INDIAN INSTITUTE OF TECHNOLOGY MADRAS

Department of Chemical Engineering

CH5350 : Applied Time Series Analysis
Solutions to Assignment #3

Marks Distribution

	Q.1	Q.2	Q.3	Q.4	Q.5
(a)	10	10	10	10	15
(b)	10	10	5	5	-
(c)	-	5 (Merits and Demerits)	5	5	-

1 ARIMA process

(a)

(i) AR(2) model

$$H(q^{-1}) = \frac{1}{1 - 0.7q^{-1} + 0.1q^{-2}}$$

$$H(q^{-1}) = \frac{1}{3}(\frac{5}{1 - 0.5q^{-1}} - \frac{2}{1 - 0.2q^{-1}})$$

This can be written as,

$$H(q^{-1}) = \sum_{0}^{\infty} \frac{5}{3}(0.5^n)q^{-1} - \sum_{0}^{\infty} \frac{2}{3}(0.2^n)q^{-1}$$

$$h[n] = \frac{5}{3}(0.5^n) - \frac{2}{3}(0.2^n)$$

(ii) ARMA(1,1) model

$$H(q^{-1}) = \frac{1 + 0.3q^{-1}}{1 - 0.6q^{-1}}$$

Again using Taylor series expansion, this can be written as,

$$H(q^{-1}) = (1 + 0.3q^{-1})(1 + 0.6q^{-1} + (0.6q^{-1})^2 + ...)$$

In parametric form,

$$h[n] = \begin{cases} (0.6)^n + 0.3(0.6)^{n-1}) & n > 0 \\ 1 & n = 0 \\ 0 & n < 0 \end{cases}$$

(b)

The data is shown in Figure 1

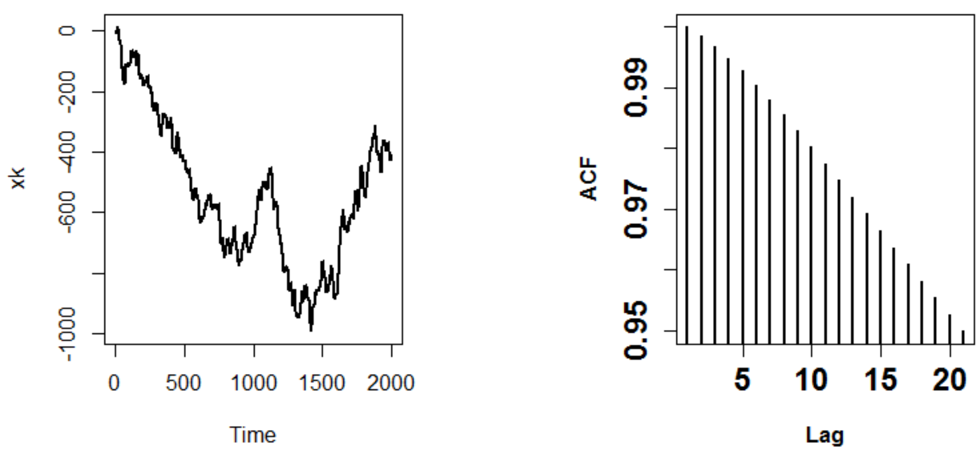


Figure 1:

From the plot, it is inferred that the data is not stationary. The ACF plot of the data is shown in Figure 1. The ACF plot shows random walk behaviour.
Unit root test is conducted on the data and results are shown below

```
adf.test(xk)


Title:
```

```
 Augmented Dickey-Fuller Unit Root Test
Test Results:

  Test regression none

  Call:
  lm(formula = z.diff ~ z.lag.1 - 1 + z.diff.lag)

  Residuals:
      Min      1Q  Median      3Q     Max
  -4.1822 -0.9460 -0.0196  0.8404  5.0079

  Coefficients:
              Estimate Std. Error t value
  z.lag.1    -4.433e-05  5.063e-05  -0.876
  z.diff.lag  9.468e-01  7.238e-03 130.809
             Pr(>|t|)
  z.lag.1       0.381
  z.diff.lag   <2e-16 ***
  ---
  Signif. codes:
  0 ?***? 0.001 ?**? 0.01 ?*? 0.05 ?.? 0.1 ? ? 1

  Residual standard error: 1.342 on 1996 degrees of freedom
  Multiple R-squared:  0.8956,Adjusted R-squared:  0.8954
  F-statistic:  8557 on 2 and 1996 DF,  p-value: < 2.2e-16

  Value of test-statistic is: -0.8756

  Critical values for test statistics:
        1pct  5pct 10pct
  tau1 -2.58 -1.95 -1.62

Description:
 Tue Oct 04 23:46:14 2016 by user: Dheeraj kumar
```

Clearly, the obtained p-value is less than that of critical value which suggests that there is a pole on unit circle. Hence, the data is differenced one time and is differenced data is shown in Figure 2

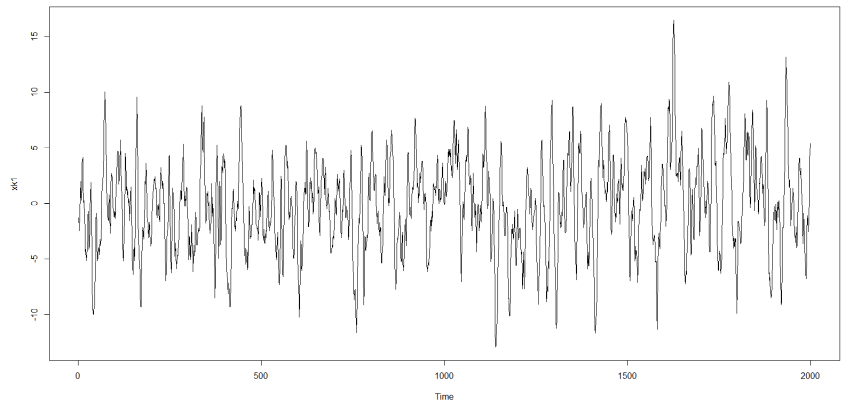


Figure 2: Time series plot of differenced data

From the plot, it is inferred that the differenced data is stationary. The ACF plot of differenced data is shown in Figure 3

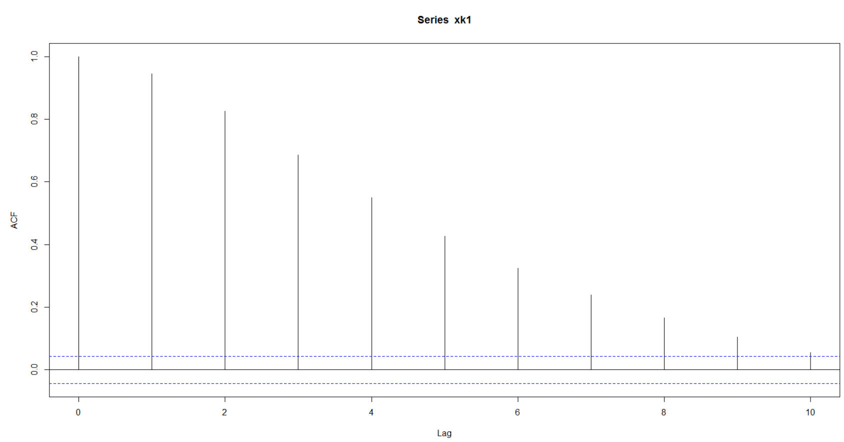


Figure 3: ACF plot of data

From the plot, it is observed that ACF is slowly decaying with time. Hence, the process can be treated as AR. THE PACF plot of differenced data is shown in Figure 4

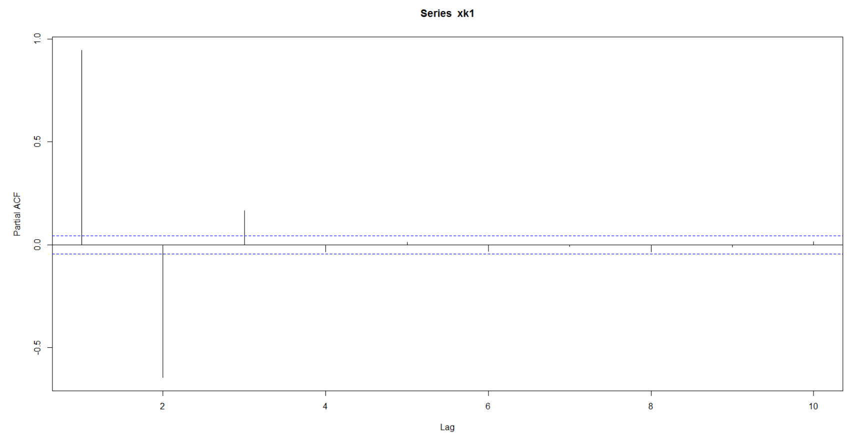


Figure 4: PACF plot of data

From the plot, it can be inferred that the process is of AR model of order 3. The process is modelled using arima command in R as
arima($x = xk$, order $=$ c$(3, 1, 0)$)

Coefficients:

	ar1	ar2	ar3
	1.6742	-0.9216	0.1742
s.e.	0.0220	0.0382	0.0220

The ACF plot of residuals is shown in Figure 5

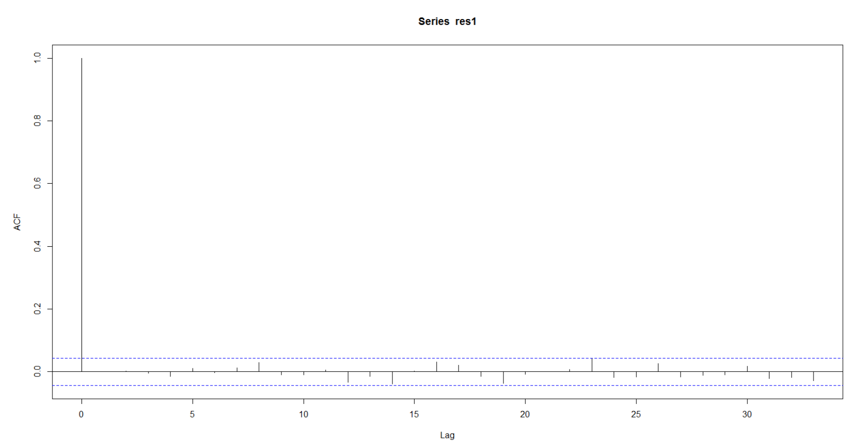


Figure 5: ACF plot of residuals

The PACF plot of the residuals is shown in Figure 6

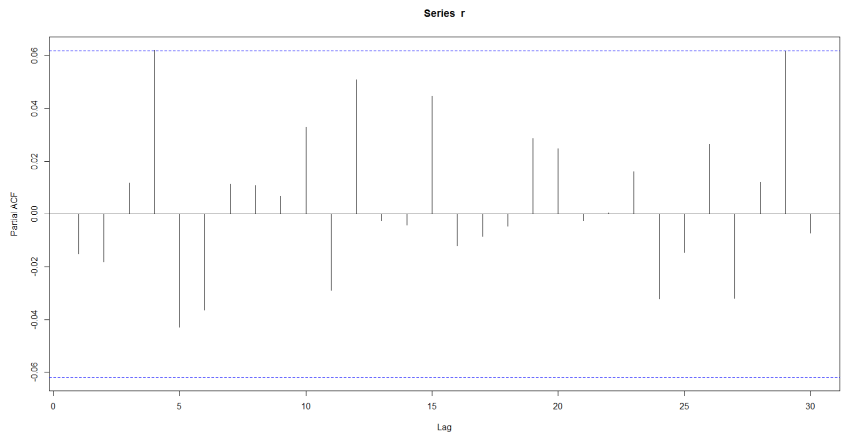


Figure 6: PACF plot of residuals

Both ACF and PACF plots resembles that of a white noise process. So, best model for the given data is

$$v[k] = 1.6742v[k-1] - 0.9216v[k-2] + 0.1742v[k-3] + e[k]$$

2 Processes with trend

Given $x[k] = \beta_0 + \beta_1 k + v[k]$ where $v[k]$ is a stationary ARMA process The above series can be modelled in two ways.

1. Fitting a linear model (in time) followed by an ARMA fit to the residuals

2. Using the differencing of the series approach (ARMA fit to the differenced series)

The advantages and limitations of both the methods are detailed below
An advantage of the linear model fit followed by ARMA model estimation is that it gives efficient estimates when the residuals are white noise. The main lmitation of this method is that if β_1 and β_0 are not estimated properly it results in non-stationarity of $v[k]$. An advantage of differencing is that it does not require estimation of additional parameters i.e. it is non-parametric. A disadvantage is that it introduces an additional zero in the $v[k]$ which is to be modelled.
The data is shown in the Figure 2.

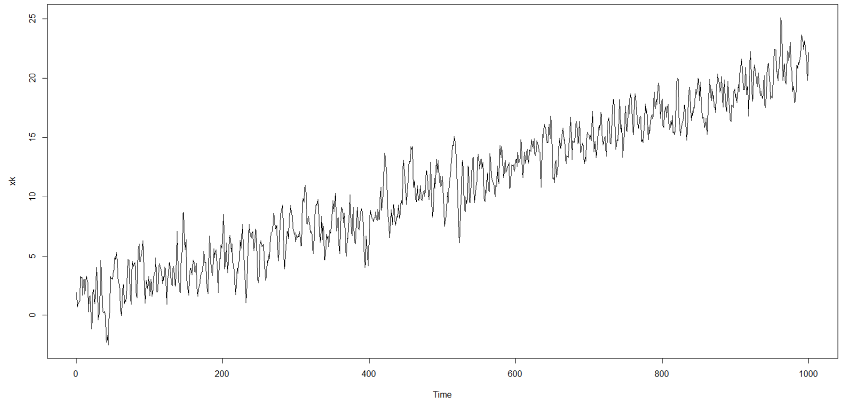


Figure 7: Time series plot of data

First method From the plot, it is evident a trend is there in data. Hence, we remove a linear trend by using lm command in R gives

$$\hat{x}[k] = 0.0214 \times k + v[k]$$

The data $v[k]$ is shown in Figure 2

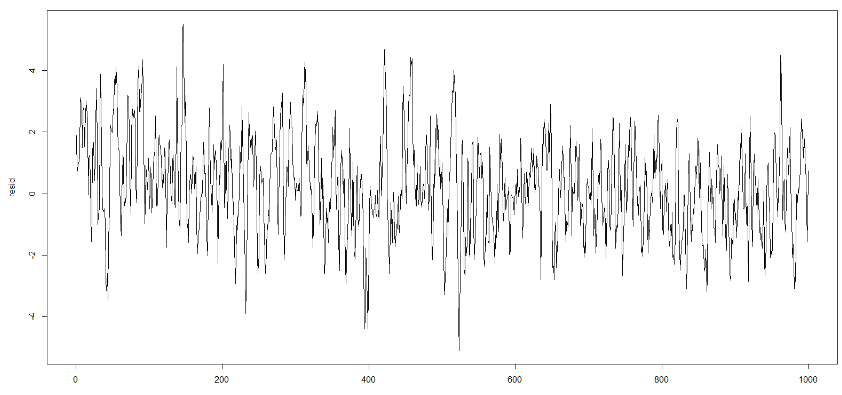


Figure 8: Residuals obtained from linear fit for data

The ACF plot of residuals $v[k]$ is shown in Figure 2.

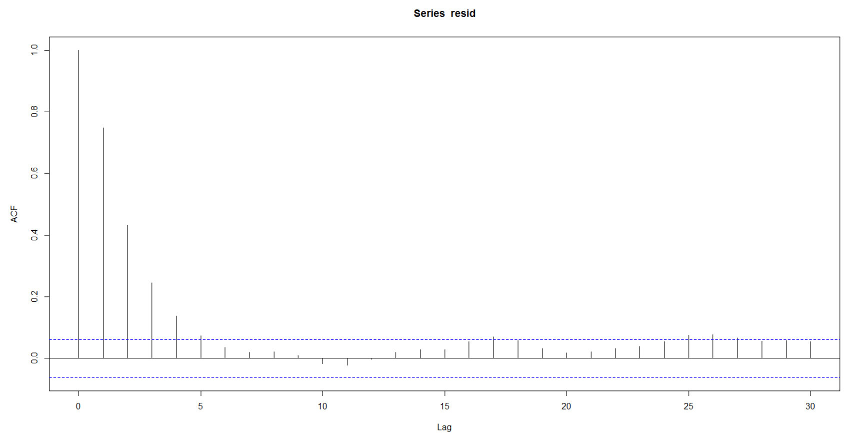
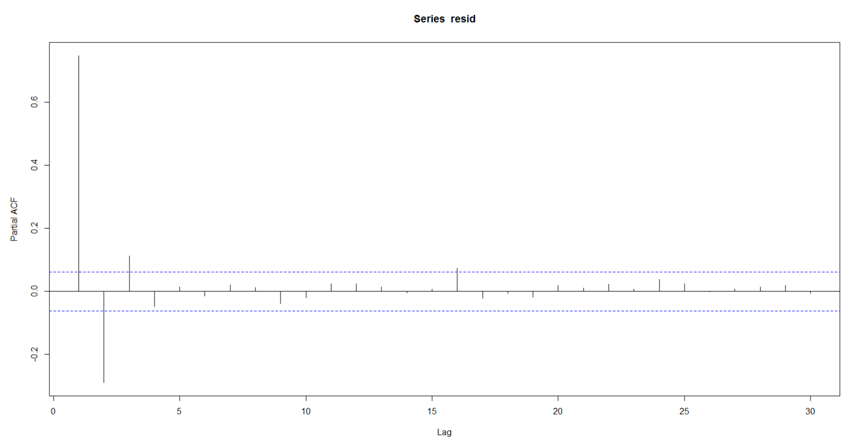


Figure 9: ACF plot of residuals

The PACF plot of residuals is shown in Figure 2



Form the plots the process looks like AR process of order 3. The process is modelled as arima($x = xk$, order $=$ c$(3, 0, 0)$)

Coefficients:

	ar1	ar2	ar3
	1.00	-0.4024	0.1143
s.e.	0.0314	0.0428	0.0314

The ACF plot of residuals is shown in Figure 2.

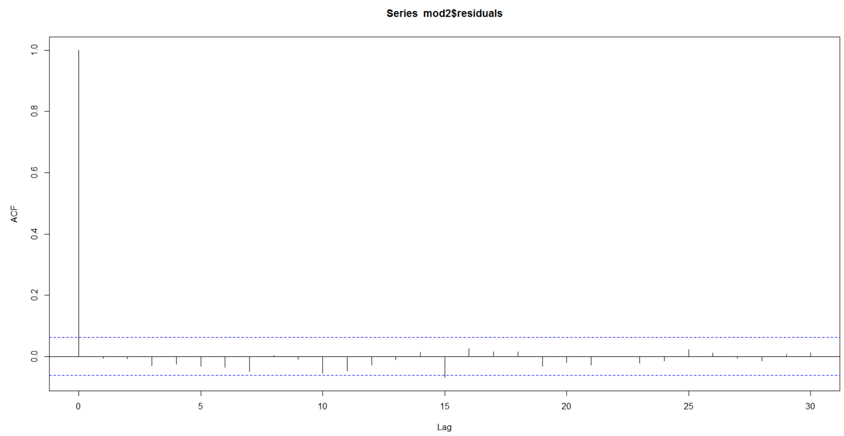


Figure 10: ACF plot of residuals

The PACF plot of residuals is shown in Figure 2.

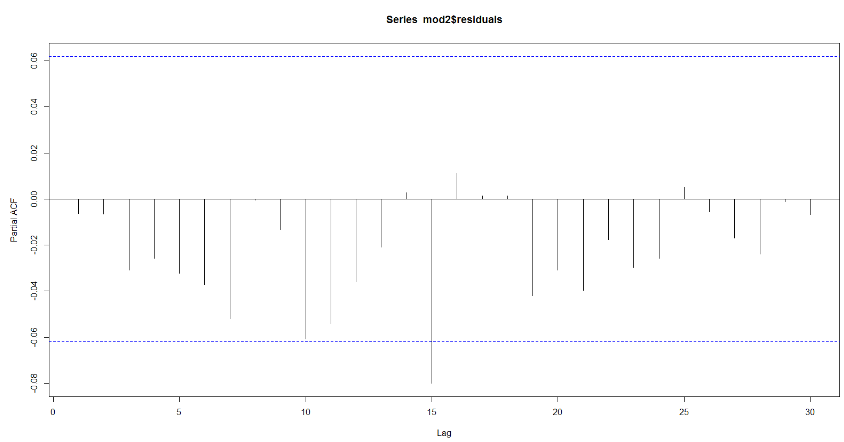


Figure 11: PACF plot of residuals

Both ACF and PACF plots resembles like of a white noise process. Hence, the final model is

$$x[k] = 0.0214 \times k + v[k] \text{ where } v[k] = 1.00v[k-1] - 0.40v[k-2] - 0.11v[k-3] + e[k]$$

Second method The data is shown in Figure 2. The series has a linear trend in it. The series is differenced one time and an ARMA model of order $(1, 0)$ is fitted for the series. The ACF of the series is shown in Figure 2.

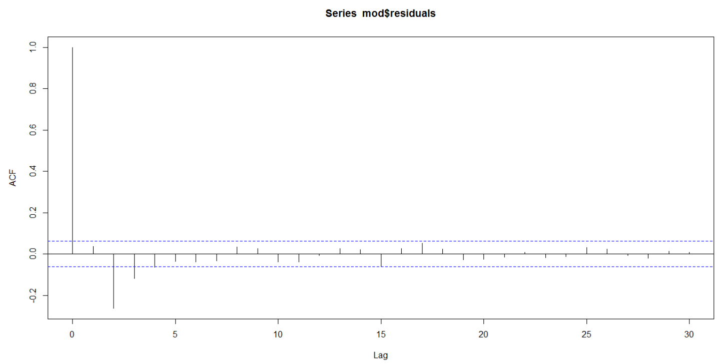


Figure 12: ACF plot of residuals

The ACF plot does not resembles like of a white noise. Up on further testing various models, ARMA model of order $(2, 1)$ gives satisfactory result. The model parameters are given as arima($x = xk$, order = c$(2, 1, 1)$)

Coefficients:

	ar1	ar2	ma1
	0.9410	-0.3104	0.9384
s.e.	0.0312	0.0306	0.0111

The ACF and PACF plot of residuals are shown in Figures 2 and 2.

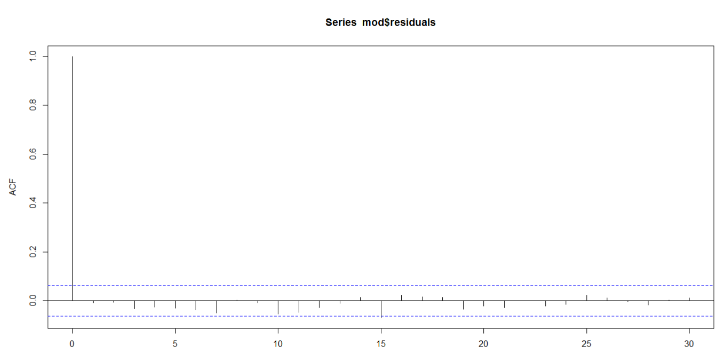


Figure 13: ACF plot of residuals

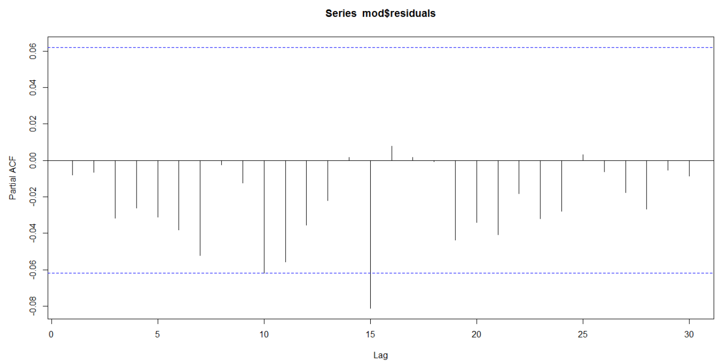


Figure 14: PACF plot of residuals

The ACF and PACF plot resembles like of a white noise. Hence, the series is modelled as

$$x[k] = 0.94x[k-1] - 0.31x[k-2] - 0.93e[k-1] + e[k]$$

3 Discrete - Time Fourier series

(a) Sketch magnitude and phase spectra

We know for a periodic signal $x[k]$ having fundamental period N_p, the Fourier coefficients are given by

$$C_n = \frac{1}{N} \sum_{k=0}^{N_p-1} x[k] \exp(-j2\pi nk)$$

The amplitude of the signal at different frequencies is given by magnitude of Fourier coefficients C_n. Similarly, the phase of the signal is given by phase of C_n. R code to calculate magnitude and phase spectra of a signal is given below

```
Np = n  # Enter the fundamental time period
k = seq(0,Np)
x = sin(2*pi*k/5)*cos(2*pi*k/3) # Given signal
c = rep (0,Np) #Initialise to zeros
for (h in 1:Np){
  for (h1 in 1:Np){
    c[h] = c[h] + (x[h1]*exp(-1i*2*pi*(h1-1)*(h-1)/(Np));
}
c[h] = c[h]/(Np);
}
plot(seq(0,1-(1/Np),1/Np),abs(c),type="l") # Magnitude plot
```

```
frame()
plot(seq(0,1-(1/Np),1/Np),Arg(c)*180/pi,type="l") # Phase plot in degrees

```

(i) $x[k] = 4\sin(\frac{\pi(k-2)}{3}k)$

The signal has fundamental period of 6 samples. The Fourier coefficients of the signal are given by

$$c_n = \frac{1}{6}\sum_{k=0}^{5} x[k]\exp(-j2\pi nk)$$

The magnitude and phase spectra of the above series are shown in Figures 3 and 3 respectively.

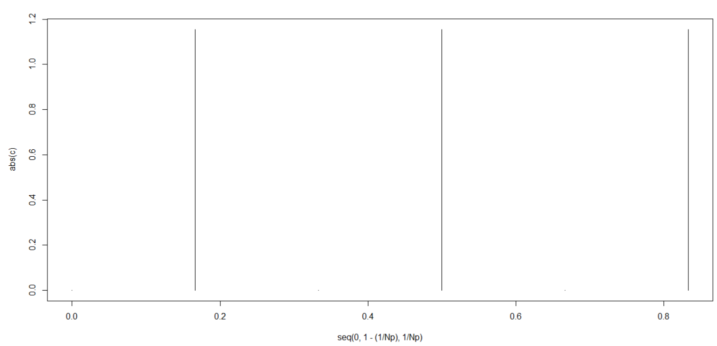


Figure 15: Magnitude plot

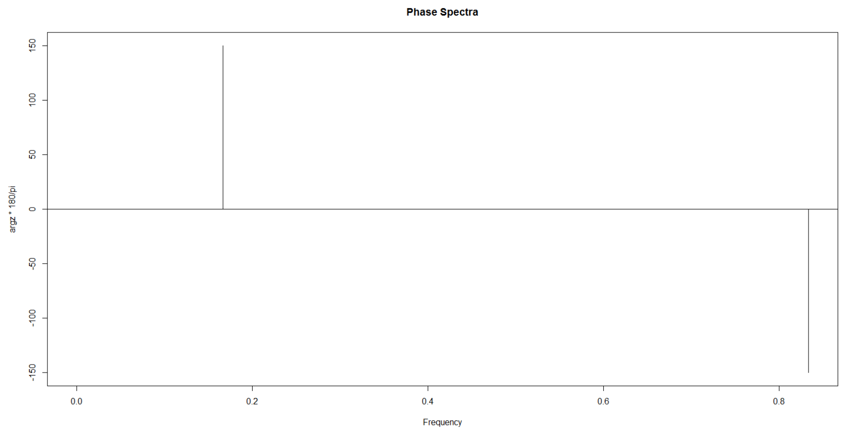


Figure 16: phase plot

(ii) $x[k] = \cos(\frac{2\pi}{3}k) + \sin(\frac{2\pi}{5}k)$

The signal has fundamental period of 15 samples. The Fourier coefficients of the signal are given by

$$c_n = \frac{1}{15}\sum_{k=0}^{14} x[k]\exp(-j2\pi nk)$$

The magnitude and phase spectra of the above series are shown in Figures 3 and 3 respectively.

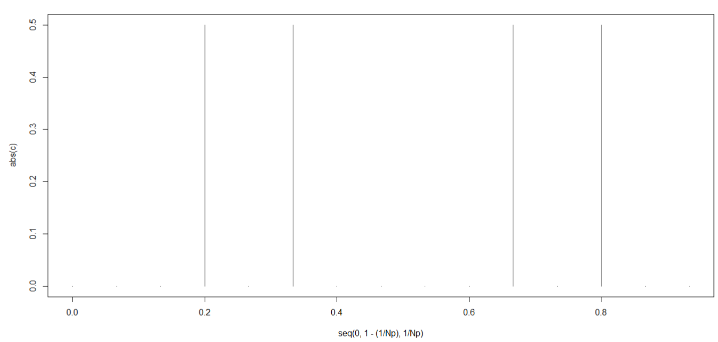


Figure 17: Magnitude plot

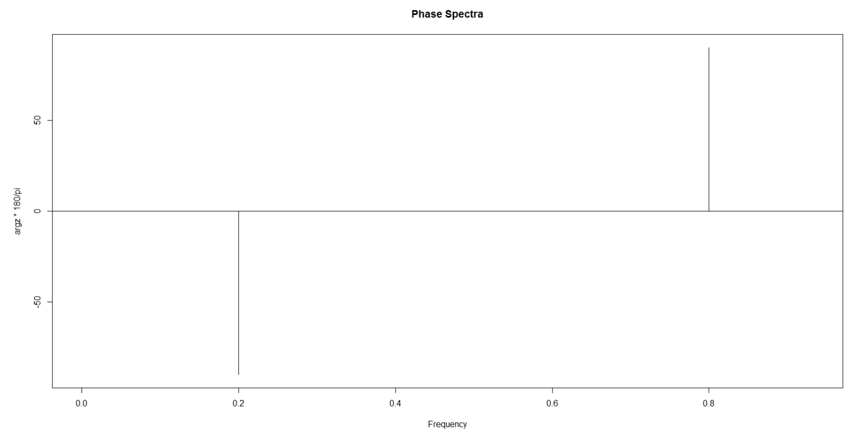


Figure 18: phase plot

(iii) $x[k] = \cos(\frac{2\pi}{3}k)\sin(\frac{2\pi}{5}k)$

The signal can be written as

$$x[k] = 0.5\left(\sin(\frac{16\pi}{15}k) - \sin(\frac{4\pi}{15}k)\right)$$

The signal has fundamental period of 15 samples. The Fourier coefficients of the signal are given by

$$c_n = \frac{1}{15}\sum_{k=0}^{14} x[k]\exp(-j2\pi nk)$$

The magnitude and phase spectra of the above series are shown in Figures 3 and 3 respectively.

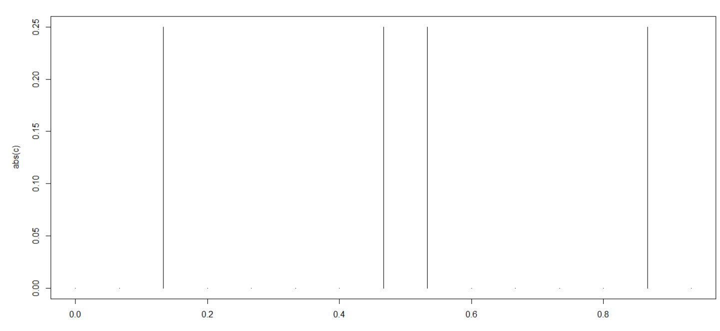


Figure 19: Magnitude plot

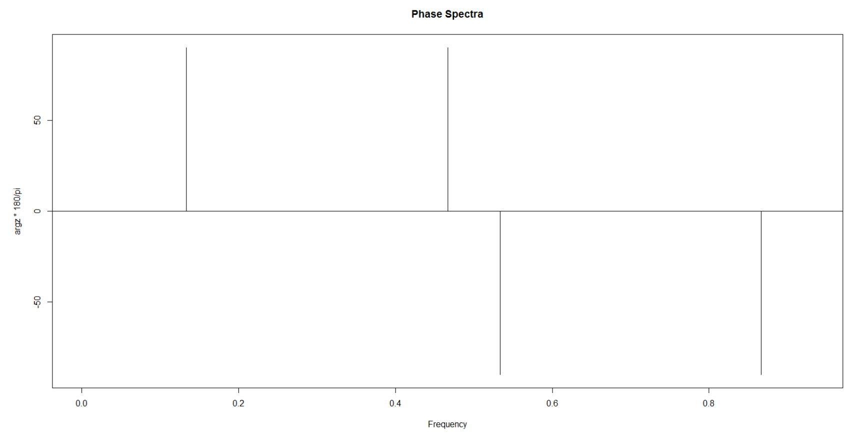


Figure 20: phase plot

(b) Given $C_n = \cos(\frac{\pi n}{4}) + \sin(\frac{3\pi n}{4})$

The fundamental period is N_p is 8 samples. The series $x[k]$ is given by

$$x[k] = \sum_{n=0}^{7} C_n \exp(j2\pi nk)$$

Calculating the coefficients in R gives

$$x[k] = \{0, 4, 4i, 0, 0, -4i, 4\} \quad \text{starting from} \quad k = 0$$

(c) Given periodic signal $x[k] = 1, 0, 1, 2, 3, 2$

The Parseval's identity for a periodic signal is given by

$$\sum_{k=0}^{N-1} |x[k]|^2 = \frac{1}{N} \sum_{n=0}^{N-1} |C_n|^2$$

where $X[n]$ is Fourier coefficient of $x[k]$. Substituting given series in the equation, we get

$$\sum_{k=0}^{5} |x[k]|^2 = 19$$

and

$$\frac{1}{6} \sum_{n=0}^{5} |X[n]|^2 = 19$$

Hence, Parseval's identity has been proved.

4 Fourier Transforms and FRF

(a)

(i) $X(0)$

From the analysis equation of the DTFT,

$$
\begin{aligned}
X(0) &= \sum_{k=-\infty}^{\infty} x[k] \\
&= \sum_{k=-2}^{2} x[k] = -1
\end{aligned}
$$

(ii) $\angle X(\omega)$

The analysis equation of the DTFT can be expanded as

$$X(\omega) = \sum_{k=-\infty}^{\infty} x[k]\cos(\omega k) - j \sum_{k=-\infty}^{\infty} x[k]\sin(\omega k)$$

Since the sequence is even, the imaginary terms cancel out.

$$\therefore \angle X(\omega) = 0$$

(iii) $X(\pi)$

From part 4,

$$X(\omega) = \sum_{k=-\infty}^{\infty} x[k]\cos(\omega k)$$

$$\therefore X(\pi) = \sum_{k=-\infty}^{\infty} (-1)^k x[k] = -9$$

(iv) $\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$

From Parseval's Theorem,

$$\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = 2\pi \sum_{k=-\infty}^{\infty} |x[k]|^2$$
$$= 38\pi$$

(b)

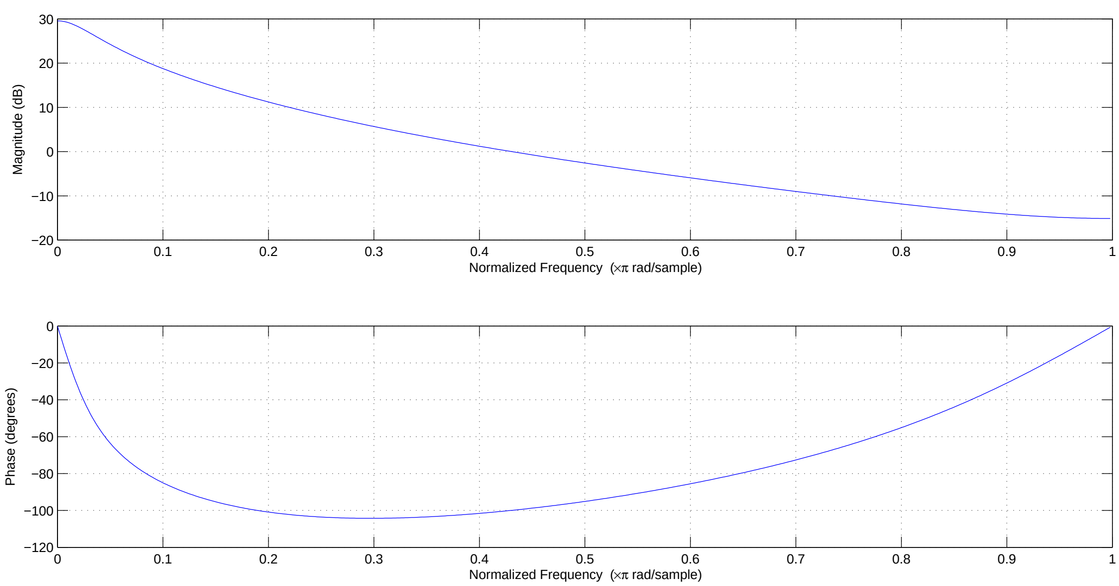
y[k] - 1.4y[k-1] + 0.45y[k-2] = u[k] + 0.5u[k-1]

$$G(z^{-1}) = \frac{1 + 0.5z^{-1}}{1 - 1.4z^{-1} + 0.45z^{-2}}$$

Frequency Response Function:

$$G(\omega) = \frac{1 + 0.5e^{-j\omega}}{1 - 1.4e^{-j\omega} + 0.45z^{-2j\omega}}$$

(c)



From the above plot, it is clear that it is a low-pass filter.

5 Spectral density of mixed process

Given $v[k] = v_1[k] + v_2[k]$, $v_1[k] = \phi_1 v_1[k-1] + e_2[k]$, $v_2[k] = e_1[k]$
Rearranging the equations,

$$v[k] = \phi_1 v[k-1] - \phi_1 e_1[k-1] + e_1[k] + e_2[k]$$

$v[k]$is not strictly ARMA(1,1) process, because of two different white noises.
As $v_1[k]$ and $v_2[k]$ are uncorrelated at all lags,

$$\sigma_{vv}[l] = \sigma_{v_1v_1}[l] + \sigma_{v_2v_2}[l]$$

multiplying by $e^{-j\omega l}$ and summing between $l = -\infty$ and $l = \infty$

$$\Phi_{vv}(\omega) = \Phi_{v_1v_1}(\omega) + \Phi_{v_2v_2}(\omega)$$

$$= \frac{1}{|1 - \phi_1 e^{-j\omega}|^2}\sigma_{e_2}^2 + \sigma_{e_1}^2$$

$$= \frac{\sigma_{e_2}^2 + \sigma_{e_1}^2 + \phi_1^2\sigma_{e_1}^2 - 2\phi_1\sigma_{e_1}^2 cos(\omega)}{|1 - \phi_1 e^{-j\omega}|^2}$$

Given

$$\Phi_{vv}(\omega) = \frac{\sigma_e^2\,|1 - \theta_1 e^{-j\omega}|^2}{|1 - \phi_1 e^{-j\omega}|^2}$$

Comparing the equations,
we have $\phi_1\sigma_{e_1}^2 = \theta_1\sigma_e^2$ and $(1+\theta_1^2)\sigma_e^2 = \sigma_{e_2}^2 + (1+\phi_1^2)\sigma_{e_1}^2$
Solving we get

$$\theta_1 = \frac{(\sigma_{e_2}^2 + (1+\phi_1^2)\sigma_{e_1}^2) \pm \sqrt{(\sigma_{e_2}^2 + (1+\phi_1)^2\sigma_{e_1}^2)(\sigma_{e_2}^2 + (1-\phi_1)^2\sigma_{e_1}^2)}}{2\phi_1\sigma_{e_1}^2}$$

$$\sigma_e^2 = \frac{2\phi_1^2\sigma_{e_2}^4}{(\sigma_{e_1}^2 + (1+\phi_1^2)\sigma_{e_2}^2) \pm \sqrt{(\sigma_{e_1}^2 + (1+\phi_1)^2\sigma_{e_2}^2)(\sigma_{e_1}^2 + (1-\phi_1)^2\sigma_{e_2}^2)}}$$