Confidence Level, Interval Estimation, Hypothesis testing, Z-test, Significance Level, Power of a test, Critical Value, Z-score

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① A candy machine makes chocolate bars that are on average 5 gm. A worker claims that the machine after maintenance no longer makes 5 gm ~~of~~ bars. Write the Null hypothesis and Alternate hypothesis.

Ans :

$H_0 : \mu = 5$ gm

$H_a : \mu \neq 5$ gm.

② A machine dispenses 80 mL of fluid in a bottle on average. An employer studies ~~a~~ 40 samples ~~of~~ and finds out the sample mean to be 78 mL. Assume standard deviation to be 2.5. ~~At~~ Write down the Null and Alternate hypothesis. At a 95% confidence level, is there enough evidence to support the idea that the machine is not working properly?

Ans :

$H_0 : \mu = 80$

$H_a : \mu \neq 80$

[Population is $N(80, (2.5)^2)$]

Sample mean, $m = 78$

Sample size, $N = 40$.

$$m \sim N\left(\mu, \frac{(2.5)^2}{N}\right)$$

$$\frac{(m-\mu)\sqrt{N}}{2.5} \sim N(0,1)$$

$$\text{Prob}\left\{-Z_{\alpha/2} < m < Z_{\alpha/2}\right\} = 0.95$$

Or, $$\text{Prob}\left\{\left(m - \frac{Z_{\alpha/2}(2.5)}{\sqrt{40}}\right) < \mu < \left(m + \frac{Z_{\alpha/2}(2.5)}{\sqrt{40}}\right)\right\} = 0.95$$

$Z_{\alpha/2} = 1.96$ from the standard Normal distribution ~~chart~~ table

$\therefore$ 95% confidence interval is $(77.2252, 78.7747)$

$$m - \frac{Z_{\alpha/2}(2.5)}{\sqrt{40}} = 77.2252$$

$$m + \frac{Z_{\alpha/2}(2.5)}{\sqrt{40}} = 78.7747$$

$H_0 : \mu = 80$ does not lie in the 95% confidence interval

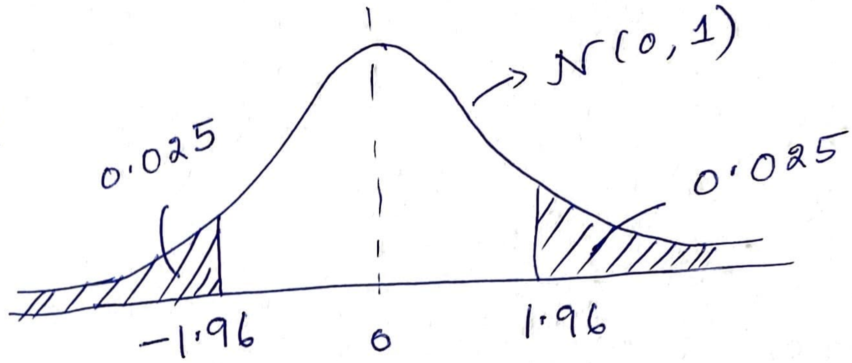
Ans : There is enough evidence to reject the Null hypothesis H_0 at a 95% confidence level.

Alternative Solution

$$Z \text{ score} = \frac{m-\mu}{\sigma/\sqrt{N}} = -5.06$$

$-5.06 < -1.96$

Therefore, we ~~should~~ will reject the Null hypothesis H_0 with 95% confidence.



③ A machine dispenses 80 mL of fluid in a bottle on average. An employer studies 40 samples and finds the sample mean to be 78 mL. Assume $\sigma = 2.5$. Design a hypothesis test such that the significance level is 0.05.

$H_0: \mu = 80$
$H_a: \mu \neq 80$

Ans

Population $\sim \mathcal{N}(80, (2.5)^2)$

Sample mean, $m \sim \mathcal{N}\left(80, \frac{(2.5)^2}{40}\right)$

$m = 78$

c : critical value

Test : Reject H_0 if $m < c$

Significance Level = ~~α = Prob $\{m < c \mid \mu = 80\}$~~

$\alpha = P\{\text{Reject } H_0 \mid H_0 \text{ is true}\}$

$= \text{Prob}\{m < c \mid \mu = 80\}$

$$= \text{Prob}\left\{ N\left(\cancel{8}\mu, \frac{(2.5)^2}{40}\right) < c \;\middle|\; \mu = 80 \right\}$$

$$= \text{Prob}\left\{ N\left(80, \frac{(2.5)^2}{40}\right) < c \right\}$$

$$= \text{Prob}\left\{ N(0, 1) < \frac{(c-80)\sqrt{40}}{2.5} \right\}$$

$$= 0.05$$

From the ~~chart~~ table,

$$\frac{(c-80)\sqrt{40}}{2.5} = -1.645$$

or, $c = 79.25$

<u>Ans:</u> Test designed : Reject H_0 if $m < 79.25$

Sample Mean $= m = 78$

Therefore the Null hypothesis H_0 is rejected with significance level $= 0.05$.

4 Repeat Prob 3 for a significance level $= 0.02$

(5) Scores in an exam is Normally distributed with standard deviation = 5.6.
A random sample of 40 scores on the exam has a mean of 32. Estimate the population mean with 80% confidence, 90% confidence, 98% confidence.

Population $\sim N(\mu, (5.6)^2)$

$N = 40.$

$m = 32.$

$m \sim N\left(\mu, \frac{(5.6)^2}{40}\right)$

From standard Normal distribution chart

Prob $\{-1.28 < Z < 1.28\} = .80$

Prob $\{-1.645 < Z < 1.645\} = 0.90$

Prob $\{-1.96 < Z < 1.96\} = 0.95$

Prob $\{-2.33 < Z < 2.33\} = 0.98$

$Z \sim N(0,1)$

(a) 80% confidence

~~Prob~~ ~~Confi~~ Confidence Interval

$= \left(m - \frac{(1.28) \times (5.6)}{\sqrt{40}}, \; m + \frac{(1.28) \times (5.6)}{\sqrt{40}} \right)$

$= (30.87, 33.13)$

(b) 90% confidence

$$\left(m - \frac{1.645 \times 5.6}{\sqrt{40}},\ m + \frac{1.645 \times 5.6}{\sqrt{40}}\right)$$

$$= (30.54,\ 33.46)$$

(c) 98% confidence level

$$\left(m - \frac{2.33 \times 5.6}{\sqrt{40}},\ m + \frac{2.33 \times 5.6}{\sqrt{40}}\right)$$

$$= (29.94,\ 34.06)$$

Confidence Level	80%	90%	98%
Critical value, $z_{\alpha/2}$	1.28	1.645	2.33
Confidence Interval	(30.87, 33.13)	(30.54, 33.46)	(29.94, 34.06)

As confidence level ↑, the ~~interva~~ interval expands.

(6) It is ~~believed~~ thought that average life span of a car battery is ≥ 2 year. An engineer believes that the value is less. Using 30 samples, the engineer finds out that average life span is 1.8. At 99% confidence level, is there enough evidence to discard the null hypothesis? Assume standard deviation = 0.15

Ans.

$$H_0 : \mu \geq 2$$

$$H_a : \mu < 2$$

$N = 30$, $m = 1.8$, $\sigma = 0.15$

This will be a 1-tail test.

Confidence Interval :

$$\left(-\infty, \quad m + \frac{2.33 \times 0.15}{\sqrt{30}}\right)$$

$$= \left(-\infty, \quad 1.8 + \frac{2.33 \times 0.15}{\sqrt{30}}\right)$$

$$= (-\infty, \quad 1.864)$$

$$\boxed{\text{Prob}\{N(0, 1) < 2.33\} = 1 - 0.01 = 0.99}$$

Ans: The engineer ~~can~~ establishes with 99% confidence level that average life span of a battery lies in the range $(-\infty, 1.864)$

or, the engineer ~~establishes~~ with 99% confidence level ~~that~~ ~~average life span~~ discards the null hypothesis, H_0.

(7) The average test score for an entire school is 75 with $\sigma = 10$. What is the probability that a random sample of 30 students scored above 80?

Population $\sim N(75, 10^2)$

$N = 30$

$m = 80$

$m \sim N\left(75, \frac{10^2}{30}\right)$

Ans

$\text{Prob}\{m > 80\}$

$= \text{Prob}\left\{N\left(75, \frac{100}{30}\right) > 80\right\}$

$= \text{Prob}\left\{N(0,1) > \frac{(80-75)\sqrt{30}}{10}\right\}$

$= \text{Prob}\{N(0,1) > 2.738\}$

$= 0.0031$.

8) Consider a population with mean μ and variance 1.

The Null and Alternative hypothesis are

$H_0 : \mu = -1$

$H_a : \mu = 1.$

~~A test~~ Sample mean on 50 samples is ~~conducted~~ computed.

Suppose the test is : Reject H_0 if sample mean is greater than -0.7.

Calculate the significance level and power of the test.

Ans

Population $\sim N(\mu, 1)$

$N = 50,$ $\quad m$: sample mean

Test : Reject H_0 if $m > -0.7$

Significance Level $= \alpha$

$= \text{Prob}\{\text{Reject } H_0 \mid H_0 \text{ is true}\}$

$= \text{Prob}\{\text{Reject } H_0 \mid \mu = -1\}$

$= \text{Prob}\{m > -0.7 \mid \mu = -1\}$

$= \text{Prob}\{N(\mu, \frac{1}{50}) > -0.7 \mid \mu = -1\}$

$= \text{Prob}\{N(-1, \frac{1}{50}) > -0.7\}$

$= \text{Prob}\{N(0, 1) > (-0.7 + 1)\sqrt{50}\}$

$= \text{Prob}\{N(0, 1) > 2.12\} = 0.0170$

$\beta = \text{Prob}\{ \text{Fail to reject } H_0 \mid H_a \text{ is true} \}$

$= \text{Prob}\{ m \leq -0.7 \mid \mu = 1 \}$

$= \text{prob}\{ N(\mu, \frac{1}{50}) \leq -0.7 \mid \mu = 1 \}$

$= \text{Prob}\{ N(1, \frac{1}{50}) \leq -0.7 \}$

$= \text{Prob}\{ N(0, 1) \leq (-0.7 - 1)\sqrt{50} \}$

$= \text{Prob}\{ N(0, 1) \leq -12.02 \}$

$\simeq 0$

$\therefore$ Power of test $= 1 - \beta \approx 1.$

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