CH5350: Applied Time-Series Analysis

Spectral Representations of Random Signals

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Opening remarks

We have learnt, until this point, how to represent deterministic signals (and systems) in the frequency-domain using Fourier series / transforms.

- Signal decomposition results in also energy / power decomposition (as the case maybe) by virtue of Parseval's relations.
- **Signal decomposition** is primarily useful in filtering and signal estimation, whereas **power / energy decomposition** is useful in detection of periodic components / frequency content of signals.
- Fourier analysis, as we have seen, is the key to characterizing the frequency response of LTI systems and analyzing their "filtering" nature,

Analysis of stochastic processes

Frequency-domain analysis of random signals is, however, not as straightforward, primarily because,

$$\boxed{\text{Fourier transforms of random signals do not exist.}}$$

- Random signals are, in general, **aperiodic**, but with infinite energy (they exist forever, by definition). They are, in fact, **power signals.**
- **Periodic** stationary random signals also exist, but the Fourier series idea cannot be applied straight away to such signals.

Does this rule out the possibility of constructing a Fourier / spectral representation of random signals?

Fourier analysis of random signals

In the frequency-domain analysis of random signals we are primarily interested in **power / energy decomposition** rather than *signal decomposition*, because we would like to characterize the random process and **not** necessarily the realization.

Stationary random signals can be of two types: (i) periodic and (ii) aperiodic. Both classes of signals are **power** signals (assuming finite variance or power).

- However, we cannot adopt the approach used for deterministic signals in quantifying their frequency-domain characteristics.

Random periodic processes

What are random periodic processes?

They are stationary processes that exhibit a periodic behaviour.

Example: Train arrival times at a station - it has a periodicity but some randomness due to minor variations in starting times or arrival times

Why do we study random periodic processes?

Random periodic processes ...contd.

Because they can model a variety of periodic phenomena, both natural and man-made, that are embedded in some stochastic processes.

How can a random process be periodic?

The periodicity of a random process is not in its value, as it was for deterministic signals, but is in a mean square sense.

It turns out that this definition is also equal to requiring that the ACVF be periodic.

Harmonic Process

Definition

A discrete-time (wide-sense) *stationary* process $\{v[k]\}$ is said to be *periodic* with period N_p if

$$E((v[k+N_p]-v[k])^2)=0 \tag{1}$$

or equivalently,

1. $\sigma_{vv}[l+N_p]=\sigma_{vv}[l], \quad \forall l \in \mathbb{Z}$ (Periodic ACVF)
2. $\sigma_{vv}[N_p]=\sigma_{vv}[0]$
3. $\Pr(v[k+N_p]=v[k])=1 \ \ \forall k \in \mathbb{Z}$

Harmonic processes

A simple possible way of constructing a random, stationary periodic signal $v[k]$ is through a linear combination of sines and cosines with **random coefficients** (or by linearly combining sines with **random amplitudes and phases**)

$$v[k]=\sum_{n=1}^{F} a_n \cos(2\pi f_n k)+b_n \sin(2\pi f_n k)$$

However, these **coefficients cannot be arbitrarily random.**

Harmonic processes

In fact, for $v[k]$ to be stationary, it is required that

a_n and b_n are independent random variables with $E(a_n) = E(b_n) = 0$ and with equal variances

With the assumption $E(a_n) = 0 = E(b_n)$, it is easy to show that:

$$\sigma_{vv}[l] = \sum_{n=1}^{F} \sigma_n^2 \cos(2\pi f_n l) \qquad \text{where} \qquad \sigma_n^2 = E(a_n^2) = E(b_n^2)$$

The ACVF of a random periodic signal is periodic with contributions from each frequency component proportional to their respective variances

DTFS for ACVF

The fact is that for this case **ACVF has a Fourier Series expansion**

Observe:

- For the deterministic case, the squared magnitude of Fourier coefficients gave the power spectrum
- For random signals, the expectation of the square of Fourier coefficients can be thought of the power spectrum of the random periodic signal

Spectral analysis of aperiodic random signals

Fourier representation of stationary aperiodic random signals can be constructed by admitting all frequency components; however, unlike in deterministic signals, the "coefficients" of combination have to be uncorrelated **random variables.** This is the idea in Wiener's **generalized harmonic analysis** (GHA).

Question: How do we define **power spectral density** for random aperiodic signals?

Three different approaches to p.s.d.

1. **Wiener's GHA:** A generalization of the Fourier analysis to the class of signals which are neither periodic nor finite-energy, aperiodic signals (e.g., $\cos\sqrt{2}k$). It is theoretically sound, but also involves the use of advanced mathematical concepts, e.g., stochastic integrals.
2. **Semi-formal approach:** Construct the spectral density as the ensemble average of the empirical spectral density of a finite-length realization in the limit as $N \to \infty$.
3. **Wiener-Khinchin relation:** One of the most fundamental results in spectral analysis of stochastic processes, it allows us to compute the spectral density as the Fourier transform of the ACVF. This is perhaps the most widely used approach.

Focus: Last two approaches and the conditions for the existence of a spectral density.

Semi-formal approach

Consider a length-N sample record of a random signal. Compute the **periodogram**, i.e., the empirical p.s.d., of the finite-length realization

$$\gamma_{vv}^{(i,N)}(\omega_n) = \frac{|V_N^{(i)}(\omega_n)|^2}{2\pi N} = \frac{1}{2\pi N}\left|\sum_{k=0}^{N-1} v^{(i)}[k]e^{-j\omega_n k}\right|^2$$

where $V_N^{(i)}(\omega_n)$ is the N-point DFT of the finite length i^{th} realization.

Semi-formal approach

The spectral density of the random signal is the **ensemble average (expectation)** of the density in the limiting case of $N \to \infty$

$$\boxed{\gamma_{vv}(\omega) = \lim_{N\to\infty} E(\gamma_{vv}^{(i,N)}(\omega_n)) = \lim_{N\to\infty} E\left(\frac{|V_N^{(i)}(\omega_n)|^2}{2\pi N}\right)}$$

The spectral density exists when the limit of average of periodogram exists.

When does the empirical definition exist?

In order to determine the conditions of existence, we begin by writing (dropping the superscript for realization)

$$|V_N(\omega_n)|^2 = V_N(\omega_n)V_N^\star(\omega_n) = \sum_{k=0}^{N-1} v[k]e^{-j\omega_n k} \sum_{m=0}^{N-1} v[m]e^{-j\omega_n m}$$

Next, take expectations and introduce a change of variable $l = k - m$ to obtain,

$$\gamma_{vv}(\omega) = \frac{1}{2\pi} \lim_{N\to\infty} \sum_{l=-(N-1)}^{N-1} f_N(l)\sigma_{vv}[l]e^{-j\omega l}, \qquad \text{where } f_N(l) = 1 - \frac{|l|}{N}$$

Conditions for existence

Now, **importantly**, assume that $\sigma_{vv}[l]$ is absolutely convergent, i.e.,

$$\sum_{l=-\infty}^{\infty} |\sigma_{vv}[l]| < \infty$$

Further, that it decays sufficiently fast, $\sum_{l=-\infty}^{\infty} |l|\sigma_{vv}[l] < \infty$.

Under these conditions, the limit converges and the p.s.d. is obtained as,

$$\boxed{\gamma_{vv}(\omega) = \sum_{l=-\infty}^{l=\infty} \sigma_{vv}[l]e^{-j\omega l}}$$

Spectral Representation / Wiener-Khinchin Theorem

W-K Theorem (Shumway and Stoffer, 2006)

Any stationary process with ACVF $\sigma_{vv}[l]$ satisfying

$$\sum_{l=-\infty}^{\infty} |\sigma_{vv}[l]| < \infty \qquad \text{(absolutely summable)}$$

has the spectral representation

$$\sigma_{vv}[l] = \int_{-\pi}^{\pi} \gamma_{vv}(\omega) e^{j\omega l}\, d\omega, \quad \text{where} \quad \gamma_{vv}(\omega) = \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} \sigma_{vv}[l] e^{-j\omega l} \quad -\pi \leq \omega < \pi$$

$\gamma_{vv}(\omega)$ is known as the **spectral density.**

W-K Theorem: Remarks

- It is one of the milestone results in the analysis of linear random processes.
- Recall that a similar version also exists for aperiodic, finite-energy, deterministic signals. The p.s.d. is replaced by e.s.d. (energy spectral density). Thus, it provides a **unified** definition for both deterministic and stochastic signals.
- It establishes a direct connection between the second-order statistical properties in time to second-order properties in frequency domain.
- The inverse result offers an alternative way of computing the ACVF of a signal.

A more general statement of the theorem unifies both classes of random signals, the ones with absolutely convergent ACVFs and the ones with periodic ACVFs (harmonic processes).

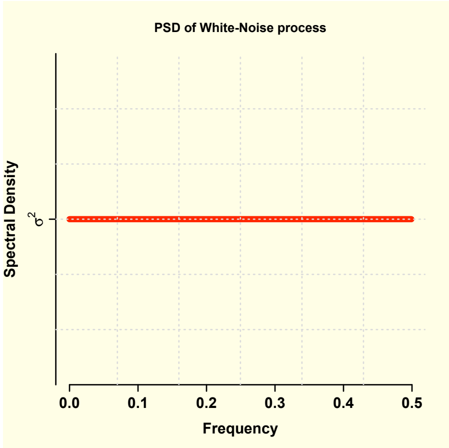
Spectral Representation of a WN process

Recall that the ACVF of WN is an impulse centered at lag $l = 0$,

The WN process is a stationary process with a constant p.s.d.

$$\gamma_{ee}(\omega) = \frac{1}{2\pi}\sum_{l=-\infty}^{\infty}\sigma_{ee}[l]e^{-j\omega l} = \frac{\sigma_e^2}{2\pi}, \quad -\pi \leq \omega \leq \pi$$

All frequencies contribute uniformly to the power of a WN process (as in white light). Hence the name.



Auto-correlated processes $\equiv$ Coloured Noise

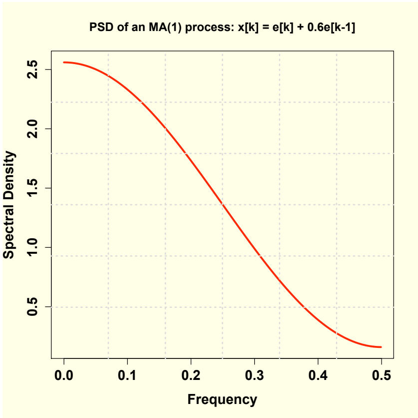
We can also examine the spectral density of AR and MA processes.

Two examples are taken up: (i) an MA(1) process and an (ii) an AR(1) process

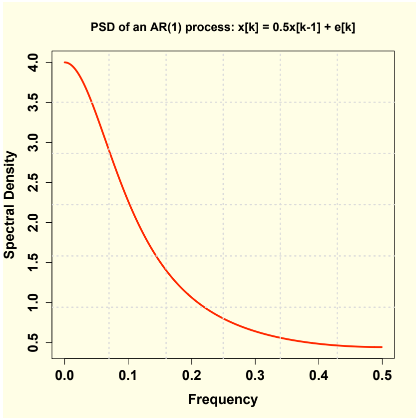
$$\sigma_{vv}[l] = \begin{cases} 1.36 & l = 0 \\ 0.6 & |l| = 1 \\ 0 & |l| \geq 2 \end{cases} \quad \text{(MA(1))}$$

$$\sigma_{vv}[l] = \frac{4}{3}(0.5)^{|l|} \; \forall l \quad \text{(AR(1))}$$

PSD of MA(1) and AR(1) processes



The spectral density is a function of the frequency unlike the "white" noise. Correlated processes therefore acquire the name *coloured* noise.



Obtaining p.s.d. from time-series models

The p.s.d. of a random process was computed using its ACVF and the W-K theorem. However, if a time-series model exists, the p.s.d. can be computed directly from the transfer function as:

$$\boxed{\gamma_{vv}(\omega) = |H(e^{-j\omega})|^2 \gamma_{ee}(\omega) = |H(e^{-j\omega})|^2 \frac{\sigma_e^2}{2\pi}} \tag{2}$$

$$\text{where} \quad H(e^{-j\omega}) = \text{DTFT}(h[k]) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k} \tag{3}$$

PSD from Model

Derivation: Start with the general definition of a linear random process

$$v[k] = \sum_{m=-\infty}^{\infty} h[m]e[k-m] \qquad \Longrightarrow \sigma_{vv}[l] = \sum_{m=-\infty}^{\infty} h[m]h[l-m]\sigma_{ee}^2$$

Taking (discrete-time) Fourier Transform on both sides yields the main result.

$$\boxed{\gamma_{vv}(\omega) = |H(e^{-j\omega})|^2\gamma_{ee}(\omega) = |H(e^{-j\omega})|^2\frac{\sigma_e^2}{2\pi}} \tag{4}$$

The p.s.d. of a linear random process is $\propto$ the squared magnitude of its FRF

A linear random process is a filter

The result that we just observed is well-known in the frequency-domain analysis of deterministic LTI systems. Drawing parallels, we introduce the terminology

$h[.]$: Impulse response of the random process

$H(e^{j\omega})$: **Frequency response function** (FRF) of the process

Essentially, **any linear random process acts like a filter**

Filtering perspective: Example

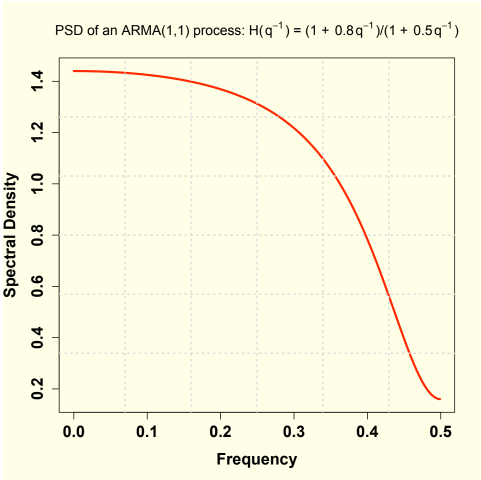
Example

An ARMA (1,1) process has the T.F.:

$$H(q^{-1}) = \frac{1 + 0.8q^{-1}}{1 + 0.5q^{-1}}$$

The p.s.d. is therefore proportional to

$$|H(e^{j\omega})|^2 = \frac{1.64 + 1.16\cos\omega}{1.25 + 0.5\cos\omega}$$



Spectral distribution function

Spectral densities can only be defined for random processes that are not periodic. However, the notion of a **spectral distribution function** can be applied to both classes of processes.

Spectral Distribution Function

The spectral distribution function of an aperiodic stationary stochastic process is defined as

$$\boxed{\Gamma(\omega) = \int_{-\pi}^{\omega} \gamma(\omega)\, d\omega} \quad \text{or} \quad \boxed{\gamma(\omega) = \frac{d\Gamma(\omega)}{d\omega}} \tag{5}$$

For periodic random processes, $\Gamma(\omega)$ is a staircase function with spikes at the frequencies.

The quantities $\Gamma(\omega)/\sigma^2$ and $\gamma(\omega)/\sigma^2$ are the **normalized spectral** distribution and density functions respectively

Unified Wiener-Khinchin theorem

Theorem (Khintchine, (1934), Wiener, (1930), and Wold, (1938))

A discrete sequence $\rho[l]$ is the auto-correlation function of a discrete-time stochastic process $v[k]$ if and only if there exists a function $F(\omega)$, such that

$$\rho[l] = \int_{-\pi}^{\pi} e^{j\omega l} dF(\omega) \qquad l \in \mathbb{Z} \tag{6}$$

where $F(\omega)$ has the properties of a (normalized) distribution function on the interval $(-\pi, \pi)$, i.e., $F(\omega)$ is right-continuous, non-decreasing, bounded on $[-\pi, \pi]$ and $F(-\pi) = 0$, $F(\pi) = 1$.

Proof is found in standard texts. See Brockwell and Davis, (1991) and Priestley, (1981).

W-K Theorem ...contd.

The function $F(.)$ in (6) is called the *normalized spectral distribution function*.

Comparing with earlier equations,

$$F(\omega) = \Gamma(\omega)/\sigma^2 \qquad \text{such that} \qquad F(\pi) = 1 \tag{7}$$

Spectral Factorization

Recall

For the causal linear time-series model

$$v[k] = \sum_{n=0}^{\infty} h[n]e[k-n] = H(q^{-1})e[k], \qquad \sum_{n=0}^{\infty} |h[n]| < \infty, \qquad e[k] \sim \text{WN}(0, \sigma_e^2) \tag{8}$$

Then, we know

$$\gamma(\omega) = \frac{\sigma_e^2}{2\pi}|H(e^{-j\omega})|^2 = \frac{\sigma_e^2}{2\pi}H(e^{-j\omega})H^{\star}(e^{-j\omega}) = \frac{\sigma_e^2}{2\pi}H(e^{-j\omega})H(e^{j\omega}) \tag{9}$$

Spectral Factorization

Spectral factorization is the inverse problem, as stated below.

Given a time-series with continuous, symmetric, non-negative spectral density $\gamma(\omega)$ that is integrable over $[-\pi, \pi]$ find a factorization of the form (9). From this viewpoint, $H(e^{-j\omega})$ is known as the **spectral factor**.

Why is spectral factorization important?

A few questions

Given a time-series, building a linear model (predictor) in (8) (or even its non-causal version) amounts to factorizing the spectral density as in (9)

Q: Under what conditions is it possible to obtain the factorization (9) and when is it **unique**? Are there any restrictions on $\gamma(\omega)$ or the spectral factor $H(e^{j\omega})$?

Recall the ACVGF, also called as the spectral density:

$$\gamma(z) = \sum_{l=-\infty}^{\infty} \sigma[l] z^{-l} \tag{10}$$

Clearly, $\gamma(\omega) = \gamma(z)|_{z=e^{-j\omega}}$.

A more general problem

Spectral factorization

Find σ^2 and $H(z)$ such that the spectral density $\gamma(z)$ in (10) can be factorized as

$$\gamma(z) = \frac{\sigma^2}{2\pi} H(z^{-1})H(z) \tag{11}$$

where

$$H(z) = \sum_{n=0}^{\infty} h[n]z^{-n}; \qquad H(z^{-1}) = \sum_{n=0}^{\infty} h[n]z^{n} \tag{12}$$

$H(z^{-1})$ is obtained by replacing every appearance of z in $H(z)$ with z^{-1}.

Remarks

1. The factorization in both forms (9) and (11), *is not unique*. If (σ_e^2, H) is a solution, then $(\alpha^2\sigma^2, H/\alpha), \quad \alpha \in \mathbb{R}$ is also a solution.
   To fix the non-uniqueness issue, we require that (recall Chapter 9)

$$h[0] = 1 \qquad \Longrightarrow \qquad H(0) = 1 \tag{13}$$

2. *Spectral factors can only be identified correctly up to a phase.* If $H(z)$ is a solution, then so is $H(z)e^{-D\omega}$. Nevertheless, spectral factorization guarantees the identification of a minimum-phase filter $H(z)$.
3. Thirdly, if $H(z)$ is a solution, then $H(z^{-1})$ is an equally likely solution.

Conditions for existence of factorization

1. A non-causal (two-sided), infinite-order, MA representation exists for all stationary processes that have continuous spectral densities. For proof, see Priestley, (1981).
2. We seek *causal* (one-sided) representations of the form (8), i.e., the IR sequence $\{h[.]\}$ is one-sided. This is guaranteed if the spectral density, satisfies the following **Paley-Wiener condition:**

$$\boxed{\int_{-\pi}^{\pi} \log \gamma(\omega)\, d\omega > -\infty} \tag{14}$$

   - Satisfied by most stationary processes with continuous PSD unless $\gamma(\omega)$ is zero over a continuous interval in frequency.
   - When $\gamma(\omega) \neq 0$ at almost all ω, the process is said to be **regular.**

Guaranteeing invertibility

Interestingly, the condition in (14) does not guarantee invertibility of the factor or an AR representation of the process.

An *invertible* spectral factor exists if and only if

> *The logarithm of the spectral density* $\log \gamma(z)$ *is analytic in the annulus*
> $\beta < |z| < 1/\beta, \;\beta < 1$

- Analytic $\Longrightarrow$ the function does not assume indeterminate values. This condition is a generalization of (14).
- Ensures that an AR representation of the process exists (see Priestley, (1981, Chapter 10)). Furthermore, it also leads to $h[0] = 1$ (the uniqueness issue)!

Putting together: Main result

Theorem (Spectral factorization)

Given a (discrete-time) stationary process whose spectral density is,

1. **Symmetric**: $\gamma(\omega) = \gamma(-\omega), \quad \omega \in [-\pi, \pi]$
2. **Non-negative**: $\gamma(\omega) \geq 0$, *(cannot be zero over an interval of frequencies)*
3. **Integrable**: $0 < \displaystyle\int_{-\pi}^{\pi} \gamma(\omega)\, d\omega < \infty$ *(finite variance)*
4. **Log-Analytic**: $\log(\gamma(z))$ *possesses derivatives of all orders in the annulus* $\beta < |z| < 1/\beta, \quad \beta < 1$

Putting together: Main result

Theorem (Spectral factorization ...contd.)

its spectral density function is factorizable as

$$\gamma(z) = e^{c_0} H(z^{-1}) H(z) \tag{15}$$

with $H(z^{-1})$ and $H(z)$ as defined in (12). Further,

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log(\gamma(\omega))\, d\omega, \qquad h[0] = 1, \qquad |\textit{zeros}(H(z))| < 1 \ \textit{invertible)} \tag{16a}$$

Rational spectral densities

When the spectral density $\gamma(\omega)$ is a rational function of trigonometric polynomials,

$$\gamma(\omega) = \frac{\alpha_0 + \sum_{r=1}^{M} \alpha_r \cos(r\omega)}{\beta_0 + \sum_{s=1}^{N} \beta_s \cos(s\omega)} \tag{17}$$

the solution to the factorization simplifies considerably because all ARMA(P, M) processes possess rational spectral densities of the form above.

Example

ARMA Model from Rational Spectral Density

Suppose a random process $v[k]$ is known to possess the spectral density

$$\gamma_{vv}(\omega) = 4\frac{1.09 + 0.6\cos\omega}{1.64 - 1.16\cos\omega}$$

By visual inspection, $\gamma_{vv}(\omega)$ can be factorized as

$$\gamma_{vv}(\omega) = 4\left(\frac{1 + 0.3e^{-j\omega}}{1 - 0.8e^{-j\omega}}\right)\left(\frac{1 + 0.3e^{j\omega}}{1 - 0.8e^{j\omega}}\right)$$

Example ... contd.

There are two solutions to the filter that generate $v[k]$, one which has zeros and poles inside the unit circle and the other which has them outside the unit circle.
We choose the one that is both causal and invertible.

$$v[k] = \frac{1 + 0.3q^{-1}}{1 - 0.8q^{-1}} e[k] \qquad e[k] \sim \text{WN}(0, 8\pi) \tag{18}$$

General scenario

Theorem

If γ is a symmetric, non-negative, continuous spectral density on $[-\pi, \pi)$, then for every $\epsilon > 0$, there exists a non-negative integer M and a polynomial

$$A(z) = \prod_{i=1}^{M} (1 - \eta_i^{-1} z) = 1 + a_1 z + a_2 z^2 + \cdots + a_p z^M \qquad |\eta_j| > 1, \;\; \forall j = 1, \cdots, M \tag{19}$$

with real-valued coefficients such that

$$|K|A(e^{-j\omega})|^2 - \gamma(\omega)| < \epsilon \qquad \forall \omega \in [-\pi, \pi] \tag{20}$$

where

$$K = \frac{1}{(1 + a_1^2 + a_2^2 + \cdots + a_M^2)} \int_{-\pi}^{\pi} \gamma(\omega) \, d\omega$$

To conclude

ARMA models can be used to model most linear stationary random processes with continuous spectral densities (to be precise, those satisfying the conditions listed in the main result).

- When the true process has rational spectral density, the ARMA model provides an exact representation.
- In other cases, an approximate model with an arbitrarily small degree of error can be constructed.

Cross-spectrum and Coherence

The cross-spectral density detects linear relationship between two series as the CCVF. Extending the W-K Theorem to the bivariate case, the cross p.s.d. of two random processes $y[k]$ and $u[k]$ is

$$\gamma_{yu}(\omega) = \text{DTFT}(\sigma_{yu}[l]) = \sum_{l=-\infty}^{\infty} \sigma_{yu}[l]e^{-j\omega l}$$

- It is a complex-valued quantity!
- $|\gamma_{yu}(\omega)|$ gives the strength of common power at that frequency
- $\angle\gamma_{yu}(\omega)$ (**phase**) is useful in estimating delays in the system
- Useful result: $\boxed{\gamma_{yu}(\omega) = H(e^{-j\omega})\gamma_{uu}(\omega)}$

Coherence

As with CCF, a normalized CPSD, known as **coherence function**, is used in practice:

$$\boxed{\kappa_{yu}(\omega) = \frac{\gamma_{yu}(\omega)}{\sqrt{\gamma_{yy}(\omega)\gamma_{uu}(\omega)}}}$$

Coherence

The magnitude of coherence function is **coherence**.

Coherence

Property of Coherence

A system is LTI if and only if coherence is unity at all frequencies

$$\kappa_{yu}(\omega) = 1 \ \forall \omega$$

Whenever $\kappa_{yu}(\omega) \neq 1$, same conclusions can be drawn as with correlation:

- The two series are probably non-linearly related at that frequency
- The "true" series are linearly related, but noise in the data could be masking the linear relationship.

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