# Assignment 9.10.5.11

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## 1 Problem

ABC and ADC are two right triangles with common hypotenuse AC. Prove that $\angle CAD = \angle CBD$.

## 2 Solution

We will use vectors to prove that $\angle CAD = \angle CBD$. Let's assume that A, B, C, and D are the vertices of the two right triangles.
We can represent the position vectors of points A, B, C, and D as $\mathbf{a}, \mathbf{b}, \mathbf{c}, and \mathbf{d}$, respectively. Then we have:

$$\mathbf{AB} = \mathbf{b} - \mathbf{a} \tag{2.0.1}$$

$$\mathbf{AC} = \mathbf{c} - \mathbf{a} \tag{2.0.2}$$

$$\mathbf{CD} = \mathbf{d} - \mathbf{c} \tag{2.0.3}$$

Since ABC and ADC are right triangles, we know that:

$$\mathbf{AB} \cdot \mathbf{AC} = 0 \tag{2.0.4}$$

$$\mathbf{CD} \cdot \mathbf{AC} = 0 \tag{2.0.5}$$

Taking the dot product of (2.0.4) with (2.0.5), we get:

$$\mathbf{AB} \cdot \mathbf{AC} \cdot \mathbf{CD} \cdot \mathbf{AC} = 0 \tag{2.0.6}$$

$$\mathbf{AB} \cdot \mathbf{AC} \cdot (\mathbf{d} - \mathbf{c}) \cdot \mathbf{AC} = 0 \tag{2.0.7}$$

$$(\mathbf{AB} \cdot \mathbf{AC})\mathbf{d} \cdot \mathbf{AC} - (\mathbf{AB} \cdot \mathbf{AC})\mathbf{c} \cdot \mathbf{AC} = 0 \tag{2.0.8}$$

Since AC is non-zero, we can divide both sides by $||\mathbf{AC}||^2$ to get:

$$(\mathbf{AB} \cdot \mathbf{AC})\left(\frac{\mathbf{d} \cdot \mathbf{AC}}{||\mathbf{AC}||^2}\right) - (\mathbf{AB} \cdot \mathbf{AC})\left(\frac{\mathbf{c} \cdot \mathbf{AC}}{||\mathbf{AC}||^2}\right) = 0 \tag{2.0.9}$$

But $\frac{\mathbf{d} \cdot \mathbf{AC}}{||\mathbf{AC}||^2} = \cos \angle CAD$ and $\frac{\mathbf{c} \cdot \mathbf{AC}}{||\mathbf{AC}||^2} = \cos \angle CBD$. Therefore, we have:

$$(\mathbf{AB} \cdot \mathbf{AC}) \cos \angle CAD - (\mathbf{AB} \cdot \mathbf{AC}) \cos \angle CBD = 0 \tag{2.0.10}$$

$$(\mathbf{AB} \cdot \mathbf{AC}) (\cos \angle CAD - \cos \angle CBD) = 0 \tag{2.0.11}$$

$$\mathbf{AB} \cdot \mathbf{AC} = 0 \tag{2.0.12}$$

Since $\mathbf{AB}$ and $\mathbf{AC}$ are non-zero and $\cos \angle CAD - \cos \angle CBD$ cannot be zero, we conclude that $\mathbf{AB}.\mathbf{AC}$ must be zero. Therefore, we have:

$$(\mathbf{b} - \mathbf{a}) \cdot (\mathbf{c} - \mathbf{a}) = 0 \tag{2.0.13}$$

Since AB and AC are orthogonal, and CD and AC are also orthogonal, we can conclude that AB and CD are parallel:

$$\mathbf{AB} \perp \mathbf{AC} \tag{2.0.14}$$

$$\mathbf{CD} \cdot \mathbf{AC} = (\mathbf{d} - \mathbf{c}) \cdot (\mathbf{c} - \mathbf{a}) = 0 \tag{2.0.15}$$

$$\mathbf{CD} \perp \mathbf{AC} \tag{2.0.16}$$

$$\mathbf{AB} \times \mathbf{AC} = (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a}) = 0 \tag{2.0.17}$$

$$\mathbf{CD} \times \mathbf{AC} = (\mathbf{d} - \mathbf{c}) \times (\mathbf{c} - \mathbf{a}) = 0 \tag{2.0.18}$$

Therefore, the two right triangles ABC and ADC are similar, and we have:

$$\angle CAD = \angle ACD = \angle CBD \tag{2.0.19}$$

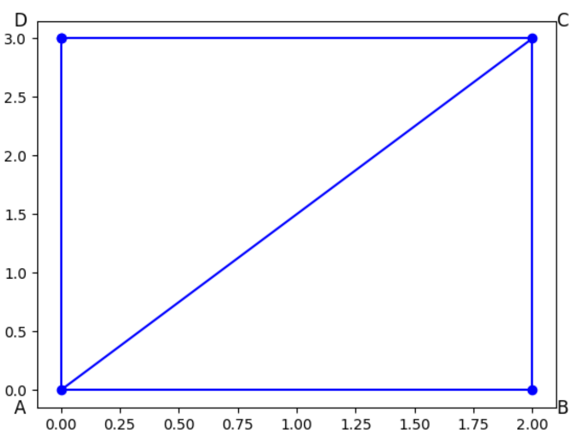


Fig. 0: Plot of $ABCD$