CH5350: Applied Time-Series Analysis

Seasonal ARIMA models

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Seasonality

Seasonality is a term, predominantly used in econometrics, to describe the **periodic** or **repetitive** effect of seasonal phenomenon on the time-evolution of a given series. It results in a "periodic" *non-stationarity* in the random process.

Examples:

- Recurring effect of festivals on sales of clothes in a store
- Repetitive effect of end-of-year vacation on number of air passengers

Seasonality is always of a **fixed** period.

Seasonality ... contd.

Definition

A series $v[k]$ that has no trend is said to be **seasonal** if its expected value is not constant, but varies in a periodic manner,

$$E(v[k]) = E(v[k+S]) \quad (1)$$

where S is the **seasonality** of $v[k]$.

Seasonality can be of two types:

1. **Deterministic:** Seasonal component is a periodic deterministic wave.
2. **Stochastic:** Seasonal component is a deterministic periodic plus zero-mean stochastic signal.

Cyclicity

Cyclicity, on the other hand, also being related to repetitive behaviour, however, is not necessarily of fixed period.

- It is natural to think of seasonal and cyclic characteristics as identical - after all, cycles imply periodic or oscillatory behaviour (in physics)
- However, unfortunately, due to certain misleading nomenclature, cyclicity in econometrics is considered different from seasonality in the sense that it can have varying periods.

Simple model for seasonality

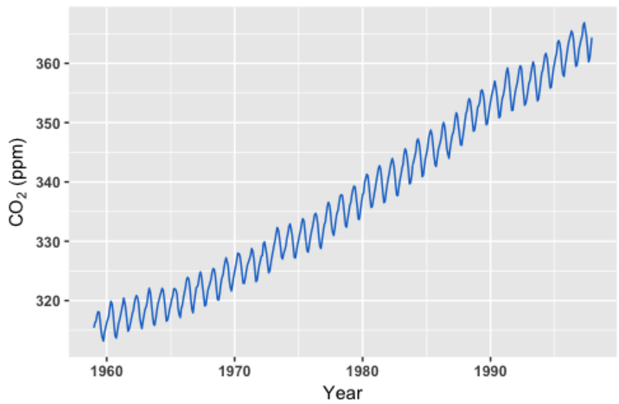
The simplest model for seasonality is that of a **deterministic** periodic wave plus stationary signal

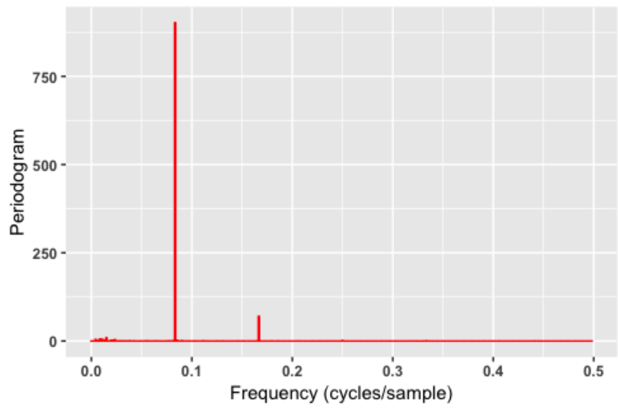
$$v[k] = x[k] + w[k] \qquad (2)$$

where $x[k+S] = x[k]$ is a deterministic waveform and $w[k]$ stems from an ARMA process.

- There can exist more than one periodic component.
- The period S may be known a priori (from domain knowledge) or can be determined through Fourier analysis (periodogram).

Example: Seasonality





Series has trend, seasonality and possibly a stationary component. Data is obtained on a monthly basis.

Periodogram (*after trend removal*) shows the presence of two seasonal components with periodicities $S_1 = 12$ ($f_1 = 1/12$) and $S_2 = 6$ ($f_2 = 1/6$) months.

Seasonality: Scenario 2

A slightly more complicated model for seasonality is

$$v[k] = x[k] + w[k] \tag{3}$$
$$x[k] = \mu_S[k] + \eta[k] \tag{4}$$

where $\mu_S[k]$ is a periodic deterministic signal (of period S) and $\eta[k]$ is a zero-mean stationary process.

Verify that the condition for seasonality (in (1)) of $v[k]$ still holds.

This model can be, in fact, simplified to the earlier simpler model, by pooling $w[k]$ and $\eta[k]$.

Seasonality: Scenario 3

In this case, the seasonal component is generated by the model

$$x[k] = x[k-S] + \eta[k] \tag{5}$$

where $\eta[k]$ is a zero-mean stationary process. This can be thought of as an **integrating effect** at the seasonal scale.

Introduce the seasonal differencing operator $\nabla_S = 1 - q^{-S}$. Then, the seasonality in (5) can be modelled as

$$\nabla_S x[k] = \eta[k] \tag{6}$$

In fact, we can apply the ∇_S operator to $v[k]$ for the earlier two scenarios so as to obtain

A generalizing model

A generalization of the three previous scenarios is that there are essentially two sub-phenomena that constitute a given random process. These two phenomena are (i) one that operates at the seasonal scale S and (ii) another that evolves at the regular observation scale (sampling interval).

An **additive** model represents $v[k]$ assumes a *parallel* configuration, while a **multiplicative** model considers a *series* arrangement of these effects.

- The multiplicative model can be viewed as if the noise driving the seasonal phenomenon is an ARIMA process at the regular scale.

SARIMA model

When the seasonal and regular effects are simultaneously present in a random process and interacting with each other, a **SARIMA** model is born.

SARIMA model

A stochastic process $v[k]$ is said to follow a (multiplicative) SARIMA(P,d,M)$_s$ $\times$(P,d,M) model if it can be represented as

$$\nabla_S^{d_s}(q^{-S})\nabla^d(q^{-1})v[k] = \frac{C_s(q^{-S})C(q^{-1})}{D_s(q^{-S})D(q^{-1})}e[k] \qquad e[k] \sim \text{WN}(0, \sigma_e^2) \tag{7}$$

where $C_s(q^{-S})$ and $D(q^{-S})$ are finite-order **monic** polynomials (of degree P_s and M_s, respectively) in the seasonal backward shift operator, q^{-S}, where d_s is the degree of seasonal differencing.

Example: SARIMA Model

SARIMA(1,1,0)$_4$×(1,0,1) model

The TF operator of a SARIMA(1,1,0)$_4$×(1,0,1) process $v[k] = H(q^{-1})e[k]$ is given by:

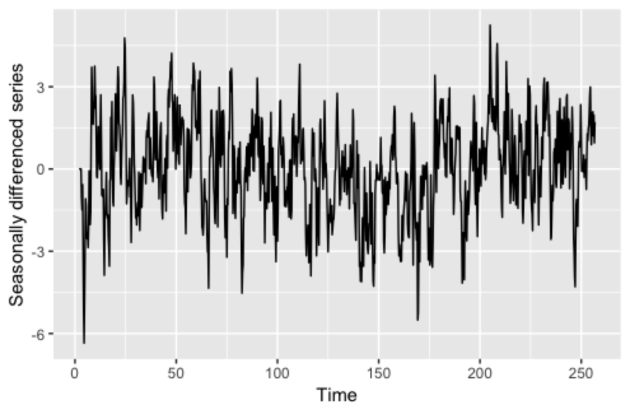
$$H(q^{-1}) = \frac{1}{(1-q^{-4})(1+d_{4s}q^{-4})}\left(\frac{1+c_1q^{-1}}{1+d_1q^{-1}}\right) \tag{8}$$

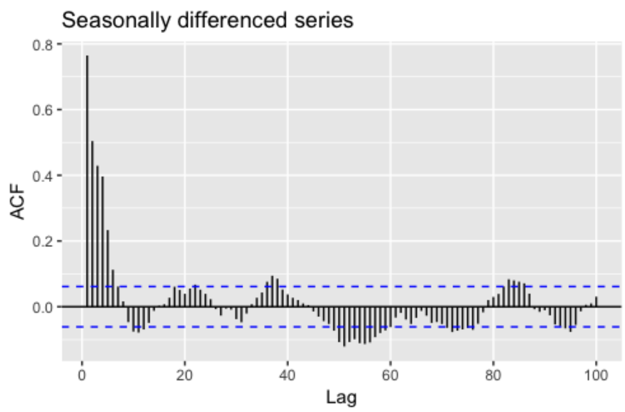
Therefore, $w[k] \triangleq \nabla_4 v[k]$ is described by the DE:

$$w[k] + d_1w[k-1] + d_{1s}w[k-4] + d_{4s}d_1w[k-5] = e[k] + c_1e[k-1] \tag{9}$$

Notice that essentially $w[k]$ follows a difference equation of high order but parametrized in a particular manner.

Simulated series





Seasonally differenced series $w[k] = \nabla_4 v[k]$.

$c_1 = 0.6,\ d_1 = 0.4,\ d_{1s} = 0.3$

ACF of the **seasonally differenced** series shows a decaying correlation (at the observation scale) with "periodic" pattern.