Maths

Unit-1

Analytic Function

A complex valued func $w = f(z)$ i.e said to be analytic at point

$$\frac{dw}{dz} = f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\partial z}$$

exists & is unique at z_0

$\boxed{\text{C-R Eq}^n}$

$$W = u + iv$$

$i^3 = -i$

$i^4 = 1$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

i^{54}

$(i^4)^{13} \times i^2$

$(1)^{13} \times -1$

eg:- $f(z) = z^2$

$\therefore z = x + iy$

$\therefore i^2 = -1$

$$f(z) = (x+iy)^2$$

$$= x^2 + 2ixy + i^2y^2$$

$$f(z) = (x^2 - y^2) + 2ixy$$

$u = x^2 - y^2$, $v = 2xy$

$\frac{\partial u}{\partial x} = 2x$, $\frac{\partial u}{\partial y} = -2y$

$\frac{\partial v}{\partial x} = 2x$ $\frac{\partial u}{\partial y} = 2y$

This is analytic

Q2. $f(z) = e^{-x}(\cos y + i \sin y)$ is analytic or not

$$f(z) = e^{-x}(\cos y + i \sin y)$$
$$= e^{-x}\cos y + i e^{-x}\sin y$$

$u = e^{-x}\cos y$, $v = e^{-x}\sin y$

$\frac{\partial u}{\partial x} = -e^{-x}\cos y$ $\quad$ $\frac{\partial v}{\partial y} = +e^{-x}\cancel{\sin y}\cos y$

$\frac{\partial u}{\partial y} = -e^{-x}\sin y$ $\quad$ $\frac{\partial v}{\partial x} = -e^{-x}\sin y$

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$$

This is not analytic

Q3: $f(z) = \sin z$

$\because z = x + iy$

$f(z) = \sin(x+iy)$

$\boxed{\because \cos iy = \cosh y \quad \sin iy = i\sinh y}$

$= \sin x . \cos iy + \cos x . \sin iy$
$= \sin x . \cosh y + \cos x . i\sinh y$

$u = \sin x . \cosh y$, $v = \cos x . \sinh y$
$\hookrightarrow (+)$

$\frac{du}{dx} = \cos x . \cosh y$ $\quad$ $\frac{du}{dy} = \sin x . \sinh y$

$\frac{dv}{dy} = \cos x \cosh y$ $\qquad$ $\frac{dv}{dx} = \sin x \cdot \sinh y$

$\frac{du}{dx} = \frac{dv}{dy}$, $\qquad$ $\frac{du}{dy} = -\frac{dv}{dx}$

Analytic

* Algebra of complex no.

$z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$

$z_1 + z_2 = x_1 + x_2 + i(y_1 + y_2)$

$z_1 - z_2 = x_1 - x_2 + i(y_1 - y_2)$

$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$

$\frac{z_1}{z_2} = \frac{(x_1 x_2 + y_1 y_2) + i(y_1 x_2 - x_1 y_2)}{x_2^2 + y_2^2}$

Polar form of a complex No.

$z = x + iy$ can be represented by point 'P'

$x = r\cos\theta$, $y = r\sin\theta$

$z = x + iy$
$= r\cos\theta + i r\sin\theta$
$= r(\cos\theta + i\sin\theta)$
$= re^{i}$

Modulus & argument of complex no.

$z = x + iy$
$|z| = \sqrt{x^2 + y^2}$
$= r$

$\arg z = \theta = \tan^{-1}\left(\frac{y}{x}\right)$

* Properties

i) $Re(z) = x = \frac{z+\bar{z}}{2}$

ii) $Im(z) = y = \frac{z-\bar{z}}{2i}$

iii) $\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$

iv) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

v) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

vi) $z\bar{z} = |\bar{z}|^2 = |z|^2$

vii) $|z_1 z_2| = |z_1||z_2|$

viii) $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

Q. Find principle and modulus value.

i) $-1 + i\sqrt{3}$

$|z| = \sqrt{(-1)^2 + (\sqrt{3})^2}$

$= \sqrt{4} = 2$

$\tan^{-1}\left(\frac{-\sqrt{3}}{1}\right) = \frac{2\pi}{3}$

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ii) $(4+2i)(-3+\sqrt{2}i)$

$(4+2i)(-3+\sqrt{2}i)$

$4(-3) + 4(\sqrt{2}i) + 2i(-3) + 2i(\sqrt{2}i)$

$-12 + 4\sqrt{2}i - 6i + 2\sqrt{2}i^2$

$\because i^2 = -1$

$= -12 + 4\sqrt{2}i - 6i - 2\sqrt{2}$

$= (-12 - 2\sqrt{2}) + i(4\sqrt{2} - 6)$

Now,

$|z| = \sqrt{(-12-2\sqrt{2})^2 + (4\sqrt{2}-6)^2}$

$= \sqrt{144 + \cancel{48\sqrt{2}} + 8 + 32 - \cancel{48\sqrt{2}} + 36}$

$= \sqrt{220} = 2\sqrt{55}$

$\tan^{-1}\left(\frac{4\sqrt{2}-6}{12-2\sqrt{2}}\right)$

Q. Express in polar form

i) $\left(\frac{2+i}{3-i}\right)^2$

$z = \frac{(2+i)^2}{(3-i)^2} = \frac{4-1+4i}{9-1+6i} = \frac{3+4i}{8-6i}$

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$$\frac{3+4i}{8-6i} \times \frac{8+6i}{8+6i} = \frac{\cancel{24} + 18i + 32i - \cancel{24}}{64+36}$$

$$= \frac{50i}{100} = \frac{1}{2}i$$

$\therefore x = 0, \; y = \frac{1}{2}$

$r = \sqrt{0 + \frac{1}{4}} = \frac{1}{2}$

$\tan^{-1}\frac{1}{2} = \frac{\pi}{2}$

$\left(\frac{2+i}{3-i}\right)^2 = \frac{1}{2}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

iii) $(1 + \sin\alpha + i\cos\alpha)$

$x = 1 + \sin\alpha, \quad y = \cos\alpha$

$r = \sqrt{(1+\sin\alpha)^2 + (\cos\alpha)^2}$

$r = \sqrt{1 + 2\sin\alpha + \sin^2\alpha + \cos^2\alpha}$

$r = \sqrt{2 + 2\sin\alpha}$

$r = \sqrt{2(1+\sin\alpha)}$

$\theta = \tan^{-1}\left(\frac{\cos\alpha}{1+\sin\alpha}\right)$

$\tan\frac{\theta}{2} = \frac{\sin\alpha}{1+\cos\alpha}$

$$\tan\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) = \frac{\cos\alpha}{1+\sin\alpha}$$

$$z = re^{i\theta}$$

$$z = \sqrt{2(1+\sin\alpha)}\, e^{i\tan^{-1}\left(\frac{\cos\alpha}{1+\sin\alpha}\right)}$$

iii) $\sqrt{-5+12i}$

$$x+iy = \sqrt{-5+12i}$$

$$x^2 - y^2 + 2ixy = 5+12i$$

$$x^2 - y^2 = -5$$

$$2xy = 12$$

$$y = \frac{6}{x}$$

Put $y = \frac{6}{x}$ is $x^2 - y^2 = -5$

$$x^2 - \frac{36}{x^2} = -5$$

$$x^4 - 36 + 5x^2 = 0$$

$$x^4 + 5x^2 - 36 = 0$$

$$(x^2-9)(x^2-4) = 0$$

$x^2 = +9$ ✗ $\quad x^2 = 4$ ✓

$$x = \pm 2$$

Q. $\dfrac{(1+i)^8(\sqrt{3}-i)^4}{(1-i)^4(\sqrt{3}+i)^8}$

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$(1+i)$ —— ①

$r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$r = \sqrt{1+1} = \sqrt{2}$, $\theta = \tan^{-1}\left(\frac{1}{1}\right)$

$= \frac{\pi}{4}$

$1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$= \sqrt{2}\, e^{i\pi/4}$

$(1-i)$ —— ②

$= \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

$= \sqrt{2}\, e^{i\frac{3\pi}{4}}$

$(\sqrt{3}+i)$ —— ③

$r = \sqrt{3+1} = 2$

$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$= \frac{\pi}{6}$

$\sqrt{3}+i = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$= 2e^{i\pi/6}$

$\sqrt{3} - i$ ——— (4)

$\sqrt{3} - i = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

$= 2e^{i\,5\pi/6}$

Now,

$$\frac{\left(\sqrt{2}\,e^{i\pi/4}\right)^8 \left(2e^{i\,5\pi/6}\right)^4}{\left(\sqrt{2}\,e^{i\,3\pi/4}\right)^4 \left(2e^{i\pi/6}\right)^8}$$

$\frac{5\pi}{\cancel{6}_3} \times \cancel{4}^2$

$$= \frac{\cancel{16}\,e^{2\pi i} \times \cancel{16}\,e^{i\frac{10\pi}{3}}}{4\,e^{i3\pi} \times \cancel{256}\,e^{i\,4\pi/3}}$$

$= \frac{1}{4}\,e^{-i\pi}\,e^{2\pi i}$

$= \frac{1}{4}\,e^{i\pi} \quad = \quad \frac{1}{4}(\cos\pi + i\sin\pi)$

Harmonic fun^c & How to find

$f(z) = u + iV \longrightarrow$ if $u = x^2 - y^2$ find v

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $\qquad$ $\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$

diff wrt x $\qquad$ diff wrt y

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$ — a $\qquad$ $\frac{\partial^2 u}{\partial y^2} = \frac{-\partial^2 v}{\partial x \partial y}$ — b add a & b

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0} \quad \& \quad \boxed{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0}$$

Q: $u = x^2 - y^2$ find v

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial u}{dy} = -2y$$

$$\frac{\partial^2 u}{\partial x^2} = 2 \qquad \frac{\partial^2 u}{\partial y^2} = -2$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

This is Harmonic function

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

$$dv = 2y\,dx + 2x\,dy$$

$$dv = 2(y\,dx + x\,dy)$$

$$\int dv = 2\int d(xy)$$

$$v = 2xy + C$$

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Q. $V(x,y) = 3x^2y - y^3$

$\frac{\partial v}{\partial x} = 6xy$ $\quad\quad$ $\frac{\partial v}{\partial y} = 3x^2 - 3y^2$

$\frac{\partial^2 v}{\partial x^2} = 6y$ $\quad\quad$ $\frac{\partial^2 v}{\partial y^2} = -6y$

$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ $\quad$ This is Harmonic func^n

$\underline{du} = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$

$\frac{\partial u}{\cancel{\partial x}} = \frac{\partial v}{\partial y} dx - \frac{\partial v}{\partial x} dy$

$\frac{du}{\cancel{dx}} = (3x^2 - 3y^2) dx - 6xy\, dy$

$= 3x^2 dx - 3(y^2 dx - 2xy\, dy)$

$\int du = 3\int x^2 dx - 3\int d(xy^2) + c$

$u = x^3 - 3xy^2 + c$

Milne Thomson Method

$f(z) = z^3 = (x + iy)^3$

$= x^3 + 3x^2 iy + 3x(iy)^2 + i^3 y^3$

$= x^3 - 3xy^2 + i(3x^2y - y^3)$

$u(x,y) = x^3 - 3xy^2$, $v = 3x^2y - y^3$

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Q. if $u = x^3 - 3xy^2$ it is real part of A.f $f(z)$ then find $f(z)$

$\frac{\partial u}{\partial x} = 3x^2 - 3y^2$ $\qquad$ $\frac{du}{dy} = -6xy$

$\phi(z,0) = 3z^2 - 0$ $\qquad$ $\phi(z,0) = 0$

$$f(z) = \int_C (3z^2 - i0)\, dz + c$$

$$\boxed{f_z = \int [\phi_1(z,0) - i\phi_2(z,0)]\, dz + c}$$

$$f(z) = \frac{\cancel{3}z^3}{\cancel{3}} + c$$

$$= z^3 + c$$

Q. if $u = 3x^2y - y^3$ & it is img. part of A.f $f(z)$ then find

$\frac{dv}{dy} = 3x^2 - 3y^2$ $\qquad$ $\frac{dv}{dx} = 6xy$

$\phi(z,0) = 3z^2$ $\qquad$ $\phi(z,0) = 0$

$$\boxed{f(z) = \int [\phi_1(z,0) + i\phi_2(z,0)]\, dz + c}$$

$$f(z) = \int_C (3z^2 - i0)\, dz + 0$$

$$f(z) = z^3 + c$$

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Q: Given $(u-v) = (x-y)(x^2+4xy+y^2)$ find A.F $f(z) = u+iv$

$$\boxed{\begin{aligned}(u-v) &= U\\ (u+v) &= V\end{aligned}}$$

$f(z) = u+iv$ —— ①

$if(z) = iu - v$ —— ②

$f(z) = u+iv$

$if(z) = iu + i^2 v$

$iu - v$

1+2

$(1+i)f(z) = (u-v) + i(u+v)$

$F(z) = U + iV$

$\therefore U = (x-y)(x^2+4xy+y^2)$

$$\frac{dU}{dx} = x^2 + 4\cancel{xy}^{\,0} + \cancel{y^2}^{\,0} + (x-y)(2x+4y)$$

$\phi_1(z,0) = z^2 + (z-0)(2z+0)$
$= 3z^2$

$$\frac{dU}{dy} = -(x^2+4xy+y^2) + (x-y)(4x+2y)$$

$\phi_2(z,0) = -(z^2) + (z-0)(4z)$
$= 3z^2$

$$\boxed{F(z) = \int(\phi_1(z,0) - i\phi_2(z,0))\,dx + c}$$

$F(z) = \int(3z^2 - i\,3z^2)\,dz + c$

$(1+i)f(z) = 3(1-i)\int z^2\,dz + c$

$f(z) = \frac{(1-i)}{(1+i)} \cdot z^3 + \frac{c}{(1+i)}$

$= \frac{(1-i)(1-i)}{1-i^2} \cdot z^3 + \frac{c}{(1+i)}$

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$$f(z) = \left(\frac{\cancel{1} - \cancel{i} + \cancel{i^2}}{\cancel{i}}\right) z^3 + \frac{c}{1+i}$$

$$f(z) = -iz^3 + K$$

Q. $u + v = \dfrac{2\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x}$ find A.F

$f(z) = u + iv$

$\therefore \cosh 2y = \dfrac{e^{2y} + e^{-2y}}{2}$

$e^{2y} + e^{-2y} = 2\cosh 2y$

$u + v = \dfrac{2\sin 2x}{2\cosh 2y - 2\cos 2x}$

$= \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

$\therefore (1+i) f(z) = (u - v) + i(u + v)$

$F(z) = U + iV$

$V = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

$\dfrac{\partial V}{\partial y} = -\dfrac{\sin 2x \cdot 2\sinh 2y}{(\cosh 2y - \cos 2x)^2}$

$\therefore \left(\dfrac{1}{y}\right) dy = \dfrac{-1}{y^2}$

$\phi(z, 0) = 0$

$d\left(\cosh 2y\right) = 2\sinh 2y$

$$\frac{\partial v}{\partial x} = \frac{(\cosh 2y - \cos 2x) \cdot 2\cos 2x - \sin 2x\,(0 + 2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$\phi(z,0) = \frac{(1 - \cos 2z) \cdot 2\cos 2z - 2\sin^2 2z}{(1 - \cos 2z)^2}$$

$$= \frac{2\cos 2z - 2\cos^2 2z - 2\sin^2 2z}{(1 - \cos 2z)^2}$$

$$= \frac{2\cos 2z - 2}{(1 - \cos 2z)^2}$$

$$= -2\frac{\cancel{(-\cos 2z + 1)}}{(1 - \cos 2z)^{\cancel{2}}}$$

$$\phi(z,0) = \frac{-2}{1 - \cos 2z}$$

Bilinear transformation

Transform of type $w = \dfrac{az+b}{cz+d}$ where a, b, c, d are real or complex const. st $ad - bc \neq 0$

Q. find BLT that maps $z = 1, i, -i$ into $w = i, 0, -i$. Hence find image of $|z| < 1$

$$\frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)} = \frac{(w - w_1)(w_2 - w_3)}{(w_1 - w_2)(w_3 - w)}$$

$z_1 = 1$, $z_2 = i$, $z_3 = -i$

$w_1 = i$, $w_2 = 0$, $w_3 = -i$

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$$= \frac{(z-1)(i+1)}{(1-i)(-1-z)} = \frac{(w-i)(\cancel{0+i})}{(\cancel{i-0})(-i-w)}$$

$$\left(\frac{z-1}{z+1}\right)\left(\frac{1+i}{1-i}\right) = \frac{w-i}{w+i}$$

divide & multiply by $\left(\frac{1+i}{1+i}\right)$

$$\left(\frac{z-1}{z+1}\right)\frac{(1+i)^2}{(1^2-i^2)} = \frac{w-1}{w+1}$$

$$\left(\frac{z-1}{z+1}\right)\frac{\cancel{2}i}{\cancel{2}} = \frac{w-1}{w+1}$$

$$(zi-i)(w+1) - (w-1)(z+1) = 0.$$

$$z[iw-1-(w-i)] = i(w+i)+(w-i)$$

$$z[(i-1)w+(i-1)] = (i+1)w-(i+1)$$

$$z = \frac{(i+1)}{(i-1)}\frac{[w-1]}{[w+1]}$$

$$= \frac{(i+1)}{(i+1)}\frac{(i+1)}{(i-1)}\frac{[w-1]}{[w+1]} = z = \frac{\cancel{2}i}{-\cancel{2}}\left[\frac{w-1}{w+1}\right]$$

$$z = i\left[\frac{1-w}{1+w}\right]$$

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Q. find BLT of $z = 1, i, -1$ and $w = 0, 1, \infty$

$$\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)} = \frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}$$

$$\frac{(z-1)(i+1)}{(1-i)(-1-z)} = \frac{(w-0)(1-w_3)}{(0-1)(w_3-w)}$$

$$\left(\frac{z-1}{z+1}\right)\frac{1+i}{(1-i)}\frac{(1+i)}{(1+i)} = \frac{w\, \cancel{w_3}\,(1/w_3 - 1)}{\cancel{w_3}\,(1 - w/w_3)}$$

$$\frac{2i(z-1)}{2(z+1)} = \frac{-w}{1}$$

$$w = \frac{-i(z-1)}{z+1}$$

$$= \frac{i(1-z)}{z+1}$$

Q. $w = \dfrac{2z+3}{z-4}$ maps the circle $\boxed{x^2+y^2-4x=0}$ into st line $4u+3$

$x^2+y^2-4x=0$

center $(2,0)$

Radius $= \sqrt{4+0-0}$.

$\therefore$ center $= (-f, -g)$

$\therefore$ radius $= \sqrt{f^2+g^2-h}$

radius $= 2$

equation of circle

$|z-c| = R$

$\therefore c = x+iy$.

$|z-2| = 2$ —— (1)

Given

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$$w = \frac{2z+3}{z-4}$$

$$w(z-4) = 2z+3$$
$$wz - 4w - 2z - 3 = 0$$
$$(w-2)z = 4w+3$$
$$\boxed{z = \frac{4w+3}{w-2}}$$

$\rightarrow$ Put in eq^n 1

$$|z-2| = 2 \Rightarrow \left|\frac{4w+3}{w-2} - 2\right| = 2$$

$$= \left|\frac{4w+3-2w+4}{w-2}\right| = 2$$

$$= \left|\frac{2w+7}{w-2}\right| = 2$$
$$= |2w+7| = 2|w-2|$$
$$= |2w+7|^2 = 4|w-2|^2$$
$$= (2w+7)(2\bar{w}+7) = 4(w-2)(\bar{w}-2)$$
$$= \cancel{4w\bar{w}} + 14(w+\bar{w}) + 49 = 4(\cancel{w\bar{w}} - 2(w+\bar{w}) + 4)$$

$$\because |w|^2 = w\bar{w} = 1$$
$$|z|^2 = z\bar{z} = 1$$

$$= 22(w+\bar{w}) + 33 = 0$$
$$2(w+\bar{w}) + 3 = 0$$

$$\therefore w = u + \cancel{iv}$$
$$+\ \bar{w} = u - \cancel{iv}$$
$$2u$$

$$= 2(2u) + 3 = 0$$
$$4u + 3 = 0 \quad \underline{H.P}$$

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Integral line

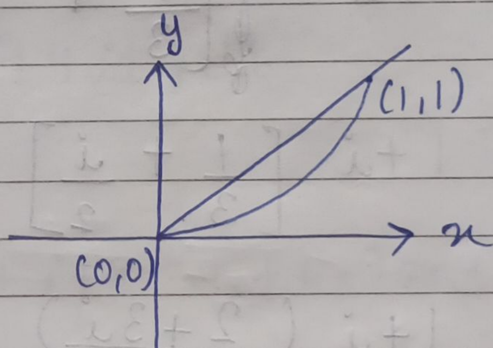
Consider $f(z)$ is a complex func defined for some z on a continous

$$\int_C f(z)\,dz = \int_C (u+iv)(dx+idy)$$

Q: find value of $\int_0^{1+i} (x^2+iy)\,dz$

i) along $y=x$, ii) along $y=x^2$

along $y=x$



$dy = dx$

$z = x+iy$
$dz = dx + idy$

$$= \int_0^{1+i} (x^2+iy)(dx+idy)$$

$$= \int_{x=0}^{1} (x^2+ix)(dx+idx)$$

$$= (1+i)\int_0^1 (x^2+ix)\,dx$$

$$= (1+i)\left[\frac{x^3}{3}\right]_0^1 + \left[\frac{ix^2}{2}\right]_0^1 = (1+i)\left[\frac{1}{3}+\frac{i}{2}\right]$$

$$(1+i)\frac{(2+3i)}{6} \Rightarrow \frac{(2-3)+5i}{6}$$

$$= \frac{-1+5i}{6}$$

$(x^2+x^2) i 2x$

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along $y = x^2$

$dy = 2x\,dx$

$$\int_{(0,0)}^{(1,1)} (x^2 + iy)(dx + i\,dy) = \int_0^1 (x^2 + ix^2)(dx + i2x\,dx)$$

$$= (1+i)\int_0^1 (x^2 + 2ix^3)\,dx$$

$$= 1+i \left[\frac{x^3}{3} + \frac{\cancel{2}ix^4}{\cancel{4}2}\right]_0^1$$

$$= 1+i \left[\frac{1}{3} + \frac{i}{2}\right]$$

$$= 1+i \left(\frac{2+3i}{6}\right) = \frac{2+3i+2i-3}{6}$$

$$= \frac{-1+5i}{6}$$

Q2. Evaluate $\int_{1-i}^{2+i} (2x + iy + i)\,dz$

along Path

1) $x = t+1$, $y = 2t^2 - 1$

2) st line $1-i$ & $2+i$

$\therefore z = x+iy$

$dz = dx + i\,dy$

Ans: $\int_{(1,-1)}^{(2,1)} (2x + iy + 1)(dx + i\,dy)$

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① along $x = t+1$, $dx = dt$
$y = 2t^2 - 1$, $dy = 4t\,dt$

$$\int_{(1,-1)}^{(2,1)} (2x + iy + 1)(dx + i\,dy) = \int$$

$$= \int_{t=0}^{1} [2(t+1) + i(2t^2-1) + 1]\,dt + 4it\,dt$$

$$= \int_0^1 [2t+3] + i(2t^2-1)]\,[1 + 4it]\,dt$$

$$= \int_0^1 [(2t+3) - 4t(2t^2-1)] + i[2t^2-1 + 4t(2t+3)]\,dt$$

$$= \int_0^1 (-8t^3 + 6t + 3) + i\,[10t^2 + 12t - 1]\,dt$$

$$= 4 + \frac{25i}{3}$$

② St line at $\underset{x_1\ \ y_1}{1-i}$ & $\underset{x_2\ \ y_2}{2+i}$

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$\frac{y-2}{1+1} = \frac{x-1}{2-1} = t \text{ (let)}$$

$(x-1) = t$
$x = t+1$

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$x = t+1$
$y = 2t+2$

$(y-2)(2-1)$
$= 2y - y - 4 + 2$
$= y - 2 = t$
$y = t+2$

Cauchy's Integral Thm:-

If a complex function f(z) analytic on a simple closed curve c then

$$\int_C f(z)\,dz = 0$$

Proof:-

Let R be the region of curve 'c'

$$\oint_C f(z)\,dz = \int_C (u+iv)(dx+idy)$$

$$= \int_C (udx - vdy) + i(udy + vdx)$$

Cauchy's Integral Formula

If f(z) is analytic within and on a closed surface 'c' & 'a' is any point within 'c' then

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)}\,dz$$

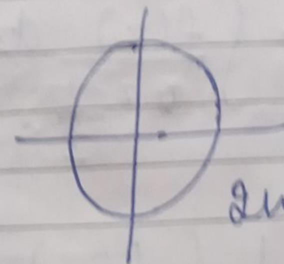
Circle

$$\int \frac{f(z)\,dz}{z-a} = 2\pi i f(a)$$

Q. $\int \frac{z^3 - 6}{3z - i} \, dz$, $|z| = 1$
radius

$3z - i = 0$

$z = i/3$, $|z| = 1/3$



Inside

Let $f(z) = z^3 - 6$.

$\frac{1}{3} \int \frac{f(z)}{z - i/3}$

$= \frac{1}{3} 2\pi i \, f(i/3)$

$= \frac{2\pi i}{3} \left(\frac{i}{3}\right)^3 - 6$

$= \frac{2\pi i}{3} \left(\frac{-i}{27} - 6\right)$

$= \frac{-2\pi i}{3} \left(\frac{i + 162}{27}\right)$

Q. $\int \frac{\cos 2\pi z}{(2z-1)(z-3)} \, dz$ $|z| = 1$

$\frac{1}{(2z-1)(z-3)} = \frac{A}{2z-1} + \frac{B}{z-3}$

$A = \frac{1}{\frac{1}{2} - 3} = \frac{-2}{5}$

$B = \frac{1}{5}$

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$$= f(z)\left(\frac{-2}{5}\frac{1}{2z-1} + \frac{1}{5}\frac{1}{(\cancel{2z}+3)}\right)dz$$

$$= \frac{-1}{5}\int \frac{f(z)}{z-\frac{1}{2}}dz + \cancel{\frac{1}{5}\int_C \frac{f(z)}{z-3}dz} \to 0$$

↳ outside

$$= -\frac{1}{5}\int \frac{f(z)}{z-\frac{1}{2}}dz \Rightarrow -\frac{1}{5}\,2\pi i f\left(\frac{1}{2}\right) + 0$$

* Singularity.

Zeroes of func

$f(z) = \cancel{\frac{z^2}{z^2}}\ \frac{z^2}{z-1}$, $z = 0$

$f(z) = \sin z$

$z = n\pi$

$f(z) = \cos z$

$z = (2n+1)\frac{\pi}{2}$

Order of zero

$f(z) = z^2$ $f''(z) = 2$

$f(0) = 0$ $f''(0) \neq 0$

$f'(z) = 2z$

$f(0) = 0$ Then the order is '2'

Regular Point -

$$f(z) = e^z$$

SINGULAR POINT:-

A point $z=a$ is called a singular point for $f(z)$ if $f(z)$ is not analytic at $z=a$

$f(z) = \frac{1}{z-1}$ is not an analytic at $z=1$

so $z=1$ is singular point

Isolated

Removable — Pole — Essential

1) Removable

$$f(z) = \frac{\sin z}{z} \; ; \; z \neq 0$$

Q. $\dfrac{e^{z}-1}{z}$

e^z series

$$\frac{\cancel{1}+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\ldots\ldots \cancel{-1}}{z}$$

$$\frac{\cancel{z}}{\cancel{z}}\left(1+\frac{z}{2!}+\frac{z^2}{3!}+\ldots\ldots\right)$$

Q. $\operatorname{cosec} z - \frac{1}{z} = f(z)$

$$f(z) = \frac{1}{\sin z} - \frac{1}{z}$$

$$f(z) = \frac{z - \sin z}{z \sin z}$$

$$= \lim_{z \to 0} \frac{z - \sin z}{z \sin z}$$

$$= \lim_{z \to 0} \frac{1 - \cos z}{1 \cdot \sin z + z \cdot \cos z}$$

$$= \lim_{z \to 0} \frac{\sin z}{\cos z - z \sin z + \cancel{\cos z}\, \cos z}$$

$$= \frac{0}{2 - 0} = 0$$

or
series expantion

$$f(z) = \frac{z - \sin z}{z \sin z}$$

$$= \frac{z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots \right)}{z \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots \right)}$$

$$\frac{\left(\frac{z^3}{3!} - \frac{z^5}{5!} + \dots \right)}{z^2 \left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots \right)}$$

$$= \frac{\cancel{z^2}}{\cancel{z^2}} \frac{\frac{z}{3!} - \frac{z^3}{5!}}{1 - \frac{z^2}{3!} + \frac{z^4}{5!}}$$

*

Polar Form Ques.

$$z = \frac{(2-3i)(1+i)}{(2+i)}$$

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

↳ Real ↳ Img.

$$\frac{(2-3i)(1+i)}{(2+i)} \times \frac{(2-i)}{(2-i)} = \frac{-9-7i}{5}$$

$$= -\frac{9}{5} - \frac{7}{5}i$$

$$|z| = \sqrt{\left(\frac{-7}{5}\right)^2 + \left(\frac{-9}{5}\right)^2} = \frac{\sqrt{130}}{5}$$

$$\theta = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{-7}{\cancel{5}} \times -\frac{\cancel{5}}{9}\right|$$

$$= \tan^{-1}\left|\frac{7}{9}\right|$$

$\pi - \theta$	θ
$-(\pi - \theta)$	$-\theta$

$$\arg(z) = -(\pi - \theta)$$

$$= -\left(\pi - \tan^{-1}\left|\frac{7}{9}\right|\right)$$

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Q. $1 - \cos\alpha + i\sin\alpha$

$x = 1 - \cos\alpha$
$y = \sin\alpha$

$r = \sqrt{(1-\cos\alpha)^2 + (\sin\alpha)^2}$

$= \sqrt{1 + \underbrace{\cos^2\alpha} - 2\cos\alpha + \underbrace{\sin^2\alpha}}$

$= \sqrt{1 + 1 - 2\cos\alpha}$
$= \sqrt{2 - 2\cos\alpha}$ ... continue.

De-Moivre's Theorem

$z = r(\cos\theta + i\sin\theta)^n$

for any real no. n

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$
$(\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta$

Taylor's Series :-

$f(z) = f(a) + \frac{z-a}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a)$

$+ \ldots \frac{(z-a)^n}{n!} f^n(a)$

Q. $f(z) = \sin z$
Expand ~~did~~ at $z = \pi/4$ $\therefore a = \pi/4$

$f(z) = \sin z$ $\quad f(\pi/4) = \frac{1}{\sqrt{2}}$

$f'(z) = \cos z$ $\quad f'(\pi/4) = \frac{1}{\sqrt{2}}$

$f''(z) = -\sin z$ $\quad f''(\pi/4) = \frac{-1}{\sqrt{2}}$

$$\sin z = \frac{1}{\sqrt{2}} + \left(z - \frac{\pi}{4}\right)\left(\frac{1}{\sqrt{2}}\right) + \frac{\left(z - \pi/4\right)^2}{2!}\left(\frac{-1}{\sqrt{2}}\right)$$

Q2. $f(z) = \frac{1}{z^2 - 4z + 3}$ abt point $z = 4$

$z - 4 = 0$

let