CH5350: Applied Time-Series Analysis

Fourier Transforms for Deterministic Signals

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Recap

- Correlation structure (predictability) of stationary processes is characterized by the auto-covariance function.
- A linear random process representation can be constructed only if the **spectral density** exists and if it satisfies the Wiener-Paley condition.
- Parametrization of the IR sequence (or the ACVF) of a linear random process leads to MA(M), AR(P) or ARMA processes.
- Every linear random process is a **filter.**
- Trend non-stationarities are handled by applying suitable **filters**.
- Seasonalities are detected by peaks in **spectral** plots.

Motivation

- What is meant by **spectral density?**
- Stationary processes with periodic ACVFs do not have a linear convolution form - how do we describe them?
- Does there exist a connection between the time-domain and frequency-domain characteristics of a (random) signal?
- Is there a method to detect periodicities embedded in noise?
- What does it mean to construct a spectral representation for a stationary random process?

Frequency-domain analysis

Frequency-domain characterizations of processes, also known as **spectral representations** offer a powerful framework for both a theoretical and practical analysis of random processes

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The term "spectral representation" stems from the term "spectrum" which, in signal analysis stands for a function of energy/power in the frequency domain.

What does a spectral representation mean?

Spectral representation provides a decomposition of the power / energy of the process in the frequency-domain.

In understanding this topic, we shall seek answers to several questions:

- What is the mathematical definition of spectrum?
- Is there a difference between energy and power of a signal?
- What is the utility of a spectral decomposition?
- What does spectral representation mathematically look like?
- Can any random process be given a spectral representation?
- What are the connections between time-domain and (frequency) spectral representations of a process?
- ...

Fourier transform is the main tool

The main tool for carrying out a frequency-domain analysis of signals / processes is the **Fourier transform.**

- In order to understand the various aspects of Fourier transforms and its applications to signal analysis, it is useful to first gain an understanding of how **deterministic processes** are treated in the frequency domain.
- Further, we shall categorize deterministic signals into four classes, namely, **continuous-time and discrete-time**, periodic and aperiodc signals.
- It is important to first understand quantities such as **energy**, **power** and their **densities** in the context of signal analysis.

Energy signal

Energy

The energy of a continuous-time signal $x(t)$ and a discrete-time signal $x[k]$ are, respectively, defined as,

$$\boxed{E_{xx} = \int_{-\infty}^{\infty} |x(t)|^2 \, dt \; ; \qquad E_{xx} = \sum_{-\infty}^{\infty} |x[k]|^2} \tag{1}$$

A signal with finite energy, i.e., $0 < E_{xx} < \infty$ is said to be an *energy signal*[a].

[a]The squared modulus is introduced to accommodate complex-valued signals.

Examples: exponentially decaying signals, all finite-duration bounded amplitude signals

Power signal

Power

The *average* power of a continuous-time signal $x(t)$ and a discrete-time signal $x[k]$ are, respectively, defined as,

$$\boxed{P_{xx} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt \; ; \qquad P_{xx} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{k=N} |x[k]|^2} \tag{2}$$

A signal with finite power, i.e., $0 < P_{xx} < \infty$ is said to be a *power signal*.

Power signal ... contd.

Examples: periodic signals, random signals

All finite-duration (and amplitude) signals have $P_{xx} = 0$. *In general, any energy signal is not a power signal and vice versa*. However, it is possible that a signal is neither an energy nor a power signal.

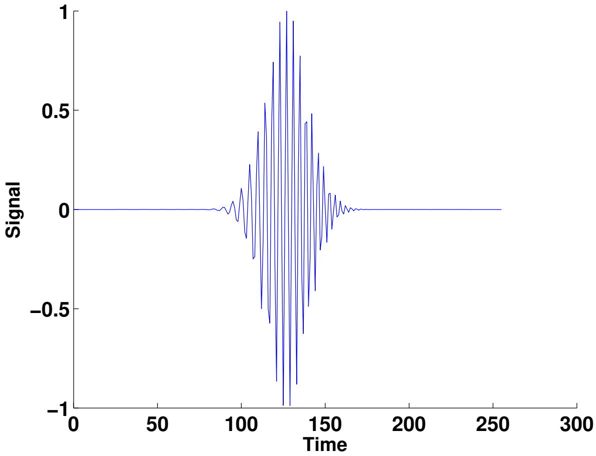
Energy Density

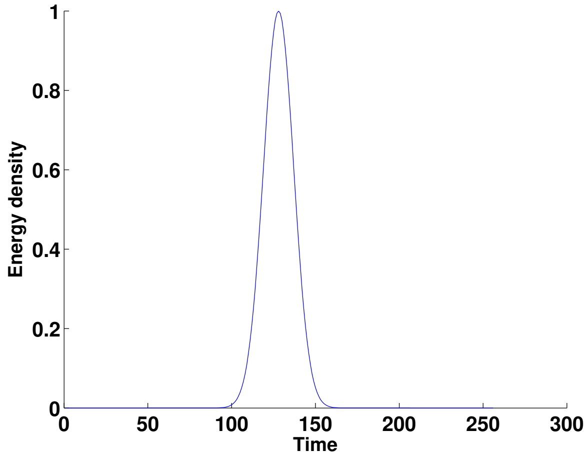
Equation (1) gives rise to the idea of an *energy density* in *time*. Drawing analogies with probability density function and densities in mechanics, the quantity

$$S_{xx}(t) = |x(t)|^2 \tag{3}$$

is termed as the energy density per unit time. It can also be thought of as an "instantaneous" power.

Energy density: Example





Power Density

Similarly, the *power density in time* can be defined as

$$\gamma_{xx}(t) = \frac{|x(t)|^2}{T} \tag{4}$$

- **For the discrete-time case, the energy and power density in time are not defined** since the time domain is not a continuum. The distribution functions exist nevertheless.

Power density ...contd.

On the other hand, we can think of energy and power densities of d.t. signals in a **transform domain**, provided that the new domain is **continuous** and that the transform is energy / power preserving.

This is the basis for defining spectral densities of c.t. and d.t. signals in the Fourier (frequency) domain.

The energy / power densities in frequency domain share a strong connection with the time-domain characteristics (properties) of the signal, specifically the **covariance functions.**

Auto-covariance functions

The ACVFs of periodic and (finite-energy) aperiodic deterministic signals are, respectively,

$$\boxed{\sigma_{x_p x_p}[l] = \frac{1}{N_p}\sum_{k=0}^{N_p-1} x_p[k]x_p[k-l]; \qquad \sigma_{xx}[l] = \sum_{k=-\infty}^{\infty} x[k]x[k-l]} \tag{5}$$

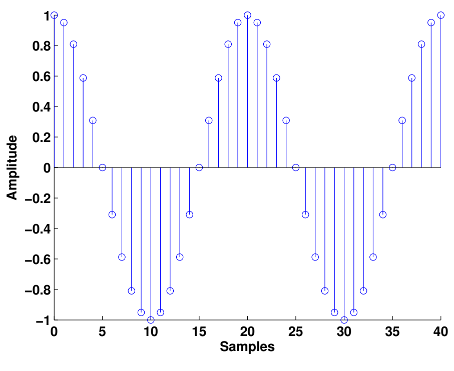
Auto-covariance functions ... contd.

- **Unlike the CCVF**, the ACVF is a **symmetric** function.
- As before, normalized versions can be defined to obtain the respective ACFs.
- **The ACF inherits the characteristics of the signal.** For instance, **the ACVF of a periodic signal is also periodic with the same period.**

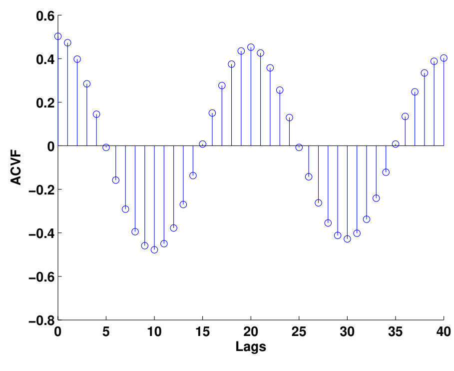
$$\boxed{\sigma_{x_p x_p}[l + N_p] = \sigma_{x_p x_p}[l]} \tag{6}$$

Example: Periodic signal

$$x_p[k] = \cos(2\pi f k) \qquad \Longrightarrow \qquad \sigma_{x_p x_p}[l] = \frac{1}{2}\cos(2\pi f l)$$



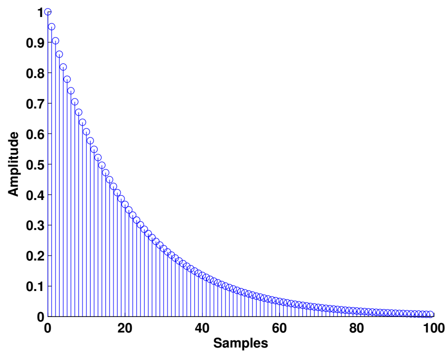
(a) Snapshot of the cosine



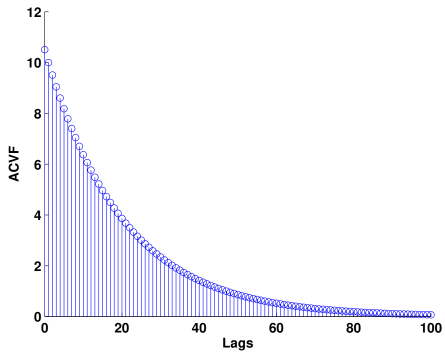
(b) ACVF of the cosine

Example: Aperiodic signal

$$x[k] = \begin{cases} e^{\alpha k}, & k \geq 0 \\ 0, & k < 0 \end{cases} \quad , \alpha < 0 \qquad \Longrightarrow \qquad \sigma_{xx}[l] = \frac{e^{\alpha l}}{1 - e^{2\alpha}}$$



(c) Exponential signal



(d) ACVF of signal

Cross-covariance function

The **cross-covariance function (CCVF) is a measure of the linear dependence** between two time-lagged (random or deterministic) signals.

- Based on the notion of **covariance**, a quantity that measures the linear dependence between two zero-lagged deterministic signals (or two random variables).
- A normalized version known as, **cross-correlation function** (CCF), is more suitable for analysis since it is invariant to the choice of units (for signals).

Caution: It is a common practice in signal processing to use the alternative terms cross-correlation and normalized cross-correlation, for CCVF and CCF, respectively.

CCVF for periodic signals

The cross-covariance function between two *zero-mean, periodic deterministic* signals $x_p[k]$ and $y_p[k]$ with a *(least) common period* N_p is defined as

$$\boxed{\sigma_{x_p y_p}[l] = \frac{1}{N_p} \sum_{k=0}^{N_p-1} x_p[k] y_p[k-l]} \tag{7}$$

CCVF for periodic signals

- The (normalized) cross-correlation function is defined as

$$\boxed{\rho_{x_p y_p}[l] = \frac{\sigma_{x_p y_p}[l]}{\sqrt{\sigma_{x_p x_p}[0] \sigma_{y_p y_p}[0]}}} \tag{8}$$

- Observe that by setting $x_p = y_p$ and $l = 0$ in (7), we obtain the **average power** of the periodic signal.

CCVF for aperiodic signals

The cross-covariance function between two *aperiodic deterministic, energy* signals $x[k]$ and $y[k]$ is defined as

$$\sigma_{xy}[l] = \sum_{k=-\infty}^{\infty} x[k]y[k-l] \tag{9}$$

CCVF for aperiodic signals

As before,

- The (normalized) cross-correlation function is defined as

$$\boxed{\rho_{xy}[l] = \frac{\sigma_{xy}[l]}{\sqrt{\sigma_{xx}[0]\sigma_{yy}[0]}}} \tag{10}$$

- Observe that by setting $x = y$ and $l = 0$ in (9), we obtain the **energy** of the aperiodic signal.

Properties and uses of CCVF

The CCVF has a few, but very useful, properties and is one of the most widely used time-domain signal analysis tools:

- The CCVF measures the linear dependence between $x[k]$ and time-shifted $y[k]$ (by l samples). This property is used in testing linear relationships between two signals.
- It is **asymmetric**, i.e., $\sigma_{xy}[l] \neq \sigma_{xy}[-l]$ (Why?).
  The asymmetric property is used in estimating time-delays between signals (by searching for peaks in the CCFs).
- The CCVF specializes to auto-covariance function (ACVF) for univariate signals, which is a widely used tool for **periodicity detection** and **echo cancellation**.

Frequency-domain (Fourier) Descriptions (Series & Transforms)

Transform: Synthesis and Analysis

Every transform consists of a

i. **Synthesis equation:** Mathematical imagination of how the signal is possibly constructed from a family of *building blocks* (atoms).

ii. **Analysis equation:** Allows us to determine "which" members of the family have participated in the signal synthesis through a decomposition.

Qs: Which atoms (functions)? Is the decomposition unique, Is perfect recovery possible?

Transforms: Analysis and Filtering

The motivation of every transform is ease of analysis in the new domain.

The type of transform and approach depends on the objective:

- **Analysis:** Starting with **signal decomposition**, one proceeds to **energy / power decomposition**.
- **Filtering:** Signal is decomposed, operation(s) is / are performed in the transform domain and finally (a modified signal is) reconstructed using the synthesis equation.

A mix of both may be required in several applications.

Fixed vs. Adaptive Basis

When the building blocks are fixed a priori and independent of the signal, the transform is said to be built on **fixed basis.**

`Examples: Fourier, Wavelet transforms.`

On the other hand, when these building blocks are derived from data, the transform is said to work with **adaptive basis.**

`Examples: Wigner-Ville distributions, Principal component analysis.`

General ideas

- **Idea**: Breakdown a signal into weighted combinations of sinusoids with different frequencies:
  - Similar to expressing signals as a combination of impulses
- **Significance**
  - Weights or coefficients are in general complex
  - Magnitude of weights give the energy or power of that frequency component in the signal - Spectral Analysis
  - Angle of coefficients give how much each sinusoid (at that frequency) in the signal is aligned with respect to the basis sinusoids. Useful in time-delay or lag estimation.

Correlation perspective

Every transform of a signal can be viewed as a correlation of that signal with the analyzing function. The coefficient of transform is the "amount" of correlation or similarity of that signal with the analyzing (basis) function.

Remarks

- The breaking down of a signal is equivalent to finding the best set of sinusoids that can explain the pattern in that signal.
- In Fourier analysis, each analyzing function is a (complex) sinusoid of a certain frequency.
- The coefficient at each frequency is a measure of similarity between the signal and the sinusoid
- The complex sinusoid is chosen so as to capture shift and magnitude in a convenient manner

Continous-time Fourier Series (CTFS)

Continuous-time periodic signals: Synthesis equation

Idea: A continuous-time periodic signal with fundamental period $T_0 = 1/F_0$ is expressed as a (linear) weighted combination of (positive and negative) harmonics:

$$\boxed{x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n F_0 t}} \qquad \text{(Fourier Series)} \qquad (11)$$

Note: The summation in (11) includes both negative and positive frequencies!

Continuous-time Fourier series: Analysis equation

The coefficient c_n, is in general **a complex quantity**, and is calculated as:

$$\boxed{c_n = \frac{1}{T_p}\int_{T_p} x(t)e^{-j2\pi nF_0t}\,dt = \frac{\langle x(t), e^{j2\pi nF_0t}\rangle_{[0,T_p]}}{\langle e^{j2\pi nF_0t}, e^{j2\pi nF_0t}\rangle_{[0,T_p]}}} \qquad (12)$$

- Generally useful in theoretical analysis of signals and systems

Continuous-time Fourier series

Variant	Synthesis / analysis	Parseval's relation (power decomposition) and signal requirements
Fourier Series	$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi nF_0t}$ $c_n \triangleq \frac{1}{T_p}\int_{T_p} x(t)e^{-j2\pi nF_0t}\,dt$	$P_{xx} = \frac{1}{T_p}\int_0^{T_p} \lvert x(t)\rvert^2\,dt = \sum_{n=-\infty}^{\infty} \lvert c_n\rvert^2$ $x(t)$ is periodic with fundamental period $T_p = 1/F_0$

Why do negative frequencies come in?

- The need for including negative frequencies is purely mathematical. Consider, for e.g.,

$$\sin(2\pi F_0 t) = \frac{1}{2j}(e^{j2\pi F_0 t} - e^{-j2\pi F_0 t})$$

  Observe that two exponentials, one with a positive frequency F_0 and the other with a negative frequency $-F_0$ are required to explain a sinusoid
- The corresponding coefficients are c_1 $(k = 1)$ and c_{-1} are $\frac{1}{2j}$ and $-\frac{1}{2j}$
- In general, Fourier series / transform involves expressing any signal as addition and subtraction of cosines / sines.

Power spectrum

Fourier series (and transform) is concerned with a **signal decomposition**, but gives rise to a more important result - the power (spectral) decomposition of the signal.

- A periodic signal has infinite energy, but finite power given by

$$P_{xx} = \frac{1}{T_p}\int_0^{T_p} |x(t)|^2 \, dt$$

Using the signal decomposition in a Fourier series, we can break up the average power into contributions from respective frequencies

Power spectral decomposition of CT periodic signals

The average power can be broken up as

$$\boxed{P_{xx} = \frac{1}{T_p}\int_0^{T_p} |x(t)|^2 \; dt = \sum_{n=-\infty}^{\infty} |c_n|^2} \qquad \text{(Parseval's relation)} \qquad (13)$$

- $P_n = |c_n|^2$ is the contribution by the n^{th} harmonic and hence known as the *power spectrum*.
- **The power in a periodic signal exists only at discrete frequencies. Hence the power spectrum is also known as line spectrum.**

Power and phase spectrum: Remarks

- Since $c_n = |c_n|e^{j\theta_n}$, c_0 represents the average component of the signal
- The power spectral density plot is independent of (or blind to) the phase
  - *Two signals having two different phases but same strengths will have identical power spectral densities*
- For a real-valued signal, $c_n^* = c_{-n} \implies$ Power spectrum of any measurement is symmetric
- As $T_p \to \infty$, $x(t)$ becomes an *aperiodic* signal and frequency spacing tends to zero.
- **Phase**: $\theta_n = \angle c_n$. A plot of θ_n vs. n shows how each frequency component is aligned w.r.t the basis functions

Example

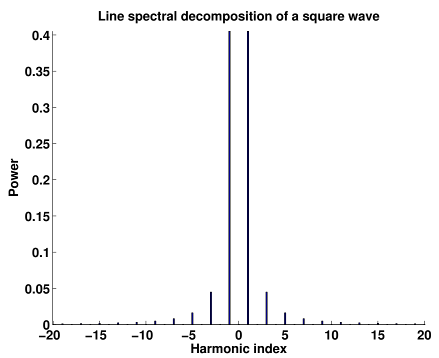
The Fourier series representation of the periodic square wave

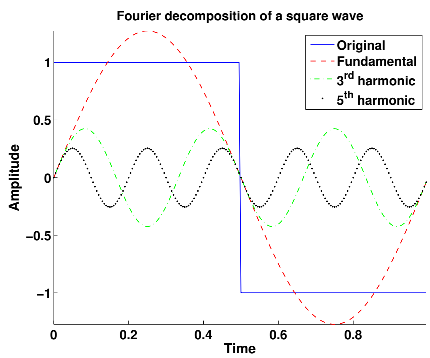
$$x(t) = \begin{cases} 1, & 0 \le t \le 1/2 \\ -1, & 1/2 < t \le 1 \end{cases} \tag{14}$$

with period $T_p = 1$ is given by the coefficients

$$\begin{aligned} c_n = \frac{1}{T_p}\int_0^1 x(t)e^{-j2\pi nt}\,dt &= \int_0^{1/2} e^{-j2\pi nt}\,dt - \int_{1/2}^{1} e^{-j2\pi nt}\,dt \\ &= j\sin\left(\frac{n\pi}{2}\right)\sin c\left(\frac{n\pi}{2}\right)e^{-jn\pi} \end{aligned}$$

Example contd.





- Line spectrum is plotted as a function of the harmonic index. Observe that it is symmetric and that only odd harmonics contribute to the signal.
- The signal decomposition shown above is purely mathematical.

Existence of Fourier series

- The coefficients c_n exist iff the signal $x(t)$ is absolutely convergent in $[0, T_p]$, i.e., $x(t) \in L^1(0, T_p)$.
- On the other hand, the series converges to $x(t)$ if it is continuous and of bounded variation in $[0, T_p]$. For discontinuous signals with finite extrema and finite number of discontinuities, the series converges to the average value of the left and right limits . This is termed as the *Gibbs phenomenon*
- Sufficiency conditions for any function $f(t)$ to possess a Fourier series expansion was established by Dirichlet and are popularly known as *Dirichlet conditions* (Priestley, 1981).

Existence of Fourier series

- A weaker requirement is that $x(t)$ has a finite 2-norm in the interval $(0, T_p)$. Then, the summation

$$x_M(t) = \sum_{n=-M}^{M} c_n e^{-j2\pi n F_0 t} \tag{15}$$

converges to $x(t)$ in the MS sense, i.e.,

$$\lim_{M \longrightarrow \infty} \int (x(t) - x_M(t))^2 \, dt = 0 \tag{16}$$

Continuous-time Fourier transform

Variant	Synthesis / analysis	Parseval's relation (energy decomposition) and signal requirements
Fourier Transform	$x(t) = \int_{-\infty}^{\infty} X(F) e^{j2\pi Ft}\, dF$ $X(F) \triangleq \int_{-\infty}^{\infty} x(t) e^{-j2\pi Ft}\, dt$	$E_{xx} = \int_{-\infty}^{\infty} \|x(t)\|^2\, dt = \int_{-\infty}^{\infty} \|X(F)\|^2\, dF$ $x(t)$ is aperiodic; $\int_{-\infty}^{\infty} \|x(t)\|\, dt < \infty$ or $\int_{-\infty}^{\infty} \|x(t)\|^2\, dt < \infty$ (finite energy, weaker requirement)

Energy spectral decomposition

The signal decomposition by Fourier transform can be shown to yield an energy decomposition in the frequency domain by virtue of Parseval's result.

$$\boxed{E_{xx} = \int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} |X(F)|^2\, dF \qquad \text{(Energy decomposition)}} \qquad (17)$$

Energy spectral density

- Thus, **energy is preserved by the transform**. A more general result is the preservation of inner products.
- The quantity $|X(F)|^2$ is a continuous function of the frequency and can be given the interpretation of an **energy spectral density**.
- Alternatively, $|X(F)|^2 dF$ measures the energy contributions of the frequency components within the band $(F, F + dF)$ to the total energy of the signal.

Example 1: Finite duration pulse

The Fourier transform of the **finite duration** rectangular pulse signal

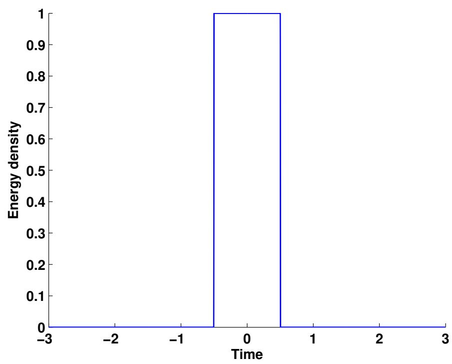
$$x(t) = A\Pi\left(\frac{t}{T}\right) = \begin{cases} A & |t| < T/2 \\ 0 & \text{otherwise} \end{cases}$$

is given by

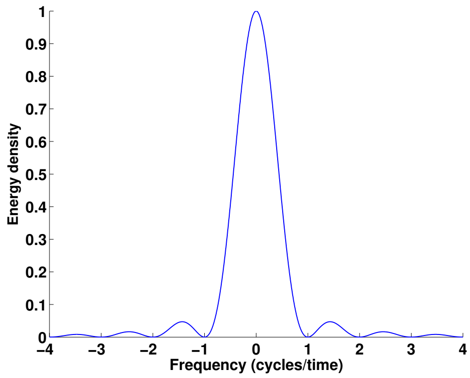
$$X(F) = \int_{-T/2}^{T/2} Ae^{-j2\pi Ft}\, dt = A\left(\left.\frac{e^{-j2\pi Ft}}{-j2\pi F}\right|_{-T/2}^{T/2}\right) = AT\frac{\sin(\pi FT)}{\pi FT} = AT\text{sinc}(\pi FT)$$

Thus, **the finite-duration pulse has an infinitely long Fourier transform.**

Example 1 contd.



(e) Energy density in time



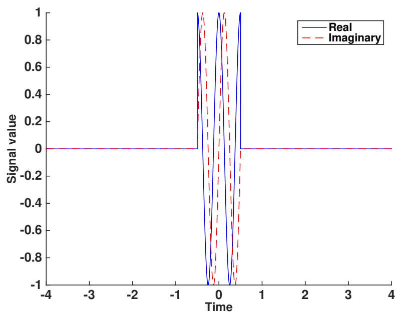
(f) Energy spectral density

Finite-duration signal has an infinitely-spread energy density in frequency.

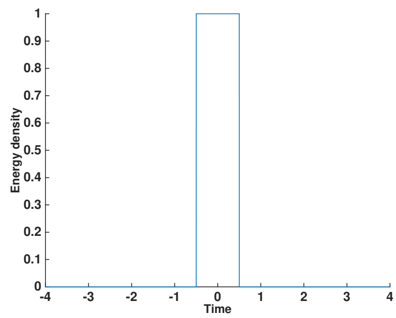
Example 2: Finite duration complex sine

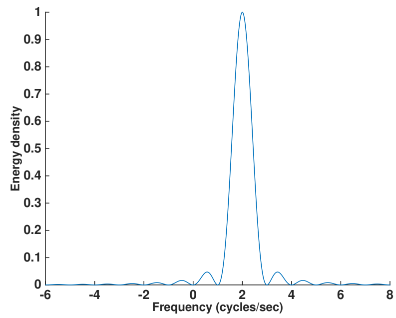
When $x(t) = \begin{cases} Ae^{j2\pi F_0 t} & |t| < T/2 \\ 0 & \text{otherwise} \end{cases}$

Therefore, $X(F) = AT\text{sinc}(\pi(F - F_0)T)$

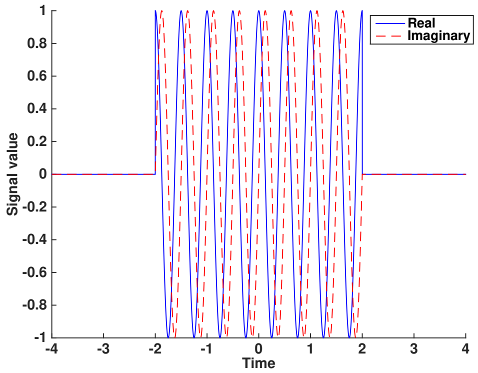


$F_0 = 2$ Hz$, T = 1, A = 1$

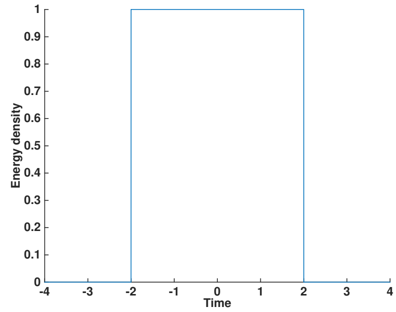


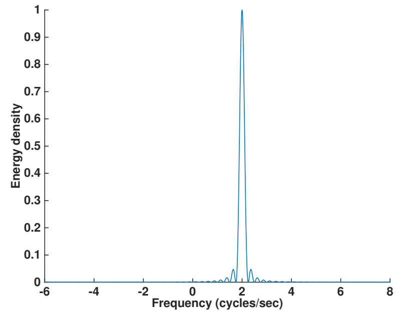


Longer finite-duration sine wave



$$F_0 = 2\,\text{Hz}, T = 4, A = 1$$





- **Longer duration** results in **narrower frequency spread** (bandwidth)

Duration and Bandwidth are tied together

All finite-duration signals have Fourier transforms that are infinitely long and vice versa. The fundamental **duration-bandwidth principle** places a lower bound on the product of the energy spreads in both domains

$$\sigma_t^2 \sigma_F^2 \geq 1/4 \tag{18}$$

where the spreads σ_t^2 and σ_F^2 are the second-order central moments of the energy densities in time and frequency, respectively (Cohen, 1994)

- The quantities σ_t and σ_F are known as the *duration* and *bandwidth*, respectively. This result has profound implications in the joint time-frequency analysis of signals.

Fourier-Stieltjes transform

The Fourier-Stieltjes transform fuses the Fourier series and transform into a single integral.

The basic idea is to re-write the synthesis equation in CTFT by introducing $dX(F) = X(F)dF$ as

$$\boxed{x(t) = \int_{-\infty}^{\infty} e^{j2\pi Ft} dX(F)} \tag{19}$$

Equation (19) is known as **Fourier-Stieltjes transform.**

Fourier-Stieltjes transform

In order to accommodate periodic functions, i.e., the Fourier series, we allow $dX(F)$ to be piecewise continuous, specifically, an impulse train function so that

$$dX(F) = \begin{cases} c_n, & F = F_n, \ n \in \mathbb{Z} \\ 0, & \text{elsewhere} \end{cases}$$

It facilitates frequency-domain representations for signals (functions) that are neither periodic nor absolutely integrable, but have bounded amplitudes, e.g., random signals.

Discrete-time Fourier Series (DTFS)

Opening remarks

The Fourier series representation for discrete-time signals has some similarities with that of continuous-time signals. Nevertheless, **certain differences exist:**

- The **period** of a discrete-time signal **is expressed in samples.**
- Discrete-time signals are unique over the frequency range $f \in [-0.5, 0.5)$ or $\omega \in [-\pi, \pi)$ (or any interval of this length).
- A discrete-time signal of fundamental period N can consist of frequency components $f = \frac{1}{N}, \frac{2}{N}, \cdots, \frac{(N-1)}{N}$ besides $f = 0$, the DC component
  - Therefore, the Fourier series representation of the **discrete-time periodic** signal contains only N complex exponential basis functions.

Fourier series for d.t. periodic signals

Given a periodic sequence $x[k]$ with period N, the Fourier series representation for $x[k]$ uses N harmonically related exponential functions

$$e^{j2\pi kn/N}, \quad k = 0, 1, \cdots, N-1$$

The Fourier series is expressed as

$$\boxed{x[k] = \sum_{n=0}^{N-1} c_n e^{j2\pi kn/N}} \tag{20}$$

Discrete-time Fourier Series

Variant	Synthesis / analysis	Parseval's relation (power decomposition) and signal requirements
Discrete-Time Fourier Series	$x[k] = \sum_{n=0}^{N-1} c_n e^{j2\pi kn/N}$ $c_n \triangleq \frac{1}{N} \sum_{k=0}^{N-1} x[k] e^{-j2\pi kn/N}$	$P_{xx} = \frac{1}{N} \sum_{k=0}^{N-1} \lvert x[k]\rvert^2 = \sum_{n=0}^{N-1} \lvert c_n\rvert^2$ $x[k]$ is periodic with fundamental period N

Example: Periodic pulse

The discrete-time Fourier representation of a periodic signal $x[k] = \{1, 1, 0, 0\}$ with period $N = 4$ is given by,

$$c_n = \frac{1}{4}\sum_{k=0}^{3} x[k]e^{-j2\pi kn/4} = \frac{1}{4}(1 + e^{-j2\pi n/4}) \qquad n = 0, 1, 2, 3$$

This gives the coefficients

$$c_0 = \frac{1}{2}; \qquad c_1 = \frac{1}{4}(1 - j); \qquad c_2 = 0; \qquad c_3 = \frac{1}{4}(1 + j)$$

Observe that $c_1 = c_3^\star$.

Power spectrum and auto-covariance function

The power spectrum of a discrete-time periodic signal and its auto-covariance function form a Fourier pair.

$$P_{xx}[n] = \frac{1}{N}\sum_{l=0}^{N-1} \sigma_{xx}[l]e^{-j2\pi ln/N} \tag{21a}$$

$$\sigma_{xx}[l] = \sum_{l=0}^{N-1} P_{xx}[n]e^{j2\pi ln/N} \tag{21b}$$

DISCRETE-TIME FOURIER TRANSFORM (DTFT)

Opening remarks

- The discrete-time aperiodic signal is treated in the same way as the continuous-time case, i.e., as an extension of the DTFS to the case of periodic signal as $N \to \infty$.
- Consequently, the frequency axis is a **continuum**.
- The synthesis equation is now an integral, but still restricted to $f \in [-1/2, 1/2)$ or $\omega \in [-\pi, \pi)$.

Discrete-Time Fourier Transform

Variant	Synthesis / analysis	Parseval's relation (energy decomposition) and signal requirements
Discrete-Time Fourier Transform	$x[k] = \int_{-1/2}^{1/2} X(f)e^{j2\pi fk}\, df$ $X(f) \triangleq \sum_{k=-\infty}^{\infty} x[k]e^{-j2\pi fk}$	$E_{xx} = \sum_{k=-\infty}^{\infty} \|x[k]\|^2 = \int_{-1/2}^{1/2} \|X(f)\|^2\, df$ $x[k]$ is aperiodic; $\sum_{k=-\infty}^{\infty} \|x[k]\| < \infty$ or $\sum_{k=-\infty}^{\infty} \|x[k]\|^2 < \infty$ (finite energy, weaker requirement)

Example: Discrete-time impulse

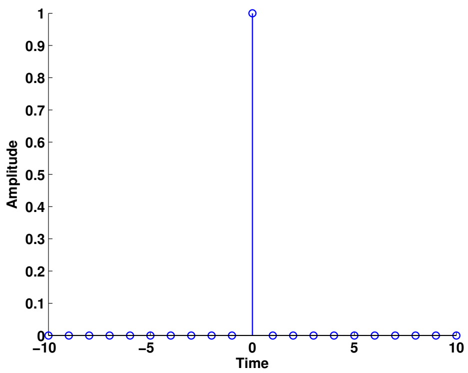
The Fourier transform of a discrete-time impulse $x[k] = \delta[n]$ (Kronecker delta) is

$$X(f) = \mathcal{F}\{\delta[n]\} = \sum_{k=-\infty}^{\infty} \delta[k]e^{-j2\pi fk} = 1 \quad \forall f \tag{22}$$

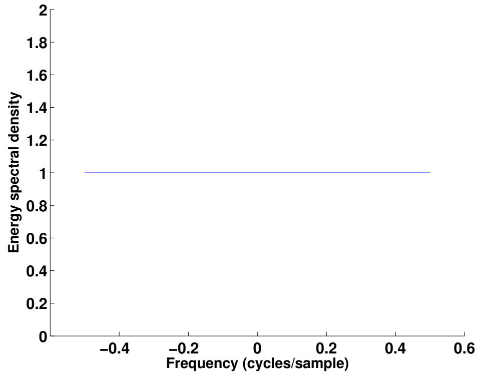
giving rise to a uniform energy spectral density

$$S_{xx}(f) = |X(f)|^2 = 1 \quad \forall f \tag{23}$$

Example: Discrete-time impulse



(g) Finite-duration pulse



(h) Energy spectral density

Example: Discrete-time finite-duration pulse

Compute the Fourier transform and the energy density spectrum of a finite-duration rectangular pulse

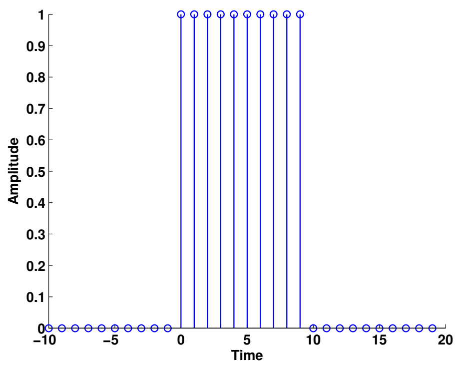
$$x[k] = \begin{cases} A, & 0 \le k \le L-1 \\ 0 & \text{otherwise} \end{cases}$$

Solution: The DTFT of the given signal is

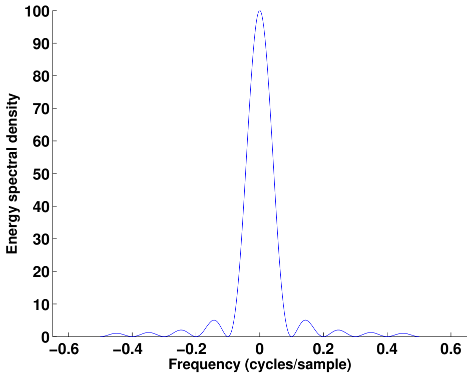
$$X(f) = \sum_{k=-\infty}^{\infty} x[k]e^{-j2\pi fk} = \sum_{k=0}^{L-1} Ae^{-j2\pi fk} = A\frac{1-e^{-j2\pi fL}}{1-e^{-j2\pi f}}$$

$$S_{xx}(f) = A^2\frac{1-\cos(2\pi fL)}{1-\cos 2\pi f}$$

Example: Discrete-time impulse contd.



(i) Finite-duration pulse



(j) Energy spectral density

Finite-length pulse and its energy spectral density for $A = 1, L = 10$.

Energy spectral density and auto-covariance function

The energy spectral density of a discrete-time aperiodic signal and its auto-covariance function form a Fourier pair.

$$S_{xx}(f) = \sum_{l=-\infty}^{\infty} \sigma_{xx}[l] e^{-j2\pi lf} \tag{24a}$$

$$\sigma_{xx}[l] = \int_{-1/2}^{1/2} S_{xx}(f) e^{j2\pi fl}\, df \tag{24b}$$

Cross-energy spectral density

In multivariable signal analysis, it is useful to define a quantity known as cross-energy spectral density,

$$\boxed{S_{x_2 x_1}(f) = X_2(f) X_1^\star(f)} \tag{25}$$

The cross-spectral density measures the linear relationship between two signals in the frequency domain, whereas the auto-energy spectral density measures linear dependencies within the observations of a signal.

Cross energy spectral density ... contd.

When $x_2[k]$ and $x_1[k]$ are the output and input of a linear time-invariant system respectively, i.e.,

$$x_2[k] = G(q^{-1}) x_1[k] = \sum_{n=-\infty}^{n=\infty} g[n] x_1[k-n] = g_1[k] \star x_1[k] \tag{26}$$

two important results emerge

$$\boxed{S_{x_2 x_1}(f) = G_1(e^{-j2\pi f}) S_{x_1 x_1}(f); \qquad S_{x_2 x_2}(f) = |G_1(e^{-j2\pi f})|^2 S_{x_1 x_1}(f)} \tag{27}$$

Summary

It is useful to summarize our observations on the spectral characteristics of different classes of signals.

i. *Continuous-time* signals have *aperiodic spectra*
ii. *Discrete-time* signals have *periodic spectra*
iii. *Periodic* signals have *discrete (line) power spectra*
iv. *Aperiodic (finite energy)* signals have *continuous energy spectra*

Continuous spectra are qualified by a *spectral density function*.

Spectral Distribution Function

In all cases, one can define an energy / power **spectral distribution function**, $\Gamma(f)$.

For **periodic** signals, we have **step-like power spectral distribution** function,
For **aperiodic** signals, we have a **smooth energy spectral distribution** function,
where one could write the **spectral density** as,

$$S_{xx}(f) = d\Gamma(f)/df \qquad \text{or} \qquad \Gamma_{xx}(f) = \int_{-1/2}^{f} S_{xx}(f)\, df \tag{28}$$

.

Properties of DTFT

Linearity property

1. **Linearity:**

If $x_1[k] \xrightarrow{\mathcal{F}} X_1(\omega)$ and $x_2[k] \xrightarrow{\mathcal{F}} X_2(\omega)$ then

$$a_1 x_1[k] + a_2 x_2[k] \xrightarrow{\mathcal{F}} a_1 X_1(f) + a_2 X_2(f)$$

The Fourier transform of a sum of discrete-time (aperiodic) signals is the respective sum of transforms.

Shift property

2. **Time shifting:**

If $x_1[k] \xrightarrow{\mathcal{F}} X_1(\omega)$ then
$x_1[k-D] \xrightarrow{\mathcal{F}} e^{-j2\pi fD}X_1(f)$

- Time-shifts result in frequency-domain modulations.
- Note that the **energy spectrum of the shifted signal remains unchanged** while the phase spectrum shifts by $-\omega k$ at each frequency.

Dual:

A shift in frequency $X(f-f_0)$ corresponds to modulation in time, $e^{j2\pi f_0 k}x[k]$.

Time reversal

3. **Time reversal:**

If $x[k] \xrightarrow{\mathcal{F}} X(\omega)$, then $x[-k] \xleftrightarrow{\mathcal{F}} X(-f) = X^\star(f)$

If a signal is folded in time, then its power spectrum remains unchanged; however, the phase spectrum undergoes a sign reversal.

Dual: The dual is contained in the statement above.

Scaling property

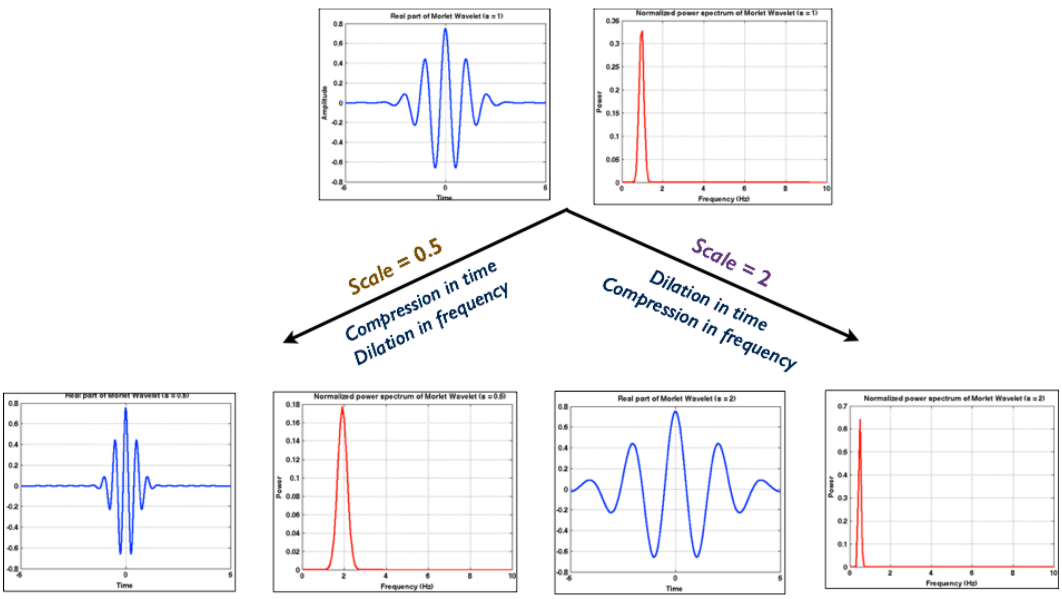
4. **Scaling:**

If $x[k] \xrightarrow{\mathcal{F}} X(\omega)$ (or $x(t) \xrightarrow{\mathcal{F}} X(F)$),
then $x\left[\dfrac{k}{s}\right] \xrightarrow{\mathcal{F}} X(sf)$ (or $x\left(\dfrac{t}{s}\right) \xrightarrow{\mathcal{F}} X(sF)$)

If $X(F)$ has a center frequency F_c, then scaling the signal $x(t)$ by a factor $\dfrac{1}{s}$ results in shifting the center frequency (of the scaled signal) to $\dfrac{F_c}{s}$

Note: For real-valued functions, it is more appropriate to refer to $|X(F)|$,

Example: Scaling a Morlet wave



Convolution

5. **Convolution Theorem:** Convolution in time-domain transforms into a product in the frequency domain.

Theorem

If $x_1[k] \xrightarrow{\mathcal{F}} X_1(\omega)$ *and* $x_2[k] \xrightarrow{\mathcal{F}} X_2(\omega)$ *and*

$$x[k] = (x_1 \star x_2)[k] = \sum_{n=-\infty}^{\infty} x_1[n]x_2[k-n]$$

$$\text{then} \;\; X(f) \triangleq \mathcal{F}\{x[k]\} = X_1(f)X_2(f)$$

This is a highly useful result in the analysis of signals and LTI systems or linear filters.

Product

6. **Dual of convolution:** Multiplication in time corresponds to convolution in frequency domain.

$$x[k] = x_1[k]x_2[k] \xrightarrow{\mathcal{F}} \int_{-1/2}^{1/2} X_1(\lambda)X_2(f-\lambda)\, d\lambda$$

- This result is useful in studying Fourier transform of windowed or finite-length signals such as **STFT and discrete Fourier transform (DFT)**.

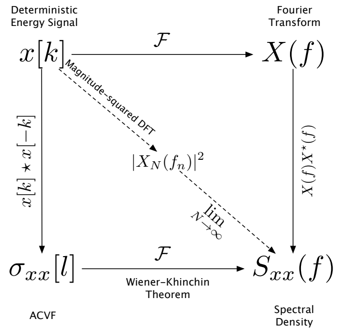
Correlation theorem

7. **Correlation Theorem** (Wiener-Khinchin theorem for deterministic signals)

Theorem

The Fourier transform of the cross-covariance function $\sigma_{x_1x_2}[l]$ is the cross-energy spectral density

$$\mathcal{F}\{\sigma_{x_1x_2}[l]\} = \sum_{l=\infty}^{\infty} \sigma_{x_1x_2}[l]e^{-j2\pi fl} = S_{x_1x_2}(f) = 2\pi S_{x_1x_2}(\omega)$$



- This result provides alternative way of computing spectral densities (esp. useful for random signals)

Discrete Fourier Transform (DFT) and Periodogram

Opening remarks

- Signals encountered in reality are not necessarily periodic.
- Computation of DTFT, i.e., the Fourier transform of discrete-time aperiodic signals, presents two difficulties in practice:
  1. **Only finite-length N measurements are available.**
  2. **DTFT can only be computed at a discrete set of frequencies.**

Computing the DTFT: Practical issues

- Can we compute the finite-length DTFT, i.e., restrict the summation to the extent observed?
- Or do we artificially extend the signal outside the observed interval? Either way what are the consequences?
- Some form of discretization of the frequency axis, *i.e.*, *sampling in frequency* is therefore inevitable.

When the DTFT is restricted to the duration of observation and evaluated on a frequency grid, we have the **Discrete Fourier Transform** (DFT)

Sampled finite-length DTFT: DFT

DFT

The discrete Fourier transform of a finite length sequence $x[k],\ k = 0, 1, \cdots, N-1$ is defined as:

$$X(f_n) = \sum_{k=0}^{N-1} x[k] e^{-j2\pi f_n k}, \tag{29}$$

The transform derives its name from the fact that it is now *discrete in both time and frequency*.

Q: What should be the grid spacing (sampling interval) in frequency?

Main result

For signal $x[k]$ of length N_l, its DTFT $X(f)$ is perfectly recoverable from its sampled version $X(f_n)$ if and only if the frequency axis is sampled uniformly at N_l points in $[-1/2, 1/2)$, i.e., iff

$$\triangle f = \frac{1}{N_l} \qquad \text{or} \qquad \triangle\omega = \frac{2\pi}{N_l} \tag{30}$$

See Proakis and Manolakis, (2005) for a proof.

N-point DFT

The resulting DFT is known as the N-point DFT with $N = N_l$. The associated analysis and synthesis equations are given by

$$\boxed{\begin{aligned} X[n] \triangleq X(f_n) &= \sum_{k=0}^{N-1} x[k] e^{-j\frac{2\pi}{N}nk} \qquad && n = 0, 1, \cdots, N-1 \qquad (31a) \\ x[k] &= \frac{1}{N}\sum_{n=0}^{N-1} X[n] e^{j\frac{2\pi}{N}kn} \qquad && k = 0, 1, \cdots, N-1 \qquad (31b) \end{aligned}}$$

Unitary DFT

It is also a common practice to use a factor $1/\sqrt{N}$ on both (31a) and (31b) to achieve symmetry of expressions.

$$\boxed{\begin{aligned} X[n] &= \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} x[k] e^{-j2\pi f_n k} \qquad && f_n = \frac{n}{N},\ n = 0, 1, \cdots, N-1 \qquad (32a) \\ x[k] &= \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} X[n] e^{j2\pi f_n k} \qquad && k = 0, 1, \cdots, N-1 \qquad (32b) \end{aligned}}$$

The resulting transforms are known as **unitary** transforms since they are norm-preserving, i.e., $||x[k]||_2^2 = ||X[n]||_2^2$.

Reconstructing $X(f)$ from $X[n]$

The reconstruction of $X(f)$ from its N-point DFT is facilitated by the following expression (Proakis and Manolakis, 2005):

$$X(f) = \sum_{n=0}^{N-1} X\left(\frac{2\pi n}{N}\right) P\left(2\pi f - \frac{2\pi n}{N}\right) \quad N \geq N_l \tag{33}$$

$$\text{where} \quad P(f) = \frac{\sin(\pi f N)}{N \sin(\pi f)} e^{-j\pi f(N-1)}$$

- Equation (33) has very close similarities to that for a continuous-time signal $x(t)$ from its samples $x[k]$ (Proakis and Manolakis, 2005).
- Further, the condition $N \geq N_l$ is similar to the requirement for avoiding aliasing.

Consequences of sampling the frequency axis

When the DTFT is evaluated at N equidistant points in $[-\pi, \pi]$, one obtains

$$\begin{aligned} X\left(\frac{2\pi}{N} n\right) &= \sum_{k=-\infty}^{\infty} x[k] e^{-j2\pi nk/N} \qquad n = 0, 1, \cdots, N-1 \\ &= \sum_{l=-\infty}^{\infty} \sum_{k=lN}^{lN+N-1} x[k] e^{-j2\pi nk/N} \\ &= \sum_{k=0}^{N-1} \sum_{l=-\infty}^{\infty} x[k - lN] e^{-j2\pi nk/N} \end{aligned} \tag{34}$$

Now, define $x_p[k] = \sum_{l=-\infty}^{\infty} x[k - lN]$, with period $N_p = N$.

Equivalence between DFT and DTFS

Then (34) appears structurally very similar to that of the coefficients of a DTFS:

$$Nc_n = \sum_{k=0}^{N-1} x_p[k]e^{-j2\pi nk/N} \tag{35}$$

The N-point DFT $X[n]$ of a sequence $\mathbf{x}_N = \{x[0], x[1], \cdots, x[N-1]\}$ is equivalent to the coefficient c_n of the DTFS of the periodic extension of $\mathbf{x}_N$. Mathematically,

$$X[n] = Nc_n, \qquad c_n = \frac{1}{N}\sum_{k=0}^{N-1} x[k]e^{-j\frac{2\pi}{N}kn} \tag{36}$$

Putting together $\cdots$

An N-point DFT *implictly assumes the given finite-length signal to be periodic with a period equal to N regardless of the nature of the original signal.*

- The basis blocks are $\cos(2\pi\frac{k}{N}n)$ and $\sin(2\pi\frac{k}{N}n)$ characterized by the index n
    - The quantity n denotes the number of cycles completed by each basis block for the duration of N samples
- DFT inherits all the properties of DTFT with the convolution property replaced by *circular convolution.*

DFT: Summary

Definition

The N-point DFT and IDFT are given by

$$X[n] = \sum_{k=0}^{N-1} x[k] e^{-j2\pi kn/N}; \qquad x[k] = \frac{1}{N}\sum_{n=0}^{N-1} X[n] e^{j2\pi kn/N}$$

- Introducing $W_N = e^{j2\pi/N}$, the above relationships are also sometimes written as

$$X[n] = \sum_{k=0}^{N-1} x[k] W_N^{-kn}; \quad x[k] = \frac{1}{N}\sum_{n=0}^{N-1} X[n] W_N^{kn}$$

Points to remember

- The frequency resolution in DFT is equal to $1/N$ or $2\pi/N$. **Increasing the length artificially by padding with zeros does not provide any new information** but can only provide a better "display" of the spectrum
- DFT is calculated assuming that the given signal $x[k]$ is periodic and therefore it is a *Fourier series* expansion of $x[k]$ in reality!
- In an N-point DFT, only $N/2 + 1$ frequencies are unique. For example, in a 1024-point DFT, only 513 frequencies are sufficient to reconstruct the original signal.

DFT in practice: FFT

- The linear transformation relationships are useful for short calculations.
- In 1960s, Cooley and Tukey developed an efficient algorithm for fast computation of DFT which revolutionized the world of spectral analysis
  - This algorithm and its subsequent variations came to be known as the Fast Fourier Transform (FFT), which is available with almost every computational package.
- The FFT algorithm reduced the number of operations from N^2 in regular DFT to the order of $N \log(N)$
- FFT algorithms are fast when N is exactly a power of 2
  - Modern algorithms are not bounded by this requirement!

R: `fft`

Power or energy spectral density?

- Practically we encounter either finite-energy aperiodic or stochastic (or mixed) signals, which are characterized by *energy* and *power* spectral density, respectively.
- However, the practical situation is that we have a finite-length signal $\mathbf{x}^N = \{x[0], x[1], \cdots, x[N-1]\}$.
- Computing the N-point DFT amounts to treating the underlying infinitely long signal $\tilde{x}[k]$ as *periodic* with period N.

Thus, strictly speaking **we have neither densities. Instead DFT always implies a power spectrum (line spectrum) regardless of the nature of underlying signal!**

Periodogram: Heuristic power spectral density

The power spectrum $P_{xx}(f_n)$ for the finite-length signal $\mathbf{x}_N$ is obtained as

$$P_{xx}(f_n) = |c_n|^2 = \frac{|X[n]|^2}{N^2} \tag{37}$$

A *heuristic* power spectral density (power per unit cyclic frequency), known as the **periodogram**, introduced by Schuster, (1897), for the finite-length sequence is used,

$$\boxed{\mathbb{P}_{xx}(f_n) \triangleq \text{PSD}(f_n) = \frac{P_{xx}(f_n)}{\triangle f} = N|c_n|^2 = \frac{|X[n]|^2}{N}} \tag{38}$$

Alternatively, $$\boxed{\mathbb{P}_{xx}(\omega_n) = \frac{1}{2\pi N}|X[n]|^2} \tag{39}$$

Routines in R

Task	Routine	Remark
Convolution	`convolve`, `conv`	Computes product of DFTs followed by inversion (`conv` from the `signal` package)
Compute IR	`impz`	Part of the `signal` package
Compute FRF	`freqz`	Part of the `signal` package
DFT	`fft`	Implements the FFT algorithm
Periodogram	`spec.pgram`, `periodogram`	Part of the `stats` and `TSA` packages, respectively

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