided to write g_α and \dot{g}_α in such way so that they can be applied vector-wise and not element-wise. Implementation of all the functions in R looks as follows:

```
g = function(y, alpha) {
  out <- numeric(length(y))

  # case 1: y >= 0, alpha != 0
  in_1 = (y >= 0 & alpha != 0)
  out[in_1] = (((y[in_1] + 1)^(alpha - 1)) / alpha)

  # case 2: y >= 0, alpha == 0
  in_2 = (y >= 0 & alpha == 0)
  out[in_2] = log(y[in_2] + 1)

  # case 3: y < 0, alpha != 2
  in_3 = (y < 0 & alpha != 2)
  out[in_3] = ( - (((-y[in_3] + 1)^(2 - alpha)) - 1) / (2 - alpha))

  # case 4: y < 0, alpha == 2
  in_4 = (y < 0 & alpha == 2)
  out[in_4] = ( - log(-y[in_4] + 1))

  return(out)
}
```



```

g_prim = function(y, alpha) {
  out <- numeric(length(y))

  # case 1: y >= 0
  in_1 = (y >= 0)
  out[in_1] = (y[in_1] + 1)^(alpha - 1)

  # case 2: y < 0
  in_2 = (y < 0)
  out[in_2] = (1 - y[in_2])^(1 - alpha)

  return(out)
}

profile_log_likelihood = function(Y, X, alpha) {
  n = length(Y)

  # transformed Y
  g_Y <- g(Y, alpha)

  # B_hat estimate
  B_hat = solve(t(X) %*% X) %*% t(X) %*% g_Y

  # Residual Sum of Squares
  RSS = sum((g_Y - X %*% B_hat)^2)

  # Constants
  constants = (-n/2)*log(2*pi) - (n/2)

  # profile log likelihood = sum of log(g_prim(y)) -n/2 log(sigma_hat^2) + constants
  out = sum(log(g_prim(Y, alpha))) - ((n/2)*log((1/n) * RSS)) + constants

  return(out)
}

```

We then load the data, add intercept term and find the ML estimate using `optim()`

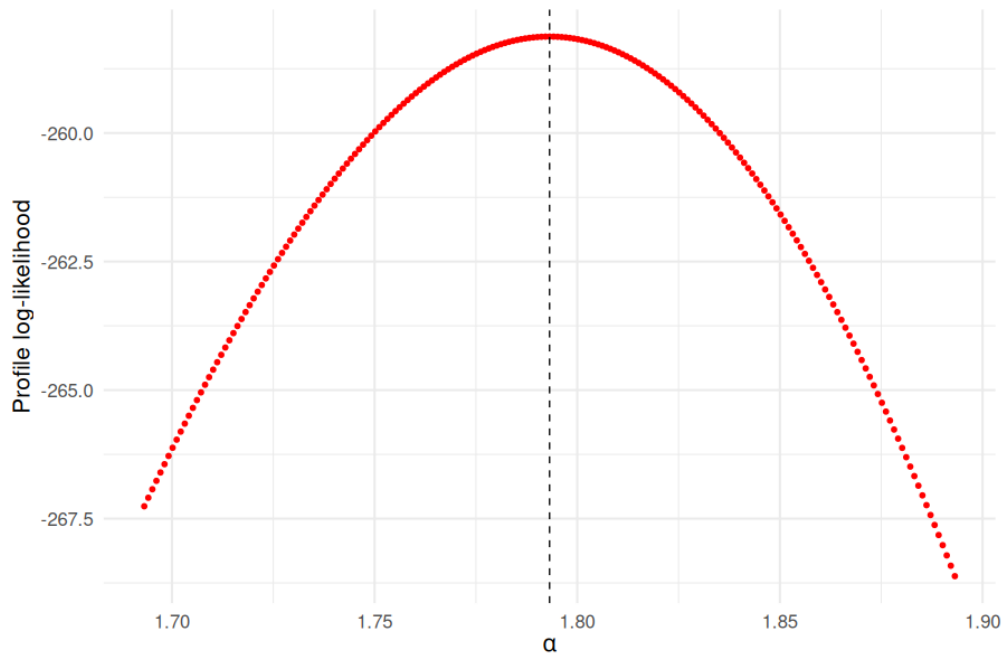
```

load("Data_Assignment2_Ex2_E2025.rdata")
df_x <- cbind(1, x)
df_y <- Y

out <- optim(
  par = 0,
  fn = function(a) {-profile_log_likelihood(df_y, df_x, a)}
)

```

We plot the values of loglik function spread in ± 0.1 around from `out$par` (which is the result of the optimization) with step of 0.001:



As we can see $\hat{\alpha} \approx 1.8$

Final transformation is then (by substituting $\alpha = 1.8$):

$$g_{\alpha}(Y) = \begin{cases} \frac{(Y+1)^{1.8} - 1}{1.8}, & y \geq 0 \\ -\frac{(-Y+1)^{0.2} - 1}{0.2}, & y < 0 \end{cases}$$