CH5350- Assignment 1

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Q.1 a) Probability of random variables

1) For two random variables x & y having joint probability density function $f(x, y)$, we can say that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\,(x, y)\,dx\,dy = 1$$

Since

$$f(x, y) = 0, x > 0y > 0$$

the above integral reduces to

$$\int_{0}^{\infty}\int_{0}^{\infty} f\,(x, y)\,dx\,dy = 1$$

Substituting the value of f(x,y), we get

$$\int_{0}^{\infty}\int_{0}^{\infty} K\frac{e^{-x/y}e^{-y}}{y}\,dx\,dy = 1$$

Integrating with respect with x we get

$$\int_{0}^{\infty} K\,e^{-y}\,dy = 1$$

and integrating over y we get

$$K = 1$$

2)

Marginal Density of y $f_y(y)$ is defined as

$$f_y(y) = \int_{-\infty}^{\infty} f\,(x, y)\,dx$$

Again,

$$f_y(y) = \int_{0}^{\infty} \frac{e^{-x/y}e^{-y}}{y}\,dx$$

Therefore,

$$f_y(y) = e^{-y} for y > 0$$

3) We know that,

$$Pr(0 < X < 1, 0.2 < Y < 0.4) = \int_{0.2}^{0.4} \int_{0}^{1} f\,(x, y)\,dx\,dy$$

To calculate the above double integration, 'adaptIntegrate' function in the 'cubature' library was used.

```
library(cubature)
f = function(x){(exp(-x[1]/x[2])*exp(-x[2]))/x[2] }   #where x[1] = x and x
[2] = y
c = adaptIntegrate(f, lowerLimit = c(0,0.2), upperLimit = c(1,0.4))
probability = c[["integral"]]
print(probability)

## [1] 0.1428716
```

4) Conditional Expectation

$$E(X|Y) = \int_{-\infty}^{\infty} x \frac{f(x, y)}{f(y)}\,dx$$

Substituting the values of $f(x, y)$ and $f(y)$, we get

$$E(X|Y) = \int_{0}^{\infty} x \frac{e^{-x/y} e^{-y}}{e^{-y} y}\,dx$$

Then integrating By Parts, we get

$$E(X|Y) = [-(x + y)e^{-x/y}]_0^{\infty}$$

Therefore

$$E(X|Y) = y$$

b) Given - two random variables X and Y are jointly normal. Assuming both X and Y have zero mean and some positive variance to avoid unnecessary complexity in the proof.

Let us define

$$\hat{Y} = \rho \frac{\sigma_y}{\sigma_x} X$$

where ρ is correlation coefficient of X and Y.

$$\overline{Y} = Y - \hat{Y}$$

Since X and Y are linear combinations of independent normal random variables, it follows that Y and $\overline{X}$ are also linear combinations of the same independent normal random variables, hence jointly normal.

Also,

$$E[X\overline{Y}] = E[XY] - E[X\,\hat{Y}] = \rho\sigma_x\sigma_y - \rho\frac{\sigma_x}{\sigma_y}\sigma_y^2 = 0$$

Therefore X and are uncorrelated and, therefore independent (property of Jointly normal random variables). Since {Y} is scalar multiple of X, it follows that {Y} and are independent. Therefore,

$$Y = \hat{Y} + \overline{Y} = \rho\frac{\sigma_y}{\sigma_x}X + \overline{Y}$$

Taking conditional expectation on both sides, we get

$$E[Y|X] = \rho\frac{\sigma_y}{\sigma_x}E[X|X] + E[\,\overline{Y}|X] = \rho\frac{\sigma_y}{\sigma_x}X = \hat{Y}$$

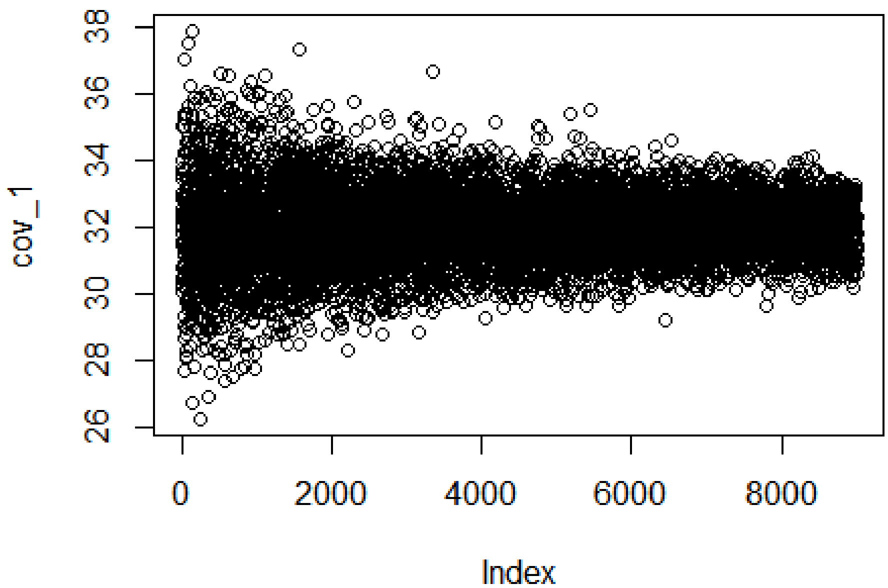
where,

$$E[\overline{Y}|X] = 0$$

Hence proved that the conditional expectation $E[Y|X]$ is a linear function of X .

Q.2) Covariance Matrix

```
# The function assumes that the
cv = function(N,m,std){
X = rnorm(N, mean = m, sd = std)
Y = Y = 2*X^2 + 4*X

X_ = sum(X)/length(X)
Y_ = sum(Y)/length(Y)

c = sum((X - X_)*(Y - Y_))/length(X)
}

sam_cov_mat = cv(1000, 1 ,2)
cov_1<-{}
for (k in 1000:10000) {

  cov_1 <-append(cov_1, cv(k,1,2))

}
plot(cov_1)
```



In the above plot we see as the number of samples tends to infinity the covariance converses to a value (here 32, shown below)

The theoretical value of covariance can be calculated as follows

$$\sigma_{xy} = E((Y - \overline{Y})(X - \overline{X}))$$

$$\sigma_{xy} = E((2X^2 + 4X - E(Y))(X - 1))$$

$$\sigma_{xy} = E((2X^2 + 4X - E(2X^2 + 4X))(X - 1))$$

$$\sigma_{xy} = E((2X^2 + 4X - 2E(X^2) - 4E(X))(X - 1))$$

$$\sigma_{xy} = E((2X^2 + 4X - 14)(X - 1))$$

$$\sigma_{xy} = E(2X^3 + 2X^2 - 18X + 14)$$

$$\sigma_{xy} = 2E(X^3) + 2E(X^2) - 18E(X) + E(14)$$

$$\sigma_{xy} = 2(3E(X^2) + 3E(X) - 1) + 10 - 18 + 14$$

$$\sigma_{xy} = 2(15 - 3 + 1) + 6$$

$$\sigma_{xy} = 32$$

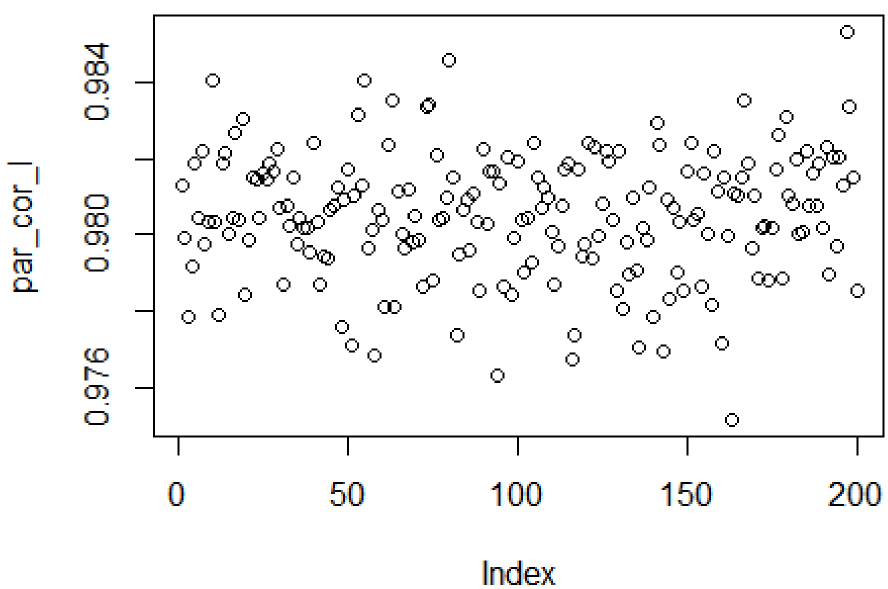
Q.3) Partial Correlation

```
# Partial Correlation

# Generating values of X, Y, Z, W, V
V = rnorm(500, mean = 0,  sd = 1)
W = rnorm(500, mean = 0, sd = 2 )
Z = rnorm(500, mean = 1, sd = 3)
X = 3*Z+W
Y = Z+2.5*X+V

# Partial Correlation by Inversion method

matw = cbind(X, Y, Z)
cov_mat = cov(matw)
inv_cov = solve(cov_mat)
par_cor = -inv_cov[2,1]/(sqrt(inv_cov[1,1])*sqrt(inv_cov[2,2]))
print(par_cor)

## [1] 0.9806906

#Partial Correlation by Linear Regression method

par_cor_lg = sqrt((cov(X,Y) - cov(X,Z)*cov(Y,Z))^2/((1 - cov(Y,Z)^2)*(1- cov
(X,Z)^2)))

print(par_cor_lg)

## [1] 0.8806168
```

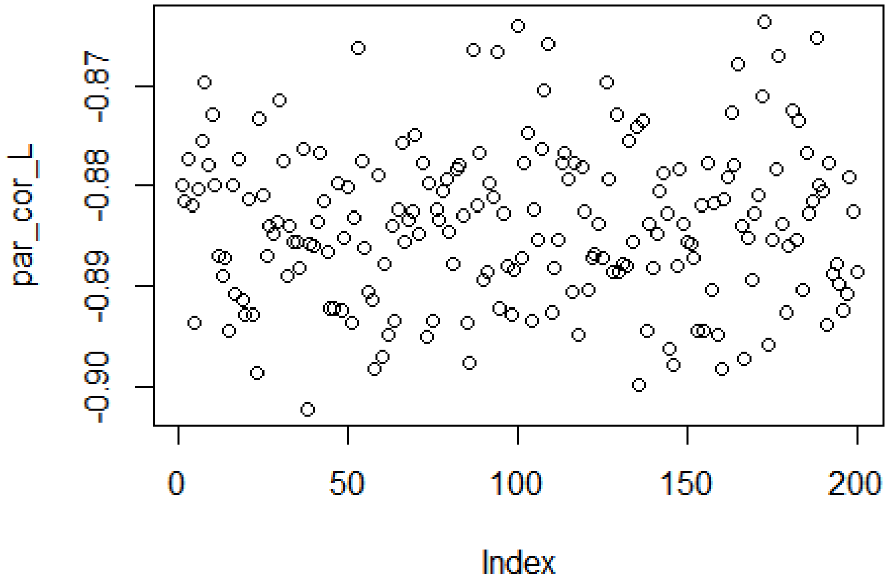
c) Repeating a and b for R = 200 Monte-Carlo simulations.

```
par_cor_I = {}          #empty vector for partial correlation for INVERSION me
thod
par_cor_L = {}         #empty vector for partial correlation by REGRESSIONN me
thod

for (R in 1:200) {
  # Generating values of X, Y, Z, W, V
V = rnorm(500, mean = 0,  sd = 1)
W = rnorm(500, mean = 0, sd = 2 )
Z = rnorm(500, mean = 1, sd = 3)
X = 3*Z+W
Y = Z+2.5*X+V

# Partial Correlation by Inversion method

matw = cbind(X, Y, Z)
cov_mat = cov(matw)
inv_cov = solve(cov_mat)
par_cor = -inv_cov[2,1]/(sqrt(inv_cov[1,1])*sqrt(inv_cov[2,2]))
```

```
par_cor_I = append(par_cor_I, par_cor)

#Partial Correlation by Linear Regression method

par_cor_lg = (cov(X,Y) - cov(X,Z)*cov(Y,Z))/sqrt(((1 - cov(Y,Z)^2)*(1- cov(X,
Z)^2)))
par_cor_L = append(par_cor_L, par_cor_lg)

}
print(sd(par_cor_L)) #Standard deviation - Regression

## [1] 0.007802059

print(sd(par_cor_I)) #Standard deviation - Inversion

## [1] 0.001645692

plot(par_cor_I)
```




```
plot(par_cor_L)
```



Comparing standard deviation we can say that, Partial correlation by Inversion method yields lower error.

Q.4) Periodicities and ACF

a) Given

$$v[k] = A cos(2\pi f_o k) + e[k]$$

Calculating expectation of the process we get

$$E(v[k]) = E(A cos(2\pi f_o k)) + E(e[k])$$

Therefore

$$E(v[k]) = A cos(2\pi f_o k)$$

Since the expectation of the signal is depended on k, we can conclude that the process is not stationary

$$R_{vv}[l] = \frac{1}{N}\sum_{l+1}^{N}(v[k]v[k-l] + e[k]v[k-l] + e[k-l]v[k] + e[k]e[k-l])$$

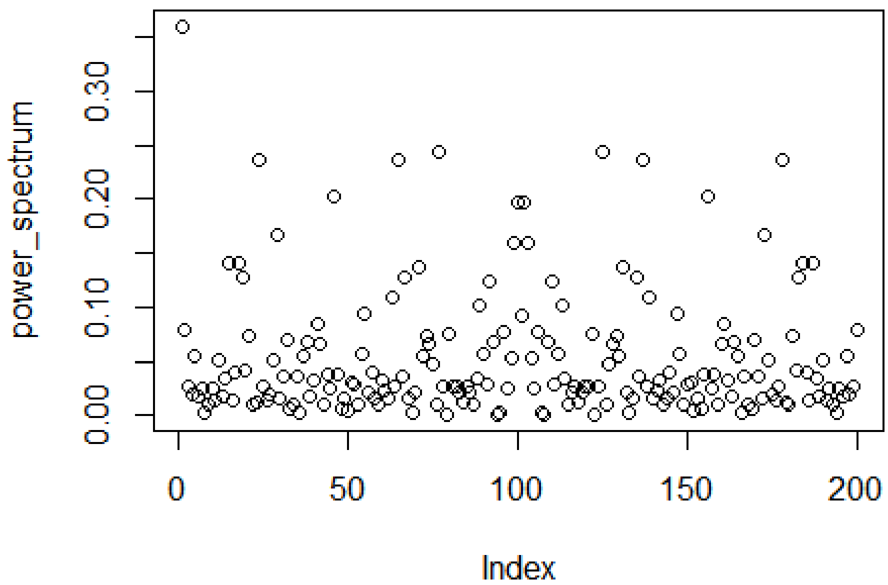
Substituting the respective values and working out the above equation we show that for large samples the left hand side of the above equation reduces down to

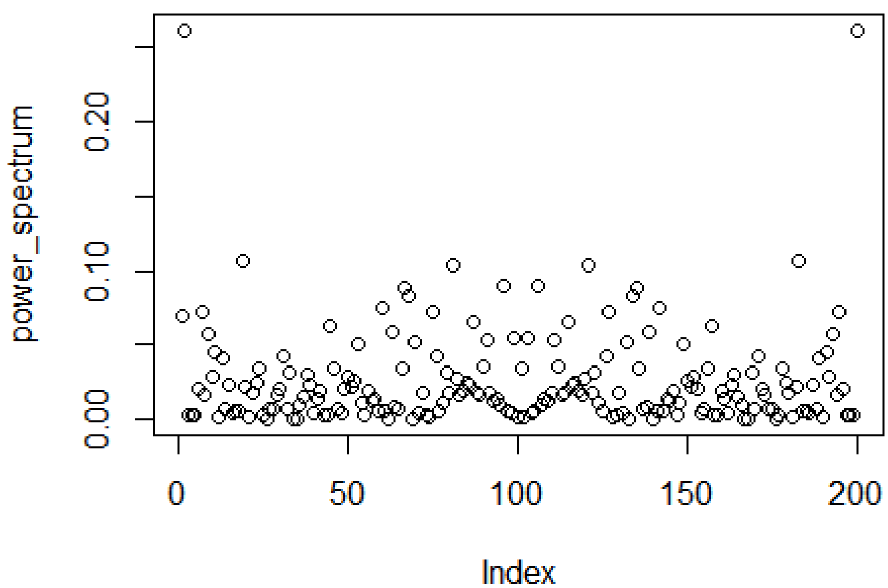
$$R_{vv}[l] = \frac{A^2}{2} cos(2\pi f_o k)$$

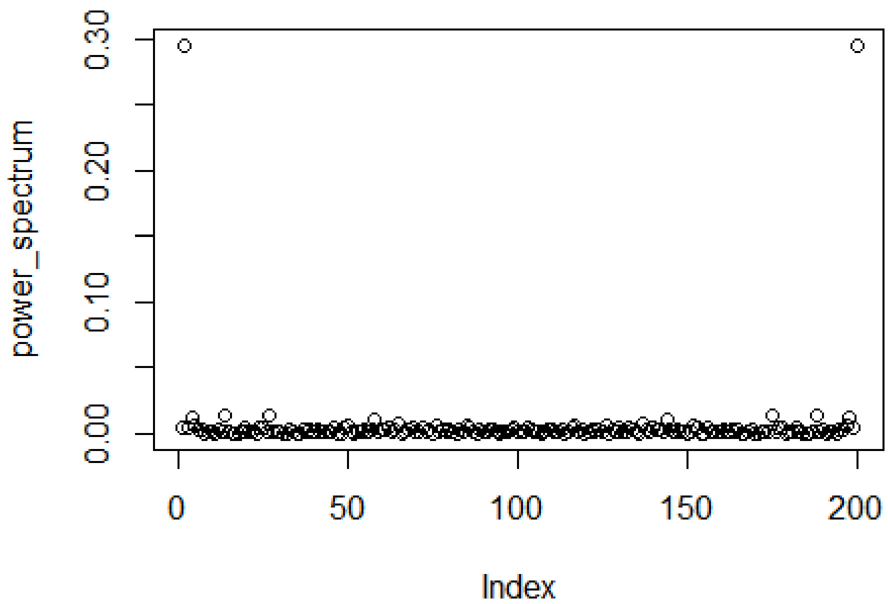
Hence proved that the time-averaged ACVF of $v[k]$ is sinusoidal with frequency f_o for large samples.

c) There is an advantage of detecting periodicity of the sine wave from it's ACF rather than $v[k]$ because in $v[k]$ the extra whitenoise factor should taken into account which makes it difficult.

```
N = 200               # Total length of the signal
t = seq(0,2*pi,,N)  #time vector
x_ = sin(0.3*pi*t)
stdx_ = sd(x_)
print(stdx_)

## [1] 0.726728

snr = c(20,10, 1, 0.1)
for (r in snr) {


std_e = sqrt(r)*stdx_       # standard deviation of white noise
e = rnorm(200, mean = 0, sd = std_e)
x = x_ + e
X_f = fft(x)
power_spectrum = abs(X_f)^2/N^2
plot(power_spectrum)
}
```


power_spectrum
0.00
0.10
0.20
0.30
0
50
100
150
200
Index


power_spectrum
0.00
0.10
0.20
0
50
100
150
200
Index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0.00
0.10
0.20
0.30
0
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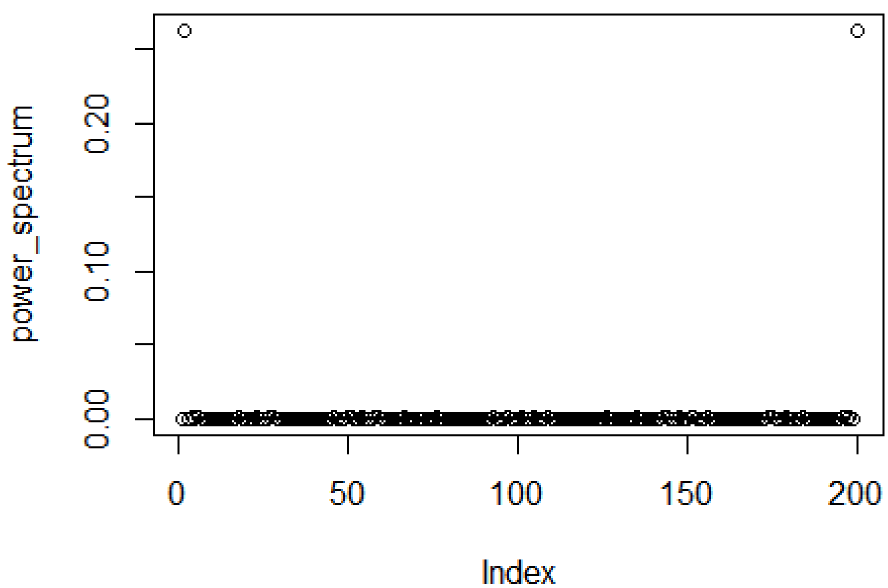
100
150
200
Index


power_spectrum
0.00
0.10
0.20
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100
150
200
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