Topological Invariance of Whitehead Torsion

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December 2024

Chapman's Proof of Topological Invariance of Whitehead Torsion

Notation:

- I_j = [-1,1], j = 1,2,3...
- $\mathbf{Q} = \prod_{j=1}^{\infty} I_j$ = Hilbert cube
- $I_k = \prod_{j=1}^{k} I_j$
- $Q_{k+1} = \prod_{j=k+1}^{\infty} I_j$
- X, Y, X', Y' denote finite CW-complexes in the following discussion unless stated otherwise.
- X $\rightleftharpoons$ Y denotes a formal deformation (expansion or cell retraction) between them.

Main theorem: $f : X \to Y$ is a simple homotopy equivalence (s.h.e.) $\iff$ $f \times 1_Q : X \times Q \to Y \times Q$ is homotopic to a homeomorphism $G : X \times Q \to Y \times Q$.

Corollary 1: Topological invariance of the Whitehead torsion - $f : X \to Y$ is a homeomorphism $\implies$ $f \times 1_Q : X \times Q \to Y \times Q$ is a homeomorphism $\implies$ $f : X \to Y$ is a s.h.e. $\implies$ $\tau(X, X') = \tau(Y, f(X'))$.

Corollary 2: $X \rightleftharpoons Y \iff X \times Q \cong Y \times Q$-
Let $G : X \times Q \cong Y \times Q$. Consider the following composition-

$$\begin{tikzcd} X \arrow[r, "i_0"] \arrow[rrr, bend right, "f"'] & X \times Q \arrow[r, "G"] & Y \times Q \arrow[r, "\pi_Y"] & Y \end{tikzcd} \tag{1}$$

where $i_0 : X \to X \times Q, x \mapsto (x, (0)_i)$. Then we notice that $f \times i_Q \sim G$. The homotopy being-

$$\begin{aligned} H &: X \times Q \times I \to Y \times Q \\ &\quad (x, q, t) \mapsto (\pi_y(G(x, t \cdot q)), (1-t) \cdot q + t \cdot \pi_Q(G(x, q))) \\ &\implies (x, q, 0) \mapsto (f(x), q) \\ \& &\implies (x, q, 1) \mapsto (\pi_Y(G(x, q)), \pi_Q(G(x, q))) = G(x, q). \end{aligned} \tag{2}$$

where we have used the convexity (more generally, contractibility) of Q. And by the Main theorem, f is a s.h.e. The other direction follows trivially.

Results from Infinite Dimensional Topology: (will be used without proof)

Prop 1($\implies$): $f : X \to Y$ is a simple homotopy equivalence (s.h.e.) $\implies$ $f \times 1_Q : X \times Q \to Y \times Q$ is homotopic to a homeomorphism $G : X \times Q \to Y \times Q$. (refer to [])

Prop 2 (Handle straightening theorem): Let M be a f.d. PL-manifold[1] (possibly $\partial M \neq \phi$) and $\alpha : \mathbb{R}^n \times Q \to M \times Q$ ($n \geq 2$) an open embedding. Then there exist

- integer k > 0
- $V \subset M \times I_k$ (a compact, PL submanifold with codim = 0)
- $G : M \times Q \to M \times Q$ (homeomorphism)

such that

- $G_{\alpha(S)} : \alpha(S) \to M \times Q = 1_{\alpha(S)}$ (where $S = (\mathbb{R}^n - B_0(2)^o \times Q)$ and $B_0(2)$ is the ball of radius 2 centered at 0).
- $G \circ \alpha(B_0(1) \times Q) = V \times Q_{k+1}$
- $\partial^{top} V$ is p.l. bicollared[2] in $M \times I_k$ ($\partial^{top} V$ referring to V's topological boundary).

Proof of the Main Theorem (Idea): We will use the fact that Q is contractible to attain a covariant homotopy functor that "translates" a homeomorphism $G : X \times Q \to Y \times Q$ to a map $G_0 : X \to Y$ and also preserves homotopy ($F \sim G \implies F_0 \sim G_0, (FG)_0 \sim F_0 G_0$). A property P (a naturally motivated construction) is then introduced- (X,Y) has property P iff $G : X \times Q \to Y \times Q \implies \tau(G_0) = 0$. If we can prove that all (X,Y) pairs have property P, then the Main theorem is proved. We will show that if M is a PL-manifold, then (X,M) has property P. This is surprisingly sufficient to prove the general case (we can perform suitable reductions)! To prove the case for (X,M), one proceeds by induction on the number r of simplices of X with dimension ≥ 2. Prop 2 will be required, along with a "Sum lemma" finally to decompose M into smaller subsets (for induction) and use the computed (by induction) torsion of $G_0 : X \to M$ restricted to the smaller subspaces to obtain the torsion of G_0 (=0).

Proof: Since Q is contractible, consider the commutative diagram (Given a G, consider the G_0 that makes the diagram commute):

$$\begin{tikzcd} \mathbf{X} \times \mathbf{Q} \arrow[r, "G"] & \mathbf{Y} \times \mathbf{Q} \arrow[d, "\pi_y"] \\ \mathbf{X} \arrow[u, "i_0"] \arrow[r, "G_0"] & \mathbf{Y} \end{tikzcd} \tag{3}$$

. This describes a functor $\tilde{\phi}$ s.t.

$$\begin{gathered} \tilde{\phi} : X \times Q \mapsto X \\ \tilde{\phi} : (G : X \times Q \mapsto Y \times Q) \mapsto (G_0 : X \mapsto Y) \end{gathered} \tag{4}$$

Observe that this functor preserves "homotopy", i.e., it satisfies

[1] Refer to Appendix[]
[2] refer to []

- $F \sim G \implies F_0 \sim G_0$
- $(FG)_0 \sim F_0 G_0$
- $f : X \to Y \implies (f \times 1)_0 = f$
- $(1_{X \times Q})_0 = 1_X$

These properties are easy to verify by composing homotopies (via maps in the diagram and getting the required resultant homotopies). We will exhibit a homotopy H_0 between F_0 and G_0, given H, a homotopy between F and G. The rest follow similarly.
Consider

$$\begin{tikzcd} X \times I \arrow[r, "i_0 \times 1"] \arrow[rrr, bend right, "H_0"'] & X \times Q \times I \arrow[r, "H"] & Y \times Q \arrow[r, "\pi_Y"] & Y \end{tikzcd} \tag{5}$$

H_0 is the required homotopy.

Property P: (X,Y) has property P iff ($G : X \times Q \cong Y \times Q \implies \tau(G_0) = 0$)

Lemma 1: (X,Y) has property P $\implies$ (Y,X) has property P.
Proof: $H : Y \cong X \iff H^{-1} : X \cong Y$. But $(H^{-1})_0$ is a homotopy inverse to $H_0 \implies^3 \tau(H_0) = 0 \iff \tau((H^{-1})_0) = 0$.

Lemma 2: (X,Y) has property P, $X \rightleftharpoons X'$ and $Y \rightleftharpoons Y' \implies$ (X',Y') has property P.
Proof: Suffices to prove this for the case Y = Y'. Let $g : X \to X'$ be a s.h.e., then by Prop 1, $\exists G : X \times Q \cong Y \times Q, G \sim g \times 1_Q$. Now, let $H : X' \times Q \cong Y \times Q$. We want to show that $\tau(H_0) = 0$. But $HG := H \circ G : X \times Q \cong Y \times Q \implies \tau((HG)_0) = 0$ and $(HG)_0 \sim H_0 G_0 \sim H_0(g \times 1_Q) \sim H_0 g$. Since g is a s.h.e. and $H_0 g$ is a s.h.e, H_0 must be a s.h.e. and so (X',Y) has property P.

Lemma 3: Let S denote a simplicial complex, M a PL-manifold. Then ($\forall S, M, (S, M)$ has property P) $\implies$ ($\forall X, Y, (X, Y)$ has property P)
Proof: If X is a finite CW-complex, then $\exists S$, a simplicial complex s.t. $X \rightleftharpoons S$ [4]. Thus, given (X,Y), a finite CW pair, $\exists S, T$ s.t. $X \rightleftharpoons S, Y \rightleftharpoons T$. Consider

$$j : T \hookrightarrow \mathbb{R}^N \qquad \text{for N large enough, a simplicial embedding.}$$

Then $\exists M$ s.t. M is a regular neighbourhood[5] of j(T), and M is a PL-manifold[6]. By definition of a regular neighbourhood, $M \rightleftharpoons j(T)$. Thus, $Y \rightleftharpoons T \cong j(T) \rightleftharpoons M$ is a s.h.e, and by Lemma 2, we're done.

Lemma 4(Sum Lemma): Let K, L, $K_1, K_2, K_0, L_1, L_2, L_0$ be finite CW complexes and $f : K \to L$ s.t.

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[3] follows from the observation that $h \sim f \circ g \implies \tau(h) = \tau(f) \cdot \tau(g)$
[4] Refer to 7.2 Cohen
[5] Appendix definition
[6] refer to Hudson

- $K = K_1 \cup K_2, L = L_1 \cup L_2$,
- $K_0 = K_1 \cap K_2, L_0 = L_1 \cap L_2$,
- f restricts to homotopy equivalences $f_1 : K_1 \to L_1, f_2 : K_2 \to L_2, f_0 : K_0 \to L_0$.

Let $j_1 : L_1 \hookrightarrow L, j_2 : L_2 \hookrightarrow L, j_0 : L_0 \hookrightarrow L$. Then

- f is a homotopy equivalence,
- $\tau(f) = j_{1*}\tau(f_1) + j_{2*}\tau(f_2) - j_{0*}\tau(f_0)$.

We will not prove this here.

Lemma 5: Let S be a connected simplicial complex and M a PL-manifold, then (S,M) has property P.
Proof:

- Let r = no. of simplices of S with dimension n ≥ 2.
- If r = 0, then S is homotopy equivalent to a point or wedge of circles. Thus $Wh(S) = 0$[7] and Property P holds.
- Let r > 0 and σ be a maximal dimensional simplex (dim = n) in S. By subdividing if necessary, let $\sigma_0 \subset \sigma^o$, be an n-simplex s.t. $\sigma - \sigma_0^o \cong^{(p.l.)} \sigma \times I$. Consider $S_0 = S - \sigma_0^o$ and consider the cell structure c on S_0 given by $c(S_0) = c(S - \sigma^o) \cup c(\partial\sigma \times I)$.
- Note that $S - \sigma^o$ has one less cell than S, so applying the induction hypothesis to $S - \sigma^o$, we have that ($S - \sigma^o$, M) has property P (given any M, a PL manifold). Since $S_0 \rightleftharpoons S - \sigma^0$, we conclude that ($S_0$, M) has property P.
- Let $H : S \times Q \cong M \times Q$, then we want to show that $\tau(H_0) = 0$.
- Let

$$\begin{gathered} \beta : (\mathbb{R}^n, B_0(1)) \cong (\sigma^o, \sigma_0), \\ \alpha = H \circ (\beta \times 1_Q) : \mathbb{R}^n \times Q \to M \times Q \quad (n \geq 2). \end{gathered} \tag{6}$$

- Note that α is an open embedding (with n ≥ 2). So $\exists$ (k, V, G)- that satisfy the properties in Prop 2.
- Now observe that if we have $G : B_0(r) \times Q \cong B_0(r) \times Q$ s.t. $G|_{\partial(B_0(r)\times Q)} = 1_{\partial(B_0(r)\times Q)}$ then $G \sim 1_{B_0(r)\times Q}$ (follows easily from the contractibility of $B_0(r) \times Q$).
- Since $G|_{\alpha(\mathbb{R}^n - B_0(2)^o)} : \alpha(\mathbb{R}^n - B_0(2)^o) \to M \times Q = 1_{\alpha(\mathbb{R}^n - B_0(2)^o)}$, we have that $G \sim 1_{M\times Q}$.
- Thus $H \sim GH \implies H_0 \sim (GH)_0$.

[7] $\pi_1(S) = 1$ or $\mathbb{Z} * \mathbb{Z} * ..\mathbb{Z} \implies Wh(S) = 0$, refer to Cohen.

- Let $\pi_k : M \times Q \to M \times I_k$ be the natural projection and $i_0^k : M \to M \times I_k$ $(m \mapsto (m, 0)$. Then we have the following diagram

$$\begin{tikzcd}
\mathbf{S} \times \mathbf{Q} \arrow[r, "GH"] & \mathbf{M} \times \mathbf{Q} \arrow[d, "\pi_M"] \arrow[dr, "\pi_k"] & \\
\mathbf{S} \arrow[u, "i_0"] \arrow[r, "H_0"] & \mathbf{M} \arrow[r, "i_0^k"] & \mathbf{M} \times \mathbf{I_k}
\end{tikzcd} \tag{7}$$

  where $H_0 \sim (GH)_0$ and $i_0^k \circ \pi_M \sim \pi_k \implies f := \pi_k \circ GH \circ i_0 \sim i_0^k H_0$.

- Let $M_0 = \overline{(M \times I_k - V)}$. Then
  - $GH(\sigma_0 \times Q) = V \times Q_{k+1}$
  - $GH(\partial\sigma_0 \times Q) = \partial V \times Q_{k+1}$ (∂- topological boundary)
  - $GH(S_0 \times Q) = M_0 \times Q_{k+1}$

  These can be easily verified using the fact that if $Y \subset X \subset \sigma$, then

$$\begin{aligned}
GH(X - Y \times Q) &= GH \circ (\beta \times 1_Q)(((\beta \times 1_Q)^{-1} X - (\beta \times 1_Q)^{-1} Y) \times Q) \\
&= GH \circ (\beta \times 1_Q)(((\beta \times 1_Q)^{-1} X \times Q) - ((\beta \times 1_Q)^{-1} Y \times Q)) \\
&= GH \circ (\beta \times 1_Q)((\beta \times 1_Q)^{-1} X \times Q) - GH \circ (\beta \times 1_Q)((\beta \times 1_Q)^{-1} Y \times Q) \\
&= GH(X \times Q) - GH(Y \times Q).
\end{aligned} \tag{8}$$

- As defined earlier, $f := \pi_k \circ GH \circ i_0 \sim i_0^k H_0$. Note that $f|\sigma_0 : \sigma_0 \to V, f|\partial\sigma_0 : \partial\sigma_0 \to \partial V, f|S_0 : S_0 \to M_0$ are homotopy equivalences (contractibility of Q, $i_0 : \sigma_0 \hookrightarrow \sigma_0 \times Q$ is a homotopy equivalence $\implies f|\sigma_0 : \sigma_0 \to V$ is a homotopy equivalence (others follow similarly)).

- Since dim $\sigma_0 \geq 2, \sigma_0$ is simply connected $\implies WH(\pi_1(\sigma_0)) = \{1\} \implies f|_{\sigma_0}$ is s.h.e. Similarly, $\partial\sigma_0$ is homotopy equivalent to a wedge of circles or a point, so $Wh(\pi(\partial\sigma_0)) = \{1\} \implies f|_{\partial\sigma_0}$ is a s.h.e.

- It suffices to now show that $f|_{S_0}$ is a s.h.e. in order to use the Sum lemma to conclude that f is a s.h.e. Consider the following commutative diagram

$$\begin{tikzcd}
\mathbf{S_0} \times \mathbf{Q} \arrow[r, "GH|_{S_0 \times Q}"] & \mathbf{M_0} \times \mathbf{Q_{k+1}} \arrow[r, "1 \times h"] \arrow[d, "\pi_M"] & \mathbf{M_0} \times \mathbf{Q} \arrow[dl, "\pi_M"] \\
\mathbf{S_0} \arrow[u, "i_0"] \arrow[r, "f|S_0"] & \mathbf{M_0} &
\end{tikzcd} \tag{9}$$

  where $h : Q_{k+1} \to Q$ is the natural homeomorphism $(..., x_i, ...) \mapsto (..., x_{i-k}, ...)$(DOUBT- is this correct?)

- Note that M_0 is a PL-manifold since ∂V is PL-bicollared. Thus, since (S_0, M) satisfies property P $\forall M$ (PL-manifolds), we have that (S_0, M_0) satisfies property P. Thus, since $GH|_{S_0 \times Q}$ is a homemorphism in the diagram, we have that $\tau(f|S_0) = 0$.

- By the Sum lemma, $\tau(f) = 0$.

- But $f \sim i_0^k H_0$ and since $i_0^k : M \to M \times I_k$ is a s.h.e., we finally conclude that $\tau(H_0) = 0$!