OPTIMAL CONTROL APPLICATIONS TO THE SLAM PROBLEM

SHRITA SINGH

Abstract. This proposal contains an introduction to the SLAM problem and a brief introduction to Pontryagin's Maximum Principle in both continuous time and discrete time. The idea is to apply ideas from optimal control based on the Maximum Principle to the SLAM problem. With this in mind, the proposal presents a brief introduction to the SLAM problem with the existing objectives, and then moves on to introduce the Pontryagin's Maximum Principle (PMP) followed by its discrete verrsion.

1. Introduction to SLAM

Being able to build a map of the environment and to simultaneously localize within this map is an essential skill for mobile robot navigation in unknown environments in the absence of external referencing systems such as GPS. Learning maps under pose uncertainty is often referred to as the simultaneous localization and mapping (SLAM) problem. Consider a mobile robot moving through an environment taking relative observations of a number of unknown landmarks using a sensor located on the robot. At a time instant k, the following quantities are

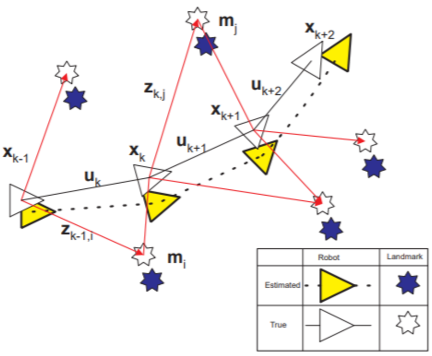

Figure 1. The SLAM Problem

defined:
x_k: The state vector describing the location and orientation of the vehicle
u_k: The control vector applied at time k-1 to drive the body to state x_k at time k
m_i : A vector describing the location of the ith landmark whose true location is assumed time invariant

Date: January 30, 2020.
Key words and phrases. Pontryagin's maximum principle, Simultaneous Location and Mapping,Pontryagin's maximum principle in discrete time.

z_{ik}: An observation of the location of the ith landmark taken from the body at time k

The Simultaneous Localisation and Mapping (SLAM) problem requires that the probability distribution

$$P(x_k, m \mid Z_{0:k}, U_{0:k}, x_0) \tag{1.1}$$

be computed for all times k. Here, $X_{0:k} = (x_0, x_1, ..., x_k)$ is the history of vehicle states, $U_{0:k} = (u_1, u_2, ..., u_k)$ is the history of control inputs, $m = (m_1, m_2, ..., m_n)$ is the set of all landmarks, and $Z_{0:k} = (z_1, z_2, ..., z_k)$ is the set of all landmark observations. This probability distribution describes the joint posterior density of the landmark locations and vehicle state (at time k) given the recorded observations and control inputs up to and including time k together with the initial state of the vehicle [DWB06]. The observation model describes the probability of making an observation z_k when the vehicle location and landmark locations are known, and is generally described in the form:

$$P(z_k \mid x_k, m) \tag{1.2}$$

The motion model for the vehicle can be described in terms of a probability distribution on state transitions in the form

$$P(x_k \mid x_{k-1}, u_k) \tag{1.3}$$

The SLAM algorithm is now implemented in a standard two-step prediction (time-update) and correction (measurement-update) form:

$$P(x_k, m \,|\, Z_{0:k-1}, U_{0:k}, x_0) = \int P(x_k \,|\, x_{k-1}, u_k) P(x_{k-1}, m \,|\, Z_{0:k-1}, U_{0:k-1}, x_0) dx_{k-1} \tag{1.4}$$

$$P(x_k, m \mid Z_{0:k}, U_{0:k}, x_0) = \frac{P(z_k \mid x_k, m) P(x_k, m|Z_{0:k-1}, U_{0:k}, x_0)}{P(z_k|Z_{0:k-1}, U_{0:k})} \tag{1.5}$$

Equations (1.4) and (1.5) provide a recursive procedure for calculating the joint posterior $P(x_k, m \mid Z_{0:k}, U_{0:k}, x_0)$ for the robot state x_k and map m at a time k based on all observations $Z_{0:k}$ and all control inputs $U_{0:k}$ up to and including time k. The recursion is a function of a vehicle model $P(x_k \,|\, x_{k1}, u_k)$ and an observation model $P(z_k \mid x_k, m)$. Thus, solutions to the probabilistic SLAM problem involve finding an appropriate representation for the observation model Equation (1.2) and motion model Equation (1.3) which allows efficient and consistent computation of the prior and posterior distributions in Equations (1.4) and (1.5) [DWB06].

The different approaches devised to solve the SLAM problem can be classified either as filtering or smoothing. Filtering approaches model the problem as an on-line state estimation where the state of the system consists in the current robot position and the map. The estimate is augmented and refined by incorporating the new measurements as they become available. A smoothing approach to SLAM involves not just the most current robot location, but the entire robot trajectory up to the current time. Here, the optimization problem associated with full SLAM can be concisely stated in terms of sparse linear algebra, which is traditionally concerned with the solution of large least-squares problems. In our proposed work, the probabilistic ideas of this section would be replaced by deterministic optimal control theory We now move on to a brief introduction to the PMP.

2. Optimal Control and Pontryagin's Maximum Principle

2.1. **Continuous Time PMP.** The optimal control problem is of the form:

$$\text{minimize } J(u) := \int_{t_0}^{t_f} L(t, x(t), u(t))dt + K(t_f, x_f) \tag{2.1}$$

$$\text{subject to } \begin{cases} \dot{x} = f(t, x, u) \\ x(t_0) = x_0 \\ x \in R^n \\ u \in U \in R^m \end{cases} \tag{2.2}$$

where x is the state taking values in R^n, u is the control input taking values in some control set $U \in R^m$, t is time, t_0 is the initial time, and x_0 is the initial state. A system is well-posed when for every choice of the initial data (t_0, x_0) and every admissible control $u(t)$, the system (2.2) has a unique solution $x(t)$ on some time interval $[t_0, t_1]$. To ensure this, we need to impose some regularity conditions on the right-hand side f and on the admissible controls u: f is continuous in t and u and $C1$ in x; f_x is continuous in t and u; and $u(t)$ is piecewise continuous as a function of t [Lib12].

For a given initial data (t_0, x_0), the behaviors are parameterized by control functions u. Thus, the cost functional assigns a cost value to each admissible control. Thus, L and K are given functions (running cost and terminal cost, respectively), t_f is the final (or terminal) time which is either free or fixed, and $x_f := x(t_f)$ is the final (or terminal) state which is either free or fixed or belongs to some given target set. The objective is to find a control u that minimizes $J(u)$ over all admissible controls (or at least over nearby controls). The set in which the controls u take values may be constrained by some practical considerations, which would make it difficult to solve the above problem using calculus of variations. In the optimal control formulation, such constraints are incorporated very naturally by working with an appropriate control set. The Pontryagin's Maximum Principle [Lib12], used to solve optimal control problems is stated below:
Let $u(t) : [t_0; t_f] \to U$ be an optimal control (in the global sense) and let $x(t)$: $[t_0; t_f] to R^n$ be the corresponding optimal state trajectory. Then there exists a function $p(t) : [t_0, t_f] \to R^n$ and a constant $p_0 = 0$ satisfying $(p_0, p(t)) \neq (0, 0)$ for all $t \in [t_0, t_f]$ and having the following properties:

(1) x and p satisfy the canonical equations:

$$\dot{x} = H_p(x, u, p, p_0) \tag{2.3}$$
$$\dot{p} = H_x(x, u, p, p_0) \tag{2.4}$$

with the boundary conditions $x(t_0) = x_0$ and $x(t_f) = x_1$, where the Hamiltonian $H : R^n \times U \times R^n \times R \to R$ is defined as

$$H(x, u, p, p_0) := \langle p, f(x, u)\rangle + p_0 L(x, u) \tag{2.5}$$

(2) For each fixed t, the function $u \to H(x(t), u, p(t), p_0)$ has a global maximum at $u = u^*(t)$, i.e.,

$$H(x^*(t), u^*(t), p^*(t), p_0^*) \geq H(x(t), u, p(t), p_0) \forall t \in [t_0, t_f], \forall u \in U \tag{2.6}$$

(3)

$$H(x^*(t), u^*(t), p^*(t), p_0^*) = 0 \forall t \in [t_0, t_f] \tag{2.7}$$

2.2. **Discrete Time PMP.** In the discrete setting, the classical optimal control problem is of the form

$$\min_{u(\cdot)} \sum_{t=0}^{N-1} f_t^0(x(t), u(t))$$
$$\text{subject to} \quad f^k(x) = 0 \quad \text{for all} \quad k = 1, 2, \ldots, s.$$

where, $f^k(x)$ for all $k = 1, \ldots, s$ are smooth functions, such that $f^i : \mathbb{R}^d \longmapsto \mathbb{R}$ and $f_t^0 : \mathbb{R}^d \times \mathbb{R}^m \longmapsto \mathbb{R}$ which gives cost per stage, objective is to minimize the total cost, satisfying the s constraints given by $f^1(x) = 0, \ldots\ f^s(x) = 0$, over the set of admissible inputs u [Bol75]. Let $\Sigma = \{x \in \mathbb{R}^d \mid f^1(x) = 0, \ldots, f^s(x) = 0\}$.
Assume $\arg\min_{x\in\Sigma} f^0(x)$ exist and $x_0 \in \arg\min_{x\in\Sigma} f^0(x)$.

Define $\begin{bmatrix} \Omega_1 = \{x \in R^d \mid f^1(x) = 0\} \\ \vdots \\ \Omega_s = \{x \in R^d \mid f^s(x) = 0\} \end{bmatrix}$ and $\Omega_0 = \{x_0\} \cup \{x \in R^d \mid f^0(x) < f^0(x_0)\}$.

Clearly, $x_0 = \arg\min_{x\in\Sigma} f^0(x)$ if and only if

$$\Omega_0 \cap \ldots \cap \Omega_s = \{x_0\}. \tag{2.8}$$

Above assertion reduces the optimal control problem to finding the condition under which (2.8) is true [Bol75]. Further details are also found in [Bol75].

References

[Bol75] V G Boltyanskii. THE METHOD OF TENTS IN THE THEORY OF EXTREMAL PROBLEMS. *Russian Mathematical Surveys*, 30(3):1–54, jun 1975.

[DWB06] Hugh Durrant-Whyte and Tim Bailey. Simultaneous localisation and mapping (slam): Part i the essential algorithms. *Robotics Automation Magazine*, 13, 01 2006.

[Lib12] Daniel Liberzon. *Calculus of Variations and Optimal Control Theory: A Concise Introduction.* Princeton University Press, Princeton, New Jersey, 2012.

Email address: shritasingh2000@gmail.com