CH5350: Applied Time-Series Analysis

Partial Auto-Correlation Functions

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Partial ACF

Recall that correlation-based measures suffer from **confounding**, *i.e.*, the common influence of a third extraneous variable can cause two variables to *appear* as correlated.

In time-series analysis, the correlation between two observations of a series (or two different series) is most likely to suffer from confounding.

- Intermediate samples can introduce an apparent correlation due to propagated effects.
- This phenomenon is particularly important for auto-regressive processes.

To understand this, revisit the ACF of an AR(1) process.

ACF of an AR(1) process

Recall the ACF of an AR(1) process $v[k] = -d_1 v[k-1] + e[k]$,

$$\rho_{vv}[l] = (-d_1)^{|l|}$$

The ACF suggests that $v[k]$ and $v[k-l]$ are correlated whereas the governing difference equation for the process clearly shows that only two successive samples $v[k]$ and $v[k-1]$ **directly** influence each other.

ACF of an AR(1) process

Q: What is the cause of this apparent correlation between samples separated by las $L > 1$?

A: The cause for this apparent correlation is the **propagated effect.** For instance, the difference equation of the process can be re-written as

$$v[k] = -d_1(-d_1 v[k-2] + e[k-1]) + e[k] = d_1^2 v[k-2] - d_1 e[k-1] + e[k]$$

Thus $v[k-2]$ appears to influence $v[k]$ **indirectly** through $v[k-1]$. The same argument can be extended to explain correlation at other lags as well.

How do we ensure ACF measures direct correlations only?

Conditioned ACF: Partial ACF

To measure the direct correlation between $v[k-l]$ and $v[k]$ we should account for the possible propagated effects of the intermediate variables $\{v[k-l+1, \cdots, v[k-1]\}$.

The procedure is illustrated for $l = 2$. The idea is to remove the presence of $v[k-1]$ in both $v[k]$ and $v[k-2]$ followed by a correlation between the respective residuals. The resulting correlation is known as **partial auto-correlation function** (PACF)

Partial ACF

Remarks:

- Partial ACF is analogous to "partial derivative" where only the effects w.r.t. a specific variable are evaluated.
- As we learnt earlier, computing partial correlation (or any other measure) is known as **conditioning** in signal processing
- The partial ACF measures **direct correlation** whereas the ACF measures **total correlation**

Procedure to compute PACF

1. Obtain the best predictor for $v[k]$ using $v[k-1]$. Denote the associated residuals by $\eta[k]$

$$\hat{v}[k|v[k-1]] = \alpha_1 v[k-1]; \qquad \eta[k] = v[k] - \alpha_1^\star v[k-1]$$

2. Obtain the best "predictor" for $v[k-2]$ using $v[k-1]$. Denote the associated residuals by $\eta[k-2]$

$$\hat{v}[k-2|v[k-1]] = \beta_1 v[k-1]; \qquad \eta[k-2] = v[k-2] - \beta_1^\star v[k-1]$$

   where $\alpha_1^\star$ and $\beta_1^\star$ are the optimal estimates of α_1 and β_1 respectively.

3. Compute $\phi_{vv}[2] = \text{corr}(\eta[k], \eta[k-2])$ to obtain the PACF of the series $v[k]$ at lag 2

Procedure to compute PACF ...contd.

The optimal estimates of α_1 and β_1 are obtained in such a way that $\eta[k]$ and $\eta[k-2]$ do not contain any (linear) effects of $v[k-1]$, *i.e.*,

$$\text{corr}(\eta[k], v[k-1]) = 0 \qquad \text{corr}(\eta[k-2], v[k-1]) = 0$$

These are also the conditions of optimality for the least squares technique. Thus, $\alpha_1^\star$ and $\beta_1^\star$ are the LS estimates.

$$\boxed{\alpha_1^\star = \rho_{vv}[1]} \qquad \boxed{\beta_1^\star = \rho_{vv}[1]}$$

PACF of an AR(1) process

To compute the PACF at lag $l = 2$, recall from the procedure (alternatively, the expression for partial correlation),

$$\begin{aligned}\phi_{vv}[2] &= \text{corr}(v[k] - \alpha_1^\star v[k-1], v[k-2] - \beta_1^\star v[k-1]) \\ &= \frac{\text{cov}(v[k] - \rho_{vv}[1]v[k-1], v[k-2] - \rho_{vv}[1]v[k-1])}{\sqrt{\text{var}(v[k] - \rho_{vv}[1]v[k-1])\text{var}(v[k-2] - \rho_{vv}[1]v[k-1])}} \\ &= \frac{\rho[2] - \rho^2[1]}{1 - \rho[1]^2}\end{aligned}$$

PACF of an AR(1) process ... contd.

- For an AR(1) process, $\rho[l] = (-d_1)^{|l|}$, therefore $\phi_{vv}[2] = 0$
- At a later stage, it will be shown that $\phi_{vv}[l] = 0$ for all lags $l \geq 2$ for an AR(1) process

The PACF for an AR(1) process falls off abruptly to zero $\forall |l| > 2$.

Note: For an MA(1) process, $\phi_{vv}[1] = -c_1(1 + c_1^2)/(1 + c_1^4 - c_1^2)$

General procedure

The general procedure to obtain PACF is given below.

1. Obtain the best predictors for $v[k]$ and $v[k-l]$ using $\{v[k-1], v[k-2], \cdots, v[k-l+1]\}$. Denote the associated residuals by $\eta[k]$ and $\eta[k-l]$ respectively

$$\eta[k] = v[k] - \sum_{j=1}^{l-1} \alpha_j^{\star} v[k-j] \qquad \eta[k-l] = v[k-l] - \sum_{j=1}^{l-1} \beta_j^{\star} v[k-l+j]$$

   where the * denote the optimal values (least squares) estimates.

2. Compute $\phi_{vv}[l] = \text{corr}(\eta[k], \eta[k-l])$ to obtain the PACF at lag l

Alternative procedure

The PACF coefficient at any lag p, $\phi_{vv}[p]$ can be shown to be the last coefficient of an AR(p) model fit to the series $v[k]$

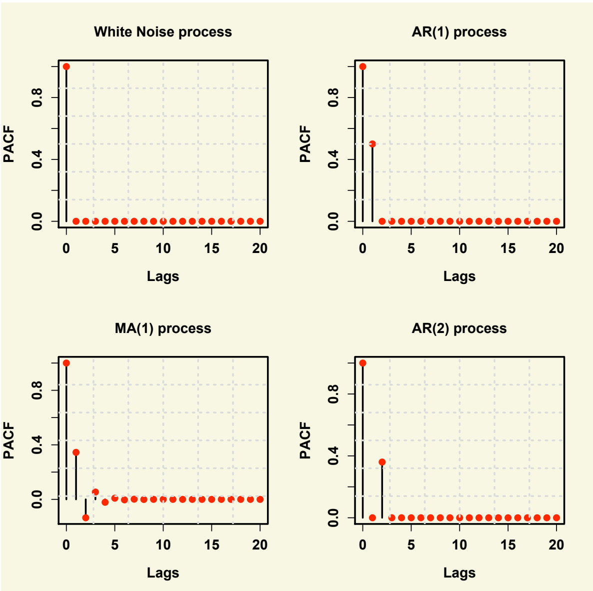
1. Fit an AR(l) model at each lag l.
2. Determine the PACF at any lag l as the last coefficient of that model.

A **recursive** algorithm due to Durbin and Levinson is used in practice to compute $\phi_{vv}[p]$ using the coefficient at $l = p-1$ and the ACF coefficients.

Remarks

- The "prediction" of $v[k-l]$ using future values is known as **backcasting**
- The PACF at lag $l = 0$ is not defined. However, to be consistent with ACF, PACF at lag $l = 0$ maybe set to unity.

Theoretical PACF: Examples



- PACF of a WN process is zero at all lags (like the ACF)
- PACF of an MA(1) process dies down exponentially (similar to that of ACF for an AR(1) process)
- Notion of PACF can be extended to handle negative lags. PACF is symmetric for stationary processes.

Summary ... contd.

- Partial ACF accounts for possible confounding in the ACF, particularly for auto-regressive processes
- The PACF and ACF measures are in some respects, **duals** of each other
  - ACF decays exponentially for an AR process while the PACF falls off abruptly after an appropriate lag for the same process
  - The above behaviour is reversed for the case of an MA process