Optimal Quantum Cloning Machines

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Outline

Definitions[1]

- Fidelity : for initial state $|\psi\rangle$ and reduced output state ρ_j :

$$\mathcal{F}_j = \langle\psi|\rho_j|\psi\rangle \, , j = 1, ..., M \tag{1}$$

- Universal QCM: QCM for which fidelity is independent of $|\psi\rangle$
- Symmetric QCM: QCM for which the fidelity is same for all clones
- Optimality : To achieve our goal is to maximize either the average fidelity $\langle\mathcal{F}\rangle = \int_S d\psi F(\psi)$ the minimal fidelity $F_{min} = min_{\psi\in S}F(\psi)$ over the states

$$\rho_A = \rho_B = \mathcal{F}\,|\psi\rangle\langle\psi| + (1-\mathcal{F})\,|\psi^{\perp}\rangle\langle\psi^{\perp}| \tag{2}$$

1-2 Symmetric UQCM (Buzek-Hillery, 1996)[2][3]

$$|0\rangle_A |R\rangle_B |\mathcal{M}\rangle \rightarrow \sqrt{2/3}\,|0\rangle\,|0\rangle\,|1\rangle - \sqrt{1/3}\,\left|\Psi^+\right\rangle|0\rangle \tag{3}$$

$$(-\,|1\rangle)_A |R\rangle_B |\mathcal{M}\rangle \rightarrow \sqrt{2/3}\,|1\rangle\,|1\rangle\,|0\rangle - \sqrt{1/3}\,\left|\Psi^+\right\rangle|1\rangle \tag{4}$$

where $|\Psi^+\rangle = \frac{1}{\sqrt{2}}[|10\rangle + |01\rangle]$, $|R\rangle$ is the blank state and $|\mathcal{M}\rangle$ is the ancilla state.

By linearity, above two relations induce the following action on the most general input state $\psi = \alpha\,|0\rangle + \beta\,|1\rangle$:

$$|\psi\rangle\,|R\rangle\,|M\rangle \rightarrow \sqrt{\frac{2}{3}}\,|\psi\rangle\,|\psi\rangle\,|\psi^\perp\rangle - \sqrt{\frac{1}{6}}[|\psi\rangle\,|\psi^\perp\rangle + |\psi^\perp\rangle\,|\psi\rangle]\,|\psi\rangle \tag{5}$$

where, $\left|\psi^\perp\right\rangle = \alpha^*\,|0\rangle - \beta^*\,|1\rangle$,

$$\rho_A = \rho_B = \frac{5}{6}\,|\psi\rangle\langle\psi| + \frac{1}{6}\,|\psi^\perp\rangle\langle\psi^\perp| \tag{6}$$

1-M Symmetric UQCM[3]

$$U_{1,M}\left|\psi\right\rangle \otimes R \otimes \mathcal{M} = \sum_{j=0}^{M-1} \alpha_j \left|(M-j)\psi, j\psi^{\perp}\right\rangle \otimes \mathcal{M}_j, \tag{7}$$

where, $\alpha_j = \sqrt{\frac{2(M-j)}{M(M+1)}}$, $\mathcal{M}_j(\psi)$ represents ancilla with $\mathcal{M}_j(\psi) \perp \mathcal{M}_k(\psi)$ for all $j \neq k$.

$$\begin{aligned}\mathcal{F}_{1,M} &= \sum_{j=0}^{M-1} Prob(jerrorsintheM-1qubits) \\ &= \sum_{j=0}^{M-1} \frac{M-j}{M}\alpha_j^2 = \frac{2M+1}{3M}\end{aligned} \tag{8}$$

1-M Classical Copying Machine[3]

- In classical machine we actually project the input state onto two (randomly chosen) orthogonal states $|\phi\rangle$ and $|\phi^{\perp}\rangle$.
- The density matrix describing the M copies, averaged over the orientation of the measuring basis $|\phi\rangle$, we get

$$\rho_{CCM} = \int d\Omega_{\phi}(|\langle\psi|\phi\rangle|^2 P_{|M\phi\rangle} + |\langle\psi|\phi^{\perp}\rangle|^2 P_{|M\phi^{\perp}\rangle}) \tag{9}$$

$$\rho_{CCM} = \sum_{s=0}^{M} \frac{2(M+1-s)}{(M+1)(M+2)} P_{|(M-s)\psi, s\psi^{\perp}\rangle} \tag{10}$$

for M $= 2$, $\mathcal{F}_{CCM} = 2/3 < \mathcal{F}_{QCM} = 5/6$

$$Tr[\rho_{QCM} - \rho_{CCM}]^2 \simeq M^{-3} \tag{11}$$

So we can see that, QCM tends to CCM as M increases.

Proof of Optimality[3]

The most general QCM acts on the input qubits $\uparrow, \downarrow$ in the following way :

$$|j\rangle\,|R\rangle \rightarrow |M-k\uparrow, k\downarrow\rangle\,|R_{jk}\rangle\,, j=\uparrow,\downarrow$$

where, R is the initial state of the QCM and the blank state, R_{jk} are the unnormalized final states of the ancilla,
Any arbitrary input qubit can be expressed as a SU(2) rotation $\mathcal{O}_{j'j}(\Omega)$ of the $\uparrow$ state.
The evolution of the arbitrary input state is then

$$|\psi\rangle\,|R\rangle = \mathcal{O}_{\uparrow j}\,|j\rangle\,|R\rangle \rightarrow |\psi_{out}\rangle \quad = \mathcal{O}_{\uparrow j}\,|M-k\uparrow, k\downarrow\rangle\,|R_{jk}\rangle \tag{12}$$

The average fidelity comes out to be of the form

$$\mathcal{F} = \left\langle R_{j'k'} \middle| R_{jk} \right\rangle A_{j'k'jk} \tag{13}$$

Proof of Optimality[3] contd . . .

- Imposing the unitary constraints and using Lagrange multipliers we extremize:

$$\mathcal{F} = \left\langle R_{j'k'} \middle| R_{jk} \right\rangle A_{j'k'jk} - \lambda(\left\langle R_{j'k'} \middle| R_{jk} \right\rangle \delta_{k'k}\delta_{j'j} - 2) \tag{14}$$

We obtain the equation

$$(A_{j'k'jk} - \lambda\delta_{k'k}\delta_{j'j}) \left| R_{jk} \right\rangle = 0 \tag{15}$$

where, λ is eigenvalues of the $A_{j'k'jk}$ and $\left| R_{jk} \right\rangle$ are eigenvectors
So, we get $\mathcal{F} = 2\lambda$
The largest eigenvalue of A is (2M + 1)/6M, therefore

$$\mathcal{F} \leq (2M + 1)/3M \tag{16}$$

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