CH5350: Applied Time-Series Analysis

Introduction to Random Processes

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Random signals

Definition 1

A (discrete-time) random signal is a function $x[k]$ whose characteristics *cannot be accurately* described by any existing mathematical function. At each point, it is characterized by a probability distribution.

- **Prediction viewpoint:** Signal is not accurately predictable.
- **Knowledge viewpoint,** Signal is always known with some error or uncertainty.

Random signals ... contd.

Definition 2

A (discrete-time) random signal

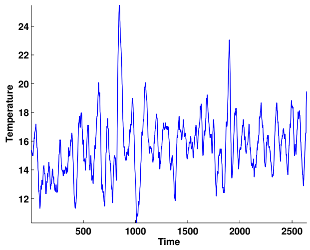
$$\mathbf{x} = \{x[1], x[2], \cdots, x[k], x[k+1], \cdots\}$$

is an *index-ordered* sequence of random variables in the temporal and/or spatial and/or frequency domain. Its characteristics can only be described by probabilistic laws and not merely by mathematical models.

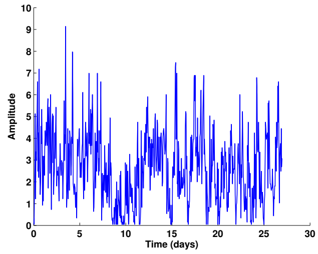
Implications

1. A random signal cannot be predicted accurately
2. At any instant the observed value is one of the many possible values.

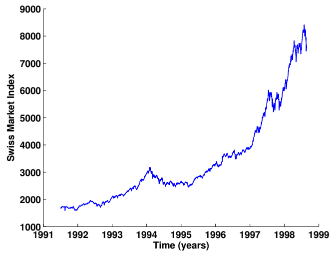
Examples



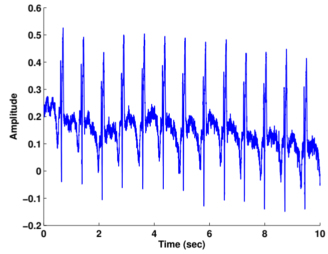
Temperature



Wind speed



Swiss SMI



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Random process

Conventional definition

The random process is the ensemble of the random signal $x[k]$, i.e., it is the collection of all possible realizations of $x[k]$.

A random process is thus **a function of both the time and realization**, and often denoted as $x[\Omega, k]$. We shall, however, for the sake of convenience refer to it as $\{x[k]\}$.

Simpler definition

The process (or a subsystem) that generates a random signal (all possibilities) is termed as the **random process**.

Remember

- A random process always exists - the existence is NOT random
- A process is not random intrinsically - it is treated as random because it is not understood well (e.g., atmospheric process can be treated as a random process)

A **deterministic process** on the other hand generates signals whose values are accurately known and predictable

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Note: The phrases "random signal" and "random process" are used interchangeably.

Stochastic processes: Predictability

- **Is there a process that is absolutely unpredictable?**
  YES. A large class of phenomena can be explained to some extent by physical reasoning. However, a class of processes do exist which are not at all predictable, either using linear or non-linear models.
- **Does randomness imply no predictability?**
  NO. There are a large class of processes which are random and are still predictable.
  Key point: it is not possible to provide accurate predictions

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Note: In some schools of thought, stochastic process has some element of predictability while a random process does not.

Deterministic processes

For completeness sake, we define deterministic signals and processes.

Deterministic signals

A deterministic signal is one which can take only one possible value at a given time. It is understood to be generated by a deterministic process, *i.e.*, whose evolution or behaviour can be predicted (modelled) accurately

Remember

- A large class of signals that we observe can be considered to contain a mix of deterministic as well as stochastic effects.

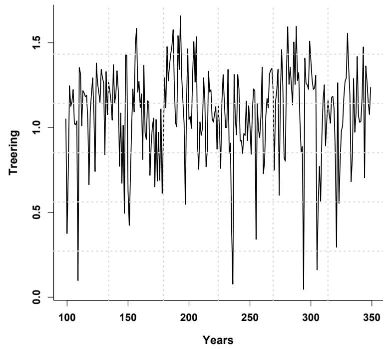
Challenges in analysis of random processes

In practice, we have only a single realization. The challenge is to be able to infer the truth from this single realization.

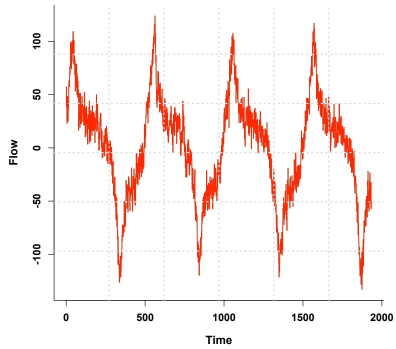
1. **Stationarity:** Invariance property of the process w.r.t. time
2. **Ergodicity:** Provision to replace ensemble averages with time averages.

Stationarity

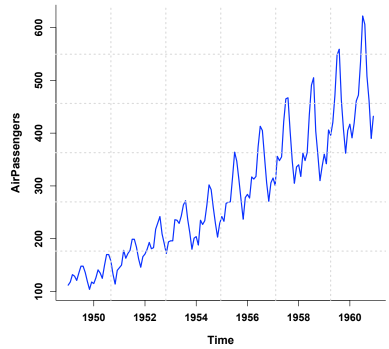
A model built from a given set of observations is useful only if the probability structure remains invariant with time. This is the condition of stationarity.



(a) Stationary



(b) Harmonic



(c) Non-stationary

Formal definition

The stationarity condition $\equiv$ requirement of a "steady-state" on the statistical properties

Strict stationarity

A random process is said to be strictly stationary if all of its statistical properties remain invariant to shifts in time. This is to say that the joint p.d.f.:

$$f(x[1], x[2]. \cdots, x[N]) = f(x[T+1], x[T+2], \cdots, x[T+N]) \quad T \in \mathcal{Z}^+, \forall N \qquad (1)$$

Processes that do not satisfy the stationarity condition are **non-stationary.**

Non-stationary process

Example

Consider the random signal $x[k] = A\cos(\omega k + \phi)$, where ϕ is a random variable with uniform distribution in $[0, \pi]$. The mean of this signal (process) is

$$E(x[k]) = E(A\cos(\omega k + \phi)) = A\int_0^\pi \cos(\omega k + \phi) f(\phi)\, d\phi = \frac{A}{\pi}\int_0^\pi \cos(\omega k + \phi)\, d\phi$$
$$= -\frac{2A}{\pi}\sin(\omega k)$$

which is a function of time. Thus, the process is **non-stationary.**

Order stationarity in distribution

A stochastic process is said to be N^{th}**-order stationary** (in distribution) if the joint distribution of N observations is invariant to shifts in time,

$$F_{X_{k_1},\cdots,X_{k_N}}(x_1,\cdots,x_N) = F_{X_{T+k_1},\cdots,X_{T+k_N}}(x_1,\cdots,x_N) \quad \forall T, k_1,\cdots,k_N \in \mathcal{Z}^+ \qquad (2)$$

where $X_{k_1},\cdots,X_{k_N}$ are the RVs associated with the observations at $k = k_1,\cdots,k_N$, respectively and $x_1,\cdots,x_N$ are any real numbers.

Special cases

1. **First-order stationarity in distribution:**

$$F_{X_{k_1}}(x_1) = F_{X_{k_1+T}}(x_1) \qquad \forall T, k_1 \in \mathcal{Z}^+ \tag{3}$$

   Every observation should fall out of the same distribution.

2. **Second-order stationarity in distribution:**

$$F_{X_{k_1},X_{k_2}}(x_1, x_2) = F_{X_{k_1+T},X_{k_2+T}}(x_1, x_2) \qquad \forall T, k_1, k_2 \in \mathcal{Z}^+ \tag{4}$$

   The distribution depends **only on the time-difference** $k_2 - k_1$.

Relaxation of strict stationarity

The requirement of strict stationarity is similar to the strict requirement of time-invariance or linearity (in deterministic processes), which are also academically convenient assumptions, but rarely satisfied in practice.

In reality, rarely will we find a process that satisfies the strict requirements of stationarity defined above.

A **weaker** requirement is that certain key statistical properties of interest such as mean, variance and a few others at least, remain invariant with time

Weak stationarity

A common relaxation, is to require invariance up to second-order moments.

Weak or wide-sense or second-order stationarity

A process is said to be weakly or wide-sense or second-order stationary if:

i. The **mean of the process is independent of time**, *i.e.*, invariant w.r.t. time.
ii. It has **finite variance**.
iii. The **auto-covariance function** of the process

$$\sigma_{xx}[k_1, k_2] = \text{cov}(X_{k_1}, X_{k_2}) = E((X_{k_1} - \mu_1)(X_{k_2} - \mu_2)) \tag{5}$$

is only a function of the "time-difference" (lag $l = k_2 - k_1$) but not the time.

On wide-sense stationarity (WSS)

Q: Under what conditions is the weak stationarity assumption justified?

Where linear models are concerned, the optimal parameter estimates are fully determined by the first- and second-order properties of the joint p.d.f.

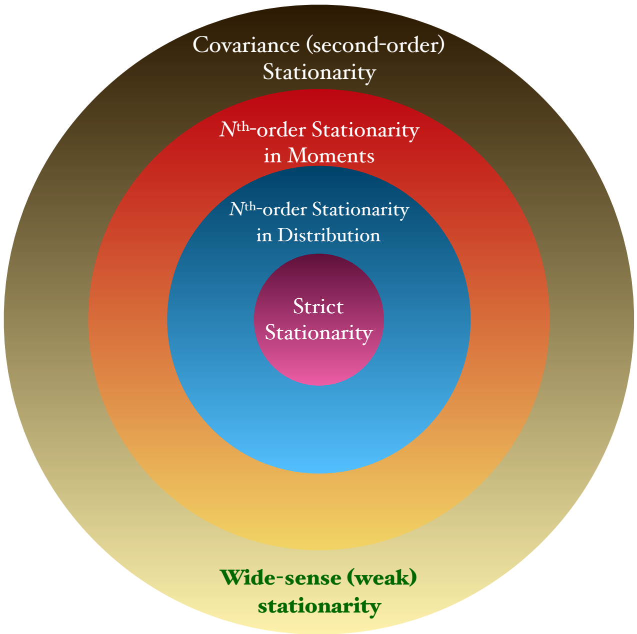
Example

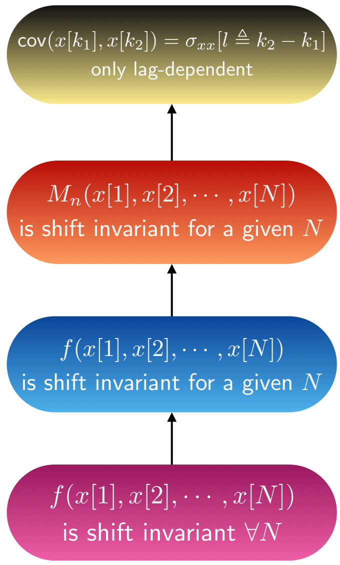
For a stationary process, suppose a linear predictor of the form

$$\hat{x}[k] = -d_1 x[k-1] - d_2 x[k-2]$$

is considered. Determine the optimal (in the m.s. prediction error sense) estimates.

Forms of stationarity: Relative standing





Gaussian WSS process

A WSS multivariate Gaussian process is also strictly stationary.

Why?

- A joint Gaussian distribution is completely characterized by the first two moments.

$$f(x[1], x[2], \cdots, x[N]; \boldsymbol{\mu}, \Sigma) = \frac{1}{\sqrt{(2\pi)^N |\Sigma|}} \exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu})\right) \quad (6)$$

where $\mathbf{x} = \begin{bmatrix} x[1] & x[2] & \cdots & x[N] \end{bmatrix}$, $\boldsymbol{\mu}$ is the vector of means and Σ is the covariance matrix. Therefore, when the first two moments remain invariant, the joint pdf also remains invariant.

Non-stationarities

Just as with non-linearities, there are different types of non-stationarities, for e.g., mean non-stationarity, variance non-stationarity, and so on. It is useful to categorize them, for working purposes, into two classes:

1. **Deterministic:** Polynomial trend, variance non-stationarity, periodic, etc.
2. **Stochastic:** Integrating type, i.e., random walk, heteroskedastic processes, etc.

We shall be particularly interested in two types of non-stationarities, namely, *trend-type* and *random walk* or *integrating* type non-stationarities, with a glimpse of *heteroskedasticity*.

Trend-type non-stationarity

A suitable mathematical model for such a process is,

$$x[k] = \mu_k + w[k] \tag{7}$$

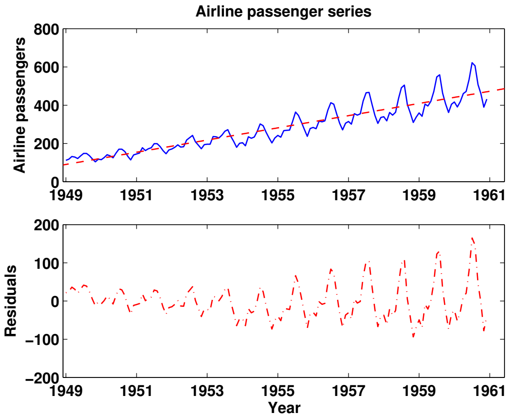
where μ_k is a polynomial function of time and $w[k]$ is a stationary process.

For example, a linear trend is modeled as $\mu_k = a + bk$.

Trend type non-stationarity ...contd.

- When the removal of a trend (e.g., linear ,quadratic) results in stationary residuals, the process is said to be **trend stationary.**
- Trends may be removed by either fitting a suitable polynomial (parametric approach) or by applying an appropriate smoothing filter (non-parametric approach).

Example: Trend-type non-stationarity



Monthly airline passenger series.

- A linear trend is fit to the series.
- Residuals show variance type nonstationarity. This is typical of a growth series.
- Several approaches possible:
  1. Trend fit + transformation + additive seasonal model
  2. Trend fit + multiplicative seasonal model
  3. Trend fit + Advanced models known as GARCH (generalized auto-regressive conditional heteroskedastic) models.

Integrating type non-stationarity

One of the most commonly encountered non-stationary processes is the *random walk* process (special case of Brownian motion).

The simplest random walk process is an *integrating* process,

$$x[k] = \sum_{n=0}^{k} e[k] \tag{8}$$

where $e[k]$ is the (unpredictable) *white-noise* affecting the process at the k^{th} instant.

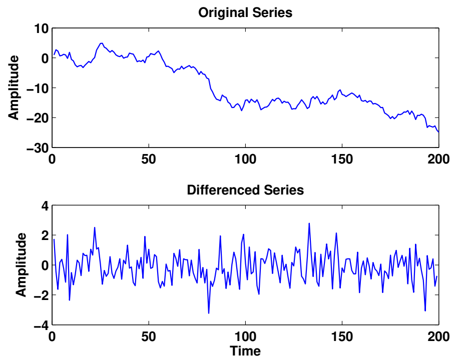
Integrating processes

- At any instant the signal is the accumulation of all shock-wave like changes from the beginning. Hence the name *integrating*.
- It is also known as a difference stationary process because

$$x[k] - x[k-1] = e[k] \tag{9}$$

Example

- Non-stationarity can be easily discerned by a visual inspection.
- The differenced series appears to be stationary.
- In general, a single degree of differencing is capable of removing a linear trend, two degrees removes quadratic trends and so on.



Top panel: $N = 200$ samples of a random walk series. Bottom plot: differenced series.

Caution

Despite its capability in handling a wide range of stationarities, the differencing approach also has potentially a few detrimental effects.

- Excessive or unnecessary differencing can lead to spurious correlations in the series.
- Amplification of noise, i.e., decrease in SNR in system identification.

Ergodicity

When the process is stationary, the second requirement on the time-series stems from the fact that in practice we work with only a single record of data.

Estimates computed from averages of time (or other domain) samples should serve as suitable representatives of the theoretical statistical properties, which are defined as averages in the outcome space (**ensemble**).

Ergodicity: Formal statement

Ergodicity

A process is said to be **ergodic** if the (time averaged) estimate **converges** to the true value (statistical average) when the number of observations $N \to \infty$.

Examples

1. A stationary i.i.d. random process is ergodic (by the strong LLN)

$$\frac{1}{N}\sum_{k=1}^{N} x[k] \xrightarrow{a.s.} E(X_k) \qquad \text{as } N \to \infty \tag{10}$$

2. A process such that $x[k] = A, \ \forall k, \ \text{s.t.} E(A) = 0$. Is it ergodic?

Remarks

- We can speak of ergodicity only when the process is stationary!
- *Loose interpretation:* **given sufficient time, the process would have unravelled nearly all possibilities that exist at any time instant (regardless of the starting point)**,
- Ergodicity is not necessarily a characteristic of the process, but also of the experiment that is carried out to obtain the time-series.
- Ergodicity is difficult to verify in practice; however, it can be ensured by a careful experimentation, and a judicious measurement mechanism.
- Finally, we shall study later a broader concept in inferencing (estimation theory) known as **consistency**, which encompasses ergodicity.

Summary

- Stationarity and Ergodicity are two prime assumptions for time-series analysis to be meaningful.
- Stationarity assumes that the process is **invariant** in the sense of joint distributions or moments.
- Covariance or weak stationarity is sufficient for developing linear models.
- Ergodicity provides the license to work with the time-averages from a single realization as suitable representatives of the ensemble averages.
- In practice, tests for stationarity (assuming ergodicity) are available. Ergodicity is hard to test - one has to know its holding a priori.